The stress state of the rock mass with spherical cavity

Vladimir Andreev*

Moscow State University of Civil Engineering, Yaroslavskoye shosse, 26, 129337, Moscow, Russia

Abstract. One of the main hypotheses accepted in the mechanics of deformable solids is the assumption of the homogeneity of materials. This means that all mechanical characteristics of the material (modulus of elasticity, Poisson's ratio, yield strength, relaxation parameters, etc.) are constant over the volume of the body, in other words, these characteristics are constants. This hypothesis makes it possible not to take into account the natural inhomogeneity of materials at the microlevel - the presence of various fractions in composite materials (concrete, fiberglass, etc.), crystal lattice defects, etc. Examples can be given when various physical phenomena (temperature field, radiation exposure, explosive impact, etc.) lead to a change in the mechanical characteristics along the body. These changes can be quite significant. So, for example, in the presence of high-gradient temperature fields, the deformation characteristics of materials at different points of the body can change dozens of times. Thus, when calculating and designing structures, it is necessary to take into account such macro heterogeneity, since it leads to a significant change in the stress-strain state of bodies. This article considers the problem associated with the continuous inhomogeneity of materials. It means such a heterogeneity that arose in the process of creating an underground cavity with the help of an explosion. In contrast to the classical mechanics of a deformable solid body, the problems of which are reduced to differential equations with constant coefficients, in the mechanics of continuously inhomogeneous bodies we deal with equations with variable coefficients, which greatly complicates their solution. In this case, depending on the type of inhomogeneity functions—functions that describe the change in mechanical characteristics along the coordinates—differential equations turn out to be significantly different.

Keywords: heterogeneity, explosion, sphere, mechanical characteristics, underground cavity

1 Introduction

In contrast to classical mechanics of a deformable solid body, the problems of which are reduced to differential equations with constant coefficients, in the mechanics of continuously inhomogeneous bodies, problems are associated with equations with variable coefficients, which greatly complicates their solutions. Depending on the type of inhomogeneity functions that describe the change in mechanical characteristics along the coordinates, differential equations are obtained significantly different. The first works published in the mid-thirties of the twentieth century were the works of Mikhlin S.G. [1], devoted to the derivation of the equations of a plane problem of the theory of elasticity of

* Corresponding author: asv@mgsu.ru

an inhomogeneous. Thus, one of the first tasks is to find a way to approximate real dependencies. The functions of inhomogeneity, which are further included in the system of equations for solving a particular problem, should, on the one hand, be quite simple, which makes it possible to obtain simpler equations, and, on the other hand, should most adequately describe the experimental data, since even small differences in the choice of approximating functions can lead to significantly different results. A significant contribution to the development of the mechanics of inhomogeneous bodies was made by Soviet and Russian scientists: Kolchin G.B. [2,3], Lekhnitsky S.G.[4], Lomakin V.A.[5], Rostovtsev N.A. [6] and many others. We should also mention the works of Polish scientists, primarily Olshak V. [7-9] and his students.

2 Methods

For an inhomogeneous medium, all the basic equations of the mechanics of a deformable solid are valid. The difference lies in the fact that in inhomogeneous bodies in the relationships connecting the components of the stress and strain tensors, the mechanical characteristics are functions of the coordinates. To solve problems in the mechanics of inhomogeneous bodies, both analytical and numerical methods are usually used. To solve the axisymmetric problem considered in this chapter in spherical coordinates, a numerical-analytical method was applied.

3 Results

3.1. Equations of a three-dimensional problem in displacements

All equations and relations used below are expressed in spherical coordinates. To derive the equilibrium equations in displacements, substituting the Cauchy relations into Hooke's law in the Lame form, we obtain expressions in terms of stress displacements:

$$
\begin{gathered}
\sigma_r = \lambda\left(\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{2u}{r} + \frac{v}{r}\operatorname{ctg}\theta + \frac{1}{r\sin\theta}\frac{\partial w}{\partial \varphi}\right) + 2\mu\frac{\partial u}{\partial r} - 3K\varepsilon_{‰};\\
\sigma_\theta = \lambda\left(\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{2u}{r} + \frac{v}{r}\operatorname{ctg}\theta + \frac{1}{r\sin\theta}\frac{\partial w}{\partial \varphi}\right) + 2\mu\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right) - 3K\varepsilon_{‰};\\
\sigma_\varphi = \lambda\left(\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{2u}{r} + \frac{v}{r}\operatorname{ctg}\theta + \frac{1}{r\sin\theta}\frac{\partial w}{\partial \varphi}\right) + 2\mu\left(\frac{u}{r} + \frac{v}{r}\operatorname{ctg}\theta + \frac{1}{r\sin\theta}\frac{\partial w}{\partial \varphi}\right) - 3K\varepsilon_{‰}; \qquad (1)\\
\tau_{r\theta} = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right); \qquad \tau_{\theta\varphi} = \mu\left[\frac{1}{r}\left(\frac{\partial w}{\partial \theta} - w\operatorname{ctg}\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v}{\partial \varphi}\right];\\
\tau_{\varphi r} = \mu\left[\frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi} + \frac{\partial w}{\partial r} - \frac{w}{r}\right].
\end{gathered}
$$

