ULTIMATE STRENGTH OF FIBRES AND FIBROUS BUNDLES

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The main models for description of fibers ultimate strength based on the probabilistic approach are discussed. For identification of fiber strength type and estimation of corresponding parameters the modified quantile regression is proposed. The bundle strength predictions based on the simplest approach of uniform share of loading and knowledge of fiber strength distribution is described. The simulation approach starting from reliability of parallel system is used as well. These predictions are used for estimation of basalt roving strength. Predicted values are compared with experimental data.

1. INTRODUCTION

Strength at break is one of basic properties of fibers. This parameter is important both for textile technologists and textiles designers. Generally it is assumed that fiber strength is in nature stochastic variable and corresponding distribution confirm to mechanisms of failure. Classical theories lead to unimodal distributions skewed obviously to the right [1].

For polymeric materials, where more types of cracks appear, the polymodal strength distribution results. Number of modal values is indicator of specific defects (obviously surface defects and volume ones) [2]. In contribution [3] the discrete spectrum of defects has been identified. By the proper statistical technique the polymodality has not been proved for modified PES, carbon, aromatic polyamides and ceramic fibers [4–5]. These fibers exhibit typically unimodal and very broad tensile strength distribution by the risk functions R(σ).

This contribution is devoted to the selection of risk function of failure R(σ) for description of the tensile breaking strength σ distribution. For parameters estimation and right model selection the method based on the order statistics and nonlinear regression is proposed. The simple models for prediction of bundle strength are discussed. These predictions are used for estimation of basalt roving strength. Predicted values are compared with experimental data.

2. STATISTICAL ANALYSIS OF FIBRES STRENGTH

The fracture of fibers can be generally described by the micro mechanical models or on the base of pure probabilistic ideas [2]. The probabilistic approach is based on these assumptions:

(i) fiber breaks at specific place with critical defect (catastrophic flaw),

(ii) defects are distributed randomly along the length of fiber (model of Poisson marked process),

(iii) fracture probabilities at individual places are mutually independent.

The cumulative probability of **non-fracture** C(V,σ) depends on the tensile stress level σ and fiber volume V. For very small body (V→ 0) no defects are present and therefore C(0,σ) = 1 is valid. For the very large body (V→ ∞) is C(∞,σ) = 0.

The simple derivation of the stress at break distribution described below is a modification of deductions of Kittl and Diaz [6]. By using of independence assumption the probability of non-fracture of body composed from volume V and volume ΔV without common points has the form

$$C(V + \Delta V,\sigma) = C(V,\sigma)\, C(\Delta V,\sigma) \qquad (1)$$

Eqn. (1) is based on the assumption of independence of non-fracture probability in volume V and in volume ΔV. By using of Taylor linearization the C(ΔV,σ) may be written as

$$C(\Delta V,\sigma) = C(0 + \Delta V,\sigma) = C(0,\sigma) + [dC(0,\sigma)/dC]\,\Delta V \qquad (2)$$

and the C(V + ΔV,σ) as

$$C(V + \Delta V,\sigma) = C(V,\sigma) + [dC(V,\sigma)/d\sigma]\Delta V \qquad (3)$$

Using eqns. (2) and (3) and the boundary condition C(0,σ) = 1, the following expression results

$$C(V + \Delta V,\sigma) = C(V,\sigma)\{1 + [dC(0,\sigma)/d\sigma]\Delta V\} = C(V,\sigma) + [dC(V,\sigma)/d\sigma]\Delta V \qquad (4)$$

After rearrangements of eqn. (4) the final form is obtained

$$\frac{dC(V,\sigma)/d\sigma}{C(V,\sigma)} = \frac{dC(0,\sigma)}{d\sigma} = -R(\sigma) \qquad (5)$$

The R(σ) is known as the specific risk function. This function is positive and monotonously increasing as C = (0,σ) must be negative. Therefore in eqn. (5) must be negative sign at the term R(σ). Integration of eqn. (7) with boundary condition C(0,σ) = 1 gives

$$C(V,\sigma) = \exp[-R(\sigma)] \qquad (6)$$

The cumulative probability of break $F(\sigma)$ is complement to the $C(V,\sigma)$. Then the distribution of stress at break is expressed as

$$F(\sigma) = 1 - \exp[-R(\sigma)] \quad (7)$$

For famous Weibull distribution [1] (model WEI3) has $R(\sigma)$ form

$$R(\sigma)] = [(\sigma - A)/B]^C \quad (8)$$

Here A is lower strength limit, B is scale parameter and C is shape parameter. For brittle materials is often assumed A = 0 (model WEI2).

