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Finding extremal eigenvalues of graphs from the Hamming and Johnson scheme

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Finding extremal eigenvalues of graphs from the Hamming and Johnson scheme

Master thesis

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Abstract

In this thesis, some extremal eigenvalues of graphs in the Hamming and Johnson scheme are studied, namely the second largest in absolute value and the smallest one. In order to visualize these eigenvalues, help with the understanding of previous known results and also find some new results, a visualization tool for the P-matrices of graphs in the Hamming and Johnson scheme was created. In the first part of this thesis, existing results on this topic from Brouwer, Cioabă, Ihringer and McGinnis [Journal of Combinatorial Theory, series B 133 (2017)] are shared, with most of their proofs worked out in detail. Next, a new theorem on the second largest eigenvalue in absolute value of graphs from the Hamming scheme is presented, along with some new observations and conjectures. Lastly, an application of the smallest eigenvalue to the max-k-cut problem is discussed, based on a paper by van Dam and Sotirov [Math. Programming 151 (2014)].

Contents

Chapter 1

Introduction and motivation

The Hamming and the Johnson scheme provide two well-known families of graphs in Algebraic Combinatorics. Hamming graphs were introduced by Richard Hamming in light of error-correcting codes during his time at Bell Labs [35]. These are graphs that have vectors in $\mathbb{F}_q^d$ as vertices and where the adjacency of two vertices depends on their Hamming distance. Johnson graphs were introduced by Selmer Johnson and consider sets instead of coordinates. In a Johnson graph, the vertices are subsets of fixed size of a set, where the adjacency of two vertices depends on the size of their intersection.

It is well known that one can derive properties from a graph based on the eigensystem of its adjacency matrix. Especially the largest, second largest (in absolute value) and smallest eigenvalue are of interest in many applications, which we will discuss later in this chapter. Since the graphs we will consider, namely graphs from the Hamming and Johnson scheme, are regular, we know what their largest eigenvalue is. Indeed, for regular graphs, the largest eigenvalue (also in absolute value) is equal to the valency of the graph. A proof of this statement can be found in Section 2.2, Theorem 2.9.

Since we know the largest eigenvalue, which is also the largest in absolute value, this thesis will focus on finding the smallest eigenvalue and the second largest eigenvalue in absolute value for graphs from the Hamming and the Johnson scheme. For ease of notation, we will sometimes refer to the latter as the 'penabsolute' eigenvalue, which is a contraction of 'absolute' and 'penultimate', as mentioned in [27]. In Sections 2.3 and 2.4, we will see that there exist closed formulas for calculating the eigenvalues of graphs from the Hamming and Johnson scheme [4]. The challenge is therefore to find out which of these eigenvalues is the smallest or second largest in absolute value, without having to calculate all of them.

One might agree that distance-regular graphs and their eigensystems are interesting and useful objects to study, regardless of their possible applications [28, Ch. 15]. In the paragraphs below, however, we will provide some additional motivation on why the smallest and penabsolute eigenvalue of the adjacency matrices of distance-regular graphs (or graphs in general) are interesting.

We start with some applications of the smallest eigenvalue of the adjacency matrix. Hoffman in [14] shows a bound on the independence number $\alpha(G)$ of a regular graph using the smallest eigenvalue of its adjacency matrix. Moreover, van Dam, Koolen and Tanaka in [28, Prop. 2.11] provide a bound on the clique number of a distance-regular graph using the same eigenvalue. Brouwer and Haemers in [5, Cor. 2] provide a condition that determines if distance-regular graphs (or k-regular k-connected graphs for some k), which are not the Petersen graph, are Hamiltonian, also using the smallest eigenvalue. In [16], Karloff uses the smallest eigenvalue of $J(2d, d, j)$ for large enough j to prove that the performance ratio of the Goemans-Williamson max-cut algorithm is precisely $\alpha = \frac{2}{\pi} \min_{0 \le \theta \le \pi} \frac{\theta}{1-\cos\theta}$, where π is the well-known mathematical constant. Previously, the performance ratio was known to be at least α. Lastly, van Dam and Sotirov in [30] use the largest Laplacian eigenvalue to find new bounds on the max-k-cut problem, with extra results on walk-regular graphs (which include distance-regular graphs) and graphs from the Hamming scheme. Some of the latter results will be studied in Chapter 6. Note that for regular graphs, the largest Laplacian eigenvalue equals the degree minus the smallest eigenvalue of the adjacency matrix of a graph.

Next, we discuss some applications of the penabsolute eigenvalue of the adjacency matrix. For ease of notation, denote the absolute value of the penabsolute eigenvalue by $|\lambda_{pen}|$ in this paragraph. A d-regular graph of n vertices is called an (n, d, λ)-*graph* if $|\lambda_{pen}| \leq \lambda$. These graphs have many interesting applications, see [26] by Sudakov, of which possibly the most well-known one is the Expander-Mixing Lemma. Moreover, a connected k-regular graph is a Ramanujan graph if $|\lambda_{pen}| \leq 2\sqrt{k-1}$, which is for instance the case for $H(4, 2, 1)$. Ramanujan graphs are known to be great expander graphs, which again have many applications (see [20]).

Since we know that penabsolute eigenvalue of regular graphs is either the smallest or the second largest eigenvalue, we also state some applications of the second largest eigenvalue of the adjacency matrix. With this eigenvalue, the algebraic connectivity can easily be calculated for regular graphs, as the algebraic connectivity equals the second smallest eigenvalue of the Laplacian. This value indicates how well connected a graph is [17]. Lastly, Koolen, Park and Yu in [18] provide a relation between the smallest and second largest eigenvalue of the adjacency matrix of a distance-regular graph, and they provide some properties of these graphs using the second largest eigenvalue.

This thesis is structured as follows. Chapter 2 will go over the required preliminaries regarding distance-regular graphs, with a special focus on graphs from the Hamming and Johnson scheme, and some preliminaries on binomial coefficients that will turn out to be very useful. Note that we expect the reader to have some preliminary knowledge on graph theory and eigensystems of matrices. In the paragraphs above, some motivation was be provided on why the smallest and penabsolute eigenvalue of distance-regular graphs (or graphs in general) are interesting. We will focus on the application to the max-k-cut problem, as shown by van Dam and Sotirov in [30], but more on this will be told in Section 6. Chapter 4 will discuss several results by Brouwer et. al. [3] about which are the smallest and penabsolute eigenvalues of graphs from the Hamming and Johnson scheme. This chapter will also provide extra details that were omitted in the original proofs from [3]. The most important results in [3] only consider large enough values of j [1], which made us wonder if there was something to say about smaller values of j. The short answer is yes, as we will see in Chapter 5. Chapter 3 will discuss the new computational tool that is used to come to conclusions about the values of j and to find new conjectures. In Chapter 6, the main conclusions of this thesis will be stated.

Lastly, it is important to note that proofs *with* a * are adaptations from or elaborations on proofs from their cited resources, whereas poofs *without* a * are constructed entirely by myself.

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[1] The meaning of this variable j will be discussed in Sections 2.3 and 2.4.

Chapter 2

Preliminaries

As noted before, some preliminary knowledge on graph theory and eigensystems of matrices are required by the reader before they continue with this thesis. The basics on distance-regular graphs, Hamming graphs and Johnson graphs are given in the next few sections, as well as some useful lemmas and identities on binomial coefficients, which we will start with.

2.1 Binomial coefficients

Binomial coefficients appear in may areas of mathematics. They are often denoted by $\binom{n}{k}$, which for integers k, n with $0 \leq k \leq n$ can be interpreted as the number of ways to choose an unordered subset of size k from a set of size n. Note that all definitions, lemmas and theorems in this section come form a book by Nienhuys and van Lint [31], unless stated otherwise. We start with some definitions.

Definition 2.1 (Falling factorial). *Let k, n be integers where $n \geq 0$, then*

$$k^{\underline{n}} = \prod_{i=0}^{n-1} (k - i) = k(k-1) \dots (k - (n-1)).$$

Using the definition of a falling factorial, we can provide two definitions for a binomial coefficient.

Definition 2.2 (Binomial coefficient). *Let k, n be integers, then*

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!} & 0 \leq k \leq n, \\ 0 & \textit{else} \end{cases} = \begin{cases} \dfrac{n^{\underline{k}}}{k!} & k, n \geq 0, \\ 0 & \textit{else.} \end{cases}$$

Note that this definition could be extended to complex numbers n, but in this thesis, only integer values of n are needed so this definition suffices. On the next few pages, we will list some lemmas on binomial coefficients that will be useful later.

Lemma 2.1 (Pascal's identity). *For integers $n, k \geq 0$, it holds that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.*

Proof. We have

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-1-k)!} + \frac{(n-1)!}{(k-1)!(n-k)!}$$

$$\begin{aligned} &= \frac{(n-1)! \cdot (n-k)}{k!(n-k)!} + \frac{(n-1)! \cdot k}{k!(n-k)!} \\ &= \frac{n!}{k!(n-k)!} \\ &= \binom{n}{k}. \end{aligned}$$

□

Lemma 2.2. *For integers a, b, c, it holds that $\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c}$.*

Proof. We have

$$\binom{a}{b}\binom{b}{c} = \frac{a!\, b!}{b!\,(a-b)!\,c!\,(b-c)!} = \frac{a!\,(a-c)!}{(a-b)!\,c!\,(b-c)!\,(a-c)!} = \binom{a}{c}\binom{a-c}{b-c}.$$

□

Proposition 2.3. *For integers a, b, c, it holds that $\binom{a}{c}\binom{b}{a} = \binom{b}{a-c}\binom{b-a+c}{c}$.*

Proof. We have

$$\binom{a}{c}\binom{b}{a} = \frac{a!b!}{c!(a-c)!a!(b-a)!} = \frac{b!}{c!(a-c)!(b-a)!} \cdot \frac{(b-a+c)!}{(b-a+c)!} = \binom{b}{a-c}\binom{d-a+c}{c}.$$

□

Lemma 2.4. *Let a, b be integers with $a, b \geq 1$. Then $\binom{a}{b} = \frac{a}{b}\binom{a-1}{b-1}$.*

Proof. We have

$$\binom{a}{b} = \frac{a!}{b!(a-b)!} = \frac{a(a-1)!}{b(b-1)!((a-1)-(b-1))!} = \frac{a}{b}\binom{a-1}{b-1}.$$

□

Lemma 2.5 (Vandermonde identity)**.** *For integers $n \geq 0$ and $b \geq a \geq 0$, it holds that*

$$\sum_{k=0}^{n} \binom{a}{k}\binom{b-a}{n-k} = \binom{b}{n}.$$

Proof. Say there are b people of which a are women and $b-a$ are men. The number of ways to construct a team of n members from these b people is $\binom{b}{n}$. This is the same as first choosing k women and then choosing $n-k$ men, where we sum up over all possible divisions of men and women, so $k = 0, ..., n$. This gives us $\sum_{k=0}^{n} \binom{a}{k}\binom{b-a}{n-k}$ possible ways to construct our team, so this sum must be equal to $\binom{b}{n}$. Note that the terms with $k > a$ give a zero term on the left hand side, which does not change the conclusion. □

Lemma 2.6. *Let $n, k \geq 0$ be integers. We can generalize Definition 2.2 for negative values of n by writing $\binom{-n}{k} = (-1)^k\binom{n+k-1}{k}$.*

Lemma 2.7. *Let $n \geq 1$ be an integer. Then $\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$.*

Note that in the following proof, the supporting text for going from line l to line $l+1$ is put after line $l+1$. This notation is used throughout the whole thesis.

Proof. We do this proof by induction on n. For $n = 1$, we have $\sum_{k=0}^{1}(-1)^k\binom{1}{k} = \binom{1}{0} - \binom{0}{0} = 0$. For the inductive step, we have

$$\begin{aligned}
\sum_{k=0}^{n}(-1)^k\binom{n}{k} &= \sum_{k=0}^{n}(-1)^k\binom{n-1}{k} + \sum_{k=0}^{n}(-1)^k\binom{n-1}{k-1} \\
&\quad \text{Pascal's identity (Lemma 2.1) applied} \\
&= (-1)^n\binom{n-1}{n} + \sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k} + (-1)^0\binom{n-1}{-1} + \sum_{k=0}^{n-1}(-1)^{k+1}\binom{n-1}{k} \\
&\quad \text{shifted the index of the second sum} \\
&= 0 \\
&\quad \text{induction hypothesis applied}
\end{aligned}$$

□

2.2 Distance-regular graphs

Distance-regular graphs are regular graphs that have some additional symmetry-related properties. They were introduced by Norman Biggs around 1974. A special case of distance-regular graphs, namely the ones with diameter two, are possibly even more well known, as these graphs are precisely the class of strongly regular graphs [28]. All definitions in this section come from a suvey by van Dam, Koolen and Tanaka [28], unless stated otherwise. We will start by giving a definition of a distance-regular graph.

Definition 2.3 (Distance-regular graph). *Let $G = (V, E)$ be a simple, connected and undirected graph with diameter d. Graph G is distance-regular if and only if there exist integers a_j, b_j, c_j for $j = 0, ..., d$ such that for every pair $x, y \in V$ with $\delta_G(x, y) = j$, all of the following statements hold:*

- *y has precisely c_j neighbors at distance $j - 1$ from x,*
- *y has precisely a_j neighbors at distance j from x,*
- *y has precisely b_j neighbors at distance $j + 1$ from x,*

where $\delta_G(x, y)$ indicates the distance between x and y.

By means of example, look at the distance-regular graph in Figure 2.1. If we take $x = X$ and $y = Y$, we see that $\delta_G(X, Y) = j = 2$. The neighbors of Y at distance $j - 1 = 1$ from X are A and B, so we should have $c_2 = 2$. Vertex Y has no neighbors at distance $j = 2$ from X, so we should have $a_2 = 0$. Lastly, the neighbor of Y at distance $j + 1 = 3$ from X is F, so we should have $b_2 = 1$. Since there is only one pair of vertices at distance two, up to graph automorphism, we conclude that this holds for all vertices at distance two. We could do this for distances zero, one and three (which is the diameter) as well, and we get the intersection numbers $c_0 = 0, a_0 = 0, b_0 = 3, c_1 = 1, a_1 = 0, b_1 = 2, c_3 = 3, a_3 = 0$ and $b_3 = 0$.

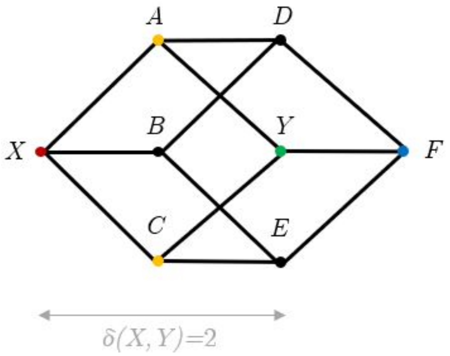

Figure 2.1: A distance-regular graph.

Next, we will note some (trivial) properties of distance-regular graphs. For every distance-regular graph G, we have

- G is regular with valency b_0, which will also be denoted by k,
- $a_j + b_j + c_j = k$ for every $j = 0, \dots, d$,
- $b_d = 0$, $c_0 = 0$, $a_0 = 0$ and $c_1 = 1$.

The array $\{b_0, \dots, b_{d-1}; c_1, \dots, c_d\}$ is called the intersection array of a distance-regular graph. Note that we do not write b_d and c_0 since they are always equal to zero, and we do not write the a_j since they can be calculated with the formula $a_j = k - b_j - c_j = b_0 - b_j - c_j$. By putting the intersection numbers in a matrix, we get the intersection matrix R, which looks like

$$R = \begin{bmatrix} 0 & b_0 & & & \\ c_1 & a_1 & b_1 & & \\ & c_2 & a_2 & \ddots & \\ & & \ddots & \ddots & b_{d-1} \\ & & & c_d & a_d \end{bmatrix},$$

where the empty spaces are filled with zeros.

Now let $G_j = (V_j, E_j)$ be the distance-j graph corresponding to G, that is,

$$V_j = V \quad \text{and} \quad E_j = \{(x,y) \in V^2 \text{ such that } \delta_G(x,y) = j\}.$$

Note that G_j is regular, but not necessarily distance-regular. If we define $G_j(x)$ for $x \in V$ as the graph induced on the vertices in V at distance 1 from x (so at distance j from x in G), we get that the number of vertices in $G_j(x)$ is constant for every $x \in V$. This follows from induction, since

$$k_0 = |V_{G_0(x)}| = 1 \text{ for every } x \in V \qquad \text{and} \qquad k_{i+1} = \frac{b_i k_i}{c_{i+1}} \text{ for } i = 0, \dots, d-1. \tag{2.1}$$

The right equality can be derived from the so-called distance-distribution diagram, which is shown in Figure 2.2.

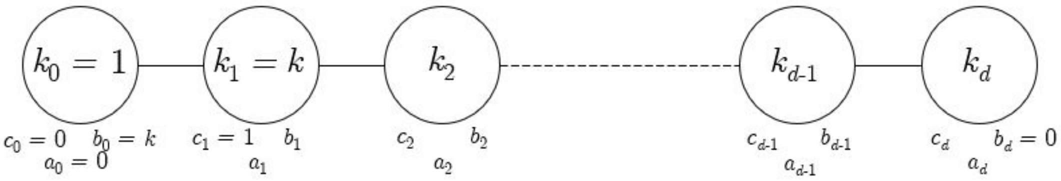

Figure 2.2: The distance-distribution diagram.

This means we have $|V| = k_0 + k_1 + \cdots + k_d$ and by formula (2.1) we have $k_{i+1} = \frac{b_0 \cdots b_i}{c_1 \cdots c_{i+1}}$. Furthermore, we know the b_i are decreasing and the c_i are increasing in i. This is summarized in the following lemma.

Lemma 2.8. *We have* $b_0 \geq b_1 \geq \cdots \geq b_{d-1}$ *and* $c_1 \leq c_2 \leq \cdots \leq c_d$.

Proof. Let $\delta(x, z) = 1$, $\delta(z, y) = i - 1$ and $\delta(x, y) = i$ for some i. The c_{i-1} neighbors of y that are at distance $i - 2$ from z are also at distance $i - 1$ from x, so $c_{i-1} \leq c_i$. The proof on b_i is similar and is left to the reader. □

We noted that the graphs G_j are regular for all j. As mentioned in the Introduction, this means that the largest eigenvalue, also in absolute value, is equal to the valency of the graph, which is $b_0 = k$. A theorem with proof summarizing this result is provided below.

Theorem 2.9. *[23] Let $G = (V, E)$ be a simple, undirected graph. The largest eigenvalue of G in absolute value equals the maximum degree if and only if G is regular. If this is the case, then the eigenvector corresponding to this eigenvalue is the all-one vector and the largest eigenvalue in absolute value is also the largest eigenvalue, absolute value omitted.*

Proof.* Let $V = \{1, \ldots, n\}$. Furthermore, let A be the adjacency matrix of graph G, let d_{max} be the maximum degree and let λ_{max} the largest eigenvalue of A in absolute value, where $v = (v_1, \ldots, v_n)$ is the eigenvector corresponding to λ_{max}. First assume G is connected. Let $j = \underset{1 \leq i \leq n}{\text{argmax}} |v_i|$. With $\mathcal{N}(j)$ we denote the vertices in the neighborhood of j. We have

$$|\lambda_{max}|\,|v_j| = |(Av)_j| = \left| \sum_{i \in \mathcal{N}(j)} v_i \right| \leq \sum_{i \in \mathcal{N}(j)} |v_i| \leq \deg(j)\,|v_j| \leq d_{max}\,|v_j|,$$

so $|\lambda_{max}| \leq d_{max}$.

Thus, if we assume $\lambda_{max} = d_{max}$, we need both $\deg(j) = d_{max}$ and $v_j = v_i$ for all $i \in \mathcal{N}(j)$. Repeating this argument for all $i \in \mathcal{N}(j)$ gives that all neighbors of i also have degree d_{max} and eigenvector entry v_j. If we continue to repeat this process until all vertices of G are covered, we get $\deg(i) = d_{max}$ and $v_i = v_j$ for all $i \in V$. This means G is regular and we have (up to a constant multiple) that $v = (1, \ldots, 1)$.

Now assume G is not connected and has m connected components. Let the vertex sets of these components be V_k such that $V = V_1 \dot{\cup} \ldots \dot{\cup} V_m$, where $\dot{\cup}$ indicates a disjoint union. Let $j_k = \underset{1 \leq i \leq n,\, i \in V_k}{\text{argmax}} |v_i|$. By the same argument as before, we need $\deg(j_k) = d_{max}$ and $v_{j_k} = v_i$ for all $i \in \mathcal{N}(j_k)$ and $1 \leq k \leq m$ in order to satisfy $\lambda_{max} = d_{max}$. Repeating this argument until all vertices of V_k are covered for every $1 \leq k \leq m$ gives that G is regular with degree d_{max}. Moreover, we can choose $v = (1, ..., 1)$.

On the other hand, if we assume that G is regular, we get $\deg(i) = d_{max}$ for all $i \in V$. This means $A \cdot (1, \ldots, 1)^T = d_{max} \cdot (1, \ldots, 1)^T$, so d_{max} is an eigenvalue with corresponding eigenvector $(1, \ldots, 1)$. Together with $|\lambda_{max}| \leq d_{max}$ from above, we know that d_{max} is the largest eigenvalue in absolute value. Note that it also is the largest eigenvalue, absolute value omitted. □

Moreover, the multiplicity of the largest eigenvalue is equal to the number of connected components of the graph [6, Prop. 1.3.7]. We end this section with a remark on the number of distinct eigenvalues of the adjacency matrix of a distance-regular graph. For a general graph G, the following lemma holds.

Lemma 2.10. *[28, Prop. 2.5] Let $G = (V, E)$ connected with diameter d, and let $\lambda_0 < \cdots < \lambda_\Delta$ be the distinct eigenvalues of the adjacency matrix A of G. Then $d + 1 \leq \Delta + 1$, where $\Delta + 1$ is the number of distinct eigenvalues of A.*

For distance-regular graphs, the inequality sign changes to an equality sign, and we know the values of these distinct eigenvalues.

Lemma 2.11. *[28, Prop. 2.6 and 2.7] Let $G = (V, E)$ connected and distance-regular with diameter d, and let $\lambda_0 < \cdots < \lambda_\Delta$ be the distinct eigenvalues of the adjacency matrix A of G. Then $d+1 = \Delta+1$, where $\Delta+1$ is the number of distinct eigenvalues of A. Moreover, the $d+1$ distinct eigenvalues of A are precisely the eigenvalues of the intersection matrix R.*

The multiplicities of these eigenvalues in A can be calculated with Biggs' formula, see [28, Thm. 2.8].

2.3 Hamming graphs

One well-known family of distance-regular graphs with classical parameters is the family of Hamming graphs. Let $q \geq 2$ and $d \geq 1$ be integers, and let Q be a set of size q. The *Hamming graph*, denoted by $H(d, q, 1)$, is the graph with vertex set Q^d where two vertices x, y share an edge if their Hamming distance equals one. That is, when the coordinates of x and y differ in exactly one place. More generally, the Hamming scheme $H(d, q)$ can be seen as the collection of graphs $H(d, q, j)$ with $0 \leq j \leq d$. These graphs have vertex set Q^d where two vertices x, y share an edge if their Hamming distance equals j. In terms of the previous section, it means that if $G = H(d, q, 1)$, then $G_j = H(d, q, j)$. Note that $H(d, q, 1)$ is distance-regular, but by Section 2.2, $H(d, q, j)$ might not be if $j \neq 1$.

One example of a Hamming graph is $H(3, 2, 1)$, which looks like a cube as can be seen in Figure 2.3 on the left. The same figure displays the Hamming graph $H(3, 3, 1)$ on the right. Note that we provide examples for small d and q since $|V| = q^d$, so the number of vertices of graphs from the Hamming scheme grows rapidly with d and q.

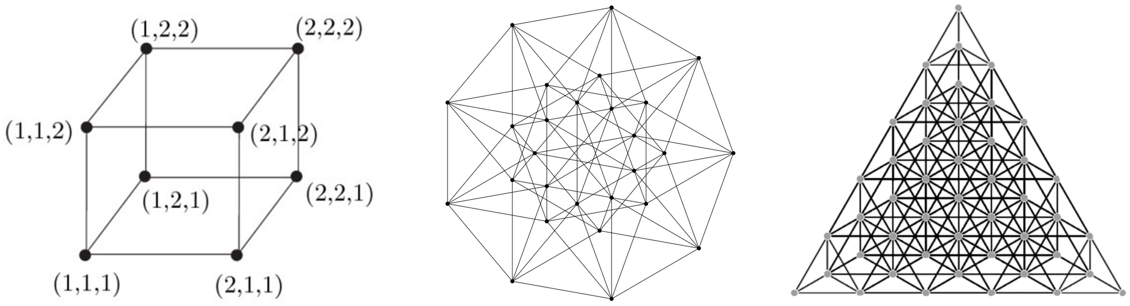

Figure 2.3: *Left:* The Hamming graph $H(3, 2, 1)$. Image taken from [22]. *Middle:* The Hamming graph $H(3, 3, 1)$. Image taken from [32]. *Right:* The Hamming graph $H(5, 4, 1)$. Image taken from [37].

Let x be a vertex of $H(d,q,j)$. Since the coordinate of x has d entries, this graph has diameter d. Furthermore, it is regular of degree $k_j = \binom{d}{j}(q-1)^j$. This is because if we want to go from vertex x to some vertex that differs exactly j places with x, we need to choose j out of d places in the coordinate of x to differ, and for each of these places we can choose one of $q-1$ values that are different from the value it had in x.

The intersection array of the Hamming graph $H(d,q,1)$ is also quite intuitive:

Lemma 2.12. *[4, Thm. 9.2.1] The intersection numbers of the graph $H(d,q,1)$ are*

$$b_j = (d-j)(q-1) \quad \textit{and} \quad c_j = j \quad \textit{for} \quad 0 \leq j \leq d.$$

Proof. Let x,y be vertices with Hamming distance j. To go from y to some vertex with Hamming distance one to y and $j-1$ to x, we choose on of j places in which the coordinate of y differs from x and change it to the value it has in x, so $c_j = j$.

To go from y to some vertex with Hamming distance one to y and $j+1$ to x, we first choose one of the $d-j$ places in which the coordinate of y matches with x. For every place, there are $q-1$ values that are different from the value it had in y. This results in $(d-j)(q-1)$ options, so $b_j = (d-j)(q-1)$. ☐

Lastly, we write down the formula for the eigenvalues of a graph from the Hamming scheme, as we will need this later on. For the sake of completeness, we also write down the multiplicity of every eigenvalue in the next two lemmas. Note that a *P-matrix* corresponding to a scheme is the matrix such that its columns correspond to the eigenvalues of the adjacency matrices of the corresponding graphs in the scheme.

Theorem 2.13 (Eigenvalues of graphs from the Hamming scheme)**.** *[1, III, Thm 2.3] Let $H(d,q)$ be a Hamming scheme. Then the eigenmatrix P of $H(d,q)$ has entries $P_{ij} = K_j(i)$, where*

$$K_j(i) = \sum_{h=0}^{j} (-1)^h (q-1)^{j-h} \binom{i}{h} \binom{d-i}{j-h}$$

for $0 \leq i,j \leq d$. Thus, the eigenvalues of graph $H(d,q,j)$ are the values in column j of P.

Lemma 2.14. *[30, p. 229] For $0 \leq i \leq d$, the multiplicity of the i^{th} eigenvalue of the graph $H(d,q,j)$ is given by*

$$m_i = \binom{d}{i}(q-1)^i.$$

Note that the first column of P equals $(1,\ldots,1)^T$, since $K_0(i) = (-1)^0(q-1)^0\binom{i}{0}\binom{d-i}{0} = 1$ for all $0 \leq i \leq d$. Also note that the first row equals $(k_1,\ldots,k_d)$, since $K_j(0) = (q-1)^j\binom{d}{j} = k_j$ for all values $0 \leq j \leq d$. Lastly, we know from Theorem 2.11 that the eigenvalues in the column corresponding to $j=1$ are all distinct.

As we noted in the Introduction, we wish to find the smallest and penabsolute [1] eigenvalue of the graphs $H(d,q,j)$. In other words, we wish to know, for given d,q,j, the values of i_{min} and i_{pen} such that $K_j(i_{min}) \leq K_j(i)$ for $0 \leq i \leq d$ and $|K_j(i_{pen})| \geq |K_j(i)|$ for $1 \leq i \leq d$. The last inequality excludes $i=0$, since we have shown in Theorem 2.9 that the valency of a regular graph (which is k_j for $H(d,q,j)$) is the largest eigenvalue, also in absolute value. Results about the values of i_{min} and i_{pen} for given d,q and j are provided in Sections 4.1 and 5.1.

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[1] Recall that by ‘penabsolute’, we mean ‘second largest in absolute value’.

Example. Consider the Hamming scheme $H(7,3)$. For the valencies of the distance-j graphs, we have

$$k_j = \binom{d}{j}(q-1)^j = \binom{7}{j}2^j, \quad \text{so} \quad (k_0, k_1, \ldots, k_7) = (1, 14, 84, 280, 560, 672, 448, 128).$$

The intersection array can be calculated with Lemma 2.12:

$$\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\} = \{14, 12, 10, 8, 6, 4, 2; 1, 2, 3, 4, 5, 6, 7\},$$

and the intersection matrix R is

$$R = \begin{bmatrix} 0 & 14 & & & & & & \\ 1 & 1 & 12 & & & & & \\ & 2 & 2 & 10 & & & & \\ & & 3 & 3 & 8 & & & \\ & & & 4 & 4 & 6 & & \\ & & & & 5 & 5 & 4 & \\ & & & & & 6 & 6 & 2 \\ & & & & & & 7 & 7 \end{bmatrix}.$$

The eigenvalues of R are $\{14, 11, 8, 5, 2, -1, -4, -7\}$, all with multiplicity one. The P-matrix of $H(7,3)$ looks like

$$P = \begin{bmatrix} 1 & 14 & 84 & 280 & 560 & 672 & 448 & 128 \\ 1 & 11 & 48 & 100 & 80 & -48 & -128 & -64 \\ 1 & 8 & 21 & 10 & -40 & -48 & 16 & 32 \\ 1 & 5 & 3 & -17 & -16 & 24 & 16 & -16 \\ 1 & 2 & -6 & -8 & 17 & 6 & -20 & 8 \\ 1 & -1 & -6 & 10 & 5 & -21 & 16 & -4 \\ 1 & -4 & 3 & 10 & -25 & 24 & -11 & 2 \\ 1 & -7 & 21 & -35 & 35 & -21 & 7 & -1 \end{bmatrix}$$

by Theorem 2.13. Note that the first column is indeed an all-one vector and that the first row is equal to $(k_0, \ldots, k_d)$. Moreover, the second column (so the one that corresponds to $j = 1$) has all distinct values, which are precisely the eigenvalues of R.

2.4 Johnson graphs

Another relevant family of distance-regular graphs with classical parameters is the class of Johnson graphs. Let $n, d \geq 1$ be integers. The Johnson graph, denoted by $J(n, d, 1)$, has all d-subsets of set of size n as its vertices, and two vertices share and edge if they meet in a $(d-1)$-set. This implies that the number of vertices is $|V| = \binom{n}{d}$.

More generally, the Johnson scheme $J(n, d)$ can be seen as the collection of graphs $J(n, d, j)$ for $0 \leq j \leq d$. These graphs have all d-subsets of a set of size n as its vertices, and two vertices share and edge if they meet in a $(d-j)$-set. As in the Hamming case, the relation to Section 2.2 is that if $G = J(n, d, 1)$, then $G_j = J(n, d, j)$, and thus $J(n, d, j)$ need not to be distance-regular for $j \neq 1$. Note that the graphs $J(n, d, j)$ and $J(n, n-d, j)$ are isomorphic, so we will assume $n \geq 2d$. If $j = d$, we call it the Kneser graph $K(n, d)$, so $J(n, d, d) = K(n, d)$.

An example of a Johnson graph is $J(5, 2, 1)$, which is the complement of the well-known Petersen graph. This graph is portrayed in Figure 2.4 on the left. The Johnson graph $J(4, 2, 1)$ is displayed on the right.

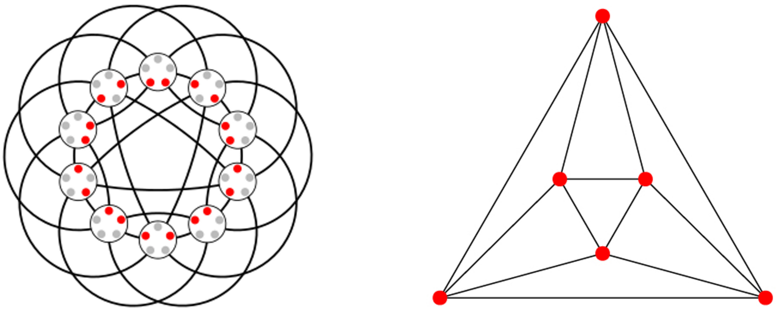


Figure 2.4: *Left:* The Johnson graph $J(5, 2, 1)$. Image taken from [33]. *Right:* The Johnson graph $J(4, 2, 1)$. Image taken from [38].

The diameter of $J(n, d, j)$ is $\min\{d, n-d\}$, which is equal to d since we assumed $n \geq 2d$. Furthermore, the graph $J(n, d, j)$ is regular of degree $k_j = \binom{d}{j}\binom{n-d}{j}$. This is because if we want to go from some set X to another set at distance j, we first delete j out of d elements in X and then add j out of $n-d$ elements in X^c. This results in a total of $\binom{d}{j}\binom{n-d}{j}$ options, so $k_j = \binom{d}{j}\binom{n-d}{j}$.

Like with the graphs from the Hamming scheme, the intersection array of a graph from the Johnson scheme is quite intuitive:

Lemma 2.15. *[4, Thm. 9.1.2] The intersection numbers of the graph $J(n, d, 1)$ are*

$$b_j = (d-j)(n-d-j) \quad \text{and} \quad c_j = j^2.$$

Proof. Let X and Y be sets that meet in a $(d-j)$-set. We have $|X| = |Y| = d$, $|X \cap Y| = d - j$ and thus $|X \backslash Y| = |Y \backslash X| = j$. To go from Y to some vertex that meets Y in a $(d-1)$-set and X in a $(d-(j-1))$-set, we delete one element in $Y \backslash X$ and add one element in $X \backslash Y$. There are $|Y \backslash X| \cdot |X \backslash Y| = j^2$ ways to do this, so $c_j = j^2$.

To go from Y to some vertex that meets Y in a $(d-1)$-set and X in a $d-(j+1))$-set, we delete one element in $X \cap Y$ and add one element in $(X \cup Y)^c$. There are $|X \cap Y| \cdot |(X \cup Y)^c| = (d-j) \cdot (n - |X \cap Y| - |X \backslash Y| - |Y \backslash X|) = (d-j)(n-d-j)$ ways to do this, so $b_j = (d-j)(n-d-j)$. $\square$

Lastly, we provide the formula for the eigenvalues of $J(n, d, j)$ and their multiplicities, as we will need this later on. For the sake of completeness, we also provide the multiplicity of the eigenvalues.

Theorem 2.16 (Eigenvalues of graphs from the Johnson scheme). *[1, III, Thm. 2.10] Let $J(n,d)$ be a Johnson scheme. Then the eigenmatrix P of $J(n,d)$ has entries $P_{ij} = E_j(i)$, where*

$$E_j(i) = \sum_{h=0}^{j} (-1)^{j-h} \binom{d-i}{h} \binom{d-h}{j-h} \binom{n-d-i+h}{h}$$

for $0 \leq i, j \leq d$. Thus, the eigenvalues of graph $J(n,d,j)$ are the values in column j of P.

Lemma 2.17. *[4, Thm. 9.1.2] For $0 \leq i \leq d$, the multiplicity of the i^{th} eigenvalue of the graph $J(n,d,j)$ is given by*

$$m_i = \binom{n}{i} - \binom{n}{i-1}.$$

Note that, like for the Hamming scheme, the first column of P equals $(1, \ldots, 1)^T$, since $E_0(i) = 1$ for all $0 \leq i \leq d$. Also note that first row equals $(k_1, \ldots, k_d)$. The last result becomes trivial when formula (4.11c) is introduced in Section 4.2. Lastly, we know from Theorem 2.11 that the eigenvalues in the column corresponding to $j = 1$ are all distinct.

From Theorem 2.16, we can derive the eigenvalues of Kneser graphs. Recall that Kneser graphs are graphs from the Johnson scheme where $j = d$.

Corollary 2.18. *[3, Prop. 3.1] The eigenvalues of the Kneser graph are*

$$P(i,d) = (-1)^i \binom{n-d-i}{d-i} = (-1)^i \binom{n-d-i}{n-2d}.$$

Proof.* Recall that the Kneser graph is a graph from the Johnson scheme with $j = d$. Therefore, we have

$$\begin{aligned} P(i,d) = E_d(i) = & \sum_{h=0}^{i} (-1)^{i-h} \binom{i}{h} \binom{d-h}{d} \binom{n-d-i+h}{n-2d} \\ & \text{formula 4.11c applied} \\ = & (-1)^i \binom{n-d-i}{n-2d} \\ & \text{term for } h = 0 \text{ is the only nonzero term} \\ = & (-1)^i \binom{n-d-i}{d-i}. \\ & \text{symmetry applied} \end{aligned}$$

□

As for graphs from the Hamming scheme, we wish to know which of the eigenvalues of $J(n,d,j)$ is the smallest and which is the penabsolute. In other words, we want to find, for given n, d, j, the values of i_{min} and i_{pen} such that $E_j(i_{min}) \leq E_j(i)$ for $0 \leq i \leq d$ and $|E_j(i_{pen})| \geq |E_j(i)|$ for $1 \leq i \leq d$. The last inequality again excludes $i = 0$ because of Theorem 2.9. Results about the values of i_{min} and i_{pen} for given n, d and j are provided in Sections 4.2 and 5.2.

Example. Consider the Johnson scheme $J(12,6)$. For the valency of the distance-j graphs, we have

$$k_j = \binom{d}{j} \binom{n-d}{j} = \binom{6}{j}^2, \quad \text{so} \quad (k_0, k_1, \ldots, k_6) = (1, 36, 225, 400, 225, 36, 1).$$

The intersection array can be calculated with Lemma 2.15:

$$\{b_0, \ldots, b_{d-1}; c_1, \ldots, c_d\} = \{36, 25, 16, 9, 4, 1; 1, 4, 9, 16, 25, 36\},$$

and the intersection matrix R is

$$R = \begin{bmatrix} 0 & 36 & & & & & \\ 1 & 10 & 25 & & & & \\ & 4 & 16 & 16 & & & \\ & & 9 & 18 & 9 & & \\ & & & 16 & 16 & 4 & \\ & & & & 25 & 10 & 1 \\ & & & & & 36 & 0 \end{bmatrix}$$

The eigenvalues of R are $\{36, 24, 14, 6, 0, -4, -6\}$, all with multiplicity one. The P-matrix of $J(12, 6)$ looks like

$$P = \begin{bmatrix} 1 & 36 & 225 & 400 & 225 & 36 & 1 \\ 1 & 24 & 75 & 0 & -75 & -24 & -1 \\ 1 & 14 & 5 & -40 & 5 & 14 & 1 \\ 1 & 6 & -15 & 0 & 15 & -6 & -1 \\ 1 & 0 & -9 & 16 & -9 & 0 & 1 \\ 1 & -4 & 5 & 0 & -5 & 4 & -1 \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 \end{bmatrix}$$

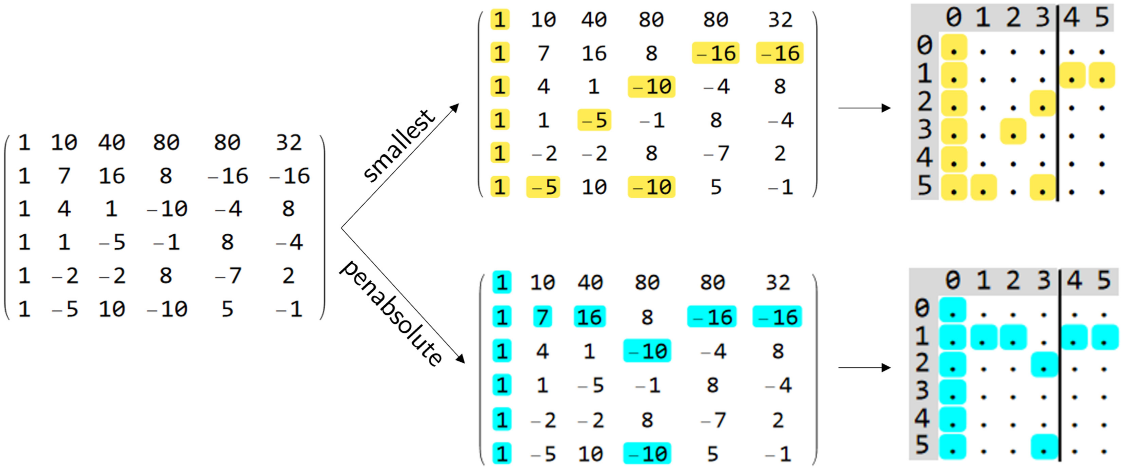
Note that the first column is indeed an all-one vector and that the first row is equal to $(k_0, \ldots, k_d)$. Moreover, the second column (so the one that corresponds to $j = 1$) has all distinct values, which are precisely the eigenvalues of R.

Chapter 3

Visualization of P-matrices

When we started reading the paper from Hamming et. al. [3], we noted that the results that are shared in the paper are quite abstract. The authors of this paper did not include any figures and very little context to visualize their results, which makes the paper difficult to read. This chapter's goal is therefore to create a tool that helps the reader to visualize the P-matrices of Hamming and Johnson schemes, which in turn helps with the understanding of the results that are studied in Chapter 4 of this thesis. This tool will also be used to find new results, as will be discussed in Chapter 5.

The first step in creating this visualization tool is calculating the P-matrix of the desired scheme, using Theorem 2.13 for the Hamming scheme and Theorem 2.16 for the Johnson scheme. An example for the Hamming scheme $H(5,3)$ can be seen in Figure 3.1 on the left. Recall that the columns are indexed by values of j and that the rows are indexed by values of i with $0 \leq i, j \leq d$. Moreover, recall that for this scheme we have $d = 5$ and $q = 3$. Next, we are interested in either the smallest or the penabsolute eigenvalue per column, so we highlight these values, as can be seen in Figure 3.1 in the center for $H(5,3)$.



$$\begin{pmatrix} 1 & 10 & 40 & 80 & 80 & 32 \\ 1 & 7 & 16 & 8 & -16 & -16 \\ 1 & 4 & 1 & -10 & -4 & 8 \\ 1 & 1 & -5 & -1 & 8 & -4 \\ 1 & -2 & -2 & 8 & -7 & 2 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{pmatrix}$$

Figure 3.1: A visual representation of the P-matrix of $H(5,3)$. Note that the columns are indexed by values of j and the rows are indexed by values of i.

Since we are only interested in the *position* of the smallest/penabsolute eigenvalue, and since these eigenvalues can get very large for bigger values of n, d or q, it makes sense to replace the entries

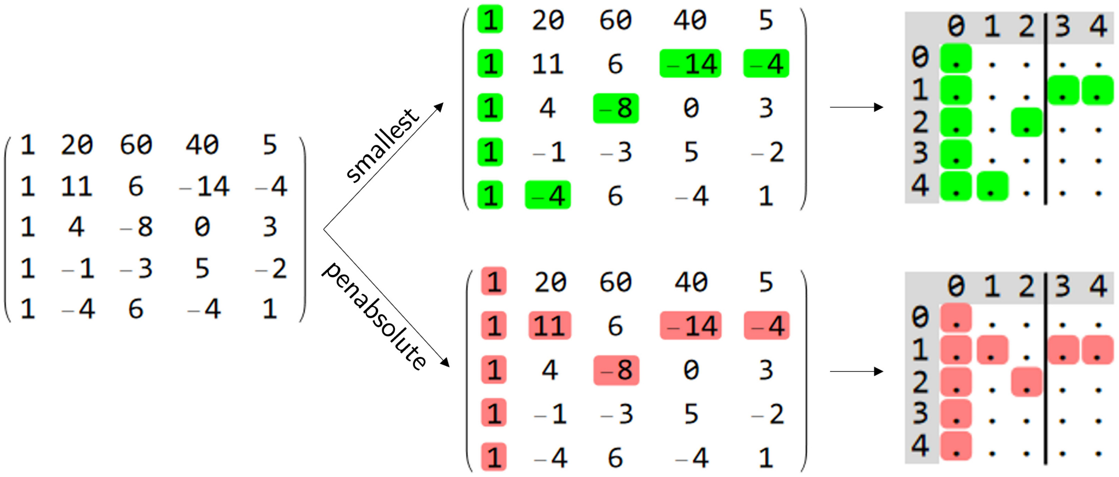


Figure 3.2: A visual representation of the P-matrix of $J(9, 4)$. Note that the columns are indexed by values of j and the rows are indexed by values of i.

of the P-matrix by dots. Additionally, we will see in Sections 4.1 and 4.2 that many theorems assume a bound on j, namely $j \geq d - \frac{d-1}{q}$ for Hamming and $j \geq \frac{d(n-d)}{n-1}$ for Johnson. Therefore it is convenient to place a black vertical line between columns $\lceil d - \frac{d-1}{q} \rceil$ and $\lceil d - \frac{d-1}{q} \rceil - 1$ and between columns $\lceil \frac{d(n-d)}{n-1} \rceil$ and $\lceil \frac{d(n-d)}{n-1} \rceil - 1$ for the Hamming and Johnson scheme respectively. To finish up we add headers, and this produces a nice visual representation of the information that we need from the desired P-matrix. This can be seen in Figure 3.1 on the right for $H(5, 3)$. In Figure 3.2, we provide an additional example for the Johnson scheme $J(9, 4)$. Recall that for this scheme we have $n = 9$ and $d = 4$.

The code used to create this visualization tool can be found in Appendix A. The rest of this thesis will include several visualizations of the form on the right in Figures 3.1 and 3.2 to help clarify the theorems shared in this thesis.

Chapter 4

Existing results

In this chapter, we will discuss results on the smallest and penabsolute eigenvalue of graphs in the Hamming and Johnson scheme that were shown by Brouwer, Cioabă, Ihringer and McGinnis in [3]. Most of the lemmas and theorems in [3] are accompanied by a proof, however, these proofs often give limited details. Therefore, and for the sake of completeness, I will include (elaborated, if necessary) proofs for all lemmas and theorems in the next two sections. This chapter also contains some new results, which can be recognized by the lack of citation. We will start with results on graphs in the Hamming scheme, after which we will discuss results on graphs in the Johnson scheme.

4.1 The Hamming case

Since this section is rather long, we start with an overview of the most important results that we will show in this subsection:

- For $j \geq d - \frac{d-1}{q}$, with the additional condition that j is even or $j = d$ if $q = 2$, we have that $K_j(1)$ is the smallest eigenvalue. *(Corollary 4.13* and *Theorem 4.14b)*
- For $q = 2$, we have that $|K_j(1)|$ is the penabsolute eigenvalue if $j \neq \frac{d}{2}$. If $j = \frac{d}{2}$, then $|K_j(2)|$ is the penabsolute eigenvalue. *(Theorem 4.11* and *Corollary 4.12)*
- For $q \geq 3$ and $j \geq d - \frac{d-1}{q}$, we have that $|K_j(1)|$ is the penabsolute eigenvalue except if $(q, d, j) = (3, 4, 3)$. In the latter case, $|K_j(3)|$ is the penabsolute eigenvalue. *(Theorem 4.14a)*

To prove these results, we will need a lot of intermediate steps in the form of lemmas, propositions and corollaries. Recall from Theorem 2.13 that the formula for the eigenvalues of graphs from the Hamming scheme is

$$K_j(i) = \sum_{h=0}^{j} (-1)^h (q-1)^{j-h} \binom{i}{h} \binom{d-i}{j-h}$$

for $0 \leq i, j \leq d$. This polynomial is also called a *Kravchuk or Krawtchouk polynomial*, introduced by Mykhailo Kravchuk in 1929 [34]. There are multiple equivalent ways to write down these polynomials. Three of them are provided in the following lemma.

