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Inhomogeneous creep equation for viscoelastic materials

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Abstract. The paper consider an inhomogeneous creep equation arising from a generalized Voigt model containing a Riemann-Liouville fractional derivative of the order $0 < \beta < 1$. The Laplace transform is used for the numerical solution. The obtained solutions are compared with experimental data of polymer concrete samples. On the basis of this comparison the conclusion about the adequacy of the numerical solution method is made, and estimates of the model parameters are given.

1 Introduction

The creep of materials is a solid body deformation change over time under a constant load. In mathematical terms, this means that the relationship between the stresses and strains of the material contains time explicitly or by means of operators.

The most important task arising in the design of new and examination of existing buildings and structures is the prediction of their service life, as well as determining the real picture of deformation of structures over time. The solution of this problem is impossible without building an adequate method of mathematical modeling of creep.

Models with standard viscous and elastic elements (the model of Maxwell, Voigt, Zener, etc.) [1] do not always correspond adequately to the experimental data, although many consist of a large number of elements and contain many parameters.

Models using fractional-order derivatives are the most suitable for describing the creep of materials with viscoelastic properties. These include, for example, some types of polymers, concretes, etc. Differential equations of fractional order arising in this case require special methods for exact or numerical solution.

The papers [2, 3] are considered fundamental for the modern theory of fractional calculus in viscoelasticity. Those ideas were developed later in many subsequent papers, e.g., [4-8] for modeling systems with damping.

An overview of creep models with viscoelastic elements is presented in [9]. In turn, the mathematical apparatus necessary for the fractional differential equations study is also continuously developing, the latest results can be found in [10].

In this paper a generalized Maxwell model under periodic loading and its corresponding inhomogeneous fractional differential equation is considered for the first time. For the numerical solution, the Laplace transform is used, with the solution image decomposed into a series. The solution itself is also obtained as a series. A good correspondence between the

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numerical solution and the experimental data for polymer concrete samples in the vibrotest has been established.

2 Inhomogeneous fractional differential equation of creep

We use a generalized Voigt model in which the viscous element is replaced by a viscoelastic one for a better creep description (Fig.1).

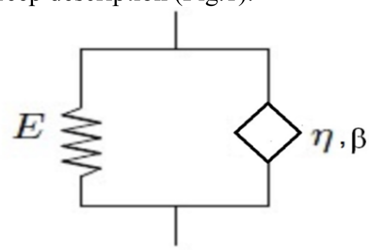

Fig. 1. Voigt generalized model.

Let us consider modeling and numerical solution of the generalized Voigt model with periodic loading. Such conditions arise naturally, for example, in the case of daily or seasonal load variations. Equation of creep under periodic (sinusoidal) load will look like [11]:

$$\sigma_0 sin(\omega t) = E\varepsilon(t) + \eta D^{\beta}\varepsilon(t) \tag{1}$$

The parameter β is determined for a particular material by experimental data, in our case it takes the value $0 < \beta < 1$. Various methods of parametric identification are described in [12]. To determine the fractional derivative of order β, we will use the Riemann-Liouville definition [13]:

$$D^{\beta}\varepsilon(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t \varepsilon(\tau)(t-\tau)^{-\beta}d\tau \tag{2}$$

3 Numerical solution method

Equation (1) has no analytical solution, approaches to numerical solution of similar equations can be found in [14-16]. Let us integrate (1) within 0 to t as done in [14]:

$$\frac{\sigma_0}{\omega}(1-\cos(\omega t)) = \int_0^t E\varepsilon(\tau)d\tau + \frac{\eta}{\Gamma(1-\beta)}\int_0^t \varepsilon(t)(t-\tau)^{-\beta}d\tau \tag{3}$$

Let us perform the Laplace transform for integral equation (3), going from original to image by the following formulas:

$$\begin{gathered} 1 \risingdotseq \frac{1}{p} \\ \cos(\omega t) \risingdotseq \frac{p}{p^2+\omega^2} \\ \int_0^t E\varepsilon(\tau)d\tau \risingdotseq \frac{F(p)}{p} \\ \int_0^t \varepsilon(t)g(t-\tau)d\tau \risingdotseq F(p)G(p) \\ t^{\beta} \risingdotseq \frac{\Gamma(1+\beta)}{p^{1+\beta}} \end{gathered} \tag{4}$$

We get the following equation for the image

$$\frac{\boldsymbol{\sigma_0}}{\boldsymbol{\omega}}\left(\frac{\mathbf{1}}{\boldsymbol{p}}-\frac{\boldsymbol{p}}{\boldsymbol{p}^2+\boldsymbol{\omega}^2}\right)=\boldsymbol{E}\frac{\boldsymbol{F}(\boldsymbol{p})}{\boldsymbol{p}}+\frac{\boldsymbol{\eta}}{\Gamma(\mathbf{1}-\boldsymbol{\beta})}\boldsymbol{F}(\boldsymbol{p})\frac{\Gamma(\mathbf{1}-\boldsymbol{\beta})}{\boldsymbol{p}^{\mathbf{1}-\boldsymbol{\beta}}} \tag{5}$$

