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SOME NEW INTEGRAL INEQUALITIES FOR NEGATIVE SUMMATION PARAMETERS

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Abstract. In this paper, we prove some Hardy type and Hardy-Steklov type integral inequalities for two negative summation parameters and we deduce some well-known results with sharp constants.

1 Introduction

G.H. Hardy stated in 1920 (see [5]) and proved in 1925 (see [6]) the following inequality

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{1.1}$$

where $p > 1$, f is a non-negative Lebesgue measurable function and $F(x) = \int_0^x f(t)dt$.

In 1928, Hardy presented a generalized form of inequality (1.1) as follows

$$\int_0^\infty \frac{F^p(x)}{x^r}dx \quad \le \quad \begin{cases} \left(\dfrac{p}{r-1}\right)^p \displaystyle\int_0^\infty x^{p-r} f^p(x)dx \quad \text{for} \quad r > 1 \\ \left(\dfrac{p}{1-r}\right)^p \displaystyle\int_0^\infty x^{p-r} f^p(x)dx \quad \text{for} \quad r < 1, \end{cases} \tag{1.2}$$

where $p > 1$, f is a non-negative Lebesgue measurable function and

$$F(x) = \begin{cases} \displaystyle\int_0^x f(t)dt \quad if \quad r > 1, \\ \displaystyle\int_x^\infty f(t)dt \quad if \quad r < 1, \end{cases}$$

for all $x > 0$.

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By putting $r = p - \gamma$ in (1.2), one can get the following inequalities

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)dt \right)^p x^\gamma dx \leq \left(\frac{p}{p-1-\gamma} \right)^p \int_0^\infty f^p(x) x^\gamma dx, \tag{1.3}$$

if $\gamma < p - 1$ and

$$\int_0^\infty \left(\frac{1}{x} \int_x^\infty f(t)dt \right)^p x^\gamma dx \leq \left(\frac{p}{\gamma - p + 1} \right)^p \int_0^\infty f^p(x) x^\gamma dx, \tag{1.4}$$

if $\gamma > p - 1$.

The period of more than ten years of research until Hardy finally discovered his inequalities (1.1) and (1.2) was described in [9]. The Hardy inequalities also play an important role in various fields of mathematics, especially in functional and spectral analysis, where one investigates properties of the Hardy operator, like boundedness and compactness (see for example [4]) and also behavior in more general function spaces.

Some weighted integral inequalities for $1 < p \leq q < \infty$ were established in [1]. Namely, the following statements (Theorems 1, 4, 2) were proved there.

Theorem 1. *Let $0 < a < b < +\infty$, $1 < p \leq q < \infty$, f be a non-negative Lebesgue measurable function on (a, b) and v be a positive Lebesgue measurable function on (a, b) such that $v, vf \in L_1(a, b)$:*

1. *If $\lambda > \frac{s-1}{p+s-1}$, then*

$$\int_a^b \frac{v(x)}{V^s(x)} H_{v,1}^p(x)dx \leq \left(\frac{\lambda p}{s-1} \right)^p \left(\int_a^b v(x)dx \right)^{1-\frac{p}{q}} \left(\int_a^b \frac{v(x)}{V^{\frac{sq}{p}}(x)} f^q(x)dx \right)^{\frac{p}{q}}, \tag{1.5}$$

where $s > 1$, $V(x) = \int_0^x v(t)dt$, $H_{v,1}(x) = \frac{1}{V(x)} \int_a^x v(t)f(t)dt$.

2. *If $\gamma < p - 1$, then*

$$\int_a^b \frac{v(x)}{V^{p-\gamma}(x)} H_{v,2}^p(x)dx \leq \left(\frac{p}{p-1-\gamma} \right)^p \left(\int_a^b v(x)dx \right)^{1-\frac{p}{q}} \left(\int_a^b \frac{v(x)}{V^{q-\frac{\gamma q}{p}}(x)} f^q(x)dx \right)^{\frac{p}{q}}, \tag{1.6}$$

where $H_{v,2}(x) = \int_a^x \frac{v(t)f(t)}{V(t)} dt$.