Lemma 4.1 (Equivalence of Kravchuk polynomials)**.** *[3, p. 92] The following three expressions for the Kravchuk polynomials are equivalent:*

$$K_j(i) = \sum_{h=0}^{j} (-1)^h (q-1)^{j-h} \binom{i}{h}\binom{d-i}{j-h} \tag{4.1a}$$

$$= \sum_{h=0}^{j} (-q)^h (q-1)^{j-h} \binom{i}{h}\binom{d-h}{j-h} \tag{4.1b}$$

$$= \sum_{h=0}^{j} (-1)^h q^{j-h} \binom{d-i}{j-h}\binom{d-j+h}{h}. \tag{4.1c}$$

Proof. The first expression follows from Theorem 2.13. We start with showing that the third expression is true.

$$\begin{aligned}
K_j(i) &= \sum_{h=0}^{j} (-1)^h (q-1)^{j-h} \binom{i}{h}\binom{d-i}{j-h} \\
&= \sum_{h=0}^{j} (-1)^h \left(\sum_{l=0}^{j-h} \binom{j-h}{l} q^l (-1)^{j-h-l}\right) \binom{i}{h}\binom{d-i}{j-h} \\
&\quad \text{binomial formula applied} \\
&= \sum_{l=0}^{j}\sum_{h=0}^{j-l} (-1)^{j-l} q^l \binom{i}{h}\binom{j-h}{l}\binom{d-i}{j-h} \\
&\quad \text{change of summation} \\
&= \sum_{h=0}^{j}\sum_{l=0}^{j-h} (-1)^{j-h} q^h \binom{i}{l}\binom{j-l}{h}\binom{d-i}{j-l} \\
&\quad \text{interchange names of } l \text{ and } h \\
&= \sum_{h=0}^{j}\sum_{l=0}^{j-h} (-1)^{j-h} q^h \binom{i}{l}\binom{d-i}{h}\binom{d-i-h}{j-l-h} \\
&\quad \text{Lemma 2.2 applied} \\
&= \sum_{k=0}^{j} (-1)^k q^{j-k} \binom{d-i}{j-k} \sum_{l=0}^{k} \binom{i}{l}\binom{d-i-j+k}{k-l} \\
&\quad \text{change of variables } k = j-h \\
&= \sum_{k=0}^{j} (-1)^k q^{j-k} \binom{d-i}{j-k}\binom{d-j+k}{k}. \\
&\quad \text{Vandermonde identity (Lemma 2.5) applied}
\end{aligned}$$

Next, we show that the second expression is equivalent to the third. We have

$$\begin{aligned}
K_j(i) &= \sum_{h=0}^{j} (-1)^h q^{j-h} \binom{d-j+h}{h}\binom{d-i}{j-h} \\
&= \sum_{h=0}^{j} (-1)^h q^{j-h} \binom{d-j+h}{h} (-1)^{j-h} \binom{j-h-d-1+i}{j-h} \\
&\quad \text{Lemma 2.6 applied}
\end{aligned}$$

$$=\sum_{h=0}^{j}(-1)^h q^{j-h}\binom{d-j+h}{h}\sum_{l=0}^{j-h}(-1)^l\binom{i}{l}(-1)^{j-h-l}\binom{j-h-d-1}{j-h-l}$$

Vandermonde identity (Lemma 2.5) applied

$$=\sum_{h=0}^{j}(-1)^h q^{j-h}\binom{d-j+h}{h}\sum_{l=0}^{j-h}(-1)^l\binom{i}{l}\binom{d-l}{j-h-l}$$

Lemma 2.6 applied again

$$=\sum_{h=0}^{j}\sum_{l=0}^{j-h}(-1)^{h+l}q^{j-h}\binom{j-l}{h}\binom{d-l}{j-l}\binom{i}{l}$$

Proposition 2.3 applied

$$=\sum_{l=0}^{j}\sum_{h=0}^{j-l}(-1)^{h+l}q^{j-l}\binom{j-h}{l}\binom{d-h}{j-h}\binom{i}{h}$$

interchange names of l and h

$$=\sum_{h=0}^{j}(-q)^h\left(\sum_{l=0}^{j-h}\binom{j-h}{l}q^{j-h-l}(-1)^l\right)\binom{i}{h}\binom{d-h}{j-h}$$

change order of summation

$$=\sum_{h=0}^{j}(-q)^h(q-1)^{j-h}\binom{i}{h}\binom{d-h}{j-h}.$$

binomial formula applied

□

We start with some properties of Kravchuk polynomials that will be helpful in later proofs.

Lemma 4.2 (Kravchuk symmetry 1)**.** *[3, p. 92] Let $K_j(i)$ be a Kravchuk polynomial. Then*

$$\frac{K_j(i)}{(q-1)^j\binom{d}{j}}=\frac{K_i(j)}{(q-1)^i\binom{d}{i}}.$$

Proof.

$$\frac{K_j(i)}{(q-1)^j\binom{d}{j}}=\sum_{h=0}^{j}\left(\frac{q}{1-q}\right)^h\frac{\binom{i}{h}\binom{d-h}{j-h}}{\binom{d}{j}}$$

expression (4.1b) applied

$$=\sum_{h=0}^{j}\left(\frac{q}{1-q}\right)^h\frac{\binom{i}{h}\binom{j}{h}}{\binom{d}{h}}$$

Lemma 2.2 applied

$$=\sum_{h=0}^{\min\{i,j\}}\left(\frac{q}{1-q}\right)^h\frac{\binom{i}{h}\binom{j}{h}}{\binom{d}{h}}.$$

since $\binom{i}{h}\binom{j}{h}=0$ for $h\geq\min\{i,j\}$

Similarly, we get

$$\frac{K_i(j)}{(q-1)^i\binom{d}{i}} = \sum_{h=0}^{\min\{i,j\}} \left(\frac{q}{1-q}\right)^h \frac{\binom{i}{h}\binom{j}{h}}{\binom{d}{h}},$$

which proves the statement. □

Lemma 4.3 (Kravchuk symmetry 2)**.** *[3, p. 92] Let $K_j(i)$ be a Kravchuk polynomial. Then*

$$K_{d-j}(i) = (-1)^{i-j}(q-1)^{d-i-j}K_j(d-i).$$

Proof. In this proof we write $K_j(i,d)$ instead of $K_j(i)$ to emphasize the value of d. This will be useful when applying induction on d later on. We first prove the statement for $j=0$ and $j=d$. For $j=0$, we get $K_d(i)=(-1)^i(q-1)^{d-i}$ and $K_0(d-i)=1$, so indeed $K_d(i)=(-1)^i(q-1)^{d-i}K_0(d-i)$. For $j=d$, we get $K_0(i)=1$ and $K_d(d-i)=(-1)^{d-i}(q-1)^i$, so indeed $K_0(i)=(-1)^{i-d}(q-1)^{-i}K_d(d-i)$.

Now assume $0<j<d$. For $K_{d-j}(i,d)$ we have

$$\begin{aligned}
K_{d-j}(i,d) =& \sum_{h=0}^{d-j}(-1)^h(q-1)^{d-j-h}\binom{i}{h}\binom{d-i}{d-j-h} \\
& \text{expression (4.1a) applied} \\
=& \sum_{h=0}^{d-j}(-1)^h(q-1)^{d-j-h}\binom{i}{h}\binom{(d-1)-i}{d-j-h} \\
& + \sum_{h=0}^{d-j}(-1)^h(q-1)^{d-j-h}\binom{i}{h}\binom{(d-1)-i}{(d-1)-j-h} \\
& \text{Pascal's triangle (Lemma 2.1) applied} \\
=& \sum_{h=0}^{(d-1)-(j-1)} (-1)^h(q-1)^{(d-1)-(j-1)-h}\binom{i}{h}\binom{(d-1)-i}{(d-1)-(j-1)-h} \\
& + (q-1)\sum_{h=0}^{d-j}(-1)^h(q-1)^{(d-1)-j-h}\binom{i}{h}\binom{(d-1)-i}{(d-1)-j-h} \\
=& K_{(d-1)-(j-1)}(i,d-1) + (q-1)\sum_{h=0}^{(d-1)-j}(-1)^h(q-1)^{(d-1)-j-h}\binom{i}{h}\binom{(d-1)-i}{(d-1)-j-h} \\
& \text{if } h=d-j\text{, then } \binom{(d-1)-i}{(d-1)-j-h} = \binom{(d-1)-i}{-1} = 0 \\
=& K_{(d-1)-(j-1)}(i,d-1) + (q-1)K_{(d-1)-j}(i,d-1).
\end{aligned}$$

Next, we look at $K_j(d-i,d)$:

$$\begin{aligned}
K_j(d-i,d) =& \sum_{h=0}^{j}(-1)^h(q-1)^{j-h}\binom{d-i}{h}\binom{i}{j-h} \\
& \text{expression (4.1a) applied} \\
=& \sum_{h=0}^{j}(-1)^h(q-1)^{j-h}\binom{(d-1)-i}{h}\binom{i}{j-h} \\
& + \sum_{h=0}^{j}(-1)^h(q-1)^{j-h}\binom{(d-1)-i}{h-1}\binom{i}{j-h}
\end{aligned}$$

$$\begin{aligned}
&\text{Pascal's triangle (Lemma 2.1) applied}\\
=&K_j((d-1)-i,d-1)+\sum_{h=1}^{j}(-1)^h(q-1)^{j-h}\binom{(d-1)-i}{h-1}\binom{i}{j-h}\\
&\text{if } h=0, \text{ then } \binom{(d-1)-i}{h-1}=\binom{(d-1)-i}{-1}=0\\
=&K_j((d-1)-i,d-1)+\sum_{h=0}^{j-1}(-1)(-1)^h(q-1)^{(j-1)-h}\binom{(d-1)-i}{h}\binom{i}{(j-1)-h}\\
&\text{moved the index}\\
=&K_j((d-1)-i,d-1)-K_{j-1}((d-1)-i,d-1).
\end{aligned}$$

Now we use induction to finish the proof. For $d=0$ (and thus $i=j=0$), we have

$$K_0(0,0)=(-1)^0(q-1)^0K_0(0,0)=K_0(0,0).$$

Using the induction hypothesis $K_{(d-1)-j}(i,d-1)=(-1)^{i-j}(q-1)^{(d-1)-i-j}K_j((d-1)-i,d-1)$, we get that for $0\leq i<d$ we have

$$\begin{aligned}
K_{d-j}(i,d)=&K_{(d-1)-(j-1)}(i,d-1)+(q-1)K_{(d-1)-j}(i,d-1)\\
&\quad\text{previously derived formula applied}\\
=&(-1)^{i-j+1}(q-1)^{d-i-j}K_{j-1}((d-1)-i,d-1)\\
&\quad+(-1)^{i-j}(q-1)^{d-i-j}K_j((d-1)-i,d-1)\\
&\quad\text{induction hypothesis applied}\\
=&(-1)^{i-j}(q-1)^{d-i-j}\left(K_j((d-1)-i,d-1)-K_{j-1}((d-1)-i,d-1)\right)\\
=&(-1)^{i-j}(q-1)^{d-i-j}K_j(d-i,d)\\
&\quad\text{previously derived formula applied}
\end{aligned}$$

and for $i=d$ we have $K_{d-j}(d,d)=(-1)^{d-j}\binom{d}{d-j}$ and $K_j(0,d)=(q-1)^j\binom{d}{j}$, so

$$K_{d-j}(d,d)=(-1)^{d-j}(q-1)^{d-d-j}K_j(d-d,d).$$

□

Lemma 4.4 (Kravchuk symmetry 3). *[3, p. 94] Let $K_j(i)$ be a Kravchuk polynomial. If $q=2$, then*

$$K_j(d-i)=(-1)^jK_j(i).$$

Proof. Using expression (4.1a) and rewriting, we get

$$\begin{aligned}
K_j(d-i)&=\sum_{h=0}^{j}(-1)^h\binom{d-i}{h}\binom{i}{j-h}\\
&=\sum_{j-h=0}^{j}(-1)^h\binom{d-i}{h}\binom{i}{j-h}\\
&=\sum_{k=0}^{j}(-1)^{j-k}\binom{d-i}{j-k}\binom{i}{k}\\
&\quad\text{change of variable } k=j-h\\
&=(-1)^jK_j(i).
\end{aligned}$$

□

Lemma 4.5 (Kravchuk recursive identity)**.** *[3, Prop. 2.1] Let $K_j(i)$ be a Kravchuk polynomial and let $1 \leq i < d$ and $1 \leq j \leq d$. Then*

$$(q-1)(d-i)K_j(i+1) - (i + (q-1)(d-i) - qj)K_j(i) + iK_j(i-1) = 0.$$

Proof. First, we look at the case $j = d$. We have $K_d(i) = (-1)^i(q-1)^{d-i}$, so we need to show

$$(q-1)(d-i)(-1)^{i+1}(q-1)^{d-i-1} - (i+(q-1)(d-i)-qj)(-1)^i(q-1)^{d-i} + i(-1)^{i-1}(q-1)^{d-i+1} = 0.$$

Dividing by $(-1)^i(q-1)^{d-i}$ on both sides and expanding the brackets gives that this expression is indeed equal to zero, so we have shown the statement is true for $j = d$.

Now assume $1 \leq i, j < d$. Like before, we write $K_j(i,d)$ instead of $K_j(i)$ if we want to emphasize the value of d. From the binomial formula, we know

$$(1+x)^i = \sum_{h=0}^{i} \binom{i}{h} x^h \quad \text{and} \quad (1+x)^{d-i} = \sum_{k=0}^{d-i} \binom{d-i}{j-k} x^{j-k},$$

where on the right we did a change of variable $k = j - h$. Now let $[x^h]f(x)$ be the coefficient of x^h in the power series expansion of some function $f(x)$. Then

$$\binom{i}{h} = [x^h](1+x)^i \quad \text{and} \quad \binom{d-i}{j-h} = [x^{j-h}](1+x)^{d-i}. \tag{4.2}$$

Furthermore, for polynomials f and g we have

$$[x^j]\left(f(x)g(x)\right) = \sum_{h=0}^{j} [x^h]f(x) \cdot [x^{j-h}]g(x), \tag{4.3}$$

and for any real constant a we have

$$a^h[x^h]f(x) = [x^h]f(ax). \tag{4.4}$$

Furthermore, let $f(x) = \sum_{k=0}^{m} \alpha_k x^k$ be a polynomial with $\alpha_k \in \mathbb{R}$. Then $f'(x) = \sum_{k=1}^{m} \alpha_k k x^{k-1}$ and

$$[x^k]f'(x) = (k+1)\alpha_{k+1} \quad \text{and} \quad [x^{k+1}]f(x) = \alpha_{k+1}. \tag{4.5}$$

We use this notation and these expressions to rewrite $K_j(i,d)$ as the coefficient belonging to x^j of some function in x:

$$\begin{aligned}
K_j(i,d) =& \sum_{h=0}^{j} (-1)^h (q-1)^{j-h} \binom{i}{h}\binom{d-i}{j-h} \\
& \text{formula (4.1a) used} \\
=& \sum_{h=0}^{j} (-1)^h [x^h](1+x)^i \cdot (q-1)^{j-h}[x^{j-h}](1+x)^{d-i} \\
& \text{formula (4.2) used} \\
=& \sum_{h=0}^{j} [x^h](1-x)^i \cdot [x^{j-h}](1+(q-1)x)^{d-i} \\
& \text{line (4.4) used} \\
=& [x^j]\left((1-x)^i(1+(q-1)x)^{d-i}\right). \\
& \text{line (4.3) used}
\end{aligned}$$

We let $g_{i,d}(x) = (1-x)^i(1+(q-1)x)^{d-i}$, so that $K_j(i,d) = [x^j]g_{i,d}(x)$. Taking the derivative of g w.r.t. x, we get

$$\begin{aligned} g'_{i,d}(x) &= -\,i(1-x)^{i-1}(1+(q-1)x)^{d-i} + (1-x)^i(d-i)(q-1)(1+(q-1)x)^{d-i-1} \\ &= -\,i\, g_{i-1,d-1}(x) + (d-i)(q-1)\, g_{i,d-1}(x), \end{aligned}$$

and thus

$$\begin{aligned} [x^j]g'_{i,d}(x) &= -i[x^j]g_{i-1,d-1}(x) + (d-i)(q-1)[x^j]g_{i,d-1}(x) \\ (j+1)[x^{j+1}]g_{i,d}(x) &= -i[x^j]g_{i-1,d-1}(x) + (d-i)(q-1)[x^j]g_{i,d-1}(x) \\ &\text{formula (4.5) used} \\ (j+1)K_{j+1}(i,d) &= -iK_j(i-1,d-1) + (d-i)(q-1)K_j(i,d-1). \end{aligned}$$

Shifting the variable j in the above expression gives

$$jK_j(i,d) = -iK_{j-1}(i-1,d-1) + (d-i)(q-1)K_{j-1}(i,d-1). \tag{4.6}$$

Furthermore, we have

$$\begin{aligned} K_j(i,d) &= [x^j](1-x)^i(1+(q-1)x)^{d-i} \\ &= [x^j](1-x)^i(1+(q-1)x)^{d-i-1}(1+(q-1)x) \\ &= [x^j]\Big((1-x)^i(1+(q-1)x)^{d-i-1} + (q-1)x(1-x)^i(1+(q-1)x)^{d-i-1}\Big) \\ &= [x^j](1-x)^i(1+(q-1)x)^{d-i-1} + (q-1)[x^j]x(1-x)^i(1+(q-1)x)^{d-i-1} \\ &= [x^j](1-x)^i(1+(q-1)x)^{d-i-1} + (q-1)[x^{j-1}](1-x)^i(1+(q-1)x)^{d-i-1} \\ &= K_j(i,d-1) + (q-1)K_{j-1}(i,d-1) \end{aligned}$$

and

$$\begin{aligned} K_j(i+1,d) &= [x^j](1-x)^{i+1}(1+(q-1)x)^{d-i-1} \\ &= [x^j](1-x)^i(1+(q-1)x)^{d-i-1}(1-x) \\ &= [x^j](1-x)^i(1+(q-1)x)^{d-i-1} - [x^j]x(1-x)^i(1+(q-1)x)^{d-i-1} \\ &= [x^j](1-x)^i(1+(q-1)x)^{d-i-1} - [x^{j-1}](1-x)^i(1+(q-1)x)^{d-i-1} \\ &= K_j(i,d-1) - [x^{j-1}](1-x)^i(1+(q-1)x)^{d-i-1} \\ &= K_j(i,d-1) - K_{j-1}(i,d-1), \end{aligned}$$

so

$$\begin{aligned} K_j(i,d) - K_j(i+1,d) &= K_j(i,d-1) + (q-1)K_{j-1}(i,d-1) \\ &\quad - K_j(i,d-1) + K_{j-1}(i,d-1) \\ &= qK_{j-1}(i,d-1). \end{aligned} \tag{4.7}$$

Bringing everything together, we get

$$\begin{aligned} jK_j(i,d) =&\; -\,iK_{j-1}(i-1,d-1) + (d-i)(q-1)K_{j-1}(i,d-1), \\ &\quad \text{formula (4.6) used} \\ 0 =&\; qjK_j(i,d) + qiK_{j-1}(i-1,d-1) - q(d-i)(q-1)K_{j-1}(i,d-1), \\ &\quad \text{multiplied with } (-q) \\ 0 =&\; qjK_j(i,d) + i\left(K_j(i-1,d) - K_j(i,d)\right) - (d-i)(q-1)\left(K_j(i,d) - K_j(i+1,d)\right), \\ &\quad \text{formula (4.7) used twice} \\ 0 =&\; (d-i)(q-1)\left(K_j(i+1,d) - K_j(i,d)\right) - (i-qj)K_j(i,d) + iK_j(i-1,d), \\ 0 =&\; (q-1)(d-i)K_j(i+1,d) - (i+(q-1)(d-i)-qj)K_j(i,d) + iK_j(i-1,d). \end{aligned}$$

□

Next, we start working our way towards the results that were mentioned in the beginning of this section. Note that the bound $j \geq d - \frac{d-1}{q}$ that appears in these results finds it origin in the following lemma. Moreover, it is assumed that $d \geq 1$ or $d \geq 2$ if $K_j(1)$ respectively $K_j(2)$ are mentioned in the lemma below, since $i \leq d$ in $K_j(i)$.

Lemma 4.6. *[3, Prop. 2.2] Let $q \geq 2$ and $0 \leq j \leq d$. Then*

(a) $K_j(1) < 0$ *if and only if* $j \geq d - \frac{d-1}{q}$,

(b) $K_j(2) = K_j(1)$ *if and only if* $j = 0$ *or* $j = d - \frac{d-1}{q}$,

(c) $K_j(2) > K_j(1)$ *if and only if* $j > d - \frac{d-1}{q}$,

(d) $K_j(2) = \frac{-1}{q-1}K_j(1)$ *if and only if* $j = (d-1)(1-\frac{1}{q})$ *or* $j = d$,

(e) *If* $d - \frac{d-1}{q} \leq j \leq d$, *then* $|K_j(2)| \leq |K_j(1)|$.

Proof.* (a) We have $d - \frac{d-1}{q} \leq j$, which can be rewritten to $(q-1)d - qj + 1 \leq 0$. Since $K_1(j) = (q-1)(d-j) - j = (q-1)d - qj$, this means $K_1(j) < 0 \Leftrightarrow d - \frac{d-1}{q} \leq j$. From Lemma 4.2 we know that $K_j(i)$ and $K_i(j)$ have the same sign, so $K_j(1) < 0 \Leftrightarrow K_1(j) < 0$.

(b) On the one hand, expression (4.1a) gives us

$$\begin{aligned}K_2(j) =&\sum_{h=0}^{2}(-1)^h(q-1)^{2-h}\binom{j}{h}\binom{d-j}{2-h}\\ &= (q-1)^2\binom{d-j}{2} - (q-1)j(d-j) + \binom{j}{2}\\ &= \frac{1}{2}(q-1)^2(d-j)(d-j-1) - (q-1)(d-j)j + \frac{1}{2}j(j-1).\end{aligned}$$

Assuming $K_j(2) = K_j(1)$, we also have

$$\begin{aligned}K_2(j) =&K_j(2)\frac{\binom{d}{2}}{\binom{d}{j}}(q-1)^{2-j}\\ &\text{Lemma 4.2 applied}\\ =&K_j(1)\frac{\binom{d}{2}}{\binom{d}{j}}(q-1)^{2-j}\\ &\text{assumption } K_j(2) = K_j(1) \text{ applied}\\ =&\frac{\binom{d}{j}}{\binom{d}{1}}(q-1)^{j-1}K_1(j)\frac{\binom{d}{2}}{\binom{d}{j}}(q-1)^{2-j}\\ &\text{Lemma 4.2 applied again}\\ =&\frac{1}{2}(d-1)(q-1)K_1(j)\\ &\text{simplified}\\ =&\frac{1}{2}(d-1)(q-1)\left((q-1)(d-j) - j\right).\\ &\text{used expression (4.1a) to calculate } K_1(j)\end{aligned}$$

These two expressions for $K_2(j)$ are equal if and only if

$$(q-1)^2(d-j)(d-1) - (q-1)^2(d-j)j - 2(q-1)(d-j)j + j(j-1) = (q-1)^2(d-j)(d-1) - (d-1)(q-1)j,$$

so if and only if

$$j\left(-(q-1)^2(d-j)-2(d-j)(q-1)+(j-1)+(d-1)(q-1)\right)=0,$$

so if and only if

$$j(jq^2-dq^2+dq-q)=0.$$

This happens precisely when $j=0$ and $j=d-\frac{d-1}{q}$.

(c) $K_1(j)$ and thus $\frac{1}{2}(d-1)(q-1)K_1(j)$ are linearly decreasing in j. In (b) we derived that

$$\begin{aligned}K_2(j)&=\frac{1}{2}(q-1)^2(d-j)(d-j-1)-(q-1)(d-j)j+\frac{1}{2}j(j-1)\\&=\frac{1}{2}(q-1)^2(d^2-d)+\left(\frac{1}{2}(q-1)^2(1-2d)-(q-1)d-\frac{1}{2}\right)j+\left(\frac{1}{2}(q-1)^2+(q-1)+\frac{1}{2}\right)j^2,\end{aligned}$$

thus $K_2(j)$ is quadratic in j with a positive leading coefficient. From above, we have that $K_2(j)=\frac{1}{2}(d-1)(q-1)K_1(j)$ if and only if $K_j(2)=K_j(1)$, so if and only if $j=0$ or $j=d-\frac{d-1}{q}$ by (b). This means that $K_j(2)>K_j(1)$ if and only if $j>d-\frac{d-1}{q}$.

(d) From (b), we get

$$K_2(j)=\frac{1}{2}(q-1)^2(d-j)(d-j-1)-(q-1)(d-j)j+\frac{1}{2}j(j-1).$$

Assuming $K_j(2)=\frac{-1}{q-1}K_j(1)$, we also have

$$\begin{aligned}K_2(j)=&K_j(2)\frac{\binom{d}{2}}{\binom{d}{j}}(q-1)^{2-j}\\&\text{Lemma 4.2 applied}\\=&-K_j(1)\frac{\binom{d}{2}}{\binom{d}{j}}(q-1)^{1-j}\\&\text{assumption } K_j(2)=\frac{-1}{q-1}K_j(1)\text{ applied}\\=&-\frac{\binom{d}{j}}{\binom{d}{1}}(q-1)^{j-1}K_1(j)\frac{\binom{d}{2}}{\binom{d}{j}}(q-1)^{1-j}\\&\text{Lemma 4.2 applied again}\\=&-\frac{1}{2}(d-1)K_1(j)\\&\text{simplified}\\=&-\frac{1}{2}(d-1)\left((q-1)(d-j)-j\right).\end{aligned}$$

These two expressions for $K_2(j)$ are equal if and only if

$$(q-1)^2(d-j)(d-j-1)-2(q-1)(d-j)j+j(j-1)=-(d-1)(q-1)(d-j)+(d-1)j,$$

so if and only if

$$(d-j)\left((q-1)^2(d-j-1)-2(q-1)j-j+(d-1)(q-1)\right)=q(d-j)\big((d-1)(q-1)-jq\big)=0.$$

This happens when $j=d$ or $j=(d-1)(1-\frac{1}{q})$. Because the expression is quadratic in j, we know there are no more solutions.

(e) From (a) we have $K_j(1) < 0$ for $d - \frac{d-1}{q} \leq j \leq d$. This means

$$|K_j(2)| \leq |K_j(1)| \quad \Leftrightarrow \quad |K_j(2)| \leq -K_j(1).$$

This is true if both $K_j(2) \geq K_j(1)$ and $K_j(2) \leq -K_j(1)$. The first inequality is true for $d - \frac{d-1}{q} \leq j \leq d$ because of (b) and (c). For the second inequality we need Lemma 4.2 again. This gives us

$$K_2(j) = K_j(2)\frac{\binom{d}{2}}{\binom{d}{j}}(q-1)^{2-j} \quad \text{and} \quad K_1(j) = K_j(1)\frac{\binom{d}{1}}{\binom{d}{j}}(q-1)^{1-j}$$

and thus

$$K_j(2) \leq -K_j(1) \quad \Leftrightarrow \quad K_2(j) \leq -\frac{1}{2}(d-1)(q-1)K_1(j).$$

From (d) we get

$$K_2(j) = \frac{-1}{q-1}K_j(1) = -\frac{1}{2}(d-1)K_1(j) \quad \Leftrightarrow \quad j = (d-1)\left(1 - \frac{1}{q}\right) \text{ or } j = d.$$

From (b), we know that $K_2(j)$ is quadratic in j with a positive leading coefficient and $K_1(j)$ is linear in j. This means that $K_2(j) + \frac{1}{2}(d-1)K_1(j)$ is quadratic in j with a positive leading coefficient, so

$$K_2(j) \leq -\frac{1}{2}(d-1)K_1(j) \quad \text{for} \quad (d-1)(1-\frac{1}{q}) \leq j \leq d$$

and thus in particular for $d - \frac{d-1}{q} \leq j \leq d$. Because $q \geq 2$, this also means

$$K_2(j) \leq -\frac{1}{2}(d-1)(q-1)K_1(j) \quad \text{for} \quad d - \frac{d-1}{q} \leq j \leq d,$$

which concludes the proof. □

The results of the previous Lemma (except (d)) are visualized in Figure 4.1.

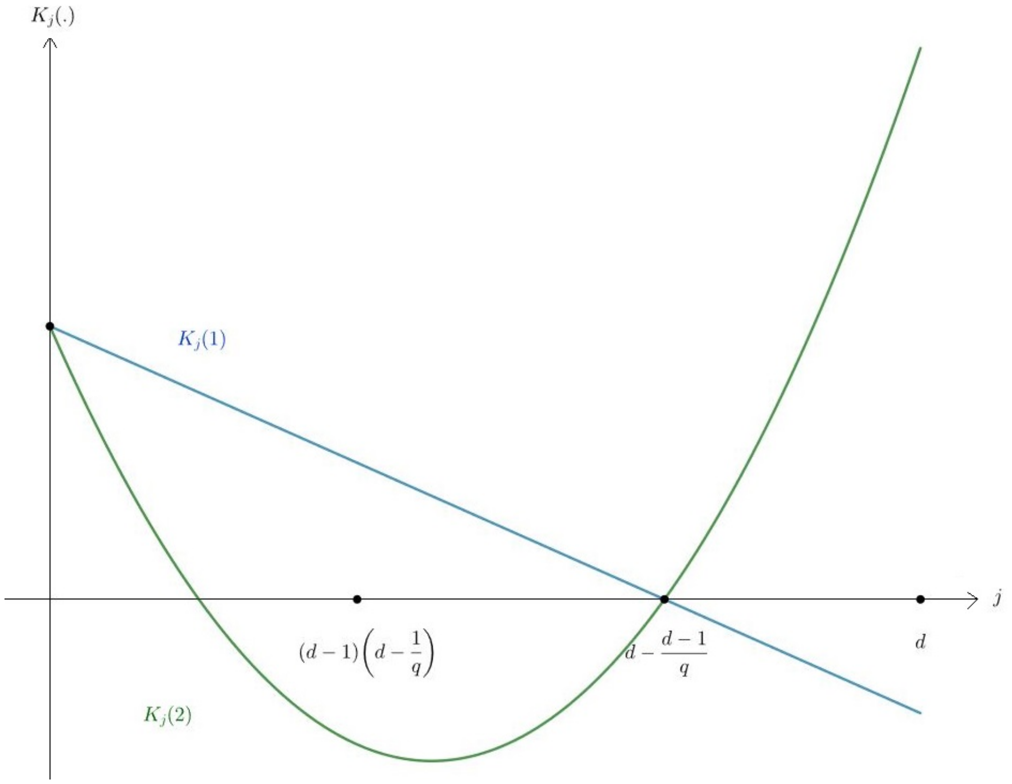


Figure 4.1: Summary of the results of Lemma 4.6.

Next, we prove three lemmas and a proposition that are necessary for proving Theorem 4.11.

Lemma 4.7. *[3, Lemma 2.3] Let $K_j(i)$ be a Kravchuk polynomial. Then*

$$|K_j(i)| \leq (q-1)^{d-i}\binom{d}{j}.$$

Proof.*

$$|K_j(i)| = \left|\sum_{h=0}^{j}(-1)^h(q-1)^{j-h}\binom{i}{h}\binom{d-i}{j-h}\right|$$

expression (4.1a) applied

$$\leq \sum_{h=0}^{j}(q-1)^{j-h}\binom{i}{h}\binom{d-i}{j-h}$$

triangle inequality

$$\leq (q-1)^{d-i}\sum_{h=0}^{j}\binom{i}{h}\binom{d-i}{j-h}$$

if $j-h > d-i$ then $\binom{d-i}{j-h} = 0$

$$= (q-1)^{d-i}\binom{d}{j}.$$

Vandermonde identity (Lemma 2.5) applied ☐

Lemma 4.8. *[3, adaptation from Lemma 2.4] Let $1 < i < d$ and $d - \frac{d-1}{q} \leq j \leq d$. If $qj \leq 2(q-1)(d-i+1)$, then*

$$|K_j(i+1)| \leq \max\{|K_j(i-1)|, |K_j(i)|\}.$$

Remark. Paper [3] used the constraint $qj \leq 2(q-1)(d-i)$ instead of $qj \leq 2(q-1)(d-i+1)$. We changed it, since our version is more convenient to use in the proof of Theorem 4.14a.

Proof.* For the sake of convenience, write $a = (q-1)(d-i+1)$ and $M = \max\{|K_j(i-1)|, |K_j(i)|\}$. We have

$$\begin{aligned}
(a-(q-1))K_j(i+1) =& (i - qj + a - (q-1))K_j(i) - iK_j(i-1) \\
& \text{Lemma 4.5 applied} \\
|a-(q-1)||K_j(i+1)| =& |(i - qj + a - q + 1)K_j(i) - iK_j(i-1)| \\
& \text{absolute value taken on both sides} \\
\leq & |i - qj + a - q + 1||K_j(i)| + i|K_j(i-1)| \\
& \text{triangle inequality applied} \\
\leq & (|i - qj + a - q + 1| + i)M.
\end{aligned}$$

The conclusion follows if $|i - qj + a - q + 1| + i \leq |a - (q-1)|$. We have

$$a - (q-1) = (q-1)(d-i+1) - (q-1) = (q-1)(d-i) > 0$$

and

$$i - qj + a - q + 1 \leq i - q\left(d - \frac{d-1}{q}\right) + (q-1)(d-i+1) - q + 1 = -i(q-2) - 1 < -i(q-2) \leq 0,$$

so the conclusion follows if

$$-i + qj - a + q - 1 + i \leq a - q + 1.$$

This happens if $qj \leq 2a - 2q + 2 \leq 2a$, which was one of the conditions from the lemma at hand. □

Proposition 4.9. *Let $1 \leq i \leq d$ and $d - \frac{d-1}{q} \leq j \leq d$. If $qj \leq 2(q-1)(d-i+1)$, then $|K_j(i)| \leq |K_j(1)|$.*

Proof. For $i = 1$ this is obvious, so let $i \geq 2$. By Lemma 4.8 and using induction on i, we can state

$$|K_j(i)| \leq \max\{|K_j(i-2)|, |K_j(i-1)|\} \leq ... \leq \max\{|K_j(1)|, |K_j(2)|\}.$$

The right hand side is equal to $|K_j(1)|$ by Lemma 4.6e. □

Lemma 4.10. *[3, Lemma 2.5] Let $q = 2$, $j < \frac{d}{2}$ and $0 < i < d$. Then*

$$\binom{d-1}{j-1} \leq \sum_{g=0}^{\lfloor \frac{j}{2} \rfloor} \binom{i}{2g}\binom{d-i}{j-2g} \leq \binom{d-1}{j}.$$

Proof.* We prove this lemma by induction on d. For the base case $d = 2$ (and thus $j = 0$, since $j < \frac{d}{2}$, and $i = 1$) we have

$$0 = \binom{1}{-1} \leq \sum_{g=0}^{0} \binom{i}{2g}\binom{2-i}{j-2g} \leq \binom{1}{0} = 1,$$

because the sum is equal to 1. By the induction hypothesis, which we can apply for $i < d-1$ and $j < \frac{d-1}{2}$, we have

$$\binom{d-2}{j-1} \leq \sum_{g=0}^{\lfloor \frac{j}{2} \rfloor} \binom{i}{2g}\binom{d-i-1}{j-2g} \leq \binom{d-2}{j}, \quad \text{and}$$
$$\binom{d-2}{j-2} \leq \sum_{g=0}^{\lfloor \frac{j-1}{2} \rfloor} \binom{i}{2g}\binom{d-i-1}{j-2g-1} \leq \binom{d-2}{j-1}.$$

Note that for $j = 0$, the statements are trivially true and note that for $g = \frac{j}{2}$ we have $\binom{i}{2g}\binom{d-i-1}{j-2g-1} = \binom{i}{j}\binom{d-i-1}{-1} = 0$, so the sum in the second inequality can be rewritten such that we get

$$\binom{d-2}{j-1} \leq \sum_{g=0}^{\lfloor \frac{j}{2} \rfloor} \binom{i}{2g}\binom{d-i-1}{j-2g} \leq \binom{d-2}{j}, \quad \text{and}$$
$$\binom{d-2}{j-2} \leq \sum_{g=0}^{\lfloor \frac{j}{2} \rfloor} \binom{i}{2g}\binom{d-i-1}{j-2g-1} \leq \binom{d-2}{j-1}.$$

Using the induction hypothesis and Pascal's formula (Lemma 2.1), we get

$$\begin{aligned} \binom{d-2}{j-1} + \binom{d-2}{j-2} \leq{}& \sum_{g=0}^{\lfloor \frac{j}{2} \rfloor} \binom{i}{2g}\binom{d-i-1}{j-2g} + \sum_{g=0}^{\lfloor \frac{j}{2} \rfloor} \binom{i}{2g}\binom{d-i-1}{j-2g-1} \leq \binom{d-2}{j} + \binom{d-2}{j-1} \\ \binom{d-1}{j-1} \leq{}& \sum_{g=0}^{\lfloor \frac{j}{2} \rfloor} \binom{i}{2g}\binom{d-i}{j-2g} \leq \binom{d-1}{j}, \end{aligned}$$

which is what we needed to show. We noted that we can't use the induction hypothesis if $i = d-1$, since the Lemma we want to prove holds for $0 < i < d$, so we need to prove this case separately. For $i = d-1$, the claim is

$$\binom{d-1}{j-1} \leq \sum_{g=0}^{\lfloor \frac{j}{2} \rfloor} \binom{d-1}{2g}\binom{1}{j-2g} \leq \binom{d-1}{j},$$

which is true because the sum equals $\binom{d-1}{j}$ if j is even and $\binom{d-1}{j-1}$ if j is odd. We also noted that we can't use the induction hypothesis if $j = \frac{d-1}{2}$, since the Lemma we want to prove holds for $j < \frac{d}{2}$. This case will be shown in the proof of Theorem 4.11a. □

Theorem 4.11. *[3, Thm. 1.2] Let $q = 2$. Then*

- (a) *if $j \neq \frac{d}{2}$, then $|K_j(i)| \leq |K_j(1)|$ for all $1 \leq i \leq d-1$,*
- (b) *if $j = \frac{d}{2}$, then $K_j(1) = 0$ and $|K_j(i)| \leq |K_j(2)|$ for all $1 \leq i \leq d-1$.*

Proof.* Firstly, note that by Lemmas 4.3 and 4.4, we have

$$K_{d-j}(i) = (-1)^{i-j}K_j(d-i) = (-1)^i K_j(i). \tag{4.8}$$

(a) This implies $|K_{d-j}(i)| = |K_j(i)|$, so we can limit our proof to the case $j < \frac{d}{2}$. Using $q = 2$ and expression (4.1a) , we get

$$K_j(1) = \sum_{h=0}^{j} (-1)^h \binom{1}{h}\binom{d-1}{j-h} = \binom{d-1}{j} - \binom{d-1}{j-1} \geq 0.$$

If we then also use Vandermonde's identity (Lemma 2.5) in the last step, we get

$$\begin{aligned} K_j(i) &= \sum_{h=0}^{j}(-1)^h\binom{i}{h}\binom{d-i}{j-h}\\ &= \sum_{\substack{h=0\\ h \text{ even}}}^{j}\binom{i}{h}\binom{d-i}{j-h} - \sum_{\substack{h=0\\ h \text{ odd}}}^{j}\binom{i}{h}\binom{d-i}{j-h}\\ &= 2\sum_{\substack{h=0\\ h \text{ even}}}^{j}\binom{i}{h}\binom{d-i}{j-h} - \sum_{h=0}^{j}\binom{i}{h}\binom{d-i}{j-h}\\ &= 2\sum_{g=0}^{\lfloor \frac{j}{2}\rfloor}\binom{i}{2g}\binom{d-i}{j-2g} - \binom{d}{j}. \end{aligned}$$

This means we have

$$\begin{array}{rcl} 2\binom{d-1}{j-1} \leq & 2\sum_{g=0}^{\lfloor \frac{j}{2}\rfloor}\binom{i}{2g}\binom{d-i}{j-2g} & \leq \; 2\binom{d-1}{j}\\ & \Updownarrow & \\ 2\binom{d-1}{j-1} \leq & K_j(i) + \binom{d}{j} & \leq \; 2\binom{d-1}{j}\\ & \text{formula derived above applied} & \\ & \Updownarrow & \\ \binom{d-1}{j-1} - \binom{d-1}{j} \leq & K_j(i) & \leq \; \binom{d-1}{j} - \binom{d-1}{j-1}\\ & \text{Pascal's formula (Lemma 2.1) applied} & \\ & \Updownarrow & \\ -K_j(1) \leq & K_j(i) & \leq \; K_j(1)\\ & \text{formula derived above applied} & \\ & \Updownarrow & \\ & |K_j(i)| & \leq \; |K_j(1)|\\ & \text{we had } K_j(1) \geq 0 & \end{array}$$

and thus

$$2\binom{d-1}{j-1} \leq 2\sum_{g=0}^{\lfloor \frac{j}{2}\rfloor}\binom{i}{2g}\binom{d-i}{j-2g} \leq 2\binom{d-1}{j} \quad \Leftrightarrow \quad |K_j(i)| \leq |K_j(1)|. \tag{4.9}$$

We first finish the proof of Lemma 4.10. For this, we need to show

$$2\binom{d-1}{j-1} \leq \; 2\sum_{g=0}^{\lfloor \frac{j}{2}\rfloor}\binom{i}{2g}\binom{d-i}{j-2g} \; \leq \; 2\binom{d-1}{j}$$

holds for $j = \frac{d-1}{2}$. This is equivalent to showing $|K_j(i)| \leq |K_j(1)|$ holds for $j = \frac{d-1}{2}$. Because of the symmetry (4.8) derived above, we can show the case $j = \frac{d+1}{2}$ instead. In other words, we can show that $|K_{\frac{d+1}{2}}(i)| \leq |K_{\frac{d+1}{2}}(1)|$ holds. Because of Lemma 4.4, we have $|K_j(d-i)| = |K_j(i)|$, so we may assume $2 \leq i \leq \frac{d}{2}$.

If $qj \leq 2(q-1)(d-i+1)$, then Proposition 4.9 gives the desired result. We have $q=2$ and $j=\frac{d+1}{2}$ and therefore need to show that inequality $d+1=qj \leq 2(q-1)(d-i+1)=2(d-i+1)$ holds. Since $d+1 \leq 2(d-i+1)$ is equivalent to $i \leq \frac{d+1}{2}$, we have that the inequality holds by the assumption $2 \leq i \leq \frac{d}{2}$, and therefore the statement is true.

Since we finished the proof of Lemma 4.10, we can use equivalence (4.9) to conclude $|K_j(i)| \leq |K_j(1)|$ holds for all $j<\frac{d}{2}$ and thus for all $j \neq \frac{d}{2}$ by the expression $|K_{d-j}(i)|=|K_j(i)|$ stated above.

(b) First, let us prove this statement for odd i. We had

$$K_{d-j}(i)=(-1)^i K_j(i).$$

This means that for $j=\frac{d}{2}$ and odd i, we get $K_{\frac{d}{2}}(i)=0$, which implies $|K_{\frac{d}{2}}(i)| \leq |K_{\frac{d}{2}}(2)|$ for i odd and $1 \leq i \leq d-1$. To prove the statement for even i, we first look at the case $i=0$:

$$K_{\frac{d}{2}}(0)=\sum_{h=0}^{\frac{d}{2}}(-1)^h\binom{0}{h}\binom{d}{\frac{d}{2}-h}=\binom{d}{\frac{d}{2}}.$$

By Lemma 4.5, we get

$$(d-i)K_{\frac{d}{2}}(i+1)+iK_{\frac{d}{2}}(i-1)=0.$$

for odd i. By taking $i=2h+1$, we get

$$(d-2h-1)K_{\frac{d}{2}}(2h+2)+(2h+1)K_{\frac{d}{2}}(2h)=0,$$

which gives

$$\frac{|K_{\frac{d}{2}}(2(h+1))|}{|K_{\frac{d}{2}}(2h)|}=\frac{2h+1}{d-2h-1}.$$

The right hand side is less or equal than 1 for $2h \leq \frac{d}{2}-1$, so

$$|K_{\frac{d}{2}}(2h)| \leq |K_{\frac{d}{2}}(2)| \quad \text{for all} \quad 1 \leq h \leq \frac{d-2}{4}.$$

By Lemma 4.4, we have $K_{\frac{d}{2}}(d-i)=(-1)^{\frac{d}{2}}K_{\frac{d}{2}}(i)$, so $|K_{\frac{d}{2}}(d-i)|=|K_{\frac{d}{2}}(i)|$, so

$$|K_{\frac{d}{2}}(2h)| \leq |K_{\frac{d}{2}}(2)| \quad \text{for all} \quad 1 \leq h \leq \frac{d-1}{2}.$$

In other words, we have $|K_{\frac{d}{2}}(i)| \leq |K_{\frac{d}{2}}(2)|$ for all i even and $1 \leq i \leq d-1$. This, together with the conclusion on odd i, proves our statement. □

Theorem 4.11 left out some interesting observations, which are summarized in the following proposition. Note that we are looking for the penabsolute eigenvalue, which is the second largest eigenvalue in absolute value that is different from $|K_j(0)|$.

Proposition 4.12. *Let $q=2$. Then*

(a) *if $j \neq \frac{d}{2}$, then $|K_j(d-1)|=|K_j(1)| \leq |K_j(0)|=|K_j(d)|$,*

(b) *if $j=\frac{d}{2}$, then $|K_j(d-2)|=|K_j(2)| \leq |K_j(0)|=|K_j(d)|$,*

(c) *if d odd, then $|K_j(d-1)|=|K_j(1)|=|K_j(2)|=|K_j(d-2)|$ for $j=\frac{d-1}{2}$ and $j=\frac{d+1}{2}$.*

Proof. (a),(b) This follows from Theorem 4.11 and symmetry (4.8), which implies $|K_{d-j}(i)| = |K_j(i)|$.

(c) We have $|K_j(d-1)| = |K_j(1)|$ and $|K_j(d-2)| = |K_j(2)|$ by symmetry (4.8) as before. Moreover, by formula (4.1a) and Pascal's formula, we have

$$K_j(1) = \binom{d-1}{j} - \binom{d-1}{j-1} = \binom{d-2}{j} + \binom{d-2}{j-1} - \binom{d-2}{j-1} - \binom{d-2}{j-2} = \binom{d-2}{j} - \binom{d-2}{j-2}$$

and

$$K_j(2) = \binom{d-2}{j} - 2\binom{d-2}{j-1} + \binom{d-2}{j-2}.$$

For $j = \frac{d+1}{2}$ we have

$$\begin{aligned} K_{\frac{d+1}{2}}(2) &= \binom{d-2}{\frac{d+1}{2}} - 2\binom{d-2}{\frac{d-1}{2}} + \binom{d-2}{\frac{d-3}{2}} \\ &= \binom{d-2}{\frac{d+1}{2}} - 2\binom{d-2}{\frac{d-3}{2}} + \binom{d-2}{\frac{d-3}{2}} \\ &\quad \text{symmetry applied} \\ &= \binom{d-2}{\frac{d+1}{2}} - \binom{d-2}{\frac{d-3}{2}} \\ &= K_{\frac{d+1}{2}}(1). \end{aligned}$$

For $j = \frac{d-1}{2}$ we have

$$\begin{aligned} K_{\frac{d-1}{2}}(2) &= \binom{d-2}{\frac{d-1}{2}} - 2\binom{d-2}{\frac{d-3}{2}} + \binom{d-2}{\frac{d-5}{2}} \\ &= \binom{d-2}{\frac{d-1}{2}} - 2\binom{d-2}{\frac{d-1}{2}} + \binom{d-2}{\frac{d-5}{2}} \\ &\quad \text{symmetry applied} \\ &= -\binom{d-2}{\frac{d-1}{2}} + \binom{d-2}{\frac{d-5}{2}} \\ &= -K_{\frac{d-1}{2}}(1). \end{aligned}$$

Thus, indeed $|K_j(2)| = |K_j(1)|$ for $j = \frac{d-1}{2}$ and $j = \frac{d+1}{2}$. □

Figure 4.2 shows some visualizations of P-matrices for $q = 2$ and $8 \leq d \leq 13$ where the highlighted boxes are the penabsolute eigenvalues per column. We also show a result on the smallest eigenvalue for the case $q = 2$.

Corollary 4.13. *[3, Cor. 1.3] Let $q = 2$ and $j \geq \frac{d+1}{2}$. Then*

(a) *$K_j(1) \leq K_j(i)$ for all $0 \leq i \leq d-1$,*

(b) *$K_j(1) \leq K_j(d)$ if and only if j is even or $j = d$.*

Proof.* (a) Note that $j \geq \frac{d+1}{2}$ and $q = 2$, so $j \neq \frac{d}{2}$ and $j \geq \frac{d+1}{q}$. By Theorem 4.11a, we have

$$|K_j(i)| \leq |K_j(1)| \quad \text{for} \quad 1 \leq i \leq d-1.$$

By Lemma 4.6a, we have $K_j(1) < 0$, so $K_j(1) \leq K_j(i) \leq -K_j(1)$ and thus $K_j(1) \leq K_j(i)$ for $1 \leq i \leq d-1$. Additionally, we know $K_j(0) = \binom{d}{j} \geq 0$, so we also have $K_j(1) \leq K_j(0)$.

