Aggregation and Definition of an Algebraic Framework for Fuzzy Time Series: An Application in the Supply-Demand Domain

Published in:
International Journal of Approximate Reasoning

Publication Status:
Published (in print/issue): 01/10/2022

DOI:
10.1016/j.ijar.2022.07.005

Document Version
Author Accepted version

Aggregation and Definition of an Algebraic Framework for Fuzzy Time Series: An Application in the Supply-Demand Domain

Luis Rodriguez-Benitez[a,*], Juan Moreno-Garcia[1], Ester del Castillo-Herrera[a], Jun Liu[c], Luis Jimenez-Linares[a]

[a] *Department of Information and System Technologies, Escuela Superior de Informática, University of Castilla-La Mancha, Ciudad Real, Spain*

[b] *Department of Information and System Technologies, Escuela de Ingeniería Industrial y Aeroespacial, University of Castilla-La Mancha, Toledo, Spain*

[c] *School of Computing, University of Ulster, Northern Ireland, UK*

Abstract

This paper proposes a method for aggregating the information contained in sets of Time Series (TS) into a Fuzzy Time Series (FTS). First, an aggregation technique is defined, which is based on the algorithm known as Kernel Density Estimation (KDE) which reconstructs the probabilistic density function of a set of points, in this case a TS. Second, to operate with FTS, an algebraic framework is created based on Zadeh's extension principle and as a result of operating with FTS, a new FTS is obtained, which allows obtaining richer information and operating under conditions of uncertainty. Finally, the operations needed to compute the membership function of any TS in the aggregated FTS are introduced too. As a use case, it is proposed to work with sets of TS in the supply and demand domain, in such a way that information regarding the satisfaction of demand over time can be extracted. The specific application domain chosen will be that of the electricity market, analysing the consumption of buildings and their self-generation of energy to obtain information about the dependence on the electricity grid.

Keywords: Fuzzy Time Series, Time Series Aggregation, Zadeh's Extension

*Corresponding author

Email address: `Luis.Rodriguez@uclm.es` (Luis Rodriguez-Benitez)

Principle, Supply-Demand Time Series, Kernel Density Estimation

1. Introduction

A TS is a set of regular time-ordered observations of a quantitative characteristic of a phenomenon measured at successive points in time. Song and Chissom [1] laid the foundations of FTS, which has been used extensively in recent years. FTS values are fuzzy sets [2, 3] and there is a relationship between current time observations and previous time observations. The use of fuzzy sets for time series modelling allows not only to incorporate the ability of fuzzy models to approximate functions but also to add the readability associated with linguistic variables that makes them more accessible to the analysis of non-experts in the field of application. Chen [4] evolved the foundational work mentioned above by dividing the universe of discourse of the TS into intervals or partitions, which would be fuzzy sets. This allows learning the behaviour of each area by extracting rules from the patterns of the series that specify the way in which the partitions relate to each other along the time dimension. As an alternative to Type-1 FTS models where the membership functions are represented by ranges of values from zero to one and therefore each element of the universe of discourse is assigned a degree of membership in that range, we propose as degree of membership the use of fuzzy sets. This makes it possible to express more information and to address problems where uncertainty appears, in the specific case of TS, this uncertainty could be associated with noise or non-stationarity. In our proposal, sets of TS are summarised into a single FTS. This reduction of the data is based on the concept of aggregation and one of the greatest advantages of TS aggregation is that it allows the visualisation of the main trend of several series in a direct way by summarising several series by a single one. In this work, the aggregation is obtained by using KDE [5, 6], a non-parametric method to estimate the probability density function of a random variable. Once the sets of TS are summarized into a FTS, a framework is established for the operation between them. For this purpose, Zadeh's extension principle [2] is used to construct operations between FTS, i.e. to define the operability between FTS. Furthermore, the computation of the membership function of any TS to the aggregated FTS is defined. As a use case, our proposal is applied to a specific type of series: series with supply and demand data for a product or good. There is a large amount of work linked to this type of series in fields as diverse as energy [7, 8, 9], mobile crowdsourced service supply and demand at a given time and space [10], balance of higher education [11], supply chain management [12]. Energy is chosen as the field of application of this proposal.

1.1. Major contributions

As major contributions of this proposal we highlight:

- Novelties with respect to time series aggregation: We build on the concepts of Song's work [1] and move away from the use of classical, mainly statistical, aggregation techniques. This allows retaining information regarding the uncertainty, with membership functions during the whole process. Moreover, by not using fuzzy numbers as membership functions, some relevant information is not lost.
- Novelties in the representation of fuzzy time series: No labels are used, as they are in most of the current techniques, and this allows working directly with the fuzzy sets.
- Novelties with respect to the definition of operations between FTS. FTS is defined as an algebraic element in such a way that instead of using classical comparisons based on statistical measures, this proposal allows determining the degree of similarity at each instant t of the series by means of a value between 0 and 1. In addition, a specific operation to compute the degree of membership of any TS to the FTS resulting from the aggregation is also proposed.
- Novelties in the way that fuzzy sets are graphically represented: The way FTS are graphically presented is associated with the own method used

to generate them in the aggregation process, that is, KDE. Membership values are represented in terms of colour and size or thickness of lines.

To summarise, it can be noted that most existing methods tend to start from a fixed set of labels, and so the data, regardless of time t, are always fuzzified on the same linguistic variable. This leads directly to a loss of information from this process. In addition, the other main contribution of our proposal is the definition of an algebraic framework that allows performing operations on the series.

1.2. Structure of this paper

The rest of this paper is organised as follows: First, Section 2 details the background. Next, Section 3 introduces the theoretical proposal of this paper. Section 4 details the use case related to the electricity market and lastly, Section 5 presents some conclusions and several lines for future research.