3. *If $\lambda > \frac{1-s}{1-s-p} > 0$, then*

$$\int_a^b \frac{v(x)}{V^s(x)} \widetilde{H}_{v,1}^p(x)dx \leq \left(\frac{\lambda p}{1-s} \right)^p \left(\int_a^b v(x)dx \right)^{1-\frac{p}{q}} \left(\int_a^b \frac{v(x)}{V^{\frac{sq}{p}}(x)} f^q(x)dx \right)^{\frac{p}{q}}, \tag{1.7}$$

where $s < 1$ *and* $\widetilde{H}_{v,1}(x) = \frac{1}{V(x)} \int_x^b v(t) f(t) dt$.

Next we note several Hardy-type integral inequalities for the case of one negative parameter (see [3], [10]). The following theorem was established in [10].

Theorem 2. *Let* $f : [a, b] \longrightarrow \mathbb{R}^+$ *and* p *is positive. Then the following is true*

$$\int_a^b \left(\frac{1}{x-a} \int_a^x f(t) dt \right)^{-p} dx \quad \leq \quad \left(\frac{p+1}{p} \right)^p \int_a^b f^{-p}(t) dt. \tag{1.8}$$

Moreover the constant $\left(\frac{p+1}{p}\right)^p$ *is the best possible.*

Bicheng Yang proved the following Hardy type integral inequalities for one negative parameter (see [3] for more details).

Theorem 3. *Let* $p < 0$, $r \in \mathbb{R}$, $r \neq 1$, $f(t) \geq 0$ *and* $\int_0^\infty t^{-r} (t f(t))^p dt < \infty$.

1. *If* $r < 1$, $F_1(x) = \int_0^x f(t) dt$, *then the inequality*

$$\int_0^\infty x^{-r} F_1^p(x) dx \quad < \quad \left(\frac{p}{r-1} \right)^p \int_0^\infty t^{-r} (t f(t))^p dt \tag{1.9}$$

   holds, where the constant $\left(\frac{p}{r-1}\right)^p$ *is the best possible.*

2. *If* $r > 1$, $F_2(x) = \int_x^\infty f(t) dt$, *then the inequality*

$$\int_0^\infty x^{-r} F_2^p(x) dx \quad < \quad \left(\frac{p}{1-r} \right)^p \int_0^\infty t^{-r} (t f(t))^p dt \tag{1.10}$$

   holds, where the constant $\left(\frac{p}{1-r}\right)^p$ *is the best possible.*

For both $p, q < 0$, some weighted integral inequalities were established in [2] for weight functions satisfying Muckenhoupt-type conditions.
In this paper, we give a version of inequalities (1.5), (1.6) and (1.7) to the case of both negative parameters p and q. We also prove some weighted integral inequalities for the Hardy-Steklov operator with both negative parameters p and q. Moreover, Theorem 2 and Theorem 3 are generalized.

2 Preliminaries

In this paper, all functions are supposed to be non-negative and measurable and if some negative power of the function appears, then we assume that the function is strictly positive a.e.

The following lemma is well known (for details one can consult [7]).

Lemma 4. *If $p < 0$, $\frac{1}{p} + \frac{1}{p'} = 1$, $f \in L_p(\Omega)$, $g \in L_{p'}(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is Lebesgue measurable set, $f(t)$, $g(t) \geq 0$, then*

$$\int_\Omega f(t)g(t)dt \geq \left(\int_\Omega f^p(t)dt\right)^{\frac{1}{p}} \left(\int_\Omega g^{p'}(t)dt\right)^{\frac{1}{p'}}, \tag{2.1}$$

where the equality holds if and only if there exists constants c and d, such they are not all zero, that

$$cf^p(t) = dg^{p'}(t),$$

almost everywhere in Ω.

Lemma 5. *Let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set $q \leq p < 0$ and f, v be non-negative Lebesgue measurable functions on Ω such that $\int_\Omega v(x)(f(x))^q dx < \infty$, $v(x) \neq 0$ and $\int_\Omega v(x)dx < \infty$. Then*

$$\int_\Omega (f(x))^p v(x)dx \leq \left(\int_\Omega (f(x))^q v(x)dx\right)^{\frac{p}{q}} \left(\int_\Omega v(x)dx\right)^{1-\frac{p}{q}}.$$

Proof. By Hölder's inequality with exponent $\frac{q}{p} \geq 1$ and its conjugate $(\frac{q}{p})' = \frac{q}{q-p}$, we get

$$\begin{aligned}\int_\Omega (f(x))^p v(x)dx &= \int_\Omega (f(x))^p (v(x))^{\frac{p}{q}} (v(x))^{1-\frac{p}{q}} dx \\ &\leq \left(\int_\Omega (f(x))^q v(x)dx\right)^{\frac{p}{q}} \left(\int_\Omega v(x)dx\right)^{1-\frac{p}{q}}.\end{aligned}$$