Substitution (1) into the equilibrium equations, taking into account the dependences of the mechanical characteristics λ,μ and r,θ,φ of the material and K on the coordinates, leads to a system of equilibrium equations in displacements [10]:

$$
\mu\nabla^2 u - \frac{2\mu}{r^2}\left(u + \frac{\partial v}{\partial \theta} + v\operatorname{ctg}\theta + \frac{1}{\sin\theta}\frac{\partial w}{\partial \varphi}\right) + 2\frac{\partial \mu}{\partial r}\frac{\partial u}{\partial r} + \frac{1}{r}\frac{\partial \mu}{\partial \theta}\left(\frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial \mu}{\partial \varphi}\left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi} + \frac{\partial w}{\partial r} - \frac{w}{r}\right) = 0; \qquad (2)
$$

$$\mu\nabla^2 v+\frac{\mu}{r^2}\left(2\frac{\partial u}{\partial\theta}-\frac{v}{\sin^2\theta}-\frac{2\operatorname{ctg}\theta}{\sin\theta}\frac{\partial w}{\partial\varphi}\right)+\frac{\partial\mu}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial\theta}+\frac{\partial v}{\partial r}-\frac{v}{r}\right)+\frac{2}{r^2}\frac{\partial\mu}{\partial\theta}\left(u+\frac{\partial v}{\partial\theta}\right)+\frac{1}{r^2\sin\theta}\frac{\partial\mu}{\partial\varphi}\left(\frac{1}{\sin\theta}\frac{\partial v}{\partial\varphi}+\frac{\partial w}{\partial\theta}-w\operatorname{ctg}\theta\right)=0; \tag{3}$$

$$\mu\nabla^2 w+\frac{\mu}{r^2\sin\theta}\left(2\frac{\partial u}{\partial\varphi}+2\operatorname{ctg}\theta\frac{\partial v}{\partial\varphi}-\frac{w}{\sin\theta}\right)+\frac{\partial\mu}{\partial r}\left(\frac{1}{r\sin\theta}\frac{\partial u}{\partial\varphi}+\frac{\partial w}{\partial r}-\frac{w}{r}\right)+\frac{1}{r}\frac{\partial\mu}{\partial\theta}\left(\frac{1}{r\sin\theta}\frac{\partial v}{\partial\varphi}+\frac{1}{r}\frac{\partial w}{\partial\theta}+w\operatorname{ctg}\theta\right)+ \tag{4}$$

$$+\frac{2}{r^2\sin\theta}\frac{\partial\mu}{\partial\varphi}\left(u+v\operatorname{ctg}\theta+\frac{1}{\sin\theta}\frac{\partial w}{\partial\varphi}\right)=0.$$

In these equalities

$$\nabla^2=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2};$$

$$3\varepsilon_{cp}=\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{2u}{r}+\frac{v}{r}\operatorname{ctg}\theta+\frac{1}{r\sin\theta}\frac{\partial w}{\partial\varphi}.$$

Equations (2-4) are the most general relations of the problem of the theory of elasticity of continuously inhomogeneous bodies in spherical coordinates. From them, one can obtain various special cases. For example, assuming $\frac{\partial}{\partial\varphi}=0$ and $w=0$, one can show that equation (8.4) is satisfied identically, and from (2) and (3) one can obtain equations corresponding to an axisymmetric torsion-free problem:

$$\mu\nabla^2 u-\frac{2\mu}{r^2}\left(u+\frac{\partial v}{\partial\theta}+v\operatorname{ctg}\theta\right)+\frac{\partial\mu}{\partial r}\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial\mu}{\partial\theta}\left(\frac{1}{r}\frac{\partial u}{\partial\theta}+\frac{\partial v}{\partial r}-\frac{v}{r}\right)=0; \tag{5}$$

$$\mu\nabla^2 v+\frac{\mu}{r^2}\left(2\frac{\partial u}{\partial\theta}-\frac{v}{\sin^2\theta}\right)+\frac{\partial\mu}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial\theta}+\frac{\partial v}{\partial r}-\frac{v}{r}\right)+\frac{2}{r^2}\frac{\partial\mu}{\partial\theta}\left(u+\frac{\partial v}{\partial\theta}\right)=0. \tag{6}$$