(b) First, we look at $K_j(d)$. By Lemma 4.4, we have

$$K_j(d) = (-1)^j K_j(0) = (-1)^j \binom{d}{j} = \begin{cases} -\binom{d}{j} & j \text{ odd and } j < d, \\ -1 & j \text{ odd and } j = d, \\ \binom{d}{j} & j \text{ even and } j < d, \\ 1 & j \text{ even and } j = d. \end{cases}$$

Now we look at $K_j(1)$. We have $K_1(j) = (q-1)d - qj = d - 2j$. Because $\frac{d+1}{2} \leq j \leq d$, this means $-d \leq K_1(j) \leq -1$. Applying Lemma 4.2 gives $K_j(1) = \frac{1}{d}\binom{d}{j}(d-2j)$, so

$$-\binom{d}{j} \leq K_j(1) \leq -\frac{1}{d}\binom{d}{j} \leq 0.$$

We can see that for j even we have $K_j(1) \leq K_j(d)$ and for j odd and $j < d$ we have $K_j(1) \geq K_j(d)$. The only case left is j odd and $j = d$. In this case we have $K_d(d) = -1$ and

$$K_d(1) = \sum_{h=0}^{d} (-1)^h \binom{1}{h}\binom{d-1}{d-h} = \binom{d-1}{d} - \binom{d-1}{d-1} = -1.$$

We can conclude that $K_j(1) \leq K_j(d)$ if and only if j is even or $j = d$. ☐

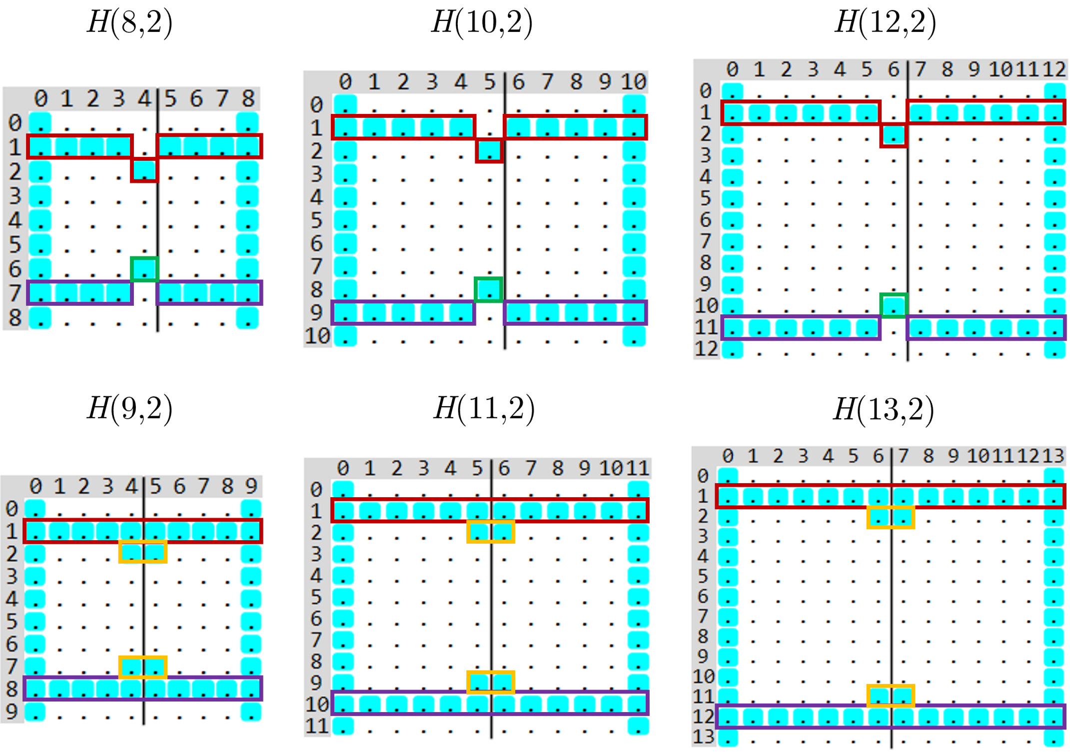
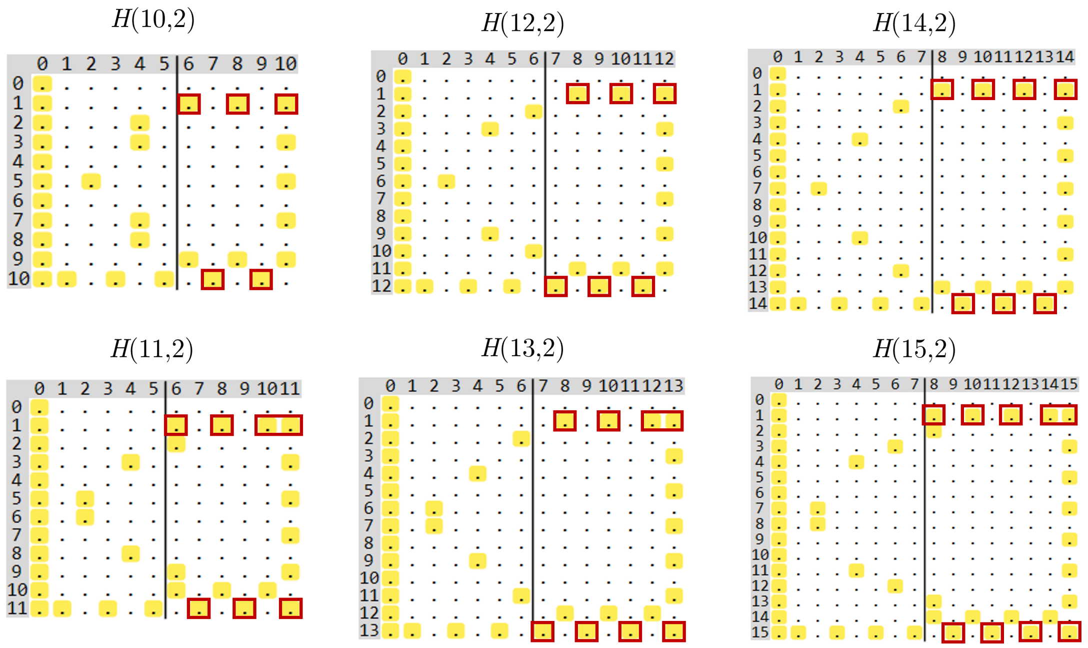

$H(8,2)$ $H(10,2)$ $H(12,2)$
$H(9,2)$ $H(11,2)$ $H(13,2)$

Figure 4.2: Visualization of the results of Theorem 4.11 and Proposition 4.12. The highlighted boxes indicate the penabsolute eigenvalues per column. The red fields indicate the results of Theorem 4.11, the purple fields of Proposition 4.12a, the green fields of Proposition 4.12b and the orange fields of Proposition 4.12c.


H(10,2) H(12,2) H(14,2)

H(11,2) H(13,2) H(15,2)

Figure 4.3: Visualization of the results of Corollary 4.13. The highlighted boxes indicate the smallest eigenvalues per column. The red fields indicate the results of Corollary 4.13.

Figure 4.3 again shows some visualizations of P-matrices for $q = 2$ and $10 \leq d \leq 15$, but this time the smallest eigenvalues per column are highlighted. The red fields indicate the results of Corollary 4.13. Note that when $q = 2$, we have $d - \frac{d-1}{q} = \frac{d+1}{2}$, to the columns on the right side of the vertical black line correspond to $j \geq \frac{d+1}{2}$.

Now consider the case $q \geq 3$.

Theorem 4.14. *[3, Thm. 1.4] Let $q \geq 3$ and $d - \frac{d-1}{q} \leq j \leq d$. Then*

(a) $|K_j(i)| \leq |K_j(1)|$ *for all* $i \geq 1$, *unless* $(q, d, i, j) = (3, 4, 3, 3)$,

(b) $K_j(1) \leq K_j(i)$ *for all* $0 \leq i \leq d$.

Proof.* (a) This proof is quite long, so we will start with an overview of the structure. First, we will consider the case $i = 2$, after which we may assume $i \geq 3$. Then we will do the cases $j = d$ and $j = d - 1$, which is where the exception $(q, d, i, j) = (3, 4, 3, 3)$ will arise. Lastly, we consider the case $d - \frac{d-1}{q} \leq j \leq d - 2$ that will be split up into two scenarios: if $qj \leq 2(q-1)(d-i+1)$ we can use Lemma 4.8 and otherwise we can use Lemma 4.7 to finish the proof.

The case $i = 2$ follows immediately from Lemma 4.6e, so we may assume $i \geq 3$ from now on. For $j = d$, we have

$$K_d(i) = \sum_{h=0}^{d} (-1)^h (q-1)^{d-h} \binom{i}{h} \binom{d-i}{d-h} = (-1)^i (q-1)^{d-i}.$$

This means $|K_d(i)| = (q-1)^{d-i} \leq (q-1)^{d-1} = |K_d(1)|$. For $j = d - 1$, we have

$$K_{d-1}(i) = \sum_{h=0}^{d-1} (-1)^h (q-1)^{d-h-1} \binom{i}{h} \binom{d-i}{d-1-h}$$

$$\begin{aligned} &= (-1)^{i-1}(q-1)^{d-i}\binom{i}{i-1}\binom{d-i}{d-i} + (-1)^{i}(q-1)^{d-i-1}\binom{i}{i}\binom{d-i}{d-1-i} \\ &= (-1)^{i-1}(q-1)^{d-i-1}(qi-d), \\ \text{so} \quad |K_{d-1}(i)| &= (q-1)^{d-i-1}|qi-d|. \end{aligned}$$

Since $j = d-1$, we know $d - \frac{d-1}{q} \leq j = d-1$ and thus $d-1 \geq q$, so $|K_{d-1}(1)| = (q-1)^{d-2}(d-q)$. We therefore want to show $|qi-d| \leq (q-1)^{i-1}(d-q)$. If we assume $qi-d<0$, then we need to show $|qi-d| = d-qi \leq (q-1)^{i-1}(d-q)$, which is equivalent to

$$\frac{d-qi}{d-q} \leq (q-1)^{i-1}.$$

The left hand side is negative for $i \geq 2$ and the right hand side is positive for $i \geq 2$, so this inequality holds. This means $|K_{d-1}(i)| \leq |K_{d-1}(1)|$ if $qi-d<0$. If this is not the case, then $qi-d \geq 0$, so we need to show $|qi-d| = qi-d \leq (q-1)^{i-1}(d-q)$. We have

$$|qi-d| = qi-d \leq q(i-1)-1 \leq (q-1)^{i-1} \leq (q-1)^{i-1}(d-q),$$

where the first and the last inequality follow from $d - \frac{d-1}{q} \leq j = d-1$. It is therefore enough to show the middle inequality, so $q(i-1)-1 \leq (q-1)^{i-1}$.

First assume $q=3$. In this case, the inequality is false for $i=3$ and true for $i=4$. Because it is true for $i=4$, it is also true for $i \geq 4$, since the left hand side is linear in i and the right hand side is exponential in i. Now assume $q=4$. In this case, the inequality is true for $i=3$ and by the same reasoning as above, it is true for all $i \geq 3$. This means we can conclude it is true for all combinations $q \geq 4$, $i \geq 3$.

The inequality $q(i-1)-1 \leq (q-1)^{i-1}$ therefore holds for all pairs (q,i) with $q,i \geq 3$ except for $(q,i) = (3,3)$. For this pair, we therefore need to ask ourselves if the inequality $|qi-d| = |9-d| \leq 4(d-3) = (q-1)^{i-1}(d-q)$ does hold.

If $d>9$, then we want $|9-d| = d-9 \leq 4(d-3)$, which is true since $d \geq 1$. If $d \leq 9$, we want $|9-d| = 9-d \leq 4(d-3)$, which is true for $d \geq 5$. This means the inequality is false for $d \leq 3$ and $d=4$. For $d \leq 3$ and $j=d-1$, the inequality $d - \frac{d-1}{q} \leq j$ doesn't hold anymore, so this case can be neglected. For the case $d=4$, the aforementioned inequality does hold, which means $(q,d,i,j) = (3,4,3,3)$ is an exception in our theorem. We can conclude that $|K_{d-1}(i)| \leq |K_{d-1}(1)|$ for all $i \geq 1$ unless $(q,d,i,j) = (3,4,3,3)$.

Lastly, we consider the case $d - \frac{d-1}{q} \leq j \leq d-2$. If $qj \leq 2(q-1)(d-i+1)$, we can use Lemma 4.9 to conclude $|K_j(i)| \leq |K_j(1)|$. This leaves us the case $qj > 2(q-1)(d-i+1)$. For $K_j(1)$, we have

$$\begin{aligned} K_j(1) &= \sum_{h=0}^{j}(-1)^h(q-1)^{j-h}\binom{1}{h}\binom{d-1}{j-h} \\ &= (q-1)^j\binom{d-1}{j} - (q-1)^{j-1}\binom{d-1}{j-1} \\ &= (q-1)^{j-1}\binom{d}{j}\left(q-1-\frac{qj}{d}\right), \end{aligned}$$

so $|K_j(1)| = (q-1)^{j-1}\binom{d}{j}\left(1+\frac{qj}{d}-q\right)$, since $d - \frac{d-1}{q} \leq j$ is equivalent to $q-1-\frac{qj}{d} \leq -\frac{1}{d}$. By Lemma 4.7, we have $|K_j(i)| \leq (q-1)^{d-i}\binom{d}{j}$, so it is enough to show that the inequality

$$(q-1)^{d-i}\binom{d}{j} \leq (q-1)^{j-1}\binom{d}{j}\left(1+\frac{qj}{d}-q\right)$$

holds, which is equivalent to

$$d \leq (q-1)^{j-d+i-1}(qj-(q-1)d).$$

To do this, we will first show that the following inequalities (I), (II) and (III) hold.

$$2^{\frac{d}{6}} \overset{(q\geq 3)}{\leq} (q-1)^{\frac{d}{6}} \overset{\text{(I)}}{<} (q-1)^{\frac{j}{4}} \overset{\text{(II)}}{\leq} (q-1)^{j-d+i-1} \overset{\text{(III)}}{\leq} (q-1)^{j-d+i-1}(qj-(q-1)d).$$

Inequality (I) follows from $j \geq d - \frac{d-1}{q} \geq d - \frac{d-1}{3} = \frac{2}{3}d + \frac{1}{3} > \frac{2}{3}d$. Inequality (II) follows from $qj > 2(q-1)(d-i+1)$, which gives us $d-i+1 < \frac{qj}{2(q-1)} = \frac{1}{2}\frac{q}{q-1}j \leq \frac{3}{4}j$. Inequality (III) follows from $qj-(q-1)d = q(j-d)+d \geq q(-\frac{d-1}{q})+d = 1$.

These inequalities imply that it is enough to show $d \leq 2^{\frac{d}{6}}$. This is the case for $d \geq 30$. We can check by computer for the cases with $d < 30$ whether the following inequality holds:

$$d \leq (q-1)^{j-d+i-1}(qj-(q-1)d). \tag{4.10}$$

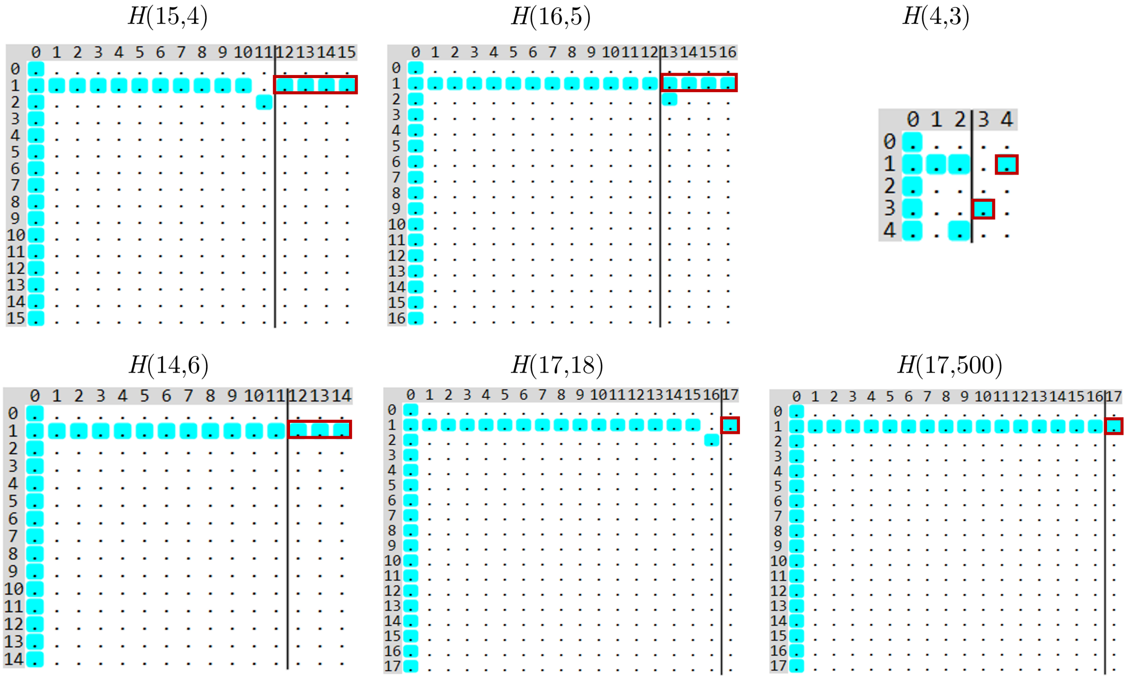
In order to do this, we need a bound on q. Recall that $d-\frac{d-1}{q} \leq j \leq d-2$, so we need $d-\frac{d-1}{q} \leq d-2$ and thus $q \leq \frac{d-1}{2}$ for there to be a feasible j. Thus, we need to check (4.10) for the finitely many cases with $d < 30$, $q \leq \frac{d-1}{2}$. It turns out that this inequality holds for $1 \leq d < 30$, see Appendix B.1 for the code used to check this. Therefore we can conclude $|K_j(i)| \leq |K_j(i)|$ for all i and $d - \frac{d-1}{q} \leq j \leq d-2$, which finishes the proof.

(b) For $i = 0$, we have that $K_j(0)$ is the valency and thus the largest eigenvalue over all $K_j(i)$, so $K_j(1) \leq K_j(0)$. For $i \geq 1$, we have $|K_j(i)| \leq |K_j(1)|$ from (a) and $K_j(1) < 0$ from Lemma 4.6a. This gives

$$K_j(1) = -|K_j(1)| \leq K_j(i) \leq |K_j(1)| = -K_j(1),$$

so $K_j(1) \leq K_j(i)$ for $i \geq 1$ too. □

Like before, Figures 4.4 and 4.5 visualize the results of Theorem 4.14a and 4.14b, respectively.


H(15,4) H(16,5) H(4,3)

H(14,6) H(17,18) H(17,500)

Figure 4.4: Visualization of the results of Theorem 4.14a. The highlighted boxes indicate the penabsolute eigenvalues per column. The red fields indicate the results of Theorem 4.14a.

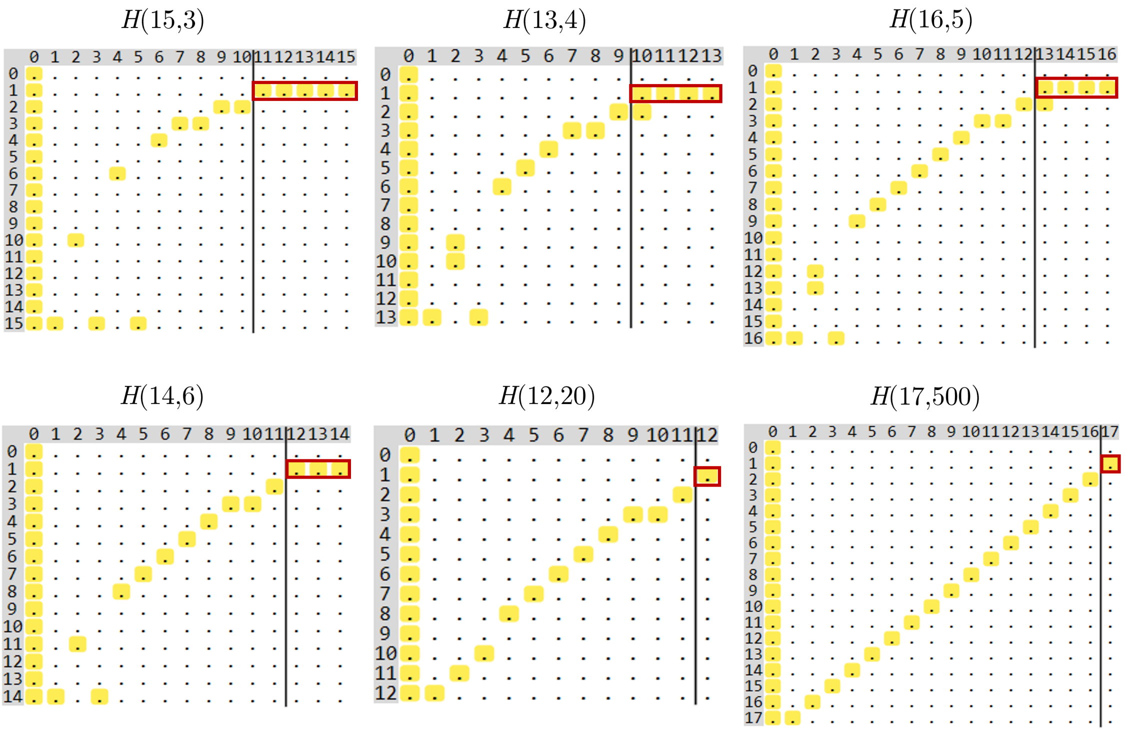


Figure 4.5: Visualization of the results of Theorem 4.14b. The highlighted boxes indicate the smallest eigenvalues per column. The red fields indicate the results of Theorem 4.14b.

With the previous theorem, we have shown all the results that were mentioned in the beginning of this section. However, [3] mentioned some additional results for large q and some results on coinciding eigenvalues. For the sake of completeness, we will mention these results here as well. We start with a short lemma that is needed to prove Lemma 4.16.

Lemma 4.15. *Let* $(a_i)_{i=0}^{n}$ *be a sequence with* $a_0 \geq a_1 \geq ... \geq a_n \geq 0$ *and let* $S = a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n$. *Then*

(a) $S \leq a_0$,

(b) $S \geq a_0 - a_1$, *and thus* $S \geq 0$.

Proof. (a) We have $S = a_0 - (a_1 - a_2) - (a_3 - a_4) - \cdots - (a_{n-1} - a_n)$ for n even and $S = a_0 - (a_1 - a_2) - (a_3 - a_4) - \cdots - (a_{n-2} - a_{n-1}) - a_n$ for n odd. Every term inside the brackets is non-negative, so in both cases we have $S \leq a_0$.

(b) We have $S = a_0 - a_1 + (a_2 - a_3) + \cdots + (a_{n-1} - a_n)$ for n odd and $S = a_0 - a_1 + (a_2 - a_3) + \cdots + (a_{n-2} - a_{n-1}) + a_n$ for n even. Every term inside the brackets is non-negative, so in both cases we have $S \geq a_0 - a_1$. Note that this also means $S \geq 0$. □

Lemma 4.16. *[3, Lemma 2.7] Let* $q > \frac{1}{4}d^2 + 1$. *Then*

(a) $K_j(i) > 0$ *for* $i \leq d - j$,

(b) $K_j(d - j + 1) < 0$,

(c) $|K_j(i)| < |K_j(d - j + 1)|$ *for* $i > d - j + 1$.

Proof.* First, we will show that the nonzero terms $(q-1)^{j-h}\binom{i}{h}\binom{d-i}{j-h}$ decrease monotonically when h increases, except when $h = i+j-d-1$. For any h, we want to show

$$(q-1)^{j-h}\binom{i}{h}\binom{d-i}{j-h} \geq (q-1)^{j-h-1}\binom{i}{h+1}\binom{d-i}{j-h-1}.$$

When $h > i$, $h > j$ or $h < i+j-d-1$, we have that $\binom{i}{h} = 0$ or $\binom{d-i}{j-h} = 0$ and that $\binom{i}{h+1} = 0$ or $\binom{d-i}{j-h-1} = 0$, so both sides are equal to zero and the inequality holds. When $h = i$ or $h = j$, the right hand side is equal to zero and the left hand side is non-negative, so here the inequality holds as well. For $h = i+j-d-1$, the inequality is flipped:

$$0 = (q-1)^{j-h}\binom{i}{i+j-d-1}\binom{d-i}{d-i+1} < (q-1)^{j-h-1}\binom{i}{i+j-d}\binom{d-i}{d-i} = (q-1)^{j-h-1}\binom{i}{i+j-d}.$$

This leaves the case $i+j-d \leq h < \min\{i,j\}$, where we can rewrite the binomial coefficients such that we can cancel them and we get

$$(q-1)(h+1)(d-i-j+h+1) \geq (i-h)(j-h),$$

which is equivalent to

$$h(q-1)(h-i-j+d) + (q-1)(2h+d-i-j+1) \geq (h^2-(i+j)h+dh) + (ij-dh).$$

Using $q \geq 2$ and $h-i-j+d \geq 0$, it is enough to show

$$(q-1)(2h+d-i-j+1) \geq ij-dh.$$

Using $i+j-d \leq h$ again, it is enough to show $(q-1)(h+1) \geq ij-dh$. We have assumed $q > \frac{1}{4}d^2+1$, so it is enough to show $\frac{1}{4}d^2(h+1) \geq ij-dh$. Using $i+j \leq d+h$ once again and also $ab \leq \frac{(a+b)^2}{4}$ for real numbers a, b, we have

$$ij-dh \leq \frac{(i+j)^2}{4} - dh \leq \frac{(d+h)^2}{4} - dh = \frac{(d-h)^2}{4} \leq \frac{d^2}{4} \leq \frac{1}{4}d^2(h+1).$$

This means that the terms $(q-1)^{j-h}\binom{i}{h}\binom{d-i}{j-h}$ indeed decrease monotonically when h increases, except when $h = i+j-d-1$. Thus, when $h \neq i+j-d-1$, we can think of $K_j(i) = \sum_{h=0}^{j}(-1)^h(q-1)^{j-h}\binom{i}{h}\binom{d-i}{j-h}$ as $S = a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n$ from Lemma 4.15 where $a_h = (q-1)^{j-h}\binom{i}{h}\binom{d-i}{j-h}$.

(a) Let $i \leq d-j$. The paragraph above implies that we can use Lemma 4.15b to conclude that the sign of $K_j(i)$ is that of its first nonzero term. For $i \leq d-j$, the first nonzero term is the term with $h = 0$, which is positive, so $K_j(i) > 0$ for $i \leq d-j$.

(b) For $i > d-j$, the first nonzero term is the term with $h = i+j-d$. Like before, we can use Lemma 4.15b to conclude that the sign of $K_j(i)$ is that of its first nonzero term. This means that for $i > d-j$, we have $K_j(i) > 0$ if $i+j-d$ is even and $K_j(i) < 0$ if $i+j-d$ is odd. Choosing $i = d-j+1$, we have that $d-j+1 > d-j$ and that $i+j-d = 1$ is odd, so $K_j(d-j+1) < 0$.

(c) Assume $q > \frac{1}{4}d^2+1$. We want to show $|K_j(i)| < |K_j(d-j+1)|$ for $i > d-j+1$. For $d = 2$, we need to show $|K_2(i)| < |K_2(1)|$ for $i > 1$. From expression (4.1a), we can derive $|K_2(i)| = (q-1)^{2-i}$, which implies the result we wanted to show. This means that from now on, we can assume $d \geq 3$.

Showing $|K_j(i)| < |K_j(d-j+1)|$ for $i > d-j+1$ is equivalent to showing $|K_j(d-j+e)| < |K_j(d-j+1)|$ for $2 \leq e \leq j$. Using lemmas 4.15a and 4.15b we find

$$|K_j(d-j+1)| \geq (q-1)^{j-1}(d-j+1) - (q-1)^{j-2}\binom{d-j+1}{2}(j-1)$$

$$\begin{aligned}&= (q-1)^{j-2}(d-j+1)\left(q-1-\frac{1}{2}(d-j)(j-1)\right)\\ &\geq \frac{1}{2}q(q-1)^{j-2}(d-j+1).\end{aligned}$$

This last step requires some more explanation. It is true if we can show $q \geq 2+(d-j)(j-1)$. Looking at the right hand side as a quadratic polynomial in j, we see that it acquires its maximum at $j = \frac{d+1}{2}$, which gives that the maximum value of the right hand side is $2+\left(\frac{d-1}{2}\right)^2 = \frac{1}{4}d^2 - \frac{1}{2}d + \frac{9}{4}$. For $d \geq 3$, the inequality is true because $\frac{1}{4}d^2 - \frac{1}{2}d + \frac{9}{4} < \frac{1}{4}d^2 + 1 < q$.

Using lemmas 4.15a and 4.15b we find

$$\begin{aligned}|K_j(d-j+e)| &\leq (q-1)^{j-e}\binom{d-j+e}{e} + (q-1)^{j-e-1}\binom{d-j+e}{e+1}(j-e)\\ &= (q-1)^{j-e-1}\binom{d-j+e}{e}\left(q-1+\frac{d-j}{e+1}(j-e)\right)\\ &\leq \frac{4}{3}q(q-1)^{j-e-1}\binom{d-j+e}{e}.\end{aligned}$$

Here, the last step also requires some more explanation. The step is true if we can show $(d-j)(j-e) \leq \left(\frac{q}{3}+1\right)(e+1)$. Using $e \geq 2$ and $q \geq 2+(d-j)(j-1)$, which we derived above, the result follows:

$$\left(\frac{q}{3}+1\right)(e+1) \geq \left(\frac{5}{3}+\frac{1}{3}(d-j)(j-1)\right)(e+1) \geq 5+(d-j)(j-1) > (d-j)(j-e).$$

The two inequalities we have derived for $|K_j(d-j+1)|$ and $|K_j(d-j+e)|$ will be essential in the rest of the proof. First, we will show the case $j=e=2$ after which we will use induction to show the case $e \geq 2, j \geq 3$. For $j=e=2$, we want to show

$$(q-1)^{2-2-1}\binom{d-2+2}{2}\left(q-1+\frac{d-2}{2+1}(2-2)\right) < \frac{1}{2}q(q-1)^{2-2}(d-2+1).$$

This inequality holds if $d < q$, which is the case since $q > \frac{1}{4}d^2+1$ and by the quadratic formula $d \leq \frac{1}{4}d^2+1$ for every d.

Now we will use induction to show the case $e \geq 2, j \geq 3$. For the base case $e=2, j\geq 3$, we want to show

$$\frac{4}{3}q(q-1)^{j-2-1}\binom{d-j+2}{2} < \frac{1}{2}q(q-1)^{j-2}(d-j+1),$$

which is equivalent to

$$d-j+2 < \frac{3}{4}(q-1).$$

If $d=3$, this implies $j=3$ and $q > \frac{1}{4}3^2+1$, so $q \geq 4$ and the inequality holds. For $d \geq 4$ we need to do something else. Note that, because $j \geq 3$ and $q < \frac{1}{4}d^2+1$, it is enough to show

$$d-1 \leq \frac{3}{16}d^2.$$

This is true for $d=4$, and since we are dealing with a quadratic inequality with positive leading coefficient, it is true for $d \geq 4$. This means we have shown the statement for the base case $e=2, j\geq 3$.

Assume now that $e, j \geq 3$. We want to show

$$\frac{4}{3}q(q-1)^{j-e-1}\binom{d-j+e}{e} < \frac{1}{2}q(q-1)^{j-2}(d-j+1),$$

which is equivalent to showing the inequality

$$\frac{d-j+e}{e}\binom{d-j+e-1}{e-1} = \binom{d-j+e}{e} < \frac{3}{8}(q-1)^{e-1}(d-j+1).$$

By the induction hypothesis, this inequality is true if

$$\frac{e(q-1)}{d-j+e} \geq 1.$$

Because $3 \leq e \leq j$ and $q > \frac{1}{4}d^2 + 1$, we can see that this is true:

$$\frac{e(q-1)}{d-j+e} \geq \frac{\frac{3}{4}d^2}{d} \geq 1.$$

With this step we have shown $|K_j(d-j+e)| < |K_j(d-j+1)|$ for all $2 \leq e \leq j$, which concludes this proof. □

Next, we show some results on coinciding eigenvalues.

Lemma 4.17. *[3, Lemma 2.8] Let $q = 2$. Then*

(a) *if j is even, then $K_j(i) = K_j(d-i)$,*

(b) *if $d = 2j$, then $K_j(i) = 0$ for all odd i,*

(c) *if $d = 2j - 1$, then $K_j(2h-1) = K_j(2h)$ for $1 \leq h \leq j-1$,*

(d) *if $d = j$, then $K_j(i) = (-1)^i$ for all i.*

Proof. (a) This follows directly from Lemma 4.4.

(b) By lemmas 4.3 and 4.4, we have

$$K_{d-j}(i) = (-1)^{i-j}K_j(d-i) = (-1)^i K_j(i).$$

(c) Using formula 4.1a, we have

$$K_j(2h) = \sum_{h=0}^{j}(-1)^h\binom{2h}{h}\binom{2j-1-2h}{j-h} = 2\sum_{h=0}^{j}(-1)^h\binom{2h-1}{h}\binom{2j-1-2h}{j-h},$$
$$K_j(2h-1) = \sum_{h=0}^{j}(-1)^h\binom{2h-1}{h}\binom{2j-2h}{j-h} = 2\sum_{h=0}^{j}(-1)^h\binom{2h-1}{h}\binom{2j-1-2h}{j-h}.$$

(d) We have

$$K_j(i) = \sum_{h=0}^{j}(-1)^h\binom{i}{h}\binom{j-i}{j-h} = (-1)^i\binom{i}{i}\binom{j-i}{j-i} = (-1)^i.$$

□

The following Lemma is an adaptation from Lemma 2.9 in [3], since the lemma in the paper was missing some cases in (b) and (c), so we added them. More specific, in (b) the cited paper had $h \geq 3$, which we changed to $h \geq 2$. In (c), the cited paper had $h \geq 2$, which we changed to $h \geq 1$. Furthermore we added the solution $d = 2i$ in (c).

Lemma 4.18. *[3, adaptation from Lemma 2.9] Let $q = 2$ and $i, j \leq \frac{d}{2}$. Then*

(a) *$K_1(i) = 0$ if and only if $d = 2i$,*

(b) *$K_2(i) = 0$ if and only if $i = \binom{h}{2}$ and $d = h^2$ for some integer $h \geq 2$,*

(c) *$K_3(i) = 0$ if and only if $d = 2i$, or $i = \frac{h(3h\pm1)}{2}$ and $d = 3h^2 + 3h + \frac{3}{2} \pm \left(h + \frac{1}{2}\right)$ for some integer $h \geq 1$,*

(d) *$K_{2h}(4h - 1) = 0$ if $d = 8h + 1$.*

Proof. (a) We have

$$K_1(i) = \sum_{h=0}^{1} (-1)^h \binom{i}{h}\binom{d-i}{1-h} = \binom{d-i}{1} - \binom{i}{1} = d - 2i,$$

so $K_1(i) = 0$ if and only if $d - 2i = 0$.

(b) We have

$$K_2(i) = \sum_{h=0}^{2} (-1)^h \binom{i}{h}\binom{d-i}{2-h} = \binom{d-i}{2} - i(d-i) + \binom{i}{2}.$$

This means

$$K_2(i) = 0 \quad \Leftrightarrow \quad (d-i)(d-i-1) - 2i(d-i) + i(i-1) = 0 \quad \Leftrightarrow \quad (d-2i)^2 = d.$$

Since d and i are integers and $(d - 2i)^2$ is a square, d should also be a square, so we can write $d = h^2$ for some integer h. This gives us $(h^2 - 2i)^2 = h^2$. Because $h \geq 0$ and $i \leq \frac{d}{2} = \frac{h^2}{2}$, this is equivalent to $h^2 - 2i = h$, which happens exactly if $i = \frac{1}{2}h(h-1) = \binom{h}{2}$. We need to choose $h \geq 2$, since otherwise $j = 2 > \frac{d}{2} = \frac{h^2}{2}$.

(c) We have

$$K_3(i) = \sum_{h=0}^{3} (-1)^h \binom{i}{h}\binom{d-i}{3-h} = \binom{d-i}{3} - i\binom{d-i}{2} + \binom{i}{2}(d-i) - \binom{i}{3}.$$

This means

$$K_3(i) = 0 \quad \Leftrightarrow \quad (d-i)(d-i-1)(d-i-2) + 3i(d-i)(2i-d) - i(i-1)(i-2) = 0.$$

Solving this equation with a computer gives $d = 2i$ or $d = \frac{1}{2}\left(3 + 4i \pm \sqrt{1 + 24i}\right)$. We need d and i to be integers, so looking at the second solution for d we see that $1 + 24i$ should be the square of some odd number.

We can write $1 + 24i = (2g + 1)^2$ for some integer g, which gives $i = \frac{1}{6}(g^2 + g)$. This means $g^2 + g$ should be divisible by 6, so divisible by 2 and by 3. It is already divisible by 2: if g is odd, then g^2 is odd so $g^2 + g$ is even and if g is even, then g^2 is even so $g^2 + g$ is even. We also need $g^2 + g$ to be divisible by 3. If $g = 3h$ for some integer h this is obvious and we get $i = \frac{9h^2+3h}{6}$. If $g = 3h + 1$ then $g^2 + g = 9h^2 + 9h + 2$, which is not divisible by 3. Lastly, If $g = 3h - 1$ then $g^2 + g = 9h^2 - 3h$, which is divisible by 3, and we get $i = \frac{9h^2-3h}{6}$. This means $i = \frac{3h(h\pm1)}{2}$ for some integer $h \geq 1$.

Substituting this expression for i into the expressions for d given above, a computer gives the results $d = 3h^2 + 3h + \frac{3}{2} \pm \left(h + \frac{1}{2}\right)$, $d = 3h^2 - 3h + \frac{3}{2} \pm \left(h - \frac{1}{2}\right)$ and $d = 2i = 3h^3 \pm 3h$. The second result can be neglected since $i > \frac{d}{2}$ here. Note that we stated before that the third result $d = 2i$ holds for any i.

(d) This proof is omitted here, but can be found in [7, thm 4.6]. □

Lemma 4.19. *[3, Lemma 2.10] Let $q \geq 2$ and $j = 2$. Then $K_j(i) = K_j(h)$ if and only if $i = h$ or $i = 2(d-1)\left(1 - \frac{1}{q}\right) + 1 - h$.*

Proof.* We can check with a computer that the given solutions are indeed solutions to $K_j(i) = K_j(h)$. Because $j = 2$, we know that $K_j(i)$ is quadratic in i, so $K_j(i) = K_j(h)$ has a maximum of 2 solutions for i, which then must be the ones given in the lemma. ☐

4.2 The Johnson case

As in the previous section, we start with an overview of the most important results that we will show in this subsection:

- $E_j(1)$ is the smallest eigenvalue if and only if $j \geq \frac{d(n-d)}{n-1}$. *(Theorem 4.32b)*
- For $j \geq \frac{d(n-d)}{n-1}$, $|E_j(1)|$ is the penabsolute eigenvalue. *(Theorem 4.32a)*

We again need several intermediate steps to get to these results. From Theorem 2.16 we know that the formula for the eigenvalues of graphs from the Johnson scheme is

$$E_j(i) = \sum_{h=0}^{j} (-1)^{j-h} \binom{d-i}{h} \binom{d-h}{j-h} \binom{n-d-i+h}{h}$$

for $0 \leq i, j \leq d$. This polynomial is also called an *Erberlein polynomial* [36], and as with the Kravchuk polynomials, there are multiple equivalent ways to write them down. An overview is provided in the following lemma.

Lemma 4.20 (Equivalence of Erberlein polynomials)**.** *[3, p. 99] The following three expressions for the eigenvalues of a graph from the Johnson scheme are equivalent:*

$$E_j(i) = \sum_{h=0}^{j} (-1)^{h} \binom{i}{h} \binom{d-i}{j-h} \binom{n-d-i}{j-h} \tag{4.11a}$$

$$= \sum_{h=0}^{j} (-1)^{j-h} \binom{d-i}{h} \binom{d-h}{j-h} \binom{n-d-i+h}{h} \tag{4.11b}$$

$$= \sum_{h=0}^{i} (-1)^{i-h} \binom{i}{h} \binom{d-h}{j} \binom{n-d-i+h}{n-d-j}. \tag{4.11c}$$

Proof. The second follows from Theorem 2.16. We will first show that this is equivalent to the first expression.

$$\begin{aligned}
E_j(i) &= \sum_{h=0}^{j} (-1)^{j-h} \binom{d-i}{h} \binom{d-h}{j-h} \binom{n-d-i+h}{h} \\
&= \sum_{h=0}^{j} \sum_{m=0}^{h} (-1)^{j-h} \binom{d-i}{h} \binom{d-h}{j-h} \binom{n-d-i}{m} \binom{h}{h-m} \\
&\quad \text{Vandermonde identity (Lemma 2.5) applied} \\
&= \sum_{h=0}^{j} \sum_{m=0}^{h} (-1)^{j-h} \binom{d-h}{j-h} \binom{n-d-i}{m} \binom{d-i}{h} \binom{h}{m}
\end{aligned}$$

symmetry applied

$$= \sum_{h=0}^{j} \sum_{m=0}^{h} (-1)^{j-h} \binom{d-h}{j-h} \binom{n-d-i}{m} \binom{d-i}{m} \binom{d-i-m}{h-m}$$

Lemma 2.2 applied on the product of the last two binomial coefficients

$$= \sum_{m=0}^{j} \binom{n-d-i}{m} \binom{d-i}{m} \sum_{h=m}^{j} (-1)^{j-h} \binom{d-h}{j-h} \binom{d-i-m}{h-m}$$

changed the order of summation

$$= \sum_{m=0}^{j} \binom{n-d-i}{m} \binom{d-i}{m} \sum_{h=m}^{j} \binom{j-d-1}{j-h} \binom{d-i-m}{h-m}$$

Lemma 2.6 applied

$$= \sum_{m=0}^{j} \binom{n-d-i}{m} \binom{d-i}{m} \sum_{h=0}^{j-m} \binom{j-d-1}{j-h-m} \binom{d-i-m}{h}$$

shifted the index of the inner sum

$$= \sum_{m=0}^{j} \binom{n-d-i}{m} \binom{d-i}{m} \binom{j-m-i-1}{j-m}$$

Vandermonde identity (Lemma 2.5) applied

$$= \sum_{h=0}^{j} \binom{n-d-i}{j-h} \binom{d-i}{j-h} \binom{h-i-1}{h}$$

change of variable $h = j - m$ applied

$$= \sum_{h=0}^{j} (-1)^{h} \binom{n-d-i}{j-h} \binom{d-i}{j-h} \binom{i}{h}.$$

Lemma 2.6 applied

Next, we show that the third expression is equivalent to the first. We have

$$E_j(i) = \sum_{h=0}^{i} (-1)^{i-h} \binom{i}{h} \binom{d-h}{j} \binom{n-d-i+h}{n-d-j}$$

$$= \sum_{h=0}^{i} \sum_{m=0}^{j} (-1)^{i-h} \binom{i}{h} \binom{i-h}{m} \binom{d-i}{j-m} \binom{n-d-i+h}{n-d-j}$$

Vandermonde identity (Lemma 2.5) applied

$$= \sum_{m=0}^{j} \binom{d-i}{j-m} \sum_{h=0}^{i} (-1)^{i-h} \binom{i}{h} \binom{i-h}{m} \binom{n-d-i+h}{n-d-j}$$

changed the order of summation

$$= \sum_{h=0}^{j} \binom{d-i}{j-h} \sum_{m=0}^{i} (-1)^{i-m} \binom{i}{i-m} \binom{i-m}{h} \binom{n-d-i+m}{n-d-j}$$

interchanged the names of h and m and used symmetry

$$= \sum_{h=0}^{j} \binom{i}{h} \binom{d-i}{j-h} \sum_{m=0}^{i} (-1)^{i-m} \binom{i-h}{m} \binom{n-d-i+m}{n-d-j}.$$

Lemma 2.2 applied

This means we still need to show

$$(-1)^h\binom{n-d-i}{j-h}=\sum_{m=0}^{i}(-1)^{i-m}\binom{i-h}{m}\binom{n-d-i+m}{n-d-j}$$

for $i+j-d\leq h\leq\min\{i,j\}$. We do this proof by induction on $n-d$. First, we check two base cases, namely $(n-d,i,j)=(k,i,k)$ and $(n-d,i,j)=(k,i,0)$ for any k and any feasible h. For the first scenario, we need to check

$$(-1)^h\binom{k-i}{k-h}=\sum_{m=0}^{i}(-1)^{i-m}\binom{i-h}{m}\binom{k-i+m}{0}.$$

The left hand side is equal to $(-1)^i$ if $h=i$ and equal to 0 if $h<i$. The right hand side can be rewritten to $(-1)^i\sum_{m=0}^{i-h}(-1)^m\binom{i-h}{m}$. This sum is equal to $(-1)^i$ for $h=i$ and equal to 0 if $h<i$ by Lemma 2.7. For the second scenario, namely $(n-d,i,j)=(k,i,0)$, we need to check

$$(-1)^h\binom{k-i}{-h}=\sum_{m=0}^{i}(-1)^{i-m}\binom{i-h}{m}\binom{k-i+m}{k}.$$

The left hand side is equal to 1 if $h=0$ and equal to 0 else. The right hand side can be rewritten to $(-1)^{i-i}\binom{i-h}{i}\binom{k-i+i}{k}$. It is equal to 1 if $h=0$ and equal to 0 else.

We checked the base cases, so now we can say

$$\begin{aligned}\sum_{m=0}^{i}(-1)^{i-m}\binom{i-h}{m}\binom{n-d-i+m}{n-d-j}&=\sum_{m=0}^{i}(-1)^{i-m}\binom{i-h}{m}\binom{(n-d-1)-i+m}{(n-d-1)-(j-1)}\\&\quad+\sum_{m=0}^{i}(-1)^{i-m}\binom{i-h}{m}\binom{(n-d-1)-i+m}{(n-d-1)-j}\\&\quad\text{Pascal's identity (Lemma 2.1) applied}\\&=(-1)^h\binom{n-d-i-1}{j-h-1}+(-1)^h\binom{n-d-i-1}{j-h}\\&\quad\text{induction hypothesis applied}\\&=(-1)^h\binom{n-d-i}{j-h}.\\&\quad\text{Pascal's identity (Lemma 2.1) applied}\end{aligned}$$

□

Next we will see some properties of the Erberlein polynomials. We write $E_j(i,n,d)$ instead of $E_j(i)$ if we want to emphasize the values of n and d.