Simplifying equation (5), we obtain an expression for the image

$$\begin{aligned}
&\frac{\sigma_0}{\eta}\left(\frac{\omega}{p^2+\omega^2}\right)=\frac{E}{\eta}F(p)+\eta F(p)p^\beta\\
&F(p)\left[p^\beta+\frac{E}{\eta}\right]=\frac{\sigma_0\omega}{\eta}\cdot\frac{1}{p^2+\omega^2}\\
&F(p)=\frac{\sigma_0\omega}{(p^2+\omega^2)(\eta p^\beta+E)}
\end{aligned} \tag{6}$$

To restore the original, let us decompose expression (6) into a series of infinitely decreasing geometric progressions in the vicinity of zero:

$$\begin{aligned}
&F(p)=\omega\cdot\frac{1}{p^{2+\beta}}\cdot\frac{1}{1+\left(\frac{\omega}{p}\right)^2}\cdot\frac{1}{1+\left(\frac{E}{\eta p}\right)^\beta}=\\
&=\omega\cdot\frac{1}{p^{2+\beta}}\cdot\sum_{n=1}^{\infty}(-1)^n\frac{\omega^{2n}}{p^{2n}}\cdot\sum_{n=1}^{\infty}(-1)^n(\frac{E}{\eta})^{\beta n}\frac{1}{p^{\beta n}}\\
&=\sum_{n=1}^{\infty}(-1)^n\frac{\omega^{2n+1}}{p^{2n+2}}\cdot\sum_{n=1}^{\infty}(-1)^n\frac{1}{p^{\beta n+\beta}}
\end{aligned} \tag{7}$$

Let's go from a product of sums to a double sum:

$$\begin{aligned}
&F(p)=\sum_{n=1}^{\infty}(-1)^n\frac{\omega^{2n+1}}{p^{2n+2}}\cdot\sum_{n=1}^{\infty}(-1)^n(\frac{E}{\eta})^{\beta n}\frac{1}{p^{\beta n+\beta}}=\\
&=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{n+k}(\frac{E}{\eta})^{\beta k}\frac{\omega^{2n+1}}{p^{2n+\beta k+2+\beta}}
\end{aligned} \tag{8}$$

Going back from the image to the original function using the formula:

$$\textstyle\sum_{n=0}^{\infty}\frac{1}{p^{n+1}}\risingdotseq\sum_{n=0}^{\infty}\frac{t^n}{\Gamma(n+1)} \tag{9}$$

We get an analytical solution of (1) as double series:

$$(t)=\textstyle\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{n+k}\cdot\omega^{2n+1}\cdot(\frac{E}{\eta})^{\beta n}\cdot\frac{t^{2n+\beta k+1+\beta}}{\Gamma(2n+\beta k+2+\beta)} \tag{10}$$

We take a finite number in each sum, for example $\mathrm{N}=20$ и $\mathrm{K}=50$ to obtain an approximate solution.

4 Numerical results and experimental data

Samples of polymer concrete based on polyester resin were taken for experimental study. Polyester resin is polyethers based on dian and dichlorohydride-1,1-dichloro-2,2 di (n-carboxyphenyl) ethylene. Although all polyester resins are similar, a wide range of mechanical properties can always be achieved by varying the basic constituents and their proportions.

Researches of samples were carried out on vibrotest, the scheme of which is resulted on Fig. 2 with parameters of loading $\sigma_0 = 20$ MPa, ω = 0,2 и ω = 0,1.

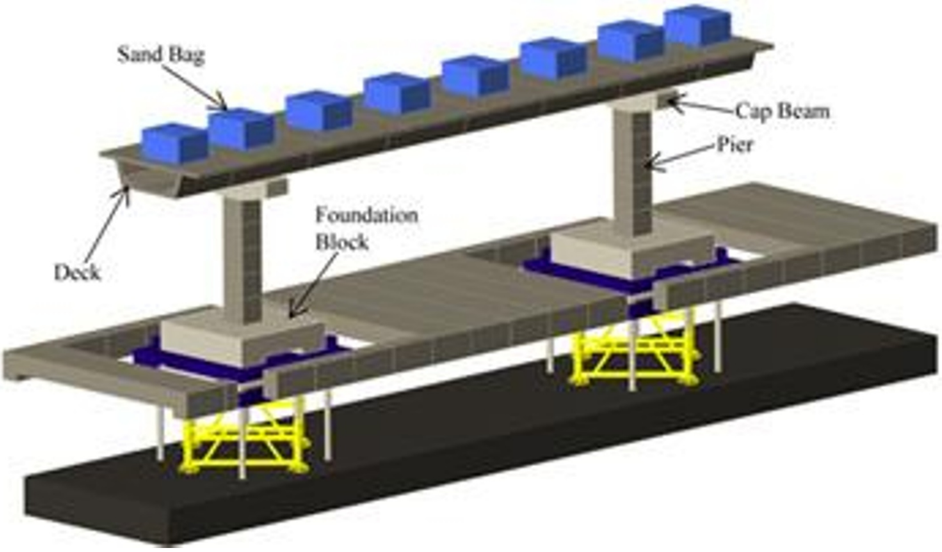

Fig. 2. Vibrotest scheme.