2. Related Work

FTS has been mainly applied to manage nonlinear problems. The following is a series of articles that through the application of fuzzy techniques allow the resolution of problems with such characteristic. As topics, there can be found enrollment [13, 14, 15], temperature [16, 17, 18], reactors [19], the concentration of pollutant gases [20, 21], tourism [22, 23, 24] and aspects related to COVID-19 disease [25, 26], among others. But undoubtedly, a large majority of applications are focused on forecasting. A survey can be found in [27] where it is highlighted how there are certain characteristics of fuzzy time series that make them more suitable for forecasting than those that can be considered as classical. In addition, it is interesting to note that there are a number of common stages in the generation of forecasting models. First, the universe of discourse is defined, then data partitioning techniques are used, followed by a process of fuzzification of the data in order to identify fuzzy logical relationships. Finally, the last step in fuzzy time series modelling is defuzzification. Many studies do not cover all stages of the model but try to optimise the model by focusing on a single stage. Thus, among the data partitioning techniques, some are based on mathematical models [4, 28, 29, 30], others on optimisation techniques, among which the particle swarm optimisation algorithms [31, 32, 33, 34] undoubtedly stand out. Also at this stage, clustering algorithms are applied, which present variations for this application [35] with respect to the classical techniques. Fuzzy Logical Relationships (FLS) are linked to FTS, as they allow a relationship to be established between a current state and a previous one. There are a large number of works that analyse this type of relationship [36, 37, 38, 39]. Finally, with respect to defuzzification, its foundations were established in the work of Song and Chisson [1, 13]. A differentiation can be made between those studies using only Type-1 fuzzy sets where each element of the universe is assigned a precise degree of membership in the range of zero to one and those using Type-2 fuzzy sets that are able to handle the incorporation of the uncertainty normally associated with noisy or non-stationary conditions. Examples of the latter include [40, 41, 42, 43, 44]. ~~As a conclusion of this bibliographical review, it can be stated that the approach presented in this work is more linked to the Type-2 approach since one of the main objectives is to retain uncertainty by means of membership functions throughout the process.~~

Finally, we would like to highlight those works that are considered to be more related to our proposal regarding the introduction of the temporal component in the context of fuzzy logic. For example, [45] develops a generalization of negation, conjunction, and disconjunction for temporal intuitionistic fuzzy sets. In [46], Zadeh's possibility theory is used as a general framework for modelling temporal knowledge imbued with imprecision or uncertainty, and in [47], temporal fuzzy sets are defined and it is shown how they can be induced from the dynamic trajectory of a physical system.

3. Aggregation of TS and definition of operations between FTS

As Song and Chissom exposited in [1], the main difference between conventional TS and FTS is that the observations in the former are real numbers while those in the latter are fuzzy sets. A FTS, denoted by $F(t)$, can be defined as a collection of fuzzy sets $f_i(t)$ over the universe of discourse $Y(t)$, with $Y(t)$ being a subset of $\mathbb{R}$, $t = (0, 1, 2, \ldots)$. In this section, first we propose an aggregation technique of TS into a FTS (Section 3.1), second we establish a theoretical framework for operating with the obtained FTS (Section 3.2) and then a way to compute the membership function of a TS in a FTS is introduced (Section 3.3).

3.1. Aggregation of a set of TS into a Fuzzy Time Series

The first goal of this work is to aggregate a set of TS into a FTS. A collection of TS is represented as $S = \{s_0, \ldots, s_{n-1}\}$, where each TS is formalised as $s_i = \{v_{(i,t_0)}, v_{(i,t_1)}, \ldots, v_{(i,t_{m-1})}\}$ where t_j represents the time for each observation. As an example associated with the use case subsequently developed in this paper, the set S could represent the collection of the hourly electrical consumption of a building over 24 hours (several Thursdays), i.e., s_i with t_j =(0, 1,. . . ,23). The actual time interval respects a correspondence with the activity of consumption, so that the day is considered to start at 5 a.m. at the same time as the activity in the building. In the case of Thursdays, this interval of 24 hours runs from 5:00 a.m. on a Thursday to 4:00 a.m. on the following Friday. Figure 1 shows 19 TS corresponding to 19 Thursdays belonging to work weeks from 06/02/2014 to 03/07/2014, where, as explained, t_0=5 a.m. and t_{23}=4 a.m. Note that the date format used is Month/Day/Year.

Let the domain be $Y(t) = \{v_{(0,t)}, \ldots, v_{(n-1,t)}\}$ with $t \in$ $(t_0, \ldots, t_{m-1})$ and $DE_t(v)$ a density function which estimates the probability that v belongs to $Y(t)$. The fuzzy set f_t over the domain $Y(t)$ is defined as the membership function $\mu_t(v)$:

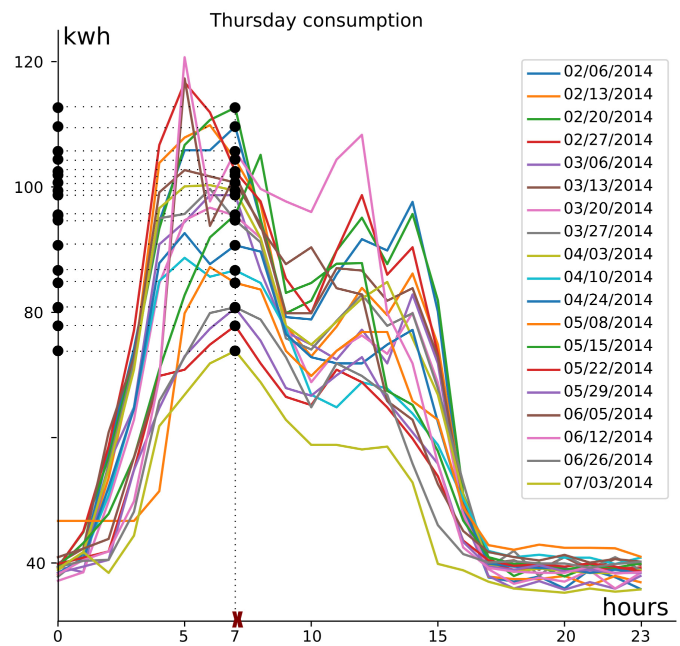

Figure 1: Thursday consumption TS and values of the domain $Y(7)$

$$\mu_t(v) = \frac{DE_t(v)}{max\{DE_t(z)|z \in Y(t)\}} \tag{1}$$

In this paper, the KDE technique [5, 6] is employed with Gaussian kernel functions to obtain the density functions DE_t. KDE expands the idea of the histogram, where each observation increases the probability density in the area where it is located, but does so in such a way that the contributions are grouped together to create a smooth continuous curve. The kernel is the function that determines how the influence of each observation is distributed. Therefore, it can have a significant impact on the estimation of the resulting density function. In the vast majority of cases, such as ours, a Gaussian kernel (normal distribution) is used. It must be noted that DE_t is used in the membership function as it reflects the distribution of values at a specific time instant and other membership functions could be used as long as they respect this assumption.