Lemma 4.21. *[3, Prop. 3.2] Let* $i,j\geq 1$. *Then*

$$E_j(i,n+2,d+1)=E_j(i-1,n,d)-E_{j-1}(i-1,n,d).$$

Proof.* We have

$$\begin{aligned}E_j(i,n+2,d+1)&=\sum_{h=0}^{j}(-1)^h\binom{i}{h}\binom{d-i+1}{j-h}\binom{n-d-i+1}{j-h}\\&\quad\text{formula 4.11a used}\end{aligned}$$

$$= \sum_{h=0}^{j} (-1)^h \binom{i-1}{h} \binom{d-i+1}{j-h} \binom{n-d-i+1}{j-h}$$
$$+ \sum_{h=0}^{j} (-1)^h \binom{i-1}{h-1} \binom{d-i+1}{j-h} \binom{n-d-i+1}{j-h}$$

Pascal's identity (Lemma 2.1) applied

$$= E_j(i-1,n,d) - \sum_{h=0}^{j-1} (-1)^h \binom{i-1}{h} \binom{d-i+1}{j-h-1} \binom{n-d-i+1}{j-h-1}$$

shifted the index and used $\binom{i-1}{-1} = 0$

$$= E_j(i-1,n,d) - E_{j-1}(i-1,n,d)$$

□

Lemma 4.22. *[3, Prop. 3.3] If $n = 2d$, then $E_{d-j}(i) = (-1)^i E_j(i)$.*

Proof. We want to show $E_{d-j}(i, 2d, d) = (-1)^i E_j(i, 2d, d)$. For the left hand side, by formula 4.11c, we have

$$E_{d-j}(i, 2d, d) = \sum_{h=0}^{i} (-1)^{i-h} \binom{i}{h} \binom{d-h}{d-j} \binom{d-i+h}{j}.$$

For the right hand side, by formula 4.11c, we have

$$E_j(i, 2d, d) = \sum_{k=0}^{i} (-1)^{i-k} \binom{i}{k} \binom{d-k}{j} \binom{d-i+k}{d-j}$$
$$= \sum_{h=0}^{i} (-1)^h \binom{i}{i-h} \binom{d-i+h}{j} \binom{d-h}{d-j}$$

flipped the index: $k = i - h$

$$= (-1)^i \sum_{h=0}^{i} (-1)^{i-h} \binom{i}{h} \binom{d-h}{d-j} \binom{d-i+h}{j}.$$

applied symmetry

□

Lemma 4.23. *[3, Prop. 3.4] Let $n = 2d+1$ and $j = \frac{d+1}{2}$ and $0 < t < \frac{d}{2}$. Then*

$$E_j(2t-1, n, d) = E_j(2t, n, d) = E_j(2t-1, n-1, d).$$

Proof. We start with showing that the first and the third expression are equal. By formula 4.11c, we have

$$E_j(2t-1, n, d) = \sum_{h=0}^{2t-1} (-1)^{2t-h-1} \binom{2t-1}{h} \binom{d-h}{j} \binom{d-2t+h+2}{j}$$

and

$$E_j(2t-1, n-1, d) = \sum_{h=0}^{2t-1} (-1)^{2t-h-1} \binom{2t-1}{h} \binom{d-h}{j} \binom{d-2t+h+1}{j-1}.$$

Using Pascal's formula (Lemma 2.1) on the rightmost binomial coefficient,

$$\binom{d-2t+h+2}{j} = \binom{d-2t+h+1}{j} + \binom{d-2t+h+1}{j-1},$$

we see that $E_j(2t-1,n,d) = E_j(2t-1,n-1,d)$ if and only if

$$\sum_{h=0}^{2t-1} (-1)^{2t-h-1} \binom{2t-1}{h} \binom{d-h}{j} \binom{d-2t+h+1}{j} = 0.$$

This sum has an even number of terms. The terms for $h = 0$ and $h = 2t-1$ cancel each other, and the same holds for $h = 1$, $h = 2t-2$ up to $h = t-1$, $h = t$. In other words, the terms $h = k$ and $h = 2t-1-k$ cancel for $k = 0, \ldots, t-1$. This is because the term with $h = k$ looks like

$$(-1)^{2t-k-1} \binom{2t-1}{k} \binom{d-k}{j} \binom{d-2t+k+1}{j}$$

and the term with $h = 2t-1-k$ looks like

$$\begin{aligned} &(-1)^{2t-(2t-1-k)-1} \binom{2t-1}{2t-1-k} \binom{d-(2t-1-k)}{j} \binom{d-2t+(2t-1-k)+1}{j} \\ &= (-1)^k \binom{2t-1}{k} \binom{d-2t+k+1}{j} \binom{d-k}{j}, \end{aligned}$$

where $2t-k-1$ is even when k is odd and vice versa. Therefore, the sum of all terms equals zero and thus $E_j(2t-1,n,d) = E_j(2t-1,n-1,d)$. Note that this result holds for all feasible j.

Next, we show that the second and the third expression are equal. By formula 4.11a, we have

$$E_j(2t-1,n,d) = \sum_{h=0}^{j} (-1)^h \binom{2t}{h} \binom{d-2t}{j-h} \binom{d-2t+1}{j-h}$$

and

$$E_j(2t-1,2d,d) = \sum_{h=0}^{j} (-1)^h \binom{2t-1}{h} \binom{d-2t+1}{-h} \binom{d-2t+1}{j-h}.$$

We have

$$\begin{aligned} E_j(2t-1,n,d) =& \sum_{h=0}^{j} (-1)^h \binom{2t-1}{h} \binom{d-2t}{j-h} \binom{d-2t+1}{j-h} \\ &+ \sum_{h=0}^{j} (-1)^h \binom{2t-1}{h-1} \binom{d-2t}{j-h} \binom{d-2t+1}{j-h} \\ &\text{Pascal's identity (Lemma 2.1) applied} \\ =& E_j(2t-1,2d,d) - \sum_{h=0}^{j} (-1)^h \binom{2t-1}{h} \binom{d-2t}{j-h-1} \binom{d-2t+1}{j-h} \\ &+ \sum_{h=0}^{j} (-1)^h \binom{2t-1}{h-1} \binom{d-2t}{j-h} \binom{d-2t+1}{j-h} \\ &\text{Pascal's identity (Lemma 2.1) applied on the left sum} \\ =& E_j(2t-1,2d,d) - \sum_{h=0}^{j} (-1)^h \binom{2t-1}{h} \binom{d-2t}{j-h-1} \binom{d-2t+1}{j-h} \\ &- \sum_{h=0}^{j} (-1)^h \binom{2t-1}{h} \binom{d-2t}{j-h-1} \binom{d-2t+1}{j-h-1} \\ &\text{Shifted the index of the lower sum} \end{aligned}$$

$$=E_j(2t-1,2d,d)-\sum_{h=0}^{j}(-1)^h\binom{2t-1}{h}\binom{d-2t}{j-h-1}\binom{d-2t+2}{j-h}.$$

Pascal's identity (Lemma 2.1) applied

Since $j=\frac{d+1}{2}$ and thus $d=2j-1$, we need to show

$$\sum_{h=0}^{j}(-1)^h\binom{2t-1}{h}\binom{2j-2t-1}{j-h-1}\binom{2j-2t+1}{j-h}=0.$$

We can replace the upper bound of the sum with $2t-1$ since the terms with $h\geq j$ and $h\geq 2t-1$ are equal to zero, so we need to show

$$\sum_{h=0}^{2t-1}(-1)^h\binom{2t-1}{h}\binom{2j-2t-1}{j-h-1}\binom{2j-2t+1}{j-h}=0.$$

Like before, the sum has an even number of terms and the terms with $h=k$ and $h=2t-1-k$ cancel for k is $0,\dots,t-1$ because the term with $h=k$ looks like

$$(-1)^k\binom{2t-1}{k}\binom{2j-2t-1}{j-k-1}\binom{2j-2t+1}{j-k}$$

and the term with $h=2t-1-k$ looks like

$$\begin{aligned}&(-1)^{2t-1-k}\binom{2t-1}{2t-1-k}\binom{2j-2t-1}{j-(2t-1-k)-1}\binom{2j-2t+1}{j-(2t-1-k)}\\&=(-1)^{2t-1-k}\binom{2t-1}{k}\binom{2j-2t-1}{j-k-1}\binom{2j-2t+1}{j-k},\end{aligned}$$

where $2t-1-k$ is even when k is odd and vice versa. Therefore, the sum of these terms equals zero and thus $E_j(2t-1,n,d)=E_j(2t-1,2d,d)$. □

Now we can start working towards the results that were mentioned in the beginning of this section. Note that the bound $j\geq\frac{d(n-d)}{n-1}$ that is assumed in these results comes from the following lemma. It is assumed that $d\geq 1$ or $d\geq 2$ if $E_j(1)$ respectively $E_j(2)$ are mentioned.

Lemma 4.24. *[3, Prop. 3.5] Let $j>0$. Then*

- (a) $E_j(1)=0$ *if and only if* $j=\frac{d(n-d)}{n}$,
- (b) $E_j(1)<0$ *if and only if* $j>\frac{d(n-d)}{n}$,
- (c) $E_j(1)=E_j(2)$ *if and only if* $j=\frac{d(n-d)}{n-1}$,
- (d) $E_j(1)<E_j(2)$ *if and only if* $j>\frac{d(n-d)}{n-1}$.

Proof.* Note that $n\geq 2d\geq 2j\geq 2$.

(a) We have

$$\begin{aligned}E_j(1)&=\sum_{h=0}^{1}(-1)^{1-h}\binom{1}{h}\binom{d-h}{j}\binom{n-d+h-1}{n-d-j}\\&=-\binom{d}{j}\binom{n-d-1}{n-d-j}+\binom{d-1}{j}\binom{n-d}{n-d-j}\end{aligned}$$

$$= \binom{d}{j}\binom{n-d}{j}\left(-\frac{j}{n-d}+\frac{d-j}{d}\right)$$
$$= \binom{d}{j}\binom{n-d}{j}\left(1-\frac{jn}{d(n-d)}\right).$$

This shows immediately that $E_j(1) = 0$ if and only if $jn = d(n-d)$.

(b) The product of the two binomial coefficients in $E_j(1)$ is positive, so $E_j(1) < 0$ if and only if $\frac{jn}{d(n-d)} > 1$, which happens exactly when $jn > d(n-d)$.

(c) Assume $j \geq 2$. The case $j = 1$ will be looked at later. From (a), we got an expression for $E_j(1)$ that we can rewrite to

$$\begin{aligned}
E_j(1) =& \binom{d-1}{j}\binom{n-d}{n-d-j} - \binom{d}{j}\binom{n-d-1}{n-d-j} \\
=& \binom{d-1}{j}\binom{n-d}{j} - \binom{d}{j}\binom{n-d-1}{j-1} \\
& \text{symmetry applied} \\
=& \frac{1}{j^2(j-1)^2}\binom{d-2}{j-2}\binom{n-d-2}{j-2}\left((d-j)(d-1)(n-d)(n-d-1) - jd(d-1)(n-d-1)\right).
\end{aligned}$$

In a similar way, we write an expression for $E_j(2)$:

$$\begin{aligned}
E_j(2) &= \sum_{h=0}^{2}(-1)^{2-h}\binom{2}{h}\binom{d-h}{j}\binom{n-d+h-2}{n-d-j} \\
&= \binom{d}{j}\binom{n-d-2}{j-2} - 2\binom{d-1}{j}\binom{n-d-1}{j-1} + \binom{d-2}{j}\binom{n-d}{j} \\
&= \frac{1}{j^2(j-1)^2}\binom{d-2}{j-2}\binom{n-d-2}{j-2} f(n,d,j)
\end{aligned}$$

where

$$f(n,d,j) := dj(d-1)(j-1) - 2j(d-j)(d-1)(n-d-1) + (d-j)(d-j-1)(n-d)(n-d-1).$$

We can then set $E_j(1) = E_j(2)$, cancel the common factor $\frac{1}{j^2(j-1)^2}\binom{d-2}{j-2}\binom{n-d-2}{j-2}$ and expand the brackets on both sides of the equality sign. Simplifying this result gives $j(n-1)(n-2) = d(n-2)(n-d)$. For $j \geq 2$ and thus $n \geq 2d \geq 2j \geq 4$, we get the result $j = \frac{d(n-d)}{n-1}$.

For $j = 1$, we get $E_1(1) = dn - d^2 - n$ and $E_1(2) = dn - d^2 - 2n + 2$ after simplification. This gives that $E_1(1) = E_1(2)$ is equivalent to $-n = -2n + 2$, so to $n = 2$. Assuming $E_1(1) = E_1(2)$, we have $n = 2$, so $d = 1$, so $1 = j = \frac{d(n-d)}{n-1}$. On the other hand, if we assume $1 = \frac{d(n-d)}{n-1}$, we know $d = 1$, $n = 2$ is the only feasible solution. This is because $d \geq 2$ would mean $n - 1 = d(n-d) \geq 2(n-d)$ and thus $2d \geq n + 1$, which is a contradiction to the inequality $n \geq 2d$. Therefore, $1 = \frac{d(n-d)}{n-1}$ implies $n = 2$ and thus $E_1(1) = E_1(2)$.

(d) For $j = 1$, we had $E_1(1) = dn - d^2 - n$ and $E_1(2) = dn - d^2 - 2n + 2$. This means $E_j(1) < E_j(2)$ if and only if $n < 2$, which is not possible since we assumed $n > 1$. We also have that $\frac{d(n-d)}{n-1} < 1 = j$ is not possible, so the statement holds for $j = 1$.

Now assume $j \geq 2$. Let $C = \frac{1}{j^2(j-1)^2}\binom{d-2}{j-2}\binom{n-d-2}{j-2}$ and note that $C > 0$. We saw that $\frac{1}{C}E_j(1)$ is linear in j and $\frac{1}{C}E_j(2)$ is quadratic in j with leading coefficient $d(d-1) + 2(d-1)(n-d-1) + (n-d)(n-d-1)$. All terms are positive, so the leading coefficient is positive. Now we know that $\frac{1}{C}\left(E_j(2) - E_j(1)\right)$ is quadratic in j with positive leading coefficient.

From (c), we got that $\frac{1}{C}\left(E_j(2) - E_j(1)\right)$ has only one zero for $j \geq 1$, namely $j = \frac{d(n-d)}{n-1}$. Also, $E_j(2) - E_j(1) = 1 - 1 = 0$ for $j = 0$. Since the two zeros of $E_j(2) - E_j(1)$ are $j = 0$ and $j = \frac{d(n-d)}{n-1}$ and the leading coefficient of $\frac{1}{C}\left(E_j(2) - E_j(1)\right)$ is positive, we have that $\frac{1}{C}\left(E_j(2) - E_j(1)\right) > 0$ if and only if $j > \frac{d(n-d)}{n-1}$, so $E_j(1) < E_j(2)$ if and only if $j > \frac{d(n-d)}{n-1}$. □

The results of Lemma 4.24 are summarized in Figure 4.6.

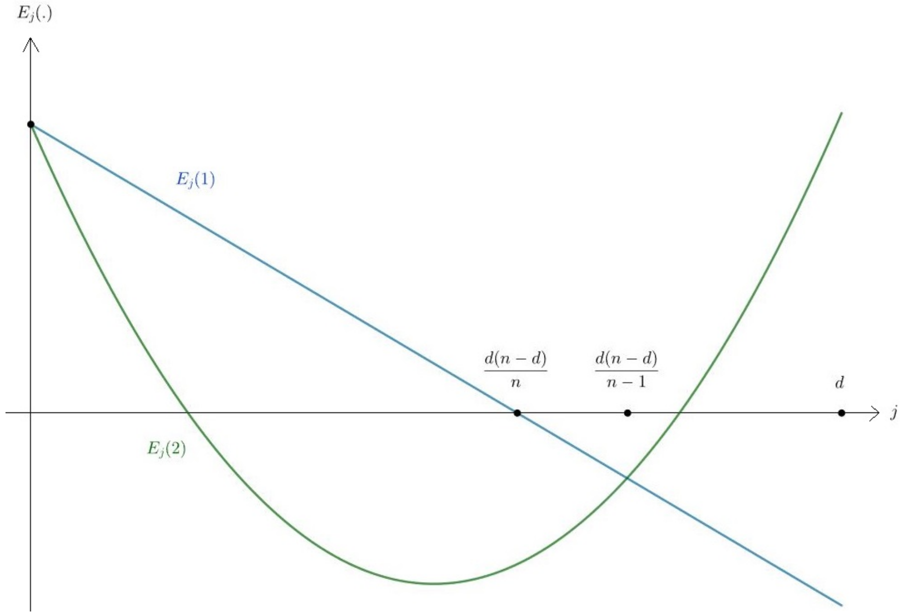


Figure 4.6: This graph summarizes the results of Lemma 4.23.

Next, we need some preliminary results from probability theory. More specifically, we need tail inequalities from the hypergeometric distribution since this will provide us with a bound that can be used in the proof of Lemma 4.26. This distribution with parameters (N, M, k) can be seen as follows: we have an urn with N balls, of which M are white and $N - M$ are black. Someone is going to draw k balls from this urn, uniformly and without replacement. The random variable X is the number of white balls among the k balls that were drawn. For this distribution, we have $\mathbb{P}(X = x) = \frac{\binom{M}{x}\binom{N-M}{k-x}}{\binom{N}{k}}$ and $\mathbb{E}[X] = \frac{kM}{N}$. The following lemma summarizes the result we need in order to prove Lemma 4.26.

Lemma 4.25 (Chvátal tail inequalities)**.** *[24][9] Let X be a random variable with*

$$X \sim hypergeometric(N, M, k).$$

Then

$$\mathbb{P}\left(X \geq \mathbb{E}[X] + tk\right) \leq e^{-2t^2k} \quad and \quad \mathbb{P}\left(X \leq \mathbb{E}[X] - tk\right) \leq e^{-2t^2k}.$$

Recall that $k_j = \binom{d}{j}\binom{n-d}{j}$ and $|V| = \binom{n}{d}$. Using Lemma 4.25, we get the following:

Lemma 4.26. *[3, adaptation from Lemma 3.6] Let $I = \left(\frac{d(n-d)}{n} - \sqrt{d}, \frac{d(n-d)}{n} + \sqrt{d}\right)$. Then* $\sum_{j \in I} k_j > \frac{8}{11}|V|$.

Remark. The inequality for k_j in [3] was not strict, but it is in fact strict.

Proof.* Let X be a random variable with $X \sim \textit{hypergeometric}(n, n-d, d)$. From the formulas provided above, we have

$$\mathbb{E}[X] = \frac{d(n-d)}{n} \quad \text{and} \quad \mathbb{P}(X = j) = \frac{\binom{d}{j}\binom{n-d}{j}}{\binom{n}{d}} = \frac{k_j}{|V|}.$$

By Lemma 4.25 we have, for any $t > 0$, the tail inequalities

$$\mathbb{P}\left(X - \mathbb{E}[X] \geq td\right) \leq e^{-2t^2 d} \quad \text{and} \quad \mathbb{P}\left(\mathbb{E}[X] - X \geq td\right) \leq e^{-2t^2 d}.$$

Substituting $t = d^{-1/2}$, this gives

$$\begin{aligned}\mathbb{P}\left(\frac{d(n-d)}{n} - \sqrt{d} < X < \frac{d(n-d)}{n} + \sqrt{d}\right) &= \mathbb{P}\left(-td < X - \mathbb{E}[X] < td\right) \\ &= 1 - \mathbb{P}\left(X - \mathbb{E}[X] \geq td\right) - \mathbb{P}\left(\mathbb{E}[X] - X \geq td\right) \\ &\geq 1 - 2e^{-2\left(d^{-1/2}\right)^2 d} \\ &= 1 - 2e^{-2} \\ &> \frac{8}{11}.\end{aligned}$$

To conclude the statement, we write

$$\frac{\sum_{j\in I} k_j}{|V|} = \sum_{j\in I} \mathbb{P}(X = j) = \mathbb{P}\left(\frac{d(n-d)}{n} - \sqrt{d} < X < \frac{d(n-d)}{n} + \sqrt{d}\right) > \frac{8}{11}.$$

□

Next, we prove some more propositions and lemmas that we need in order to prove the results that were mentioned in the beginning of this section.

Proposition 4.27. *Let $j_0 = \frac{d(n-d)}{n}$. Then $\operatorname*{argmax}_{0\leq j\leq d} \binom{d}{j}\binom{n-d}{j}$ is equal to $\lfloor j_0 \rfloor$ or $\lceil j_0 \rceil$.*

Remark. It is *not* necessarily true that the argmax is equal to the integer closest to j_0. Take for example $d = 13$ and $n = 100$. Then $j_0 \approx 11.3$, so j_0 is closest to $\lfloor j_0 \rfloor$. However, we have $\binom{13}{11}\binom{100-13}{11} \approx 2.18 \cdot 10^{15}$ and $\binom{13}{12}\binom{100-13}{12} \approx 2.31 \cdot 10^{15}$, so $\binom{d}{\lfloor j_0 \rfloor}\binom{n-d}{\lfloor j_0 \rfloor} < \binom{d}{\lceil j_0 \rceil}\binom{n-d}{\lceil j_0 \rceil}$.

On the other hand we can take $d = 3$ and $n = 10$. Then $j_0 \approx 2.1$, so j_0 is again closest to $\lfloor j_0 \rfloor$. In this case, however, we have $\binom{3}{2}\binom{10-3}{2} = 63$ and $\binom{3}{3}\binom{10-3}{3} = 35$, so here $\binom{d}{\lfloor j_0 \rfloor}\binom{n-d}{\lfloor j_0 \rfloor} > \binom{d}{\lceil j_0 \rceil}\binom{n-d}{\lceil j_0 \rceil}$.

Proof. It is sufficient to show that for all feasible $m > 0$, the following two statements hold:

$$\binom{d}{\lceil j_0 \rceil}\binom{n-d}{\lceil j_0 \rceil} > \binom{d}{\lceil j_0 \rceil + m}\binom{n-d}{\lceil j_0 \rceil + m} \quad \text{and} \quad \binom{d}{\lfloor j_0 \rfloor}\binom{n-d}{\lfloor j_0 \rfloor} > \binom{d}{\lfloor j_0 \rfloor - m}\binom{n-d}{\lfloor j_0 \rfloor - m}.$$

Since the proofs of these two statements are very similar, we will only show the first one. We have

$$\begin{aligned}\frac{\binom{d}{\lceil j_0 \rceil}\binom{n-d}{\lceil j_0 \rceil}}{\binom{d}{\lceil j_0 \rceil + m}\binom{n-d}{\lceil j_0 \rceil + m}} &= \frac{(\lceil j_0 \rceil + m)!^2\,(d - \lceil j_0 \rceil - m)!\,(n - d - \lceil j_0 \rceil - m)!}{\lceil j_0 \rceil!^2\,(d - \lceil j_0 \rceil)!\,(n - d - \lceil j_0 \rceil)!} \\ &= \prod_{k=1}^{m} \frac{(\lceil j_0 \rceil + k)^2}{(d - \lceil j_0 \rceil - (k-1))(n - d - \lceil j_0 \rceil - (k-1))} \\ &= \prod_{k=1}^{m} \frac{\lceil j_0 \rceil^2 + 2k\lceil j_0 \rceil + k^2}{d(n-d) - n\lceil j_0 \rceil - n(k-1) + \lceil j_0 \rceil^2 + 2(k-1)\lceil j_0 \rceil + (k-1)^2}\end{aligned}$$

$$= \prod_{k=1}^{m} \frac{\lceil j_0 \rceil^2 + 2k\lceil j_0 \rceil + k^2}{(\lceil j_0 \rceil^2 + 2k\lceil j_0 \rceil + k^2) + (nj_0 - (n+2)\lceil j_0 \rceil - (n+2)k + n + 1)}.$$

It is therefore enough to show $nj_0 - (n+2)\lceil j_0 \rceil - (n+2)k + n + 1 < 0$ for all k for the first statement. We have

$$nj_0 - (n+2)\lceil j_0 \rceil - (n+2)k + n + 1 \leq -2j_0 - (n+2)k + n + 1 < -2j_0 < 0.$$

The proof of the second statement is left to the reader. □

Recall from Lemma 2.17 that $m_i = \binom{n}{i} - \binom{n}{i-1}$.

Proposition 4.28. *We have $m_i \geq m_3$ for $i \geq 3$ except if $(n, i) = (8, 4)$, $(9, 4)$, $(10, 5)$ or $(12, 6)$.*

Remark. Note that this means we have $m_i \geq m_3$ for $i \geq 7$.

Proof. We have

$$m_i = \binom{n}{i} - \binom{n}{i-1} = \left(1 - \frac{i}{n-i+1}\right)\binom{n}{i} = \left(\frac{n-2i+1}{n-i+1}\right)\binom{n}{i}$$

and thus $m_3 = \frac{n(n-1)(n-5)}{6}$. This means we first want to show

$$\binom{n}{i} \geq \frac{n(n-1)(n-5)(n-i+1)}{6(n-2i+1)}$$

for $i \geq 7$. Using $i \geq 7$ and $n \geq 2d \geq 2i \geq 14$, we get that this inequality holds:

$$\begin{aligned} \binom{n}{i} \geq \binom{n}{7} &= \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{7!} \\ &\geq \frac{n(n-1)(n-5)(n-6)}{6} \\ &\geq \frac{n(n-1)(n-5)(n-i+1)}{6(n-2i+1)}. \end{aligned}$$

For fixed i and $n \to \infty$, we see that this inequality holds as well. Therefore we can check the cases $i = 4, 5, 6$ by hand easily. The cases where the inequality does *not* hold are $(n, i) = (8, 4)$, $(9, 4)$, $(10, 5)$ and $(12, 6)$. □

Lemma 4.29. *[3, Lemma 3.7] Let $j_0 = \frac{d(n-d)}{n}$ and let $j_0 \leq j < j_0 + \frac{3}{2}$. If $\frac{d(n-d)}{n-1} \leq j < d$ and $i \geq 3$, then $|E_j(i)| \leq |E_j(1)|$.*

Proof.* This proof consists of five main steps:

1. Show that $\frac{n}{j_0^2} < 1 + \frac{3}{2d}$ for $d \geq 10$ and $n \geq 73$.
2. Show that $\frac{k_{j-1}}{k_j} < 3$ for $d \geq 10$ and $n \geq 73$.
3. Show that $\frac{v}{k_j} < \frac{n-5}{6}$ for $d \geq 10$ and $n \geq 73$.
4. Show that $E_j(i)^2 \leq E_j(1)^2$ for $d \geq 10$, $n \geq 73$ and $i \geq 3$.
5. Show that $E_j(i)^2 \leq E_j(1)^2$ for $n < 73$ and $i \geq 3$.

Step 1. Note that $\frac{n}{j_0^2} = \frac{n^3}{d^2(n-d)^2}$. For there to exist a feasible j, we need $\frac{d(n-d)}{n-1} \leq d-1$. This is true for $n \leq d^2 - d + 1$, so for $n - d \leq (d-1)^2$. We have $\frac{n}{j_0^2} = \frac{n^3}{d^2(n-d)^2}$ and

$$\frac{\partial}{\partial n} \frac{n^3}{d^2(n-d)^2} = \frac{3n^2(n-d) - 2n^3}{d^2(n-d)^3} = \frac{n^2(n-3d)}{d^2(n-d)^3},$$

so $\frac{n}{j_0^2}$ is decreasing in n for $n \leq 3d$ and increasing in n for $n \geq 3d$. For $d \geq 10$ and $n \geq 73$, this means it is maximal for n as large as possible, so $n = d^2 - d + 1$. This gives

$$\frac{n}{j_0^2} = \frac{n^3}{d^2(n-d)^2} \leq \frac{(d^2-d+1)^3}{d^2(d-1)^4} = \frac{((d-1)^2+d)^3}{d^2(d-1)^4}.$$

Expanding the numerator of the right hand side gives the following:

$$\frac{((d-1)^2+d)^3}{d^2(d-1)^4} = \frac{(d-1)^2}{d^2} + \frac{3}{d} + \frac{3}{(d-1)^2} + \frac{d}{(d-1)^4} = 1 + \frac{1}{d} + \left(\frac{1}{d^2} + \frac{3}{(d-1)^2} + \frac{d}{(d-1)^4}\right).$$

The part between the brackets decreases faster in d than $\frac{1}{2d}$. We found by computer that for $d = 9$, the part between the brackets is approximately equal to 0.061 and $\frac{1}{2d} \approx 0.055$. For $d = 10$, the part between the brackets is approximately equal to 0.049 and $\frac{1}{2d} = 0.05$. Therefore, for $d \geq 10$, the part between the brackets is less than $\frac{1}{2d}$, so we have $\frac{n}{j_0^2} < 1 + \frac{3}{2d}$ for $d \geq 10$.

Step 2. By formula (2.1) and Lemma 2.15, we have

$$\frac{k_{j-1}}{k_j} = \frac{c_j}{b_{j-1}} = \frac{j^2}{(d-j+1)(n-d-j+1)}.$$

We want to show $\frac{k_{j-1}}{k_j} < 3$, which is equivalent to showing

$$LHS(j) := d(n-d) - n(j-1) + (j-1)^2 - \frac{1}{3}j^2 = \frac{2}{3}j^2 - (n+2)j + (n+1+d(n-d)) > 0.$$

The left hand side, which we will refer to with $LHS(j)$ from now on, is quadratic in j with a positive leading coefficient. By setting the derivative of $LHS(j)$ w.r.t. j to zero, we find that the minimum of $LHS(j)$ is attained in $j = \frac{3}{4}(n+2)$. We have

$$j_0 + \frac{3}{2} = d(n-d) \cdot \frac{1}{n} + \frac{3}{2} \leq \frac{n^2}{4} \cdot \frac{1}{n} + \frac{3}{2} < \frac{3}{4}(n+2),$$

where the first inequality comes from the fact that $d(n-d)$ is maximal at $n = 2d$. The line above implies $LHS(j)$ is decreasing in j for $j_0 \leq j < j_0 + \frac{3}{2}$. This means it is enough to show

$$LHS\left(j_0 + \frac{3}{2}\right) = nj_0 - n\left(j_0 + \frac{1}{2}\right) + \left(j_0 + \frac{1}{2}\right)^2 - \frac{1}{3}\left(j_0 + \frac{3}{2}\right)^2 > 0,$$

where we used $nj_0 = d(n-d)$. Simplifying, we see that it is enough to show $\frac{4}{3}j_0^2 > n+1$. From step 1, we got $\frac{n}{j_0^2} < \frac{2d+3}{2d}$, so $\frac{4}{3}j_0^2 > \frac{8nd}{6d+9}$. This means it is enough to show the inequality

$$\frac{8nd}{6d+9} \geq n+1 = \frac{6nd+9n+6d+9}{6d+9},$$

so it is enough to show $2nd - 9n - 6d - 9 \geq 0$. The derivative of the left hand side w.r.t. n is equal to $2d - 9$, which is positive for $d \geq 10$, so this function is increasing in n. Since $n \geq 2d$, it is therefore enough to show $4d^2 - 24d - 9 > 0$, which is true for $d \geq 10$. This means we have shown $\frac{k_{j-1}}{k_j} < 3$ for $d \geq 10$.

Step 3. By Lemma 4.26, we know

$$\sum_{l\in(j_0-\sqrt{d},j_0+\sqrt{d})} k_l > \frac{8}{11}|V|.$$

We have shown in Proposition 4.27 that the maximum of k_l is obtained in k_{j_1}, where j_1 is either $\lfloor j_0 \rfloor$ or $\lceil j_0 \rceil$. This means $2\sqrt{d}k_{j_1} > \frac{8}{11}|V|$ and thus $\frac{|V|}{k_{j_1}} < \frac{11}{4}\sqrt{d}$. For this j_1 we have $j_1 \leq \lceil j_0 \rceil \leq j$ and $j - j_1 < j_0 - j_1 + \frac{3}{2} < \frac{5}{2}$, so $j - j_1 \leq 2$, which gives $j - 2 \leq j_1 \leq j$.

If $j_1 = j - 1$, we have $k_{j_1} = k_{j-1} < 3k_j$ by step 2. If $j_1 = j - 2$, we can use formula (2.1) and Lemma 2.8 to conclude

$$\frac{k_{j-2}}{k_{j-1}} = \frac{c_{j-1}}{b_{j-2}} \leq \frac{c_j}{b_{j-1}} = \frac{k_{j-1}}{k_j} < 3,$$

so $k_{j_1} = k_{j-2} < 9k_j$. This means that for any possible value of j_1, we have $k_{j_1} < 9k_j$ and thus $\frac{|V|}{k_j} < \frac{99}{4}\sqrt{d}$. To prove the statement of step 3, it is enough to show $\frac{99}{4}\sqrt{d} \leq \frac{n-5}{6}$ for all $n \geq 2d$. Plugging in $n \geq 2d$ gives us the inequality

$$\frac{99}{4}\sqrt{\frac{n}{2}} \leq \frac{n-5}{6},$$

which is true for $n \geq 11037$. For $n < 11037$, we checked that $\frac{|V|}{k_j} < \frac{n-5}{6}$ for all feasible values of n, d and j by computer. The code that was used to check this can be found in appendix B.2. From this we can conclude that $\frac{v}{k_j} < \frac{n-5}{6}$ for $d \geq 10$.

Step 4. On the one hand, the sum of the squares of the eigenvalues of some matrix equals the trace of this matrix squared. For the graph $J(n, d, j)$, this translates to

$$|V|k_j = \sum_{i=0}^{d} m_i E_j(i)^2,$$

where the right hand side is greater than or equal to $m_i E_j(i)^2$ for any i. Because $n \geq 2d \geq 20$, we can use Lemma 4.28 to find

$$E_j(i)^2 \leq \frac{|V|k_j}{m_3} = \frac{6|V|k_j}{n(n-5)(n-1)}.$$

On the other hand, we have

$$E_j(1)^2 = \left(\binom{n}{j}\binom{n-d}{j}\left(1 - \frac{nj}{d(n-d)}\right)\right)^2 = k_j^2\left(1 - \frac{j}{j_0}\right)^2 = k_j^2\left(\frac{j_0 - j}{j_0}\right)^2.$$

We can use our knowledge on j to estimate the right hand side:

$$j_0 - j \leq \frac{d(n-d)}{n} - \frac{d(n-d)}{n-1} = \frac{-j_0}{n-1} \quad \Rightarrow \quad (j_0 - j)^2 \geq \frac{j_0^2}{(n-1)^2},$$

which then gives $E_j(1)^2 \geq \frac{k_j^2}{(n-1)^2}$. This means it suffices to show

$$\frac{k_j^2}{(n-1)^2} \geq \frac{6|V|k_j}{n(n-1)(n-5)} \quad \Leftrightarrow \quad \frac{|V|}{k_j} \leq \frac{n(n-5)}{6(n-1)},$$

which follows from step 3. We have shown now that $E_j(i)^2 \leq E_j(1)^2$ or equivalently $|E_j(i)| \leq |E_j(1)|$ for $d \geq 10$ and $i \geq 3$.

Step 5. For $n \leq 73$, We checked if $E_j(i)^2 \leq E_j(1)^2$ for all feasible combinations of (n, d, j, i) by computer. The result used to check this can be found in appendix B.3. It turns out that $E_j(i)^2 \leq E_j(1)^2$ for $n \leq 73$ and $i \geq 3$ as well. For $d \leq 9$, we know $n \leq d^2 - d + 1 \leq 73$, so all cases for $d \leq 9$ have been checked as well, and thus we can conclude that $|E_j(i)| \leq |E_j(1)|$ for all n, d and $i \geq 3$. □

Lemma 4.30. *[3, Lemma 3.8] Let $(j-1)(n+1) \geq d(n-d)$. Then*

$$E_j(0) + |E_{j-1}(1)| + |E_j(1)| \leq E_{j-1}(0).$$

Proof.* We have $E_j(0) = k_j$ and

$$E_j(1) = \binom{d}{j}\binom{n-d}{j}\left(1 - \frac{jn}{d(n-d)}\right).$$

Moreover, we have $(j-1)(n+1) = jn-(n-j)-1 \geq d(n-d)$, so $jn \geq d(n-d)+(n-j)+1 > d(n-d)$ and $j \geq 2$. We want to show $E_j(0) + |E_{j-1}(1)| + |E_j(1)| \leq E_{j-1}(0)$, which is equivalent to

$$\binom{d}{j}\binom{n-d}{j}+\binom{d}{j-1}\binom{n-d}{j-1}\left|1-\frac{(j-1)n}{d(n-d)}\right|+\binom{d}{j}\binom{n-d}{j}\left|1-\frac{jn}{d(n-d)}\right| \leq \binom{d}{j-1}\binom{n-d}{j-1}.$$

Dividing by $\binom{d}{j-1}\binom{n-d}{j-1}$ on both sides and simplifying, this means we need to show

$$\frac{d-j+1}{j}\cdot\frac{n-d-j+1}{j}\cdot\left(1+\left|1-\frac{jn}{d(n-d)}\right|\right)+\left|1-\frac{(j-1)n}{d(n-d)}\right| \leq 1.$$

Since $jn \geq d(n-d)$, we know that the argument of the leftmost absolute value is negative and we can rewrite our inequality to

$$\frac{n(d-j+1)(n-d-j+1)}{jd(n-d)}+\left|1-\frac{(j-1)n}{d(n-d)}\right| \leq 1.$$

If $(j-1)n \geq d(n-d)$, we need to show

$$\frac{n(d-j+1)(n-d-j+1)}{jd(n-d)} - 1 + \frac{(j-1)n}{d(n-d)} \leq 1,$$

which, after simplification, is equivalent to

$$(d-j+1)(n-d-j+1)+j(j-1) \leq \frac{2jd(n-d)}{n},$$

which is again equivalent to

$$d(n-d)(n-2j) \leq n(j-1)(n-2j+1).$$

We know $n(j-1)(n-2j+1) = (j-1)(n^2-2jn+n)$ and by assumption $d(n-d)(n-2j) \leq (j-1)(n+1)(n-2j) = (j-1)(n^2-2jn+n-2j)$, so indeed $d(n-d)(n-2j) \leq n(j-1)(n-2j+1)$ and therefore $E_j(0) + |E_{j-1}(1)| + |E_j(1)| \leq E_{j-1}(0)$ for $(j-1)n \geq d(n-d)$.

If $(j-1)n < d(n-d)$, we need to show

$$\frac{n(d-j+1)(n-d-j+1)}{jd(n-d)} + 1 - \frac{(j-1)n}{d(n-d)} \leq 1,$$

which, after simplification, is equivalent to $(d-j+1)(n-d-j+1) \leq j(j-1)$, which is again equivalent to $d(n-d) \leq (j-1)(n+1)$, and this holds by assumption. □

Proposition 4.31. *[3, Prop. 3.9] Let $d \geq 1$. Then the smallest eigenvalue of $K(n,d)$ is $E_d(1)$. Moreover, $E_d(1)$ is the second largest eigenvalue in absolute value.*

Proof. Note that $j = d$ for the Kneser graph. From Proposition 2.18, we know

$$E_d(i) = (-1)^i\binom{n-d-i}{n-2d}.$$

The value of i for which $E_d(i)$ is smallest is the value of i for which i is odd and $\binom{n-d-i}{n-2d}$ is as large as possible, so i is as small as possible. This happens when $i = 1$, so $E_d(1)$ is the smallest eigenvalue of $K(n, d)$.

The value of i for which $|E_d(i)|$ is largest is the value of i for which $\binom{n-d-i}{n-2d}$ is as large as possible, so i is as small as possible. This means $|E_d(0)|$ is the largest eigenvalue in absolute value, and $|E_d(1)|$ is the second largest eigenvalue in absolute value. □

Finally, we get to the theorem that shows the results from the beginning of this section.

Theorem 4.32. *[3, Thm. 3.10] Let $j > 0$. Then*

(a) $|E_j(i)| \leq |E_j(1)|$ *for* $1 \leq i \leq d$ *if* $j(n-1) \geq d(n-d)$,

(b) $E_j(1) \leq E_j(i)$ *for all* $0 \leq i \leq d$ *if and only if* $j(n-1) \geq d(n-d)$.

Remark. Note from Theorem 2.9 that the valency is the largest eigenvalue, also in absolute value, so $|E_j(1)| < |E_j(0)| = k_j$.

Proof.* (a) For $i = 2$, this theorem holds by Lemmas 4.24c and 4.24d for every d. When $d = 1$, we have that $j(n-1) \geq d(n-d)$ only holds when $j = d = 1$, so the theorem holds by Proposition 4.31. When $d = 2$, we look at the cases $j = 1$ and $j = 2$ separately. For $j = 1$, the inequality $j(n-1) \geq d(n-d)$ only holds when $n \leq 3$, which is not possible since $n \geq 2d = 4$. For $j = 2$, the theorem again holds by Proposition 4.31. Since the theorem is trivial for $i = 1$, we can assume $i, d \geq 3$ from now on.

When $j = d$, the theorem holds by Proposition 4.31. When $\frac{d(n-d)}{n-1} \leq j \leq \frac{d(n-d)}{n-3}$ and $j < d$, we have $j > \frac{d(n-d)}{n} = j_0$ and $j < \frac{d(n-d)}{n} + \frac{3}{2} = j_0 + \frac{3}{2}$, since

$$j < \frac{d(n-d)}{n-3} \quad \Leftrightarrow \quad nj - 3j < d(n-d) \quad \Rightarrow \quad nj - 3d < d(n-d) \quad \Rightarrow \quad nj - \frac{3}{2}n < d(n-d).$$

This means that the theorem holds by Lemma 4.24. We are left with the case $j \geq \frac{d(n-d)}{n-3}$, which we will show using induction on d. We checked the base case $d = 2$ already. Now assume $|E_j(i)| \leq |E_j(1)|$ holds for $d-1$ and all i, j, n satisfying the conditions. More specifically, assume

$$|E_j(i-1, n-2, d-1)| \leq |E_j(1, n-2, d-1)| \text{ and } |E_{j-1}(i-1, n-2, d-1)| \leq |E_{j-1}(1, n-2, d-1)|.$$

This means we have

$$\begin{aligned}
|E_j(i, n, d)| &= |E_j(i-1, n-2, d-1) - E_{j-1}(i-1, n-2, d-1)| \\
&\quad \text{Proposition 4.21 applied} \\
&\leq |E_j(i-1, n-2, d-1)| + |E_{j-1}(i-1, n-2, d-1)| \\
&\quad \text{triangle inequality applied} \\
&\leq |E_j(1, n-2, d-1)| + |E_{j-1}(1, n-2, d-1)| \\
&\quad \text{induction hypothesis applied} \\
&\leq E_{j-1}(0, n-2, d-1) - E_j(0, n-2, d-1) \\
&\quad \text{Lemma 4.30 applied} \\
&\leq |E_{j-1}(0, n-2, d-1) - E_j(0, n-2, d-1)| \\
&= |E_j(1, n, d)|. \\
&\quad \text{Proposition 4.21 applied}
\end{aligned}$$

Note that we may apply Lemma 4.30 because $j(n-1) \geq d(n-d) \Leftrightarrow (j-1)((n-2)+1) \geq (d-1)((n-2)-(d-1))$. The induction hypothesis can be applied because we have

$$j((n-2)-1) = j(n-3) \geq d(n-d) > d(n-d) - n + 1$$

$$
\begin{aligned}
&= (n-d-1)(d-1) = (d-1)((n-2)-(d-1)), \text{ and} \\
(j-1)((n-2)-1) &= j(n-3) - n + 3 \geq d(n-d) - n + 3 > d(n-d) - n + 1 \\
&= (n-d-1)(d-1) = (d-1)((n-2)-(d-1)).
\end{aligned}
$$

(b) If $E_j(1) \leq E_j(i)$ for all i, then also $E_j(1) \leq E_j(2)$, so by propositions 4.24c and 4.24d we have $j(n-1) \geq d(n-d)$. If $j(n-1) \geq d(n-d)$, then $jn > d(n-d)$, so $E_j(1) < 0$ by Proposition 4.24. This, together with (a) and the fact that $E_j(0) = \binom{d}{j}\binom{n-d}{j} \geq 0$ proves the statement. □

Figures 4.7 and 4.8 show some visualizations of P-matrices. The results of Theorem 4.32 are indicated with red fields. Recall that every column on the right of the black vertical line corresponds to $j \geq \frac{d(n-d)}{n-1}$.

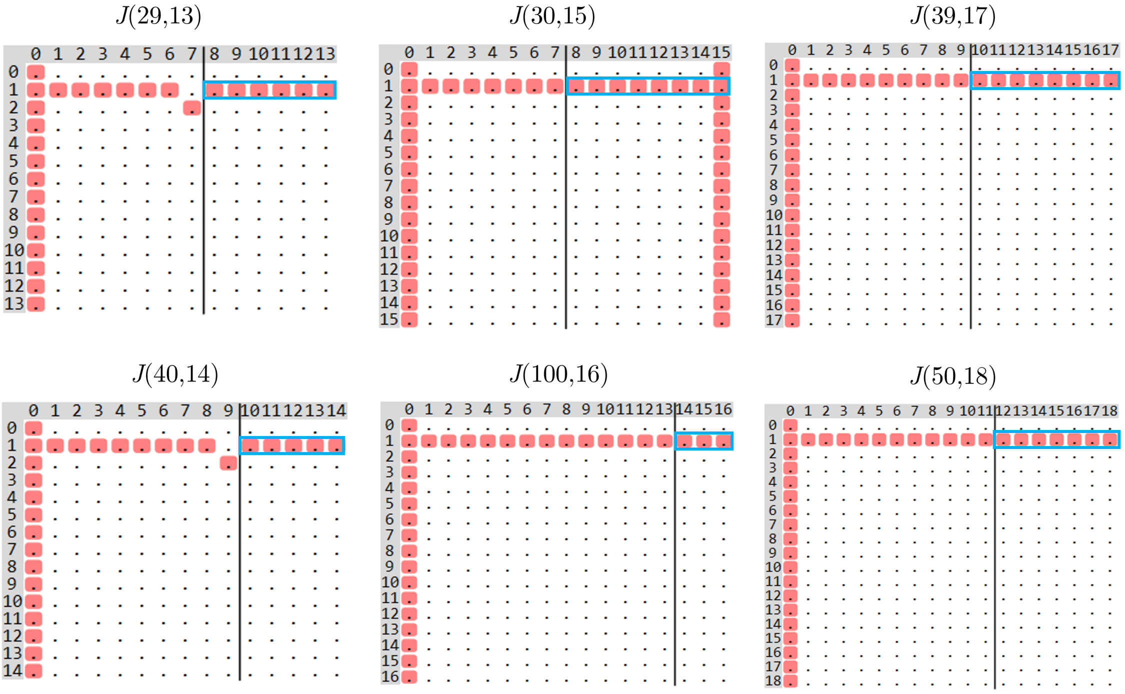


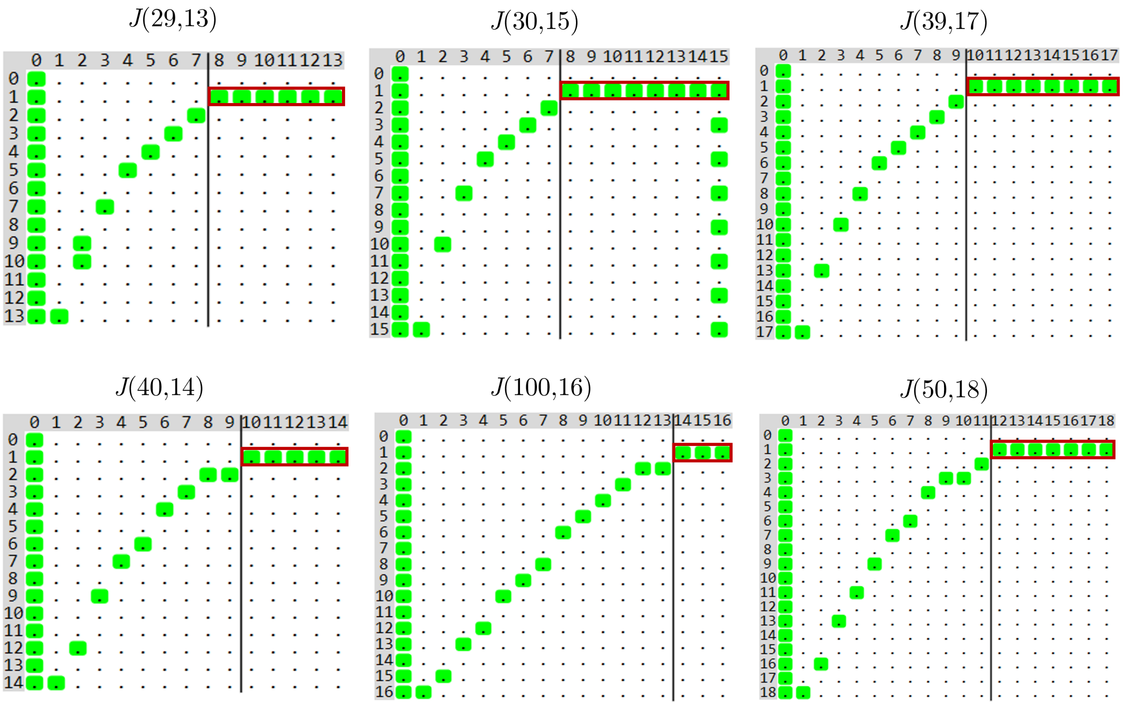
Figure 4.7: Visualization of the results of Theorem 4.32a. The highlighted boxes indicate the penabsolute eigenvalues per column. The blue fields indicate the results of Theorem 4.32.

As we noted in Chapter 1, the smallest eigenvalue of $J(2d, d, j)$ for large enough d was used to show that the performance ratio of the Goemans-Williamson algorithm for the max-cut problem was tight. The result that was needed to prove that this ratio is tight is summarized in the following corollary.