□

Lemma 6. *Let $-\infty < q \leq p < 0$ and f, g, v be non-negative measurable functions on (a, b) such that $V(x) = \int_0^x v(t)dt$. If $s \in \mathbb{R}, s \neq 1$, then*

$$\int_a^b \frac{v(x)}{V^s(x)} g^p(f(x))dx \leq \left(\int_a^b v(x)dx\right)^{1-\frac{p}{q}} \left(\int_a^b \frac{v(x)}{V^{\frac{sq}{p}}(x)} g^q(f(x))dx\right)^{\frac{p}{q}}. \tag{2.2}$$

Proof. By applying Lemma 5, we obtain inequality (2.2). □

3 Main results

All the integrals introduced in this work are assumed to be convergent and v is a positive weight function. Let $0 \leq a < b \leq +\infty$, $V(x) = \int_0^x v(t)dt$ and we assume that $0 \times \infty = 0$.

Theorem 7. *Let* $-\infty < q \leq p < 0$, $0 \leq a < b \leq +\infty$, f *be a non-negative Lebesgue measurable function on* (a, b) *and* v *be a positive Lebesgue measurable function on* (a, b). *If* $0 < s < 1$, *then*

$$\int_a^b \frac{v(x)}{V^s(x)} H_{v,1}^p(x)dx \leq \left(\frac{p}{p+s-1}\right)^p \left(\int_a^b v(x)dx\right)^{1-\frac{p}{q}} \left(\int_a^b \frac{v(x)}{V^{\frac{sq}{p}}(x)} f^q(x)dx\right)^{\frac{p}{q}}, \tag{3.1}$$

where $H_{v,1}(x) = \dfrac{1}{V(x)} \displaystyle\int_a^x v(t)f(t)dt$.

Proof. Note that for almost all $x \in (a, b)$

$$(H_{v,1})'(x) = \frac{1}{V(x)} \left[v(x)f(x) - v(x)H_{v,1}(x)\right].$$

Since the functions $V^{1-s}(x)$ and $H_{v,1}^p(x)$ are absolutely continuous on $[a, b]$, then by applying the integration by parts in the left hand side of (3.1), we obtain

$$\begin{aligned}\int_a^b \frac{v(x)}{V^s(x)} H_{v,1}^p(x)dx &= \left[\frac{H_{v,1}^p(x)}{(1-s)V^{s-1}(x)}\right]_a^b - \frac{p}{s-1}\int_a^b \frac{v(x)}{V^s(x)} H_{v,1}^p(x)dx \\ &+ \frac{p}{s-1}\int_a^b \frac{v(x)f(x)}{V^s(x)} H_{v,1}^{p-1}(x)dx.\end{aligned}$$

As $s < 1$ and $H_{v,1}(b) \geq 0$, we get

$$\frac{p+s-1}{s-1}\int_a^b \frac{v(x)}{V^s(x)} H_{v,1}^p(x)dx \geq \frac{p}{s-1}\int_a^b \frac{v(x)f(x)}{V^s(x)} H_{v,1}^{p-1}(x)dx.$$

Consequently by using Hölder's inequality (2.1) for $p < 0$, we have

$$\int_a^b \frac{v(x)}{V^s(x)} H_{v,1}^p(x)dx \geq \left(\frac{p}{p+s-1}\right)\left(\int_a^b \frac{v(x)}{V^s(x)} H_{v,1}^p(x)dx\right)^{\frac{1}{P'}} \left(\int_a^b \frac{v(x)f^p(x)}{V^s(x)}dx\right)^{\frac{1}{P}}.$$

Thus

$$\left(\int_a^b \frac{v(x)}{V^s(x)} H_{v,1}^p(x)dx\right)^{\frac{1}{P}} \geq \left(\frac{p}{p+s-1}\right)\left(\int_a^b \frac{v(x)f^p(x)}{V^s(x)}dx\right)^{\frac{1}{P}},$$

then

$$\int_a^b \frac{v(x)}{V^s(x)} H_{v,1}^p(x)dx \leq \left(\frac{p}{p+s-1}\right)^p \int_a^b \frac{v(x)f^p(x)}{V^s(x)}dx.$$