For instance, in Figure 2 it is shown how the fuzzy set f_7 is obtained within its membership function $\mu_7(v)$ being the domain $Y(7)$=(109.63, 104.4, 112.65, 102.47, 98.69, 100.69, 95.42, 94.68, 99.4 , 86.68, 90.68, 84.69, 95.56, 77.84, 80.68, 101.89, 105.68, 80.8, 73.84) that is the consumption at instant t_7 for the 19 TS.

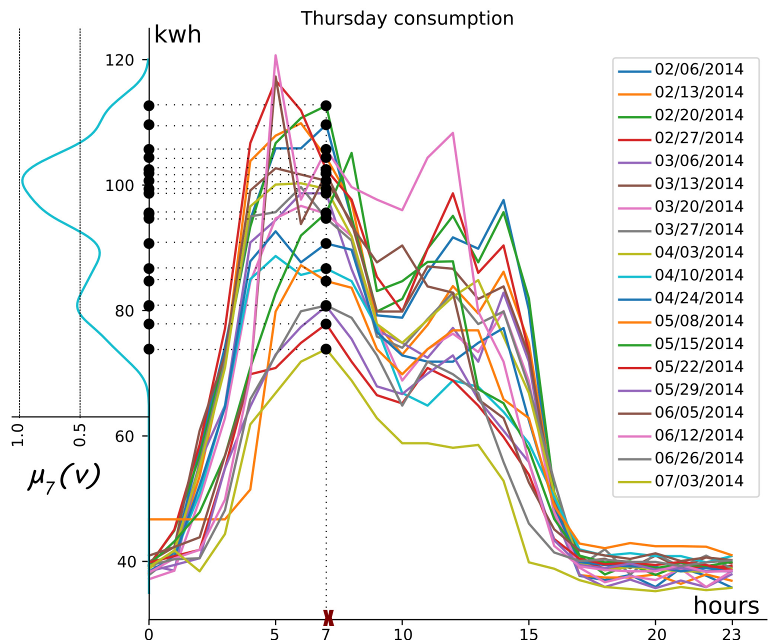

Figure 2: Set of Thursdays' consumption with the fuzzy set f_7 and its membership function $\mu_7(v)$

Now, once a fuzzy set f_t has been obtained for every instant t, there can be defined the FTS over the collection S: $FTS(S) = \{f_{t_0}, f_{t_1}, \ldots, f_{t_{m-1}}\}$. This definition is extended to an arbitrary set T with $t \subset T$ such that if $r \in T$, its membership function f_r is

$$\mu_r(v) = \begin{cases} \frac{DE_r(v)}{max\{DE_r(z)|z\in Y(r)\}} & r \in t \\ \mu_{t_{down}}(v)(1-\alpha) + \alpha\mu_{t_{up}}(v) & t_0 \leq r \leq t_{m-1}, r \notin t \\ 0 & r < t_0 \ or \ r > t_m \end{cases} \quad (2)$$

with $t_{down} = max_j\{j < r|j \in t\}$, $t_{up} = min_i\{i > r|i \in t\}$ and $\alpha = \frac{r-t_{down}}{t_{up}-t_{down}}$.

In short, if r is outside the range of values of t, it is 0, if r is in t, it is the value of the fuzzy set, and if r is between two values of t, it is the linear interpolation between them. Now, we try clarify this with an example. If we observe Figure 1, in these time series, the set of possible values for t is $\{0, 1, ..., 23\}$ and so the first case of the equation is applied when r takes a value in this set. For instance, to compute $\mu_5(v)$, $\mu_{22}(v)$, etc. The second case occurs for values between 0 and 23 not belonging to the set. For instance, for $\mu_{5.3}(v)$, $\mu_{7.5}(v)$, etc. This case is necessary because the time series may be continuous and because, in addition, the time instants for one series can be different than those for another series. Therefore, in a formal definition, this situation must be considered, although it must be emphasised that in our examples, as in our use case, the time instants for all series are the same. Finally, the third case takes into account values lower than 0 and higher than 23. In this case, for example $\mu_{33}(v)$, the membership function is always equal to 0.

As an example, Figure 3 shows the time series of electricity consumption for 19 working Thursdays, where the colour and width of the vertical lines indicate the degree of membership of the fuzzy sets f_t. To clarify this representation, in Figure 4 the membership functions $\mu_7(v)$ and $\mu_{10}(v)$ are shown. It can be observed how the values of membership closer to 1 for $\mu_7(v)s$ are around 100 kwh, while for $\mu_{10}(v)$, they are around 75kwh.

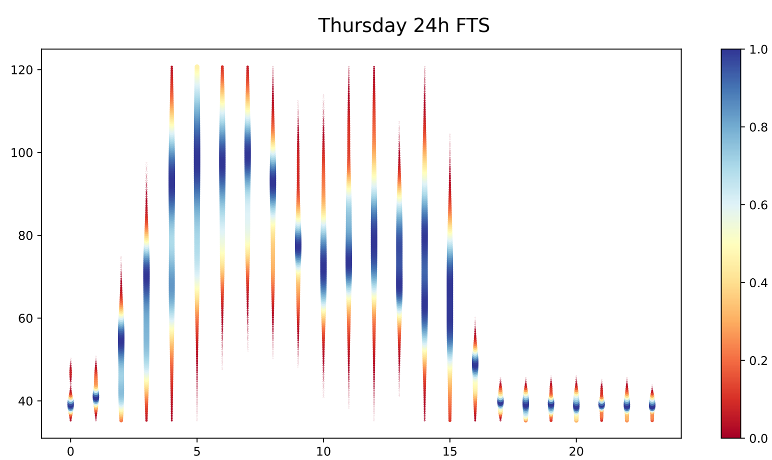

Figure 3: FTS result of the aggregation of the TS with the consumption of 19 Thursdays

