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Abstract

An extremal element in a Lie algebra $\mathfrak{g}$ is a non-zero element $x \in \mathfrak{g}$ such that $[x, [x, \mathfrak{g}]] \subseteq \mathbb{F}x$, and we denote by $E(\mathfrak{g})$ the set of extremal elements of $\mathfrak{g}$. If, however, $[x, [x, \mathfrak{g}]]$ is 0-dimensional, then x is called a sandwich element. The extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of a Lie algebra $\mathfrak{g}$ is the point-line geometry whose point set $\mathcal{E}$ is the set $E(\mathfrak{g})$ of extremal elements of $\mathfrak{g}$ and whose line set $\mathcal{L}$ is the set of 2-dimensional subspaces of $\mathfrak{g}$ spanned by two commuting and linearly independent extremal elements all of whose elements are again extremal.

The finitary orthogonal Lie algebra $\mathfrak{fso}(V, f)$ for some possibly infinite-dimensional vector space V over a field $\mathbb{F}$ and some non-degenerate symmetric bilinear form $f : V \times V \to \mathbb{F}$ on V is a classical linear Lie algebra whose extremal geometry is isomorphic to the orthogonal geometry $\Gamma_{\mathcal{O}}(V, f)$, whose point set is the set of all 2-dimensional totally f-isotropic subspaces of V and whose line set is the set of all subsets of the point set whose elements are the 2-dimensional totally f-isotropic subspaces of V all of which are contained in a fixed 3-dimensional totally f-isotropic subspace of V and containing a fixed 1-dimensional totally f-isotropic subspace of V. If V is finite-dimensional, the extremal geometry of $\mathfrak{so}(V, f)$ is isomorphic to a root shadow space of type $BC_{n,2}$ or $D_{n+1,2}$ $(n \geq 3)$.

Preceded by an extensive exhibition of relevant theory and many examples, in this thesis we first prove that the finitary orthogonal Lie algebra $\mathfrak{fso}(V, f)$, for some possibly infinite-dimensional vector space V over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, and some non-degenerate symmetric bilinear form $f : V \times V \to \mathbb{F}$ on V such that f has Witt index at least three, is uniquely determined by its extremal geometry, up to isomorphism. In particular, we prove that $\mathfrak{fso}(V, f) \cong \mathfrak{g}$ for any possibly infinite-dimensional simple Lie algebra $\mathfrak{g}$ without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements whose extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ is isomorphic to the orthogonal geometry $\Gamma_{\mathcal{O}}(V, f)$.

If the singular rank of the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of a Lie algebra $\mathfrak{g}$ is finite, it is isomorphic to a root shadow space of classical or exceptional type, although it is possible for $\Gamma_{\mathfrak{g}}$ to have infinite singular rank, for example if $\Gamma_{\mathfrak{g}}$ is isomorphic to the Siegel geometry $\Gamma_{\mathcal{S}}$ with V infinite-dimensional. Therefore, in this thesis we next prove that the singular rank of the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of a simple Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements is finite if $\Gamma_{\mathfrak{g}}$ is not isomorphic to a root shadow space of classical type. Specifically, we show that $\Gamma_{\mathfrak{g}}$ is isomorphic to a root shadow space of type $E_{6,2}$, $E_{7,1}$, $E_{8,8}$ or $F_{4,1}$.

Contents

Chapter 1

Introduction

The author of this thesis would like to express his sincerest gratitude to prof. dr. H. Cuypers for his guidance and expertise during the realisation of this thesis.

1.1 Main results of this thesis

Initiated by Lie in the nineteenth century with the aim of using symmetry to classify spaces in terms of their geometry, in particular the space of ordinary differential equations, Lie theory is of a topological nature having manifolds and one-parameter subgroups as its elementary concepts and knowing many applications in mathematical physics. Central in Lie theory is the correspondence between Lie groups and Lie algebras via the exponential map, thereby establishing a connection between topology, group theory and abstract algebra.

A Lie algebra $\mathfrak{g}$ is a vector space over a field $\mathbb{F}$ with a bilinear alternating operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket satisfying the Jacobi identity

$$[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$$

for all $x, y, z \in \mathfrak{g}$. Of particular importance for the study of Lie algebras is the classification of all simple Lie algebras, initiated by Killing at the end of the nineteenth century. Over the field of complex numbers $\mathbb{C}$, this eventually led to the discovery of, on the one hand, the classical Lie algebras attributed to the infinite families A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$ and D_n $(n \geq 4)$, and, on the other hand, the exceptional Lie algebras of types E_6, E_7, E_8, F_4 and G_2. The classical Lie algebras correspond to the classical groups $\mathrm{SL}_{n+1}(\mathbb{C})$, $\mathrm{SO}_{2n+1}(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{C})$ and $\mathrm{SO}_{2n}(\mathbb{C})$, so they are accordingly denoted by $\mathfrak{sl}_{n+1}(\mathbb{C})$, $\mathfrak{so}_{2n+1}(\mathbb{C})$, $\mathfrak{sp}_{2n}(\mathbb{C})$ and $\mathfrak{so}_{2n}\mathbb{C})$, respectively. The next step would be a similar classification over fields different from $\mathbb{C}$; this goal was achieved over a half a century later by Chevalley and Dickson for finite fields, giving rise to the modular Lie algebras. Many more results over fields of characteristic zero have been obtained by Lie, Killing, Engel, Cartan and Weyl, see e.g. [1].

The aforementioned types of the classical and exceptional Lie algebras also appear in geometry, specifically in the geometric representation of real reflection groups; they are exactly the diagram types of all affine finite Coxeter groups as introduced and classified by Coxeter [14] in the first half of the twentieth century. Moreover, these types are fundamental in the theory of buildings, developed by Tits [8] in the second half of the twentieth century. The combination of Coxeter groups and spherical buildings brings about root shadow spaces and root filtration spaces, at the basis of which lies a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$, in which $\mathcal{P}$ is a possibly infinite set of elements called points and $\mathcal{L}$ is a set of subsets of $\mathcal{P}$ each of size at least two called lines. A root shadow space can be obtained from a shadow space of the geometry of a building of Weyl type, whereas a root filtration space is identified by a quintuple of disjoint symmetric relations on the Cartesian product of its point set that satisfy certain properties. A characterisation of and a connection between root shadow spaces and root filtration spaces has been made by Cohen and Ivanyos [19]. Their findings play a fundamental role in the classification of Lie algebras generated by their extremal elements.

An extremal element in a Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ is a non-zero element $x \in \mathfrak{g}$ such that

$$[x, [x, y]] \subseteq \mathbb{F}x$$

for all $y \in \mathfrak{g}$. If, however, $[x, [x, \mathfrak{g}]]$ is 0-dimensional, then we call x a sandwich element instead. Examples of extremal elements are long root elements of classical Lie algebras, but also inner ideals of modules for Lie algebras as introduced for the first time by Faulkner [21] in the late twentieth century. Extremal elements pave the way for the characterisation of simple Lie algebras without sandwich elements that they generate. Key to this characterisation is the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of a Lie algebra $\mathfrak{g}$ whose point set $\mathcal{E}$ is the set of 1-dimensional subspaces $\mathbb{F}x$ of extremal elements $x \in \mathfrak{g}$ and whose line set $\mathcal{L}$ is the set of 2-dimensional subspaces $\mathbb{F}x + \mathbb{F}y$ with $x, y \in \mathfrak{g}$ extremal and linearly independent such that $[x, y] = 0$ and every non-zero element of $\mathbb{F}x + \mathbb{F}y$ is extremal. In particular, since $\Gamma_{\mathfrak{g}}$ turns out to be a root filtration space, the natural question arises whether $\Gamma_{\mathfrak{g}}$ uniquely determines $\mathfrak{g}$, up to isomorphism. Finding the answer to this question is motivated by the types of the classical and exceptional Lie algebras, on the one hand, coinciding with the types of Coxeter groups and spherical buildings, on the other hand.

Recent interest and studies in the field of Lie algebra and geometry have resulted in an almost complete answer to this question, with at its core the work of Cohen and Ivanyos [5, 19], which, in turn, is based on the work of Kasikova and Shult [25]. Specifically, we have the following theorem due to Cuypers and Fleischmann [13].

Theorem 1.1.1. *Let $\mathfrak{g}$ be a simple Lie algebra generated by its set of extremal elements. If the extremal geometry of $\mathfrak{g}$ is the root shadow space of a spherical building of rank at least 3, then $\mathfrak{g}$ is, up to isomorphism, uniquely determined by its extremal geometry.*

The types of the spherical buildings that the above theorem pertains to are the exceptional types E_6, E_7, E_8 and F_4, but also the classical types BC_n $(n \geq 3)$ and D_n $(n \geq 4)$.

Buildings of type A_n ($n \geq 2$) have been covered by Roberts [11] and Shpectorov et al. [9]. Note, however, that Theorem 1.1.1 only applies in case the building is spherical and if the extremal geometry of $\mathfrak{g}$ contains lines. This raises the question whether a similar result can be obtained if the building is not spherical or if the extremal geometry of $\mathfrak{g}$ does not contain lines.

In [20], Cuypers and Fleischmann show that $\mathfrak{g}$ is isomorphic to the finitary symplectic Lie algebra $\mathfrak{fsp}(V, f)$ for some possibly infinite-dimensional vector space V over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, and some non-degenerate symplectic form $f : V \times V \to \mathbb{F}$ if the extremal geometry of $\mathfrak{g}$ does not contain lines. To remedy the absence of lines, it is possible to use a particular second degree field extension $\mathbb{K}$ of $\mathbb{F}$ such that $\mathfrak{g} \otimes_{\mathbb{F}} \mathbb{K}$ does contain lines. Note that this is not possible if $\mathfrak{g} \cong \mathfrak{fsp}(V, f)$. In case the extremal geometry of $\mathfrak{g} \otimes_{\mathbb{F}} \mathbb{K}$ is isomorphic to the root shadow space of a non-spherical building of type A_n ($n \geq 1$), it is shown by Cuypers and Oostendorp [23] that $\mathfrak{g} \otimes_{\mathbb{F}} \mathbb{K}$ is isomorphic to the projective finitary special unitary Lie algebra $\mathfrak{pfsu}(V, f)$ for some possibly infinite-dimensional vector space V, $\dim(V) \geq 4$, over $\mathbb{K}$ and some non-degenerate Hermitian form $f : V \times V \to \mathbb{F}$. In addition, they show that $\mathfrak{g}$ is isomorphic to the projective finitary special linear Lie algebra $\mathfrak{pfsl}(V)$ for some possibly infinite-dimensional vector space V, $\dim(V) \geq 3$, over $\mathbb{F}$ if the extremal geometry of $\mathfrak{g}$ is isomorphic to the root shadow space of a non-spherical building of type A_n ($n \geq 1$). Now all that remains to extend Theorem 1.1.1 is to consider the case in which $\mathfrak{g}$ is an infinite-dimensional Lie algebra whose extremal geometry is isomorphic to the root shadow space of a non-spherical building of type BC_n ($n \geq 3$) or D_n ($n \geq 4$).

Corresponding to types BC_n ($n \geq 3$) and D_n ($n \geq 4$) is the orthogonal Lie algebra $\mathfrak{so}(V, f)$ for some finite-dimensional vector space V over a field $\mathbb{F}$ and some non-degenerate symmetric bilinear form $f : V \times V \to \mathbb{F}$. Denoting by V^* the dual space of V, the linear maps

$$t_{v,\varphi} : \begin{array}{ccc} V & \to & V \\ w & \mapsto & \varphi(w)v \end{array},$$

where $v \in V$ and $\varphi \in V^*$, also referred to as infinitesimal transvections, are extremal elements of the finitary general linear Lie algebra $\mathfrak{fgl}(V)$. Upon defining $f_v \in V^*$ with $v \in V$ to be the linear map given by

$$f_v : \begin{array}{ccc} V & \to & V \\ w & \mapsto & f(v, w), \end{array}$$

the infinitesimal Siegel transvections $s_{v,w} := t_{v,f_w} - t_{w,f_v}$ are extremal in the finitary orthogonal Lie algebra $\mathfrak{fso}(V, f)$ and span it linearly.

The extremal geometry $\Gamma_{\mathfrak{fso}(V,f)} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{fso}(V, f)$, with V now possibly infinite-dimensional, is isomorphic to the orthogonal geometry $\Gamma_{\mathcal{O}}(V, f)$. This orthogonal geometry can be realised using f-isotropic vectors in V and totally f-isotropic subspaces of V, which are vectors $v \in V$ such that $f(v, v) = 0$ and subspaces $W \subseteq V$ such that $f(w, w') = 0$ for all $w, w' \in W$, respectively, with the dimension of a maximal totally f-isotropic subspace of V being called the Witt index of f. Its point set is the set of all 2-dimensional totally f-isotropic subspaces of V and its line set is the set of all subsets of the point set whose elements are the 2-dimensional totally f-isotropic subspaces V all of which are contained in a fixed 3-dimensional totally f-isotropic subspace of V and containing a fixed 1-dimensional totally f-isotropic subspace of V. If V is finite-dimensional, the orthogonal geometry $\Gamma_{\mathcal{S}}$ is a root shadow space of a building of type BC_n or D_{n+1} ($n \geq 3$), the distinction between both types originating from whether the quadratic form $Q : V \to \mathbb{F}$ associated to f is non-split, respectively split.

The root shadow space of a building of type BC_n or D_{n+1} ($n \geq 3$) can in general be obtained from a non-degenerate polar space of rank n, which is a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ satisfying the Buekenhout-Shult axiom [16]. Polar spaces of rank at least three, including infinite rank, have been studied and classified by Tits [8], Veldkamp [12] and Pasini et al. [22] in the twentieth century, and their classification plays an important role in the characterisation of $\mathfrak{fso}(V, f)$.

This characterisation is one of the main subjects of this thesis; with the help of a thorough exposition of relevant background theory and examples, we will work towards proving the following theorem as an extension of Theorem 1.1.1.

Theorem 1.1.2. *Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements. If the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{g}$ is isomorphic to the orthogonal geometry $\Gamma_{\mathcal{O}}(V, f)$ of some vector space V over $\mathbb{F}$ and some non-degenerate symmetric bilinear form $f : V \times V \to \mathbb{F}$ on V such that f has Witt index at least three, then $\mathfrak{g}$ is isomorphic to the finitary orthogonal Lie algebra $\mathfrak{fso}(V, f)$.*

As mentioned before, Cohen and Ivanyos have established a fundamental connection between and characterisation of root filtration spaces and root shadow spaces in [19], relying largely on the findings from Kasikova and Shult [18]. Specifically, Cohen and Ivanyos have proven the following theorem.

Theorem 1.1.3. *Let $(\mathcal{P}, \mathcal{L})$ be a non-degenerate root filtration space. If the singular rank of $(\mathcal{P}, \mathcal{L})$ is finite, then $(\mathcal{P}, \mathcal{L})$ is isomorphic to a root shadow space of type $A_{n,\{1,n\}}$ ($n \geq 2$), $BC_{n,2}$ ($n \geq 3$), $D_{n,2}$ ($n \geq 4$), $E_{6,2}$, $E_{7,1}$, $E_{8,8}$, $F_{4,1}$ or $G_{2,1}$.*

The labeling of the nodes in the Coxeter diagrams corresponding to these root shadow spaces is as in [4]. A consequence of the above theorem is then that the nature of the extremal geometry $\Gamma_{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ is completely determined if $\Gamma_{\mathfrak{g}}$, viewed as a root filtration space, is non-degenerate having finite singular rank. However, it is entirely possible that $\Gamma_{\mathfrak{g}}$ has infinite singular rank, for example if $\Gamma_{\mathfrak{g}}$ is isomorphic with a root shadow space of classical type.

This observation gives rise to the question whether $\Gamma_{\mathfrak{g}}$ can only have infinite singular rank if $\Gamma_{\mathfrak{g}}$ is isomorphic to a root shadow space of classical type. The answer to this question turns out to be affirmative, as characterised by the second theorem we will prove in this thesis.

Theorem 1.1.4. *Let* $\mathfrak{g}$ *be a simple Lie algebra over a field* $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, *without sandwich elements generated by its set* $E(\mathfrak{g})$ *of extremal elements. If the extremal geometry* $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ *of* $\mathfrak{g}$ *is not isomorphic to a root shadow space of classical type, then* $\Gamma_{\mathfrak{g}}$ *has finite singular rank.*

In particular, $\Gamma_{\mathfrak{g}}$ *will have singular rank three, four, five or seven.*

1.2 Structure of this thesis

In Chapter 2, we introduce Lie algebras as well as the most important concepts and theory relevant for our study. The basic theory surrounding Lie algebras is covered in Section 2.1, and the classical linear Lie algebras and some of their properties will be examined in Section 2.2.

In Chapter 3, we move on to extremal elements of Lie algebras in Section 3.1, in which we will prove some identities involving extremal elements and introduce the extremal form of a Lie algebra. In Section 3.2, we combine the theory developed in the previous sections to characterise the classical linear Lie algebras by their extremal elements using tensor spaces and infinitesimal transvections.

Chapter 4 is dedicated to point-line geometries and related theory in the area of geometry that we require for our proof of Theorem 1.1.2. We start with the basic theory of point-line geometry, in particular polar spaces and projective spaces, in Section 4.1, after which we will discuss geometries and chamber systems in Section 4.2 as a means of introducing Coxeter systems and buildings. We treat root systems in Section 4.3, and in combination with Section 4.1 and Section 4.2 we then define root shadow spaces and root filtration spaces in Section 4.4, respectively Section 4.5. In the final two sections of Chapter 4, Section 4.6 and Section 4.7, we will examine polar spaces and root filtration spaces in more detail by discussing their classification and embeddability in projective spaces, respectively.

In Chapter 5, we will combine the theory from Chapter 2 and Chapter 4 in Section 5.1 to introduce the extremal geometry of a Lie algebra and to derive some of the properties of the extremal geometry of a Lie algebra.

Chapter 6 is devoted to our proof of Theorem 1.1.2. First, however, we will discuss local systems of Lie algebras in Section 6.1, which we require to prove Theorem 1.1.2 in case $\mathfrak{g}$ is infinite-dimensional. In Section 6.2 and Section 6.3, our proof of Theorem 1.1.2 will be given. This is done in two parts; the first part, covered in Section 6.2, focuses on $\mathfrak{g}$ being finite-dimensional by detailing the structure of $\mathfrak{fso}(V, f)$ and mostly serves as a basis for the second part in Section 6.3, in which the case of $\mathfrak{g}$ being infinite-dimensional will be covered.

Finally, in Chapter 7, we will prove Theorem 1.1.4. In Section 7.1, we introduce abstract root subgroups and discuss several related point-line geometries. In particular, by geometrically approaching the relevant theory of abstract root subgroups, which are group-theoretical in nature, we are able to establish a connection between abstract root subgroups and the extremal geometry of a Lie algebra. In Section 7.2, our proof of Theorem 1.1.4 will be given.

Chapter 2

Lie algebras

In this chapter, we thoroughly explore the basic theory of Lie algebras. In addition, we discuss non-degenerate reflexive sesquilinear forms and their classification as a means of introducing the classical linear Lie algebras.

The theory discussed in Section 2.1 is derived from [1], whereas Section 2.2 is based on [13, 15, 20, 23].

2.1 Basic theory of Lie algebras

The definition of a Lie algebra has already been given in the second paragraph of Section 1.1, but we formally present it here for the sake of completeness.

Definition 2.1.1 (Lie algebra & Lie subalgebra)**.** *Let* $\mathbb{F}$ *be a field. A* ***Lie algebra*** *is a vector space* $\mathfrak{g}$ *over* $\mathbb{F}$ *equipped with a bilinear alternating form* $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *such that the Jacobi identity*

$$[[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$$

is satisfied for all $x, y, z \in \mathfrak{g}$*. A subspace* $\mathfrak{h}$ *of* $\mathfrak{g}$ *is called a* ***Lie subalgebra*** *of* $\mathfrak{g}$ *if* $[x, y] \in \mathfrak{h}$ *for all* $x, y \in \mathfrak{h}$*.*

The bilinear alternating form $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ defining a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$ is called the *Lie bracket*. Note that alternativity of the Lie bracket, together with bilinearity, implies anti-symmetry; for all $x, y \in \mathfrak{g}$, we have

$$0 = [x+y, x+y] = [x,x] + [x,y] + [y,x] + [y,y] = [x,y] + [y,x] \implies [x,y] = -[y,x].$$

If moreover the characteristic of $\mathbb{F}$ is different from 2, then anti-symmetry also implies alternativity, because $[x,x] = -[x,x] \implies 2[x,x] = 0 \implies [x,x] = 0$ for all $x \in \mathfrak{g}$.

An important Lie algebra is described in the following example.