By applying inequality (2.2) to the right hand side of last inequality, we obtain (3.1). □

Remark 8. By putting $s = p - \gamma$ in (2.2), we get

$$\int_a^b \frac{v(x)g^p(f(x))}{V^{p-\gamma}}dx \leq \left(\int_a^b v(x)dx\right)^{1-\frac{p}{q}} \left(\int_a^b \frac{v(x)g^q(f(x))}{V^{q-\frac{\gamma q}{p}}}dx\right)^{\frac{p}{q}}. \quad (3.2)$$

Theorem 9. *Let $-\infty < q \leq p < 0$ and f, v be non-negative Lebesgue measurable functions on $(0, \infty)$.*

If $\gamma > p - 1$, then

$$\int_a^b \frac{v(x)H_{v,2}^p(x)}{V^{p-\gamma}(x)}dx \leq \left(\frac{p}{p-1-\gamma}\right)^p \left(\int_a^b v(x)dx\right)^{1-\frac{p}{q}} \left(\int_a^b \frac{v(x)f^q(x)}{V^{q-\frac{\gamma q}{p}}(x)}dx\right)^{\frac{p}{q}}, \quad (3.3)$$

where $H_{v,2}(x) = \int_a^x \frac{v(t)f(t)}{V(t)}dt$.

Proof. We note that

$$(H_{v,2})'(x) = \frac{v(x)f(x)}{V(x)}.$$

We integrate by parts in the left hand side of (3.3), thus

$$\begin{aligned}\int_a^b \frac{v(x)H_{v,2}^p(x)}{V^{p-\gamma}(x)}dx &= \left[\frac{H_{v,2}^p}{(1-p+\gamma)V^{p-\gamma-1}(x)}\right]_a^b + \frac{p}{p-\gamma-1}\int_a^b \frac{v(x)f(x)H_{v,2}^{p-1}(x)}{V^{p-\gamma}(x)}dx \\ &= \frac{H_{v,2}^p(b)}{(1-p+\gamma)V^{p-\gamma-1}(b)} + \frac{p}{p-\gamma-1}\int_a^b \frac{v(x)f(x)H_{v,2}^{p-1}(x)}{V^{p-\gamma}(x)}dx.\end{aligned}$$

Assumptions imply that $\frac{p}{p-\gamma-1} > 0$ and $H_{v,2}(b) \geq 0$, then

$$\int_a^b \frac{v(x)H_{v,2}^p(x)}{V^{p-\gamma}(x)}dx \geq \frac{p}{p-\gamma-1}\int_a^b \frac{v(x)f(x)H_{v,2}^{p-1}(x)}{V^{p-\gamma}(x)}dx.$$

By applying Hölder's inequality (2.1), we have

$$\int_a^b \frac{v(x)H_{v,2}^p(x)}{V^{p-\gamma}(x)}dx \geq \frac{p}{p-\gamma-1}\left(\int_a^b \frac{v(x)H_{v,2}^p(x)}{V^{p-\gamma}(x)}dx\right)^{\frac{1}{p'}} \left(\int_a^b \frac{v(x)f^p(x)}{V^{p-\gamma}(x)}dx\right)^{\frac{1}{p}},$$

therefore

$$\int_a^b \frac{v(x)H_{v,2}^p(x)}{V^{p-\gamma}(x)}dx \leq \left(\frac{p}{p-\gamma-1}\right)^p \int_a^b \frac{v(x)f^p(x)}{V^{p-\gamma}(x)}dx.$$

Finally by using inequality (3.2), we obtain (3.3). □

Let $q = p$, $v(x) = 1$, $a = 0$ in (3.1), then we get the following corollary.

Corollary 10. *Let $p < 0$, f non-negative Lebesgue measurable function on $(0, \infty)$, $s < 1$, then*

$$\int_a^b x^{-s-p} F_1^p(x)dx \leq \left(\frac{p}{p+s-1}\right)^p \left(\int_a^b x^{-s} f^p(x)dx\right), \tag{3.4}$$

where $F_1(x) = \int_0^x f(t)dt$.

Remark 11. Let $s = r - p$, $a = 0$ and $b = \infty$ in (3.4). Since $s < 1$, $r - p < 1$, then $r < p + 1 < 1$ and we get inequality (1.9) with sharp constant $\left(\frac{p}{r-1}\right)^p$.