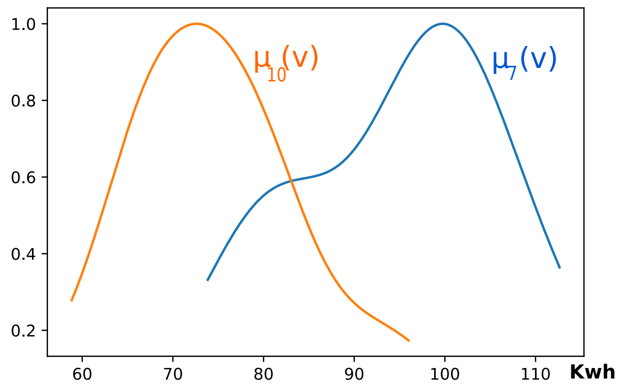

Figure 4: Components of the FTS $\mu_7(v)s$ and $\mu_{10}(v)$

3.2. Definition of the operations with FTS

Let A and B be two FTS with membership functions $\mu_A(t,x),\ x \in D_x$ and $\mu_B(t,y),\ y \in D_y$ and $T : D_x \times D_y :\rightarrow D_z$ be a function defined over the domain $D_x \times D_y$ with values in $D_z = T^{-1}(x,y)$. The extension of the function T to the domain of the FTS is made by defining $\tilde{T}(A,B) = Z$, where Z is a new FTS with membership function $\mu_Z(t,z),\ z \in D_z$ that follows the extension principle proposed by Zadeh, such that:

$$\mu_Z(t,z) = \bigvee_{x \in D_x, y \in D_y} \{\mu_A(t,x) \bigwedge \mu_B(t,y) | T(x,y) = z\} \tag{3}$$

where $\bigvee$ is a t-conorm and $\bigwedge$ a t-norm.

As an example of an operation, consider T, the division of FTS with t-conorm the maximum and t-norm the minimum. An example of this operation is shown in Figure 5. The FTS A represents the consumption of the building during 19 Thursdays while the FTS B corresponds to the generation using a photovoltaic infrastructure of 20 solar panels of 220 watts simulated with the library pvlib [48] ~~photovoltaic panels during these 19 Thursdays~~ ~~photovoltaic panels during these 19 Thursdays~~. The resulting FTS Z is obtained after dividing A by B as described in Equation 3. In this case the meaning of the division is the number of photovoltaic installations needed to satisfy the consumption of this building. So, the coordinate axis is not kwh, but the number of such installations. Therefore, it can be concluded that considering the photovoltaic infrastructure described above, to satisfy the energy demand during the period when the solar panels are capable of producing electricity, sunshine hours, a maximum of 40 installations would be needed, at dawn and dusk, while 22 installations would be sufficient during the central hours of the day, when the maximum peak of electricity generation is reached. It should be noted that t_0=5 a.m. and t_{23}=4 a.m. so the generation hours range from 7 a.m. to 6 p.m. in this particular example

3.3. Membership of a Time Series in a Fuzzy Time Series.

It is of interest to know if a TS can be considered as the prototype series of the aggregate set of series. For this, it is necessary to define a membership function of a TS to the FTS resulting from the aggregation.

Let $s' = v_{t_0}, \ldots, v_{t_{m-1}}$ be a TS defined on the same domain as the set S. Then the mean degree of membership of this series in the $FTS(S)$ is defined as follows:

$$\mu_{FTS(S)}(s') = \frac{1}{m} \sum_{i=0}^{m-1} \mu_{t_i}(v_{t_i}) \qquad (4)$$

4. Use Case: Characterisation of the electricity consumption of a building.

An example of the use of the proposed theory will now be presented. Subsequently, to demonstrate its ability to describe and synthesize FTS, as well as its operation, we will present two ways of characterising the patterns of the consumption of electricity of a public building based on an analysis of its consumption over the 153 days from Monday 02/03/2014 to Sunday 07/06/2014. To do so, Section 4.1 will present an example that compares the consumption series between different buildings. Then Section 4.2 will illustrate the search for a representative of the building's consumption. In addition, to strengthen the applicability in the context of supply and demand, a study of the adequacy of the generation and demand, also linked to public buildings, and the results of a

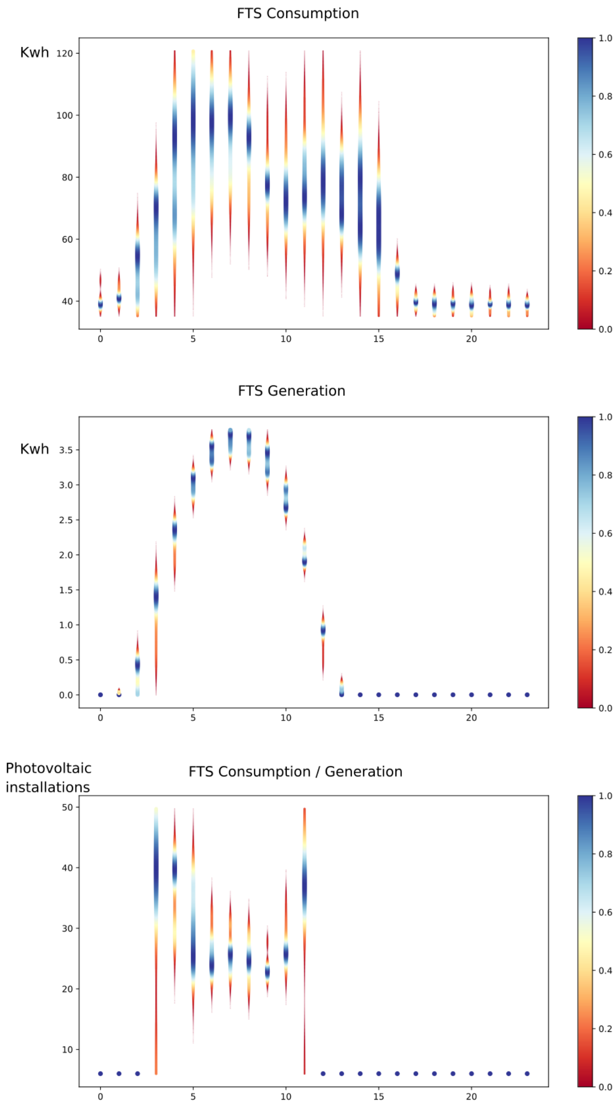

Figure 5: FTS obtained after dividing the Consumption of electricity by the Generation of electricity during 19 Thursdays

simulation of a generation using only photovoltaic panels, will be presented in Section 4.3.

4.1. Study of consumption on each day of the week, average consumption on workdays and weekends.

To carry out this study, we separate the set D (which has the data for 153 days) into 7 disjoint subsets: one for each day of the week.

$$D = d_{Monday} \cup \ldots \cup d_{Sunday} \tag{5}$$

In order to characterise the consumption of each day of the week, we obtain for each subset of D its $FTS(d_i)$, for $i \in \{Monday, Tuesday, \ldots, Sunday\}$, following the method described in detail in Section 3.1. This yields seven FTS, as shown in Figure 6. If these pictures are analysed, two clearly separate patterns of consumption can be identified.