Weibull models are physically incorrect due to unsatisfactory upper limit of strength $C(\propto,\sigma) = 0$. To overcome this limitation Kies [7] proposed more general risk function (model KIES) in the form

$$R(\sigma)] = [(\sigma - A)/(A_1 - \sigma)]^C \quad (9)$$

Here A_1 is upper strength limit. For brittle materials is again assumed A = 0 (model KIES2). Occasionally the single Weibull distribution is inconsistent with experimental data. A multi-risk model is then used for analysis of strength distribution. For a bimodal distribution (fracture is result of two distinct kinds of defects) with zero lower limiting strength the risk function is

$$R(\sigma)] = [(\sigma/B) + (\sigma/B_1)]^C \quad (10)$$

Generalization of Kies risk function has been proposed by Phani [8] (model PHA5)

$$R(\sigma) = \frac{[(\sigma - A)/B_1]^D}{[(A_1 - \sigma)/B_1]^C} \quad (11)$$

In this equation are C and D two shape parameters. It can be proved that the B and B_1 cannot be independently estimated. Therefore, the constraint $B_1 = 1$ is used in sequel. Simplified version of eqn. (4) has A = 0 (model PHA4). For well-known Gumbell distribution (GUMB) is $R(\sigma)$ expressed as

$$R(\sigma) = \exp[(\sigma - A)/B] \quad (12)$$

The selection of right $R(\sigma)$ depends critically on the estimated number of modes and on the presence or absence of non zero lower limiting strength.

3. ESTIMATION OF R(σ) TYPE AND PARAMETERS

Main aim of the statistical analysis of strength data σ_i i = 1,...N is specification of R(s) and estimate of its parameters. Owing to their special structure the parameters of Weibull type distributions can be estimated by using of the maximum likelihood, quantile based and moment based methods. Sometimes is attractive to combine these and other methods for simplification of estimation process. We propose quantile based methods for their simplicity Methods of this type use the so-called order statistics $\sigma_{(i)}$. Denote that $\sigma_{(i)}$ # $\sigma_{(i+1)}$ i = 1,...N–1. It is well known that $\sigma_{(i)}$ values are rough estimates of sample quantile function for probabilities [9]

$$P_i = F(\sigma_{(i)}) = \frac{i - 0.5}{N + 0.25} \quad (13)$$

By using of eqn. (8) and order statistics $\sigma_{(i)}$ the parameter estimation problem can be converted to the nonlinear regression task [10].

So-called Weibull transformation method uses the rearrangement of eqn. (8) for order statistics

$$\ln[R(\sigma_{(i)})] = \ln[-\ln(1 - P_i)] \quad (14)$$

The parameter estimates of $R(\sigma)$ model can be then obtained by nonlinear least squares, i.e., by minimizing of criterion

$$S(a) = \sum_{i=1}^{N} (y_i - \ln(R(\sigma_i))^2 \quad (15)$$

where $y_i = \ln[-\ln(1 - P_i)]$. Denote that graph of y_i on the $\ln(\sigma_{(i)})$ is so-called Weibull plot. This plot is for two parameter Weibull distribution straight line but for three parameter the concave curve results.

Strictly speaking, this method is based on the incorrect assumption that the y_i are uncorrelated random variables with constant variance. More logical is to use the estimated sample quantiles $\sigma_{(i)}$ as explained quantities. Corresponding least squares criterion for the quantile regression has the form

$$S(a) = \sum_{i=1}^{N} [\sigma_{(i)} - Q(Z_i)]^2 \quad (16)$$

where $Z_i = \exp(y_i)$ and $Q(Z_i)$ is theoretical quantile function. For three parameter Weibull distribution is $Q(Z_i)$ expressed as [9]

$$Q(Z_i) = A + BZ_i^{1/C} \quad (17)$$

For three parameter Kies model is valid

$$Q(Z_i) = \frac{A + A_i Z_i^{1/C}}{1 + Z_i^{1/C}} \quad (18)$$

and for Gumbell one is

$$Q(Z_i) = A + B\ln(Z_i) \quad (19)$$

According to the roughness of $\sigma_{(i)}$ and their no constant variances the special weights can be defined [9].