Remark 12. If in inequality (1.9) of Theorem 1.3, we put $r = p$, then we deduce that under assumptions $a = 0, b = \infty$, inequality (1.8) is a particular case of (1.9).

Theorem 13. *Let $-\infty < q \leq p < 0$ and f, v be non-negative Lebesgue measurable functions on $(0, \infty)$.*
If $s > 1 - p$, then

$$\int_a^b \frac{v(y)}{V^s(y)} \tilde{H}_{v,1}^p(y)dy \leq \left(\frac{p}{1-s-p}\right)^p \left(\int_a^b v(y)dy\right)^{1-\frac{p}{q}} \left(\int_a^b \frac{v(y)}{V^{\frac{sq}{p}}(y)} f^q(y)dy\right)^{\frac{p}{q}}, \tag{3.5}$$

where $\tilde{H}_{v,1}(x) = \frac{1}{V(x)} \int_x^b v(t)f(t)dt$.

Proof. Note that for almost all $x \in (a, b)$

$$(\tilde{H}_{v,1})'(x) = \frac{1}{V(x)} \left[v(x)f(x) + v(x)\tilde{H}_{v,1}(x)\right].$$

By applying the integration by parts in the left hand side of (3.5), we obtain

$$\begin{aligned} \int_a^b \frac{v(x)}{V^s(x)} \tilde{H}_{v,1}^p(x)dx &= \left[\frac{\tilde{H}_{v,1}^p(x)}{(1-s)V^{s-1}(x)}\right]_a^b + \frac{p}{1-s} \int_a^b \frac{v(x)}{V^s(x)} \tilde{H}_{v,1}^p(x)dx \\ &+ \frac{p}{1-s} \int_a^b \frac{v(x)f(x)}{V^s(x)} \tilde{H}_{v,1}^{p-1}(x)dx. \end{aligned}$$

The rest is similar to the proof of Theorem 13. □

Let $p = q$, $v(x) = 1$, $b = \infty$ in (3.5), then we obtain the following inequality.

$$\int_a^b x^{-s-p} F_2^p(x)dx \leq \left(\frac{p}{1-s-p}\right)^p \left(\int_a^b x^{-s} f^p(x)dx\right), \tag{3.6}$$

where $F_2(x) = \int_x^\infty f(t)dt$.

Remark 14. Let $s = r - p$, $a = 0$ and $b = \infty$ in (3.6), thus we get inequality (1.10) with sharp constant $(\frac{p}{1-r})^p$.

Theorem 15. *Let $-\infty < q \leq p < 0$ and f, v be non-negative Lebesgue measurable functions on $(0, \infty)$. If $0 < s < 1$, then*

$$\int_0^b \frac{v(x)}{V^s(x)}(Tf)^p(x)dx \leq$$

$$\left(\frac{p}{p+s-1}\right)^p \left(\int_0^b \frac{v(x)}{V^{\frac{sq}{p}}(x)}|K(x)|^q dx\right)^{\frac{p}{q}} \left(\int_0^b v(x)dx\right)^{1-\frac{p}{q}}, \tag{3.7}$$

where

$$(Tf)(x) = \frac{1}{V(x)} \int_{\theta x}^{\vartheta x} v(t)f(t)dt$$

and

$$K(x) = \frac{\vartheta v(\vartheta x)f(\vartheta x) - \theta v(\theta x)f(\theta x)}{v(x)}; \vartheta > 0, \theta > 0 \quad \text{and} \quad \vartheta > \theta.$$

Proof. We note that

$$\begin{aligned} (Tf)'(x) &= \frac{-v(x)}{V(x)}(Tf)(x) + \frac{\vartheta v(\vartheta x)f(\vartheta x) - \theta v(\theta x)f(\theta x)}{V(x)} \\ &= \frac{-v(x)}{V(x)}(Tf)(x) + \frac{v(x)}{V(x)}K(x). \end{aligned} \tag{3.8}$$