Example 2.1.2. Let $n \geq 0$ be an integer and denote by $M_n(\mathbb{R})$ the vector space of $n \times n$ matrices with entries in $\mathbb{R}$. We equip it with the form $[\cdot,\cdot] : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $[A, B] = AB - BA$ with $A, B \in M_n(\mathbb{R})$. Bilinearity follows from

$$[\lambda A+\mu B, C] = (\lambda A+\mu B)C-C(\lambda A+\mu B) = \lambda(AC-CA)+\mu(BC-CB) = \lambda[A, C]+\mu[B, C]$$

and a similar computation for the second argument of $[\cdot,\cdot]$ with $\lambda, \mu \in \mathbb{R}$ and $A, B, C \in M_n(\mathbb{R})$. We have $[A, A] = A^2 - A^2 = 0$ for all $A \in M_n(\mathbb{R})$, and

$$\begin{aligned}
&[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = [A, BC - CB] + [B, CA - AC] + [C, AB - BA] \\
&= A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B + C(AB - BA) \\
&\quad - (AB - BA)C \\
&= ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB + CAB - CBA \\
&\quad - ABC + BAC \\
&= (ABC - ABC) + (CBA - CBA) + (BCA - BCA) + (ACB - ACB) \\
&\quad + (CAB - CAB) + (BAC - BAC) \\
&= 0,
\end{aligned}$$

for all $A, B, C \in M_n(\mathbb{R})$ shows that the bilinear alternating form $[\cdot,\cdot]$ turns $M_n(\mathbb{R})$ into a Lie algebra.

Let V be a vector space over a field $\mathbb{F}$. An *endomorphism* of V is a linear transformation from V to itself. The set of all endomorphisms of V is denoted by $\mathrm{End}(V)$. Abstractly, we can define the Lie algebra from the above example as follows.

Definition 2.1.3 (General linear Lie algebra)**.** *Let V be a possibly infinite-dimensional vector space over a field $\mathbb{F}$. The **general linear Lie algebra** of V, denoted by $\mathfrak{gl}(V)$, is the vector space of linear transformations $\varphi : V \to V$ in $\mathrm{End}(V)$ equipped with the bilinear alternating form $[\cdot,\cdot] : \mathfrak{gl}(V) \times \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ given by $[\varphi, \psi] = \varphi\psi - \psi\varphi$ with $\varphi, \psi \in \mathrm{End}(V)$.*

Depending on the emphasis we wish to put on the dimension n of the corresponding vector space V and the underlying field $\mathbb{F}$, we will also use the notation $\mathfrak{gl}_n(\mathbb{F})$ for the general linear Lie algebra. As a Lie subalgebra of $\mathfrak{gl}(V)$ we have the *finitary general linear Lie algebra*, denoted by $\mathfrak{fgl}(V)$, consisting of all linear transformations $\varphi \in \mathfrak{gl}(V)$ such that $\dim(\varphi(V)) < \infty$. Observe that $\mathfrak{fgl}(V) \subsetneq \mathfrak{gl}(V)$ if and only if V is infinite-dimensional.

The Lie bracket of the general linear Lie algebra is called the *commutator bracket*. Upon fixing a basis of V, $\dim(V) < \infty$, we may represent the elements in $\mathrm{End}(V)$ as $n \times n$ matrices with entries in $\mathbb{F}$. A basis of $\mathfrak{gl}(V)$ then clearly consists of the matrices $E_{i,j}$ having a one in position (i, j) and zeros elsewhere, so that $\mathfrak{gl}(V)$ is n^2-dimensional. In particular, given $E_{i,j}, E_{k,l} \in \mathfrak{gl}(V)$ distinct with $1 \leq i, j, k, l \leq n$, we have

$$[E_{i,j}, E_{k,l}] = E_{i,j}E_{k,l} - E_{k,l}E_{i,j} = \delta_{j,k}E_{i,l} - \delta_{l,i}E_{k,j},$$

in which δ is the Kronecker delta function.

An endomorphism in $\mathfrak{gl}(V)$ that plays an important role in the study of Lie algebras is the following.

Definition 2.1.4 (Adjoint representation)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ with Lie bracket $[\cdot,\cdot]$, and let $\mathfrak{gl}(\mathfrak{g})$ be the general linear Lie algebra of $\mathfrak{g}$. The **adjoint representation** of $\mathfrak{g}$ is the map*

$$\begin{array}{rccc} \mathrm{ad}: & \mathfrak{g} & \to & \mathfrak{gl}(\mathfrak{g}) \\ & x & \mapsto & \mathrm{ad}_x, \end{array}$$

with $x \in \mathfrak{g}$, in which ad_x is the endomorphism of V given by $\mathrm{ad}_x(y) \mapsto [x,y]$ for all $y \in \mathfrak{g}$.

Another important Lie algebra related to the general linear Lie algebra and the adjoint representation is given in the lemma below.

Lemma 2.1.5. *Let V be a vector space over a field $\mathbb{F}$ endowed with a bilinear form $V \times V \to V$ given by $(v,w) \mapsto vw$ with $v,w \in V$. Then $\mathrm{Der}(V) = \{\varphi \in \mathfrak{gl}(V) \mid \varphi(vw) = \varphi(v)w + v\varphi(w)\}$ is a Lie subalgebra of $\mathfrak{gl}(V)$. Moreover, if V is itself a Lie algebra $\mathfrak{g}$, then $\mathrm{ad}_x \in \mathrm{Der}(\mathfrak{g})$ for all $x \in \mathfrak{g}$.*

Proof. Following Definition 2.1.1, we show that $[\varphi,\psi] \in \mathrm{Der}(V)$ for all $\varphi,\psi \in \mathrm{Der}(V)$. To this extent, let $\varphi,\psi \in \mathrm{Der}(V)$, then

$$\begin{aligned} [\varphi,\psi](vw) &= (\varphi\psi - \psi\varphi)(vw) = \varphi(\psi(vw)) - \psi(\varphi(vw)) \\ &= \varphi(\psi(v)w + v\psi(w)) - \psi(\varphi(v)w + v\varphi(w)) \\ &= \varphi(\psi(v)w) + \varphi(v\psi(w)) - \psi(\varphi(v)w) - \psi(v\varphi(w)) \\ &= (\varphi\psi)(v)w + \psi(v)\varphi(w) + \varphi(v)\psi(w) + v(\varphi\psi)(w) \\ &\quad - (\psi\varphi)(v)w - \varphi(v)\psi(w) - \psi(v)\varphi(w) - v(\psi\varphi)(w) \\ &= (\varphi\psi - \psi\varphi)(v)w + v(\varphi\psi - \psi\varphi)(w) = [\varphi,\psi](v)w + v[\varphi,\psi](w), \end{aligned}$$

so it follows that $[\varphi,\psi] \in \mathrm{Der}(V)$. Note that, in case V is a Lie algebra $\mathfrak{g}$, the bilinear form is given by $(x,y) \mapsto [x,y]$ with $x,y \in \mathfrak{g}$ so that $\mathrm{Der}(\mathfrak{g}) = \{\varphi \in \mathfrak{gl}(V) \mid \forall x,y \in \mathfrak{g} : \varphi([x,y]) = [\varphi(x),y] + [x,\varphi(y)]\}$. Now let $x \in \mathfrak{g}$, then by using Definition 2.1.1 and Definition 2.1.4 we find

$$\begin{aligned} \mathrm{ad}_x([y,z]) &= [x,[y,z]] = -[z,[x,y]] - [y,[z,x]] = [[x,y],z] + [y,[x,z]] \\ &= [\mathrm{ad}_x(y),z] + [y,\mathrm{ad}_x(z)], \end{aligned}$$

hence $\mathrm{ad}_x \in \mathrm{Der}(\mathfrak{g})$. □

The Lie subalgebra $\mathrm{Der}(V)$ of $\mathfrak{gl}(V)$ is also known as the *derived algebra* of V. The adjoint representation is an example of a *Lie algebra representation*, which is a homomorphism from a Lie algebra $\mathfrak{g}$ to its general linear Lie algebra $\mathfrak{gl}(\mathfrak{g})$. Homomorphisms between Lie algebras, as well as other types of morphisms, are defined as follows.

Definition 2.1.6 (Lie algebra morphisms)**.** *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two Lie algebras over a field $\mathbb{F}$ with respective Lie brackets $[\cdot,\cdot]_{\mathfrak{g}}$ and $[\cdot,\cdot]_{\mathfrak{g}'}$. A* ***homomorphism*** *is a map $\varphi : \mathfrak{g} \to \mathfrak{g}'$ such that $\varphi([x,y]_{\mathfrak{g}}) = [\varphi(x),\varphi(y)]_{\mathfrak{g}'}$ for all $x, y \in \mathfrak{g}$. If $\ker(\varphi)$ is trivial, then φ is called a* ***monomorphism****, and if im(φ) $= \mathfrak{g}'$, then φ is called a* ***epimorphism****. If both $\ker(\varphi)$ is trivial and im(φ) $= \mathfrak{g}'$, then φ is a* ***isomorphism****, in which case $\mathfrak{g}$ and $\mathfrak{g}'$ are isomorphic as Lie algebras, denoted by $\mathfrak{g} \cong \mathfrak{g}'$.*

A Lie algebra isomorphism from a Lie algebra $\mathfrak{g}$ to itself is called a *Lie algebra automorphism.* These automorphisms form a group under ordinary function composition, denoted by $\mathrm{Aut}(\mathfrak{g})$. Within this group of automorphisms, we distinguish between different types of automorphisms, as characterised by the following definition.

Definition 2.1.7 (Ad-nilpotent & Inner automorphism)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ and let $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ be its adjoint representation. An element $x \in \mathfrak{g}$ is* ***ad-nilpotent*** *if $(\mathrm{ad}_x)^k = 0$ with $k > 0$ minimal. An* ***inner automorphism*** *of $\mathfrak{g}$ is an automorphism of $\mathfrak{g}$ of the form*

$$e^{\mathrm{ad}_x}(y) = \sum_{i=0}^{k-1} \frac{(\mathrm{ad}_x)^i(y)}{i!}$$

with $y \in \mathfrak{g}$ and $x \in \mathfrak{g}$ ad-nilpotent and k the smallest integer such that $(\mathrm{ad}_x)^k = 0$.

The group generated by the inner automorphisms of a Lie algebra $\mathfrak{g}$, denoted by $\mathrm{Int}(\mathfrak{g})$, forms a subgroup of $\mathrm{Aut}(\mathfrak{g})$ under ordinary function composition. In particular, we have the following result.

Corollary 2.1.8. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. Further let $\mathrm{Aut}(\mathfrak{g})$ be group of automorphisms of $\mathfrak{g}$, and let $\mathrm{Int}(\mathfrak{g})$ be the group of inner automorphisms of $\mathfrak{g}$. Then $\mathrm{Int}(\mathfrak{g}) \trianglelefteq \mathrm{Aut}(\mathfrak{g})$.*

Proof. Let $e^{\mathrm{ad}_x} \in G$ with x ad-nilpotent such that $k > 0$ is the smallest integer satisfying $(\mathrm{ad}_x)^k = 0$. We show that $\varphi e^{\mathrm{ad}_x} \varphi^{-1} \in \mathrm{Int}(\mathfrak{g})$ for all $\varphi \in \mathrm{Aut}(\mathfrak{g})$. To this extent, let $\varphi \in \mathrm{Aut}(\mathfrak{g})$ be arbitrary. For all $y \in \mathfrak{g}$ and $1 \leq i \leq k-1$, we then obtain by using Definition 2.1.6 that

$$\begin{aligned}(\varphi(\mathrm{ad}_x)^i\varphi^{-1})(y) &= \varphi([x,[x,\ldots,[x,\varphi^{-1}(y)]\ldots]]) \\ &= [\varphi(x),[\varphi(x),\ldots,[\varphi(x),y]\ldots]] = (\mathrm{ad}_{\varphi(x)})^i(y)\end{aligned}$$

so that $\varphi(\mathrm{ad}_x)^i\varphi^{-1} = (\mathrm{ad}_{\varphi(x)})^i$. But then

$$(\varphi e^{\mathrm{ad}_x}\varphi^{-1})(y) = \left(\varphi \sum_{i=0}^{k-1} \frac{(\mathrm{ad}_x)^i}{i!}\varphi^{-1}\right)(y) = \sum_{i=0}^{k-1} \frac{(\mathrm{ad}_{\varphi(x)})^i(y)}{i!} = e^{\mathrm{ad}_{\varphi(x)}},$$

hence $\varphi e^{\mathrm{ad}_x}\varphi^{-1} = e^{\mathrm{ad}_{\varphi(x)}} \in \mathrm{Int}(\mathfrak{g})$. □

The quotient group $\mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$ is referred to as the group of *outer automorphisms*. The importance of the adjoint representation is further emphasised by the following lemma.

Lemma 2.1.9. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. Then the adjoint representation of $\mathfrak{g}$ is a Lie algebra homomorphism between $\mathfrak{g}$ and $\mathrm{Der}(\mathfrak{g})$.*

Proof. By Lemma 2.1.5, $\mathrm{Der}(\mathfrak{g})$ is a Lie algebra and $\mathrm{ad}_x \in \mathrm{Der}(\mathfrak{g})$ for all $x \in \mathfrak{g}$, hence the adjoint representation of $\mathfrak{g}$ is indeed a morphism of Lie algebras. It remains to show that $\mathrm{ad}_{[x,y]}(z) = [\mathrm{ad}_x, \mathrm{ad}_y](z)$ for all $z \in \mathfrak{g}$. Using Definition 2.1.1, we obtain

$$\begin{aligned}\mathrm{ad}_{[x,y]}(z) &= [[x,y],z] = -[z,[x,y]] = [y,[z,x]] + [x,[y,z]] = [x,[y,z]] - [y,[x,z]] \\ &= \mathrm{ad}_x(\mathrm{ad}_y(z)) - \mathrm{ad}_y(\mathrm{ad}_x(z)) = [\mathrm{ad}_x, \mathrm{ad}_y](z)\end{aligned}$$

for all $z \in \mathfrak{g}$, so $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{g})$ is a Lie algebra homomorphism. □

Recall from group theory the concepts of center, centraliser and normaliser. These notions also exist in the theory of Lie algebras and are defined as follows.

Definition 2.1.10 (Centraliser, Center & Normaliser)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. The* ***centraliser*** *of a subset $\mathfrak{h} \subseteq \mathfrak{g}$ is the set $C_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid \forall y \in \mathfrak{h} : [x,y] = 0\}$. If $\mathfrak{h} = \mathfrak{g}$, we obtain the set $Z(\mathfrak{g}) = C_{\mathfrak{g}}(\mathfrak{g})$, called the* ***center*** *of $\mathfrak{g}$. The* ***normaliser*** *of a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is the set $N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid \forall y \in \mathfrak{h} : [x,y] \in \mathfrak{h}\}$.*

A Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is called *self-normalising* if $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$. In the special case that $\mathfrak{g} = Z(\mathfrak{g})$, we obtain an *abelian* Lie algebra. Abelian Lie algebras can be obtained from arbitrary Lie algebras, as demonstrated by the following example.

Example 2.1.11. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. Take $x \in \mathfrak{g}$ to be an arbitrary element and consider the space $\mathbb{F}x = \langle x \rangle$ spanned by x. Then $\mathbb{F}x \subseteq \mathfrak{g}$ is clearly a Lie subalgebra of $\mathfrak{g}$. Now let $\lambda x, \mu x \in \mathbb{F}x$ with $\lambda, \mu \in \mathbb{F}$ distinct, then by bilinearity and alternativity of the Lie bracket we have $[\lambda x, \mu x] = \lambda\mu[x,x] = 0$, hence $\mathbb{F}x$ is an abelian Lie algebra.

The notions introduced in Definition 2.1.10 have the following properties, which makes them useful for the study of Lie algebras.

Lemma 2.1.12. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. Then $C_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$ for all subsets $\mathfrak{h} \subseteq \mathfrak{g}$, and $N_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$ for all Lie subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$.*

Proof. In accordance with Definition 2.1.1, we show that $[x,y]$ lies in the centraliser or normaliser of some subset or Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, respectively. First, let $x, y \in C_{\mathfrak{g}}(\mathfrak{h})$, and let $z \in \mathfrak{h}$ be arbtirary, then $[[x,y],z] = -[z,[x,y]] = [y,[z,x]] + [x,[y,z]]$ by Definition 2.1.1. Since $x, y \in C_{\mathfrak{g}}(\mathfrak{h})$ and $z \in \mathfrak{h}$, we have $[x,z] = [z,x] = 0$ and $[y,z] = [z,y] = 0$, hence

$[y,[z,x]]+[x,[y,z]]=[y,0]+[x,0]=0$ so that $[[x,y],z]=0$. It follows that $[x,y]\in C_{\mathfrak{g}}(\mathfrak{h})$, so $C_{\mathfrak{g}}(\mathfrak{h})$ is a Lie subalgebra of $\mathfrak{g}$.

Now let $x,y\in N_{\mathfrak{g}}(\mathfrak{h})\subseteq\mathfrak{g}$ and let $z\in\mathfrak{h}$ be arbitrary. By the above, we have $[[x,y],z]=[y,[z,x]]+[x,[y,z]]$. Because $x,y\in N_{\mathfrak{g}}(\mathfrak{h})$ and $z\in\mathfrak{h}$, we have $[x,z],[z,x],[y,z],[z,y]\in\mathfrak{h}$, but then also $[y,[z,x]],[x,[y,z]]\in\mathfrak{h}$. Consequently, $[[x,y],z]\in\mathfrak{h}$, which shows that $[x,y]\in N_{\mathfrak{g}}(\mathfrak{h})$ so that $N_{\mathfrak{g}}(\mathfrak{h})$ is a Lie subalgebra of $\mathfrak{g}$. □

As a consequence of Lemma 2.1.12 in combination with Definition 2.1.10, the center $Z(\mathfrak{g})$ of a Lie subalgebra $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$.

A final important class of subspaces of Lie algebras that we will discuss here is the class of ideals. Their definition is the following, analogous to how ideals are defined in ring and field theory.

Definition 2.1.13 (Ideal & Simplicity)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. An **ideal** of $\mathfrak{g}$ is a subspace $\mathfrak{i}\subseteq\mathfrak{g}$ such that $[x,i]\in\mathfrak{i}$ for all $x\in\mathfrak{g}$ and $i\in\mathfrak{i}$. If $\mathfrak{g}$ is non-abelian and contains no non-trivial ideals, then $\mathfrak{g}$ is said to be **simple**.*

Recall that the finitary general linear Lie algebra $\mathfrak{fgl}(V)$ of the general linear Lie algebra $\mathfrak{gl}(V)$ for some vector space V over a field $\mathbb{F}$ is a proper Lie subalgebra if and only if V is infinite-dimensional. The following example shows that $\mathfrak{fgl}(V)$ is also an ideal of $\mathfrak{gl}(V)$.

Example 2.1.14. Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. The center $Z(\mathfrak{g})$ of $\mathfrak{g}$ is a non-trivial ideal of $\mathfrak{g}$; indeed, for all $x\in\mathfrak{g}$ and $z\in Z(\mathfrak{g})$ we have $[z,x]=0$ by Definition 2.1.10, and clearly $0\in Z(\mathfrak{g})$. If $\mathfrak{g}$ is simple, then this forces $Z(\mathfrak{g})=0$, for otherwise $Z(\mathfrak{g})=\mathfrak{g}$ in which case $\mathfrak{g}$ would be abelian, contradicting Definition 2.1.13. The non-trivial subspace $\mathfrak{i}=[\mathfrak{g},\mathfrak{g}]$ is also an ideal of $\mathfrak{g}$, which follows immediately from $\mathfrak{i}$ being a subspace of $\mathfrak{g}$. The ideal $\mathfrak{i}$ is referred to as the *derived algebra* of $\mathfrak{g}$. If $\mathfrak{g}$ is simple, we have $\mathfrak{i}=\mathfrak{g}$, for otherwise $[\mathfrak{g},\mathfrak{g}]=0$ in which case $\mathfrak{g}$ would be abelian, again contradicting Definition 2.1.13.

Another example is the following. Given an infinite-dimensional vector space V over a field $\mathbb{F}$, the finitary general linear Lie algebra $\mathfrak{fgl}(V)\subseteq\mathfrak{gl}(V)$ is a proper ideal. Indeed, for all $\varphi\in\mathfrak{fgl}(V)$ and $\psi\in\mathfrak{gl}(V)$, we have

$$\dim([\varphi,\psi](V))=\dim((\varphi\psi-\psi\varphi)(V))\leq\dim(\varphi(\psi(V)))+\dim(\psi(\varphi(V)))<\infty$$

because $\psi(V)\subseteq V$ and $\dim(\varphi(V))<\infty$, hence $[\varphi,\psi]\in\mathfrak{fgl}(V)$.

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. Recall from Lemma 2.1.5 that $\mathrm{ad}_x\in\mathrm{Der}(\mathfrak{g})$ for all $x\in\mathfrak{g}$. The following lemma relates ideals to the adjoint representation.