1. Consumption on workdays, where, after an increase in consumption has started, the first local maximum is reached in the middle of the morning, falling to a local minimum at lunchtime, rising again to another local maximum in the afternoon, and falling to the minimum consumption, which will be maintained during the night and in the early hours of the morning.
2. Weekend consumption, where both Saturday and Sunday show a stable consumption that could be considered as the minimum consumption of the building. It should be noted that the existence of some sporadic activity on Saturday mornings means that the range of uncertainty is greater during these hours than on Sundays.

Having obtained a FTS for each day of the week, the two FTS, one reflecting the average electricity consumption for workdays and the other for the weekend, can be calculated by using Equationss 6 and 7, respectively. The seven FTS for each day and the two FTS with the average are shown in Figure 6. Note that the scale for weekdays is different from that for the weekends, as the consumption is much higher on weekdays, and using the same scale would not allow correctly observing the consumption on Saturday and Sunday.

$$FTS_{Workdays} = \frac{1}{5}(\sum_{i \in \{Monday, \ldots, Friday\}} FTS(d_i)) \tag{6}$$

$$FTS_{Weekend\_days} = \frac{1}{2}(FTS(d_{Saturday}) + FTS(d_{Sunday})) \tag{7}$$

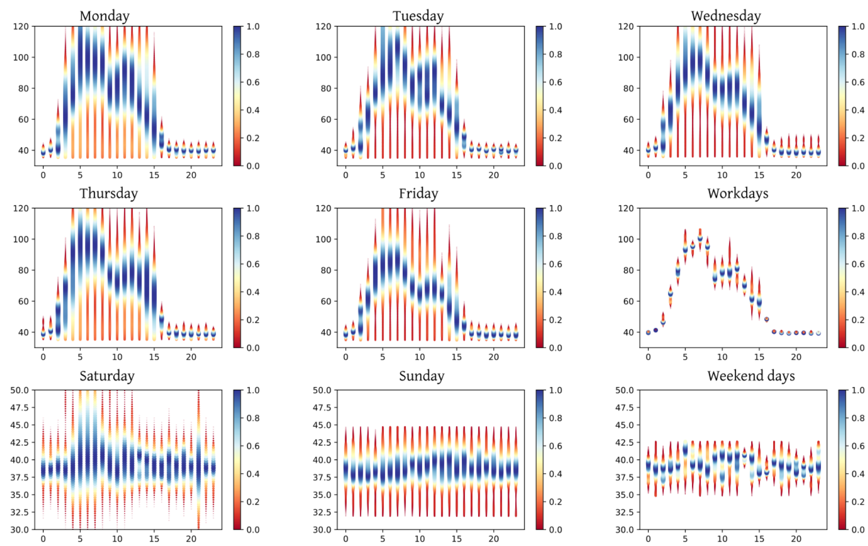

Figure 6: Seven daily FTS, and the two FTS with the average for workdays and weekends

4.2. Establishing consumption patterns

As can be seen in Section 4.1, there are clearly two weekly consumption patterns: one for workdays and one for weekends. To verify that the use of the membership value of a $\mu_{FTS}(s)$ series can be used to categorise any patterns, a clustering algorithm will be presented. It is based on concepts of Voronoi, where the representative of each region is the FTS constructed with the series of that region and the distance function is the membership function. To this end, Algorithm 1 takes as input the collection of TS S and the number of clusters or groups, k. It gives as output k sets of FTS $\{c_0, \ldots, c_{k-1}\}$, where $FTS(c_i)$ is the representative of cluster c_i. The initialisation randomly assigns each series in the collection S to some set c_i. Subsequently, for each c_i, a representative $FTS(c_i)$

is obtained. The algorithm checks the membership of every TS of every set c_i in the representative of each set. In the event that the maximum membership is not that of the FTS that is the representative of the cluster to which the series belongs, it will be assigned to the cluster in which its membership to the prototype of that cluster is the maximum. Once the membership of all series in all clusters has been determined, the prototypes FTS of each cluster are recalculated and the process is repeated. The algorithm ends when there are no changes in the compositions of the clusters.

Algorithm 1: Clustering Algorithm

input : $S = \{s_0, \ldots, s_m\}$, %Collection of Time Series;
$k \in \mathbb{N}$ %Number of clusters to be obtained
output: $C = \{c_0, \ldots, c_{k-1}\}$ %Clusters obtained;
$F = \{FTS(c_0), \ldots, FTS(c_{k-1})\}$ %Cluster representative;

%Split S in k random subsets $c_i | i = 0, \ldots, k-1$;
$C \longleftarrow \{c_0, \ldots, c_{k-1}\}$;
%Compute the FTS representative;
$F \longleftarrow \{FTS(c_0), \ldots, FTS(c_{k-1})\}$;
$changes \longleftarrow |S|$;
%Algorithm ends if the composition of the cluster doesn't change;
while $changes > 0$ **do**
    $changes \longleftarrow 0$;
    $C' \longleftarrow \{c'_0 = \emptyset, \ldots, c'_{k-1} = \emptyset\}$;
    for $i \longleftarrow 0$ **to** k-1 **do**
        for $s \in c_i$ **do**
            %Compute the maximum membership of a TS to a representative;
            $j \longleftarrow max_j\{\mu_{FTS(c_j)}(s)\}$;
            %Assign the series to a cluster depending on the maximum membership;
            $c'_j \longleftarrow c'_j \cup \{s\}$;
            %Determine if the TS has been assigned to a different cluster;
            if $i \neq j$ **then**
                changes $\longleftarrow$ changes +1;
    %Overwrite the set C with the new organisation of the cluster;
    $C \longleftarrow C'$;
    %Compute the representative FTS;
    $F \longleftarrow \{FTS(c_0), \ldots, FTS(c_{k-1})\}$;

As an example, we will present an execution of Algorithm 1 with $k = 2$ and the initial set of 153 days with their electricity consumption series every 24 h. In the initialisation, lines 1–5, the set of series is randomly divided into two groups $C = \{c_0, c_1\}$, defining the initial FTS $F = \{FTS(c_0), FTS(c_1)\}$, as shown in Figure 7, where it can be seen how these FTS show mixed workdays and weekends showing simultaneously high stays at consumption between 80 kwh and 100 kwh and at the same time at levels of electricity consumption that are around 40 kwh. After four iterations of the main loop, lines 6–16, the sets $C = \{c_0, c_1\}$ do not change their composition, that is, there is no longer any exchange of series between them. This yields two sets, as shown in Figure 8, which clearly reflect the two expected consumption patterns: weekday consumption, represented by $FTS(c_1)$, versus weekend day consumption, represented by $FTS(c_0)$.