Corollary 4.33. *[3, Cor. 3.11] If $j > \frac{d}{2}$ and $n = 2d$, then $E_j(1)$ is the smallest eigenvalue of $J(2d, d, j)$ and the second largest in absolute value.*

Proof. If $j > \frac{d}{2}$, then $j \geq \frac{d+1}{2}$ and we have $\frac{d+1}{2} \geq \frac{d^2}{2d-1} = \frac{d(n-d)}{n-1}$ for $d \geq 1$. The statement then follows directly from Theorem 4.32. □

We end this section with an additional result from [3] for large n.


J(29,13) J(30,15) J(39,17)

J(40,14) J(100,16) J(50,18)

Figure 4.8: Visualization of the results of Theorem 4.32b. The highlighted boxes indicate the smallest eigenvalues per column. The red fields indicate the results of Theorem 4.32b.

Proposition 4.34. *[3, Prop. 3.12] Let d fixed and n sufficiently large. Then*

(a) *$E_j(i) \geq 0$ for $i+j \leq d$,*

(b) *$E_j(i)$ has sign $(-1)^{i+j-d}$ for $i+j \geq d$,*

(c) *The smallest eigenvalue of $J(n,d,j)$ is $E_j(d-j+1)$ for every $j>0$.*

*Proof**. We have

$$E_j(i) = \sum_{h=0}^{j} (-1)^h \binom{i}{h} \binom{d-i}{j-h} \binom{n-d-i}{j-h}.$$

This means that for d fixed and $n \to \infty$, we have that the nonzero terms of $E_j(i)$ decrease monotonically in absolute value, so the sign of $E_j(i)$ is that of its first nonzero term by Lemma 4.15b. Furthermore we have that the first nonzero term of $E_j(i)$ is dominant, meaning it is much larger in absolute value than the other terms.

The middle binomial coefficient is nonzero for $i+j-d \leq h \leq j$. For $i+j \leq d$, the first nonzero term is the term for $h=0$, which is $\binom{d-i}{j}\binom{n-d-i}{j}$. This term is positive, so for n large enough, $E_j(i)$ is positive. For $i+j \geq d$, the first nonzero term is the term for $h = i+j-d$, which is $(-1)^{i+j-d}\binom{i}{i+j-d}\binom{n-d-i}{d-i}$. This term has sign $(-1)^{i+j-d}$, so this will also be the sign of $E_j(i)$ for n large enough.

The smallest eigenvalue of $J(n,d,j)$ will be one with a negative sign, so $i+j \geq d$ must hold and $i+j-d$ must be odd. Note that this means $i+j-d \geq 1$. For $i+j \geq d$, we have that the first nonzero term is the term for $h = i+j-d$, so

$$\frac{|E_j(i-1)|}{|E_j(i)|} \approx \frac{\binom{i-1}{i-1+j-d}\binom{d-i+1}{j-(i-1+j-d)}\binom{n-d-i+1}{j-(i-1+j-d)}}{\binom{i}{i+j-d}\binom{d-i}{j-(i+j-d)}\binom{n-d-i}{j-(i+j-d)}}$$

$$\begin{aligned}
&\text{the first nonzero term of } E_j(i) \text{ is dominant}\\
&= \frac{\binom{i-1}{i+j-d-1}\binom{n-d-i+1}{d-i+1}}{\binom{i}{i+j-d}\binom{n-d-i}{d-i}}\\
&= \frac{(i+j-d)(n-d-i+1)}{i(d-i+1)}\\
&\geq \frac{n-d-i+1}{i(d-i+1)}\\
&\qquad i+j-d \geq 1 \text{ applied}\\
&> 1\\
&\qquad \text{for } d \text{ fixed and } n \text{ large enough,}
\end{aligned}$$

so $|E_j(i)|$ is decreasing in i. This means $|E_j(i)|$ is largest when i is smallest, so $E_j(i)$ is smallest when i is smallest, $i+j \geq d$ and $i+j-d$ is odd. This happens when $i = d-j+1$, so $E_j(d-j+1)$ is the smallest eigenvalue of $J(n,d,j)$. ☐

The result of Proposition 4.34c is shown in Figure 4.9. More on this result, like for what value of n we have that $E_j(d-j+1)$ is the smallest eigenvalue, can be found in Section 5.2.

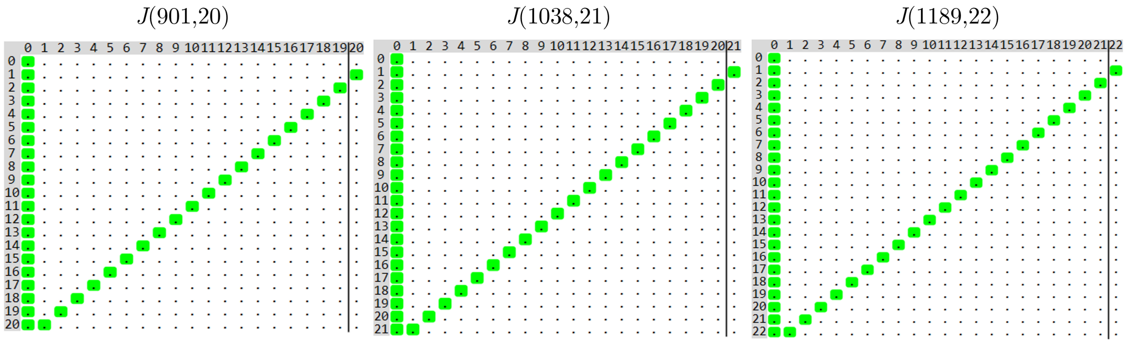

$J(901,20)$ $J(1038,21)$ $J(1189,22)$

Figure 4.9: Visualization of some P-matrices of graphs from the Johnson scheme. The highlighted boxes indicate the smallest eigenvalues per column.

Chapter 5

New results

It might have occurred to the reader when looking at the figures in Chapter 4, that the visualizations of the P-matrices of Johnson and Hamming schemes are quite structured, even for smaller j. By looking at many different of those visualizations, it was possible to come up with new conjectures regarding the smallest and penabsolute eigenvalue of Johnson and Hamming graphs. The focus of this chapter lies on penabsolute eigenvalues of Hamming graphs. Note that all results in this chapter are new.

5.1 The Hamming case

5.1.1 The smallest eigenvalue

We start with some observations on the smallest eigenvalue of graphs from the Hamming scheme when $q = 2$. Firstly, we have that for $q = 2$ and odd j, the value $K_j(d)$ is the smallest eigenvalue.

Proposition 5.1. *Let* $q = 2$ *and* j *odd. Then* $K_j(d) \leq K_j(i)$ *for* $0 \leq i \leq d$.

Proof. By Lemma 4.4, we have $K_j(d) = (-1)^j K_j(0)$. We know that $K_j(0)$ is positive and is the largest eigenvalue in absolute value, so for odd j, $K_j(d)$ must be the smallest eigenvalue. □

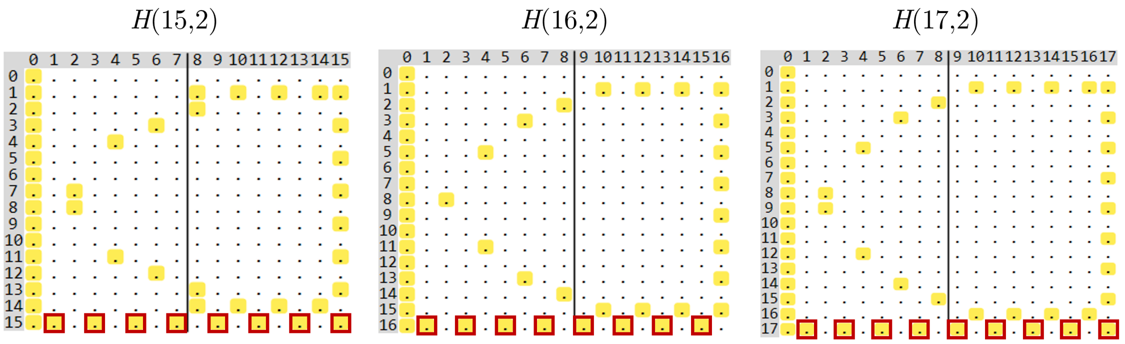


Figure 5.1: Visualization of the results of Proposition 5.1. The highlighted boxes indicate the smallest eigenvalues per column. The red fields indicate the results of the proposition.

Figure 5.1 shows the results of Proposition 5.1. Note that the vertical black lines in the figure are placed between columns $\left\lceil d - \frac{d-1}{q} \right\rceil$ and $\left\lceil d - \frac{d-1}{q} \right\rceil - 1$, so in the case of $q = 2$ this means $j < \frac{d+1}{2}$ for all columns on the left of the black vertical line. The figure suggests that for those values of j, the eigenvalue $K_j(1)$ is never the smallest. This would mean that the bound $j \geq \frac{d+1}{2}$ from Corollary 4.13 is tight. Moreover, the figure suggests that for $j = \lceil \frac{d+1}{2} \rceil - 1$, the smallest eigenvalue is either $K_j(2)$ or $K_j(d)$, depending on whether j is even or odd. We state these ideas in the following conjecture and proposition.

Conjecture 5.2. *Let $q = 2$. For $0 < j < \frac{d+1}{2}$, $K_j(1)$ is not the smallest eigenvalue.*

Test cases. This conjecture was tested by computer for $2 \leq d \leq 400$. The code that was used to test these cases can be found in Appendix B.4.

For $j = \lceil \frac{d+1}{2} \rceil - 1$, we prove Conjecture 5.2 and something more.

Proposition 5.3. *Let $q = 2$. Then the bound $j \geq \frac{d+1}{2}$ from Corollary 4.13 is tight, or in other words: $K_j(1)$ is not the smallest eigenvalue for $j = \lceil \frac{d+1}{2} \rceil - 1$. Moreover, for $j = \lceil \frac{d+1}{2} \rceil - 1$ it holds*

(a) *if j even, then $K_j(2)$ is the smallest eigenvalue,*

(b) *if j is odd, then $K_j(d)$ is the smallest eigenvalue.*

Proof. (a) If j even and d even, then $j = \lceil \frac{d+1}{2} \rceil - 1 = \frac{d}{2}$. From Theorem 4.11b and Proposition 4.12b, we know that $|K_{\frac{d}{2}}(2)|$ is the penabsolute eigenvalue, and from (4.8) we know $K_{\frac{d}{2}}(i) = 0$ for odd i. By showing that $K_{\frac{d}{2}}(2) < 0$, we have proven that $K_{\frac{d}{2}}(2)$ is indeed the smallest eigenvalue:

$$\begin{aligned} K_{\frac{d}{2}}(2) &= \sum_{h=0}^{\frac{d}{2}} (-1)^h \binom{2}{h} \binom{d-2}{\frac{d}{2} - h} \\ &\quad \text{formula (4.1a) applied} \\ &= \binom{d-2}{\frac{d}{2}} - 2\binom{d-2}{\frac{d-2}{2}} + \binom{d-2}{\frac{d-4}{2}} \\ &= 2\binom{d-2}{\frac{d}{2}} - 2\binom{d-2}{\frac{d-2}{2}} \\ &\quad \text{symmetry applied} \\ &= 2 \cdot \frac{\frac{d-2}{2}}{\frac{d-2}{2} + 1} \binom{d-2}{\frac{d-2}{2}} - 2\binom{d-2}{\frac{d-2}{2}} \\ &< 0. \end{aligned}$$

If j even and d odd, then $j = \lceil \frac{d+1}{2} \rceil - 1 = \frac{d-1}{2}$. Also, $|K_{\frac{d-1}{2}}(1)| = |K_{\frac{d-1}{2}}(2)|$ by Proposition 4.12c. From Theorem 4.11a and Proposition 4.12a, we know that $|K_{\frac{d-1}{2}}(1)|$ is the penabsolute eigenvalue. By showing $K_{\frac{d-1}{2}}(2) < 0$ and $K_{\frac{d-1}{2}}(1) \geq 0$, we have proven that $K_{\frac{d-1}{2}}(2)$ is indeed the smallest eigenvalue. We have

$$\begin{aligned} K_{\frac{d-1}{2}}(1) &= \binom{d-1}{\frac{d-1}{2}} - \binom{d-1}{\frac{d-1}{2} - 1} \\ &= \binom{d-1}{\frac{d-1}{2}} - \frac{\frac{d-1}{2}}{\frac{d-1}{2} + 1} \binom{d-1}{\frac{d-1}{2}} \\ &> 0 \end{aligned}$$

and

$$\begin{aligned}
K_{\frac{d-1}{2}}(2) &= \binom{d-2}{\frac{d-1}{2}} - 2\binom{d-2}{\frac{d-1}{2}-1} + \binom{d-2}{\frac{d-1}{2}-2} \\
&= \binom{d-2}{\frac{d-1}{2}-2} - \binom{d-2}{\frac{d-1}{2}} \\
&\quad \text{symmetry applied} \\
&= \frac{d-3}{d+1}\binom{d-2}{\frac{d-1}{2}-1} - \binom{d-2}{\frac{d-1}{2}} \\
&= \frac{d-3}{d+1}\binom{d-2}{\frac{d-1}{2}} - \binom{d-2}{\frac{d-1}{2}} \\
&\quad \text{symmetry applied} \\
&< 0.
\end{aligned}$$

(b) If j odd, then the statement follows directly from Proposition 5.1. □

We now focus on the case $q \geq 3$. Some visualizations of P-matrices from the Hamming scheme are shown in Figure 5.2.

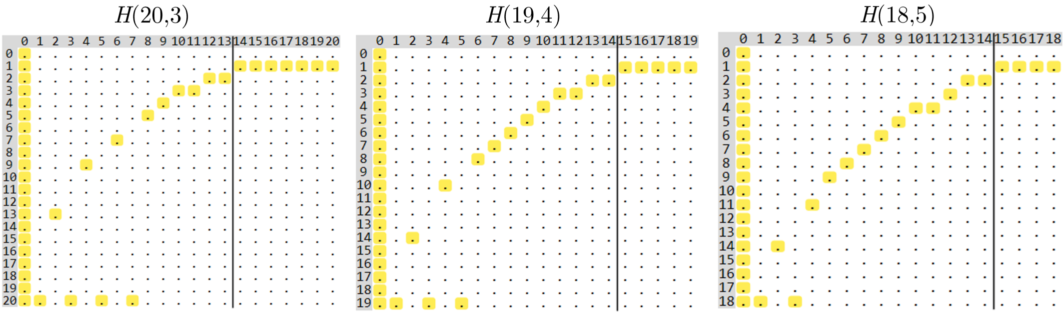

H(20,3) H(19,4) H(18,5)

Figure 5.2: Visualization of some P-matrices of graphs from the Hamming scheme. The highlighted boxes indicate the smallest eigenvalues per column.

We see that, like in the case $q = 2$, the bound $j \geq d - \frac{d-1}{q}$ in Theorem 4.14b seems to be tight. This is summarized in the following conjecture.

Conjecture 5.4. *Let $q \geq 3$. Then the bound $j \geq d - \frac{d-1}{q}$ from Theorem 4.14b is tight. Moreover, for $j < d - \frac{d-1}{q}$, $K_j(1)$ is not the smallest eigenvalue.*

Test cases. This conjecture was tested for all pairs (d, q) with $2 \leq d \leq 200$, $3 \leq q \leq 50$ and $2 \leq d \leq 50$, $50 \leq q \leq 500$. The code that was used to test these cases can be found in Appendix B.5.

What can also be seen in Figure 5.2 is that for q relatively small compared to d, the value i for which $K_j(i)$ is the smallest eigenvalue seems to be decreasing for j larger than a certain value, say $V(d, q)$. For $j < V(d, q)$, $K_j(d)$ is the smallest eigenvalue for odd j, just like for the case with $q = 2$.

For q slightly larger compared to d, we see that the value i for which $K_j(i)$ is the smallest eigenvalue seems to be decreasing for all j. This can be seen in Figure 5.3. Of course it is still possible that the bound $j \geq V(d, q)$ from before plays a role, but that $V(d, q)$ is close to zero for q large enough compared to d.

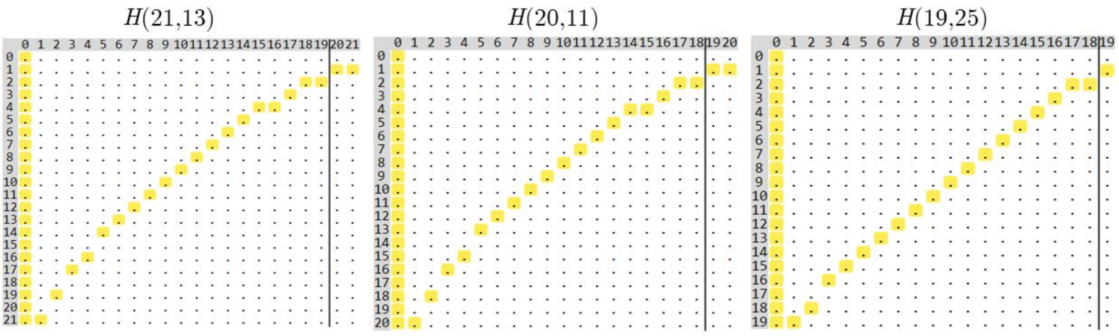


Figure 5.3: Visualization of some P-matrices of graphs from the Hamming scheme. The highlighted boxes indicate the smallest eigenvalues per column.

For q even larger compared to d, we see that the value i for which $K_j(i)$ is the smallest eigenvalue seems to be strictly decreasing for all j. This would mean that $K_j(d-j+1)$ is the smallest eigenvalue for all $j \geq 1$. This can be seen in Figure 5.4.

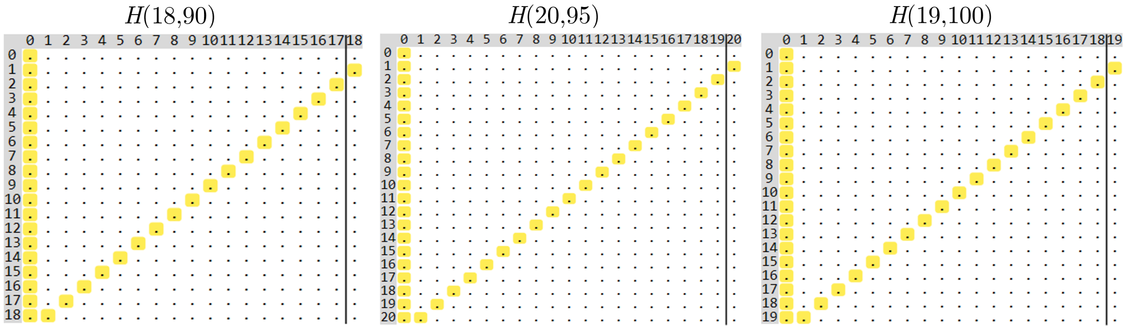


Figure 5.4: Visualization of some P-matrices of graphs from the Hamming scheme. The highlighted boxes indicate the smallest eigenvalues per column.

The smallest value q such that $K_j(d-j+1)$ is the smallest eigenvalue for all $j \geq 1$ for a particular value of d is shown in Table 5.1. For example, when $d = 8$ we have that for $n \geq 12$, the value $K_j(d-j+1)$ is the smallest eigenvalue for all $j \geq 1$.

d	3	4	5	6	7	8	9	10	11	12	13	14	15	16	
q	3	4	5	7	9	12	15	18	22	26	30	35	40	45	
d	20	25	26	27	28	29	30	40	50	60	70	80	90	100	
q	70	109	118	127	136	146	156	277	433	623	848	1107	1401	1730	

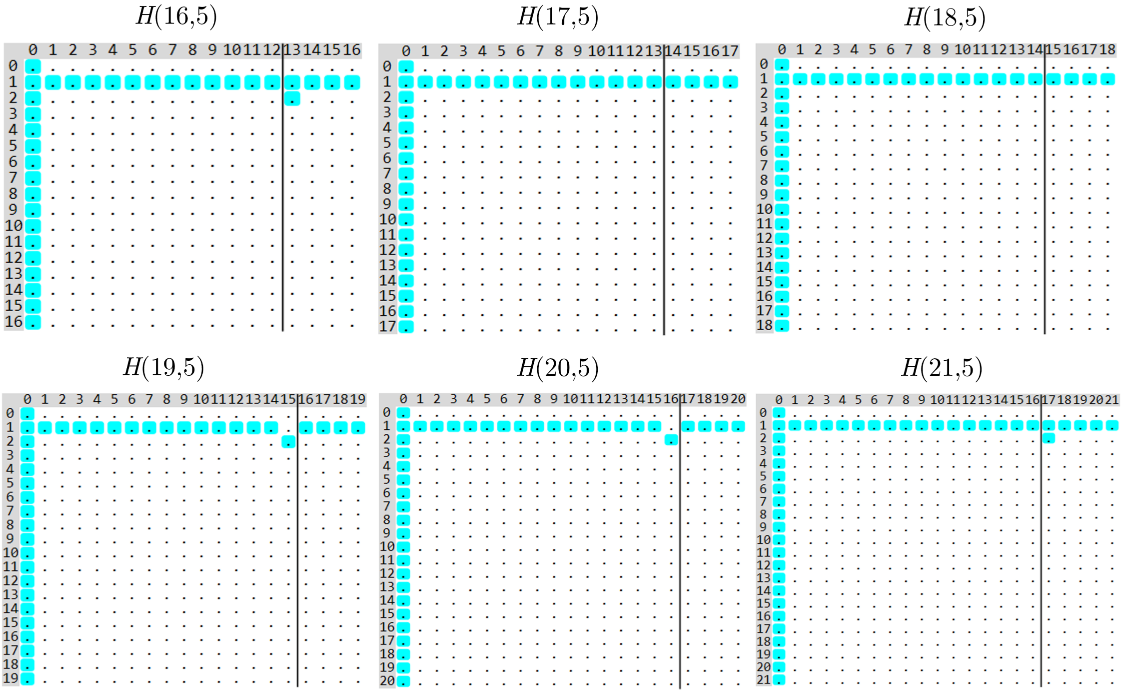
Table 5.1: Overview of the minimum value q per value of d for which $K_j(d-j+1)$ is the smallest eigenvalue for all $j \geq 1$.

5.1.2 The penabsolute eigenvalue

The case $q = 2$ is fully described in Chapter 4.1, so we focus on $q \geq 3$. Figure 5.5 displays some visualizations of P-matrices of graphs from the Hamming scheme, all with $q = 5$. It seems that $|K_j(1)|$ is the penabsolute eigenvalue for all j, except for the column just before the black vertical line. For this value of j, which is $j = \left\lceil d - \frac{d-1}{q} \right\rceil - 1$, we have that sometimes $|K_j(1)|$ and sometimes $|K_j(2)|$ is the penabsolute eigenvalue. After looking at many different visualizations with varying values of d and q, we started to believe that for $j = \left\lceil d - \frac{d-1}{q} \right\rceil - 1$, the following holds.

$$\begin{aligned} |K_j(2)| \text{ is the penabsolute eigenvalue} \quad &\Leftrightarrow \quad d = 0 \,(\text{mod } q) \text{ or } d = q-1 \,(\text{mod } q), \\ |K_j(1)| \text{ is the penabsolute eigenvalue} \quad &\Leftrightarrow \quad d \neq 0 \,(\text{mod } q) \text{ and } d \neq q-1 \,(\text{mod } q). \end{aligned}$$

In fact, Figure 5.5 confirms this belief, as for $d = 20 = 0\,(\text{mod } 5)$ and $d = 19 = 5-1\,(\text{mod } 5)$ we have $|K_j(2)|$ is the penabsolute eigenvalue in the column just before the black line. For the other values of d, we have $d \neq 0\,(\text{mod } 5)$ and $d \neq 5-1\,(\text{mod } 5)$, so here $|K_j(1)|$ is the penabsolute eigenvalue in the column just before the black line.


H(16,5) H(17,5) H(18,5)

H(19,5) H(20,5) H(21,5)

Figure 5.5: Visualization of some P-matrices of graphs from the Hamming scheme. The highlighted boxes indicate the penabsolute eigenvalues per column.

All observations from the paragraphs above can be summarized in the following conjecture:

Conjecture 5.5. *Let* $q \geq 3$ *and* $j < d - \frac{d-1}{q}$. *Then*

(a) *if* $d = 0\,(\text{mod } q)$ *or* $d = q-1\,(\text{mod } q)$, *then* $|K_j(1)| \geq |K_j(i)|$ *for* $1 \leq i \leq d$, $j < \lceil d - \frac{d-1}{q} \rceil - 1$, *and* $|K_j(2)| \geq |K_j(i)|$ *for* $1 \leq i \leq d$, $j = \lceil d - \frac{d-1}{q} \rceil - 1$,

(b) *if* $d \neq 0\,(\text{mod } q)$ *and* $d \neq q-1\,(\text{mod } q)$, *then* $|K_j(1)| \geq |K_j(i)|$ *for* $1 \leq i \leq d$.

The rest of this subsection will be devoted to proving Conjecture 5.5. We start with showing the relation between $|K_j(1)|$ and $|K_j(2)|$ for various values of j. For $j = \lceil d - \frac{d-1}{q} \rceil - 1$ we have the following.

Proposition 5.6. *Let* $q \geq 3$ *and* $j = \lceil d - \frac{d-1}{q} \rceil - 1$. *Then*

(a) *if* $d = 0\,(mod\ q)$ *or* $d = q-1\,(mod\ q)$, *then* $|K_j(1)| \leq |K_j(2)|$,

(b) *if* $d \neq 0\,(mod\ q)$ *and* $d \neq q-1\,(mod\ q)$, *then* $|K_j(1)| \geq |K_j(2)|$.

Proof. Let $x \in \{0, \dots, q-1\}$ such that $d + x = 0\,(\text{mod}\ q)$. This means that for $x = 0$ we have $d = 0\,(\text{mod}\ q)$, for $x = 1$ we have $d = q-1\,(\text{mod}\ q)$ and for $2 \leq x \leq q-1$ we have $d \neq 0\,(\text{mod}\ q)$ and $d \neq q-1\,(\text{mod}\ q)$. Thus, for $x = 0$ and $x = 1$ we want to show $|K_j(1)| \leq |K_j(2)|$ and for $2 \leq x \leq q-1$ we want to show $|K_j(1)| \geq |K_j(2)|$.

We have $K_j(1) \geq 0$ by Lemma 4.6a. Therefore, showing $|K_j(1)| = K_j(1) \leq |K_j(2)|$ is equivalent to showing either $K_j(2) \geq K_j(1)$ or $K_j(2) \leq -K_j(1)$. Since $K_j(2) < K_j(1)$ by Lemma 4.6b and 4.6c, we still need to show $K_j(2) \leq -K_j(1)$ for $x = 0$ and $x = 1$. On the other hand, showing $|K_j(1)| = K_j(1) \geq |K_j(2)|$ is equivalent to showing both $K_j(2) \leq K_j(1)$ and $K_j(2) \geq -K_j(1)$. Since $K_j(2) \leq K_j(1)$ by Lemma 4.6c, we still need to show $K_j(2) \geq -K_j(1)$ for $2 \leq x \leq q-1$.

From Lemma 4.2, we have

$$K_2(j) = K_j(2)\frac{\binom{d}{2}}{\binom{d}{j}}(q-1)^{2-j} \quad \text{and} \quad K_1(j) = K_j(1)\frac{\binom{d}{1}}{\binom{d}{j}}(q-1)^{1-j}$$

and thus

$$K_j(2) \leq -K_j(1) \quad \Leftrightarrow \quad K_2(j) \leq -\frac{1}{2}(d-1)(q-1)K_1(j)$$

and

$$K_j(2) \geq -K_j(1) \quad \Leftrightarrow \quad K_2(j) \geq -\frac{1}{2}(d-1)(q-1)K_1(j).$$

Using formula (4.1a), we get

$$K_1(j) = (q-1)(d-j) - j \ \text{ and} \tag{5.1}$$

$$K_2(j) = \frac{1}{2}(q-1)^2(d-j)(d-j-1) - (q-1)(d-j)j + \frac{1}{2}j(j-1). \tag{5.2}$$

Next, we determine j in terms of d, q and x such that we can substitute j in the formulas for $K_1(j)$ and $K_2(j)$ given above. Recall that $d + x = 0\,(\text{mod}\ q)$. We have

$$j = \left\lceil d - \frac{d-1}{q} \right\rceil - 1 = d - \left\lfloor \frac{d+x-(x+1)}{q} \right\rfloor - 1 = d - \frac{d+x}{q} + \left\lceil \frac{x+1}{q} \right\rceil - 1 = d - \frac{d+x}{q}.$$

Substituting this into the formulas (5.1) for $K_1(j)$ and (5.2) for $K_2(j)$ gives

$$K_1(j) = (q-1) \cdot \frac{d+x}{q} - d + \frac{d+x}{q} = x$$

and

$$\begin{aligned}
2K_2(j) &= (q-1)^2 \cdot \frac{d+x}{q}\left(\frac{d+x}{q} - 1\right) - 2(q-1) \cdot \frac{d+x}{q}\left(d - \frac{d+x}{q}\right) \\
&\quad + \left(d - \frac{d+x}{q}\right)\left(d - \frac{d+x}{q} - 1\right) \\
&= (d+x)^2 - q(d+x) - \frac{2(d+x)^2}{q} + 2(d+x) + \frac{(d+x)^2}{q^2} - \frac{d+x}{q} - 2d(d+x) \\
&\quad + \frac{2(d+x)^2}{q} + \frac{2d(d+x)}{q} - \frac{2(d+x)^2}{q^2} + d^2 - \frac{2d(d+x)}{q} - d + \frac{(d+x)^2}{q^2} + \frac{d+x}{q}
\end{aligned}$$

$$\begin{aligned}&= d^2 + 2dx + x^2 - dq - qx + 2d + 2x - 2d^2 - 2dx + d^2 - d\\ &= x^2 + 2x - dq - qx + d.\end{aligned}$$

For $x = 0$ and $x = 1$, we aim to show

$$\begin{aligned}K_j(2) \leq -K_j(1) &\Leftrightarrow K_2(j) \leq -\frac{1}{2}(d-1)(q-1)K_1(j)\\ &\Leftrightarrow x^2 + 2x - dq - qx + d \leq -(d-1)(q-1)x\\ &\Leftrightarrow x^2 + 3x - 2qx - dx + dqx - dq + d \leq 0.\end{aligned}$$

For $x = 0$, we have $x^2 + 3x - 2qx - dx + dqx - dq + d = d - dq \leq 0$. For $x = 1$, we have $x^2 + 3x - 2qx - dx + dqx - dq + d = 1 + 3 - 2q + dq - dq = 4 - 2q \leq 0$. With these statements we have proven that $|K_j(1)| \leq |K_j(2)|$ if $d = 0 \pmod q$ or $d = q - 1 \pmod q$, so part (a) of the proposition.

Now consider the case $2 \leq x \leq q - 1$. We wanted to show

$$\begin{aligned}K_j(2) \geq -K_j(1) &\Leftrightarrow x^2 + 3x - 2qx - dx + dqx - dq + d \geq 0\\ &\Leftrightarrow x^2 + (3 - 2q + (q-1)d)x + (d - dq) \geq 0.\end{aligned}$$

The coefficient of x is positive for $d \geq 2$, since

$$3 - 2q + (q-1)d \geq 3 - 2q + 2(q-1) = 1 > 0,$$

so $x(3 - 2q + (q-1)d) \geq 2(3 - 2q + (q-1)d)$ Thus, for $d \geq 4$, we have

$$\begin{aligned}x^2 + (3 - 2q + (q-1)d)x + (d - dq) &\geq 4 + (3 - 2q + (q-1)d)2 + (d - dq)\\ &= 10 - 4q + qd - d\\ &= 6 + d(q-1) - 4q + 4\\ &= 6 + (d-4)(q-1)\\ &> 0.\end{aligned}$$

We look at the cases $d = 2$ and $d = 3$ separately. Write $K_j(i, d)$ to emphasize the value of d. For $d = 2$, we want $d = 2 \neq 0 \pmod q$ and $d = 2 \neq q - 1 \pmod q$, so $q \neq 2$ and $q \neq 3$, so $q \geq 4$. Also, we have

$$j = \left\lceil d - \frac{d-1}{q} \right\rceil - 1 = \left\lceil 2 - \frac{1}{q} \right\rceil - 1 = 1.$$

By formula (4.1a) we have $K_1(1, 2) = q - 2$ and $K_1(2, 2) = -2$, so indeed $|K_1(1, 2)| \geq |K_1(2, 2)|$ for $q \geq 4$. For $d = 3$, we want $d = 3 \neq 0 \pmod q$ and $d = 3 \neq q - 1 \pmod q$, so $q \neq 3$ and $q \neq 4$, so $q \geq 5$. Also, we have

$$j = \left\lceil d - \frac{d-1}{q} \right\rceil - 1 = \left\lceil 3 - \frac{2}{q} \right\rceil - 1 = 2.$$

By formula (4.1a) we have $K_2(1, 3) = q^2 - 4q + 3$ and $K_2(2, 3) = -2q + 3$. For $q \geq 5$, this means $|K_2(1, 3)| = q^2 - 4q + 3$ and $|K_2(2, 3)| = 2q - 3$. Since

$$q^2 - 4q + 3 - 2q + 3 = q^2 - 6q + 6 = (q - 3 + \sqrt{3})(q - 3 - \sqrt{3})$$

and $3 + \sqrt{3} \approx 4.732$, we have indeed $|K_2(1, 3)| \geq |K_2(2, 3)|$ for $q \geq 5$. This concludes the proof of part (b) of the proposition. □

For $j < d - \frac{d-1}{q} - 1$, we have the following.

Proposition 5.7. *Let* $q \geq 2$ *and* $j < d - \frac{d-1}{q} - 1$. *Then* $|K_j(2)| \leq |K_j(1)|$.

Proof. By Lemma 4.6a, we know $K_j(1) \geq 0$. Therefore we need to show $|K_j(2)| \leq K_j(1)$, which is equivalent to showing both $K_j(2) \leq K_j(1)$ and $K_j(2) \geq -K_j(1)$. The first inequality follows from Lemma 4.6c. To show the second inequality, we first note that by Lemma 4.3, we have

$$K_2(j) = K_j(2)\frac{\binom{d}{2}}{\binom{d}{j}}(q-1)^{2-j} \quad \text{and} \quad K_1(j) = K_j(1)\frac{\binom{d}{1}}{\binom{d}{j}}(q-1)^{1-j}.$$

This means $K_j(2) \geq -K_j(1)$ is equivalent to showing $K_2(j) \geq -\frac{1}{2}(d-1)(q-1)K_1(j)$. Note that it also means that $K_1(j) \geq 0$. By Lemma 4.6d, we have

$$K_2(j) = \frac{-1}{q-1}K_j(1) = -\frac{1}{2}(d-1)K_1(j) \quad \Leftrightarrow \quad j = (d-1)\left(1-\frac{1}{q}\right) \text{ or } j = d.$$

Note that $(d-1)\left(1-\frac{1}{q}\right) = d - \frac{d-1}{q} - 1$. We have shown before, for example in the proof of Lemma 4.6b, that $K_2(j)$ is quadratic in j with positive leading coefficient and $K_1(j)$ is linear in j. This means $K_j(2) > -\frac{1}{2}(d-1)K_1(j)$ for $j < d - \frac{d-1}{q} - 1$. Since $q \geq 2$ and $K_1(j) \geq 0$, this implies $K_j(2) > -\frac{1}{2}(d-1)(q-1)K_1(j)$ and thus $K_j(2) \geq -K_j(1)$. Together with the inequality $K_j(2) \leq K_j(1)$ we have shown before, this gives $|K_j(2)| \leq |K_j(1)|$. ☐

Next, we present some intermediate results that we will need to complete the proof of Conjecture 5.5.

Proposition 5.8. *Let $1 < i < d$. If $i \leq \frac{qj}{2}$ when $i+(q-1)(d-i)-qj \geq 0$ or if $qj \leq 2(q-1)(d-i)$ when $i+(q-1)(d-i)-qj < 0$, then*

$$|K_j(i+1)| \leq \max\{|K_j(i-1)|, |K_j(i)|\}.$$

Proof. Let $M = \max\{|K_j(i-1)|, |K_j(i)|\}$. We have

$$\begin{aligned} (q-1)(d-i)K_j(i+1) &= (i+(q-1)(d-i)-qj)K_j(i) - iK_j(i-1), \text{ so} \\ &\text{Lemma 4.5 applied} \\ |(q-1)(d-i)||K_j(i+1)| &\leq |i+(q-1)(d-i)-qj||K_j(i)| + i|K_j(i-1)|, \text{ so} \\ &\text{triangle inequality applied} \\ (q-1)(d-i)|K_j(i+1)| &\leq (|i+(q-1)(d-i)-qj|+i)M. \end{aligned}$$

The conclusion follows if $|i+(q-1)(d-i)-qj|+i \leq (q-1)(d-i)$. If $i+(q-1)(d-i)-qj \geq 0$, then it suffices to show $i+(q-1)(d-i)-qj+i \leq (q-1)(d-i)$, which happens exactly when $i \leq \frac{qj}{2}$. If $i+(q-1)(d-i)-qj < 0$, then it suffices to show $-i-(q-1)(d-i)+qj+i \leq (q-1)(d-i)$, which happens exactly when $qj \leq 2(q-1)(d-i)$. ☐

For $j = \lceil d - \frac{d-1}{q} \rceil - 1$, we can prove Conjecture 5.5.

Lemma 5.9. *Let $1 \leq i \leq d$, $q \geq 3$ and $j = \lceil d - \frac{d-1}{q} \rceil - 1$. Then*

(a) *if $d = 0 \,(mod\ q)$ or $d = q-1 \,(mod\ q)$, then $|K_j(i)| \leq |K_j(2)|$,*

(b) *if $d \neq 0 \,(mod\ q)$ and $d \neq q-1 \,(mod\ q)$, then $|K_j(i)| \leq |K_j(1)|$.*

Proof. Since this proof is rather long, we start with an overview of the steps.

1. Show the statement for $i \leq \frac{qj}{2}$, $i+(q-1)(d-i)-qj \geq 0$ and for $qj \leq 2(q-1)(d-i)$, $i+(q-1)(d-i)-qj < 0$ using Proposition 5.8,

2. Show the statement for $qj > 2(q-1)(d-i)$, $i+(q-1)(d-i)-qj < 0$ by considering the cases $d \neq 0 \,(\text{mod } q)$, $d \neq q-1 \,(\text{mod } q)$ and d else separately for d, q large enough,
3. Show that the case $i > \frac{qj}{2}$, $i+(q-1)(d-i)-qj \geq 0$ is not feasible,
4. Check some cases by hand for small d, q.

Note that we may assume $i \geq 3$ since the cases $i = 1$ and $i = 2$ are either trivial or shown in Proposition 5.6.

Step 1. If $i \leq \frac{qj}{2}$ when $i+(q-1)(d-i)-qj \geq 0$ or if $qj \leq 2(q-1)(d-i)$ when $i+(q-1)(d-i)-qj < 0$, then we can use Proposition 5.8 and induction on i to prove the statement. For $i = 1$ and $i = 2$, the statement follows from Proposition 5.6. For larger i, we have

$$|K_j(i)| \leq \max\{|K_j(i-2)|, |K_j(i-1)|\} \leq \cdots \leq \max\{|K_j(1)|, |K_j(2)|\},$$

and the statement follows from Proposition 5.6.

Step 2. Now let $i+(q-1)(d-i)-qj < 0$ and $qj > 2(q-1)(d-i)$. We first show $i \geq \frac{d+1}{2}$. From $j = \lceil d - \frac{d-1}{q} \rceil - 1$, we know $j < d - \frac{d-1}{q}$, so $qj < dq - d + 1$, so $qj \leq dq - d$. This means

$$\begin{aligned} 2(q-1)(d-i) < qj \leq dq - d&, \text{ so} \\ 2(q-1)(d-i) < d(q-1)&, \text{ so} \\ \frac{d}{2} < i&, \text{ so} \\ i \geq \frac{d+1}{2}&. \end{aligned}$$

Now we use Lemma 4.7 to finish the proof. As in the proof of Lemma 5.6, let $x \in \{0, \ldots, q-1\}$ such that $d + x = 0 \,(\text{mod } q)$. This means that for $x = 0$ we have $d = 0 \,(\text{mod } q)$, for $x = 1$ we have $d = q-1 \,(\text{mod } q)$ and for $2 \leq x \leq q-1$ we have $d \neq 0 \,(\text{mod } q)$ and $d \neq q-1 \,(\text{mod } q)$. Thus, for $x = 0$ and $x = 1$ we want to show $|K_j(i)| \leq |K_j(2)|$ and for $2 \leq x \leq q-1$ we want to show $|K_j(i)| \leq |K_j(1)|$. Moreover, we have

$$j = \left\lceil d - \frac{d-1}{q} \right\rceil - 1 = d - \left\lfloor \frac{d+x-(x+1)}{q} \right\rfloor - 1 = d - \frac{d+x}{q} + \left\lceil \frac{x+1}{q} \right\rceil - 1 = d - \frac{d+x}{q}$$

and thus $q - 1 - \frac{qj}{d} = \frac{x}{d} \geq 0$.

First, let $2 \leq x \leq q-1$. Then, using Lemma 4.7, we want to show $|K_j(i)| \leq (q-1)^{d-i}\binom{d}{j} \leq |K_j(1)|$. By formula (4.1a), we have

$$|K_j(1)| = (q-1)^{j-1}\binom{d}{j}\left|q-1-\frac{qj}{d}\right| = (q-1)^{j-1}\binom{d}{j}\left(q-1-\frac{qj}{d}\right),$$

thus it is sufficient to show

$$(q-1)^{d-i}\binom{d}{j} \leq (q-1)^{j-1}\binom{d}{j}\left(q-1-\frac{qj}{d}\right),$$

which is equivalent to

$$d \leq (q-1)^{i+j-d-1}(dq-d-qj).$$

Using $j = d - \frac{d+x}{q}$ and thus $qj = qd - d - x$, we have

$$(q-1)^{i+j-d-1}(dq-d-qj) = x(q-1)^{i-\frac{d+x}{q}-1}.$$

Moreover, using $2 \leq x \leq q-1$ and $q \geq 3$ and $i \geq \frac{d+1}{2}$, we have

$$i - \frac{d+x}{q} - 1 \geq \frac{d+1}{2} - \frac{d}{q} - \frac{x}{q} - 1 \geq \frac{d}{2} - \frac{d}{q} - \frac{q-1}{q} - \frac{1}{2} = \frac{d}{2} - \frac{d}{q} + \frac{1}{q} - \frac{3}{2} \geq \frac{d}{6} - \frac{3}{2}.$$

This means

$$(q-1)^{i+j-d-1}(dq-d-qj) = x(q-1)^{i-\frac{d+x}{q}-1} \geq 2 \cdot 2^{\frac{d}{6}-\frac{3}{2}} = 2^{\frac{d}{6}-\frac{1}{2}},$$

which is greater than or equal to d for $d \geq 34$.

We can assume $d \geq 3$, since the case $d = 2$ follows from Lemma 5.6, so for $3 \leq d < 34$, we need to check $|K_j(i)| \leq |K_j(1)|$ by hand. To do this, we first need to provide a bound on q, as we cannot check the before mentioned inequality by hand for an infinite number of q's. For $d \geq 4$, we have

$$i - \frac{d+x}{q} - 1 \geq \frac{d+1}{2} - \frac{d+x}{q} - 1 = \frac{dq+q-2d-2x-2q}{2q} \geq \frac{4(q-2)-q+2-2q}{2q} = \frac{1}{2} - \frac{3}{q}.$$

This means

$$(q-1)^{i+j-d-1}(dq-d-qj) = x(q-1)^{i-\frac{d+x}{q}-1} \geq 2 \cdot (q-1)^{\frac{1}{2}-\frac{3}{q}},$$

which is greater than or equal to 34 for $q \geq 323$. For $d = 3$ we have $q \geq d$, so $x = q-3$, so $i - \frac{d+x}{q} - 1 \geq \frac{d+1}{2} - \frac{d+q-3}{q} - 1 = \frac{3+1}{2} - \frac{3+q-3}{q} - 1 = 0$, so $x(q-1)^{i-\frac{d+x}{q}-1} \geq 3$ for $x \geq 3$ and all q. For $x = 2$, we have $q = 5$. This means we need to check $|K_j(i)| \leq |K_j(1)|$ by hand for all combinations (d,q,j,i) with $d \neq 0 \pmod{q}$, $d \neq q-1 \pmod{q}$, $4 \leq d < 34$ and $3 \leq q < 323$, and all combinations with $(d,q) = (3,5)$.

Now, let $x = 1$. We have $j = d - \frac{d+1}{q}$ and we want to show $|K_j(i)| \leq (q-1)^{d-i}\binom{d}{j} \leq |K_j(2)|$ using Lemma 4.7. Using formula (4.1a) and the definition of a binomial coefficient, we get

$$\begin{aligned}
K_j(2) &= \frac{1}{d(d-1)}(q-1)^{j-2}\binom{d}{j}\big((d-j-1)(d-j)(q-1)^2 - 2j(q-1)(d-j) + j(j-1)\big) \\
&= \frac{1}{d(d-1)}(q-1)^{j-2}\binom{d}{j}\Bigg(\left(\frac{d+1}{q} - 1\right)\frac{d+1}{q}(q-1)^2 - 2\left(d - \frac{d+1}{q}\right)(q-1)\frac{d+1}{q} \\
&\quad + \left(d - \frac{d+1}{q}\right)\left(d - \frac{d+1}{q} - 1\right)\Bigg) \\
&= \frac{1}{d(d-1)}(q-1)^{j-2}\binom{d}{j}(3+d-q-dq), \text{ so} \\
|K_j(2)| &= \frac{1}{d(d-1)}(q-1)^{j-2}\binom{d}{j}|3+d-q-dq| \\
&= \frac{1}{d(d-1)}(q-1)^{j-2}\binom{d}{j}|-(q-1)(d+1)+2| \\
&= \frac{1}{d(d-1)}(q-1)^{j-2}\binom{d}{j}\left((q-1)(d+1)-2\right).
\end{aligned}$$

Thus, we want to show

$$(q-1)^{d-i}\binom{d}{j} \leq \frac{1}{d(d-1)}(q-1)^{j-2}\binom{d}{j}\left((q-1)(d+1)-2\right),$$

which is equivalent to showing

$$d(d-1) \leq (q-1)^{i+j-d-2}\left((q-1)(d+1)-2\right).$$

Since $(q-1)(d+1)-2=(q-1)d+q-3 \geq d(q-1)$, this reduces to showing

$$d-1 \leq (q-1)^{i+j-d-1}.$$

We have

$$i+j-d-1 \geq \frac{d+1}{2}-\frac{d+1}{q}-1 \geq \frac{d+1}{2}-\frac{d+1}{3}-1=\frac{d}{6}-\frac{5}{6}$$

and thus

$$(q-1)^{i+j-d-1} > 2^{\frac{d}{6}-\frac{5}{6}},$$

which is greater than or equal to $d-1$ for $d \geq 37$. Like before, we need to check the cases $3 \leq d < 37$ by hand, for which we need a bound on q. We have

$$i+j-d-1 \geq \frac{d+1}{2}-\frac{d+1}{q}-1=\frac{(d+1)(q-2)}{2q}-1 \geq \frac{4q-8}{2q}-1=1-\frac{4}{q},$$

so

$$(q-1)^{i+j-d-1} > (q-1)^{1-\frac{4}{q}},$$

which is greater than or equal to $37-1=36$ for $q \geq 51$. This means we need to check $|K_j(i)| \leq |K_j(2)|$ by computer for all combinations (d,q,j,i) with $d=q-1 \,(\mathrm{mod}\, q)$, $3 \leq q < 51$ and $3 \leq d < 37$.