Integrating by parts in the left-hand side of (3.7), we get

$$\begin{aligned} \int_0^b \frac{v(x)}{V^s(x)}(Tf)^p(x)dx &= \left[\frac{(Tf)^p(x)}{(1-s)V^{s-1}(x)}\right]_0^b + \frac{p}{(1-s)} \int_0^b \frac{v(x)}{V^s(x)}(Tf)^p(x)dx \\ &- \frac{p}{(1-s)} \int_0^b \frac{v(x)K(x)}{V^s(x)}(Tf)^{p-1}(x)dx, \end{aligned}$$

since $s < 1$ and $(Tf(b) \geq 0$, we obtain

$$\frac{p+s-1}{s-1} \int_0^b \frac{v(x)}{V^s(x)}(Tf)^p(x)dx \geq \frac{p}{s-1} \int_0^b \frac{v(x)K(x)}{V^s(x)}(Tf)^{p-1}(x)dx,$$

then

$$\int_0^b \frac{v(x)}{V^s(x)}(Tf)^p(x)dx \geq \frac{p}{p+s-1} \int_0^b \frac{v(x)K(x)}{V^s(x)}(Tf)^{p-1}(x)dx.$$

By using Lemma 2.1, for $K(x)$, $(Tf)(x) \geq 0$, we have

$$\int_0^b \frac{v(x)K(x)}{V^s(x)}(Tf)^{p-1}(x)dx$$

$$\geq \left(\int_0^b \frac{v(x)|K(x)|^p}{V^s(x)}dx\right)^{\frac{1}{p}} \left(\int_0^b \frac{v(x)}{V^s(x)}(Tf)^{(p-1)p'}(x)dx\right)^{\frac{1}{p'}}$$

$$\geq \left(\int_0^b \frac{v(x)|K(x)|^p}{V^s(x)}dx\right)^{\frac{1}{p}} \left(\int_0^b \frac{v(x)}{V^s(x)}(Tf)^{p}(x)dx\right)^{\frac{1}{p'}}.$$

Consequently

$$\int_0^b \frac{v(x)}{V^s(x)}(Tf)^p(x)dx \geq \frac{p}{p+s-1}\left(\int_0^b \frac{v(x)|K(x)|^p}{V^s(x)}dx\right)^{\frac{1}{p}} \left(\int_0^b \frac{v(x)}{V^s(x)}(Tf)^{p}(x)dx\right)^{\frac{1}{p'}},$$

thus

$$\left(\int_0^b \frac{v(x)}{V^s(x)}(Tf)^p(x)dx\right)^p \leq \left(\int_0^b \frac{v(x)}{V^s(x)}(Tf)^p(x)dx\right)^{p-1} \int_0^b \frac{v(x)|K(x)|^p}{V^s(x)}dx.$$

Therefore

$$\int_0^b \frac{v(x)}{V^s(x)}(Tf)^p(x)dx \leq \left(\frac{p}{p+s-1}\right)^p \int_0^b \frac{v(x)|K(x)|^p}{V^s(x)}dx.$$

Finally by applying (2.2) to the right hand side of last inequality, we obtain (3.7). □

Remark 16. *If in inequality (3.1) $a=0$, we obtain the following integral inequality*

$$\int_0^b \frac{v(x)}{V^s(x)}H^p_{v,1}(x)dx \leq \left(\frac{p}{p+s-1}\right)^p \left(\int_0^b v(x)dx\right)^{1-\frac{p}{q}} \left(\int_0^b \frac{v(x)}{V^{\frac{sq}{p}}(x)}f^q(x)dx\right)^{\frac{p}{q}}. \quad (3.9)$$

Now if in inequality (3.7) we put $\theta=0$, $\vartheta=1$, we get (3.9), then we can deduce, that Theorem 15 is a generalization of Theorem 7 with $a=0$.

If we set $v(x)=1$, $\theta=\frac{1}{2}$ and $\vartheta=1$ in Theorem 15, we obtain the following corollary.

Corollary 17. *Let $-\infty < q \leq p < 0$ and f, v be non-negative Lebesgue measurable functions on $(0,\infty)$. If $s<1$, then*

$$\int_0^b \frac{v(x)}{V^s(x)}(T_1f)^p(x)dx \leq \left(\frac{p}{p+s-1}\right)^p \left(\int_0^b \frac{v(x)}{V^{\frac{sq}{p}}(x)}|K_1(x)|^q dx\right)^{\frac{p}{q}} \times \left(\int_0^b v(x)dx\right)^{1-\frac{p}{q}}, \quad (3.10)$$

where

$$K_1(x) = f(x) - \frac{1}{2}f(\frac{1}{2}x),$$
$$(T_1f)(x) = \frac{1}{x}\int_{\frac{1}{2}x}^x v(t)f(t)dt.$$

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