Considering bodies of a spherical shape (solid or hollow), the boundary conditions in stresses for an axisymmetric problem can be written as follows:

$$r=a,\quad \begin{matrix}\sigma_r=-p_a;\\ \tau_{r\theta}=q_a;\end{matrix}\qquad r=b,\quad \begin{matrix}\sigma_r=-p_b;\\ \tau_{r\theta}=q_b,\end{matrix} \tag{7}$$

where a and b are, respectively, the radii of the inner and outer surfaces of the thick-walled hollow ball (in particular cases, it can be $a=0$ and $b\to\infty$); p_a, p_b – normal, q_a, q_b – tangential surface loads.

3.2. Numerical-analytical method of solution axisymmetric problem

The system of differential equations of the considered problem is described by equalities (5), (6). From equalities (1) one can obtain the corresponding expressions for stresses:

$$\sigma_r=\lambda\left(\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{2u}{r}+\frac{v}{r}\operatorname{ctg}\theta\right)+2\mu\frac{\partial u}{\partial r};$$

$$\sigma_\theta=\lambda\left(\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{2u}{r}+\frac{v}{r}\operatorname{ctg}\theta\right)+2\mu\left(\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{u}{r}\right); \tag{8}$$

$$\sigma_\varphi=\lambda\left(\frac{\partial u}{\partial r}+\frac{1}{r}\frac{\partial v}{\partial\theta}+\frac{2u}{r}+\frac{v}{r}\operatorname{ctg}\theta\right)+2\mu\left(\frac{u}{r}+\frac{v}{r}\operatorname{ctg}\theta\right);\qquad \tau_{r\theta}=\mu\left(\frac{\partial v}{\partial r}-\frac{v}{r}+\frac{1}{r}\frac{\partial u}{\partial\theta}\right).$$

We will look for the solution of equations (5), (6) in the form of expansions in Fourier series in Legendre polynomials:

$$u(r,\theta)=\sum_{n=0}^{\infty}u_n(r)P_n(\cos\theta);\qquad v(r,\theta)=\sum_{n=1}^{\infty}v_n(r)\frac{dP_n(\cos\theta)}{d\theta}, \tag{9}$$

where $P_n(\cos\theta)$ is the Legendre polynomial of the n-th degree, which is the solution of the equation [10, 11]:

$$\frac{d^2 P_n(\cos\theta)}{d\theta^2} + \frac{dP_n(\cos\theta)}{d\theta}\text{ctg}\theta + n(n+1)P_n(\cos\theta) = 0. \tag{10}$$

For integer values of *n*, the Legendre polynomials form a complete orthogonal system of functions in the interval $0 \le \theta \le \pi$, so that

$$\int_0^\pi P_n(\cos\theta)\cdot P_m(\cos\theta)\cos\theta d\theta = \begin{cases} 0 \; at \; n \ne m; \\ \dfrac{2}{2n+1} \; at \; n = m. \end{cases}$$

It is known from the theory of special functions [10, 11] that the expansion of a function in a Fourier series in Legendre polynomials has the same properties as any expansion in a Fourier series, for example, in trigonometric functions.
To satisfy the boundary conditions (7) corresponding to the axisymmetric problem, the surface loads should be represented in the form of series:

$$\begin{pmatrix} p_a \\ p_b \end{pmatrix} = \sum_{n=0}^{\infty}\begin{pmatrix} p_{a,n} \\ p_{b,n} \end{pmatrix} P_n(\cos\theta); \qquad \begin{pmatrix} q_a \\ q_b \end{pmatrix} = \sum_{n=1}^{\infty}\begin{pmatrix} q_{a,n} \\ q_{b,n} \end{pmatrix}\frac{dP_n(\cos\theta)}{d\theta}. \tag{11}$$

The expansion coefficients are determined by the formulas:

$$\begin{pmatrix} p_{a,n} \\ p_{b,n} \end{pmatrix} = \frac{2n+1}{2}\int_{-1}^{1}\begin{pmatrix} p_a(t) \\ p_b(t) \end{pmatrix}\cdot P_n(t)dt;$$

$$\begin{pmatrix} q_{a,n} \\ q_{b,n} \end{pmatrix} = \frac{1}{\int_{-1}^{1}\left(\dfrac{dP_n(t)}{d\theta}\right)^2 dt}\int_{-1}^{1}\begin{pmatrix} q_a(t) \\ q_b(t) \end{pmatrix}\cdot\frac{dP_n(t)}{d\theta}dt.$$