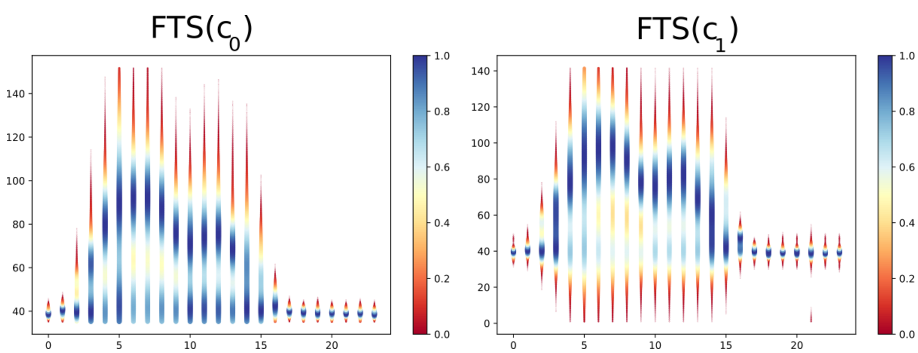

Figure 7: Initial random clustering

4.3. Adequacy of Supply and Demand

The purpose of this example is to determine what generation capacity would satisfy the consumption of a public building. This determination will be based on using subtraction as operation (Section 3.2). We start from the FTS representing the consumption of a building during the 153 days, previously used, as well as from the FTS of the generation during the same period as shown in Figure 9.

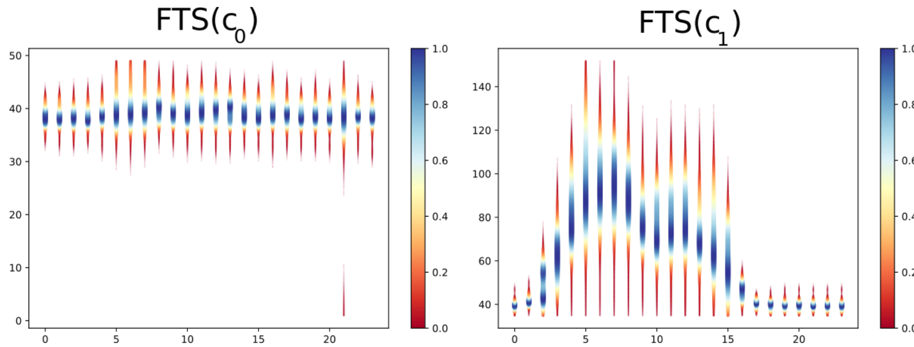

Figure 8: Final iteration.

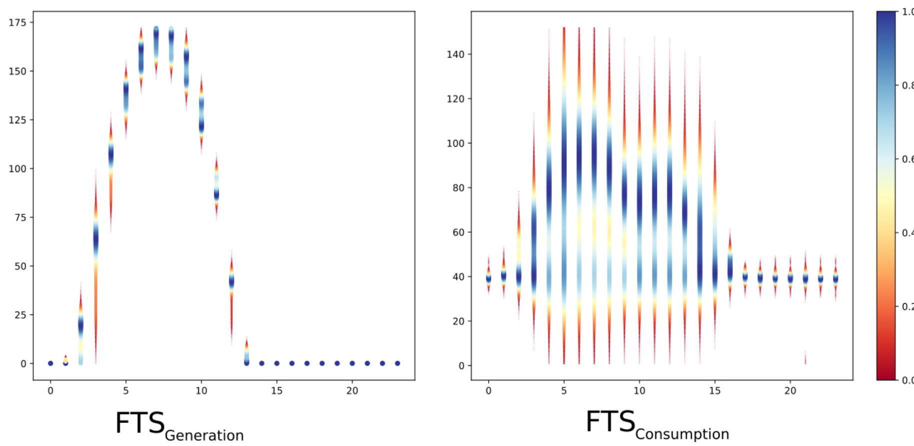

Figure 9: 24 hours generation and consumption.

Energy self-sufficiency is calculated as $FTS_{Self-Suff} = FTS_{consumption} - FTS_{generation}$. In Figure 10, analysing $FTS_{Self-Suff}$, it can be observed how negative values appears when electric energy production is higher than consumption and positive values on the contrary.

In order to better differentiate between these situations, we define two new FTS, both shown in Figure 10. $FTS_{Surplus}$ representing the excess of production over consumption (Equation 8) and FTS_{Gap} which shows that generation is less than consumption (Equation 9).

$$FTS_{Surplus} = \begin{cases} |FTS_{Self-Suff}| & FTS_{Self-Suff} \leq 0 \\ 0 & FTS_{Self-Suff} > 0 \end{cases} \tag{8}$$

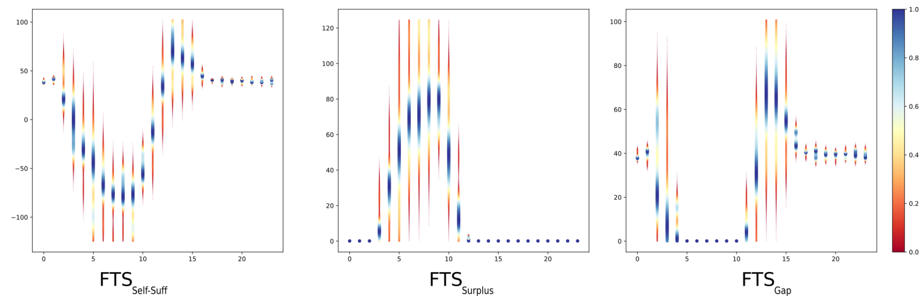

Figure 10: Electrical energy self-sufficiency.

$$FTS_{Gap} = \begin{cases} FTS_{Self-Suff} & FTS_{Self-Suff} > 0 \\ 0 & FTS_{Self-Suff} \leq 0 \end{cases} \quad (9)$$

5. Conclusions and Future Work

We can conclude that the technique that has been developed allows working at different levels with FTS. Firstly, a method has been proposed for the aggregation of time series into a FTS. This method does not follow the classical techniques based on statistical measures. Subsequently, a framework for carrying out FTS operations has been defined. As examples of these operations, division and averaging have been studied. The last theoretical contribution is the definition of a function for calculating the membership of a TS in a FTS. At a more experimental level, a new graphical representation of FTS has been developed, and the results of several experiments presented, to visualise the use of the FTS within a very specific set of time series in the supply–demand domain, more specifically, the electricity consumption and generation in public buildings.