Lemma 2.1.15. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ and let $\mathrm{ad}:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g})$ be its adjoint representation. Denote by $\mathrm{ad}_{\mathfrak{g}}$ the Lie subalgebra of $\mathrm{Der}(\mathfrak{g})$ containing all elements $\mathrm{ad}_x\in\mathrm{Der}(\mathfrak{g})$ with $x\in\mathfrak{g}$. Then $\mathrm{ad}_{\mathfrak{g}}$ is an ideal of $\mathrm{Der}(\mathfrak{g})$.*

Proof. We prove that $[\varphi, \mathrm{ad}_x] \in \mathrm{ad}_{\mathfrak{g}}$ for all $\varphi \in \mathrm{Der}(\mathfrak{g})$ and $x \in \mathfrak{g}$. To this extent, let $\varphi \in \mathrm{Der}(\mathfrak{g})$ and $x \in \mathfrak{g}$ be arbitrary, then for all $y \in \mathfrak{g}$ we have

$$[\varphi, \mathrm{ad}_x](y) = (\varphi \mathrm{ad}_x - \mathrm{ad}_x \varphi)(y) = \varphi([x, y]) - [x, \varphi(y)] = [\varphi(x), y] = \mathrm{ad}_{\varphi(x)}(y)$$

so that $[\varphi, \mathrm{ad}_x] = \mathrm{ad}_{\varphi(x)} \in \mathrm{ad}_{\mathfrak{g}}$. We conclude that $\mathrm{ad}_{\mathfrak{g}} \subseteq \mathrm{Der}(\mathfrak{g})$ is an ideal. □

The elements in the ideal $\mathrm{ad}_{\mathfrak{g}}$ of $\mathrm{Der}(\mathfrak{g})$ from the above lemma are called the *inner derivations* of $\mathrm{Der}(\mathfrak{g})$. The Lie subalgebra of inner derivations of $\mathrm{Der}(\mathfrak{g})$ is denoted by $\mathrm{Inn}(\mathfrak{g})$.

Clearly, if $\mathfrak{i}, \mathfrak{j} \subseteq \mathfrak{g}$ are two distinct proper ideals of a Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$, then so are $\mathfrak{i} + \mathfrak{j}$ and $\mathfrak{i} \cap \mathfrak{j}$, but also $[\mathfrak{i}, \mathfrak{j}]$. The *quotient Lie algebra* of $\mathfrak{g}$ and a proper ideal $\mathfrak{i} \subseteq \mathfrak{g}$ is the Lie algebra $\mathfrak{g}/\mathfrak{i} = \{x + \mathfrak{i} \mid x \in \mathfrak{g}\}$ with Lie bracket $[x + \mathfrak{i}, y + \mathfrak{i}] = [x, y] + \mathfrak{i}$. Induced by the quotient algebra $\mathfrak{g}/\mathfrak{i}$ is the *canonical map* $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$ given by $x \mapsto x + \mathfrak{i}$ with $x \in \mathfrak{g}$. It is a Lie algebra epimorphism; surjectivity follows from construction, and the identity $\pi([x, y]) = [x, y] + \mathfrak{i} = [x + \mathfrak{i}, y + \mathfrak{i}] = [\pi(x), \pi(y)]$ turns π into a Lie algebra homomorphism. Exemplary quotient Lie algebras are $\mathfrak{pgl}(V) = \mathfrak{gl}(V)/Z(\mathfrak{gl}(V))$ if V is a finite-dimensional vector space over a field $\mathbb{F}$, but also $\mathfrak{gl}(V)/\mathfrak{fgl}(V)$ in case V is infinite-dimensional.

If $\mathfrak{g}'$ is a second Lie algebra and $\varphi : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra homomorphism, then $\ker(\varphi) \subseteq \mathfrak{g}$ is an ideal of $\mathfrak{g}$; indeed, for all $x \in \mathfrak{g}$ and $i \in \ker(\varphi)$, we have $\varphi([x, i]) = [\varphi(x), \varphi(i)] = [\varphi(x), 0] = 0$. Additionally, $\mathrm{im}(\varphi) \subseteq \mathfrak{g}'$ is a Lie subalgebra of $\mathfrak{g}'$. The homomorphism theorems from group theory extend naturally to Lie algebras.

Proposition 2.1.16. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be Lie algebras over a field $\mathbb{F}$. Then*

(i) *for all Lie algebra homomorphisms $\varphi : \mathfrak{g} \to \mathfrak{g}'$ we have $\mathfrak{g}/\ker(\varphi) \cong \mathrm{im}(\varphi)$. In particular, for any ideal $\mathfrak{i} \subseteq \mathfrak{g}$ such that $\mathfrak{i} \subseteq \ker(\varphi)$, there exists a unique map $\psi : \mathfrak{g}/\mathfrak{i} \to \mathfrak{g}'$ such that $\varphi(x) = \psi(\pi(x))$ for all $x \in \mathfrak{g}$,*

(ii) *for all ideals $\mathfrak{i}, \mathfrak{j} \subseteq \mathfrak{g}$ such that $\mathfrak{i} \subset \mathfrak{j}$, the subspace $\mathfrak{j}/\mathfrak{i} \subseteq \mathfrak{g}/\mathfrak{i}$ is an ideal and $\mathfrak{g}/\mathfrak{j} \cong (\mathfrak{g}/\mathfrak{i})/(\mathfrak{j}/\mathfrak{i})$,*

(iii) *for all ideals $\mathfrak{i}, \mathfrak{j} \subseteq \mathfrak{g}$, we have $(\mathfrak{i} + \mathfrak{j})/\mathfrak{j} \cong \mathfrak{i}/(\mathfrak{i} \cap \mathfrak{j})$.*

Proof. For (i), let $\varphi : \mathfrak{g} \to \mathfrak{g}'$ be a Lie algebra homomorphism and let $\mathfrak{i} \subseteq \mathfrak{g}$ be an ideal such that $\mathfrak{i} \subseteq \ker(\varphi)$. Then the map $\psi : \mathfrak{g}/\mathfrak{i} \to \mathfrak{g}'$ given by $x + \mathfrak{i} \mapsto \varphi(x)$ with $x \in \mathfrak{g}$ is readily seen to be an isomorphism between $\mathfrak{g}/\mathfrak{i}$ and $\mathrm{im}(\varphi)$. Additionally, it is a Lie algebra homomorphism, since

$$\psi([x + \mathfrak{i}, y + \mathfrak{i}]) = \psi([x, y] + \mathfrak{i}) = \varphi([x, y]) = [\varphi(x), \varphi(y)] = [\psi(x + \mathfrak{i}), \psi(y + \mathfrak{i})].$$

In particular, we have $\mathfrak{g}/\ker(\varphi) \cong \mathrm{im}(\varphi)$ by taking $\mathfrak{i} = \ker(\varphi)$.

For (ii), let $\mathfrak{i}, \mathfrak{j} \subseteq \mathfrak{g}$ be ideals such that $\mathfrak{i} \subset \mathfrak{j}$, then we have $[x + \mathfrak{i}, j + \mathfrak{i}] = [x, j] + \mathfrak{i} \in \mathfrak{j}/\mathfrak{i}$ for all $x \in \mathfrak{g}$ and $j \in \mathfrak{j}$. The map $\varphi : \mathfrak{g}/\mathfrak{i} \to \mathfrak{g}/\mathfrak{j}$ given by $x + \mathfrak{i} \mapsto x + \mathfrak{j}$ with $x \in \mathfrak{g}$ is surjective such that $\ker(\varphi) = \mathfrak{j}/\mathfrak{i}$. By Proposition 2.1.16(i), we then have $(\mathfrak{g}/\mathfrak{i})/(\mathfrak{j}/\mathfrak{i}) = (\mathfrak{g}/\mathfrak{i})/\ker(\varphi) \cong \mathrm{im}(\varphi) = \mathfrak{g}/\mathfrak{j}$.

For (iii), let $\mathfrak{i}, \mathfrak{j} \subseteq \mathfrak{g}$ be ideals and let $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{j}$ be the canonical map from $\mathfrak{g}$ to $\mathfrak{g}/\mathfrak{j}$ and consider $\pi|_{\mathfrak{i}}$, i.e. its restriction to $\mathfrak{i}$. Then $\ker(\pi|_{\mathfrak{i}}) = \mathfrak{i} \cap \mathfrak{j}$ and $\mathrm{im}(\pi|_{\mathfrak{i}}) = (\mathfrak{i}+\mathfrak{j})/\mathfrak{j}$, hence by Proposition 2.1.16(i) we have $\mathfrak{i}/(\mathfrak{i} \cap \mathfrak{j}) = \mathfrak{i}/\ker(\pi|_{\mathfrak{i}}) \cong \mathrm{im}(\pi|_{\mathfrak{i}}) = (\mathfrak{i}+\mathfrak{j})/\mathfrak{j}$. □

We continue our discussion of ideals of Lie algebras by introducing certain sequences of ideals as follows.

Definition 2.1.17 (Derived series & Solvability)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. The **derived series** of $\mathfrak{g}$ is the sequence of ideals of $\mathfrak{g}$ given recursively by $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$ with $i \geq 0$. If there exists an $n \geq 0$ such that $\mathfrak{g}^{(n)} = 0$, then $\mathfrak{g}$ is said to be **solvable**.*

Example 2.1.18. Abelian Lie algebras are clearly solvable, whereas simple Lie algebras are not. Now consider the Lie algebra from Example 2.1.2 restricted to only the strictly upper-triangular matrices, which we will denote by $\mathfrak{n}(n, \mathbb{R})$. Note that these matrices will only have non-zero entries in positions (i, j), $1 \leq i, j \leq n$, satisfying $i < j$. To see that $\mathfrak{n}(n, \mathbb{R})$ is solvable, it suffices to observe that the elements of $(\mathfrak{n}(n, \mathbb{R}))^{(k)}$, $k \geq 0$, only contain non-zero entries in positions (i, j), $1 \leq i, j \leq n$, such that $j - i \geq 2^k$. But then certainly $(\mathfrak{n}(n, \mathbb{R}))^{(m)} = 0$ for $m \geq 0$ such that $2^m \geq n$, as then $j - i \geq 2^m \geq n$ for all $1 \leq i, j, \leq n$ so that no element of $(\mathfrak{n}(n, \mathbb{R}))^{(m)}$ contains non-zero entries.

Consider also the Lie subalgebra obtained from Example 2.1.2 obtained by only taking the upper-triangular matrices, which we will denote by $\mathfrak{t}(n, \mathbb{R})$. By observing that $[\mathfrak{t}(n, \mathbb{R}), \mathfrak{t}(n, \mathbb{R})] = \mathfrak{n}(n, \mathbb{R})$, it immediately follows that $\mathfrak{t}(n, \mathbb{R})$ is solvable as well. In particular, we have $(\mathfrak{t}(n, \mathbb{R}))^{(m+1)} = 0$ with m as in the previous paragraph.

Some simple and useful properties of solvable Lie algebras are the following.

Proposition 2.1.19. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. Then*

(i) *all homomorphic images of $\mathfrak{g}$ as well as all Lie subalgebras of $\mathfrak{g}$ are solvable if $\mathfrak{g}$ is solvable,*

(ii) *$\mathfrak{g}$ is solvable if $\mathfrak{i} \subseteq \mathfrak{g}$ is a solvable ideal such that $\mathfrak{g}/\mathfrak{i}$ is solvable,*

(iii) *$\mathfrak{i} + \mathfrak{j} \subseteq \mathfrak{g}$ is a solvable ideal whenever $\mathfrak{i}, \mathfrak{j} \subseteq \mathfrak{g}$ are solvable ideals.*

Proof. For (i), we clearly have $\mathfrak{h}^{(i)} \subseteq \mathfrak{g}^{(i)}$ for any Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$, showing that $\mathfrak{h}$ is solvable whenever $\mathfrak{g}$ is. Now let $\mathfrak{g}'$ be the homomorphic image of $\mathfrak{g}$ under some Lie algebra epimorphism $\varphi : \mathfrak{g} \to \mathfrak{g}'$. For any $i \geq 0$, we have $\varphi(\mathfrak{g}^{(i+1)}) = \varphi([\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]) = [\varphi(\mathfrak{g}^{(i)}), \varphi(\mathfrak{g}^{(i)})]$. By induction, we then obtain $\varphi(\mathfrak{g}^{(i)}) = (\mathfrak{g}')^{(i)}$ for all $i \geq 0$, hence $\mathfrak{g}'$ is solvable whenever $\mathfrak{g}$ is.

For (ii), let $\mathfrak{i} \subseteq \mathfrak{g}$ be a solvable ideal such that $\mathfrak{g}/\mathfrak{i}$ is solvable and consider the canonical map $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$. By Proposition 2.1.19(i), we have $\pi(\mathfrak{g}^{(i)}) = (\mathfrak{g}/\mathfrak{i})^{(i)}$ for all $i \geq 0$. Solvability of $\mathfrak{g}/\mathfrak{i}$ then implies that there exists an $n \geq 0$ such that $\pi(\mathfrak{g}^{(n)}) = 0$, hence $\mathfrak{g}^{(n)} \subseteq \ker(\pi) = \mathfrak{i}$. But $\mathfrak{i}$ is solvable by assumption, which proves that $\mathfrak{g}$ is solvable as well.

For (iii), let $\mathfrak{i}, \mathfrak{j} \subseteq \mathfrak{g}$ be solvable ideals. Recall from our proof of Proposition 2.1.16(iii) the canonical map $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{j}$ as well as its restriction to $\mathfrak{i}$, denoted by $\pi|_{\mathfrak{i}}$. By Proposition 2.1.19(i), we know that $\mathrm{im}(\pi|_{\mathfrak{i}}) = (\mathfrak{i} + \mathfrak{j})/\mathfrak{j}$ is solvable. But then $\mathfrak{j} \subseteq \mathfrak{i} + \mathfrak{j}$ is an ideal such that $(\mathfrak{i} + \mathfrak{j})/\mathfrak{j}$ is solvable, showing that $\mathfrak{i} + \mathfrak{j}$ is solvable by Proposition 2.1.19(ii). □

As a consequence of Proposition 2.1.19(iii), every Lie algebra $\mathfrak{g}$ contains a unique inclusion-wise maximal solvable ideal. This ideal is called the *radical* of $\mathfrak{g}$ and denoted by $\mathrm{rad}(\mathfrak{g})$. If $\mathrm{rad}(\mathfrak{g}) = 0$, then $\mathfrak{g}$ is said to be *semi-simple*. Consequently, simple Lie algebras are semi-simple and the quotient algebra $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semi-simple.

An important theorem on solvable Lie algebras is the following.

Theorem 2.1.20 (Lie's theorem)**.** *Let $\mathfrak{g}$ be a solvable Lie subalgebra of $\mathfrak{gl}(V)$ for some finite-dimensional vector space V over a field $\mathbb{F}$ and let $\mathfrak{t}(V, \mathbb{F})$ be the Lie subalgebra of $\mathfrak{gl}(V)$ of all upper-triangular matrices. Then there exists a basis of V such that $\mathfrak{g} \subseteq \mathfrak{t}(V, \mathbb{F})$.*

Proof. See Section 4.1 of [1]. □

Closely related to the notion of solvability of a Lie algebra $\mathfrak{g}$ in terms of the corresponding sequence of ideals used to define it is the concept of nilpotency as given below.

Definition 2.1.21 (Descending central series & Nilpotency)**.** *The **descending central series** of $\mathfrak{g}$ is the sequence of ideals of $\mathfrak{g}$ given recursively by $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$ with $i \geq 0$. If there exists an $n \geq 0$ such that $\mathfrak{g}^n = 0$, then $\mathfrak{g}$ is said to be **nilpotent**.*

Example 2.1.22. Abelian Lie algebras are clearly nilpotent, whereas simple Lie algebras are not. Consider again the Lie algebra $\mathfrak{n}(n, \mathbb{R})$ from the previous example. In this case, we see that the elements $(\mathfrak{n}(n, \mathbb{R}))^k$, $k \geq 0$, only contain non-zero entries in positions (i, j), $1 \leq i, j \leq n$, satisfying $j - i \geq k + 1$. It follows that $(\mathfrak{n}(n, \mathbb{R}))^m = 0$ for $m \geq 0$ such that $m + 1 \geq n$, as then $j - i \geq m + 1 \geq n$ for all $1 \leq i, j \leq n$ so that no element of $(\mathfrak{n}(n, \mathbb{R}))^m$ contains non-zero entries.

However, the Lie algebra $\mathfrak{t}(n, \mathbb{R})$ from the previous example is not nilpotent. This follows from the observation that $[\mathfrak{t}(n, \mathbb{R}), \mathfrak{n}(n, \mathbb{R})] = \mathfrak{n}(n, \mathbb{R})$ so that $(\mathfrak{t}(n, \mathbb{R}))^k = (\mathfrak{t}(n, \mathbb{R}))^1 = \mathfrak{n}(n, \mathbb{R})$ for all $k \geq 1$.

As before, we list some simple and useful properties of nilpotent Lie algebras.

Proposition 2.1.23. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. Then*

(i) *all homomorphic images of $\mathfrak{g}$ as well as all Lie subalgebras of $\mathfrak{g}$ are nilpotent if $\mathfrak{g}$ is nilpotent,*

(ii) $\mathfrak{g}$ *is nilpotent if* $\mathfrak{g}/Z(\mathfrak{g})$ *is nilpotent,*

(iii) $Z(\mathfrak{g}) \neq 0$ *if* $\mathfrak{g}$ *is nilpotent and non-zero.*

Proof. For (i), everything we have said in our proof of Proposition 2.1.19(i), but applied to the descending central series instead of the derived series, holds true.

For (ii), suppose that $\mathfrak{g}/Z(\mathfrak{g})$ is nilpotent, i.e. $(\mathfrak{g}/Z(\mathfrak{g}))^n = 0$ for some $n \geq 0$. Consequently, we have $\mathfrak{g}^n \subseteq Z(\mathfrak{g})$ so that $\mathfrak{g}^{n+1} = [\mathfrak{g}, \mathfrak{g}^n] \subseteq [\mathfrak{g}, Z(\mathfrak{g})] = 0$, showing that $\mathfrak{g}$ is nilpotent.

For (iii), assume that $\mathfrak{g}$ is nilpotent and non-zero. Let $n \geq 0$ be minimal such that $\mathfrak{g}^n = 0$. In particular, we then have $[\mathfrak{g}, \mathfrak{g}^{n-1}] = \mathfrak{g}^n = 0$, hence $\mathfrak{g}^{n-1} \subseteq Z(\mathfrak{g})$. By minimality of n, we clearly have $\mathfrak{g}^{n-1} \neq 0$ so that $Z(\mathfrak{g}) \neq 0$. □

Whereas ad-nilpotency is used for elements of a Lie algebra $\mathfrak{g}$ regarding its adjoint representation, nilpotency relates to the Lie algebra $\mathfrak{g}$ itself. The two notions are related through the following theorem.

Theorem 2.1.24 (Engel's theorem)**.** *Let* $\mathfrak{g}$ *be a Lie algebra over a field* $\mathbb{F}$. *Then every* $x \in \mathfrak{g}$ *is ad-nilpotent if and only if* $\mathfrak{g}$ *is nilpotent.*

Proof. Since $(\mathrm{ad}_x)^i(y) \in \mathfrak{g}^i$ for all $x, y \in \mathfrak{g}$ and $i \geq 0$, nilpotency of $\mathfrak{g}$ immediately implies ad-nilpotency of every $x \in \mathfrak{g}$. For the converse, see Section 3.3 of [1]. □

We finish our discussion of solvable and nilpotent Lie algebras with the following corollary, which relates both concepts to one another.

Corollary 2.1.25. *Let* $\mathfrak{g}$ *be a Lie algebra over a field* $\mathbb{F}$. *Then* $\mathfrak{g}$ *is solvable if and only if* $[\mathfrak{g}, \mathfrak{g}]$ *is nilpotent.*

Proof. First, assume that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. Since $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$, then clearly $\mathfrak{g}^0 = \mathfrak{g}$ is also nilpotent. By further observing that $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] \subseteq [\mathfrak{g}, \mathfrak{g}^{(i)}]$ for all $i \geq 0$, it follows that $\mathfrak{g}^{(i)} \subseteq \mathfrak{g}^i$ for all $i \geq 0$ by induction. But then nilpotency of $\mathfrak{g}$ implies solvability of $\mathfrak{g}$.