Substitution of relations (9) into formulas (8) allows, using equation (10), to obtain representations in the form of series for stresses included in the boundary conditions:

$$\sigma_r = \sum_{n=0}^{\infty}\left[(\lambda+2\mu)u_n' + \frac{2\lambda}{r}u_n - \frac{\lambda n(n+1)}{r}v_n - 3Kg_n\right]P_n(\cos\theta);$$

$$\tau_{r\theta} = \sum_{n=1}^{\infty}\mu\left(v_n' - \frac{v_n}{r} + \frac{u_n}{r}\right)\frac{dP_n(\cos\theta)}{d\theta}.$$

Here and below, the prime denotes differentiation with respect to the radius.
Using relations (9) from the equilibrium equations (5), (6) we obtain a series of systems (for each n) of two ordinary differential equations for the functions $u_n(r)$ and $v_n(r)$:

$$\begin{aligned}(\lambda+2\mu)u_n'' &+ \left[\frac{2(\lambda+2\mu)}{r} + (\lambda+2\mu)'\right]\cdot u_n' - \left[\frac{n(n+1)\mu + 2(\lambda+2\mu)}{r^2} + \frac{2\lambda'}{r}\right]\cdot u_n + \\ &+ n(n+1)\left[-\frac{\lambda+\mu}{r}v_n' + \left(\frac{\lambda+3\mu}{r^2} - \frac{\lambda'}{r}\right)v_n\right] = 0;\end{aligned} \tag{12}$$

$$\mu v_n'' + \left(\frac{2\mu}{r} + \mu'\right)v_n' - \left[n(n+1)\frac{\lambda+2\mu}{r^2} + \frac{\mu'}{r}\right]v_n' + \frac{\lambda+\mu}{r}u_n' + \left[\frac{2(\lambda+2\mu)}{r^2} + \frac{\mu'}{r}\right]u_n = 0. \tag{13}$$

Equations (12), (13) must be supplemented with boundary conditions for the functions and , which are written as:

$$r=\begin{pmatrix}a\\b\end{pmatrix},\qquad \begin{aligned}&(\lambda+2\mu)u_n'+\frac{2\lambda}{r}u_n-\frac{\lambda n(n+1)}{r}v_n=\begin{pmatrix}-p_{a,n}\\-p_{b,n}\end{pmatrix};\\&\mu\left(v_n'-\frac{v_n}{r}+\frac{u_n}{r}\right)=\begin{pmatrix}q_{a,n}\\q_{b,n}\end{pmatrix}.\end{aligned}\tag{14}$$

Thus we have obtained a number of boundary value problems (for each n) described inside the domain by differential equations (12), (13), and on the surface by relations (14). The question of choosing the number of members of the Fourier series, as in the problems considered above in polar and cylindrical coordinates, should be decided on the basis of an analysis of the expansions of surface loads in Fourier series using formulas (11).

Taking into account the arbitrary nature of the dependences of the mechanical characteristics of the material on the radius, the solution of the obtained one-dimensional boundary value problems should be carried out numerically.

It should be noted that a similar numerical-analytical method for calculating thick-walled radially inhomogeneous spherical shells was also developed in [11] using other resolving functions.

3.3. Numerical solution algorithm

For the numerical solution of one-dimensional boundary value problems, two second-order ordinary differential equations (12), (13) with respect to the functions $u_n(r)$ and $v_n(r)$ are reduced to a system of four first-order equations:

$$\frac{dY_n}{dr}=A_nY_n+F_n\tag{15}$$

where is a vector of unknowns of length 4, while

$$y_{1n}=u_n;\quad y_{2n}=u_n';\quad y_{3n}=v_n;\quad y_{4n}=v_n';$$

A is a matrix of coefficients of the system with a size of 4 x 4, the coefficients of which are equal to:

$$a_{11}=0;\qquad a_{12}=1;\qquad a_{13}=a_{14}=0;$$

$$a_{21}=\frac{n(n+1)\mu}{r^2(\lambda+2\mu)}+\frac{2}{r^2}+\frac{2\lambda'}{r(\lambda+2\mu)};\qquad a_{22}=-\frac{2}{r}-\frac{\lambda'+2\mu'}{\lambda+2\mu};$$

$$a_{23}=-n(n+1)\left[\frac{\lambda+3\mu}{r^2(\lambda+2\mu)}-\frac{\lambda'}{r(\lambda+2\mu)}\right];\quad a_{24}=n(n+1)\frac{\lambda+\mu}{r(\lambda+2\mu)};$$