Lastly, let $x=0$. We have $j=d-\frac{d}{q}$ and we want to show $|K_j(i)| \leq (q-1)^{d-i}\binom{d}{j} \leq |K_j(2)|$ using Lemma 4.7 like before. Using formula (4.1a) and the definition of a binomial coefficient, we get

$$\begin{aligned}
K_j(2) &= (q-1)^j\binom{d-2}{j}-2(q-1)^{j-1}\binom{d-2}{j-1}+(q-1)^{j-2}\binom{d-2}{j-2} \\
&= \frac{1}{d(d-1)}(q-1)^{j-2}\binom{d}{j}\big((d-j-1)(d-j)(q-1)^2-2j(q-1)(d-j)+j(j-1)\big) \\
&= \frac{1}{d(d-1)}(q-1)^{j-2}\binom{d}{j} \\
&\quad \cdot\left(\left(\frac{d}{q}-1\right)\frac{d}{q}(q-1)^2-2\left(d-\frac{d}{q}\right)(q-1)\frac{d}{q}+\left(d-\frac{d}{q}\right)\left(d-\frac{d}{q}-1\right)\right) \\
&= \frac{1}{d-1}(q-1)^{j-2}\binom{d}{j}(1-q), \text{ so} \\
|K_j(2)| &= \frac{1}{d-1}(q-1)^{j-1}\binom{d}{j}.
\end{aligned}$$

Thus, we want to show

$$(q-1)^{d-i}\binom{d}{j} \leq \frac{1}{d-1}(q-1)^{j-1}\binom{d}{j},$$

which is equivalent to showing

$$d-1 \leq (q-1)^{i+j-d-1}.$$

We have

$$i+j-d-1 \geq \frac{d+1}{2}-\frac{d}{q}-1 \geq \frac{d}{2}-\frac{d}{3}-\frac{1}{2}=\frac{d}{6}-\frac{1}{2}$$

and thus

$$(q-1)^{i+j-d-1} > 2^{\frac{d}{6}-\frac{1}{2}},$$

which is greater than or equal to $d-1$ for $d \geq 34$. Like before, we need to check the cases $3 \leq d < 34$ by hand, for which we need a bound on q. We have

$$i+j-d-1 \geq \frac{d+1}{2}-\frac{d}{q}-1=\frac{d(q-2)}{2q}-\frac{1}{2} \geq \frac{3q-6}{2q}-\frac{1}{2}=1-\frac{3}{q},$$

so

$$(q-1)^{i+j-d-1} > (q-1)^{1-\frac{3}{q}},$$

which is greater than or equal to $34 - 1 = 33$ for $q \geq 44$. However, since $x = 0$ and thus $d = 0 \,(\mathrm{mod}\ q)$, we need $q \leq d$. This means we need to check $|K_j(i)| \leq |K_j(2)|$ by computer for all combinations (d, q, j, i) with $d = 0 \,(\mathrm{mod}\ q)$, $3 \leq q < 34$ and $3 \leq d < 34$.

Step 3. The last case we need to consider is $i + (q-1)(d-i) - qj \geq 0$ and $i > \frac{qj}{2}$. Note that $j = d - \frac{d+x}{q}$, so for $i \geq 3$ we have

$$\begin{aligned} i + (q-1)(d-i) - qj &= i + qd - qi - d + i - qd + d + x \\ &= 2i - qi + x \\ &\leq -i(q-2) + q - 1 \\ &= (q-2)(1-i) + 1 \\ &\leq -2(q-2) + 1 \\ &< 0. \end{aligned}$$

Thus, for $j = d - \frac{d+x}{q}$ this case is not feasible.

Step 4. The last step is to check all cases with $i + (q-1)(d-i) - qj < 0$ and $qj > 2(q-1)(d-i)$ that need to be checked by a computer. These cases are:

- $3 \leq d < 34$, $3 \leq q < 34$ for $d = 0 \,(\mathrm{mod}\ q)$,
- $3 \leq d < 37$, $3 \leq q < 51$ for $d = q - 1 \,(\mathrm{mod}\ q)$, and
- $4 \leq d < 34$, $3 \leq q < 323$ and $(d, q) = (3, 5)$ for $d \neq 0 \,(\mathrm{mod}\ q)$, $d \neq q - 1 \,(\mathrm{mod}\ q)$.

The lemma was checked by computer for all pairs (d, q) with $3 \leq d < 37$ and $3 \leq q < 323$, which covers all pairs that were mentioned above. Appendix B.6 shows the code used to check these cases. The lemma turns out to be true for these cases, which completes the proof of this lemma. □

For $j < d - \frac{d-1}{q} - 1$, we split the proof into four cases:

Case 1: $i \leq \frac{qj}{2}$ and $i + (q-1)(d-i) - qj \geq 0$,

Case 2: $qj \leq 2(q-1)(d-i)$ and $i + (q-1)(d-i) - qj < 0$,

Case 3: $qj > 2(q-1)(d-i)$ and $i + (q-1)(d-i) - qj < 0$,

Case 4: $i > \frac{qj}{2}$ and $i + (q-1)(d-i) - qj \geq 0$.

Cases 1 and 2 can be shown using Proposition 5.8, which we will get back to in the proof of Lemma 5.16. Case 3 is shown in the following lemma.

Lemma 5.10. *Let $q \geq 3$, $j < d - \frac{d-1}{q} - 1$ and $qj > 2(q-1)(d-i)$. Then $|K_j(i)| \leq |K_j(1)|$ for $1 \leq i \leq d$.*

Proof. We want to use Lemma 4.7 to prove the statement. In the proof of Lemma 5.9, we have seen that for showing $|K_j(i)| \leq (q-1)^{d-i}\binom{d}{j} \leq |K_j(1)|$ it is sufficient to show

$$d \leq (q-1)^{i+j-d-1}(dq - d - qj).$$

We first derive a bound on $i + j - d - 1$. From $qj > 2(q-1)(d-i)$, we get

$$\begin{aligned} 2\frac{q-1}{q}i + j &> 2\frac{q-1}{q}d, \quad \text{so} \\ i + \frac{q}{2(q-1)}j &> d, \quad \text{so} \\ i + \left(1 - \frac{q-2}{2(q-1)}\right)j &> d, \quad \text{so} \\ i + j - d - 1 &> \frac{q-2}{2(q-1)}j - 1. \end{aligned}$$

We split up the rest of the proof into two cases, namely $j \leq \frac{q-2}{q}d$ and $j > \frac{q-2}{q}d$. First, consider the case $j \leq \frac{q-2}{q}d$. We have

$$dq - d - qj = q(d-j) - d \geq q\left(\frac{2}{q}d\right) - d = d.$$

Moreover, we have

$$i + j - d - 1 > \frac{q-2}{2(q-1)}j - 1 \geq -1, \quad \text{so} \quad i + j - d - 1 \geq 0.$$

Together, this gives

$$d \leq dq - d - qj = (q-1)^0(dq - d - qj) \leq (q-1)^{i+j-d-1}(dq - d - qj),$$

which is what we wanted to show.

Next, consider the case $j > \frac{q-2}{q}d$. We have

$$i + j - d - 1 > \frac{q-2}{2(q-1)}j - 1 > \frac{q-2}{2(q-1)}\frac{q-2}{q}d - 1 = \frac{(q-2)^2}{2q(q-1)}d - 1.$$

Let $f(q) := \frac{(q-2)^2}{2q(q-1)}$. We have

$$\frac{\partial f}{\partial q} = \frac{(q-2)(3q-2)}{2q^2(q-1)^2} > 0,$$

so f is increasing in q for $q \geq 3$. This means that f attains its minimum value at the smallest feasible q, which is $q = 3$. Thus,

$$i + j - d - 1 > f(q)d - 1 \geq f(3)d - 1 = \frac{d}{12} - 1.$$

Moreover, from $j < d - \frac{d-1}{q} - 1$ we get $dq - d - qj \geq q$, so it suffices to show the first inequality in

$$d \leq 3 \cdot 2^{\frac{d}{12}-1} \leq q(q-1)^{\frac{d}{12}-1} \leq (q-1)^{i+j-d-1}(dq - d - qj).$$

The first inequality is true for $d \geq 66$, so we need to show that $|K_j(i)| \leq |K_j(1)|$ for $3 \leq d \leq 65$ by computer. To do this, we need a bound on q. Note that we had $dq - d - qj \geq q$ and $i + j - d - 1 \geq 0$, so it suffices to show the second inequality in

$$d \leq 65 \leq q = (q-1)^0 q \leq (q-1)^{i+j-d-1}(dq - d - qj).$$

This inequality is trivially true for $q \geq 65$, so it suffices to show $|K_j(i)| \leq |K_j(1)|$ by computer for $3 \leq d, q \leq 65$ and all feasible values of i, j. The code to calculate these cases can be found in Appendix B.7. It turns out all cases that needed to be checked by computer are true, which finishes the proof of this lemma. □

This leaves us case 4, namely $i > \frac{qj}{2}$ when $i + (q-1)(d-i) - qj \geq 0$. For showing this case, we first need some intermediate results.

Proposition 5.11. *Let $i + (q-1)(d-i) - qj \geq 0$, $i > \frac{qj}{2}$ and $q \geq 4$. Then $j \leq \frac{3}{8}d - \frac{1}{8}$.*

Proof. We have $i + (q-1)(d-i) - qj \geq 0$ and thus $i \leq \frac{q-1}{q-2}d - \frac{q}{q-2}j$. Moreover, we have $i > \frac{qj}{2}$, thus $i \geq \frac{qj+1}{2}$. Thus, for there to be a feasible i, we need

$$\frac{qj+1}{2} \leq \frac{q-1}{q-2}d - \frac{q}{q-2}j.$$

This can be rewritten to

$$j \leq \frac{2(q-1)}{q^2}d - \frac{q-2}{q^2} =: f(q).$$

We have

$$\frac{\partial f}{\partial q} = \frac{4-2q}{q^3}d - \frac{4-q}{q^3},$$

which is negative for $q \geq 4$ since for $q \geq 4$ we have

$$(4-2q)d - (4-q) = 4d - 4 - q(2d-1) \leq 4d - 4 - 4(2d-1) = -4d < 0.$$

This means $f(q)$ is decreasing in q and thus takes its largest value at the smallest feasible value of q, which is $q = 4$. This means we have $j \leq f(4) \leq f(q)$ with $f(4) = \frac{3}{8}d - \frac{1}{8}$. □

Proposition 5.12. *Let $K_j(i)$ be a Kravchuk polynomial and $j > 0$. Then*

$$|K_j(i)| \leq (q-1)^{j-1}\left(\binom{d}{j} + (q-2)\binom{d-i}{j}\right).$$

Proof. We have

$$\begin{aligned}
|K_j(i)| &\leq \sum_{h=0}^{j}(q-1)^{j-h}\binom{i}{h}\binom{d-i}{j-h} \\
&\qquad \text{triangle inequality applied on formula (4.1a)} \\
&= (q-1)^j\binom{d-i}{j} + \sum_{h=1}^{j}(q-1)^{j-h}\binom{i}{h}\binom{d-i}{j-h} \\
&\leq (q-1)^j\binom{d-i}{j} + (q-1)^{j-1}\sum_{h=1}^{j}\binom{i}{h}\binom{d-i}{j-h} \\
&= (q-1)^j\binom{d-i}{j} + (q-1)^{j-1}\left(\binom{d}{j} - \binom{d-i}{j}\right) \\
&\qquad \text{Vandermonde identity (Lemma 2.5) applied} \\
&= (q-1)^{j-1}\left(\binom{d}{j} + (q-2)\binom{d-i}{j}\right).
\end{aligned}$$

□

Now we show a result similar to Proposition 5.12, but where the upper bound is slightly tighter.

Proposition 5.13. *Let $q = 3$, $j \geq 2$ and $K_j(i)$ a Kravchuk polynomial. Then*

$$|K_j(i)| \leq 2^{j-2}\left(3\binom{d-i}{j} + i\binom{d-i}{j-1} + \binom{d}{j}\right).$$

Proof. We have

$$
\begin{aligned}
|K_j(i)| &\leq \sum_{h=0}^{j} 2^{j-h}\binom{i}{h}\binom{d-i}{j-h} \\
&\quad \text{triangle inequality applied on formula (4.1a)} \\
&= 2^j\binom{d-i}{j} + i\cdot 2^{j-1}\binom{d-i}{j-1} + \sum_{h=2}^{j} 2^{j-h}\binom{i}{h}\binom{d-i}{j-h} \\
&\leq 2^j\binom{d-i}{j} + i\cdot 2^{j-1}\binom{d-i}{j-1} + 2^{j-2}\sum_{h=2}^{j}\binom{i}{h}\binom{d-i}{j-h} \\
&= 2^j\binom{d-i}{j} + i\cdot 2^{j-1}\binom{d-i}{j-1} + 2^{j-2}\left(\binom{d}{j} - \binom{d-i}{j} - i\binom{d-i}{j-1}\right) \\
&\quad \text{Vandermonde identity (Lemma 2.5) applied} \\
&= 2^{j-2}\left(4\binom{d-i}{j} + 2i\binom{d-i}{j-1} + \binom{d}{j} - \binom{d-i}{j} - i\binom{d-i}{j-1}\right) \\
&= 2^{j-2}\left(3\binom{d-i}{j} + i\binom{d-i}{j-1} + \binom{d}{j}\right).
\end{aligned}
$$

□

Finally, we prove case 4, namely $i > \frac{qj}{2}$ when $i + (q-1)(d-i) - qj \geq 0$. This case is divided over two lemmas. The first one considers $q \geq 4$ and the second covers the case $q = 3$.

Lemma 5.14. *Let $q \geq 4$, $i + (q-1)(d-i) - qj \geq 0$, $i > \frac{qj}{2}$ and $j < d - \frac{d-1}{q} - 1$. Then $|K_j(i)| \leq |K_j(1)|$ for $1 \leq i \leq d$.*

Proof. Note that we may assume $j > 0$ since $K_0(i) = 1$ for all i. By formula (4.1a), we have

$$|K_j(1)| = (q-1)^{j-1}\binom{d}{j}\left|q - 1 - \frac{qj}{d}\right| = (q-1)^{j-1}\binom{d}{j}\left(q - 1 - \frac{qj}{d}\right),$$

where the last equality follows from $j < d - \frac{d-1}{q} - 1$, so $q - 1 - \frac{qj}{d} > \frac{q-1}{d} > 0$. We want to show $|K_j(i)| \leq |K_j(1)|$ using Proposition 5.12. Thus, we want to show the second inequality in

$$|K_j(i)| \leq (q-1)^{j-1}\left(\binom{d}{j} + (q-2)\binom{d-i}{j}\right) \leq (q-1)^{j-1}\binom{d}{j}\left(q - 1 - \frac{qj}{d}\right) = |K_j(1)|.$$

This is equivalent to showing

$$(q-2)\binom{d-i}{j} \leq \binom{d}{j}\left(q - 2 - \frac{qj}{d}\right). \tag{5.3}$$

Note that $i > \frac{qj}{2}$, so $\frac{qj}{d} < \frac{2i}{d}$, so

$$q - 2 - \frac{qj}{d} > q - 2 - \frac{2i}{d} = \frac{(q-2)d - 2i}{d} \geq \frac{2d - 2i}{d} \geq 0,$$

where the second-to-last inequality follows from $q \geq 4$.

First, assume $j > d - i$. Then the left hand side of (5.3) equals zero and the right hand side is positive, so for $j > d - i$ we have that (5.3) holds. Now assume $j \leq d - i$. In this case, we have

$$\binom{d-(i-1)}{j} = \frac{d-i+1}{d-i-j+1}\binom{d-i}{j} > \binom{d-i}{j},$$

so $\binom{d-i}{j}$ is decreasing in i. Since $i > \frac{qj}{2} \geq \frac{4\cdot 1}{2} = 2$, we need $i \geq 3$, so $\binom{d-i}{j} \leq \binom{d-3}{j}$. This means it is sufficient to show

$$(q-2)\binom{d-3}{j} \leq \binom{d}{j}\left(q-2-\frac{qj}{d}\right). \tag{5.4}$$

Using the definition of a binomial coefficient, we can write

$$\binom{d-3}{j} = \frac{(d-j)(d-j-1)(d-j-2)}{d(d-1)(d-2)}\binom{d}{j}.$$

This would make it sufficient to show

$$\frac{(d-j)(d-j-1)(d-j-2)}{d(d-1)(d-2)} \leq 1 - \frac{qj}{(q-2)d}. \tag{5.5}$$

Since $q \geq 4$, we have $\frac{qj}{(q-2)d} \leq \frac{2j}{d}$, so it suffices to show

$$\frac{(d-j)(d-j-1)(d-j-2)}{d(d-1)(d-2)} + \frac{2j}{d} = \frac{(d-j)(d-j-1)(d-j-2)+2j(d-1)(d-2)}{d(d-1)(d-2)} \leq 1. \tag{5.6}$$

We have

$$\begin{aligned}(d-j)(d-j-1)(d-j-2)+2j(d-1)(d-2) = d^3 &- 3d^2 - 3jd^2 + 2d + 6jd + 3j^2d \\ &- 2j - 3j^2 - j^3 + 2jd^2 - 6jd + 4j\end{aligned}$$

and

$$d(d-1)(d-2) = d^3 - 3d^2 + 2d,$$

thus it suffices to show

$$-jd^2 + 3j^2d + 2j - 3j^2 - j^3 \leq 0 \quad \Leftrightarrow \quad f(j) := -j^2 + 3jd - 3j + 2 - d^2 \leq 0. \tag{5.7}$$

We have

$$\frac{\partial f}{\partial j} = 3d - 3 - 2j \geq 0,$$

so f is increasing in j. which means it is biggest when j attains its biggest feasible value, which is smaller or equal than $j = \frac{3}{8}d - \frac{1}{8}$ by Proposition 5.11. Therefore it suffices to show $f\left(\frac{3}{8}d - \frac{1}{8}\right) \leq 0$. We have

$$\begin{aligned}f\left(\frac{3}{8}d-\frac{1}{8}\right) &= -\left(\frac{3}{8}d-\frac{1}{8}\right)^2 + 3\left(\frac{3}{8}d-\frac{1}{8}\right)d - 3\left(\frac{3}{8}d-\frac{1}{8}\right) + 2 - d^2 \\ &= -\frac{1}{64}d^2 - \frac{90}{64}d + \frac{151}{64}\end{aligned}$$

Since this is a quadratic function in d with negative leading coefficient which equals zero for $d = -45 \pm 8\sqrt{34} \approx -45 \pm 46.65$, we have that the above equation is negative for $d \geq 2$, which finishes the proof. ☐

Finally, we get to case 4 for $q = 3$.

Lemma 5.15. *Let $q = 3$, $i + (q-1)(d-i) - qj \geq 0$, $i > \frac{qj}{2}$ and $j < d - \frac{d-1}{q} - 1$. Then $|K_j(i)| \leq |K_j(1)|$ for $1 \leq i \leq d$.*

Proof. The case $i = 2$ was shown in Proposition 5.7, so assume $i \geq 3$. We start with writing down the assumptions for $q = 3$. From $i + (q-1)(d-i) - qj \geq 0$ we get $i \leq 2d - 3j$ and from $i > \frac{qj}{2}$ we get $i \geq \frac{3j+1}{2}$. This also means $2d - 3j \geq \frac{3j+1}{2}$, so $d \geq \frac{9j+1}{4} > 2j$. Moreover, from $j < d - \frac{d-1}{q} - 1$ we get $3j \leq 2d - 3$.

By formula (4.1a), we have

$$|K_j(1)| = 2^{j-1}\binom{d}{j}\left|2-\frac{3j}{d}\right| = 2^{j-1}\binom{d}{j}\left(2-\frac{3j}{d}\right),$$

where the last equality follows from $3j \leq 2d-3$, so $2-\frac{3j}{d} > \frac{3}{d} > 0$. Note that we may assume $j > 0$ since $K_0(i) = 1$ for all i. For the case $j = 1$ we have $|K_1(1)| = 2d-3$ and from Proposition 5.12 we have $|K_1(i)| \leq d+(q-2)(d-i) = 2d-i$. Since we assumed $i \geq 3$, we have

$$|K_1(i)| \leq 2d-i \leq 2d-3 = |K_1(1)|,$$

which proves the statement for $j = 1$. Thus, from now on we may assume $j \geq 2$, $i \geq 3$.

We want to show $|K_j(i)| \leq |K_j(1)|$ using Proposition 5.13. Thus, we want to show the second inequality in

$$|K_j(i)| \leq 2^{j-2}\left(3\binom{d-i}{j}+i\binom{d-i}{j-1}+\binom{d}{j}\right) \leq 2^{j-1}\binom{d}{j}\left(2-\frac{3j}{d}\right) = |K_j(1)|.$$

This is equivalent to showing

$$3\binom{d-i}{j}+i\binom{d-i}{j-1} \leq \frac{3(d-2j)}{d}\binom{d}{j}. \tag{5.8}$$

For $j = 2$, the inequality is true. Note that $i \geq \frac{3\cdot 2+1}{2}$, so $i \geq 4$.

$$\begin{aligned}3\binom{d-i}{2}+i\binom{d-i}{1} &= \frac{3(d-i)(d-i-1)}{2}+i(d-i)\\ &= \frac{(3(d-1)-i)(d-i)}{2}\\ &\leq \frac{3(d-1)(d-4)}{2}\\ &= \frac{3(d-4)}{d}\frac{d(d-1)}{2}\\ &= \frac{3(d-4)}{d}\binom{d}{2},\end{aligned}$$

so assume $j \geq 3$ and thus $i \geq \frac{3\cdot j+1}{2} \geq \frac{3\cdot 3+1}{2} = 5$ from now on.

First, consider the case $j-1 > d-i$. In this case, the left hand side of (5.8) equals zero and the right hand side is greater or equal than zero, so for $j-1 > d-i$ the inequality holds. Now consider $j-1 = d-i$. Recall that we had $j \geq 3$ and $d > 2j$, which gives

$$\begin{aligned}3\binom{j-1}{j}+(d-j+1)\binom{j-1}{j-1} &= d-j+1 < d-1 < \frac{3}{2}(d-1)\\ &\leq \frac{3}{2}(d-2j)(d-1) = \frac{3(d-2j)}{d}\binom{d}{2} < \frac{3(d-2j)}{d}\binom{d}{j}.\end{aligned}$$

Lastly, for $j-1 < d-i$, so for $i \leq d-j$, we want to show the inequality

$$3\binom{d-i}{j}+i\binom{d-i}{j-1} = \left(3+\frac{ij}{d-i-j+1}\right)\binom{d-i}{j} \leq \frac{3(d-2j)}{d}\binom{d}{j},$$

which is equivalent to showing the inequality in

$$\prod_{k=0}^{i-1}\frac{d-j-k}{d-k} = \frac{\binom{d-i}{j}}{\binom{d}{j}} \leq \frac{3(d-2j)(d-i-j+1)}{d(3(d-i-j+1)+ij)} = \frac{3(d-2j)(d-i-j+1)}{d(3(d-j+1)+i(j-3))}. \tag{5.9}$$

The left hand side $LHS(i)$ of (5.9) is decreasing in i. The numerator of the right hand side is decreasing in i and the denominator is increasing in i, so the right hand side $RHS(i)$ of (5.9) is also decreasing in i.

To prove the inequality in (5.9), we want to know which side decreases faster, so we calculate the following.

$$\begin{aligned} r_{LHS}(i) &= \frac{LHS(i)}{LHS(i+1)} = \prod_{k=0}^{i-1} \frac{d-j-k}{d-k} \cdot \prod_{k=0}^{i} \frac{d-k}{d-j-k} = \frac{d-i}{d-j-i}, \\ r_{RHS}(i) &= \frac{RHS(i)}{RHS(i+1)} = \frac{d-i-j+1}{3(d-i-j+1)+ij} \cdot \frac{3(d-i-j)+(i+1)j}{d-i-j}. \end{aligned}$$

We claim $r_{LHS}(i) > r_{RHS}(i)$ for all i. Recall that we have $i \leq d-j$.

$$\begin{aligned} \frac{3j-3}{3(d-i-j+1)} &> \frac{j-3}{3(d-i-j+1)+ij}, \text{ so} \\ 1+\frac{j-1}{d-i-j+1} &> 1+\frac{j-3}{3(d-i-j+1)+ij}, \text{ so} \\ \frac{d-i-j+1+j-1}{d-i-j+1} &> \frac{3(d-i-j+1)+ij+j-3}{3(d-i-j+1)+ij}, \text{ so} \\ \frac{d-i}{d-i-j+1} &> \frac{3(d-i-j)+(i+1)j}{3(d-i-j+1)+ij}, \text{ so} \\ \frac{d-i}{d-j-i} &> \frac{(d-i-j+1)(3(d-i-j)+(i+1)j)}{(d-i-j)(3(d-i-j+1)+ij)}, \text{ so} \\ r_{LHS}(i) &> r_{RHS}(i). \end{aligned}$$

We want to show (5.9), thus $LHS(i) \leq RHS(i)$ for all $i \geq 5$. If the base case $LHS(5) \leq RHS(5)$ holds, then we can complete the proof of the statement by induction:

$$\frac{LHS(i)}{LHS(i+1)} > \frac{RHS(i)}{RHS(i+1)} \quad \Rightarrow \quad LHS(i) \cdot RHS(i+1) > RHS(i) \cdot LHS(i+1)$$

together with $LHS(i) \leq RHS(i)$ gives $LHS(i+1) < RHS(i+1)$.

Thus, it suffices to show that $LHS(5) \leq RHS(5)$ holds. We have

$$\begin{aligned} LHS(5) &= \frac{(d-j)(d-j-1)(d-j-2)(d-j-3)(d-j-4)}{d(d-1)(d-2)(d-3)(d-4)}, \\ RHS(5) &= \frac{3(d-2j)(d-j-4)}{d(3(d-j-4)+5j)} = \frac{3(d-2j)(d-j-4)}{d(3d+2j-12)}, \end{aligned}$$

thus we want to show

$$\frac{(d-j)(d-j-1)(d-j-2)(d-j-3)}{(d-1)(d-2)(d-3)(d-4)} \leq \frac{3(d-2j)}{3d+2j-12},$$

which is equivalent to showing

$$3(d-2j)(d-1)(d-2)(d-3)(d-4) - (3d+2j-12)(d-j)(d-j-1)(d-j-2)(d-j-3) \geq 0.$$

Expanding the brackets gives that we want to show

$$\begin{aligned} &\big(72d - 150d^2 + 105d^3 - 30d^4 + 3d^5 - 144j + 300dj - 210d^2j + 60d^3j - 6d^4j\big) \\ &\quad - \big(72d - 150d^2 + 105d^3 - 30d^4 + 3d^5 - 72j + 270dj - 260d^2j + 90d^3j \\ &\quad - 10d^4j - 120j^2 + 205dj^2 - 90d^2j^2 + 10d^3j^2 - 50j^3 + 30dj^3 - 5dj^4 + 2j^5\big) \end{aligned}$$

$$\begin{aligned}
&= -2j^5 + 5dj^4 + (-30d+50)j^3 + (-10d^3 + 90d^2 - 205d + 120)j^2 \\
&\quad + (4d^4 - 30d^3 + 50d^2 + 30d - 72)j \\
&\geq 0.
\end{aligned}$$

Dividing by j on both sides and rearranging terms gives that we want to show

$$\left(-2j^4 + 5dj^3 - 10d^3j + 4d^4\right) - (30d-50)j^2 + (90d^2 - 205d + 120)j - 30d^3 + 50d^2 + 30d - 72 \geq 0.$$

Before we continue showing above relation, we show another bound on j. We derived before that $i \geq \frac{3j+1}{2}$, meaning $j \leq \frac{2i-1}{3}$. Moreover, we had $i \leq d - j$, thus

$$j \leq \frac{2i-1}{3} \leq \frac{2(d-j)-1}{3} \quad \Leftrightarrow \quad j \leq \frac{2d-1}{5}.$$

Now let $g(j) := -2j^4 + 5dj^3 - 10d^3j + 4d^4$. Using $j < \frac{d}{2}$, we have

$$\frac{\partial g}{\partial j} = -8j^3 + 15dj^2 - 10d^3 < -8j^3 + \frac{15}{4}d^3 - 10d^3 = -8j^3 - \frac{25}{4}d^3 < 0,$$

so g is decreasing in j, meaning that g takes its smallest value at the largest value of j, which is $j = \frac{2d-1}{5}$ as we have shown above. Thus, we have

$$\begin{aligned}
g(j) \geq g\left(\frac{2d-1}{5}\right) &= -2\left(\frac{2d-1}{5}\right)^4 + 5d\left(\frac{2d-1}{5}\right)^3 - 10d^3\left(\frac{2d-1}{5}\right) + 4d^4 \\
&= \frac{168}{625}d^4 + \frac{1014}{625}d^3 + \frac{102}{625}d^2 - \frac{9}{625}d - \frac{2}{625}.
\end{aligned}$$

Furthermore, let $h(j) := -(30d-50)j^2 + (90d^2 - 205d + 120)j$. Recall that we have $i \geq 5$, so also $d \geq 5$. Thus, we have $30d - 50 \geq 0$ and also $90d^2 - 205d + 120 \geq 0$ since the discriminant $205^2 - 4 \cdot 90 \cdot 120 = -1175$ is negative and the leading coefficient 90 is positive. Thus, $h(j)$ is quadratic in j with negative leading coefficient and has a zero at $j = 0$. The maximum value of h is attained at j^*, which is defined as

$$j^* := \frac{-(90d^2 - 205d + 120)}{-2(30d-50)} = \frac{18d^2 - 41d + 24}{12d - 20} > \frac{18d^2 - 41d}{12d} = \frac{3}{2}d - \frac{41}{12} > 0.$$

Moreover, $\frac{3}{2}d - \frac{41}{12}$ is greater than or equal to $\frac{2d-1}{5}$ for $d \geq 3$, so we have $0 \leq j < \frac{2d-1}{5} < j^*$. This means that $h(j)$ is increasing in j for $0 \leq j < \frac{2d-1}{5}$, so the minimum value of $h(j)$ in this interval is attained at $j = 0$, which gives $h(j) \geq h(0) = 0$. Figure 5.6 shows the graph of $h(j)$.

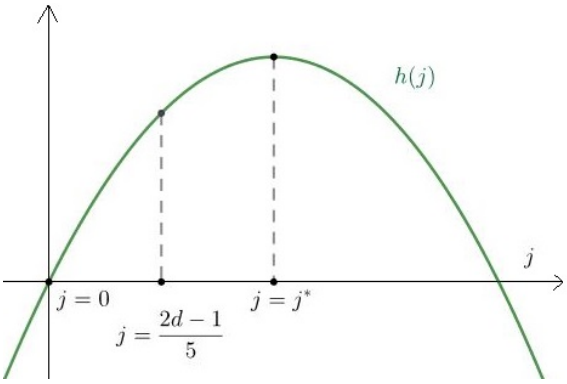


Figure 5.6: Graph of the function $h(j)$ from the proof of Lemma 5.15.

Recall that we wanted to show

$$g(j) + h(j) - 30d^3 + 50d^2 + 30d - 72 \geq 0. \tag{5.10}$$

We have

$$\begin{aligned} g(j) + h(j) - 30d^3 + 50d^2 + 30d - 72 &\geq \frac{168}{625}d^4 + \frac{1014}{625}d^3 + \frac{102}{625}d^2 - \frac{9}{625}d - \frac{2}{625} \\ &\quad - 30d^3 + 50d^2 + 30d - 72 \\ &= \frac{1}{625}(168d^4 - 17736d^3 + 31352d^2 + 18741d - 45002), \end{aligned}$$

thus it suffices to show

$$f(d) := 168d^4 - 17736d^3 + 31352d^2 + 18741d - 45002 \geq 0. \tag{5.11}$$

The leading coefficient of f is positive, and by using numeric solving methods on a computer we find that $f(d)$ has only two real zeros, namely $d \approx -1.132$ and $d \approx 103.763$. Thus, the polynomial $f(d)$ is greater than or equal to zero for $d \geq 104$. Figure 5.7 shows the graph of $f(d)$.

Thus, we have finished the proof of Lemma 5.15 for $d \geq 104$. We need to check $|K_j(i)| \leq |K_j(1)|$ with $q = 3$, $3 \leq d \leq 103$ by computer for all feasible values (i, j). The code used to do these computations is shown in Appendix B.8. It turns out this inequality holds for the aforementioned values of d, q, j, i, which concludes the proof of this lemma. ☐

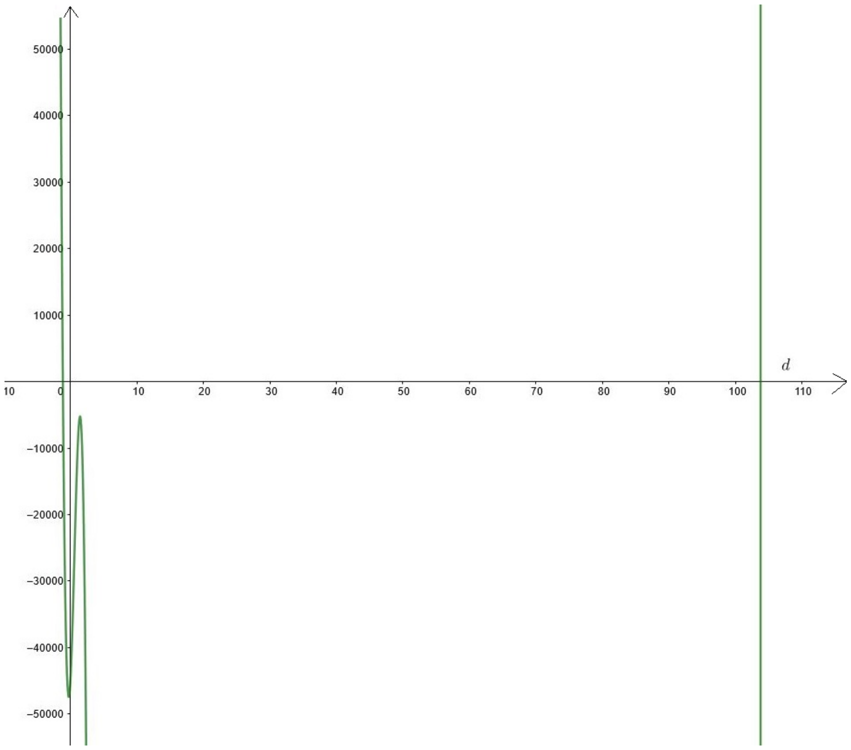


Figure 5.7: Graph of the function $f(d)$ from the proof of Lemma 5.15.

We can now summarize all four cases in the following lemma.

Lemma 5.16. *Let $q \geq 3$ and $j < d - \frac{d-1}{q} - 1$. Then $|K_j(i)| \leq |K_j(1)|$ for $1 \leq i \leq d$.*

Proof. First, assume $i \leq \frac{qj}{2}$, $i+(q-1)(d-i)-qj \geq 0$ or $qj \leq 2(q-1)(d-i)$, $i+(q-1)(d-i)-qj < 0$. If either of these cases hold, we can use Proposition 5.8 and induction on i to prove the statement, like we did in the proof of Lemma 5.9. For $i = 1$ and $i = 2$, the statement follows from Proposition 5.7. For larger i, we have

$$|K_j(i)| \leq \max\{|K_j(i-2)|, |K_j(i-1)|\} \leq \cdots \leq \max\{|K_j(1)|, |K_j(2)|\},$$

and the statement follows from Proposition 5.7.

Now assume $qj > 2(q-1)(d-i)$, $i+(q-1)(d-i)-qj < 0$. This case was shown in Lemma 5.10. Lastly, assume $i > \frac{qj}{2}$, $i+(q-1)(d-i)-qj \geq 0$. This case was shown in Lemma 5.15 for $q = 3$ and in Lemma 5.14 for $q \geq 4$. ☐

Lastly, we see that with Lemmas 5.9 and 5.16, we have shown Conjecture 5.5 for all relevant values of j. Thus, we can now state with full confidence that this conjecture is in fact a theorem.

~~**Conjecture.**~~ **Theorem 5.5.** *Let* $q \geq 3$ *and* $j < d - \frac{d-1}{q}$. *Then*

(a) *if* $d = 0\,(mod\ q)$ *or* $d = q-1\,(mod\ q)$, *then* $|K_j(1)| \geq |K_j(i)|$ *for* $1 \leq i \leq d$, $j < \lceil d - \frac{d-1}{q} \rceil - 1$, *and* $|K_j(2)| \geq |K_j(i)|$ *for* $1 \leq i \leq d$, $j = \lceil d - \frac{d-1}{q} \rceil - 1$,

(b) *if* $d \neq 0\,(mod\ q)$ *and* $d \neq q-1\,(mod\ q)$, *then* $|K_j(1)| \geq |K_j(i)|$ *for* $1 \leq i \leq d$.

Proof. The theorem follows directly from lemmas 5.9 and 5.16. ☐

Figure 5.8 shows the results of Theorem 5.5, this time for $q = 4$ and various values of d.

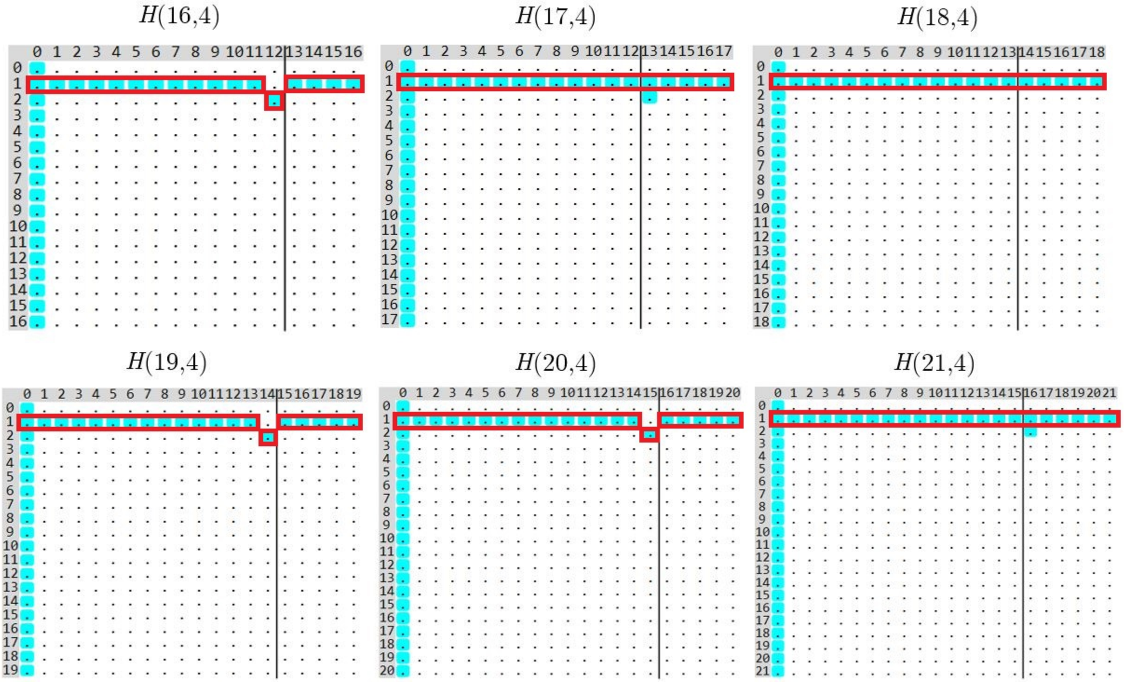


Figure 5.8: Visualization of the results of Theorem 5.5. The highlighted boxes indicate the pen-absolute eigenvalues per column. The red fields indicate the results of the theorem.

5.2 The Johnson case

5.2.1 The smallest eigenvalue

In Section 5.1, we noted that for graphs from the Hamming scheme, the bound $j \geq d - \frac{d-1}{q}$ seems to be tight in Corollary 4.13 and Theorem 4.14b. For graphs from the Johnson scheme, we have already shown in Theorem 4.32b that the bound $j \geq \frac{d(n-d)}{n-1}$ is tight. This can be seen in Figure 5.9. What can also be seen in that the value of i for which $E_j(i)$ is the smallest eigenvalue seems to be decreasing for $j > 0$.

Moreover, we see in Figure 5.10 that for q very large compared to d, the value of i for which $E_j(i)$ is the smallest eigenvalue seems to be strictly decreasing for $j > 0$. This would mean that for n large enough, we have that $E_j(d - j + 1)$ is the smallest eigenvalue for all $j \geq 1$, which was noted already in Proposition 4.34c. Table 5.2 shows the smallest value n such that $E_j(d - j + 1)$ is the smallest eigenvalue for all $j \geq 1$ for a particular value of d. For example, when $d = 3$ we have that for $n \geq 8$, the value $E_j(d - j + 1)$ is the smallest eigenvalue for all $j \geq 1$.

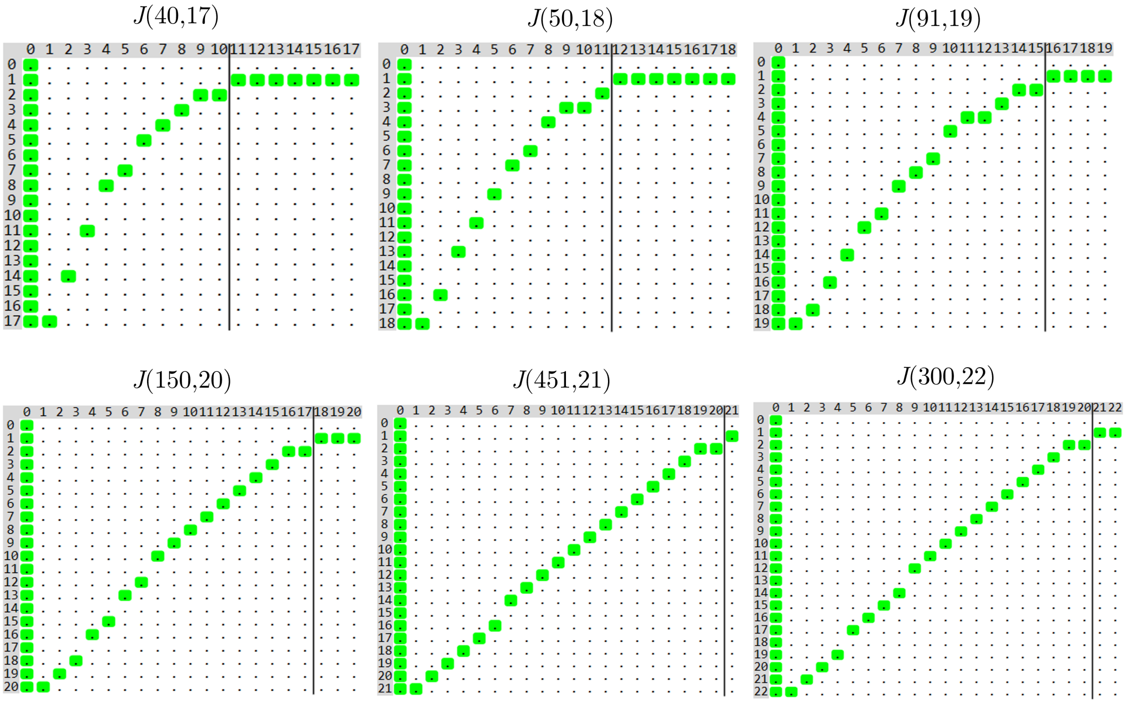


Figure 5.9: Visualization of some P-matrices of graphs from the Johnson scheme. The highlighted boxes indicate the smallest eigenvalues per column.

d	3	4	5	6	7	8	9	10	15	20	21	22
n	8	14	21	34	50	70	96	129	396	901	1038	1189
d	23	24	25	26	27	30	35	40	45	50	75	100
n	1351	1529	1724	1931	2156	2935	4616	6847	9694	13243	44169	104113

Table 5.2: Overview of the minimum value n per value of d for which $E_j(d - j + 1)$ is the smallest eigenvalue for all $j \geq 1$.

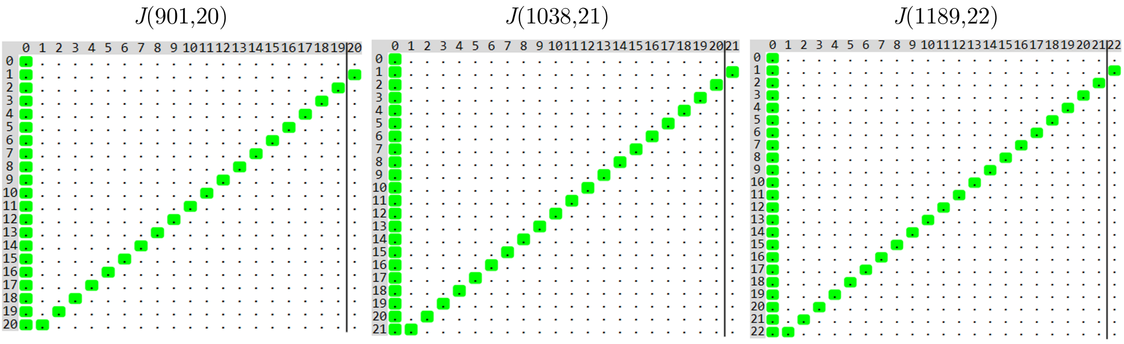


Figure 5.10: Visualization of some P-matrices of graphs from the Johnson scheme. The highlighted boxes indicate the smallest eigenvalues per column.

5.2.2 The penabsolute eigenvalue

As can be seen in Figure 5.11, the penabsolute eigenvalues of graphs from the Johnson scheme behave similar to those of graphs from the Hamming scheme (see Figure 5.8 for graphs from the Hamming scheme). It seems that $|E_j(1)|$ is the penabsolute eigenvalue for all j, except for the column just before the black vertical line.

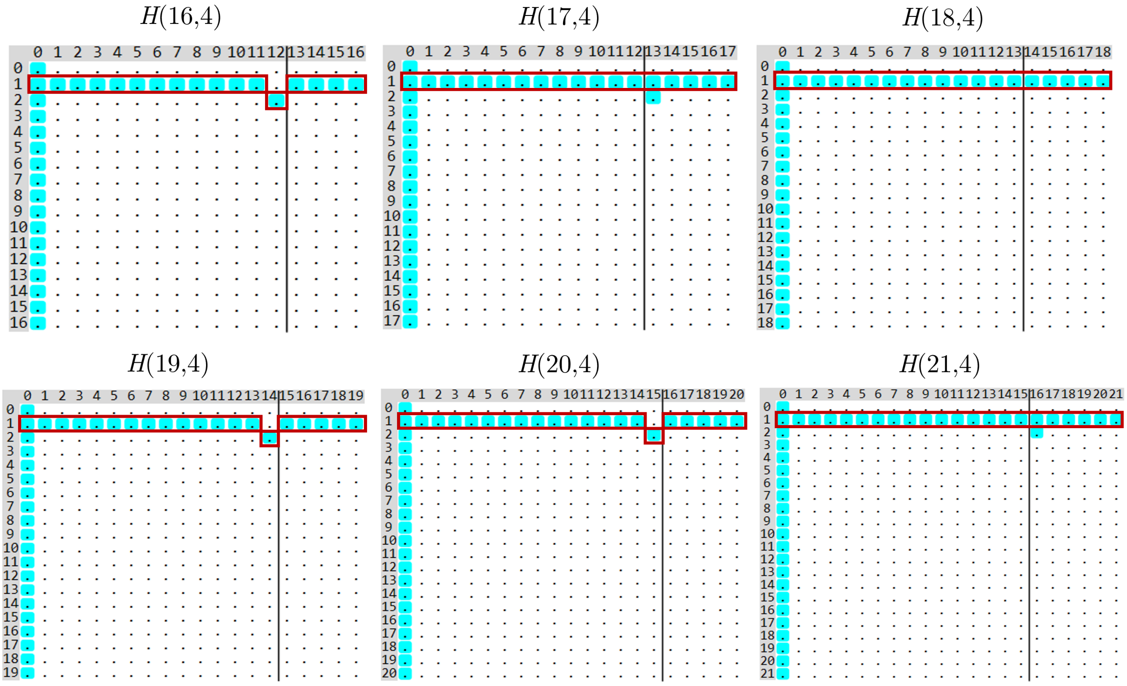


Figure 5.11: Visualization of some P-matrices of graphs from the Johnson scheme. The highlighted boxes indicate the penabsolute eigenvalues per column.

For the column just before the black vertical line, which represents $j = \left\lceil \frac{d(n-d)}{n-1} \right\rceil - 1$ for Johnson, we have that sometimes $|E_j(1)|$ and sometimes $|E_j(2)|$ is the penabsolute eigenvalue. However, in contrast to the Hamming case, we did not manage to predict when $|E_j(1)|$ or when $|E_j(2)|$ is the penabsolute eigenvalue for this particular value of j for all n, d. An overview can be found in

Table 5.3. For n large compared to d however, namely $n \geq d^2 + d + 1$, it seemed that $|E_j(1)|$ was the penabsolute eigenvalue for $j = \left\lceil \frac{d(n-d)}{n-1} \right\rceil - 1$. This can also be seen in Table 5.3. We include these observations in the following two conjectures.

Conjecture 5.17. *For* $j = \left\lceil \frac{d(n-d)}{n-1} \right\rceil - 1$*, we have that*

(a) $|E_j(1)|$ *or* $|E_j(2)|$ *is the penabsolute eigenvalue for* $1 \leq i \leq d$*,*

(b) *for* $n \geq d^2 + d + 1$*,* $|E_j(1)|$ *is the penabsolute eigenvalue for* $1 \leq i \leq d$*, and this bound is tight, meaning that* $|E_j(2)|$ *is the penabsolute eigenvalue for* $n = d^2 + d$*.*

Test cases. (a) This conjecture was tested for all pairs (n, d) with $2 \leq d \leq 200$, $2d \leq n \leq 400$. The code that was used to test these cases can be found in Appendix B.9.