Next, assume that $\mathfrak{g}$ is solvable. Denote by $\mathrm{ad}_{\mathfrak{g}}$ the Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$ containing the matrices of ad_x with $x \in \mathfrak{g}$. Then solvability of $\mathfrak{g}$ implies solvability of $\mathrm{ad}_{\mathfrak{g}}$, hence by Theorem 2.1.20 we have $\mathrm{ad}_{\mathfrak{g}} \subseteq \mathfrak{t}(\mathfrak{g}, \mathbb{F})$. By recalling from our proof of Lemma 2.1.9 that $[\mathrm{ad}_x, \mathrm{ad}_y] = \mathrm{ad}_{[x,y]}$, we obtain

$$\mathrm{ad}_{[\mathfrak{g},\mathfrak{g}]} = [\mathrm{ad}_{\mathfrak{g}}, \mathrm{ad}_{\mathfrak{g}}] \subseteq [\mathfrak{t}(\mathfrak{g}, \mathbb{F}), \mathfrak{t}(\mathfrak{g}, \mathbb{F})] = \mathfrak{n}(\mathfrak{g}, \mathbb{F}).$$

But we have seen in Example 2.1.22 that $\mathfrak{n}(\mathfrak{g}, \mathbb{F})$ is nilpotent, hence by the above every element in $[\mathfrak{g}, \mathfrak{g}]$ is ad-nilpotent. Finally, by Theorem 2.1.24, we conclude that $[\mathfrak{g}, \mathfrak{g}]$ itself is nilpotent. □

For the final part of this section, we will focus more on semi-simplicity of Lie algebras. Note, however, that the notion of semi-simplicity also applies to endomorphisms of vector spaces. In particular, given a vector space V over a field $\mathbb{F}$, we say that $\varphi \in \mathfrak{gl}(V)$ is *semi-simple* if φ is diagonalisable. Similarly, the notion of nilpotency also applies to endomorphisms of vector spaces. We say that $\varphi \in \mathfrak{gl}(V)$ is *nilpotent* if $\varphi^n = 0$ for some integer $n \geq 0$. This leads to the following useful decomposition of endomorphisms due to Jordan and Chevalley.

Definition 2.1.26 (Jordan-Chevalley decomposition)**.** *Let V be a vector space over a field $\mathbb{F}$. The **Jordan-Chevalley decomposition** of $\varphi \in \mathfrak{gl}(V)$ is the unique decomposition of φ as the sum of a semi-simple endomorphism $\varphi_s \in \mathfrak{gl}(V)$ and a nilpotent endomorphism $\varphi_n \in \mathfrak{gl}(V)$ such that $[\varphi_s, \varphi_n] = 0 = [\varphi_n, \varphi_s]$.*

Given an endomorphism $\varphi \in \mathfrak{gl}(V)$ and its unique Jordan-Chevalley decomposition $\varphi = \varphi_s + \varphi_n$ with commuting semi-simple $\varphi_s \in \mathfrak{gl}(V)$ and nilpotent $\varphi_n \in \mathfrak{gl}(V)$, we clearly see that $\mathrm{ad}_{\varphi_s} \in \mathfrak{gl}(\mathfrak{gl}(V))$ is semi-simple and $\mathrm{ad}_{\varphi_n} \in \mathfrak{gl}(\mathfrak{gl}(V))$ is nilpotent. Moreover, we have $[\mathrm{ad}_{\varphi_s}, \mathrm{ad}_{\varphi_n}] = \mathrm{ad}_{[\varphi_s,\varphi_n]} = 0$ so that $\mathrm{ad}_{\varphi} = \mathrm{ad}_{\varphi_s} + \mathrm{ad}_{\varphi_n}$ is the unique Jordan-Chevalley decomposition of $\mathrm{ad}_{\varphi} \in \mathfrak{gl}(\mathfrak{gl}(V))$.

The Jordan-Chevalley decomposition is key to proving the following theorem due to Cartan.

Theorem 2.1.27 (Cartan's criterion)**.** *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$ for some finite-dimensional vector space V over a field $\mathbb{F}$. Then $\mathfrak{g}$ is solvable if xy is traceless for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$.*

Proof. See Section 4.3 of [1]. □

We return to semi-simple Lie algebras and introduce the following definition due to Killing for the purpose of their study.

Definition 2.1.28 (Killing form)**.** *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. The **Killing form** of $\mathfrak{g}$ is the bilinear symmetric form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ given by $(x, y) \mapsto tr(\mathrm{ad}_x \mathrm{ad}_y)$ with $x, y \in \mathfrak{g}$, in which $tr : \mathfrak{gl}(\mathfrak{g}) \to \mathbb{F}$ is the linear map that sends an endomorphism in $\mathfrak{gl}(\mathfrak{g})$ to the sum of its diagonal entries.*

The Killing form κ is also associative. To see this, first observe that

$$\begin{aligned} tr([x,y]z) &= tr(xyz - yxz) = tr(xyz) - tr(y(xz)) = tr(xyz) - tr((xz)y) \\ &= tr(xyz - xzy) = tr(x[y,z]) \end{aligned}$$

for all $x, y, z \in \mathfrak{gl}(\mathfrak{g})$. Consequently, for $\mathrm{ad}_x, \mathrm{ad}_y, \mathrm{ad}_z \in \mathfrak{gl}(\mathfrak{g})$ with $x, y, z \in \mathfrak{g}$, we obtain

$$\begin{aligned} \kappa([x,y],z) &= tr(\mathrm{ad}_{[x,y]}\mathrm{ad}_z) = tr([\mathrm{ad}_x, \mathrm{ad}_y]\mathrm{ad}_z) = tr(\mathrm{ad}_x[\mathrm{ad}_y, \mathrm{ad}_z]) = \\ &= tr(\mathrm{ad}_x \mathrm{ad}_{[y,z]}) = \kappa(x, [y,z]). \end{aligned}$$

The Killing form κ is said to be *non-degenerate* if its radical $\mathrm{rad}(\kappa) = \{x \in \mathfrak{g} \mid \forall y \in \mathfrak{g} : \kappa(x, y) = 0\}$ is trivial and *degenerate* otherwise. Associativity of κ then implies that $\mathrm{rad}(\kappa) \subseteq \mathfrak{g}$ is an ideal; indeed, for all $x \in \mathfrak{g}$ and $r \in \mathrm{rad}(\kappa)$, we have $\kappa([r, x], y) = \kappa(r, [x, y]) = 0$ for all $y \in \mathfrak{g}$. This leads to the following important theorem relating the Killing form to semi-simple Lie algebras.

Theorem 2.1.29. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ and let $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ be the Killing form of $\mathfrak{g}$. Then $\mathfrak{g}$ is semi-simple if and only if κ is non-degenerate.*

Proof. First, assume that $\mathfrak{g}$ is semi-simple, or equivalently, $\mathrm{rad}(\mathfrak{g}) = 0$. Clearly, we have $\kappa(x, y) = tr(\mathrm{ad}_x \mathrm{ad}_y) = 0$ for all $x \in \mathrm{rad}(\kappa)$ and $y \in [\mathrm{rad}(\kappa), \mathrm{rad}(\kappa)] \subseteq \mathfrak{g}$. Consequently, the Lie subalgebra $\mathrm{ad}_{\mathrm{rad}(\kappa)} \subseteq \mathfrak{gl}(\mathfrak{g})$, which consists of all elements $\mathrm{ad}_x \in \mathfrak{gl}(\mathfrak{g})$ with $x \in \mathrm{rad}(\kappa)$, is solvable by Theorem 2.1.27, hence $\mathrm{rad}(\kappa)$ is solvable as well. But $\mathrm{rad}(\kappa) \subseteq \mathfrak{g}$ is an ideal, therefore $\mathrm{rad}(\kappa) \subseteq \mathrm{rad}(\mathfrak{g}) = 0$, showing that κ is non-degenerate.

Next, assume that κ is non-degenerate, i.e. $\mathrm{rad}(\kappa) = 0$, and let $\mathrm{rad}(\mathfrak{g})$ be the maximal solvable ideal of $\mathfrak{g}$. Then

$$[(\mathrm{rad}(\mathfrak{g}))^{(n-1)}, (\mathrm{rad}(\mathfrak{g}))^{(n-1)}] = (\mathrm{rad}(\mathfrak{g}))^{(n)} = 0$$

for some $n \geq 0$ minimal. In particular, $\mathfrak{i} = \mathrm{rad}(\mathfrak{g})^{(n-1)} \subseteq \mathfrak{g}$ is a non-zero abelian ideal because of minimality of n. By further noting that

$$[\mathfrak{i}, [\mathfrak{g}, [\mathfrak{i}, [\mathfrak{g}, \mathfrak{g}]]]] \subseteq [\mathfrak{i}, [\mathfrak{g}, [\mathfrak{i}, \mathfrak{g}]]] \subseteq [\mathfrak{i}, [\mathfrak{g}, \mathfrak{i}]] \subseteq [\mathfrak{i}, \mathfrak{i}] = 0,$$

we deduce that $(\mathrm{ad}_i \mathrm{ad}_x)^2 = 0$ for all $i \in \mathfrak{i}$ and $x \in \mathfrak{g}$. Consequently, $\kappa(i, x) = tr(\mathrm{ad}_i \mathrm{ad}_x) = 0$ for all $i \in \mathfrak{i}$ and $x \in \mathfrak{g}$, showing that $\mathfrak{i} \subseteq \mathrm{rad}(\kappa)$. But κ is non-degenerate, forcing $\mathfrak{i} = 0$. This, in turn, contradicts that $\mathfrak{i}$ is non-zero, so we conclude that $\mathfrak{g}$ cannot contain any solvable ideals. It follows that $\mathfrak{g}$ is semi-simple. □

We finish this section with an important characterisation of semi-simple Lie algebras and their ideals.

Theorem 2.1.30. *Let $\mathfrak{g}$ be a semi-simple Lie algebra over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) = 0$. Then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and there exist simple ideals $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ of $\mathfrak{g}$, viewed as Lie subalgebras of $\mathfrak{g}$, with $n \geq 1$ finite such that $\mathfrak{g} = \bigoplus_{k=1}^{n} \mathfrak{g}_k$. In particular, for every simple ideal $\mathfrak{i} \subseteq \mathfrak{g}$ we have $\mathfrak{i} = \mathfrak{g}_i$ for some $1 \leq i \leq n$.*

Moreover, all ideals and homomorphic images of $\mathfrak{g}$ are semi-simple, and for every ideal $\mathfrak{j} \subseteq \mathfrak{g}$ there exists an index set J such that $\mathfrak{j} = \sum_{j \in J} \mathfrak{g}_j$.

Proof. See Section 5.2 of [1]. □

2.2 Classical linear Lie algebras

In the previous section, we have already discussed the general linear Lie algebra $\mathfrak{gl}(V)$ with V a finite-dimensional vector space over a field $\mathbb{F}$. Its name can be explained by its close relation to the *general linear group*, denoted by $\mathrm{GL}(V)$, consisting of all invertible endomorphisms of V. An important (normal) subgroup of $\mathrm{GL}(V)$ is the *special linear group*, denoted by $\mathrm{SL}(V)$, containing all endomorphisms in $\mathrm{GL}(V)$ with determinant 1. This leads us to the second important linear Lie algebra.

Definition 2.2.1 (Special linear Lie algebra)**.** *Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and let $\mathfrak{gl}(V)$ be the general linear Lie algebra of V. The* ***special linear Lie algebra*** *is the Lie subalgebra $\mathfrak{sl}(V)$ of $\mathfrak{gl}(V)$ consisting of the traceless endomorphisms in $\mathfrak{gl}(V)$.*

For the same reasons as listed for the general linear Lie algebra, we will also use the notation $\mathfrak{sl}_n(\mathbb{F})$ for the special linear Lie algebra. Note that the special linear Lie algebra is an ideal of the general linear Lie algebra. If V is infinite-dimensional, we obtain the *finitary special linear Lie algebra* consisting of all traceless linear transformations $\varphi \in \mathfrak{sl}(V)$ such that $\dim(\varphi(V)) < \infty$, denoted by $\mathfrak{fsl}(V)$. In particular, $\mathfrak{fsl}(V)$ is an ideal of $\mathfrak{fgl}(V)$. By $\mathfrak{psl}(V)$ we denote the quotient Lie algebra $\mathfrak{sl}(V)/Z(\mathfrak{sl}(V))$ with V finite-dimensional. Observe that $Z(\mathfrak{sl}(V)) = \{0\}$ if and only if $\mathrm{char}(\mathbb{F}) \nmid \dim(\mathfrak{sl}(V))$.

Example 2.2.2. Let V be a 2-dimensional vector space over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$. By definition, $\mathfrak{gl}(V)$ will be 4-dimensional with standard basis $\{E_{1,1}, E_{1,2}, E_{2,1}, E_{2,2}\}$, in which $E_{i,j} \in \mathfrak{gl}(V)$, $1 \leq i, j \leq 2$, is the matrix having a one in position (i, j) and zeros elsewhere. It is easy to see that the traceless matrices in $\mathfrak{gl}(V)$ are of the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

so that $\mathfrak{sl}(V) = \mathfrak{sl}_2(\mathbb{F})$ has standard basis $\{E_{1,1} - E_{2,2}, E_{1,2}, E_{2,1}\}$. Consequently, $\mathfrak{sl}_2(\mathbb{F})$ is 3-dimensional. We write $x = E_{1,2}$, $y = E_{2,1}$ and $h = E_{1,1} - E_{2,2}$ and observe that $[x, y] = h$, $[h, x] = 2x$ and $[h, y] = -2y$. Conversely, any triple $\{x, y, h\}$ satisfying $[x, y] = h$, $[h, x] = 2x$ and $[h, y] = -2y$ is called an $\mathfrak{sl}_2$*-triple*. Observe that $[\mathfrak{sl}_2(\mathbb{F}), \mathfrak{sl}_2(\mathbb{F})] = \mathfrak{sl}_2(\mathbb{F})$ by the above relations, so $\mathfrak{sl}_2(\mathbb{F})$ is a non-solvable Lie algebra.

We finish this example by showing that $\mathfrak{sl}_2(\mathbb{F})$ is simple. Clearly, $\mathfrak{sl}_2(\mathbb{F})$ is non-zero and non-abelian. Now suppose that $\mathfrak{i} \subseteq \mathfrak{sl}_2(\mathbb{F})$ is a non-zero ideal and let $ax + by + ch \in \mathfrak{i}$ with $a, b, c \in \mathbb{F}$. Clearly, if one of x, y or h belongs to $\mathfrak{i}$, then all of x, y and h belong to $\mathfrak{i}$, forcing $\mathfrak{i} = \mathfrak{sl}_2(\mathbb{F})$. But we have $\mathfrak{i} \ni [x, [x, ax + by + ch]] = [x, bh - 2cx] = -2bx$ and $\mathfrak{i} \ni [y, [y, ax + by + ch]] = [y, a[y, x] + b[y, y] + c[y, h]] = [y, -ah + 2cy] = -a[y, h] + 2c[y, y] = -2ay$, hence $x \in \mathfrak{i}$ and/or $y \in \mathfrak{i}$ if $b \neq 0$ and/or $a \neq 0$, respectively, whereas $h \in \mathfrak{i}$ if $a = b = 0$. In any case, we have $\mathfrak{i} = \mathfrak{sl}_2(\mathbb{F})$, so we conclude that $\mathfrak{sl}_2(\mathbb{F})$ has no non-trivial ideals from which its simplicity follows. We emphasise that the assumption $\mathrm{char}(\mathbb{F}) \neq 2$ is necessary for simplicity of $\mathfrak{sl}_2(\mathbb{F})$.

The standard basis $\{E_{11} - E_{22}, E_{12}, E_{21}\}$ for $\mathfrak{sl}(V)$ as in the example above can easily be extended to the standard basis of $\mathfrak{sl}(V)$ in case V is an n-dimensional vector space over a field $\mathbb{F}$ with $n \geq 2$ an integer. This standard basis is constituted of the matrices in $\{E_{ij} \in \mathfrak{gl}(V) \mid 1 \leq i \neq j \leq n\} \cup \{E_{ii} - E_{i+1,i+1} \in \mathfrak{gl}(V) \mid 1 \leq i \leq n-1\}$, of which there are $(n^2 - n) + (n-1) = n^2 - 1$ in total. Accordingly, $\mathfrak{sl}(V)$ will be $(n^2 - 1)$-dimensional.

Recall from elementary field and ring theory that a *division ring* or *skew field* is a non-zero and generally non-commutative ring $\mathbb{K}$ in which every non-zero element has a multiplicative inverse. Its *opposite* is the division ring $\mathbb{K}^{\mathrm{opp}}$ in which the order of multiplication in $\mathbb{K}$ is reversed. An *anti-automorphism* of a division ring $\mathbb{K}$ is a bijection $\sigma : \mathbb{K} \to \mathbb{K}^{\mathrm{opp}}$ such that $(\lambda + \mu)^\sigma = \lambda^\sigma + \mu^\sigma$ and $(\lambda\mu)^\sigma = \mu^\sigma \lambda^\sigma$ for all $\lambda, \mu \in \mathbb{K}$. Note that we have written the images of λ and μ under σ as λ^σ and μ^σ instead of $\sigma(\lambda)$ and $\sigma(\mu)$; since multiplication in a division ring $\mathbb{K}$ is not necessarily commutative, a distinction has to be made between multiplication from the left and from the right in any vector space V over $\mathbb{K}$. Here and in the remainder of this section, we will opt for multiplication from the right.

Before we are able to discuss the classical linear Lie algebras, we require some theory on reflexive sesquilinear forms. We start with the following definition.

Definition 2.2.3 (Sesquilinear form). *Let $\sigma : \mathbb{K} \to \mathbb{K}^{\mathrm{opp}}$ be an anti-automorphism of a division ring $\mathbb{K}$ and let V be a possibly infinite-dimensional vector space over $\mathbb{K}$. A form $f : V \times V \to \mathbb{K}$ on V is called **sesquilinear** if f is biadditive and $f(\lambda v, \mu w) = \lambda f(v, w)\mu^\sigma$ for all $v, w \in V$ and $\lambda, \mu \in \mathbb{K}$.*

Given a σ-sesquilinear form $f : V \times V \to \mathbb{K}$ on V with V a possibly infinite-dimensional vector space over a division ring $\mathbb{K}$ and $\sigma : \mathbb{K} \to \mathbb{K}^{\mathrm{opp}}$ an anti-automorphism, two vectors $v, w \in V$ are said to be *perpendicular* if $f(v, w) = 0$, which we denote by $v \perp w$. The set of vectors perpendicular to a subset $W \subseteq V$ is $W^\perp = \{v \in V \mid \forall w \in W : f(w, v) = 0\}$. In particular, if we have $W = V$, we obtain the set $V^\perp$ called the *radical* of f, denoted by $\mathrm{rad}(f)$. The σ-sesquilinear form f is said to be *non-degenerate* if $\mathrm{rad}(f) = \{0\}$ and *degenerate* otherwise.

Example 2.2.4. Let $\sigma : \mathbb{K} \to \mathbb{K}^{\mathrm{opp}}$ be an anti-automorphism of a division ring $\mathbb{K}$ and consider again the vector space $M_n(\mathbb{K})$ of $n \times n$ matrices with entries over $\mathbb{K}$ where $n \geq 0$ is an integer from Example 2.1.2. Given a matrix $A \in M_n(\mathbb{K})$ with entries a_{ij}, $1 \leq i, j \leq n$, we denote by $A^\sigma \in M_n(\mathbb{K})$ the matrix with entries a_{ij}^σ, $1 \leq i, j \leq n$. Let the form $f : V \times V \to \mathbb{K}$ be given by $f(A, B) = tr(AB^\sigma)$ with $A, B \in M_n(\mathbb{K})$, in which $tr : M_n(\mathbb{K}) \to \mathbb{K}$ is the linear map that sends a matrix $A \in M_n(\mathbb{K})$ with entries a_{ij}, $1 \leq i, j \leq n$ to $\sum_{i=1}^n a_{ii}$.

For all $A, B \in M_n(\mathbb{K})$ with respective entries a_{ij} and b_{ij}, $1 \leq i, j \leq n$, the matrix $(A+B)^\sigma \in M_n(\mathbb{K})$ has entries $(a_{ij}+b_{ij})^\sigma = a_{ij}^\sigma + b_{ij}^\sigma$, $1 \leq i, j \leq n$, hence $(A+B)^\sigma = A^\sigma + B^\sigma$.

Additionally, for every $\lambda \in \mathbb{K}$, the matrix λA has entries λa_{ij}, $1 \leq i,j \leq n$, so that the matrix $(\lambda A)^\sigma \in M_n(\mathbb{K})$ has entries $(\lambda a_{ij})^\sigma = a_{ij}^\sigma \lambda^\sigma$, $1 \leq i,j \leq n$, therefore $(\lambda A)^\sigma = A^\sigma \lambda^\sigma$. But then by linearity of tr, we have for all $A, B, C, D \in M_n(\mathbb{K})$ and $\lambda, \mu \in \mathbb{K}$ that

$$\begin{aligned} f(A+B, C+D) &= tr((A+B)(C+D)^\sigma) = tr((A+B)(C^\sigma + D^\sigma)) \\ &= tr(AC^\sigma) + tr(AD^\sigma) + tr(BC^\sigma) + tr(BD^\sigma) \\ &= f(A,C) + f(A,D) + f(B,C) + f(B,D), \\ f(\lambda A, \mu B) &= tr((\lambda A)(\mu B)^\sigma) = tr(\lambda(AB^\sigma)\mu^\sigma) = \lambda tr(AB^\sigma)\mu^\sigma = \lambda f(A,B)\mu^\sigma, \end{aligned}$$

showing that f is a sesquilinear form on $M_n(\mathbb{K})$.