$$a_{31}=a_{32}=a_{33}=0;\qquad a_{34}=1;$$

$$a_{41}=-\frac{2(\lambda+2\mu)}{r^2\mu}-\frac{\mu'}{r\mu};\qquad {}_{42}=-\frac{\lambda+\mu}{r\mu};$$

$$a_{43}=n(n+1)\frac{\lambda+2\mu}{r^2\mu}+\frac{\mu'}{r\mu};\qquad a_{44}=-\frac{2}{r}-\frac{\mu'}{\mu};$$

is the vector of the right side of length 4, the components of which are expressed by the equalities:

$$f_1=f_3=0;\qquad f_2=\frac{3(Kg_n)'-R_n}{\lambda+2\mu};\qquad f_4=-\frac{T_n}{\mu}.$$

Boundary conditions (14) can also be written in matrix form:

$$r = a, b; \qquad B_n Y_n = \Phi_n, \tag{16}$$

where

$$B_n = \begin{vmatrix} b_{11} & b_{12} & b_{13} & 0 \\ b_{21} & 0 & b_{23} & b_{24} \end{vmatrix}; \qquad \Phi_n = \begin{vmatrix} \varphi_1 \\ \varphi_2 \end{vmatrix}.$$

In these equalities, the nonzero elements of the matrix B and the vector are determined by the equalities:

$$b_{11} = \frac{2\lambda}{r}; \quad b_{12} = (\lambda + 2\mu); \quad b_{13} = -\frac{\lambda n(n+1)}{r}; \quad b_{21} = \frac{\mu}{r}; \quad b_{23} = -\frac{\mu}{r}; \quad b_{24} = \mu;$$

$$\varphi_1 = 3Kg_n + \begin{pmatrix} -p_{a,n} \\ -p_{b,n} \end{pmatrix}; \quad \varphi_2 = \begin{pmatrix} q_{a,n} \\ q_{b,n} \end{pmatrix}.$$

The numerical solution of the boundary value problem described by equation (15) with boundary conditions (14) can be carried out using the matrix orthogonal sweep method.

4 Results

Consider the axisymmetric deformation problem radially inhomogeneous array with a spherical cavity radius a. On fig. 1. the calculation scheme is shown. Let us cut out from the array a sphere of radius $b >> a$, whose center coincides with the center of the cavity. The resulting thick-walled hollow ball is loaded with external normal (p_b) and tangential (q_b) loads corresponding to the repulsion of the medium:

where γ is the specific gravity of the material; H is the depth of the cavity. Body forces R and Θ are expressed in terms of equalities:

$$p_b(\theta) = \gamma(H - b\cos\theta)\left(\frac{\nu}{1-\nu} + \frac{1-2\nu}{1-\nu}\cos^2\theta\right);$$
$$q_b(\theta) = \frac{\gamma}{2}(H - b\cos\theta)\frac{1-2\nu}{1-\nu}\sin 2\theta, \tag{17}$$

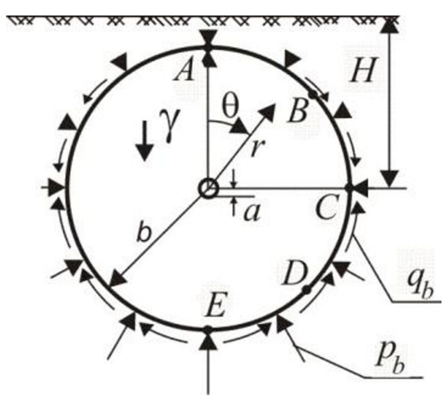

Fig. 1. Calculation scheme of an array with a spherical cavity

$$R = -\gamma\cos\theta; \qquad \Theta = \gamma\sin\theta. \tag{18}$$

In some cases, the self-weight of the cut out part of the array can be neglected. In this case, the cut array turns out to be unbalanced, however, the accepted convention does not significantly affect the results.
The radial inhomogeneity of the material is due to the centrally symmetric temperature field. In the case of a stationary regime, the temperature distribution in the array is described by the dependence

$$T(r) = \frac{1}{b-a}\left[(T_a - T_b)\frac{ab}{r} + T_b b - T_a a\right], \tag{19}$$

where T_a and T_b are the temperatures of the inner and outer surfaces of the sphere, respectively. Such a task corresponds, for example, to storage in a cavity of a product that tends to self-heat. Note that for $b \to \infty$, equality (19) transforms into formula

$$T = \frac{T_a - T_0}{r}a + T_0,$$

in which T_a – cavity contour temperature and T_0 – in the array before the explosion.
To solve the problem under consideration, the numerical-analytical method described in p.p. 2, 3.
Consider the question of choice *N*– the number of terms in the series (9) necessary to obtain sufficiently accurate results. First of all, the value *N* is determined by expansions of surface and volume loads in Fourier series. In this case, both loads are exactly represented by partial sums of series in Legendre polynomials.
The direct form of the Legendre polynomials and their derivatives is represented by the equalities [10]:

$$P_n = A_n \sum_{k=0}^{[n/2]} B_{nk} t^{n-2k}; \qquad \frac{dP_n}{dt} = A_n \sum_{k=0}^{[(n-1)/2]} C_{nk} t^{n-2k-1},$$

where

$$A_n = \frac{1}{2^n n!}; \qquad B_{nk} = \frac{n!(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!}; \qquad C_{nk} = \frac{(-1)^n(2n-2k)!}{k!(n-k)!(n-2k-1)!};$$

$t = \cos\theta$; square brackets mean the integer part of the number in them.
In accordance with the above formulas for the first four n, the Legendre polynomials have the form:

$$P_0 = 1; \quad P_1 = t; \quad P_2 = \frac{1}{2}(3t^2 - 1); \quad P_3 = \frac{1}{2}(5t^3 - 3t) \tag{20}$$

Representing, according to (11), loads p_b and q_b in the form

$$p_b = p_{b,0}P_0 + p_{b,1}P_1 + p_{b,2}P_2 + p_{b,3}P_3;$$

$$q_b = q_{b,1}\frac{dP_1}{d\theta} + q_{b,2}\frac{dP_2}{d\theta} + q_{b,3}\frac{dP_3}{d\theta}$$

and $q_{b,n}$ comparing the resulting expressions with equalities (17), we can find the coefficients $p_{bn,}$ ($n = 0,1,2,3$):

$$p_{b,0} = \frac{1+\nu}{1-\nu}\cdot\frac{\gamma H}{3}; \quad p_{b,1} = -\frac{3-\nu}{1-\nu}\cdot\frac{\gamma b}{5}; \quad p_{b,2} = \frac{1-2\nu}{1-\nu}\cdot\frac{2\gamma H}{3}; \quad p_{b,3} = -\frac{1-2\nu}{1-\nu}\cdot\frac{2\gamma b}{5};$$

$$q_{b,1} = \frac{1-2\nu}{1-\nu}\cdot\frac{\gamma b}{5}; \quad q_{b,2} = -\frac{1-2\nu}{1-\nu}\cdot\frac{\gamma H}{3}; \quad q_{b,3} = \frac{1-2\nu}{1-\nu}\cdot\frac{2\gamma b}{15}.$$

The remaining coefficients for are $n \geq 4$ equal to zero.

Similarly, comparing representations for R and Θ with formulas (18), taking into account (20), we find

$R_1 = T_1 = -\gamma$,at the same time $R_0 = 0$ and $R_n = T_n = 0$ для $n \geq 2$.

Note that, in contrast to the problems in which the numerical-analytical method was used, where the accuracy of the numerical solution was mainly determined by two parameters: *N* – the number of terms of the Fourier series and *M* – the number of steps into which the integration interval was divided, in this case, the accuracy depends only on *M*, since the finite sums of the series exactly satisfy the boundary conditions.

Thus, the solution of the problem under consideration can be obtained by numerically solving 4 boundary value problems (for n=0,1,2,3) described by the matrix differential equation (15) with boundary conditions (16). In this case, the solution of the problem for $n = 0$ is simplified, since $v_0(r) \equiv 0$.

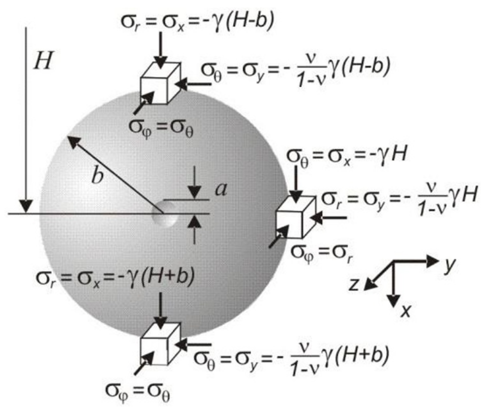

Fig. 2. To the determination of stresses on the outer surface of a spherical array

The calculation was carried out using the MOPVU program, in Fortran IV.

Let us consider an example of calculation, when the array is only under the action of external surface loads $p_b(\theta)$and $q_b(\theta)$, and the inhomogeneity of the material is due only to the explosive effect. The obtained results in comparison with some analytical data allow us to determine the required values of the number of steps (a,b) , into which the segment M is divided, and the radius of the outer surface of the cut array b .