For selection of right risk function the statistical criteria for selection of the optimal regression model form can be used [10]. To distinguish between models with various number of parameters M the Akaike information criterion AIC is suitable

$$AIC = N \ln\left[\frac{S(a^*)}{N-M}\right] + 2M \qquad (20)$$

where S(a*) is minimal value of S(a). The best model is considered to be that for which this criterion reaches a minimum. The predictive ability of regression type models may be examined by the mean quadratic error of prediction

$$MEP = \frac{1}{N}\sum_{i=1}^{N}[y_i - f(x_i, a_{(-i)})]^2 \qquad (21)$$

where $f(x_i, a)$ is model function. Parameters $a_{(-i)}$ are least squares estimates when all points except the i-th one were used. Criterion MEP is equal to the mean of the squared predictive residuals [10]. The best model with maximum predictive ability reaches a minimum of MEP

4. BUNDLE STRENGTH

Let us consider a fibrous system where *n* fibers (or filaments) form a parallel bundle with no interaction between individual fibers. Daniels [11] developed theory to estimate the maximum strength of bundles using order statistics $\sigma_{(i)}$. The maximum strength of bundle made from N fibers would be defined by relation

$$(N - i + 1)\sigma_{(i)} \geq (N - i)\sigma_{(i+1)} \qquad (22))$$

Peirce [12] examined five models in relation to the strength of bundles. His second model requires uniform tension among the fibers and is based on the distribution of breaking load. Maximum load *P* occurs when number *M* fibers of the *n* ones breaks. Let the fibers have ultimate strength distribution characterized by probability density function **(pdf)** p(s) and cumulative probability function **(cdf)** F(σ). For large *n* is then valid

$$\frac{n-M}{n} = 1 - F(z) \quad \frac{P}{n} = z[1 - F(z)] \\ z = [1 - F(\sigma)]/p(\sigma) \qquad (23)$$

For the Weibull distribution (see eqn. (5)) is valid

$$z = BC^{-(1/C)} \qquad (24)$$

Daniels [11] extended Pierce's work (fibers have the same elongation characteristics, and share the load equally). The strength distribution of bundle is approaching to the normal distribution for large *n* independently on the distribution of fibers probability density function. The expected bundle strength is

$$E(\sigma_B) = nz[1 - F(\sigma)] \qquad (25)$$

and the standard deviation is

$$D(\sigma_B) = z\sqrt{F(z)n[1 - F(z)]} \qquad (26)$$

Here z is the value maximizing σ[1 – F(σ)]. For Weibull distribution is z defined by eqn. (24). Form limiting normal distribution of bundle strength and Weibull cdf of fibers is then mean bundle strength

$$E(\sigma_B) = BC^{-(1/C)}\exp(-1/C) \qquad (26)$$

Harter [13] provided an exact formula for expectation of Weibull order statistics in the form of series

$$E(\sigma_{(i)}, B, C, n) = Bn\binom{n-1}{i-1}\Gamma(1 + 1/C) \times \\ \times \sum_{j=0}^{i-1}\binom{i-1}{j}\frac{(-1)^{1+J-1}}{(n-j)^{1+1/C}} \qquad (27)$$

These mean values can be substituted into relation (22) instead of values $\sigma_{(i)}$ and the maximum bundle strength is value fulfilling this inequality.

Simulation based computation of bundle strength based on the reliability defines bundle as system composed from parallel-organized units. The reliability is understood as a resistance of the system against a load applied to it. It is assumed that reliability is tested in such a way that the load increases from 0 to the level causing the failure of all units or up to maximal load. Further it is assumed that the experiment is relatively fast, so that the time of duration of the load does not influence the survival. The standard survival analysis approach and counting processes models are used, however, instead of time-to-failure, the breaking load of fibers is variable of interest. The concept and relevant theory of counting processes is described in the book [14]. Let the survival of fibers is described by i.i.d. random variables U_j j = 1..m with distribution given by f(u), F(u), h(u), H(u) denoting the density, distribution function, hazard function and cumulative hazard function, respectively. It is assumed that at each moment the force applied to the fiber is divided equally among the (unbroken) ones. The global force stretching the fiber is observed. However, as the break of fiber leads to an immediate re-distribution of the force to the other fibers (so that to the abrupt increase of the force affecting each individual fibers), the consequence can be the break of several of remaining fibers. For such a set of fibers broken practically at the same moment the precise level of the strength causing the break of some of them is actually not know. Thus, a part of data is interval-censored. If the sufficient number of fibers is observed the sufficiently large set of uncensored data are registered. Let the bundle of *n* identical and independent fibers are tested. Denote by U_i random variables - survivals, by N_i (u), I_i(u) related individual counting and indicator processes for the i-th filament (i = 1...n). Further denote