If a σ-sesquilinear form f additionally satisfies $f(v,w) = 0 \iff f(w,v) = 0$ for all $v, w \in V$, then the form is *reflexive*. If moreover there exists an $\epsilon \in \mathbb{K}^*$ such that $f(w,v) = \epsilon f(v,w)^\sigma$ for all $v, w \in V$, then the form f is instead referred to as a (σ,ϵ)-sesquilinear form. Such a pair (σ,ϵ) may be special, as described in the following definition.

Definition 2.2.5 (Admissible pair)**.** *Let $\sigma : \mathbb{K} \to \mathbb{K}^{\text{opp}}$ be an anti-automorphism of a division ring $\mathbb{K}$ and let $\epsilon \in \mathbb{K}^*$. The pair (σ,ϵ) is said to be an* ***admissible pair*** *if $\lambda^{\sigma^2} = \epsilon^{-1}\lambda\epsilon$ for all $\lambda \in \mathbb{K}$ and $\epsilon^\sigma = \epsilon^{-1}$.*

Given an admissible pair (σ,ϵ), we note that $\epsilon \in Z(\mathbb{K})$ implies $\lambda^{\sigma^2} = \epsilon\lambda\epsilon^{-1} = \lambda$ for all $\lambda \in \mathbb{K}$ so that σ is an involution. Conversely, if $\sigma^2 = \text{id}_{\mathbb{K}}$, then $\lambda = \epsilon\lambda\epsilon^{-1} \iff \lambda\epsilon = \epsilon\lambda$ for all $\lambda \in \mathbb{K}$, whence $\epsilon \in Z(\mathbb{K})$. The following proposition relates reflexive sesquilinear forms to admissible pairs.

Proposition 2.2.6. *Let $\sigma : \mathbb{K} \to \mathbb{K}^{\text{opp}}$ be an anti-automorphism of a division ring $\mathbb{K}$ and let $f : V \times V \to \mathbb{K}$ be a non-degenerate σ-sesquilinear form on V with V a possibly infinite-dimensional vector space over $\mathbb{K}$. Then f is reflexive if and only if there exists an $\epsilon \in \mathbb{K}^*$ such that f is a (σ,ϵ)-sesquilinear form with (σ,ϵ) an admissible pair.*

Proof. The 'only if'-part of the proposition is immediate, so assume that f is reflexive. For fixed non-zero $w \in V$, consider the forms $\varphi_w : V \to \mathbb{K}$ and $\psi_w : V \to \mathbb{K}$ given by $\varphi_w(v) = f(v,w)$ and $\psi_w(v) = f(w,v)^{\sigma^{-1}}$, respectively, with $v \in V$. Since

$$\begin{aligned} \varphi_w(\lambda u + \mu v) &= f(\lambda u + \mu v, w) = \lambda f(u,w) + \mu f(v,w) = \lambda\varphi_w(u) + \mu\varphi_w(v), \\ \psi_w(\lambda u + \mu v) &= f(w, \lambda u + \mu v)^{\sigma^{-1}} = (f(w,u)\lambda^\sigma + f(w,v)\mu^\sigma)^{\sigma^{-1}} = \\ &= \lambda f(w,u)^{\sigma^{-1}} + \mu f(w,v)^{\sigma^{-1}} = \lambda\psi_w(u) + \mu\psi_w(v), \end{aligned}$$

for all $u, v \in V$ and $\lambda, \mu \in \mathbb{K}$, the forms φ_w and ψ_w are linear, and they moreover satisfy $\ker(\varphi_w) = \ker(\psi_w)$ by reflexivity of f. Consequently, we may assume that the equations $f(v,w) = 0$ and $f(w,v)^{\sigma^{-1}} = 0$ in v are equivalent up to scaling. In particular, there exists an $\epsilon_w \in \mathbb{K}^*$ depending on w such that for all $v \in V$ we have

$$f(w,v)^{\sigma^{-1}} = f(v,w)\epsilon_w^{\sigma^{-1}} \iff f(w,v) = (f(w,v)^{\sigma^{-1}})^\sigma = (f(v,w)\epsilon_w^{\sigma^{-1}})^\sigma = \epsilon_w f(v,w)^\sigma.$$

We show now that $\epsilon_w \in \mathbb{K}^*$ is independent of w. So, let $w \neq w' \in V$ be arbitrary, then for all $v \in V$ we have

$$\begin{aligned} 0 &= f(w+w',v) - f(w,v) - f(w',v) = \epsilon_{w+w'} f(v,w+w')^\sigma - \epsilon_w f(v,w)^\sigma - \epsilon_{w'} f(v,w')^\sigma \\ &= (f(v,w+w')\epsilon_{w+w'}^{\sigma^{-1}})^\sigma - (f(v,w)\epsilon_w^{\sigma^{-1}})^\sigma - (f(v,w')\epsilon_{w'}^{\sigma^{-1}})^\sigma \\ &= f(v, \epsilon_{w+w'}^{\sigma^{-2}}(w+w') - \epsilon_w^{\sigma^{-2}} w - \epsilon_{w'}^{\sigma^{-2}} w')^\sigma \implies (\epsilon_{w+w'}^{\sigma^{-2}} - \epsilon_w^{\sigma^{-2}})w + (\epsilon_{w+w'}^{\sigma^{-2}} - \epsilon_{w'}^{\sigma^{-2}})w' = 0 \end{aligned}$$

by non-degeneracy of f. On the one hand, if w and w' are linearly independent, then the above implies $\epsilon_{w+w'}^{\sigma^{-2}} - \epsilon_w^{\sigma^{-2}} = \epsilon_{w+w'}^{\sigma^{-2}} - \epsilon_{w'}^{\sigma^{-2}} = 0$ from which it follows that $\epsilon_w = \epsilon_{w'}$. On the other hand, if w and w' are linearly dependent, there exists a $w'' \in V$ such that w'' is linearly independent with w and consequently with w'. Then again by the above, we have $\epsilon_w = \epsilon_{w''}$ and $\epsilon_{w'} = \epsilon_{w''}$ so that $\epsilon_w = \epsilon_{w'}$. We conclude that $\epsilon := \epsilon_w$ is independent of w, therefore f is a (σ, ϵ)-sesquilinear form.

It remains to show that (σ, ϵ) is an admissible pair. To this extent, let $\lambda \in \mathbb{K}$, then we can find $v, w \in V$ such that $f(v,w) = \lambda$. Consequently, we obtain

$$\lambda = f(v,w) = \epsilon f(w,v)^\sigma = \epsilon(\epsilon f(v,w)^\sigma)^\sigma = \epsilon f(v,w)^{\sigma^2}\epsilon^\sigma = \epsilon\lambda^{\sigma^2}\epsilon^\sigma.$$

For $\lambda = 1$, we find $1 = \epsilon\epsilon^\sigma$ so that $\epsilon^\sigma = \epsilon^{-1}$. But then $\lambda = \epsilon\lambda^{\sigma^2}\epsilon^\sigma = \epsilon\lambda^{\sigma^2}\epsilon^{-1} \iff \lambda^{\sigma^2} = \epsilon^{-1}\lambda\epsilon$, showing that (σ, ϵ) is an admissible pair. □

As a consequence, we may identify reflexive sesquilinear forms by (σ, ϵ)-sesquilinear forms with (σ, ϵ) an admissible pair. In order to classify reflexive sesquilinear forms, we make use of the notion of proportionality. Given two sesquilinear forms f and f', we say that f' is *proportional* to f if there exists an $\alpha \in \mathbb{K}^*$ such that $f'(v,w) = \alpha f(v,w)$ for all $v, w \in V$. We may also call f' proportional to f by α if we wish to specify the factor of proportionality $\alpha \in \mathbb{K}^*$. Observe that proportionality is an equivalence relation, so for the classification of reflexive sesquilinear forms it suffices to do so up to proportionality.

On the one hand, if $\sigma = \mathrm{id}_\mathbb{K}$, then $\epsilon = \epsilon^{-1}$ so that $\epsilon = \pm 1$. If $\epsilon = 1$, we obtain a *symmetric* form, whereas we obtain an *anti-symmetric* form if $\epsilon = -1$. On the other hand, if $\sigma \neq \mathrm{id}_\mathbb{K}$ and $\epsilon = \pm 1$, we have $\epsilon \in Z(\mathbb{K})$, forcing σ to be an involution. In this case, we obtain a *Hermitian* form for $\epsilon = 1$ and an *skew-Hermitian* form for $\epsilon = -1$. The remaining cases are covered by the following lemma.

Lemma 2.2.7. *Let f be a non-degenerate (σ, ϵ)-sesquilinear form with admissible pair (σ, ϵ) on a possibly infinite dimensional vector space V over a division ring $\mathbb{K}$. Then for all $\alpha \in \mathbb{K}^*$ there exists an admissible pair (τ, η) satisfying $\lambda^\tau = (\alpha^{-1}\lambda\alpha)^\sigma$ for all $\lambda \in \mathbb{K}$ and $\eta = \alpha\epsilon\alpha^{-\sigma}$ such that the form αf is (τ, η)-sesquilinear. Moreover, if $\sigma \neq \mathrm{id}_\mathbb{K}$, then f is proportional to a skew-Hermitian form on V.*

Proof. Let $\alpha \in \mathbb{K}^*$ be arbitrary, which we may clearly assume to satisfy $\alpha \neq 1$. By Proposition 2.2.6, the form f is reflexive. Clearly, the form αf will then also be reflexive.

Again by Proposition 2.2.6, there exists an admissible pair (τ, η) such that the form $f' := \alpha f$ is a (τ, η)-sesquilinear form. In particular, we have $f'(w, v) = \eta f(v, w)^\tau$ for all $v, w \in V$, but then

$$\begin{aligned}\eta f'(v, w)^\tau &= f'(w, v) = \alpha f(w, v) = \alpha(\epsilon f(v, w)^\sigma) = \alpha\epsilon(\alpha^{-1} f'(v, w))^\sigma \\ &= \alpha\epsilon f'(v, w)^\sigma \alpha^{-\sigma} = (\alpha\epsilon\alpha^{-\sigma})(\alpha^\sigma f'(v, w)^\sigma \alpha^{-\sigma}).\end{aligned}$$

for all $v, w \in V$. Consequently, we have $\lambda^\tau = \alpha^\sigma \lambda^\sigma \alpha^{-\sigma} = (\alpha^{-1}\lambda\alpha)^\sigma$ for all $\lambda \in \mathbb{K}$ and $\eta = \alpha\epsilon\alpha^{-\sigma}$. (Notice that the choice $\lambda^\tau = \lambda^\sigma \alpha^{-\sigma}$ and $\eta = \alpha\epsilon$ results in τ being an anti-automorphism of $\mathbb{K}$ if and only if $\alpha = 1$, since $(\lambda\mu)^\tau = (\lambda\mu)^\sigma \alpha^{-\sigma} = \mu^\sigma \lambda^\sigma \alpha^{-\sigma} = \mu^\sigma \alpha^{-\sigma} \lambda^\sigma \alpha^{-\sigma} = \mu^\tau \lambda^\tau \iff \alpha^{-\sigma} = 1 \iff \alpha = 1$ for all $\lambda, \mu \in \mathbb{K}$.) The pair (τ, η) is indeed admissible; for all $\lambda \in \mathbb{K}$, we have

$$\begin{aligned}\lambda^{\tau^2} &= (\alpha^{-1}(\alpha^{-1}\lambda\alpha)^\sigma\alpha)^\sigma = \alpha^\sigma(\alpha^{-1}\lambda\alpha)^{\sigma^2}\alpha^{-\sigma} = \alpha^\sigma(\epsilon^{-1}(\alpha^{-1}\lambda\alpha)\epsilon)\alpha^{-\sigma} \\ &= (\alpha^\sigma\epsilon^{-1}\alpha^{-1})\lambda(\alpha\epsilon\alpha^{-\sigma}) = (\alpha\epsilon\alpha^{-\sigma})^{-1}\lambda(\alpha\epsilon\alpha^{-\sigma}) = \eta^{-1}\lambda\eta, \\ \eta^\tau &= \alpha^\sigma\eta^\sigma\alpha^{-\sigma} = \alpha^\sigma(\alpha\epsilon\alpha^{-\sigma})^\sigma\alpha^{-\sigma} = \alpha^\sigma\alpha^{-\sigma^2}\epsilon^\sigma\alpha^\sigma\alpha^{-\sigma} = \alpha^\sigma(\epsilon^{-1}\alpha^{-1}\epsilon)\epsilon^{-1} \\ &= \alpha^\sigma\epsilon^{-1}\alpha^{-1} = (\alpha\epsilon\alpha^{-\sigma})^{-1} = \eta^{-1}.\end{aligned}$$

This settles the first part of the lemma.

For the second part, we show that there exists an $\alpha \in \mathbb{K}^*$ such that the form $f' = \alpha f$ is skew-Hermitian; the result then will follow from symmetry of proportionality. By the above, it will suffice to find $\alpha \in \mathbb{K}^*$ such that $\eta = -1$. So, let $\sigma \neq \mathrm{id}_{\mathbb{K}}$ and assume first that $\epsilon \neq 1$. For $\alpha = 1 - \epsilon^{-1} \in \mathbb{K}^*$, we obtain

$$\eta = (1 - \epsilon^{-1})\epsilon(1 - \epsilon^{-1})^{-\sigma} = (\epsilon - 1)((1 + \epsilon^{-1})^\sigma)^{-1} = -(1 - \epsilon)(1 - \epsilon)^{-1} = -1,$$

hence f' is a skew-Hermitian form on V. Next, let $\epsilon = 1 \in Z(\mathbb{K})$. Then $\sigma^2 = \mathrm{id}_{\mathbb{K}}$ and we may choose $\lambda \in \mathbb{K}^*$ such that $\lambda^\sigma \neq \lambda$, which exists because $\sigma \neq \mathrm{id}_{\mathbb{K}}$. By setting $\alpha := \lambda - \lambda^\sigma \in \mathbb{K}$, we find

$$\eta = (\lambda - \lambda^\sigma)\epsilon(\lambda - \lambda^\sigma)^{-\sigma} = -(\lambda^\sigma - \lambda)(\lambda^\sigma - \lambda^{\sigma^2})^{-1} = -(\lambda^\sigma - \lambda)(\lambda^\sigma - \lambda)^{-1} = -1$$

so that f' is again a skew-Hermitian form on V. □

As a consequence of the above lemma, every non-degenerate anti-Hermitian form is proportional to a non-degenerate Hermitian form. This leads to the following classification of non-degenerate reflexive sesquilinear forms.

Theorem 2.2.8. *Let $f : V \times V \to \mathbb{K}$ be a non-degenerate reflexive sesquilinear form on a possible infinite-dimensional vector space V over a division ring $\mathbb{K}$. Then f is proportional to a symmetric bilinear form, an anti-symmetric bilinear form or a Hermitian form.*

Proof. By Proposition 2.2.6, f is a (σ,ϵ)-sesquilinear form with admissible pair (σ,ϵ). As we have already seen, f is either symmetric or anti-symmetric if $(\sigma,\epsilon)=(\mathrm{id}_{\mathbb{K}},1)$ or $(\sigma,\epsilon)=(\mathrm{id}_{\mathbb{K}},-1)$, respectively. In particular, f will be bilinear because $\sigma=\mathrm{id}_{\mathbb{K}}$. If $\sigma\neq\mathrm{id}_{\mathbb{K}}$, then f is proportional to a Hermitian form by Lemma 2.2.7, in which case $\sigma^2=\mathrm{id}_{\mathbb{K}}$. □

A sesquilinear form $f:V\times V\to\mathbb{K}$ on a possibly infinite-dimensional vector space V over a division ring $\mathbb{K}$ satisfying $f(v,v)=0$ for all $v\in V$ is called *alternating*. Analogous to our discussion of alternativity and anti-symmetry of the Lie bracket at the start of Section 2.1, the form f is anti-symmetric if it is alternating, whereas the converse is true if and only if $\mathrm{char}(\mathbb{K})\neq 2$. In case f is both alternating and anti-symmetric, we call f a *symplectic* form. Necessarily, we must then have $\mathrm{char}(\mathbb{K})\neq 2$, which we will assume throughout the rest of this section.

We have the following lemma connecting non-degenerate reflexive sesquilinear forms to Lie algebras, which we require to define the classical linear Lie algebras.

Lemma 2.2.9. *Let V be a possibly infinite-dimensional vector space over a field $\mathbb{F}$ and let $\mathfrak{gl}(V)$ be the general linear Lie algebra of V. Then for every non-degenerate reflexive sesquilinear form $f:V\times V\to\mathbb{F}$ on V, the subspace*

$$\mathfrak{h}=\{\varphi\in\mathfrak{gl}(V)\mid\forall x,y\in V: f(\varphi(x),y)+f(x,\varphi(y))=0\}$$

of $\mathfrak{gl}(V)$ is a Lie subalgebra of $\mathfrak{gl}(V)$.

Proof. Let $\varphi,\psi\in\mathfrak{h}$, i.e. $f(\varphi(x),y)+f(x,\varphi(y))=0$ and $f(\psi(x),y)+f(x,\psi(y))=0$. Then

$$\begin{aligned}0&=f(\varphi(x),\psi(y))-f(\varphi(x),\psi(y))+f(\psi(x),\varphi(y))-f(\psi(x),\varphi(y))\\&=-f(\psi(\varphi(x)),y)+f(x,\varphi(\psi(y)))-f(x,\psi(\varphi(y)))+f(\varphi(\psi(x)),y)\\&=f((\varphi\psi)(x),y)-f((\psi\varphi)(x),y)+f(x,(\varphi\psi)(y))-f(x,(\psi\varphi)(y))\\&=f((\varphi\psi-\psi\varphi)(x),y)+f(x,(\varphi\psi-\psi\varphi)(y))=f([\varphi,\psi](x),y)+f(x,[\varphi,\psi](y)),\end{aligned}$$

hence $[\varphi,\psi]\in\mathfrak{h}$. We conclude that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{gl}(V)$ by Definition 2.1.1. □

In combination with Theorem 2.2.8, Lemma 2.2.9 then brings us to the following definition.

Definition 2.2.10 (Classical linear Lie algebras)**.** *Let V be a possibly infinite-dimensional vector space over a field $\mathbb{F}$ and let $\mathfrak{gl}(V)$ be the general linear Lie algebra of V. The Lie subalgebra $\{\varphi\in\mathfrak{gl}(V)\mid\forall x,y\in V: f(\varphi(x),y)+f(x,\varphi(y))=0\}$ of $\mathfrak{gl}(V)$ from* Lemma 2.2.9 *induced by a non-degenerate reflexive sesquilinear form $f:V\times V\to\mathbb{F}$ on V is called*

(i) *the* ***symplectic Lie algebra*** *if f is a symplectic bilinear form, denoted by $\mathfrak{sp}(V,f)$,*

(ii) *the* ***orthogonal Lie algebra*** *if f is a symmetric bilinear form, denoted by $\mathfrak{so}(V,f)$,*

(iii) *the* ***unitary Lie algebra*** *if f is a Hermitian form, denoted by* $\mathfrak{u}(V, f)$.

We emphasise that the unitary Lie algebra $\mathfrak{u}(V, f)$ is a Lie algebra over the subfield $\mathbb{F}_\sigma = \{\lambda \in \mathbb{F} \mid \lambda^\sigma = \lambda\} \subseteq \mathbb{F}$, in which $\sigma : \mathbb{F} \to \mathbb{F}^{\text{opp}}$ is the involutory anti-automorphism corresponding to Hermitian form $f : V \times V \to \mathbb{F}$.