Considering that at $a << b << H$ volume forces can be neglected, we will carry out the calculation without taking into account the self-weight of the cut-out array ($R = \Theta = 0$). In the absence of temperature influence, temperature inhomogeneity can be ignored. The modulus of elasticity, depending on *r,* is changed by the formula:

$$E(r) = E_0 \left[1 + (k-1) \left(\frac{a}{r} \right)^m \right]. \tag{21}$$

Here *k* and *m* are parameters that allow an increase or decrease in the modulus of elasticity. The accuracy of the results obtained can be estimated by comparing the calculated stresses at some points of the surface with analytical values, which partially follow from the boundary conditions, as well as from the solution of the problem of loading a solid mass, since at a sufficient distance from the cavity, the stress concentration near the cavity and the influence of local inhomogeneity can be neglected.

In table 1. formulas for stresses at characteristic points are given (see Fig. 2).

Table 1. Formulas for stresses on the outer surface cut array (*r=b*)

Points	θ	σ_r	σ_θ	σ_φ	$\tau_{r\theta}$
A	0°	$-\gamma(H-b)$	$-k^*\gamma(H-b)$	$-k^*\gamma(H-b)$	-
B	45°	$-p_b(\theta)$	-	-	$q_b(\theta)$
C	90°	$-k^*\gamma H$	$-\gamma H$	$-k^*\gamma H$	-
D	135°	$-p_b(\theta)$	-	-	$q_b(\theta)$
E	180°	$-\gamma(H+b)$	$-k^*\gamma(H+b)$	$-k^*\gamma(H=b)$	-

In table. 2 shows the results of calculating the stresses σ_θ and σ_φ at points *A, C* and *E* by the numerical-analytical method for homogeneous (in the formula (21) $k_1=1$) and inhomogeneous ($k_1=0{,}5$) materials, as well as according to the analytical formulas of Table1. At the same time, it should be noted that the formulas given in Table 1, in accordance with the assumptions made above, are valid for both homogeneous and inhomogeneous materials. The calculation was carried out with the following values of the initial data: $a=25m$; $b=250m$; $H=1200m$; $\gamma=25\,kN/m^3$; $E_0=2\cdot10^4 MPa$; $\nu=0{,}23$, $M=100$.

Table 2. Comparison of stresses calculated by numerical - analytical method, with analytical values

			Points		
Stress Method Material	Method Material	Material	A	C	E
	numerical	homogeneous	7,09	30	10,82
$-\sigma_\theta$, MPa	analytical	inhomogeneous	7,06	29,99	10,52
		any option	7,35	30,01	10,85
	numerical	homogeneous	7,09	8,96	10,82
$-\sigma_\varphi$, MPa MPa	analytical	inhomogeneous	7,06	8,95	10,52
		any option	7,35	8,89	10,85

Some differences in the given stress values can be explained by the fact that when calculating σ_θ and σ_φ, the functions u and v are numerically differentiated, which always leads to a decrease in accuracy.

In general, it can be noted that with the numerical-analytical method of calculation, the choice of the ratio $b/a=10$ and the division of the segment (a,b) into 100 steps gives quite satisfactory results.

On fig. 3 shows diagrams of normal stresses built along the horizontal radius ($\theta=90°$) and along the contour of the cavity.

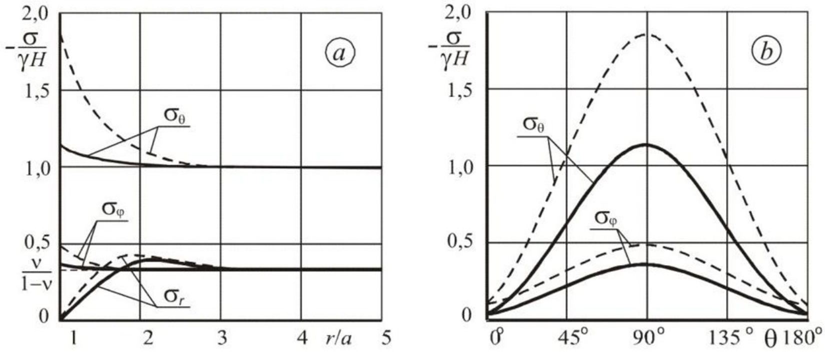

Fig. 3. Stress diagrams in an array with a spherical cavity:

a - along the radius at $\theta = 90^\circ$;

b - along the angular coordinate at и $r = a$;

- - - - homogeneous material; ——— inhomogeneous material

As in the one-dimensional problem of calculating an array with a spherical cavity, as well as in the plane problem of calculating an array with a cylindrical hole, in the zone closest to the cavity, significant differences are observed in the stress values in homogeneous and inhomogeneous arrays. In the presence of a spherical cavity, stress decay occurs faster than in the case of a cylindrical hole. It can be noted that at $r \geq 3a$, the stresses in homogeneous and inhomogeneous arrays coincide. Approximately at the same distance, the stress diagrams approach the asymptotic values corresponding to the stresses on the outer surface of the cut out spherical region. Hence, we can conclude that the assumption about the local influence on the stress state of the spherical concentrator and inhomogeneity is quite reasonable. The change in stresses along the contour of the cavity also qualitatively agrees with the results obtained for an array with a cylindrical hole.