$$N(u)=\sum_{i=1}^{n} N_i(u) \quad I(u)=\sum_{i=1}^{n} I_i(u) \qquad (28)$$

The common estimator of the cumulative hazard function is the Nelson-Aalen one

$$\hat{H}_N(u)=\int_0^u \frac{dN(v)}{I(v)} \qquad (29)$$

where is set 0/0 = 0. The ability of the estimator to approximate well the true H(u) depends on the indicator processes for all values of strength u in the interval of interest. Proof of asymptotic uniform consistency and asymptotic normality of this estimator is derived in [15].

5. EXAMPLE

Basalt rocks from VESTANY hill were used as a raw material. The roving contained 280 single filaments were prepared. Mean fineness of roving was 45 tex. Diameter of filament was 8.63 [μm]. The individual basalt filaments removed from roving were tested. The loads at break were measured under standard conditions at sample length 10 mm. Load data were transformed to the stresses at break σ_i [GPa]. The sample of 50 stresses at break values was used for evaluation of the R(σ) functions and estimate of their parameters. Model proposed by Phani (eqn. (11)) leads to the parameter A without physical sense. Model PHA4 is more realistic but the shape estimates are very high. Kies type models (eqn. (9)) are here not better that three parameter Weibull one. The differences between MEP for WEI3 and WEI 2 are very small and therefore the simpler WEI2 has been selected. Parameters of this model are B = 3.01[GPa] and C = 1.83. The mean strength value for WEI2 is 2.67 GPa.

For roving strength measurements the TIRATEST 2300 machine was used. The 50 samples of strengths P_i were collected. These values were recalculated to stress at break values σ_i **[GPa]**. The strength distribution of tempered multifilament roving was nearly normal with parameters: mean σ_p = **1.02 GPa** and variance s^2 **= 0.0075 [GPa]^2**.These parameters were estimated as sample arithmetic mean and sample variance.

Bundle strength predicted from eqn. (26) is E(σ_B) = **1**. This value is very close to experimental one. From practical point of view is probably experimental value too small because the part of fibers was crushed in jaws of testing machine. Number of broken fibers at break computed from eqn (24) is **M = 118.**

The proposed simulation based model was used for prediction of the survival of bundle when the survival distribution of fibers is Weibull with known parameters. Though the overall survival can be derived from the order statistics distribution, its computation is generally complicated. The Monte Carlo simulation has been therefore used. Based on the 3000 simulations for model WEI2 the mean value **ES(σ) = 2.21 GPa** and standard deviation SS(s) = 0.22 have been computed. These values seem to be more realistic in comparison with asymptotic results.

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7. REFERENCES

[1] Weibull W.: J.Appl. Mech. **18**, 293 (1951)
[2] Goda K., Fukunaga H.: J.Mater.Sci. **21**, 4475 (1986)
[3] Baranova S.A. et. all..: Acta Polymerica **36**, 385 (1985)
[4] Militký J., Kovacic V.: Proc. Conf. IMTEX'95, Lodz, May 1995
[5] Militký J et, all: Modified Polyester Fibers, Elsevier 1992
[6] Kittl P., Díaz G.: Res. Mechanica **24**, 99 (1988)
[7] Kies J. A.: NRL Rept 5093,Naval Research Lab., Washington DC (1958)
[8] Phani K. K.: J. Mater. Sci. **23**, 2424 (1988)
[9] Meloun M., Militký J., Forina M.: Chemometrics for Analytic Chemistry vol I, Statistical Data Analysis, Ellis Horwood, Chichester 1992
[10] Meloun M., Militký J., Forina M.: Chemometrics for Analytic Chemistry vol II, Regression and related Methods, Ellis Horwood, Hempstead 1994
[11] Daniels H.E.: Proc. Roy. Soc. London **A183**, 405, (1945)
[12] Peirce F.T.:J. Text. Inst. **17**, 355 (1928)
[13] Harter H.L. : Order Statistics and their use, US governement Printing Office 1970
[14] Anderson P.K., Borgan O., Gill R.D. and Keiding N.: Statistical models based on Counting Processes, Springer New York 1993
[15] Volf P., Linka A.: Two applications of Counting Processes, Rept. UTIA Praha No, 1935, 1998