Future work under consideration is obtaining linguistic descriptions of these FTS, which could be based on the ideas of some of the literature on forecasting described in Section 2. Moreover, along these lines, it would be of interest to combine these descriptions by means of linguistic operators, where temporality should play a significant role, in line with the work of Dubois [46].

The use case presented in this paper is an example of the applicability of the method in real applications. With the aim of validating the correctness and robustness of this proposal, it is of interest to extend the proposal to other applications, provided that one can achieve their objective or purpose based on the representation and the algebraic framework of operation defined here. Another possible line of research could incorporate seasonality. This an element that may be critical in some applications but has not yet been considered in the proposed use case. It is of interest to study how to incorporate it and whether in this case it is necessary to work in the time domain and not in the domain of values as is done in this paper. Finally, as pattern recognition is of great interest, we can consider as future work the search for a technique within this field that best allows us to perform a pattern search analysis on our proposed representation of series for the search for trends, etc., as well as to establish a comparative framework with other pattern search methods to test the possibilites of our proposal. "

6. Acknowledgments

We are grateful for support by Grants TRA2016-76914-C3-2-P and PID2020-112967GB-C32, PID2020-112967GB-C33 funded by MCIN/AEI/ 10.13039/501100011033 and by *ERDF A way of making Europe*.

References

[1] Q. Song, B. S. Chissom, Fuzzy time series and its models, Fuzzy Sets and Systems 54 (3) (1993). doi:10.1016/0165-0114(93)90372-O.

[2] L. Zadeh, Fuzzy sets, Information and Control 8 (3) (1965) 338–353. doi:10.1016/S0019-9958(65)90241-X.

[3] L. A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning—I, Information Sciences 8 (3) (1975). doi:10.1016/0020-0255(75)90036-5.

[4] S. M. Chen, Forecasting enrollments based on fuzzy time series, Fuzzy Sets and Systems 81 (3) (1996). doi:10.1016/0165-0114(95)00220-0.

[5] E. Parzen, On Estimation of a Probability Density Function and Mode, The Annals of Mathematical Statistics 33 (3) (1962). doi:10.1214/aoms/1177704472.

[6] M. Kristan, A. Leonardis, D. Skočaj, Multivariate online kernel density estimation with Gaussian kernels, Pattern Recognition 44 (10-11) (2011). doi:10.1016/j.patcog.2011.03.019.

[7] L. Dannecker, R. Schulze, M. Böhm, W. Lehner, G. Hackenbroich, Context-aware parameter estimation for forecast models in the energy domain, in: Lecture Notes in Computer Science (in subseries: Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics), Vol. 6809, 2011. doi:10.1007/978-3-642-22351-8{_}33.

[8] L. Dannecker, M. Böhm, W. Lehner, G. Hackenbroich, Forcasting evolving time series of energy demand and supply, in: Lecture Notes in Computer Science (in subseries: Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics), Vol. 6909, 2011. doi:10.1007/978-3-642-23737-9{_}22.

[9] C. Krome, J. Höft, V. Sander, Clustering time series applied to energy markets, Energy Informatics 2 (2019). doi:10.1186/s42162-019-0076-0.

[10] A. Ben Said, A. Erradi, Deep-Gap: A deep learning framework for forecasting crowdsourcing supply–demand gap based on imaging time series and residual learning, in: Proceedings of the International Conference on Cloud Computing Technology and Science, Vol. Dec. 2019, 2019. doi:10.1109/CloudCom.2019.00048.

[11] W. J. Qu, G. X. Tan, F. Xu, Z. H. Liu, An analysis for demand forecast and dynamic balance of supply and demand in higher education, in: Proceedings - 2011 4th International Conference on Information Management,

Innovation Management and Industrial Engineering, ICIII 2011, Vol. 2, 2011. doi:10.1109/ICIII.2011.262.

[12] S. M. Burney, S. M. Ali, Sales Forecasting for Supply Chain Demand Management - A Novel Fuzzy Time Series Approach, in: MACS 2019 - 13th International Conference on Mathematics, Actuarial Science, Computer Science and Statistics, Proceedings, 2019. doi:10.1109/MACS48846.2019.9024810.

[13] Q. Song, B. S. Chissom, Forecasting enrollments with fuzzy time series - Part I, Fuzzy Sets and Systems 54 (1) (1993). doi:10.1016/0165-0114(93)90355-L.

[14] S. T. Li, Y. C. Cheng, Deterministic fuzzy time series model for forecasting enrollments, Computers and Mathematics with Applications 53 (12) (2007). doi:10.1016/j.camwa.2006.03.036.

[15] N. H. Shafii, R. Alias, S. R. Shamsuddin, D. S. Mohd Nasir, Fuzzy Time Series for Projecting School Enrolment in Malaysia, Journal of Computing Research and Innovation 6 (1) (2021). doi:10.24191/jcrinn.v6i1.180.

[16] S. M. Chen, J. R. Hwang, Temperature prediction using fuzzy time series, IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics 30 (2) (2000). doi:10.1109/3477.836375.

[17] S. Yusuf, A. Mohammad, A. Hamisu, A novel two-factor high order fuzzy time series with applications to temperature and futures exchange forecasting, Nigerian Journal of Technology 36 (4) (2018). doi:10.4314/njt.v36i4.18.

[18] V. Vamitha, M. Jeyanthi, S. Rajaram, T. Revathi, Temperature Prediction Using Fuzzy Time Series and Multivariate Markov Chain, International Journal of Fuzzy Mathematics and Systems 2 (3) (2012).

[19] M. Versaci, F. C. Morabito, Fuzzy time series approach for disruption prediction in Tokamak reactors, IEEE Transactions on Magnetics 39 (3 I) (2003). doi:10.1109/TMAG.2003.810365.

[20] Y. Alyousifi, M. Othman, A. Husin, U. Rathnayake, A new hybrid fuzzy time series model with an application to predict PM10 concentration, Ecotoxicology and Environmental Safety 227 (2021). doi:10.1016/j.ecoenv.2021.112875.