(b) We tested whether $|E_j(2)|$ is the penabsolute eigenvalue for $n = d^2 + d$ and $3 \leq d \leq 200$. Furthermore, we tested whether $|E_j(1)|$ is the penabsolute eigenvalue for $d^2 + d + 1 \leq n \leq 10000$, $3 \leq d \leq 9$ and for $d^2 + d + 1 \leq n \leq d^3$, $10 \leq d \leq 200$. The code used to test these cases can be found in Appendix B.10.

Conjecture 5.18. *For* $j < \left\lceil \frac{d(n-d)}{n-1} \right\rceil - 1$*, we have that* $|E_j(1)|$ *is the penabsolute eigenvalue for* $1 \leq i \leq d$*.*

Test cases. This conjecture was tested for all pairs (n, d) with $2 \leq d \leq 200$, $2d \leq n \leq 400$. The code that was used to test these cases can be found in Appendix B.11.

$n\downarrow$ $d\rightarrow$	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
6	1																	
7	1																	
8	2	2																
9	2	2																
10	2	1	1															
11	2	1	1															
12	2	1	2	2														
13	1	1	2	2														
14	1	2	2	1	1													
15	1	2	1	1	1													
16	1	2	1	1	2	2												
17	1	2	1	2	2	2												
18	1	2	1	2	1	1	1											
19	1	2	1	2	1	1	1											
20	1	2	1	2	1	2	2	2										
21	1	1	1	1	1	2	2	2										
22	1	1	2	1	1	2	1	1	1									
23	1	1	2	1	2	2	1	1	1									
24	1	1	2	1	2	1	1	2	2	2								
25	1	1	2	1	2	1	1	2	2	2								
26	1	1	2	1	2	1	2	2	1	1	1							
27	1	1	2	1	2	1	2	1	1	1	1							
28	1	1	2	1	1	1	2	1	1	2	2	2						
29	1	1	2	1	1	1	2	1	2	2	2	2						
30	1	1	2	1	1	2	1	1	2	2	1	1	1					
31	1	1	1	1	1	2	1	1	2	1	1	1	1					
32	1	1	1	2	1	2	1	2	2	1	1	2	2	2				
33	1	1	1	2	1	2	1	2	1	1	2	2	2	2				
34	1	1	1	2	1	2	1	2	1	1	2	2	1	1	1			
35	1	1	1	2	1	2	1	2	1	2	2	1	1	1	1			
36	1	1	1	2	1	1	1	1	1	2	1	1	1	2	2	2		
37	1	1	1	2	1	1	1	1	1	2	1	1	2	2	2	2		
38	1	1	1	2	1	1	2	1	2	2	1	2	2	1	1	1	1	
39	1	1	1	2	1	1	2	1	2	1	1	2	2	1	1	1	1	
40	1	1	1	2	1	1	2	1	2	1	1	2	1	1	2	2	2	2
41	1	1	1	2	1	1	2	1	2	1	2	2	1	1	2	2	2	2
42	1	1	1	2	1	1	2	1	2	1	2	1	1	2	2	1	1	1
43	1	1	1	1	1	1	2	1	2	1	2	1	1	2	1	1	1	1
44	1	1	1	1	2	1	2	1	1	1	2	1	2	2	1	1	2	2
45	1	1	1	1	2	1	1	1	1	1	1	1	2	1	1	2	2	2
46	1	1	1	1	2	1	1	1	1	2	1	1	2	1	1	2	2	1
47	1	1	1	1	2	1	1	2	1	2	1	2	2	1	2	2	1	1
48	1	1	1	1	2	1	1	2	1	2	1	2	1	1	2	1	1	1
49	1	1	1	1	2	1	1	2	1	2	1	2	1	1	2	1	1	2
50	1	1	1	1	2	1	1	2	1	2	1	2	1	2	2	1	2	2
51	1	1	1	1	2	1	1	2	1	2	1	2	1	2	1	1	2	2
52	1	1	1	1	2	1	1	2	1	1	1	1	1	2	1	1	2	1
53	1	1	1	1	2	1	1	2	1	1	1	1	1	2	1	2	2	1

Table 5.3: Columns are values of d and rows are values of n. Note that $n \geq 2d$. Entries are 1 if $|E_j(1)|$ is the penabsolute eigenvalue and entries are 2 if $|E_j(2)|$ is the penabsolute eigenvalue for $j = \left\lceil \frac{d(n-d)}{n-1} \right\rceil - 1$. Cells are green if $n \geq d^2 + d + 1$, linking to Conjecture 5.17b.

Chapter 6

An application to the max-k-cut problem

In Chapter 1, the paper [30] by van Dam and Sotirov was mentioned as an application of the relevance of finding the smallest eigenvalue of the adjacency matrix of a graph. This chapter will discuss [30] in more detail. In [30], van Dam and Sotirov attempt to find better bounds for the max-k-cut problem using the largest Laplacian eigenvalue of a graph. As $L = D - A$, where D is a matrix with the degrees of the vertices on its diagonal and zeros elsewhere, this comes down to using the smallest eigenvalue of the adjacency matrix of the graph to find better bounds for the max-k-cut problem. Note that in this thesis so far, we only considered regular graphs where $D = sI$ for valency s.[1] However, [30] considers general, possibly non-regular graphs except if stated otherwise.

To understand the use of the largest Laplacian eigenvalue for bounds on the max-k-cut problem, we first need to define this problem. The input of the max-k-cut problem is a graph $G = (V, E)$ with $|V| = n$ and a positive integer $2 \leq k \leq n$. The question is to partition V into at most k disjoint parts $V = V_1 \cup \cdots \cup V_k$ such that the total number of edges in between parts is maximal. Consider for example a bipartite graph $G = (V = A \cup B, E)$ and $k = 2$. For this input, the problem is trivial: just take $V_1 = A$ and $V_2 = B$, and all edges of G will be edges in between parts. However, in general this problem is NP-hard, even for $k = 2$ [12]. When $k = 2$, the problem is sometimes referred to as simply the max-cut problem in the literature.

There are many known applications for the max-k-cut problem. The ones mentioned in [30] are VLSI design (combining millions of MOS transistors onto a single chip to create an integrated circuit) [2, 8], frequency planning [11], finding properties of spin glasses [2], digital-analogue converters [21], sports team scheduling [19] and fault test generation [15].

To solve the max-k-cut problem, it is useful to provide a formal problem definition. Let $G = (V, E)$ with $|V| = n$. An integer programming formulation for the problem is the following.

$$
\begin{aligned}
\text{(IP1)} \quad \max \quad & \frac{1}{2} \operatorname{tr}\left(X^T L X\right) && (6.1)\\
\text{s.t.} \quad & X\,\underline{1}_k = \underline{1}_n, && (6.2)\\
& x_{ij} \in \{0,1\} \text{ for } 1 \leq i \leq n, 1 \leq j \leq k. && (6.3)
\end{aligned}
$$

---

[1] Note that we use s for the valency instead of the previously used k, since k is already taken in "max-k-cut".

We can think of $X \in \{0,1\}^{n \times k}$ as the incidence matrix of a partition of V, where

$$x_{ij} = \begin{cases} 1 & i \text{ is in part } j, \\ 0 & \text{else.} \end{cases}$$

Every vertex should be in precisely one partition, which can be translated to $X \, \underline{1}_k = \underline{1}_n$, where $\underline{1}_k$ is the all-one vector of size k. The objective that we wish to maximize corresponds to the number of edges in between parts V_m with $1 \leq m \leq k$. We have

$$\operatorname{tr}\left(X^T L X\right) = \sum_{m=1}^{k} \sum_{l=1}^{n} x_{lm} \sum_{j=1}^{n} x_{jm} L_{jl}.$$

For every part V_m, we look at all vertices l that are in V_m (then $x_{lm} = 1$). For these vertices l, we add up L_{jl} if vertex j is in the same part as vertex l. We have

$$\begin{aligned} \sum_{j=1}^{n} x_{jm} L_{jl} &= x_{lm} L_{ll} + \sum_{j=1,\, j \neq l}^{n} x_{jm} L_{jl} \\ &= \deg(l) + (\text{number of vertices in part } m \text{ that are neighbors of } l) \cdot (-1) \\ &= \text{number of vertices outside part } m \text{ that are neighbors of } l, \end{aligned}$$

thus, the objective that we want to maximize counts the number of vertices in between parts. The $\frac{1}{2}$ from (6.1) is there because every edge is counted twice in the above reasoning. Since solving (IP1) is NP-hard [12], it makes sense to formulate the following semidefinite relaxation of (IP1) [25]:

$$\begin{aligned} \text{(SDP1)} \quad \max \quad & \frac{1}{2} \operatorname{tr}(LY) && (6.4) \\ \text{s.t.} \quad & \operatorname{diag}(Y) = \underline{1}_n, && (6.5) \\ & kY - J_n \text{ is PSD}, && (6.6) \\ & y_{ij} \geq 0 \text{ for } 1 \leq i,j \leq n. && (6.7) \end{aligned}$$

This relaxation is the only known SDP relaxation for the max-k-cut problem so far [2] that could be solved when $k > 5$ and $n > 50$ [29, p. 3]. Note that J_n is the all-one matrix of size $n \times n$. In the case that $y_{ij} \in \{0,1\}$ for all i,j in an optimal partition, we can think of Y as $Y = XX^T$, so

$$y_{ij} = \begin{cases} 1 & i,j \text{ are in the same part,} \\ 0 & \text{else.} \end{cases}$$

This is why we need constraint (6.5). Constraint (6.7) is the relaxed form of constraint (6.3). The reason that we need constraint (6.6) can be found in [25]. Because (SDP1) is a relaxation of (IP1), we have $\text{OPT}_{(\text{SDP1})} \geq \text{OPT}_{(\text{IP1})}$.

To derive a new upper bound for the max-k-cut problem using Laplacian eigenvalues, we need to introduce something called the *Lapacian algebra* $\mathcal{L}$. This is a matrix $*$-algebra [10], which is a set of matrices that is closed under addition, scalar multiplication, matrix multiplication and taking conjugate transposes. Let $0 = \mu_0 \leq \mu_1 < \cdots < \mu_m = \mu_{\max}(L)$ and $\lambda_0 \leq \lambda_1 < \cdots < \lambda_m = \lambda_{\max}(A)$ be the distinct eigenvalues of L and A respectively, where the first inequality is not strict to provide for the case that our graph is disconnected. Note that, since our graph is not necessarily regular, we cannot say $\lambda_0 = s$.

Next, let U_i be a matrix whose columns form an orthonormal basis of the eigenspace corresponding to μ_i, and let $F_i = U_i U_i^T$. We know that (a constant multiple of) an eigenvector corresponding to

---

[2] The article [29] where this claim was made was published in 2014, so it would be possible that someone found a better SDP relaxation somewhere after this article was published.

$\mu_0 = 0$ is the all-one vector, so we take $U_0 = \frac{1}{\sqrt{n}} \underline{1}_n$, and if the graph is disconnected, the columns of U_1 contain all other eigenvectors corresponding to the smallest eigenvalue. The matrices F_i, which are called idempotents of $\mathcal{L}$, form a basis of the Laplacian algebra $\mathcal{L}$. Thus, we have $\mathcal{L} = \langle \{F_0, \ldots, F_m\} \rangle$. These idempotents have some properties, which are summarized in the following lemma. Note that 0_n is the all-zero matrix of size $n \times n$.

Lemma 6.1. *[30] Let F_i be idempotents of the Laplacian algebra for $0 \leq i \leq m$. Then the following statements hold.*

(a) $\sum_{i=0}^m F_i = I$,

(b) $\sum_{i=0}^m \mu_i F_i = L$,

(c) $F_i F_j = \begin{cases} F_i & \textit{if } i = j, \\ 0_n & \textit{else} \end{cases}$,

(d) $F_0 = \frac{1}{n} J_n$.

Proof. (a) We have $\sum_{i=0}^m F_i = \sum_{i=0}^m U_i U_i^T = I$ if and only if $\left(\sum_{i=0}^m U_i U_i^T\right) \underline{z} = \underline{z}$ for all $\underline{z} \in \mathbb{R}^n$. Let $\{v_1, \ldots, v_n\}$ be all eigenvectors of L. Then for all $z \in \mathbb{R}^n$, there are numbers $\beta_1, \ldots, \beta_n \in \mathbb{R}$ such that $\underline{z} = \sum_{j=1}^n \beta_j v_j$. Therefore

$$\left(\sum_{i=0}^m U_i U_i^T\right) \underline{z} = \left(\sum_{i=0}^m U_i U_i^T\right) \left(\sum_{j=1}^n \beta_j v_j\right) \overset{(\star)}{=} \left(\sum_{j=1}^n \beta_j v_j\right) = \underline{z}.$$

If v_j is not a column of U_i, then $U_i^T v_j = \underline{0}$, so also $U_i U_i^T v_j = \underline{0}$. If v_j is a column of U_i, then $U_i^T v_j$ is a unit vector with the one in place k if and only if v_j is the k^{th} column of U_i, thus $U_i U_i^T v_j = v_j$. This explains the equality sign with a $(\star)$ in the equation above.

(b) Let $\{v_1, \ldots, v_n\}$ be all eigenvectors of L again. We have $L v_j = \mu_l v_j$ if and only if v_j is an eigenvector corresponding to the eigenvector μ_l. Moreover, we have

$$\left(\sum_{i=0}^m \mu_i F_i\right) v_j = \sum_{i=0}^m \mu_i U_i U_i^T v_j = \mu_l v_j = L v_j$$

by the same reasoning as in the last paragraph of the proof of (a). For all $z \in \mathbb{R}^n$ there are numbers $\beta_1, \ldots, \beta_n \in \mathbb{R}$ such that $\underline{z} = \sum_{j=1}^n \beta_j v_j$. Thus, for all $\underline{z}$ we have

$$\left(\sum_{i=0}^m \mu_i F_i\right) \underline{z} = \left(\sum_{i=0}^m \mu_i F_i\right) \left(\sum_{j=1}^n \beta_j v_j\right) = L \left(\sum_{j=1}^n \beta_j v_j\right) = L\underline{z},$$

thus $\sum_{i=0}^m \mu_i F_i = L$.

(c) Let $\{v_1, \ldots, v_n\}$ be all eigenvectors of L again. Since these eigenvectors are pairwise orthonormal, and the columns of U_i consist of the vectors v_j, we have $U_i^T U_j = \begin{cases} I & i = j, \\ 0_m & i \neq j. \end{cases}$ Therefore

$$F_i F_j = U_i U_i^T U_j U_j^T = \begin{cases} U_i I U_j^T & i = j, \\ U_i 0_m U_j^T & i \neq j. \end{cases} = \begin{cases} F_i & i = j, \\ 0_n & i \neq j. \end{cases}$$

(d) We have $F_0 = U_0 U_0^T = \left(\frac{1}{\sqrt{n}}\right)^2 \underline{1} \cdot \underline{1}^T = \frac{1}{n} J_n$. □

The crux of the new bound on the max-k-cut problem [30, p. 221] lies in the following theorem. Note that the data matrices of the problem are all matrices that appear in the SDP/IP formulation of the problem except for the variable matrix. In the case of (SDP1), this would be L and J_n since Y is the variable matrix.

Theorem 6.2. *[29] Let (SDP) be a semi-definite optimization problem. If a matrix $*$-algebra contains the data matrices of (SDP) and the identity matrix, then the optimization of (SDP) can be restricted to feasible points in the matrix $*$-algebra.*

Thus, we want to create a semi-definite optimization problem such that our Laplacian algebra $\mathcal{L}$ contains all data-matrices of that problem. The problem (SDP1) that was mentioned before does not satisfy these conditions, as not all matrices in $\mathcal{L}$ have constant diagonal (think of L for non-regular matrices) while (6.5) states $\operatorname{diag}(Y) = \underline{1}_n$, and matrices in $\mathcal{L}$ can have negative entries while (6.7) states $y_{ij} \geq 0$. Fortunately, we can relax (SDP1) further to a problem that does satisfy the conditions of Theorem 6.2.

$$\begin{aligned} \text{(SDP2)} \quad \max \quad & \frac{1}{2}\operatorname{tr}(LY) && (6.8)\\ \text{s.t.} \quad & \operatorname{tr}(Y) = n, && (6.9)\\ & kY - J_n \text{ is PSD.} && (6.10) \end{aligned}$$

By Theorem 6.2, we may restrict the optimization of (SDP2) to feasible points in $\mathcal{L}$, which gives us the following result.

Theorem 6.3. *[30] Let G, L, n, k, $\mu_{\max}$ and* (SDP2) *as defined before. Then*

$$max_{(\text{SDP2})} = \frac{n(k-1)}{2k} \cdot \mu_{\max}(L).$$

*Proof**. Because of Theorem 6.2, there exists an optimal solution Y from (SDP2) in $\mathcal{L}$. Thus, since $\{F_0, \dots, F_m\}$ is a basis of $\mathcal{L}$, we may assume there exist $\alpha_0, \dots, \alpha_m \in \mathbb{R}$ such that $Y = \sum_{i=0}^{m} \alpha_i F_i$. Now let $\{z_0, \dots, z_{n-1}\}$ be the orthonormal eigenvectors of L. We know that $kY - J_n$ is PSD, so for all z_j we know

$$\begin{aligned} 0 \leq z_j^T(kY - J_n)z_j &= kz_j^T Y z_j - z_j^T J_n z_j \\ &= kz_j^T \left(\sum_{i=0}^{m} \alpha_i F_i\right) z_j - \left(\sum_{l=1}^{n} z_{j_l}\right)^2 \\ &= k\sum_{i=0}^{m} \alpha_i \left(z_j^T U_i\right)\left(z_j^T U_i\right)^T - \left(\sum_{l=1}^{n} z_{j_l}\right)^2 \\ &\overset{(\star)}{=} k\alpha_j \left(z_j^T U_j\right)\left(z_j^T U_j\right)^T - \left(\sum_{l=1}^{n} z_{j_l}\right)^2 \\ &= k\alpha_j - \left(\sum_{l=1}^{n} z_{j_l}\right)^2, \end{aligned}$$

where $(\star)$ once again comes from the fact that $z_j U_i = \underline{0}$ if z_j is not a column of U_i, and $z_j U_i^T$ is a unit vector with a one on place k if and only if z_j is the k^{th} column of U_i.

For $j = 0$, we have $z_0 = \frac{1}{\sqrt{n}}\underline{1}_n$, so by the previous derivation $k\alpha_0 - \left(n \cdot \frac{1}{\sqrt{n}}\right)^2 = k\alpha_0 - n \geq 0$. This gives

$$0 \leq \frac{k\alpha_0}{n} - 1 = k - 1 - \frac{k}{n}(n - \alpha_0)$$

$$
\begin{aligned}
&= k - 1 - \frac{k}{n}(\operatorname{tr}(Y) - \alpha_0 \operatorname{tr}(F_0)) \\
&\qquad \text{since } F_0 = \frac{1}{n}J_n, \text{ so } \operatorname{tr}(F_0) = 1 \\
&= k - 1 - \frac{k}{n}(\sum_{i=0}^{m} \alpha_i \operatorname{tr}(F_i) - \alpha_0 \operatorname{tr}(F_0)) \\
&= k - 1 - \frac{k}{n}\sum_{i=1}^{m} \alpha_i \operatorname{tr}(F_i),
\end{aligned}
$$

which holds if and only if

$$\sum_{i=1}^{m} \alpha_i \operatorname{tr}(F_i) \leq \frac{n(k-1)}{k}.$$

For $j > 0$, we have $0 \leq k\alpha_j - \left(\sum_{l=1}^{n} z_{jl}\right)^2 \leq k\alpha_j$, so $\alpha_j \geq 0$. In total, this gives

$$
\begin{aligned}
\mathrm{OPT}_{(\mathrm{SDP2})} &= \frac{1}{2}\operatorname{tr}(LY) \\
&= \frac{1}{2}\operatorname{tr}\left(\sum_{i=0}^{m} \mu_i \alpha_i F_i\right) \\
&\qquad \text{used Lemmas 6.1b and 6.1c} \\
&= \frac{1}{2}\sum_{i=0}^{m} \mu_i \alpha_i \operatorname{tr}(F_i) \\
&= \frac{1}{2}\mu_0 \alpha_0 \operatorname{tr}(F_0) + \frac{1}{2}\sum_{i=1}^{m} \mu_i \alpha_i \operatorname{tr}(F_i) \\
&\leq \frac{1}{2}\mu_{\max}(L)\sum_{i=1}^{m} \alpha_i \operatorname{tr}(F_i) \\
&\qquad \text{since } \mu_0 = 0 \text{ and } \alpha_j \geq 0 \text{ for } j > 0 \\
&\leq \frac{n(k-1)}{2k} \cdot \mu_{\max}(L).
\end{aligned}
$$

□

To summarize the previous results, we write

$$\mathrm{OPT}_{\text{max-}k\text{-cut}} = \mathrm{OPT}_{(\mathrm{IP1})} \leq \mathrm{OPT}_{(\mathrm{SDP1})} \leq \mathrm{OPT}_{(\mathrm{SDP2})} = \frac{n(k-1)}{2k} \cdot \mu_{\max}(L). \tag{6.11}$$

Thus, we have found a new upper bound for the max-k-cut problem using the largest Laplacian eigenvalue of a graph.

The paper of van Dam and Sotirov notes that for some graphs, some of the inequalities in (6.11) are actually equalities. For two of these mentioned graph types, namely graphs from the Hamming scheme and walk-regular graphs, we want to highlight some results. First, we look at graphs from the Hamming scheme:

Theorem 6.4. *[30] Let $G = H(d, q, j)$ be a graph from the Hamming scheme and let* (SDP1) *and* (SDP2) *be the above mentioned relaxations of the max-k-cut problem. If $K_j(1)$ is the smallest eigenvalue of the adjacency matrix of G, then $OPT_{(\mathrm{SDP1})} = OPT_{(\mathrm{SDP2})}$.*

For the proof of Theorem 6.4, see [30]. Note that since graphs from the Hamming scheme are regular and the valency is given by $K_j(0)$, we have $\mathrm{OPT}_{(\mathrm{SDP1})} = \mathrm{OPT}_{(\mathrm{SDP2})} = \frac{n(k-1)}{2k} \cdot (K_j(0) -$

$K_j(1)$) when $K_j(1)$ is indeed the smallest eigenvalue. From Corollary 4.13 and Theorem 4.14b, we know that $K_j(1)$ is the smallest eigenvalue for $j \geq d - \frac{d-1}{q}$, with the additional condition that j even or $j = d$ when $q = 2$. We summarize this in the following corollary.

Corollary 6.5. *Let $G = H(d,q,j)$ and let* (SDP1) *and* (SDP2) *be the above mentioned relaxations of the max-k-cut problem. Moreover, let $j \geq d - \frac{d-1}{q}$, with the additional condition that j even or $j = d$ when $q = 2$. Then $OPT_{(\mathrm{SDP1})} = OPT_{(\mathrm{SDP2})} = \frac{n(k-1)}{2k} \cdot (K_j(0) - K_j(1))$.*

Proof. Note that $d - \frac{d-1}{q} = \frac{d+1}{2}$ for $q = 2$. By Corollary 4.13 and Theorem 4.14b, we have that $K_j(1)$ is the smallest eigenvalue for $j \geq d - \frac{d-1}{q}$, with the additional condition that j even or $j = d$ when $q = 2$. ☐

When $k = q$, we have an extra result:

Theorem 6.6. *[30] Let $G = H(d,q,j)$ be a graph from the Hamming scheme and let $k = q$ in the max-k-cut problem. If $K_j(1)$ is the smallest eigenvalue of the adjacency matrix of G, then $OPT_{\textit{max-q-cut}} = \frac{n(q-1)}{2q} \cdot (K_j(0) - K_j(1))$.*

Proof.* We already know from (6.11) that for the max-q-cut problem on the graph $H(d,q,j)$, we have $\mathrm{OPT}_{\text{max-}q\text{-cut}} \leq \frac{n(q-1)}{2q} \cdot (K_j(0) - K_j(1))$. Thus, we are done if we manage to find a vertex partition that results in $\frac{n(q-1)}{2q} \cdot (K_j(0) - K_j(1))$ edges in between parts.

Let $V = \{1, \ldots, q\}^d$, so $n = |V| = q^d$. We need to construct a partition $V = V_1 \cup \cdots \cup V_q$ as follows. Put vertex $v \in V$ in part V_l if and only if the first coordinate of v equals l. This way, the number of vertices from some vertex v to a vertex in another part is $\binom{d-1}{j-1}(q-1)^{j-1}$. Moreover, the number of vertices in a part is q^{d-1} and the number of ways to select two out of q parts is $\binom{q}{2}$. Thus, the total number of edges in between parts is

$$\begin{aligned}
\binom{q}{2} q^{d-1} \binom{d-1}{j-1} (q-1)^{j-1} &= \frac{q(q-1)}{2} \frac{n}{q} \binom{d-1}{j-1} (q-1)^{j-1} \\
&= \frac{n(q-1)}{2q} q \left(\frac{1}{q} \binom{d}{j} (q-1)^j - \left(\frac{1}{q} \binom{d}{j} (q-1)^j - (q-1)^{j-1} \binom{d-1}{j-1} \right) \right) \\
&= \frac{n(q-1)}{2q} \left(\binom{d}{j} (q-1)^j - \left(\binom{d}{j} (q-1)^j - q(q-1)^{j-1} \binom{d-1}{j-1} \right) \right) \\
&= \frac{n(q-1)}{2q} \left(K_j(0) - K_j(1) \right). \\
&\quad \text{used formula (4.1b)}
\end{aligned}$$

☐

Example. Consider the graph $G = H(3,2,2)$ with $d = 3$, $q = 2$ and $j = 2$, which can be seen in Figure 6.1 on the left. We have $j \geq \frac{d+1}{2} = \frac{3+1}{2}$ and j even, so from Corollary 4.13 we know that $K_j(1)$ is the smallest eigenvalue. This means that Theorem 6.6 can be applied to the max-2-cut problem on G, meaning that if we partition the vertex set V of G in two disjunct parts, namely V_1 and V_2, that the maximum number of edges between parts is equal to $\frac{n(q-1)}{2q} \cdot (K_j(0) - K_j(1))$.

From Section 2.3 we know that

$$n = |V| = q^d = 2^3 = 8 \quad \text{and} \quad K_j(0) = (q-1)^j \binom{d}{j} = (2-1)^2 \binom{3}{2} = 3.$$

Moreover, from formula (4.1a) we know

$$K_j(1) = (q-1)^j \binom{d-1}{j} - (q-1)^{j-1} \binom{d-1}{j-1} = (2-1)^2 \binom{3-1}{2} - (2-1)^{2-1} \binom{3-1}{2-1} = -1.$$

This means that the maximum number of edges between parts should be

$$\frac{n(q-1)}{2q}\cdot(K_j(0)-K_j(1))=\frac{8(2-1)}{2\cdot 2}(3-(-1))=8.$$

Following the proof of Theorem 6.6, we divide the vertices based on their first coordinate. All vertices that have first coordinate 0 are in V_1 and all vertices that have first coordinate 1 are in V_2. This is shown in Figure 6.1 on the right. The figure also shows that the number of edges in between parts, which are colored in red, is equal to 8, which was the maximum number of edges between parts according to Theorem 6.6.

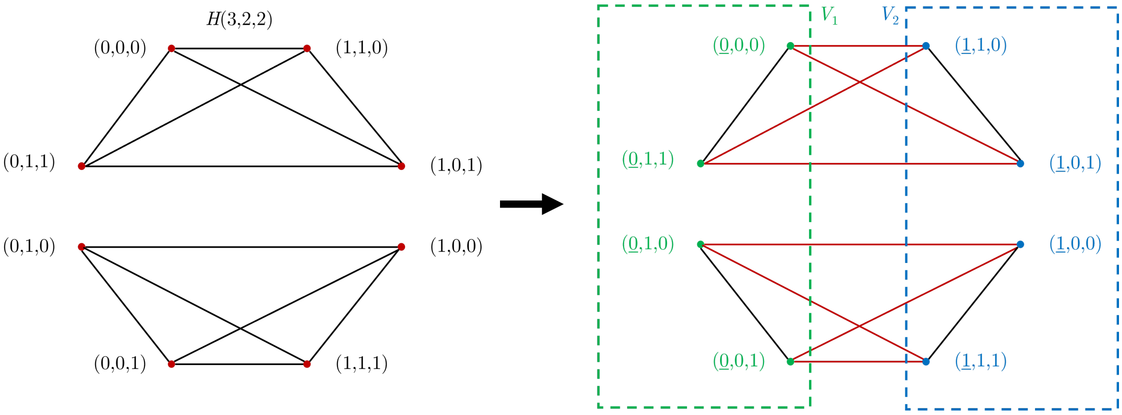


Figure 6.1: *Left:* The graph $H(3,2,2)$. *Right:* The same graph, but vertices are divided into two parts. All green vertices are in V_1 and all blue vertices are in V_2. The edges between parts are colored in red.

Now we move on to another type of graphs where an equality arises in (6.11), namely *walk-regular graphs*. These are graphs for which the number of walks of length l from a vertex to itself does not depend on the chosen vertex, but only on l [13]. Note that all distance-regular graphs are walk-regular, so the statement below also holds for distance-regular graphs. For $k = 2$, we have the following statement.

Theorem 6.7. *[30] Let G be a walk-regular graph. Then for $k = 2$ we have $OPT_{(\text{SDP1})} = OPT_{(\text{SDP2})}$.*

For the proof, see [30].

We end this section with another result using the bound derived in Theorem 6.3. Note that the chromatic number of a graph, denoted by $\chi(G)$, is the smallest number of colors needed to color the vertices of a graph such that the colors of any two adjacent vertices are different. If it is possible to find a partition $V = V_1 \cup \cdots \cup V_k$ such that $\text{OPT}_{\text{max-}k\text{-cut}} = |E|$, then this partition corresponds to a feasible coloring of G using k colors. That is, where there every part V_i for $1 \le i \le k$ corresponds to a different color. In this case we have $\chi(G) \le k$. If it is not possible to find a k-partition that results in $\text{OPT}_{\text{max-}k\text{-cut}} = |E|$, then $\chi(G) > k$. Thus, given a graph $G(V, E)$ and an integer k, we have

$$\begin{aligned}&\text{if } \text{OPT}_{\text{max-}k\text{-cut}} = |E|, \quad \text{then} \quad \chi(G) \le k,\\ &\text{if } \text{OPT}_{\text{max-}k\text{-cut}} < |E|, \quad \text{then} \quad \chi(G) \ge k+1.\end{aligned}$$

Using this reasoning, we get to the following theorem. Recall that $n = |V|$. Moreover, note that G in the following theorem is not necessarily regular.

Theorem 6.8. *[30] Let $G = (V, E)$ be a graph with Laplacian matrix L. Then*

$$\chi(G) \geq 1 + \frac{2|E|}{n\mu_{\max}(L) - 2|E|}.$$

Proof.* Let $k = \left\lceil \frac{2|E|}{n\mu_{\max}(L)-2|E|} \right\rceil$. Then

$$k < 1 + \frac{2|E|}{n\mu_{\max}(L) - 2|E|} \Leftrightarrow kn\mu_{\max}(L) - 2k|E| < n\mu_{\max}(L) \Leftrightarrow \frac{(k-1)n}{2k}\mu_{\max}(L) < |E|,$$

thus by (6.11), we have $\mathrm{OPT}_{\text{max-}k\text{-cut}} < |E|$, and thus $\chi(G) \geq k + 1 \geq 1 + \frac{2|E|}{n\mu_{\max}(L)-2|E|}$. ☐

Note that for regular graphs with valency s, we have $s = \frac{2|E|}{n}$, $L = sI - A$ and thus

$$\mu_{\max}(L) = s - \lambda_{\min}(A) = \frac{2|E|}{n} - \lambda_{\min}(A).$$

This means that the bound on the chromatic number can be simplified to

$$\chi(G) \geq 1 + \frac{2|E|}{n\mu_{\max}(L) - 2|E|} = 1 + \frac{sn}{n\lambda_{\min}(A)} = 1 + \frac{s}{\lambda_{\min}(A)},$$

which corresponds to the well-known Hoffman bound [14] on the chromatic number of a regular graph.

Chapter 7

Conclusion

In this thesis, we looked at the eigenvalues of the adjacency matrix of graphs from the Hamming and the Johnson scheme. Chapter 2 includes expressions for the eigenvalues $K_j(i)$ of graphs from the Hamming scheme and $E_j(i)$ of graphs from the Johnson scheme, which were provided in Theorems 2.13 and 2.16 respectively. The graphs we consider are all regular and we know that the eigenvalues $K_j(0)$ and $E_j(0)$ are equal to the graph valency. Thus, we know from Theorem 2.9 that these eigenvalues are the largest, also in absolute value. The goal of this thesis was to find out for which i the eigenvalues $K_j(i)$ and $E_j(i)$ are smallest or penabsolute, without having to calculate them for all $1 \leq i \leq d$. Recall that with 'penabsolute' we mean 'second largest in absolute value'.

The paper [3] contains several theorems on this topic, which were discussed in detail in Chapter 4. Chapter 5 discusses some new results and observations, including a new theorem on the penabsolute eigenvalue of graphs from the Hamming scheme (Thm. 5.5). The tables on the next page summarize the most important results that were discussed in Chapters 4 and 5. The second to last column in the tables mentions the theorems in which the aforementioned result can be found. Note that the results in red are new conjectures, for which no proof was found in this thesis. However, these conjectures were thoroughly computationally tested, as can be seen in the test results after the statement of every conjecture in Chapter 5. The statements in green are new results that were proven in this thesis. The last column of the two tables on the next page refers to a figure where the aforementioned result is illustrated using some examples.

It would not have been possible to formulate these new results and observations without the visualization tool that was discussed in Chapter 3. This tool allowed us to illustrate all relevant information from the so-called P-matrix of a Hamming or Johnson scheme. Several examples of these visualizations can be found in Chapters 4 and 5.

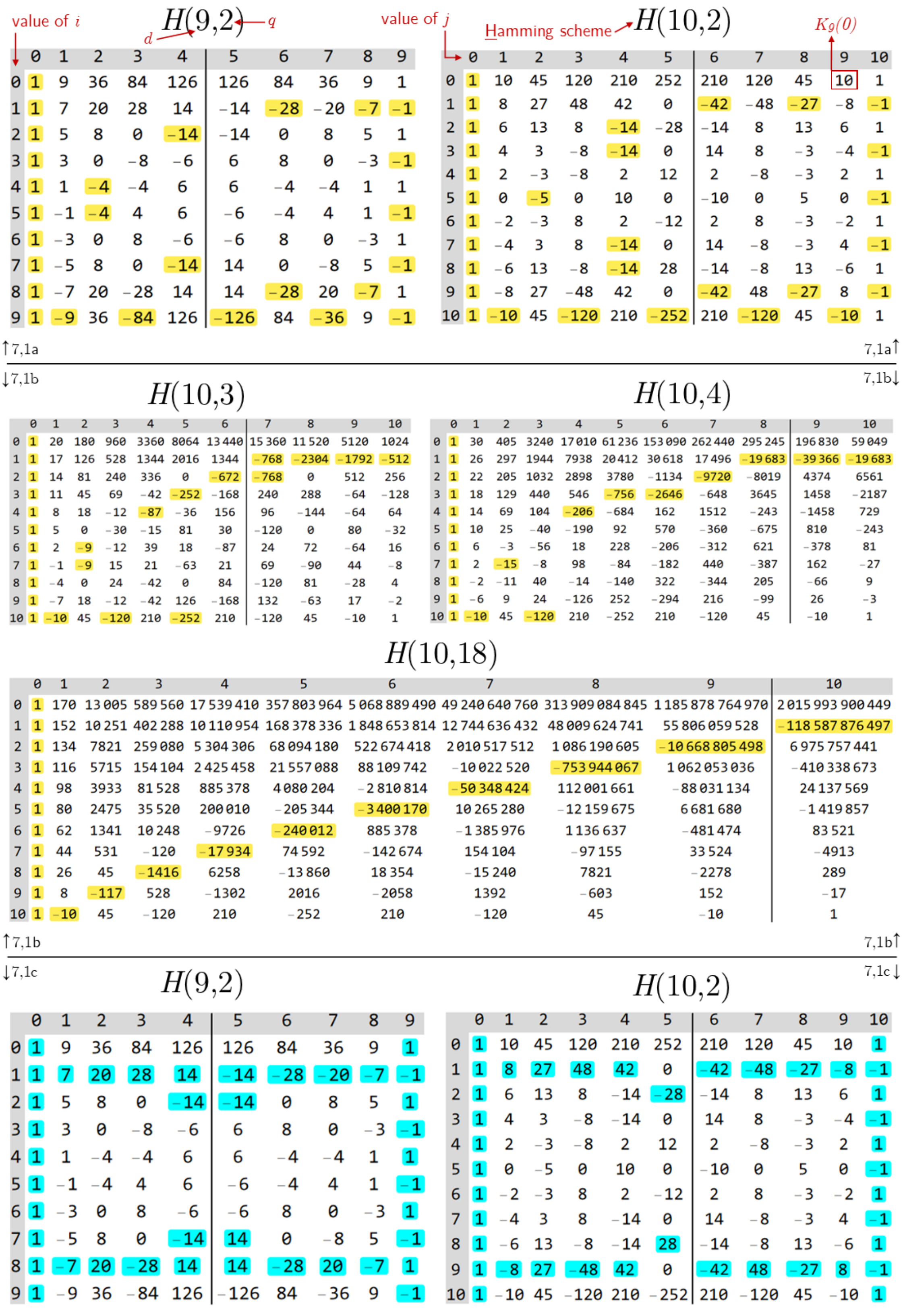
Chapter 6 focuses on one of the applications of the results investigated in this thesis, namely the application to the max-k-cut problem. In particular, we followed the paper [30] to show new bounds on the max-k-cut problem using the smallest eigenvalue of the adjacency matrix of a graph. Special attention is given to distance-regular graphs and graphs from the Hamming scheme.

The Hamming case

Smallest	$q = 2$	• For j odd, $K_j(d)$ is the smallest eigenvalue, • For $j \geq \frac{d+1}{2}$, $K_j(1)$ is the smallest eigenvalue for j even or $j = d$, • For $0 < j < \frac{d+1}{2}$, $K_j(1)$ is not the smallest eigenvalue.	Prop. 5.1 Cor. 4.13 Conj. 5.2	Fig. 7.1a
	$q \geq 3$	• For $j \geq d - \frac{d-1}{q}$, $K_j(1)$ is the smallest eigenvalue, • For $j < d - \frac{d-1}{q}$, $K_j(1)$ is not the smallest eigenvalue.	Thm. 4.14b Conj. 5.4	Fig. 7.1b
Penabsolute	$q = 2$	• Eigenvalues $\lvert K_j(0)\rvert = \lvert K_j(d)\rvert$ are the largest in absolute value, • For $j \neq \frac{d}{2}$, $\lvert K_j(1)\rvert = \lvert K_j(d-1)\rvert$ is the penabsolute eigenvalue, • For $j = \frac{d}{2}$, $\lvert K_j(2)\rvert = \lvert K_j(d-2)\rvert$ is the penabsolute eigenvalue, • If d odd, then $\lvert K_j(d-1)\rvert = \lvert K_j(1)\rvert = \lvert K_j(2)\rvert = \lvert K_j(d-2)\rvert$ for $j = \frac{d-1}{2}$ and $j = \frac{d+1}{2}$.	Thm. 2.9, Prop. 4.12a,b Prop. 4.12a Prop. 4.12b Prop. 4.12c	Fig. 7.1c
	$q \geq 3$	• Eigenvalue $\lvert K_j(0)\rvert$ is largest in absolute value, • For $j \neq \lceil d - \frac{d-1}{q} \rceil - 1$, $\lvert K_j(1)\rvert$ is the penabsolute eigenvalue, except for $(q, d, j) = (3, 4, 3)$, for which $\lvert K_j(3)\rvert$ is the penabsolute eigenvalue, • For $j = \lceil d - \frac{d-1}{q} \rceil - 1$, $\lvert K_j(2)\rvert$ is the penabsolute eigenvalue if $d = 0 \pmod{q}$ or $d = q - 1 \pmod{q}$, and $\lvert K_j(1)\rvert$ is the penabsolute eigenvalue if $d \neq 0 \pmod{q}$ and $d \neq q - 1 \pmod{q}$.	Thm. 2.9 Thm. 4.14a, Thm. 5.5 Thm. 5.5	Fig. 7.2a

The Johnson case

Smallest	• Eigenvalue $E_j(1)$ is smallest if and only if $j \geq \frac{d(n-d)}{n-1}$.	Thm. 4.32b	Fig. 7.2b
Penabsolute	• For $j \geq \frac{d(n-d)}{n-1}$, $\lvert E_j(1)\rvert$ is the penabsolute eigenvalue, • For $j = \left\lceil \frac{d(n-d)}{n-1} \right\rceil - 1$, $\lvert E_j(1)\rvert$ or $\lvert E_j(2)\rvert$ is the penabsolute eigenvalue, • For $j < \left\lceil \frac{d(n-d)}{n-1} \right\rceil - 1$, $\lvert E_j(1)\rvert$ is the penabsolute eigenvalue.	Thm. 4.32a Conj. 5.17 Conj. 5.18	Fig. 7.2c



$H(9,2)$

	0	1	2	3	4	5	6	7	8	9
0	**1**	9	36	84	126	126	84	36	9	1
1	**1**	7	20	28	14	-14	**-28**	-20	**-7**	**-1**
2	**1**	5	8	0	**-14**	-14	0	8	5	1
3	**1**	3	0	-8	-6	6	8	0	-3	**-1**
4	**1**	1	**-4**	-4	6	6	-4	-4	1	1
5	**1**	-1	**-4**	4	6	-6	-4	4	1	**-1**
6	**1**	-3	0	8	-6	-6	8	0	-3	1
7	**1**	-5	8	0	**-14**	14	0	-8	5	**-1**
8	**1**	-7	20	-28	14	14	**-28**	20	**-7**	1
9	**1**	**-9**	36	**-84**	126	**-126**	84	**-36**	9	**-1**

$H(10,2)$

	0	1	2	3	4	5	6	7	8	9	10
0	**1**	10	45	120	210	252	210	120	45	10	1
1	**1**	8	27	48	42	0	**-42**	-48	**-27**	-8	**-1**
2	**1**	6	13	8	**-14**	-28	-14	8	13	6	1
3	**1**	4	3	-8	**-14**	0	14	8	-3	-4	**-1**
4	**1**	2	-3	-8	2	12	2	-8	-3	2	1
5	**1**	0	**-5**	0	10	0	-10	0	5	0	**-1**
6	**1**	-2	-3	8	2	-12	2	8	-3	-2	1
7	**1**	-4	3	8	**-14**	0	14	-8	-3	4	**-1**
8	**1**	-6	13	-8	**-14**	28	-14	-8	13	-6	1
9	**1**	-8	27	-48	42	0	**-42**	48	**-27**	8	**-1**
10	**1**	**-10**	45	**-120**	210	**-252**	210	**-120**	45	**-10**	1

$H(10,3)$

	0	1	2	3	4	5	6	7	8	9	10
0	**1**	20	180	960	3360	8064	13 440	15 360	11 520	5120	1024
1	**1**	17	126	528	1344	2016	1344	**-768**	**-2304**	**-1792**	**-512**
2	**1**	14	81	240	336	0	**-672**	**-768**	0	512	256
3	**1**	11	45	69	-42	**-252**	-168	240	288	-64	-128
4	**1**	8	18	-12	**-87**	-36	156	96	-144	-64	64
5	**1**	5	0	-30	-15	81	30	-120	0	80	-32
6	**1**	2	**-9**	-12	39	18	-87	24	72	-64	16
7	**1**	-1	**-9**	15	21	-63	21	69	-90	44	-8
8	**1**	-4	0	24	-42	0	84	-120	81	-28	4
9	**1**	-7	18	-12	-42	126	-168	132	-63	17	-2
10	**1**	**-10**	45	**-120**	210	**-252**	210	-120	45	-10	1

$H(10,4)$

	0	1	2	3	4	5	6	7	8	9	10
0	**1**	30	405	3240	17 010	61 236	153 090	262 440	295 245	196 830	59 049
1	**1**	26	297	1944	7938	20 412	30 618	17 496	**-19 683**	**-39 366**	**-19 683**
2	**1**	22	205	1032	2898	3780	-1134	**-9720**	-8019	4374	6561
3	**1**	18	129	440	546	**-756**	**-2646**	-648	3645	1458	-2187
4	**1**	14	69	104	**-206**	-684	162	1512	-243	-1458	729
5	**1**	10	25	-40	-190	92	570	-360	-675	810	-243
6	**1**	6	-3	-56	18	228	-206	-312	621	-378	81
7	**1**	2	**-15**	-8	98	-84	-182	440	-387	162	-27
8	**1**	-2	-11	40	-14	-140	322	-344	205	-66	9
9	**1**	-6	9	24	-126	252	-294	216	-99	26	-3
10	**1**	**-10**	45	**-120**	210	-252	210	-120	45	-10	1

$H(10,18)$

	0	1	2	3	4	5	6	7	8	9	10
0	**1**	170	13 005	589 560	17 539 410	357 803 964	5 068 889 490	49 240 640 760	313 909 084 845	1 185 878 764 970	2 015 993 900 449
1	**1**	152	10 251	402 288	10 110 954	168 378 336	1 848 653 814	12 744 636 432	48 009 624 741	55 806 059 528	**-118 587 876 497**
2	**1**	134	7821	259 080	5 304 306	68 094 180	522 674 418	2 010 517 512	1 086 190 605	**-10 668 805 498**	6 975 757 441
3	**1**	116	5715	154 104	2 425 458	21 557 088	88 109 742	-10 022 520	**-753 944 067**	1 062 053 036	-410 338 673
4	**1**	98	3933	81 528	885 378	4 080 204	-2 810 814	**-50 348 424**	112 001 661	-88 031 134	24 137 569
5	**1**	80	2475	35 520	200 010	-205 344	**-3 400 170**	10 265 280	-12 159 675	6 681 680	-1 419 857
6	**1**	62	1341	10 248	-9726	**-240 012**	885 378	-1 385 976	1 136 637	-481 474	83 521
7	**1**	44	531	-120	**-17 934**	74 592	-142 674	154 104	-97 155	33 524	-4913
8	**1**	26	45	**-1416**	6258	-13 860	18 354	-15 240	7821	-2278	289
9	**1**	8	**-117**	528	-1302	2016	-2058	1392	-603	152	-17
10	**1**	**-10**	45	-120	210	-252	210	-120	45	-10	1

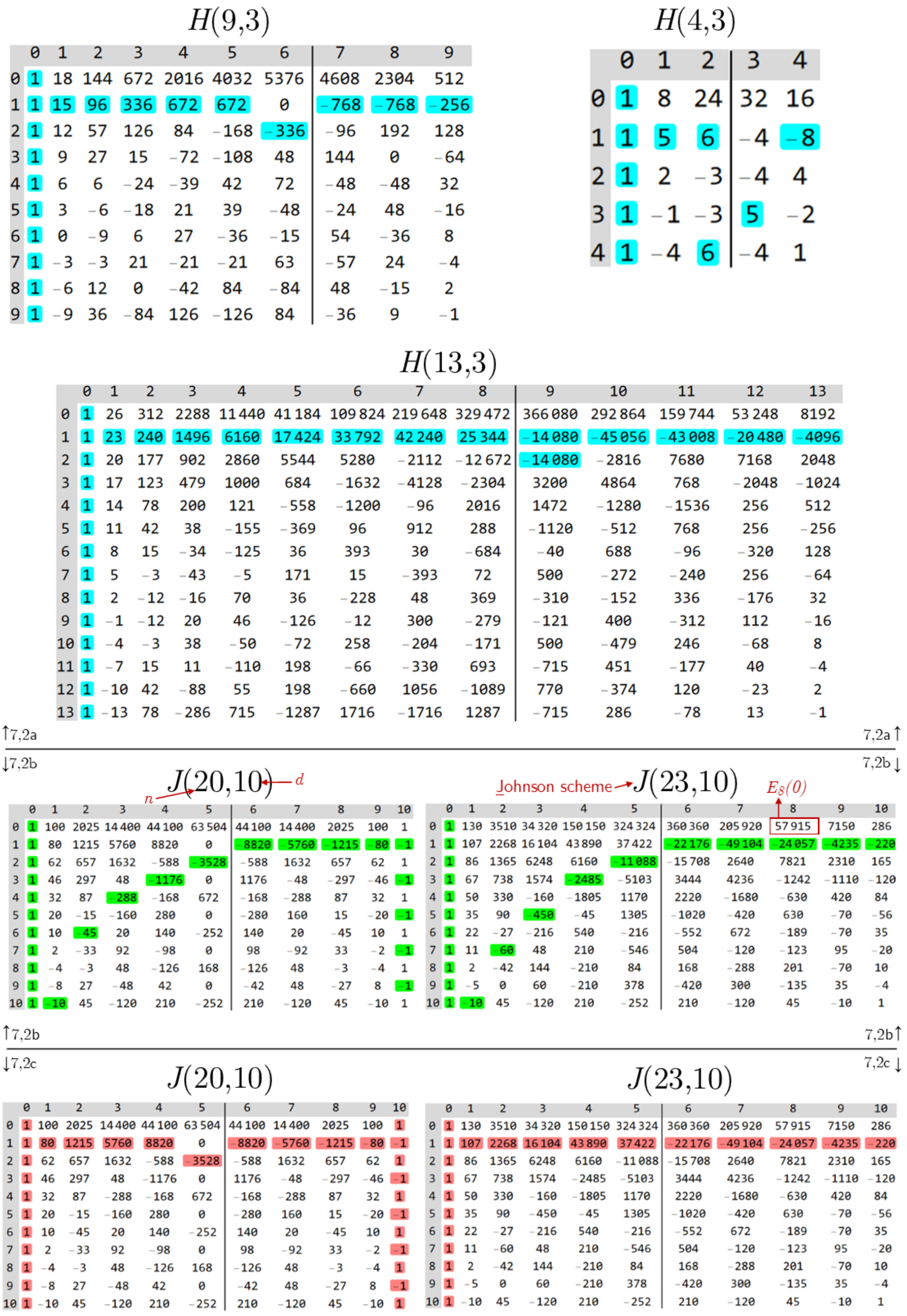
$H(9,2)$

	0	1	2	3	4	5	6	7	8	9
0	**1**	9	36	84	126	126	84	36	9	**1**
1	**1**	**7**	**20**	**28**	**14**	**-14**	**-28**	**-20**	**-7**	**-1**
2	**1**	5	8	0	**-14**	**-14**	0	8	5	**1**
3	**1**	3	0	-8	-6	6	8	0	-3	**-1**
4	**1**	1	-4	-4	6	6	-4	-4	1	**1**
5	**1**	-1	-4	4	6	-6	-4	4	1	**-1**
6	**1**	-3	0	8	-6	-6	8	0	-3	**1**
7	**1**	-5	8	0	**-14**	**14**	0	-8	5	**-1**
8	**1**	**-7**	**20**	**-28**	**14**	**14**	**-28**	**20**	**-7**	**1**
9	**1**	-9	36	-84	126	-126	84	-36	9	**-1**

$H(10,2)$

	0	1	2	3	4	5	6	7	8	9	10
0	**1**	10	45	120	210	252	210	120	45	10	**1**
1	**1**	**8**	**27**	**48**	**42**	0	**-42**	**-48**	**-27**	**-8**	**-1**
2	**1**	6	13	8	-14	**-28**	-14	8	13	6	**1**
3	**1**	4	3	-8	-14	0	14	8	-3	-4	**-1**
4	**1**	2	-3	-8	2	12	2	-8	-3	2	**1**
5	**1**	0	-5	0	10	0	-10	0	5	0	**-1**
6	**1**	-2	-3	8	2	-12	2	8	-3	-2	**1**
7	**1**	-4	3	8	-14	0	14	-8	-3	4	**-1**
8	**1**	-6	13	-8	-14	**28**	-14	-8	13	-6	**1**
9	**1**	**-8**	**27**	**-48**	**42**	0	**-42**	**48**	**-27**	**8**	**-1**
10	**1**	-10	45	-120	210	-252	210	-120	45	-10	**1**

Figure 7.1: Some P-matrices of Hamming schemes. The highlighted boxes indicate the smallest eigenvalues per column for 7.1a,b (top, middle) and the penabsolute eigenvalues per column for 7.1c (bottom).