The calculation of mathematical model was carried out at the following parameters $\eta = 135$ МПа $\cdot\, c^{\beta}$, E = 20,35 МПа, β = 0,5 и β = 0,8 which correspond to the given samples of polymer concrete. To confirm the adequacy of calculations, the value β = 0 was added, which corresponds to the obvious analytical periodic solution.

Fig. 3-4 shows the results of numerical calculations by formula (10) at different values of parameters, performed in the Mathcad system.

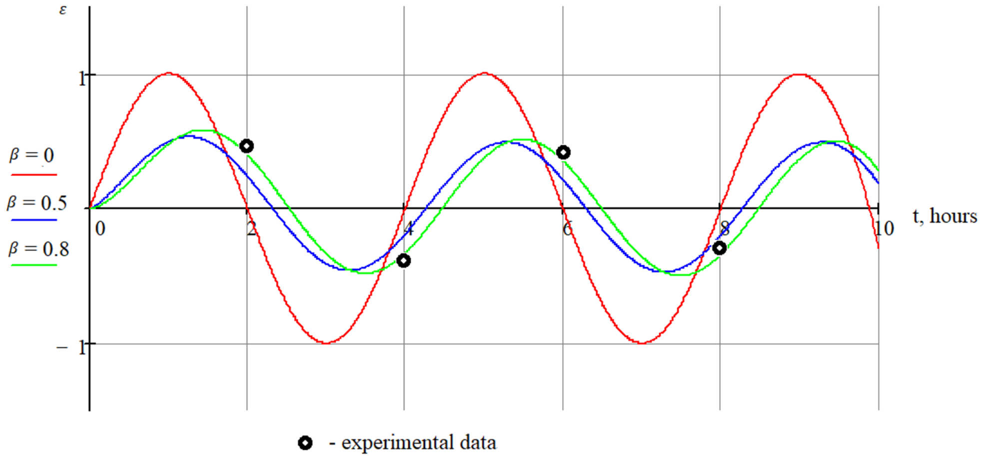

Fig. 3. Numerical solution (ω = 0,2) and experimental vibrotest data.

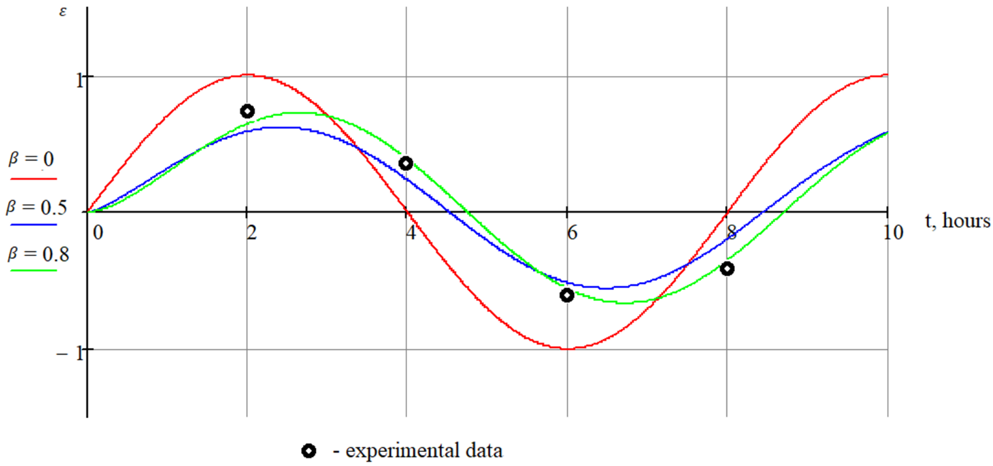

Fig. 4. Numerical solution (ω = 0,1) and experimental vibrotest data.

5 Conclusions

The paper considers a creep model of a viscoelastic material with a periodic load and the corresponding fractional differential equation with a fractional-order derivative of the Riemann-Liouville type. Using the Laplace transform, an image of the solution was obtained and the solution itself was obtained in the form of a series. Numerical experiment has demonstrated good agreement with experimental data for polymer concrete samples, which testifies to adequacy of the model and sufficient accuracy of the solution.

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