We will also use the notations $\mathfrak{so}_n(\mathbb{F}, f)$, $\mathfrak{sp}_n(\mathbb{F}, f)$ and $\mathfrak{u}_n(\mathbb{F}, f)$ if V is n-dimensional over $\mathbb{F}$. If V is infinite-dimensional, we additionally have the finitary classical linear Lie algebras $\mathfrak{fso}(V, f)$, $\mathfrak{fsp}(V, f)$ and $\mathfrak{fu}(V, f)$, obtained by intersecting $\mathfrak{so}(V, f)$, $\mathfrak{sp}(V, f)$ and $\mathfrak{u}(V, f)$ with $\mathfrak{fgl}(V, f)$. The quotient Lie algebras of $\mathfrak{so}(V, f)$, $\mathfrak{sp}(V, f)$ and $\mathfrak{u}(V, f)$ by their centers are denoted by $\mathfrak{pso}(V, f)$, $\mathfrak{psp}(V, f)$ and $\mathfrak{pu}(V, f)$, respectively. Lastly, we have the special unitary Lie algebra $\mathfrak{su}(V, f)$, obtained by intersecting $\mathfrak{u}(V, f)$ with the special linear Lie algebra $\mathfrak{sl}(V, f)$. We will later see that $\mathfrak{so}(V, f), \mathfrak{sp}(V, f) \subseteq \mathfrak{sl}(V, f)$, i.e. the special orthogonal Lie algebra and special symplectic Lie algebra coincide with the orthogonal Lie algebra and symplectic Lie algebra, respectively.

We finish this section with determining the standard bases and dimensions of the classical linear Lie algebras $\mathfrak{sp}(V, f)$, $\mathfrak{so}(V, f)$ and $\mathfrak{u}(V, f)$ as defined in Definition 2.2.10. To do so, we introduce *sesquilinear spaces*, which are pairs (V, f) in which f is a sesquilinear form on a vector space V. In particular, we will construct bases of these sesquilinear spaces as a means of determining the *Gram matrix* of f, which is the matrix $(f(v_i, v_j))_{i,j \in I}$ in which $\{v_i\}_{i \in I}$ is a basis of V indexed by some index set I. We start with the following definition.

Definition 2.2.11 (Isotropy, Anisotropy & Hyperbolic pair)**.** *Let $f : V \times V \to \mathbb{K}$ be a sesquilinear form on a possibly infinite-dimensional vector space V over a division ring $\mathbb{K}$. A vector $v \in V$ is called* ***f-isotropic*** *if $f(v, v) = 0$ and* ***f-anisotropic*** *otherwise. A pair $\{v, w\} \subseteq V$ is called a* ***hyperbolic pair*** *if $f(v, v) = f(w, w) = 0$ and $f(v, w) \neq 0$.*

The concept of isotropy extends naturally to subspaces of a vector space V; given a subspace $W \subseteq V$, then W is said to be *totally f-isotropic* if $f(w, w') = 0$ for all $w, w' \in W$. The dimension of a maximal totally f-isotropic subspace of V is an invariant, called the *Witt index of f*. Further note that a hyperbolic pair necessarily spans a two-dimensional subspace of V, which we will refer to as a *hyperbolic space*, and that for all hyperbolic pairs $\{v, w\} \subseteq V$ we may harmlessly assume $f(v, w) = 1$ by scaling either v or w.

Example 2.2.12. We specify the setting from Example 2.2.4 by letting $M_2(\mathbb{R})$ be the vector space of 2×2 matrices with real entries. Furthermore, we let $f : M_2(\mathbb{R}) \times M_2(\mathbb{R}) \to \mathbb{R}$ be the bilinear symmetric from on $M_2(\mathbb{R})$ given by $f(A, B) = tr(AB)$ with $A, B \in M_n(\mathbb{R})$. Then for all $A \in M_2(\mathbb{R})$ with entries a_{ij}, $1 \leq i, j \leq 2$, we obtain

$$\begin{aligned} f(A, A) &= tr\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\right) = tr\left(\begin{pmatrix} a_{11}^2 + a_{12}a_{21} & a_{11}a_{12} + a_{12}a_{22} \\ a_{11}a_{21} + a_{21}a_{22} & a_{22}^2 + a_{12}a_{21} \end{pmatrix}\right) \\ &= a_{11}^2 + a_{22}^2 + 2a_{12}a_{21} = (a_{11} + a_{22})^2 - 2(a_{11}a_{22} - a_{12}a_{21}) = (tr(A))^2 - 2\det(A). \end{aligned}$$

Consequently, a matrix $A \in M_2(\mathbb{K})$ is f-isotropic if and only if $(tr(A))^2 = 2\det(A)$. It immediately follows that matrices with odd trace or negative determinant are f-anisotropic.

In terms of finding hyperbolic pairs in sesquilinear spaces (V, f), the following lemma will prove useful for our investigation of these sesquilinear spaces.

Lemma 2.2.13. *Let $f : V \times V \to \mathbb{K}$ be a sesquilinear form on a possibly infinite-dimensional vector space V over a division ring $\mathbb{K}$ with $\operatorname{char}(\mathbb{K}) \neq 2$ such that $V \setminus \operatorname{rad}(f)$ contains at least one isotropic vector. For every $v \in V \setminus \operatorname{rad}(f)$, there is a $w \in V$ such that $f(w, v) = 1$.*

In particular, if f is a non-degenerate symplectic, symmetric or Hermitian form with involutory anti-automorphism $\sigma : \mathbb{K} \to \mathbb{K}^{\text{opp}}$, then there exist f-isotropic $v, w \in V$ such that $\{v, w\} \subseteq V$ is a hyperbolic pair.

Proof. For the first assertion, let $v \in V \setminus \operatorname{rad}(f)$ so that there exists a $w' \in V$ such that $f(w', v) \neq 0$. For $w := f(w', v)^{-1}w' \in V$, we then obtain $f(w, v) = f(f(w', v)^{-1}w', v) = f(w', v)^{-1}f(w', v) = 1$.

For the second assertion, first assume that f is a symplectic form. Since then every vector $v \in V$ is f-isotropic, the result follows immediately from the first assertion. Next, assume f is a Hermitian form and denote by $\mathbb{K}_\sigma = \{\lambda \in \mathbb{K} \mid \lambda^\sigma = \lambda\}$ the subfield of $\mathbb{K}$ consisting of the elements fixed by σ. Now let $v \in V$ be f-isotropic and let $w' \in V$ be such that $f(w', v) \neq 0$, which exists because f is non-degenerate. In particular, we may assume that $f(w', v) = 1$ by the first assertion. Additionally, we have $f(v, w') = f(w', v)^\sigma = 1^\sigma = 1$ and $-2^{-1}f(w', w') \in \mathbb{K}_\sigma$ because $1 \in \mathbb{K}_\sigma$ and $f(w', w') = f(w', w')^\sigma$. If w' is f-isotropic, the result follows immediately from the first assertion, so assume that w' is f-anisotropic. For $w = -2^{-1}f(w', w')v + w'$, we obtain $f(w, v) = f(-2^{-1}f(w', w')v + w', v) = -2^{-1}f(w', w')f(v, v) + f(w', v) = 1$ so that $f(v, w) = f(w, v)^\sigma = 1^\sigma = 1$ and

$$
\begin{aligned}
f(w, w) &= f(-2^{-1}f(w', w')v + w', -2^{-1}f(w', w')v + w') \\
&= f(-2^{-1}f(w', w')v, w') + f(w', -2^{-1}f(w', w')v) + f(w', w') \\
&= -2^{-1}f(w', w')f(v, w') - f(w', v)(2^{-1}f(w', w'))^\sigma + f(w', w') \\
&= -2^{-1}f(w', w') - 2^{-1}f(w', w') + f(w', w') = 0,
\end{aligned}
$$

showing that the pair $\{v, w\} \subseteq V$ is hyperbolic. Lastly, in case f is a symmetric form, we have $\mathbb{K}_\sigma = \mathbb{K}$ because now $\sigma = \operatorname{id}_{\mathbb{K}}$, and everything we have said previously remains true. □

As a consequence of the first assertion of the above lemma, we have the decomposition $V = \langle v, w\rangle \oplus \langle v, w\rangle^\perp$ with $v, w \in V$ not necessarily isotropic such that $f(v, w) = 1$. Moreover, the restriction of a non-degenerate sesquilinear form f on V to the subspace $\langle v, w\rangle^\perp = \{u \in V \mid f(v, u) = f(w, u) = 0\}$ will again be non-degenerate.

We require one last definition pertaining to hyperbolic pairs in relation to bases of vector spaces.

Definition 2.2.14 (Hyperbolic basis)**.** *Let $f : V \times V \to \mathbb{K}$ be a sesquilinear form on a possibly infinite-dimensional vector space V over a division ring $\mathbb{K}$. A **hyperbolic basis** of V is a basis of V consisting of hyperbolic pairs $\{v_i, w_i\}_{i \in \mathcal{I}}$ indexed by some index set I, i.e.*

$$f(v_i, v_j) = f(w_i, w_j) = 0 \quad \text{and} \quad f(v_i, w_j) = \delta_{i,j}$$

for all $i, j \in I$ with δ the Kronecker delta function.

We now proceed with our investigations of the sesquilinear spaces (V, f) with f a non-degenerate reflexive sesquilinear form on a finite-dimensional vector space V over a field $\mathbb{F}$. Specifically, we will be working towards decomposing V into certain subspaces from which we can easily deduce the Gram matrix of f.

first, consider a symplectic space (V, f). Alternativity of f then implies that every vector $v \in V$ is f-isotropic, hence every two linearly independent vectors $v, w \in V$ with $f(v, w) \neq 0$ constitute a hyperbolic pair $\{v, w\} \subseteq V$. In particular, we then have $f(v, v) = f(w, w) = 0$ and $f(w, v) = 1 = -f(v, w)$ by Lemma 2.2.13. The Gram matrix of f on the subspace $\langle v, w \rangle \subseteq V$ will then be

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

with respect to the basis $\{v, w\}$ of $\langle v, w \rangle$. By proceeding inductively on the subspace $\langle v, w \rangle^\perp \subseteq V$, we deduce that the Gram matrix of f will be block diagonal, its blocks being the 2×2 matrix given above. Necessarily, V must be even-dimensional, say $\dim(V) = 2n$ with $n \geq 1$ an integer. Upon permuting the elements constituting the hyperbolic pairs that form a hyperbolic basis of V, it follows that the Gram matrix of f has the form

$$\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

with $2m = n$. We then obtain the symplectic bilinear form

$$f(v, w) = v^\top \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} w$$

with $v, w \in V$. We now wish to express the elements of $\mathfrak{sp}(V, f)$ in terms of matrices instead of linear transformations, as we did for $\mathfrak{gl}(V)$ and $\mathfrak{sl}(V)$. To this extent, denote by A_φ the matrix corresponding to an arbitrary linear transformation $\varphi \in \mathfrak{gl}(V)$, and similarly denote by A_f the matrix that defines the symplectic bilinear form above. Then

$$\begin{aligned} 0 &= f(\varphi(v), w) + f(v, \varphi(w)) = \varphi(v)^\top A_f w + v^\top A_f \varphi(w) \\ &= (A_\varphi v)^\top A_f w + v^\top A_f (A_\varphi w) = v^\top A_\varphi^\top A_f w + v^\top A_f A_\varphi w \\ &= v^\top (A_\varphi^\top A_f + A_f A_\varphi) w \iff A_\varphi^\top A_f + A_f A_\varphi = 0. \end{aligned}$$

Consequently, we may equivalently write $\mathfrak{sp}(V,f) = \{\varphi \in \mathfrak{gl}(V) \mid A_\varphi^\top A_f + A_f A_\varphi = 0\} \subseteq \mathfrak{gl}(V)$.

Corollary 2.2.15. *Let V be an n-dimensional vector space over a field $\mathbb{F}$ with $n \geq 2$ an integer and let $\mathfrak{sp}(V,f)$ be the symplectic Lie algebra on V over $\mathbb{F}$ for some non-degenerate alternating bilinear form $f : V \times V \to \mathbb{F}$ on V. Then $\mathfrak{sp}(V,f)$ has dimension $\binom{n+1}{2}$.*

Proof. By the above, V must be even-dimensional, so $n = 2m$ for some integer $m \geq 1$. Write $\mathfrak{sp}_{2m}(\mathbb{F},f) = \{\varphi \in \mathfrak{gl}_{2m}(\mathbb{F}) \mid A_\varphi^\top \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} + \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} A_\varphi = 0\}$, in which I_m is the $m \times m$ identity matrix and A_φ is the matrix corresponding to linear transformation $\varphi \in \mathfrak{gl}_{2m}(\mathbb{F})$. By letting $A_\varphi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with A, B, C and D matrices of size $m \times m$ having entries in $\mathbb{F}$, we find

$$0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^\top \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} + \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} C - C^\top & A^\top + D \\ -(A + D^\top) & B^\top - B \end{pmatrix},$$

hence $C = C^\top$, $B = B^\top$ and $A = -D^\top$. Now denote by $E_{i,j}$ the matrix having a one in position (i,j) and zeros elsewhere, then

$$\begin{aligned}
\left\langle \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\rangle &= \langle E_{i,j} - E_{m+j,m+i} \mid 1 \leq i,j \leq m \rangle, \\
\left\langle \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \right\rangle &= \langle E_{i,m+i} \mid 1 \leq i \leq m \rangle \cup \langle E_{i,m+j} + E_{j,m+i} \mid 1 \leq i < j \leq m \rangle, \\
\left\langle \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \right\rangle &= \langle E_{m+i,i} \mid 1 \leq i \leq m \rangle \cup \langle E_{m+i,j} + E_{m+j,i} \mid 1 \leq i < j \leq m \rangle,
\end{aligned}$$

and it is readily seen that these spans are mutually independent standard bases of $\left\langle \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right\rangle$, $\left\langle \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \right\rangle$ and $\left\langle \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \right\rangle$, hence together they form the standard basis of $\mathfrak{sp}_{2m}(\mathbb{F},f) = \left\langle \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\rangle$. Consequently, $\mathfrak{sp}(V,f)$ has dimension

$$(m^2) + \left(m + \sum_{k=1}^{m} (m-k) \right) + \left(m + \sum_{k=1}^{m} (m-k) \right) = m(2m+1) = \binom{n+1}{2}.$$

□

Notice that the condition $A = -D^\top$ for elements $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{sp}(V,f)$ results in $tr\left(\begin{pmatrix} A & B \\ C & D \end{pmatrix}\right) = tr(A+D) = tr(D - D^\top) = 0$ so that $\mathfrak{sp}(V,f) \subseteq \mathfrak{sl}(V)$.

Next, consider a symmetric space (V,f) and assume that the field $\mathbb{F}$ on which V is defined is algebraically closed. By inductively extending a basis $\{v_1, \ldots, v_n\} \subset V$ to a basis $\{v_1, \ldots, v_{i+1}\} \subset V$ with $v_{i+1} \in \langle v_1, \ldots, v_i \rangle^\perp$, we deduce that the Gram matrix of f is diagonal. Moreover, by writing $f(v_i, v_i) = \lambda_i \in \mathbb{F}^*$ for all $1 \leq i \leq \dim(V)$ and by letting $\mu_i \in \mathbb{F}^*$, $1 \leq i \leq \dim(V)$ be a square root of λ_i, which exists as $\mathbb{F}$ is assumed to be algebraically closed, we find $f(\mu_i^{-1} v_i, \mu_i^{-1} v_i) = (\mu_i^2)^{-1}\lambda_i = \lambda_i^{-1}\lambda_i = 1$. But then the Gram matrix of f is the identity, so we obtain the symmetric form

$$f(v, w) = v^\top w,$$

with $v, w \in V$. Computations similar to those in the symplectic case show that $\mathfrak{so}(V, f) = \{\varphi \in \mathfrak{gl}(V) \mid A_\varphi^\top + A_\varphi = 0\} \subseteq \mathfrak{gl}(V)$, in which A_φ is the matrix corresponding to a linear transformation $\varphi \in \mathfrak{gl}(V)$.

Corollary 2.2.16. *Let V be an n-dimensional vector space over an algebraically closed field $\mathbb{F}$ with $n \geq 1$ an integer and let $\mathfrak{so}(V, f)$ be the orthogonal Lie algebra on V over $\mathbb{F}$ for some non-degenerate symmetric bilinear form $f : V \times V \to \mathbb{F}$ on V. Then $\mathfrak{so}(V, f)$ has dimension $\binom{n}{2}$.*

Proof. Write $\mathfrak{so}_n(\mathbb{F}, f) = \{\varphi \in \mathfrak{gl}_n(\mathbb{F}) \mid A_\varphi^\top + A_\varphi = 0\}$, in which A_φ is the matrix corresponding to linear transformation $\varphi \in \mathfrak{gl}_n(\mathbb{F})$. Then $\varphi \in \mathfrak{so}_n(\mathbb{F}, f)$ if and only if $A_\varphi = -A_\varphi^\top$, i.e. A_φ is skew-symmetric. It is readily seen that A_φ is spanned by the linearly independent matrices $E_{i,j} - E_{j,i}$ with $1 \leq i < j \leq n$, in which $E_{i,j}$ is the matrix having a one in position (i, j) and zeros elsewhere. It follows that the dimension of $\mathfrak{so}(V, f)$ equals

$$\sum_{k=1}^{n}(n-k) = \frac{n(n-1)}{2} = \binom{n}{2}.$$

□

As before, we see that the condition $A_\varphi = -A_\varphi^\top$ for an element $\varphi \in \mathfrak{so}(V, f)$ implies that the elements of $\mathfrak{so}(V, f)$ are traceless so that $\mathfrak{so}(V, f) \subseteq \mathfrak{sl}(V)$.

Lastly, consider a Hermitian space (V, f). As long as there exist f-isotropic vectors in V, which come in pairs by Lemma 2.2.13, we obtain hyperbolic pairs $\{v, w\} \subseteq V$ such that $f(v, w) = 1 = f(w, v)$ so that the Gram matrix of f restricted to $\langle v, w\rangle \subseteq V$ has the form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to the basis $\{v, w\}$ of $\langle v, w\rangle$. As before, by proceeding inductively on the subspace $\langle v, w\rangle^\perp \subseteq V$, the Gram matrix of f restricted to the $2m$-dimensional subspace of V, $2m \leq n$, spanned by its isotropic vector with respect to some hyperbolic basis is

$$\begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}.$$

We are now left with a k-dimensional subspace $U \subseteq V$, $k = n - 2m$, containing only anisotropic vectors in V. Since $f(v, v) = f(v, v)^\sigma$ for all $v \in V$, we see that $0 \neq f(u, u) \in$

$\mathbb{K}_\sigma = \{\lambda \in \mathbb{K} \mid \lambda^\sigma = \lambda\}$ with $u \in U$ anisotropic. The Gram matrix of f restricted to U with respect to a basis $\{u_1, \dots, u_k\}$, which we may assume to be orthogonal by inductively extending a basis $\{u_1, \dots, u_i\}$ to a basis $\{u_1, \dots, u_{i+1}\}$ with $u_{i+1} \in \langle u_1, \dots, u_i \rangle^\perp$, will then be diagonal with entries $f(u_1, u_1), \dots, f(u_k, u_k)$. In summary, the Gram matrix of f on V will have the form

$$\begin{pmatrix} 0 & I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & \Lambda_k \end{pmatrix},$$

with Λ_k the diagonal matrix with entries $\lambda_i \in \mathbb{K}_\sigma$, $1 \leq i \leq k$. In particular, we may assume that $\Lambda_k = \lambda I_k$ with $\lambda \in \mathbb{K}_\sigma$ after scaling appropriately. We then obtain the Hermitian form

$$f(v, w) = v^\top \begin{pmatrix} 0 & I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & \lambda I_k \end{pmatrix} w^\sigma$$

with involutory anti-automorphism $\sigma : \mathbb{F} \to \mathbb{F}^{\mathrm{opp}}$. Now let A_φ be the matrix corresponding to a linear transformation $\varphi \in \mathfrak{gl}(V)$ and consider the vectors $(A_\varphi w)^\sigma$ and $(w^\sigma)^\top (A_\varphi^\sigma)^\top$ with $w \in V$. By denoting the entries of A_φ by a_{ij}, $1 \leq i, j \leq n$, and those of w by w_i, $1 \leq i \leq n$, we obtain

$$((A_\varphi w)^\sigma)_i = \left(\sum_{j=1}^n a_{ij} w_j \right)^\sigma = \sum_{j=1}^n w_j^\sigma a_{ij}^\sigma = \sum_{i=1}^n w_i^\sigma a_{ji}^\sigma = \left((w^\sigma)^\top (A_\varphi^\sigma)^\top \right)_j,$$

hence the entries of $(A_\varphi w)^\sigma$ and $(w^\sigma)^\top (A_\varphi^\sigma)^\top$ coincide. However, since i runs over the rows of $(A_\varphi w)^\sigma$ and j runs over the columns of $(w^\sigma)^\top (A_\varphi^\sigma)^\top$, we have $(A_\varphi w)^\sigma = \left((w^\sigma)^\top (A_\varphi^\sigma)^\top \right)^\top$ as vectors. It follows that $(A_\varphi w)^\sigma = A_\varphi^\sigma w^\sigma$. Then

$$\begin{aligned} 0 &= f(\varphi(v), w) + f(v, \varphi(w)) = \varphi(v)^\top A_f w^\sigma + v^\top A_f \varphi(w)^\sigma \\ &= (A_\varphi v)^\top A_f w^\sigma + v^\top A_f (A_\varphi w)^\sigma = v^\top A_\varphi^\top A_f w^\sigma + v^\top A_f A_\varphi^\sigma w^\sigma \\ &= v^\top (A_\varphi^\top A_f + A_f A_\varphi^\sigma) w^\sigma \iff A_\varphi^\top A_f + A_f A_\varphi^\sigma = 0. \end{aligned}$$

so that $\mathfrak{u}(V) = \{\varphi \in \mathfrak{gl}(V) \mid A_\varphi^\top A_f + A_f A_\varphi^\sigma = 0\} \subseteq \mathfrak{gl}(V)$.