[21] M. Z. Khan, M. F. Khan, Application of ANFIS, ANN and fuzzy time series models to CO2 emission from the energy sector and global temperature increase, International Journal of Climate Change Strategies and Management 11 (5) (2019). doi:10.1108/IJCCSM-01-2019-0001.

[22] R. Aliyev, S. Salehi, R. Aliyev, Development of fuzzy time series model for hotel occupancy forecasting, Sustainability (Switzerland) 11 (3) (2019). doi:10.3390/su11030793.

[23] K. H. Huarng, T. H. K. Yu, L. Moutinho, Y. C. Wang, Forecasting tourism demand by fuzzy time series models, International Journal of Culture, Tourism, and Hospitality Research 6 (4) (2012). doi:10.1108/17506181211265095.

[24] R. C. Tsaur, T. C. Kuo, The adaptive fuzzy time series model with an application to Taiwan's tourism demand, Expert Systems with Applications 38 (8) (2011). doi:10.1016/j.eswa.2011.01.059.

[25] S. Xian, Y. Cheng, Pythagorean fuzzy time series model based on Pythagorean fuzzy c-means and improved Markov weighted in the prediction of the new COVID-19 cases, Soft Computing 25 (22) (2021). doi:10.1007/s00500-021-06259-2.

[26] N. Kumar, H. Kumar, A novel hybrid fuzzy time series model for prediction of COVID-19 infected cases and deaths in India, ISA Transactions (2021). doi:10.1016/j.isatra.2021.07.003.

[27] M. Bose, K. Mali, Designing fuzzy time series forecasting models: A survey, International Journal of Approximate Reasoning 111 (2019). doi:10.1016/j.ijar.2019.05.002.

[28] K. Huarng, Effective lengths of intervals to improve forecasting in fuzzy time series, Fuzzy Sets and Systems 123 (3) (2001). doi:10.1016/S0165-0114(00)00057-9.

[29] K. Huarng, Heuristic models of fuzzy time series for forecasting, Fuzzy Sets and Systems 123 (3) (2001). doi:10.1016/S0165-0114(00)00093-2.

[30] T. A. Jilani, S. M. A. Burney, A refined fuzzy time series model for stock market forecasting, Physica A: Statistical Mechanics and its Applications 387 (12) (2008). doi:10.1016/j.physa.2008.01.099.

[31] C. H. Aladag, U. Yolcu, E. Egrioglu, A. Z. Dalar, A new time invariant fuzzy time series forecasting method based on particle swarm optimization, Applied Soft Computing Journal 12 (10) (2012). doi:10.1016/j.asoc.2012.05.002.

[32] S. M. Chen, P. Y. Kao, TAIEX forecasting based on fuzzy time series, particle swarm optimization techniques and support vector machines, Information Sciences 247 (2013). doi:10.1016/j.ins.2013.06.005.

[33] S. H. Cheng, S. M. Chen, W. S. Jian, Fuzzy time series forecasting based on fuzzy logical relationships and similarity measures, Information Sciences 327 (2016). doi:10.1016/j.ins.2015.08.024.

[34] P. Singh, B. Borah, Forecasting stock index price based on M-factors fuzzy time series and particle swarm optimization, International Journal of Approximate Reasoning 55 (3) (2014). doi:10.1016/j.ijar.2013.09.014.

[35] P. D'Urso, E. A. Maharaj, Autocorrelation-based fuzzy clustering of time series, Fuzzy Sets and Systems 160 (24) (2009). doi:10.1016/j.fss.2009.04.013.

[36] C. H. Cheng, G. W. Cheng, J. W. Wang, Multi-attribute fuzzy time series method based on fuzzy clustering, Expert Systems with Applications 34 (2) (2008). doi:10.1016/j.eswa.2006.12.013.

[37] R. C. Tsaur, J. C. O Yang, H. F. Wang, Fuzzy relation analysis in fuzzy time series model, Computers and Mathematics with Applications 49 (4) (2005). doi:10.1016/j.camwa.2004.07.014.

[38] H. L. Wong, Y. H. Tu, C. C. Wang, Application of fuzzy time series models for forecasting the amount of Taiwan export, Expert Systems with Applications 37 (2) (2010). doi:10.1016/j.eswa.2009.06.106.

[39] W. Wangming, Fuzzy reasoning and fuzzy relational equations, Fuzzy Sets and Systems 20 (1) (1986). doi:10.1016/S0165-0114(86)80032-X.

[40] C. F. Liu, C. Y. Yeh, S. J. Lee, Application of type-2 neuro-fuzzy modeling in stock price prediction, Applied Soft Computing Journal 12 (4) (2012). doi:10.1016/j.asoc.2011.11.028.

[41] K. Huarng, H. K. Yu, A type 2 fuzzy time series model for stock index forecasting, Physica A: Statistical Mechanics and its Applications 353 (1-4) (2005). doi:10.1016/j.physa.2004.11.070.

[42] N. S. Bajestani, A. Zare, Forecasting TAIEX using improved type 2 fuzzy time series, Expert Systems with Applications 38 (5) (2011). doi:10.1016/j.eswa.2010.10.049.

[43] A. Khosravi, S. Nahavandi, D. Creighton, D. Srinivasan, Interval type-2 fuzzy logic systems for load forecasting: A comparative study, IEEE Transactions on Power Systems 27 (3) (2012). doi:10.1109/TPWRS.2011.2181981.

[44] M. H. Zarandi, M. R. Faraji, M. Karbasian, Interval type-2 fuzzy expert system for prediction of carbon monoxide concentration in mega-cities, Applied Soft Computing Journal 12 (1) (2012). doi:10.1016/j.asoc.2011.08.043.

[45] F. Kutlu, F. Tuğrul, M. Çitil, Introduction to temporal intuitionistic fuzzy approximate reasoning, Communications Faculty Of Science University of Ankara Series A1Mathematics and Statistics (2019). doi:10.31801/cfsuasmas.540529.

[46] D. Dubois, H. Prade, Processing Fuzzy Temporal Knowledge, IEEE Transactions on Systems, Man and Cybernetics 19 (4) (1989). doi:10.1109/21.35337.

[47] B. R. Kosanovic, L. F. Chaparro, M. Sun, R. J. Sclabassi, Physical system modeling using temporal fuzzy sets, 1994. doi:10.1109/ijcf.1994.375073.

[48] W. F. Holmgren, C. W. Hansen, M. A. Mikofski, pvlib python: a python package for modeling solar energy systems, Journal of Open Source Software 3 (29) (2018) 884. doi:10.21105/joss.00884.