$H(9,3)$

	0	1	2	3	4	5	6	7	8	9
0	**1**	18	144	672	2016	4032	5376	4608	2304	512
1	**1**	**15**	**96**	**336**	**672**	**672**	0	**-768**	**-768**	**-256**
2	**1**	12	57	126	84	-168	**-336**	-96	192	128
3	**1**	9	27	15	-72	-108	48	144	0	-64
4	**1**	6	6	-24	-39	42	72	-48	-48	32
5	**1**	3	-6	-18	21	39	-48	-24	48	-16
6	**1**	0	-9	6	27	-36	-15	54	-36	8
7	**1**	-3	-3	21	-21	-21	63	-57	24	-4
8	**1**	-6	12	0	-42	84	-84	48	-15	2
9	**1**	-9	36	-84	126	-126	84	-36	9	-1

$H(4,3)$

	0	1	2	3	4
0	**1**	8	24	32	16
1	**1**	**5**	**6**	-4	**-8**
2	**1**	2	-3	-4	4
3	**1**	-1	-3	**5**	-2
4	**1**	-4	**6**	-4	1

$H(13,3)$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	**1**	26	312	2288	11 440	41 184	109 824	219 648	329 472	366 080	292 864	159 744	53 248	8192
1	**1**	**23**	**240**	**1496**	**6160**	**17 424**	**33 792**	**42 240**	**25 344**	**-14 080**	**-45 056**	**-43 008**	**-20 480**	**-4096**
2	**1**	20	177	902	2860	5544	5280	-2112	-12 672	**-14 080**	-2816	7680	7168	2048
3	**1**	17	123	479	1000	684	-1632	-4128	-2304	3200	4864	768	-2048	-1024
4	**1**	14	78	200	121	-558	-1200	-96	2016	1472	-1280	-1536	256	512
5	**1**	11	42	38	-155	-369	96	912	288	-1120	-512	768	256	-256
6	**1**	8	15	-34	-125	36	393	30	-684	-40	688	-96	-320	128
7	**1**	5	-3	-43	-5	171	15	-393	72	500	-272	-240	256	-64
8	**1**	2	-12	-16	70	36	-228	48	369	-310	-152	336	-176	32
9	**1**	-1	-12	20	46	-126	-12	300	-279	-121	400	-312	112	-16
10	**1**	-4	-3	38	-50	-72	258	-204	-171	500	-479	246	-68	8
11	**1**	-7	15	11	-110	198	-66	-330	693	-715	451	-177	40	-4
12	**1**	-10	42	-88	55	198	-660	1056	-1089	770	-374	120	-23	2
13	**1**	-13	78	-286	715	-1287	1716	-1716	1287	-715	286	-78	13	-1

$J(20,10)$

	0	1	2	3	4	5	6	7	8	9	10
0	**1**	100	2025	14 400	44 100	63 504	44 100	14 400	2025	100	1
1	**1**	80	1215	5760	8820	0	**-8820**	**-5760**	**-1215**	**-80**	**-1**
2	**1**	62	657	1632	-588	**-3528**	-588	1632	657	62	1
3	**1**	46	297	48	**-1176**	0	1176	-48	-297	-46	**-1**
4	**1**	32	87	**-288**	-168	672	-168	-288	87	32	1
5	**1**	20	-15	-160	280	0	-280	160	15	-20	**-1**
6	**1**	10	**-45**	20	140	-252	140	20	-45	10	1
7	**1**	2	-33	92	-98	0	98	-92	33	-2	**-1**
8	**1**	-4	-3	48	-126	168	-126	48	-3	-4	1
9	**1**	-8	27	-48	42	0	-42	48	-27	8	**-1**
10	**1**	**-10**	45	-120	210	-252	210	-120	45	-10	1

$J(23,10)$

	0	1	2	3	4	5	6	7	8	9	10
0	**1**	130	3510	34 320	150 150	324 324	360 360	205 920	57 915	7150	286
1	**1**	107	2268	16 104	43 890	37 422	**-22 176**	**-49 104**	**-24 057**	**-4235**	**-220**
2	**1**	86	1365	6248	6160	**-11 088**	-15 708	2640	7821	2310	165
3	**1**	67	738	1574	**-2485**	-5103	3444	4236	-1242	-1110	-120
4	**1**	50	330	-160	-1805	1170	2220	-1680	-630	420	84
5	**1**	35	90	**-450**	-45	1305	-1020	-420	630	-70	-56
6	**1**	22	-27	-216	540	-216	-552	672	-189	-70	35
7	**1**	11	**-60**	48	210	-546	504	-120	-123	95	-20
8	**1**	2	-42	144	-210	84	168	-288	201	-70	10
9	**1**	-5	0	60	-210	378	-420	300	-135	35	-4
10	**1**	**-10**	45	-120	210	-252	210	-120	45	-10	1

$J(20,10)$

	0	1	2	3	4	5	6	7	8	9	10
0	**1**	100	2025	14 400	44 100	63 504	44 100	14 400	2025	100	**1**
1	**1**	**80**	**1215**	**5760**	**8820**	0	**-8820**	**-5760**	**-1215**	**-80**	**-1**
2	**1**	62	657	1632	-588	**-3528**	-588	1632	657	62	**1**
3	**1**	46	297	48	-1176	0	1176	-48	-297	-46	**-1**
4	**1**	32	87	-288	-168	672	-168	-288	87	32	**1**
5	**1**	20	-15	-160	280	0	-280	160	15	-20	**-1**
6	**1**	10	-45	20	140	-252	140	20	-45	10	**1**
7	**1**	2	-33	92	-98	0	98	-92	33	-2	**-1**
8	**1**	-4	-3	48	-126	168	-126	48	-3	-4	**1**
9	**1**	-8	27	-48	42	0	-42	48	-27	8	**-1**
10	**1**	-10	45	-120	210	-252	210	-120	45	-10	**1**

$J(23,10)$

	0	1	2	3	4	5	6	7	8	9	10
0	**1**	130	3510	34 320	150 150	324 324	360 360	205 920	57 915	7150	286
1	**1**	**107**	**2268**	**16 104**	**43 890**	**37 422**	**-22 176**	**-49 104**	**-24 057**	**-4235**	**-220**
2	**1**	86	1365	6248	6160	-11 088	-15 708	2640	7821	2310	165
3	**1**	67	738	1574	-2485	-5103	3444	4236	-1242	-1110	-120
4	**1**	50	330	-160	-1805	1170	2220	-1680	-630	420	84
5	**1**	35	90	-450	-45	1305	-1020	-420	630	-70	-56
6	**1**	22	-27	-216	540	-216	-552	672	-189	-70	35
7	**1**	11	-60	48	210	-546	504	-120	-123	95	-20
8	**1**	2	-42	144	-210	84	168	-288	201	-70	10
9	**1**	-5	0	60	-210	378	-420	300	-135	35	-4
10	**1**	-10	45	-120	210	-252	210	-120	45	-10	1

Figure 7.2: Some P-matrices of Hamming and Johnson schemes. The highlighted boxes indicate the penabsolute eigenvalues per column for 7.2a,c (top, bottom) and the smallest eigenvalues per column for 7.2b (middle).

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Appendix A

Code for the calculation and visualization of P-matrices

For this thesis, I used Wolfram Mathematica to write a package of functions that are useful for calculating eigenvalues and visualizing P-matrices. Below, I will first list all functions and their output, after which I will share the code of the package. Note that in the table, $\{c_1, c_2, \dots\}$ is used to indicate a list of integers where $0 < c_i \leq d$ for Hamming and $0 < c_i \leq n$ for Johnson.

Function	**Input**	**Output**
`CalcEigH`	d, q, j, i	The eigenvalue $K_j(i)$ of the Hamming graph $H(d, q, j)$.
`CalcEigJ`	n, d, j, i	The eigenvalue $E_j(i)$ of the Johnson graph $J(n, d, j)$.
`EigH`	d, q	The P-matrix of the Hamming scheme $H(d, q)$.
`EigJ`	n, d	The P-matrix of the Johnson scheme $J(n, d)$.
`EigHfx`	d, q, j	List of eigenvalues of $H(d, q, j)$.
`EigJfx`	n, d, j	List of eigenvalues of $J(n, d, j)$.
`EigHa`	d, q	The P-matrix of $H(d, q)$ where the penabsolute eigenvalue(s) per column are highlighted.
`EigHs`	d, q	The P-matrix of $H(d, q)$ where the smallest eigenvalue(s) per column are highlighted.
`EigJa`	n, d	The P-matrix of $J(n, d)$ where the penabsolute eigenvalue(s) per column are highlighted.
`EigJs`	n, d	The P-matrix of $J(n, d)$ where the smallest eigenvalue(s) per column are highlighted.
`EigHac`	d, q,$\{c_1, c_2, \dots\}$	Same as `EigHa`, but row and column headers are added, and a black vertical line is drawn between columns c_i and $c_i - 1$ for every c_i in the input list. This list may be empty.
`EigHsc`	d, q,$\{c_1, c_2, \dots\}$	Same as `EigHs`, but row and column headers are added, and a black vertical line is drawn between columns c_i and $c_i - 1$ for every c_i in the input list. This list may be empty.
`EigJac`	n, d,$\{c_1, c_2, \dots\}$	Same as `EigJa`, but row and column headers are added, and a black vertical line is drawn between columns c_i and $c_i - 1$ for every c_i in the input list. This list may be empty.
`EigJsc`	n, d,$\{c_1, c_2, \dots\}$	Same as `EigJs`, but row and column headers are added, and a black vertical line is drawn between columns c_i and $c_i - 1$ for every c_i in the input list. This list may be empty.

`DefjH`	d, q	The ceiling of $d - \frac{d-1}{q}$.
`DefjJ`	n, d	The ceiling of $\frac{d(n-d)}{n-1}$.
`HsmX`	$d, q, \{c_1, c_2, \dots\}$	Same as `EigHsc`, but all (highlighted) numbers are replaced with (highlighted) dots, and the value `DefjH[d,q]` is automatically added to the input list. The vertical line for `DefjH[d,q]` will be drawn in black, while the others will be red.
`HabX`	$d, q, \{c_1, c_2, \dots\}$	Same as `EigHac`, but all (highlighted) numbers are replaced with (highlighted) dots, and the value `DefjH[d,q]` is automatically added to the input list. The vertical line for `DefjH[d,q]` will be drawn in black, while the others will be red.
`JsmX`	$n, d, \{c_1, c_2, \dots\}$	Same as `EigJsc`, but all (highlighted) numbers are replaced with (highlighted) dots, and the value `DefjJ[n,d]` is automatically added to the input list. The vertical line for `DefjJ[n,d]` will be drawn in black, while the others will be red.
`JabX`	$n, d, \{c_1, c_2, \dots\}$	Same as `EigJac`, but all (highlighted) numbers are replaced with (highlighted) dots, and the value `DefjJ[n,d]` is automatically added to the input list. The vertical line for `DefjJ[n,d]` will be drawn in black, while the others will be red.

The objects shown in Figures 3.1 and 3.2 of Chapter 3 were also created using these functions. The following table gives an overview of the functions used to create these objects.

Figure	**Position**	**Function**
3.1	left	`EigH[5,3]`
3.1	top middle	`EigHs[5,3]`
3.1	bottom middle	`EigHa[5,3]`
3.1	top right	`HsmX[5,3,{}]`
3.1	bottom right	`HabX[5,3,{}]`
3.2	left	`EigJ[9,4]`
3.2	top middle	`EigJs[9,4]`
3.2	bottom middle	`EigJa[9,4]`
3.2	top right	`JsmX[9,4,{}]`
3.2	bottom right	`JabX[9,4,{}]`

Finally, the code where all functions are defined is given below.

```
BeginPackage["ThesisPackage`CalcEigenvalues`"]
CalcEigH::usage="CalcEigH[d,q,j,i]. Calculate specific eigenvalue for Hamming graph."
CalcEigJ::usage="CalcEigJ[n,d,j,i]. Calculate specific eigenvalue for Johnson graph."

EigH::usage="EigH[d,q]. Calculate P-matrix for Hamming graph."
EigJ::usage="EigJ[n,d]. Calculate P-matrix for Johnson graph."

EigHfx::usage="EigHfx[d,q,j]. Calculate eigenvalues for H[d,q,j]."
EigJfx::usage="EigJfx[n,d,j]. Calculate eigenvalues for J[n,d,j]."

EigHa::usage="EigHa[d,q]. Calculate P-matrix for Hamming graph and highlight the second largest in abs value per column."
EigHs::usage="EigHs[d,q]. Calculate P-matrix for Hamming graph and highlight the smallest value per column."
```

```
EigJa::usage="EigJa[n,d]. Calculate P-matrix for Johnson graph and highlight the
↪  second largest in abs value per column."
EigJs::usage="EigJs[n,d]. Calculate P-matrix for Johnson graph and highlight the
↪  smallest value per column."

EigHac::usage="EigHac[d,q,{c,c,...}]. Calculate P-matrix for Hamming graph and
↪  highlight the second largest in abs value per column. Last input is list of
↪  integers for column dividers."
EigHsc::usage="EigHsc[d,q,{c,c,...}]. Calculate P-matrix for Hamming graph and
↪  highlight the smallest value per column. Last input is list of integers for
↪  column dividers."
EigJac::usage="EigJac[n,d,{c,c,...}]. Calculate P-matrix for Johnson graph and
↪  highlight the second largest in abs value per column. Last input is list of
↪  integers for column dividers."
EigJsc::usage="EigJsc[n,d,{c,c,...}]. Calculate P-matrix for Johnson graph and
↪  highlight the smallest value per column. Last input is list of integers for
↪  column dividers."

DefjH::usage="DefjH[d,q]. Ceiling of d-(d-1)/q, for which existing theorems hold
↪  for Hamming graphs."
DefjJ::usage="DefjJ[n,d]. Ceiling of d(n-d)/(n-1), for which existing theorems
↪  hold for Johnson graphs."

HsmX::usage="HsmX[d,q,{c,c,...}]. Calculate P-matrix for Hamming graph and
↪  highlight the smallest value per column. Numbers are replaced by dots for
↪  better overview. Last input is list of integers for column dividers."
HabX::usage="HabX[d,q,{c,c,...}]. Calculate P-matrix for Hamming graph and
↪  highlight the second largest in abs value per column. Numbers are replaced
↪  by dots for better overview. Last input is list of integers for column
↪  dividers."
JsmX::usage="JsmX[n,d,{c,c,...}]. Calculate P-matrix for Johnson graph and
↪  highlight the smallest value per column. Numbers are replaced by dots for
↪  better overview. Last input is list of integers for column dividers."
JabX::usage="JabX[n,d,{c,c,...}]. Calculate P-matrix for Johnson graph and
↪  highlight the second largest in abs value per column. Numbers are replaced
↪  by dots for better overview. Last input is list of integers for column
↪  dividers."

Begin["Private`"]
CalcEigH[d_,q_,j_,i_]:=Sum[(-1)^h * (q-1)^(j-h) *
↪  Binomial[i,h]*Binomial[d-i,j-h],{h,0,j}]

CalcEigJ[n_,d_,j_,i_]:=Sum[(-1)^{i-h}*Binomial[i,h]*
↪  Binomial[d-h,j]*Binomial[n-d-i+h,n-d-j],{h,0,i}]

EigH[d_,q_]:=(funcEigH[i_,j_]:=CalcEigH[d,q,j,i];
K=Array[funcEigH,{d+1,d+1},{0,0}];
MatrixForm[K])

EigJ[n_,d_]:=(funcEigJ[i_,j_]:=First[CalcEigJ[n,d,j,i]];
M=Array[funcEigJ,{d+1,d+1},{0,0}];
MatrixForm[M])

EigHfx[d_,q_,j_]:=(funcEigH2[i_]:=CalcEigH[d,q,j,i];
```

```
L=Array[funcEigH2,d+1,0];
MatrixForm[L])

EigJfx[n_,d_,j_]:=(funcEigJ2[i_]:=First[CalcEigJ[n,d,j,i]];
L2=Array[funcEigJ2,d+1,0];
MatrixForm[L2])

EigHs[d_,q_]:=(funcEigH[i_,j_]:=CalcEigH[d,q,j,i];
K=Transpose[Array[funcEigH,{d+1,d+1},{0,0}]];
For[i=1,i<=Length[K], i++,
        min = Min[K[[i]]]; (*find minimum of the column i*)
        pos = Position[K[[i]],min]; (*give list of position(s) in which this
 ↪  minimum appears *)
        K[[i]]=ReplacePart[K[[i]],pos->Highlighted[min, FrameMargins->1,
 ↪  ContentPadding->False]]; (*highlight these positions*)
        ];
MatrixForm[Transpose[K]])

EigHa[d_,q_]:=(funcEigH[i_,j_]:=CalcEigH[d,q,j,i];
K=Transpose[Array[funcEigH,{d+1,d+1},{0,0}]];
For[i=1,i<=Length[K], i++,
        If[Length[DeleteDuplicates[Abs[K[[i]]]]]!=1, (*first check if all values
 ↪  in column are equal in abs value*)
                min = DeleteDuplicates[Sort[Abs[K[[i]]]]][[-2]], (*if not, get
 ↪  the second largest in abs value*)
                min=Min[Abs[K[[i]]]]
        ];
        pos1=Position[K[[i]],-min]; (*give list of position(s) in which this
 ↪  value appears*)
        pos2=Position[K[[i]],min];
        K[[i]]=ReplacePart[K[[i]],pos1->Highlighted[-min, FrameMargins->1,
 ↪  ContentPadding->False, Background->Cyan]];(*highlight these positions*)
    K[[i]]=ReplacePart[K[[i]],pos2->Highlighted[min, FrameMargins->1,
 ↪  ContentPadding->False, Background->Cyan]];];
MatrixForm[Transpose[K]])

EigJs[n_,d_]:=(funcEigJ[i_,j_]:=First[CalcEigJ[n,d,j,i]];
K=Transpose[Array[funcEigJ,{d+1,d+1},{0,0}]];
For[i=1,i<=Length[K], i++,
        min = Min[K[[i]]];
        pos = Position[K[[i]],min];
        K[[i]]=ReplacePart[K[[i]],pos->Highlighted[min, FrameMargins->1,
 ↪  ContentPadding->False, Background->Green]];
        ];
MatrixForm[Transpose[K]])

EigJa[n_,d_]:=(funcEigJ[i_,j_]:=First[CalcEigJ[n,d,j,i]];
K=Transpose[Array[funcEigJ,{d+1,d+1},{0,0}]];
For[i=1,i<=Length[K], i++,
        If[Length[DeleteDuplicates[Abs[K[[i]]]]]!=1,
                min = DeleteDuplicates[Sort[Abs[K[[i]]]]][[-2]],
                min=Min[Abs[K[[i]]]]
        ];
        pos1 = Position[K[[i]],-min];
```

```
        pos2=Position[K[[i]],min];
        K[[i]]=ReplacePart[K[[i]],pos1->Highlighted[-min, FrameMargins->1,
 ↪  ContentPadding->False, Background->Pink]];
        K[[i]]=ReplacePart[K[[i]],pos2->Highlighted[min, FrameMargins->1,
 ↪  ContentPadding->False, Background->Pink]];];
MatrixForm[Transpose[K]])

EigHac[d_,q_,c_]:=(K1=EigHa[d,q][[1]];
K2=Join[{Range[-1,First[Dimensions[K1]]-1]},Join[Transpose[{Range[0,
 ↪  First[Dimensions[K1]]-1]}],K1,2]];(*add headers to rows and columns*)
Part[K2,1,1]="";(*left upper corner of header should be empty*)
cols1=c+2;(*add 2 to list of integers for column dividors, because of header and
 ↪  because mathematica is 1-based*)
cols2=ConstantArray[False,First[Dimensions[K2]+2]];(*make list of repeated
 ↪  "false"*)
cols2[[cols1]]=True;(*change values in this list to true if we want a column
 ↪  dividor there*)
Grid[K2,Background->{{LightGray},{LightGray}},Dividers->{cols2}])(*make headers
 ↪  grey and add column dividors*)

EigHsc[d_,q_,c_]:=(K1=EigHs[d,q][[1]];
K2=Join[{Range[-1,First[Dimensions[K1]]-1]},Join[Transpose[{Range[0,
 ↪  First[Dimensions[K1]]-1]}],K1,2]];
Part[K2,1,1]="";
cols1=c+2;
cols2=ConstantArray[False,First[Dimensions[K2]+2]];
cols2[[cols1]]=True;
Grid[K2,Background->{{LightGray},{LightGray}},Dividers->{cols2}])

EigJac[n_,d_,c_]:=(K1=EigJa[n,d][[1]];
K2=Join[{Range[-1,First[Dimensions[K1]]-1]},Join[Transpose[{Range[0,
 ↪  First[Dimensions[K1]]-1]}],K1,2]];
Part[K2,1,1]="";
cols1=c+2;
cols2=ConstantArray[False,First[Dimensions[K2]+2]];
cols2[[cols1]]=True;
Grid[K2,Background->{{LightGray},{LightGray}},Dividers->{cols2}])

EigJsc[n_,d_,c_]:=(K1=EigJs[n,d][[1]];
K2=Join[{Range[-1,First[Dimensions[K1]]-1]},Join[Transpose[{Range[0,
 ↪  First[Dimensions[K1]]-1]}],K1,2]];
Part[K2,1,1]="";
cols1=c+2;
cols2=ConstantArray[False,First[Dimensions[K2]+2]];
cols2[[cols1]]=True;
Grid[K2,Background->{{LightGray},{LightGray}},Dividers->{cols2}])

DefjH[d_,q_]:=Ceiling[d-(d-1)/q]

DefjJ[n_,d_]:=Ceiling[d (n-d)/(n-1)]

HsmX[d_,q_,c_]:=(K1=EigHs[d,q][[1]];
K3=Replace[K1,_?IntegerQ->".",{2,3}];(*replace all integers in the P-matrix from
 ↪  EigHs to dots*)
```

```
K2=Join[{Range[-1,First[Dimensions[K3]]-1]},Join[Transpose[{Range[0,
    First[Dimensions[K3]]-1]}],K3,2]];(*add headers like in EigHsc*)
Part[K2,1,1]="";
cols1=c+2;
cols2=ConstantArray[False,First[Dimensions[K2]+2]];
cols2[[cols1]]=Red;(*the column dividers from the input will be red*)
cols2[[DefjH[d,q]+2]]=True; (*the default divider will be black. +2 because of
    header and 1-based mathematica*)
Grid[K2,Background->{{LightGray},{LightGray}}, Dividers->{cols2}, BaseStyle->10,
    Spacings->{0.1,0}, ItemSize->All])

HabX[d_,q_,c_]:=(K1=EigHa[d,q][[1]];
K3=Replace[K1,_?IntegerQ->".",{2,3}];
K2=Join[{Range[-1,First[Dimensions[K3]]-1]},Join[Transpose[{Range[0,
    First[Dimensions[K3]]-1]}],K3,2]];
Part[K2,1,1]="";
cols1=c+2;
cols2=ConstantArray[False,First[Dimensions[K2]+2]];
cols2[[cols1]]=Red;
cols2[[DefjH[d,q]+2]]=True;
Grid[K2,Background->{{LightGray},{LightGray}}, Dividers->{cols2}, BaseStyle->10,
    Spacings->{0.1,0}, ItemSize->All])

JsmX[n_,d_,c_]:=(K1=EigJs[n,d][[1]];
K3=Replace[K1,_?IntegerQ->".",{2,3}];
K2=Join[{Range[-1,First[Dimensions[K3]]-1]},Join[Transpose[{Range[0,
    First[Dimensions[K3]]-1]}],K3,2]];
Part[K2,1,1]="";
cols1=c+2;
cols2=ConstantArray[False,First[Dimensions[K2]+2]];
cols2[[cols1]]=Red;
cols2[[DefjJ[n,d]+2]]=True;
Grid[K2,Background->{{LightGray},{LightGray}}, Dividers->{cols2}, BaseStyle->10,
    Spacings->{0.1,0}, ItemSize->All])

JabX[n_,d_,c_]:=(K1=EigJa[n,d][[1]];
K3=Replace[K1,_?IntegerQ->".",{2,3}];
K2=Join[{Range[-1,First[Dimensions[K3]]-1]},Join[Transpose[{Range[0,
    First[Dimensions[K3]]-1]}],K3,2]];
Part[K2,1,1]="";
cols1=c+2;
cols2=ConstantArray[False,First[Dimensions[K2]+2]];
cols2[[cols1]]=Red;
cols2[[DefjJ[n,d]+2]]=True;
Grid[K2,Background->{{LightGray},{LightGray}}, Dividers->{cols2}, BaseStyle->10,
    Spacings->{0.1,0}, ItemSize->All])

End[]
EndPackage[]
```

Appendix B

Code for checking results and testing conjectures by computer

Like before, the code in this part is written for Wolfram Mathematica.

B.1 Checking cases for Theorem 4.14a

We need to check if $|K_j(1)| \leq |K_j(i)|$ under the restrictions $1 \leq d < 30$, $3 \leq i \leq d$, $q \geq 3$, $d - \frac{d-1}{q} \leq j \leq d-2$ and $qj > 2(q-1)(d-i+1)$. Because we need $d - \frac{d-1}{q} \leq d-2$ for there to be a feasible j, we can add an extra restriction $q \leq \frac{d-1}{2}$. Also, since we need $3 \leq \frac{d-1}{2}$ for there to be a feasible q, we can restrict d further to $d \geq 7$. Lastly, we know $j \geq d - \frac{d-1}{q} \geq 7 - \frac{d-1}{\left(\frac{d-1}{2}\right)} = 5$.

In the proof of 4.14a we noted that it is enough to show that the inequality

$$d \leq (q-1)^{j-d+i-1}(qj-(q-1)d)$$

holds. The following code outputs the number of feasible combinations and a list of all feasible combinations (q, d, i, j) for which the aforementioned inequality is *false*. Note that we don't check $|K_j(1)| \leq |K_j(i)|$ directly for all combinations, since this code would run significantly slower than the code below, which can be executed within seconds.

```
Input
f[q_, d_, i_, j_] := (q - 1)^(j - d + i - 1) *(q *j - (q - 1)*d);
falseCombinations = {};
nrFeasibleCombinations = 0;
For[d = 7, d < 30, d++,
  For[i = 3, i <= d, i++,
    For[q = 3, q <= (d - 1)/2, q++,
      For[j = 5, j <= d - 2, j++,
        If[d - (d - 1)/q <= j && j*q > 2 *(q - 1)*(d - i + 1),
          nrFeasibleCombinations++;
          If[d > f[q, d, i, j],
           AppendTo[ falseCombinations, {"q=", q, "d=", d, "i=", i, "j=", j}]];
          ];
        ];
      ];
```

```
    ];
  ];
Print["Combinations for which the inequality is false:", falseCombinations,
      "\n Nr. of feasible combinations: ", nrFeasibleCombinations];
```

Output

```
Combinations for which the inequality is false:
    {{q=,3,d=,7,i=,5,j=,5},{q=,3,d=,10,i=,6,j=,7},{q=,3,d=,10,i=,7,j=,7},
    {q=,3,d=,13,i=,8,j=,9},{q=,3,d=,16,i=,9,j=,11},{q=,3,d=,19,i=,11,j=,13},
    {q=,3,d=,22,i=,12,j=,15}}
Nr of feasible combinations: 3881
```

For the $3881 - 7 = 3894$ combinations for which $d \leq (q-1)^{j-d+i-1}(qj-(q-1)d)$ holds, we already know that $|K_j(1)| \leq |K_j(i)|$. For the seven combinations for which $d > (q-1)^{j-d+i-1}(qj-(q-1)d)$ holds, we need to check by computer if the inequality $|K_j(1)| \leq |K_j(i)|$ holds.

q	d	i	j	$\lvert K_j(i)\rvert$	$\lvert K_j(1)\rvert$	$\lvert K_j(i)\rvert \leq \lvert K_j(1)\rvert$?
3	7	5	5	21	48	True
3	10	6	7	24	768	True
3	10	7	7	69	768	True
3	13	8	9	310	14 080	True
3	16	9	11	1 596	279 552	True
3	19	11	13	2 352	5 849 088	True
3	22	12	15	64 704	127 008 768	True

The table shows that this is indeed the case.

B.2 Checking cases for Lemma 4.29, step 3

For $73 \leq n \leq 11036$ and $d \geq 10$, we want to check if $\frac{|V|}{k_j} \leq \frac{n-5}{6}$ for all feasible values of n, d, j. The following code outputs the number of feasible combinations and a list of all feasible combinations (n, d, j) for which the inequality is *false*.

Input

```
f[n_, d_, j_] := Binomial[n, d]/(Binomial[d, j]*Binomial[n - d, j]) - (n - 5)/6;
falseCombinations = {};
nrFeasibleCombinations = 0;
For[n = 73, n <= 11036, n++,
  For[d = 10, d <= n/2, d++,
    j0 = d (n - d)/n;
    j0Ceil = Ceiling[j0];
    If[Element[j0 + 3/2, Integers], jmax = j0 + 3/2 - 1, jmax = Floor[j0 + 3/2]];
    For[j = j0Ceil, j <= jmax, j++,
     If[j >= d (n - d)/(n - 1) && j < d,
       nrFeasibleCombinations++;
       If[f[n, d, j] >= 0,AppendTo[falseCombinations, {"n=", n, "d=", d, "j=", j}]
        ];
       ];
      ];
```

```
    ];
  ];
Print["False combinations:", falseCombinations,
      "\n Nr. of feasible combinations: ", nrFeasibleCombinations];
```

Output

```
False combinations:{}
Nr. of feasible combinations: 39355058
```

After checking all 39 355 058 feasible cases, we see that there are no combinations for which the inequality is false. Therefore, we may conclude that $\frac{|V|}{k_j} \leq \frac{n-5}{6}$ for $73 \leq n < 11037$ and $d \geq 10$. This computation took a few hours, which is much longer than the average computation time of the programs in the other appendices.

B.3 Checking cases for Lemma 4.29, step 5

We need to check if $E_j(i)^2 \leq E_j(1)^2$ for $n \leq 73$ and $i \geq 3$ for all feasible values of n, d, j, i. The value of d can be restricted from below by $d \geq i \geq 3$. Also, we know $n \geq 2d \geq 2i \geq 6$.

The following code outputs the number of feasible combinations and a list of all feasible combinations (n, d, j, i) for which $E_j(i)^2 \leq E_j(1)^2$ is *false*.

Input

```
g[n_, d_, j_, i_] := Sum[(-1)^h Binomial[i, h]*Binomial[d - i, j - h]
                      *Binomial[n - d - i, j - h], {h, 0, j}];
falseCombinations = {};
nrFeasibleCombinations = 0;
For[n = 6, n <= 73, n++,
  For[d = 3, d <= n/2, d++,
   j0 = d (n - d)/n;
   j0Ceil = Ceiling[j0];
   If[Element[j0 + 3/2, Integers], jmax = j0 + 3/2 - 1, jmax = Floor[j0 + 3/2]];
   For[j = j0Ceil, j <= jmax, j++,
    If[j >= d (n - d)/(n - 1) && j < d,
      For[i = 3, i <= d, i++,
        nrFeasibleCombinations++;
        g1sqrt = g[n, d, j, 1]^2;
        If[g[n, d, j, i]^2 > g1sqrt,
         AppendTo[falseCombinations, {"n=", n, "d=", d, "j=", j, "i=", i}];
         ];
        ];
      ];
    ];
   ];
  ];
Print["False combinations:", falseCombinations,
      "\n Nr. of feasible combinations: ", nrFeasibleCombinations];
```

Output

```
False combinations:{}
Nr. of feasible combinations: 17060
```

We can see that the list of false combinations is again empty, and therefore we may conclude $E_j(i)^2 \leq E_j(1)^2$ for $n \leq 73$ and $i \geq 3$.

B.4 Testing Conjecture 5.2

The code to test Conjecture 5.2 can be found below. We tested the conjecture for $2 \leq d \leq 400$.

Input

```
dmax = 400;
falseComb = {};
nrComb = 0;
For[d = 2, d <= dmax, d++,
  jmax = Ceiling[(d + 1)/2] - 1;
  For[j = 1, j <= jmax, j++,
   thisj = False;
   K1 = CalcEigH[d, 2, j, 1];
   For[i = 2, i <= d, i++,
     If[CalcEigH[d, 2, j, i] < K1, thisj = True];
     nrComb++;
     ];
   If[thisj == False, AppendTo[falseComb, {"d=", d, "j=", j}]];
   ];
  ];
Print["nr of combinations tested: ", nrComb];
Print["pairs for which conj is false: ", falseComb];
```

Output

```
nr of combinations tested: 10646700
pairs for which conj is false: {}
```

We see that the list of pairs for which the conjecture is false is empty, so the conjecture holds for $2 \leq d \leq 400$.

B.5 Testing Conjecture 5.4

We want to check Conjecture 5.4 for pairs (d, q) with $2 \leq d \leq 200$, $3 \leq q \leq 50$ and $2 \leq d \leq 50$, $50 \leq q \leq 500$. The code for the pairs with $2 \leq d \leq 200$, $3 \leq q \leq 50$ is the following.

Input

```
dmax = 200;
qmax = 50;
falseComb = {};
nrComb = 0;
For[d = 2, d <= dmax, d++,
```

```
  For[q = 3, q <= qmax, q++,
    jmax = Ceiling[d - (d - 1)/q] - 1;
    For[j = 1, j <= jmax, j++,
     thisj = False;
     K1 = CalcEigH[d, q, j, 1];
     For[i = 2, i <= d, i++,
      If[CalcEigH[d, q, j, i] < K1, thisj = True];
      nrComb++;
      ];
     If[thisj == False, AppendTo[falseComb, {"d=", d, "j=", j}]];
     ];
    ];
  ];
Print["nr of combinations tested: ", nrComb];
Print["pairs for which conj is false: ", falseComb];
```

Output

```
nr of combinations tested: 119 551 090
pairs for which conj is false: {}
```

The code for the pairs $2 \leq d \leq 50$, $50 \leq q \leq 500$ is the following.

Input

```
dmax = 50;
qmax = 500;
falseComb = {};
nrComb = 0;
For[d = 2, d <= dmax, d++,
  For[q = 50, q <= qmax, q++,
    jmax = Ceiling[d - (d - 1)/q] - 1;
    For[j = 1, j <= jmax, j++,
     thisj = False;
     K1 = CalcEigH[d, q, j, 1];
     For[i = 2, i <= d, i++,
      If[CalcEigH[d, q, j, i] < K1, thisj = True];
      nrComb++;
      ];
     If[thisj == False, AppendTo[falseComb, {"d=", d, "j=", j}]];
     ];
    ];
  ];
Print["nr of combinations tested: ", nrComb];
Print["pairs for which conj is false: ", falseComb];
```

Output

```
nr of combinations tested: 18 231 224
pairs for which conj is false: {}
```

In both cases, the list of pairs for which the conjecture is false is empty, so we may conclude that the conjecture holds for all above-mentioned pairs (d, q).

B.6 Checking cases for Lemma 5.9

To finish the proof of Lemma 5.9, it suffices to check this lemma by computer for all pairs (d, q) with $3 \leq d \leq 36$ and $3 \leq q \leq 322$. The following code does this.

Input

```
upperBoundq = 322;
upperBoundd = 36;
falseList = {};
nrTested = 0;
For[q = 3, q <= upperBoundq, q++,
  rangeForB = Range[1, q - 2]; (*if d mod q is 0 or q-1, we have Lemma 5.9a,
↪  else 5.9b*)
  For[d = 1, d <= upperBoundd, d++,
   j = Ceiling[d - (d - 1)/q] - 1;
   trueqd = True;
   If[MemberQ[rangeForB, Mod[d, q]], (*if d mod q is not 0 or q-1...*)
    EigHi1 = Abs[CalcEigH[d, q, j, 1]]; (*...then check with |K_j(1)|*)
    For[i = 1, i <= d, i++,
     If[Abs[CalcEigH[d, q, j, i]] > EigHi1, trueqd = False];
     nrTested++;
     ],
    EigHi2 = Abs[CalcEigH[d, q, j, 2]]; (*...else check with |K_j(2)|*)
    For[i = 1, i <= d, i++,
     If[Abs[CalcEigH[d, q, j, i]] > EigHi2, trueqd = False];
     nrTested++;
     ];
    ];
   If[trueqd == False, AppendTo[falseList, {"d=", d, "q=", q}]];
   ];
  ];
Print["nr of combinations tested: ", nrTested];
Print["pairs for which lemma is false: ", falseList];
```

Output

```
nr of combinations tested: 213120
pairs for which lemma is false: {}
```

We see that the list with pairs (d, q) for which the lemma is false is empty, so we can conclude that the lemma is indeed true for pairs (d, q) with $3 \leq d \leq 39$ and $3 \leq q \leq 414$.

B.7 Checking cases for Lemma 5.10

To finish the proof of Lemma 5.10, we need to calculate by computer whether $|K_j(i)| \leq |K_j(1)|$ for $3 \leq d, q \leq 65$, $j < d - \frac{d-1}{q} - 1$, $qj > 2(q-1)(d-i)$ and $1 \leq i \leq d$. The following code checks these cases.

Input

```
dmax = 65;
qmax = 65;
falseComb = {};
nrComb = 0;
For[d = 3, d <= dmax, d++,
  For[q = 3, q <= qmax, q++,
    jmax = Ceiling[d - (d - 1)/q] - 2;
    For[j = 1, j <= jmax, j++,
     For[i = 2, i <= d, i++,
       If[q*j > 2*(q - 1)*(d - i),
         nrComb++;
         If[Abs[CalcEigH[d, q, j, i]] > Abs[CalcEigH[d, q, j, 1]],
          AppendTo[falseComb, {d, q, j, i}]];
         ];
       ];
     ];
    ];
  ];
Print["nr of combinations tested: ", nrComb];
Print["pairs for which lemma is false: ", falseComb];
```

Output

```
nr of combinations tested: 1398102
pairs for which lemma is false: {}
```

We see that the list of pairs for which the lemma is false is empty, so we can conclude that indeed $|K_j(i)| \leq |K_j(1)|$ for the values mentioned above.

B.8 Checking cases for Lemma 5.15

To finish the proof of Lemma 5.15, we need to calculate by computer whether $|K_j(i)| \leq |K_j(1)|$ for $q = 3$, $3 \leq d \leq 103$ and the conditions in the statement of Lemma 5.15. The following code checks these cases. Note that we check more cases than is necessary, since we don't require $i + (q-1)(d-i) - qj \geq 0$ and $i > \frac{qj}{2}$.

Input

```
dmax = 103;
falseComb = {};
nrComb = 0;
q = 3;
For[d = 3, d <= dmax, d++,
  jmax = Ceiling[d - (d - 1)/q] - 2;
  For[j = 1, j <= jmax, j++,
   K1 = Abs[CalcEigH[d, q, j, 1]];
   For[i = 3, i <= d, i++,
    If[Abs[CalcEigH[d, q, j, i]] > K1,
     AppendTo[falseComb, {"d=", d, "j=", j, "i=", i}]];
    nrComb++;
    ];
```

```
  ];
 ];
Print["nr of combinations tested: ", nrComb];
Print["pairs for which lemma is false: ", falseComb];
```

```
Output
```

```
nr of combinations tested: 232356
pairs for which lemma is false: {}
```

We see that the list of pairs for which the lemma is false is empty, so we can conclude that indeed $|K_j(i)| \leq |K_j(1)|$ for the values mentioned above.

B.9 Testing Conjecture 5.17a

We want to test Conjecture 5.17 for all pairs (n, d) with $2 \leq d \leq 200$, $2d \leq n \leq 400$. The code used to check this is the following.

```
Input
```

```
dmax = 200;
nmax = 400;
falseComb = {};
nrComb = 0;
For[d = 2, d <= dmax, d++,
  For[n = 2*d, n <= nmax, n++,
    j = Ceiling[d*(n - d)/(n - 1)] - 1;
    E1 = Abs[CalcEigJ[n, d, j, 1]];
    E2 = Abs[CalcEigJ[n, d, j, 2]];
    thisdn = True;
    For[i = 1, i <= d, i++,
     If[Abs[CalcEigJ[n, d, j, i]] > E1 &&
       Abs[CalcEigJ[n, d, j, i]] > E2, thisdn = False];
     nrComb++;
     ];
    If[thisdn == False, AppendTo[falseComb, {"n=", n, "d=", d}]];
    ];
  ];
Print["nr of combinations tested: ", nrComb];
Print["pairs for which conj is false: ", falseComb];
```

```
Output
```

```
nr of combinations tested: 2 686 301
pairs for which conj is false: {}
```

We see that the list of pairs for which the conjecture is false is empty, so the conjecture holds for $2 \leq d \leq 200$, $2d \leq n \leq 400$.

B.10 Testing Conjecture 5.17b

We already know from Appendix B.9 that for $j = \left\lceil \frac{d(n-d)}{n-1} \right\rceil - 1$, the penabsolute eigenvalue is either $|E_j(1)|$ or $|E_j(2)|$. The code to test whether $|E_j(2)|$ is the penabsolute eigenvalue for $n = d^2 + d$, $3 \leq d \leq 200$ is the following.

Input

```
falseComb = {};
nrComb = 0;
For[d = 3, d <= 200, d++,
  n = d^2 + d;
  j = Ceiling[d*(n - d)/(n - 1)] - 1;
  E1 = Abs[CalcEigJ[n, d, j, 1]];
  E2 = Abs[CalcEigJ[n, d, j, 2]];
  If[E2 < E1, AppendTo[falseComb, d]];
  nrComb++;
  ];
Print["nr of combinations tested: ", nrComb];
Print["pairs for which conj is false: ", falseComb];
```

Output

```
nr of combinations tested: 198
pairs for which conj is false: {}
```

The code to test whether $|E_j(1)|$ is the penabsolute eigenvalue for $d^2 + d + 1 \leq n \leq 10000$, $3 \leq d \leq 10$ is the following.

Input

```
falseComb = {};
nrComb = 0;
For[d = 3, d <= 10, d++,
  nmin = d^2 + d + 1;
  For[n = nmin, n <= 10000, n++,
   j = Ceiling[d*(n - d)/(n - 1)] - 1;
   E1 = Abs[CalcEigJ[n, d, j, 1]];
   E2 = Abs[CalcEigJ[n, d, j, 2]];
   If[E2 > E1, AppendTo[falseComb, d]];
   nrComb++;
   ];
  ];
Print["nr of combinations tested: ", nrComb];
Print["pairs for which conj is false: ", falseComb];
```

Output

```
nr of combinations tested: 79568
pairs for which conj is false: {}
```

The code to test whether $|E_j(1)|$ is the penabsolute eigenvalue for $d^2+d+1 \leq n \leq d^3$, $10 \leq d \leq 200$ is the following.

Input

```
falseComb = {};
nrComb = 0;
For[d = 10, d <= 200, d++,
  nmin = d^2 + d + 1;
  nmax = d^3;
  For[n = nmin, n <= nmax, n++,
   j = Ceiling[d*(n - d)/(n - 1)] - 1;
   E1 = Abs[CalcEigJ[n, d, j, 1]];
   E2 = Abs[CalcEigJ[n, d, j, 2]];
   If[E2 > E1, AppendTo[falseComb, d]];
   nrComb++;
   ];
  ];
Print["nr of combinations tested: ", nrComb];
Print["pairs for which conj is false: ", falseComb];
```

Output

```
nr of combinations tested: 401301505
pairs for which conj is false: {}
```

We see that for all three pieces of code, there are no pairs for which the conjecture is false.

B.11 Testing Conjecture 5.18

We want to test Conjecture 5.18 for all pairs (n, d) with $2 \leq d \leq 200$, $2d \leq n \leq 400$. The code used to check this is the following.

Input

```
dmax = 200;
nmax = 400;
falseComb = {};
nrComb = 0;
For[d = 2, d <= dmax, d++,
  For[n = 2*d, n <= nmax, n++,
    jmax = Ceiling[d*(n - d)/(n - 1)] - 1;
    For[j = 1, j <= jmax, j++,
     thisj = True;
     E1 = Abs[CalcEigJ[n, d, j, 1]];
     For[i = 2, i <= d, i++,
      If[Abs[CalcEigJ[n, d, j, i]] > E1, thisj = False];
      nrComb++;
      ];
     If[thisj == False, AppendTo[falseComb, {"n=", n, "d=", d}]];
     ];
    ];
  ];
Print["nr of combinations tested: ", nrComb];
Print["pairs for which conj is false: ", falseComb];
```

Output

```
nr of combinations tested: 165 578 229
pairs for which conj is false: {}
```

We see that the list of pairs for which the conjecture is false is empty, so the conjecture holds for $2 \leq d \leq 200$, $2d \leq n \leq 400$.