Corollary 2.2.17. *Let V be an n-dimensional vector space over a field $\mathbb{F}$ with $n \geq 1$ an integer and let $\mathfrak{u}(V, f)$ be the unitary Lie algebra of V for some non-degenerate Hermitian form $f : V \times V \to \mathbb{F}$. Then $\mathfrak{u}(V, f)$ has dimension n^2.*

Proof. Let $n = 2m+k$ and write $\mathfrak{u}_n(\mathbb{F}, f) = \{\varphi \in \mathfrak{gl}_n(\mathbb{F}) \mid A_\varphi^\top \begin{pmatrix} 0 & I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & \beta I_k \end{pmatrix} + \begin{pmatrix} 0 & I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & \beta I_k \end{pmatrix} A_\varphi^\sigma = 0\}$ with A_φ the matrix corresponding to linear transformation $\varphi \in \mathfrak{gl}_n(\mathbb{F})$. By letting

$\varphi = \left(\begin{smallmatrix} A & B & C \\ D & E & F \\ X & Y & Z \end{smallmatrix}\right)$, in which A, B, D and E are $m \times m$ matrices, C, F, $X^\top$ and $Y^\top$ are $m \times k$ matrices and Z is a $k \times k$ matrix, all having entries in $\mathbb{F}$, we obtain

$$\begin{aligned} 0 &= \begin{pmatrix} A & B & C \\ D & E & F \\ X & Y & Z \end{pmatrix}^\top \begin{pmatrix} 0 & I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & \beta I_k \end{pmatrix} + \begin{pmatrix} 0 & I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & \beta I_k \end{pmatrix} \begin{pmatrix} A & B & C \\ D & E & F \\ X & Y & Z \end{pmatrix}^\sigma \\ &= \begin{pmatrix} D^\sigma + D^\top & A^\top + E^\sigma & F^\sigma + \beta X^\top \\ A^\sigma + E^\top & B^\sigma + B^\top & C^\sigma + \beta Y^\top \\ F^\top + \beta X^\sigma & C^\top + \beta Y^\sigma & \beta(Z^\top + Z^\sigma) \end{pmatrix}, \end{aligned}$$

therefore $A^\top = -E^\sigma$, $B^\sigma = -B^\top$, $D^\sigma = -D^\top$, $F^\sigma = -\beta X^\top$, $C^\sigma = -\beta Y^\top$ and $Z^\top = -Z^\sigma$. By multiplying A_φ with a scalar $\alpha \in \mathbb{F}$ such that $\alpha^\sigma = -\alpha$, we may assume that all signs are positive. We will, however, keep the equation $A^\top = -E^\sigma$. Now we distinguish between the entries of A_φ fixed by σ and not fixed by σ. In the former case, we have the equations $A^\top = -E$, $B = B^\top$, $D = D^\top$, $F = \beta X^\top$, $C = \beta Y^\top$ and $Z^\top = Z$, whereas in the latter case we have the equations $\gamma A^\top = -\gamma^\sigma E$, $\gamma^\sigma B = \gamma B^\top$, $\gamma^\sigma D = \gamma D^\top$, $\gamma^\sigma F = \beta\gamma X^\top$, $\gamma^\sigma C = \beta\gamma Y^\top$ and $\gamma Z^\top = \gamma^\sigma Z$, with $\gamma \in \mathbb{F}$ such that $\gamma^\sigma \neq \pm\gamma$. Upon fixing such a $\gamma \in \mathbb{F}$, we then find

$$\begin{aligned}
\left\langle \left(\begin{smallmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \right\rangle &= \langle E_{i,j} - E_{m+j,m+i} \mid 1 \leq i,j \leq m \rangle \cup \langle \gamma E_{i,j} - \gamma^\sigma E_{m+j,m+i} \mid 1 \leq i,j \leq m \rangle, \\
\left\langle \left(\begin{smallmatrix} 0 & B & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \right\rangle &= \langle E_{i,m+i} \mid 1 \leq i \leq m \rangle \cup \langle E_{i,m+j} + E_{j,m+i} \mid 1 \leq i < j \leq m \rangle \\
&\quad \cup \langle \gamma E_{i,m+j} + \gamma^\sigma E_{j,m+i} \mid 1 \leq i < j \leq m \rangle, \\
\left\langle \left(\begin{smallmatrix} 0 & 0 & 0 \\ D & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \right\rangle &= \langle E_{m+i,i} \mid 1 \leq i \leq m \rangle \cup \langle E_{m+i,j} + E_{m+j,i} \mid 1 \leq i < j \leq m \rangle, \\
&\quad \cup \langle \gamma E_{m+i,j} + \gamma^\sigma E_{m+j,i} \mid 1 \leq i < j \leq m \rangle, \\
\left\langle \left(\begin{smallmatrix} 0 & 0 & C \\ 0 & 0 & 0 \\ 0 & Y & 0 \end{smallmatrix}\right) \right\rangle &= \langle E_{2m+i,m+j} + \beta E_{j,2m+i} \mid 1 \leq i \leq k, 1 \leq j \leq m \rangle, \\
&\quad \cup \langle \gamma E_{2m+i,m+j} + \beta\gamma^\sigma E_{j,2m+i} \mid 1 \leq i \leq k, 1 \leq j \leq m \rangle, \\
\left\langle \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & F \\ X & 0 & 0 \end{smallmatrix}\right) \right\rangle &= \langle E_{2m+i,j} + \beta E_{m+j,2m+i} \mid 1 \leq i \leq k, 1 \leq j \leq m \rangle, \\
&\quad \cup \langle \gamma E_{2m+i,j} + \beta\gamma^\sigma E_{m+j,2m+i} \mid 1 \leq i \leq k, 1 \leq j \leq m \rangle, \\
\left\langle \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Z \end{smallmatrix}\right) \right\rangle &= \langle E_{2m+i,2m+i} \mid 1 \leq i \leq k \rangle \cup \langle E_{2m+i,2m+j} + E_{2m+j,2m+i} \mid 1 \leq i < j \leq k \rangle, \\
&\quad \cup \langle \gamma E_{2m+i,2m+j} + \gamma^\sigma E_{2m+j,2m+i} \mid 1 \leq i < j \leq k \rangle
\end{aligned}$$

to be mutually independent standard bases that together form the standard basis of

$\mathfrak{u}_n(\mathbb{F}, f)$. We then deduce that $\mathfrak{u}(V, f)$ has dimension

$$(2m^2) + 2\left(m + 2\sum_{l=1}^{m}(m-l)\right) + (4km) + \left(k + 2\sum_{l=1}^{k}(k-l)\right) = (2m+k)^2 = n^2.$$

□

Chapter 3

Extremal elements of Lie algebras

This chapter serves as an introduction to extremal elements in Lie algebras. Specifically, we will discuss the basic theory of extremal elements of Lie algebras as a means of characterising the classical linear Lie algebras by their extremal elements.

The theory presented in Section 3.1 and Section 3.2 is based on [13, 15, 20, 23].

3.1 Basic theory of extremal elements

We formally define an extremal element of a Lie algebra for the sake of completeness even though we have already done so in the fourth paragraph of Section 1.1.

Definition 3.1.1 (Extremal element)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. An element $0 \neq x \in \mathfrak{g}$ is said to be **extremal** if there exists a map $g_x : \mathfrak{g} \to \mathbb{F}$ such that*

$$(\mathrm{ad}_x)^2(y) = [x, [x, y]] = 2g_x(y)x$$

for all $y \in \mathfrak{g}$, and if the Premet identities

$$(\mathrm{ad}_{[x,y]})([x, z]) = [[x, y], [x, z]] = g_x([y, z])x + g_x(z)[x, y] - g_x(y)[x, z]$$

and

$$(\mathrm{ad}_x \mathrm{ad}_y \mathrm{ad}_x)(z) = [x, [y, [x, z]]] = g_x([y, z])x - g_x(z)[x, y] - g_x(y)[x, z]$$

hold for all $y, z \in \mathfrak{g}$.

Consequently, we have $(\mathrm{ad}_x)^2(\mathfrak{g}) \subseteq \mathbb{F}x$ for extremal $x \in \mathfrak{g}$. The map $g_x : \mathfrak{g} \to \mathbb{F}$ is called the *extremal form* of x, which we will discuss later in this chapter. By $E(\mathfrak{g})$ we denote the set of all extremal elements of $\mathfrak{g}$. Note that it follows immediately that extremal elements are ad-nilpotent of order at most 3; indeed, for extremal $x \in \mathfrak{g}$ we have $(\mathrm{ad}_x)^3(\mathfrak{g}) = [x, (\mathrm{ad}_x)^2(\mathfrak{g})] \subseteq [x, \mathbb{F}x] = \mathbb{F}[x, x] = 0$. Further note that the Premet identities ensure that not all elements in $\mathfrak{g}$ are extremal if $\mathrm{char}(\mathbb{F}) = 2$; without the Premet identities, an element $0 \neq x \in \mathfrak{g}$ would be extremal if there exists a map $g_x : \mathfrak{g} \to \mathbb{F}$ such that $[x, [x, y]] = 2g_x(y)x = 0$ for all $y \in \mathfrak{g}$ because $\mathrm{char}(\mathbb{F}) = 2$, so taking $g_x : \mathfrak{g} \to \mathbb{F}$ to be identically zero for all $0 \neq x \in \mathfrak{g}$ would then show that all $0 \neq x \in \mathfrak{g}$ are extremal.

Example 3.1.2. Recall from Example 2.2.2, in which we showed simplicity of $\mathfrak{sl}_2(\mathbb{F})$ if $\mathrm{char}(\mathbb{F}) \neq 2$, that the triple $\{x, y, h\} = \{E_{1,2}, E_{2,1}, E_{1,1} - E_{2,2}\}$ satisfying $[x, y] = h$, $[h, x] = 2x$ and $[h, y] = -2y$ is a basis of $\mathfrak{sl}_2(\mathbb{F})$, in which $E_{i,j} \in \mathfrak{gl}_2(\mathbb{F})$, $1 \leq i, j \leq 2$, is the matrix having a one in position (i, j) and zeros elsewhere. Now let $ax + by + ch \in \mathfrak{sl}_2(\mathbb{F})$ with $a, b, c \in \mathbb{F}$ and consider first the element $\lambda h \in \mathfrak{sl}_2(\mathbb{F})$ with $\lambda \in \mathbb{F}^*$. Then

$$[\lambda h, [\lambda h, ax + by + ch]] = \lambda^2[h, 2ax - 2ay] = 4\lambda^2 ax + 4\lambda^2 by,$$

hence no element in $\mathbb{F}h$ will be extremal in $\mathfrak{sl}_2(\mathbb{F})$ by linear independence of x, y and h. Now let $z = \lambda x + \mu y \in \mathfrak{sl}_2(\mathbb{F})$ with $\lambda, \mu \in \mathbb{F}$ not both zero, then

$$\begin{aligned}
[z, [z, ax + by + ch]] &= [\lambda x + \mu y, \lambda[x, ax + by + ch]] + [\lambda x + \mu y, \mu[y, ax + by + ch]] \\
&= [\lambda x + \mu y, \lambda(bh - 2cx)] + [\lambda x + \mu y, \mu(-ah + 2cy)] \\
&= \lambda^2[x, bh] + \lambda\mu[y, bh - 2cx] + \lambda\mu[x, -ah + 2cy] + \mu^2[y, -ah] \\
&= -2\lambda^2 bx + 2\lambda\mu by + 2\lambda\mu ch + 2\lambda\mu ax + 2\lambda\mu ch - 2\mu^2 ay.
\end{aligned}$$

It follows that z is extremal if and only if $4\lambda\mu ch = 0$ if and only if $\lambda = 0$ or $\mu = 0$. If $\lambda = 0$, then $z = \mu y$ and $[\mu y, [\mu y, ax + by + ch]] = -2\mu^2 ay$, hence every element in $\mathbb{F}y$ is extremal with extremal form $g_{\mu y} : \mathfrak{g} \to \mathbb{F}$ given by $g_{\mu y}(ax + by + ch) = -\mu a$. If $\mu = 0$, then $z = \lambda x$ and $[\lambda x, [\lambda x, ax + by + ch]] = -2\lambda^2 bx$ so that every element in $\mathbb{F}x$ is extremal with extremal form $g_{\lambda x} : \mathfrak{g} \to \mathbb{F}$ given by $g_{\lambda x}(ax + by + ch) = -\lambda b$.

In the above example, we concluded that all elements in $\mathbb{F}x$ and $\mathbb{F}y$ are extremal in $\mathfrak{sl}_2(\mathbb{F})$ without verifying the Premet identities. However, the Premet identities turn out to be superfluous in case $\mathrm{char}(\mathbb{F}) \neq 2$, as characterised by the following proposition.

Proposition 3.1.3. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ and let $x \in \mathfrak{g}$. Then $x \in E(\mathfrak{g})$ if there exists a map $g_x : \mathfrak{g} \to \mathbb{F}$ such that $[x, [x, y]] = 2g_x(y)x$ for all $y \in \mathfrak{g}$.*

Proof. We show that the Premet identities follow from the identity $[x, [x, y]] = 2g_x(y)x$ for all $y \in \mathfrak{g}$. By applying the Jacobi identity from Definition 2.1.1 thrice, we obtain

$$\begin{aligned}
[[x, y], [x, z]] &= -[x, [z, [x, y]]] - [z, [[x, y], x]] = -[x, [z, [x, y]]] + [z, [x, [x, y]]] \\
&= -[x, -[y, [z, x]] - [x, [y, z]]] + [z, 2g_x(y)x] \\
&= [x, [y, [z, x]]] + [x, [x, [y, z]]] - 2g_x(y)[x, z] \\
&= -[[z, x], [x, y]] - [y, [[z, x], x]] + 2g_x([y, z])x - 2g_x(y)[x, z] \\
&= -[[x, y], [x, z]] - [y, [x, [x, z]]] + 2g_x([y, z])x - 2g_x(y)[x, z] \\
&= -[[x, y], [x, z]] + 2g_y(z)[x, y] + 2g_x([y, z])x - 2g_x(y)[x, z]
\end{aligned}$$

for all $y, z \in \mathfrak{g}$, from which the first Premet identity follows by adding $[[x, y], [x, z]]$ to both sides and dividing by 2.

Again by using the Jacobi identity, we next obtain

$$\begin{aligned}[x, [y, [x, z]] &= -[[x, z], [x, y]] - [y, [[x, z], x]] = [[x, y], [x, z]] - [[x, [x, z]], y] \\ &= [[x, y], [x, z]] - 2g_x(z)[x, y],\end{aligned}$$

hence the second Premet identity follows from the first Premet identity. □

Although any $x \in E(\mathfrak{g})$ satisfies $[x, [x, \mathfrak{g}]] = 0$ if $\mathrm{char}(\mathbb{F}) = 2$, if $\mathrm{char}(\mathbb{F}) \neq 2$ it could still be the case that $[x, [x, \mathfrak{g}]] = 0$ with $x \in \mathfrak{g}$ extremal. This gives rise to the following definition.

Definition 3.1.4 (Sandwich element)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. An element $0 \neq x \in \mathfrak{g}$ is called a* ***sandwich*** *element if*

$$(\mathrm{ad}_x)^2(y) = [x, [x, y]] = 0 \quad \text{and} \quad (\mathrm{ad}_x \mathrm{ad}_y \mathrm{ad}_x)(z) = [x, [y, [x, z]]] = 0$$

for all $y, z \in \mathfrak{g}$.

A sandwich element $x \in \mathfrak{g}$ is clearly extremal, and we may choose the extremal form $g_x : \mathfrak{g} \to \mathbb{F}$ of x to be identically zero. Furthermore, the additional condition $[x, [y, [x, z]]] = 0$ for all $y, z \in \mathfrak{g}$ ensures that extremal elements are not necessarily sandwich if $\mathrm{char}(\mathbb{F}) = 2$. This distinction shows the existence of non-sandwich extremal elements, which we call *pure* extremal elements. In the remainder of this section, whenever we mention extremal elements we mean pure extremal elements unless stated otherwise.

We continue our discussion of extremal elements by listing some additional properties. In particular, we provide some relations between extremal elements of Lie algebras.

Corollary 3.1.5. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. If $x, y \in E(\mathfrak{g})$, then $g_x(y) = g_y(x)$ and $g_x([y, z]) + g_y([x, z]) = 0$ for all $z \in \mathfrak{g}$.*

Proof. First assume that $[x, y] = 0$. Then it follows from the first Premet identity that

$$0 = [[x, y], [x, z]] = g_x([y, z])x + g_x(z)[x, y] - g_x(y)[x, z] = g_x([y, z])x - g_x(y)[x, z]$$

for all $z \in \mathfrak{g}$, hence either $g_x(y) = 0$ and $g_x([y, z]) = 0$ because $x \neq 0$, or $g_x(y) \neq 0$ and $[x, z] = (g_x(y))^{-1} g_x([y, z])x$, in which case $[x, [x, z]] = 0$. Specifically, in the latter case we have $[x, \mathfrak{g}] \subseteq \mathbb{F}x$ and $[x, [x, \mathfrak{g}]] = 0$. But then $[x, [\mathfrak{g}, [x, \mathfrak{g}]]] \subseteq \mathbb{F}[x, [x, \mathfrak{g}]] = 0$ implies that x is a sandwich element by Definition 3.1.4 so that the extremal form g_x is identically zero. Regardless, we will have $g_x(y) = 0$ and consequently $g_x([y, z]) = 0$. By interchanging x and y in the above, we also conclude that $g_y(x) = 0$ and $g_y([x, z]) = 0$. Thus, $g_x(y) = g_y(x)$ and $g_x([y, z]) + g_y([x, z]) = 0$ if $[x, y] = 0$.