5 Conclusions

Summing up the results of the article, it should be noted that the questions of linear and nonlinear mechanics of inhomogeneous bodies have been of interest to scientists for almost 90 years. At the beginning of the article, the authors are mentioned - the initiators of the development of the mechanics of inhomogeneous bodies. The volume of the article does not allow listing scientists who work in the field of mechanics of inhomogeneous bodies. At the end of the conclusions, a small list of works [11-20] of Russian scientists who are currently devoting their efforts to the development of the mechanics of inhomogeneous bodies is given.

Reference

1. Mikhlin S.G. Plane problem of the theory of elasticity. - Tr. seismic Institute of the Academy of Sciences of t USSR. 1935. No. 65. -84 p.
2. Kolchin G.B. Calculation of structural elements from elastic inhomogeneous materials. - Chisinau: Kartya Moldovenyaske, 1971. - 172 p.
3. Kolchin G.B. On the applicability of the iterative method in problems of the theory of elasticity inhomogeneous bodies. //Applied mathematics and programming. - Chisinau: AN MSSR, 1969,
4. Lekhnitsky S.G. Theory of elasticity of an anisotropic body. - M.:Nauka, 1977, - 415 p.

5. Lomakin V.A. Theory of elasticity of inhomogeneous bodies. - M.:MGU, 1976. - 368 p.
6. Rostovtsev N.A. On the theory of elasticity of inhomogeneous bodies //PMM. 1964. T. 28. Issue. 4. S. 601 - 611.
7. Olshak V., Rykhlevsky Y., Urbanovsky V. Theory of plasticity of inhomogeneous bodies. - M.: Mir, 1964.-156 p.
8. Olszak W., Urbanowski W., Rychlewski J. Elastic-plastic thick-walled inhomogeneous cylinder under internal pressure and longitudinal force //Arch. down. stack. 1955. Vol. VII. № 3. pp. 315 - 336.
9. Olszak W., Urbanowski W., Elastic-plastic thick-walled spherical shell made of heterogeneous material subjected to internal and external pressure //Engineering dissertations. 1956. vol.IV. № 1. pp. 23 - 41.
10. Andreev V.I. Elastic-plastic equilibrium of a hollow thick-walled ball from an inhomogeneous material // Sat. tr. Moscow. eng.-build. in-t. 1981. No. 157. pp. 78 - 87.
11. Kozlov M.L. Approximate closed solutions of the axisymmetric problem of deformation of an inhomogeneous cylinder // Izv. universities. Ser. mechanical engineering. 1977. No. 11. S. 5-11.
12. Andreev V.I. Some problems and methods of mechanics of inhomogeneous bodies, M., DIA Publishing House, 2002, 288 p.
13. Andreev V.I. Mechanics of inhomogeneous bodies, M., YURAYT, 2015, 256 p.
14. Anastasiadi G.P., Silnikov M.V. Heterogeneity and performance of steel St. Petersburg: Polygon, 2002. - 624 p.
15. Shmorgun V. G. Formation of structural-mechanical heterogeneity in layered metal and intermetallic composites created using complex technologies, Abstract of the dissertation for the degree of Doctor of Technical Sciences, Volgograd, 2007
16. Pasnichenko P. G., Gumbarov A. D. Nonlinear problems of structural mechanics (08.05.01) Construction of unique buildings and structures, Krasnodar, KubSAU 2019
17. Yaziev B.M. Nonlinear creep of continuously inhomogeneous cylinders.Diss. Ph.D. Sciences. - M., 1990. -171 p.
18. Ostrovsky G. M., Applied mechanics of inhomogeneous media / St. Petersburg: Nauka, 2000, 358.
19. Ostashev V.E. Investigation of wave propagation in inhomogeneous media using the representation of the field in the form of a series of backscattering multiplicity Abstract. dis. for the competition scientist step. cand. Phys.-Math. Sciences: (01.04.03) 17.05.2011
20. Mozgaleva M.L., Orlov V.N. Mathematical modeling, numerical methods and software packages, M., Publishing house MISI - MGSU, 2020, 40 p.