Next, assume that $[x,y] \neq 0$ and consider the expression $g_x([y,[y,z]])x - g_y(z)[x,[x,y]]$ with $z \in \mathfrak{g}$ arbitrary. If $\mathrm{char}(\mathbb{F}) = 2$, then we obtain $g_x([y,[y,z]])x - g_y(z)[x,[x,y]] = g_x(0)x - g_y(z)\cdot 0 = 0$, and if $\mathrm{char}(\mathbb{F}) \neq 2$, then we have

$$\begin{aligned} 2g_x([y,[y,z]])x - 2g_y(z)[x,[x,y]] &= [x,[x,[y,[y,z]]]] - [x,[x,2g_y(z)y]] \\ &= [x,[x,[y,[y,z]]]] - [x,[x,[y,[y,z]]]] = 0 \end{aligned}$$

so that $g_x([y,[y,z]])x - g_y(z)[x,[x,y]] = 0$. Next, we rewrite $[x,[y,[x,[y,z]]]]$ by applying both the second Premet identity for $x \in E(\mathfrak{g})$ to $[y,z] \in \mathfrak{g}$ and ad_x to the second Premet identity for $y \in E(\mathfrak{g})$ applied to $z \in \mathfrak{g}$. This yields

$$\begin{aligned} 0 = [x,[y,[x,[y,z]]]] - [x,[y,[x,[y,z]]]] &= (\mathrm{ad}_x\mathrm{ad}_y\mathrm{ad}_x)([y,z]) - \mathrm{ad}_x([y,[x,[y,z]]]) \\ &= g_x([y,[y,z]])x - g_x([y,z])[x,y] - g_x(y)[x,[y,z]] \\ &\quad - [x, g_y([x,z])y - g_y(z)[y,x] - g_y(x)[y,z]] \\ &= g_x([y,[y,z]])x - g_x([y,z])[x,y] - g_x(y)[x,[y,z]] \\ &\quad - g_y([x,z])[x,y] - g_y(z)[x,[x,y]] + g_y(x)[x,[y,z]] \\ &= (g_y(x) - g_x(y))[x,[y,z]] - (g_x([y,z]) + g_y([x,z]))[x,y], \end{aligned}$$

from which we deduce that $g_y(x) = g_x(y)$ immediately implies $g_x([y,z]) + g_y([x,z]) = 0$. If $\mathrm{char}(\mathbb{F}) \neq 2$, then

$$0 = [[x,y],[x,y]] = [x,[y,[y,x]]] - [[x,[x,y]],y] = 2(g_y(x) - g_x(y))[x,y]$$

by the Jacobi identity, hence $g_y(x) = g_x(y)$. If $\mathrm{char}(\mathbb{F}) = 2$, we make use of the observation that we may assume $[x,[y,z]]$ and $[x,y]$ to be linearly dependent; indeed, if $[x,[y,z]]$ and $[x,y]$ are linearly independent, then $(g_y(x) - g_x(y))[x,[y,z]] - (g_x([y,z]) + g_y([x,z]))[x,y] = 0$ implies $g_y(x) - g_x(y) = 0$ and $g_x([y,z]) + g_y([x,z]) = 0$. Since interchanging x and y gives us the same identity, linear dependence of $[x,[y,z]]$ and $[x,y]$ is equivalent to $[x,[y,\mathfrak{g}]] + [y,[x,\mathfrak{g}]] \subseteq \mathbb{F}[x,y]$. As $[x,[x,\mathfrak{g}]] = [y,[y,\mathfrak{g}]] = 0$, upon applying ad_x to both sides we obtain

$$0 = \mathbb{F}[x,[x,y]] \supseteq [x,[x,[y,\mathfrak{g}]]] + [x,[y,[x,\mathfrak{g}]]] = [x,[y,[x,\mathfrak{g}]]],$$

hence the second Premet identity yields

$$0 = [x,[y,[x,\mathfrak{g}]]] = g_x([y,\mathfrak{g}])x - g_x(\mathfrak{g})[x,y] - g_x(y)[x,\mathfrak{g}],$$

from which we deduce that $[x,\mathfrak{g}]$ is contained in the subspace $\mathbb{F}x + \mathbb{F}[x,y] \subseteq \mathfrak{g}$. In particular, $[\mathbb{F}x + \mathbb{F}[x,y], \mathbb{F}x + \mathbb{F}[x,y]] = \mathbb{F}[x,[x,y]] = 0$ so that $\mathbb{F}x + \mathbb{F}[x,y]$ is an abelian Lie subalgebra of $\mathfrak{g}$, and as a consequence, so is $[x,\mathfrak{g}]$. Combining the first and second Premet identities now gives $0 = [[x,\mathfrak{g}],[x,\mathfrak{g}]] = [x,[\mathfrak{g},[x,\mathfrak{g}]]] + 2g_x(\mathfrak{g})[x,y] = [x,[\mathfrak{g},[x,\mathfrak{g}]]]$, from which we conclude that x is a sandwich element by Definition 3.1.4. Therefore, its extremal form is identically zero and we have $g_x(y) = 0$. Since all we have said remains true after interchanging x and y, we also find $g_y(x) = 0$, thus $g_y(x) = g_x(y)$ and the corollary follows. $\square$

Corollary 3.1.6. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. Then*

(i) *for all $x, y \in E(\mathfrak{g})$ and $z \in \mathfrak{g}$ we have*

$$[[x, y], [x, [y, z]]] = 2g_x(y)g_y(z)x + g_x([y, z])[x, y] - g_x(y)[x, [y, z]],$$

(ii) *for all $x, y \in E(\mathfrak{g})$ and $z \in \mathfrak{g}$ we have*

$$\begin{aligned}[[x, y], [[x, y], z]] = 2g_x(y)g_y(z)x - g_x(y)[x, [y, z]] + 2g_x(y)g_x(z)y - g_x(y)[y, [x, z]]\\ + g_x([y, z])[x, y] - g_y([x, z])[x, y],\end{aligned}$$

(iii) *for all $x, y, z \in E(\mathfrak{g})$ we have*

$$\begin{aligned}[[x, [y, z]], [y, [x, z]]] = -g_x([y, z])g_y(z)x - g_x([y, z])g_x(z)y - g_x(y)g_x([y, z])z\\ - 2g_x(z)g_y(z)[x, y] + 2g_x(y)g_y(z)[x, z] - 2g_x(y)g_x(z)[y, z].\end{aligned}$$

Proof. For (i), we first observe that $g_x([y, [y, z]]) = g_x(2g_y(z)y) = 2g_x(y)g_y(z)$ for all $z \in \mathfrak{g}$ since $y \in E(\mathfrak{g})$. Now apply the first Premet identity for $x \in E(\mathfrak{g})$ to $[y, z] \in \mathfrak{g}$ to obtain

$$\begin{aligned}[[x, y], [x, [y, z]]] &= g_x([y, [y, z]])x + g_x([y, z])[x, y] - g_x(y)[x, [y, z]]\\ &= 2g_x(y)g_y(z)x + g_x([y, z])[x, y] - g_x(y)[x, [y, z]].\end{aligned}$$

For (ii), by the Jacobi identity we have

$$[[x, y], [[x, y], z]] = [[x, y], [y, [z, x]]] + [[x, y], [x, [y, z]]] = [[y, x], [y, [x, z]]] + [[x, y], [x, [y, z]]],$$

after which we may apply Corollary 3.1.6(i) to both expressions. In combination with Corollary 3.1.5, this yields

$$\begin{aligned}[[x, y], [[x, y], z]] &= 2g_x(z)g_y(x)y + g_y([x, z])[y, x] - g_y(x)[y, [x, z]]\\ &\quad + 2g_x(y)g_y(z)x + g_x([y, z])[x, y] - g_x(y)[x, [y, z]]\\ &= 2g_x(y)g_y(z)x - g_x(y)[x, [y, z]] + 2g_x(y)g_x(z)y - g_x(y)[y, [x, z]]\\ &\quad + g_x([y, z])[x, y] - g_y([x, z])[x, y].\end{aligned}$$

For (iii), again by the Jacobi identity we have

$$[x, [y, z]], [y, [x, z]]] = [[x, z], [y, [x, [y, z]]]] - [y, [[x, z], [x, [y, z]]]].$$

Using both Premet identities and Corollary 3.1.5, the first expression simplifies to

$$\begin{aligned}
[[x,z],[y,[x,[y,z]]]] &= [[x,z], g_y([x,z])y - g_y(z)[y,x] - g_y(x)[y,z]] \\
&= g_y([x,z])[[x,z],y] - g_y(z)[[x,z],[y,x]] - g_y(x)[[x,z],[y,z]] \\
&= g_x([y,z])[y,[x,z]] + g_y(z)[[x,z],[x,y]] - g_y(x)[[z,x],[z,y]] \\
&= g_x([y,z])[y,[x,z]] + g_y(z)(g_x([z,y])x + g_x(y)[x,z] - g_x(z)[x,y]) \\
&\quad - g_x(y)(g_z([x,y])z + g_z(y)[z,x] - g_z(x)[z,y]) \\
&= g_x([y,z])[y,[x,z]] - g_x([y,z])g_y(z)x + g_x(y)g_y(z)[x,z] \\
&\quad - g_x(z)g_y(z)[x,y] - g_x([y,z])g_x(y)z + g_x(y)g_y(z)[x,z] \\
&\quad - g_x(y)g_x(z)[y,z]
\end{aligned}$$

and the second expression simplifies to

$$\begin{aligned}
[y,[[x,z],[x,[y,z]]]] &= [y, g_x([z,[y,z]])x + g_x([y,z])[x,z] - g_x(z)[x,[y,z]] \\
&= g_x([z,[z,y]])[x,y] + g_x([y,z])[y,[x,z]] - g_x(z)[y,[x,[y,z]]] \\
&= g_x(2g_z(y)z)[x,y] + g_x([y,z])[y,[x,z]] \\
&\quad - g_x(z)(g_y([x,z])y - g_y(z)[y,x] - g_y(x)[y,z]) \\
&= 2g_x(z)g_y(z)[x,y] + g_x([y,z])[y,[x,z]] \\
&\quad + g_x([y,z])g_x(z)y - g_x(z)g_y(z)[x,y] + g_x(z)g_y(x)[y,z].
\end{aligned}$$

Combining both expressions then results in

$$\begin{aligned}
[x,[y,z]],[y,[x,z]]] &= [[x,z],[y,[x,[y,z]]]] - [y,[[x,z],[x,[y,z]]]] \\
&= g_x([y,z])[y,[x,z]] - g_x([y,z])g_y(z)x + g_x(y)g_y(z)[x,z] \\
&\quad - g_x(z)g_y(z)[x,y] - g_x([y,z])g_x(y)z + g_x(y)g_y(z)[x,z] \\
&\quad - g_x(y)g_x(z)[y,z] - 2g_x(z)g_y(z)[x,y] - g_x([y,z])[y,[x,z]] \\
&\quad - g_x([y,z])g_x(z)y + g_x(z)g_y(z)[x,y] - g_x(z)g_y(x)[y,z] \\
&= -g_x([y,z])g_y(z)x - g_x([y,z])g_x(z)y - g_x([y,z])g_x(y)z \\
&\quad - 2g_x(z)g_y(z)[x,y] + 2g_x(y)g_y(z)[x,z] - 2g_x(y)g_x(z)[y,z].
\end{aligned}$$

□

Recall from Definition 2.1.7 that the set of inner automorphisms $\mathrm{Int}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ consists of all automorphisms of g of the form

$$e^{\mathrm{ad}_x}(y) = \sum_{i=0}^{k-1} \frac{(\mathrm{ad}_x)^i(y)}{i!}$$

with $y \in \mathfrak{g}$ and $x \in \mathfrak{g}$ ad-nilpotent. Since any $x \in E(\mathfrak{g})$ is ad-nilpotent of order at most 3, we obtain

$$e^{\lambda \cdot \mathrm{ad}_x}(y) = y + \lambda[x,y] + \frac{1}{2} \cdot \lambda[x, \lambda[x,y]] = y + \lambda[x,y] + \lambda^2 \cdot \frac{1}{2}[x,[x,y]] = y + \lambda[x,y] + \lambda^2 g_x(y)x.$$

for all $x \in E(\mathfrak{g})$ and $\lambda \in \mathbb{F}$. For ease of notation, we will write $\exp(x,\lambda) := e^{\lambda \cdot \mathrm{ad}_x}$, and it is easy to see that $\exp(x,\lambda) = \exp(\lambda x, 1)$ by linearity of the Lie bracket in its first entry.

Proposition 3.1.7. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. Then $\exp(x,\lambda)$ is a Lie algebra homomorphism from $\mathfrak{g}$ to itself for all $x \in E(\mathfrak{g})$ and $\lambda \in \mathbb{F}$. Moreover, we have $\exp(x,\lambda+\mu) = \exp(x,\lambda)\exp(x,\mu)$ for all $x \in E(\mathfrak{g})$ and $\lambda,\mu \in \mathbb{F}$, and $\exp(x,\lambda)^n = \exp(x,n\lambda)$ for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{F}$ if $\mathrm{char}(\mathbb{F}) \neq 2$.*

Proof. Let $x \in E(\mathfrak{g})$ and $\lambda \in \mathbb{F}$ be arbitrary, then using the Jacobi identity and the first Premet identity we find for all $y, z \in \mathfrak{g}$ that

$$\begin{aligned}
[\exp(x,\lambda)(y), \exp(x,\lambda)(z)] &= [y + \lambda[x,y] + \lambda^2 g_x(y)x, z + \lambda[x,z] + \lambda^2 g_x(z)x] \\
&= [y,z] + \lambda([y,[x,z]] + [[x,y],z]) + \lambda^4 g_x(y)g_x(z)[x,x] \\
&\quad + \lambda^2([[x,y],[x,z]] - g_x(z)[x,y] + g_x(y)[x,z]) \\
&\quad + \lambda^3(g_x(y)[x,[x,z]] - g_x(z)[x,[x,y]]) \\
&= [y,z] + \lambda[x,[y,z]] + \lambda^2 g_x([y,z])x = \exp(x,\lambda)([y,z]),
\end{aligned}$$

from which we conclude that $\exp(x,\lambda)$ is a Lie algebra homomorphism from $\mathfrak{g}$ to itself.

To prove the second assertion of the lemma, first observe that $\exp(x,\lambda)(y) = y + \lambda[x,y]$ for all $x \in E(\mathfrak{g})$, $\lambda \in \mathbb{F}$ and $y \in \mathfrak{g}$ if $\mathrm{char}(\mathbb{F}) = 2$. It then follows for all $y \in \mathfrak{g}$ that

$$\begin{aligned}
(\exp(x,\lambda)\exp(x,\mu))(y) &= \exp(x,\lambda)(y + \mu[x,y]) = y + \mu[x,y] + \lambda[x, y + \mu[x,y]] \\
&= y + (\lambda+\mu)[x,y] + \lambda\mu[x,[x,y]] = \exp(x,\lambda+\mu)
\end{aligned}$$

for all $x \in E(\mathfrak{g})$ and $\lambda,\mu \in \mathbb{F}$ so that $\exp(x,\lambda+\mu) = \exp(x,\lambda)\exp(x,\mu)$. On the other hand, if $\mathrm{char}(\mathbb{F}) \neq 2$, then $0 = [x,[x,x]] = 2g_x(x)x$ implies $g_x(x) = 0$ for all $x \in E(\mathfrak{g})$, and $0 = 2g_x(y)[x,x] = [x, 2g_x(y)x] = [x,[x,[x,y]]] = 2g_x([x,y])x$ implies $g_x([x,y]) = 0$ for all $y \in \mathfrak{g}$. We then find for all $y \in \mathfrak{g}$ that

$$\begin{aligned}
(\exp(x,\lambda)\exp(x,\mu))(y) &= \exp(x,\lambda)(y + \mu[x,y] + \mu^2 g_x(y)x) \\
&= y + \mu[x,y] + \mu^2 g_x(y)x + \lambda[x, y + \mu[x,y] + \mu^2 g_x(y)x] \\
&\quad \lambda^2 g_x(y + \mu[x,y] + \mu^2 g_x(y)x)x \\
&= y + (\lambda+\mu)[x,y] + (\lambda^2 g_x(y) + \lambda\mu[x,[x,y]] + \mu^2 g_x(y))x \\
&\quad + \lambda\mu^2 g_x(y)[x,x] + \lambda^2\mu g_x([x,y])x + \lambda^2\mu^2 g_x(x)g_x(y)x \\
&= y + (\lambda+\mu)[x,y] + (\lambda^2 + 2\lambda\mu + \mu^2)g_x(y)x \\
&= y + (\lambda+\mu)[x,y] + (\lambda+\mu)^2 g_x(y)x = \exp(x,\lambda+\mu)(y)
\end{aligned}$$

for all $x \in E(\mathfrak{g})$ and $\lambda, \mu \in \mathbb{F}$, hence $\exp(x, \lambda + \mu) = \exp(x, \lambda)\exp(x, \mu)$.

We prove that $\exp(x, \lambda)^n = \exp(x, n\lambda)$ for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{F}$ by induction on n. For $n \leq 1$, there is nothing to prove, so assume that $\exp(x, \lambda)^n = \exp(x, n\lambda)$ for some $n \geq 1$. Then for all $z \in \mathfrak{g}$ we obtain

$$\begin{aligned}
\exp(x, \lambda)^{n+1}(z) &= \exp(x, \lambda)(\exp(x, \lambda)^n(z)) = \exp(x, \lambda)(\exp(x, n\lambda)(z)) \\
&= \exp(x, \lambda)(z + n\lambda[x, z] + n^2\lambda^2 g_x(z)x) = z + n\lambda[x, z] + n^2\lambda^2 g_x(z)x \\
&\quad + \lambda[x, z + n\lambda[x, z] + n^2\lambda^2 g_x(z)x] + \lambda^2 g_x(z + n\lambda[x, z] + n^2\lambda^2 g_x(z)x)x \\
&= z + n\lambda[x, z] + n^2\lambda^2 g_x(z)x + \lambda[x, z] + n\lambda^2[x, [x, z]] + \lambda^2 g_x(z)x, \\
&= z + (n+1)\lambda[x, z] + (n+1)^2\lambda^2 g_x(z)x = \exp(x, (n+1)\lambda)(z)
\end{aligned}$$

where we have used $g_x(x) = 0$ and $2g_x([x, z]) = g_x([x, z]) + g_x([x, z]) = 0 \iff g_x([x, z]) = 0$ by Corollary 3.1.5 because $\mathrm{char}(\mathbb{F}) \neq 2$. By induction, it now follows that $\exp(x, \lambda)^n = \exp(x, n\lambda)$ for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{F}$. □

For any $x \in E(\mathfrak{g})$, denote by $\mathrm{Exp}(x)$ the set of all automorphisms $\exp(x, \lambda)$ with $\lambda \in \mathbb{F}$. Observe that $\mathrm{Exp}(x) = \mathrm{Exp}(\lambda x)$ for all $\lambda \in \mathbb{F}$ because $\exp(x, \lambda) = \exp(\lambda x, 1)$, as we have seen previously.

Although we have stated without proof in Definition 2.1.7 that $\exp(x, 1)$ is an inner automorphism of $\mathfrak{g}$ for every ad-nilpotent $x \in \mathfrak{g}$, we will provide a proof of this claim in the general case for $\exp(x, \lambda)$ with $x \in E(\mathfrak{g})$ and $\lambda \in \mathbb{F}$.

Corollary 3.1.8. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$. For any $x \in E(\mathfrak{g})$, the set $\mathrm{Exp}(x) = \{\exp(x, \lambda) \mid \lambda \in \mathbb{F}\}$ forms a subgroup of $\mathrm{Int}(\mathfrak{g})$ isomorphic to the additive group of $\mathbb{F}$.*

Proof. Let $x \in E(\mathfrak{g})$. It is clear that $\exp(x, 0)(y) = y$ for all $y \in \mathfrak{g}$ so that $\exp(x, 0)$ will be the identity element of $\mathrm{Exp}(x)$. Furthermore, we have $\exp(x, \lambda)\exp(x, \mu) = \exp(x, \lambda + \mu) \in \mathrm{Exp}(x)$ by Proposition 3.1.7 so that $\mathrm{Exp}(x)$ is closed under function composition, and again by Proposition 3.1.7 we have $\exp(x, \lambda)\exp(x, -\lambda) = \exp(x, 0)$ so that $\exp(x, -\lambda) \in \mathrm{Exp}(x)$ is the unique inverse of $\exp(x, \lambda) \in \mathrm{Exp}(x)$ with $\lambda \in \mathbb{F}$. We conclude that $\mathrm{Exp}(x)$ forms a group under ordinary function composition. It also immediately follows that $\exp(x, \lambda) \in \mathrm{Int}(\mathfrak{g})$ for all $\lambda \in \mathbb{F}$; indeed, $\exp(x, \lambda)$ is a Lie algebra homomorphism from $\mathfrak{g}$ to itself by Proposition 3.1.7, and if $\exp(x, \lambda)(y) = \exp(x, \lambda)(z)$ with $y, z \in \mathfrak{g}$, then $y = \exp(x, -\lambda)(\exp(x, \lambda)(y)) = \exp(x, -\lambda)(\exp(x, \lambda)(z)) = z$ so that $\exp(x, \lambda)$ is injective, while for every $y \in \mathfrak{g}$ we have $\exp(x, \lambda)(\exp(x, -\lambda)(y)) = \exp(x, 0)(y) = y$ so that $\exp(x, \lambda)$ is surjective, showing that $\exp(x, \lambda)$ is bijective.

The isomorphism between $\mathrm{Exp}(x)$ and the additive group of $\mathbb{F}$ is provided by the map $\varphi : \mathrm{Exp}(x) \to \mathbb{F}$ given by $\exp(x, \lambda) \mapsto \lambda$. Indeed, the map φ is readily seen to be both injective and surjective, and for all $\lambda, \mu \in \mathbb{F}$ we have

$$\varphi(\exp(x, \lambda)\exp(x, \mu)) = \varphi(\exp(x, \lambda + \mu)) = \lambda + \mu = \varphi(\exp(x, \lambda)) + \varphi(\exp(x, \mu))$$

so that φ is a group homomorphism. □

We finish this section with a discussion of the extremal form $g_x : \mathfrak{g} \to \mathbb{F}$ with $x \in E(\mathfrak{g})$. Consider first the following lemma.

Lemma 3.1.9. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ generated by its set $E(\mathfrak{g})$ of extremal elements. Then $\mathfrak{g}$ is spanned linearly by $E(\mathfrak{g})$.*

Proof. Recall from Definition 2.1.21 the descending central series of $\mathfrak{g}$, given recursively by $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$ with $i \geq 0$. By induction on i, we will show that every $z \in \mathfrak{g}^i$, $i \geq 0$, can be written as a linear combination of elements in $\mathfrak{g}$; this suffices as it will cover all elements in $\mathfrak{g}$. For $i = 0$, we have $\mathfrak{g}^0 = \langle E(\mathfrak{g}) \rangle$ so the claim is immediate. Now suppose that every $z \in \mathfrak{g}^k$ can be written as a linear combination of extremal elements of $\mathfrak{g}$ for some $k \geq 1$. For every $z \in \mathfrak{g}^{k+1} = [\mathfrak{g}, \mathfrak{g}^k]$, there exist $x \in \mathfrak{g}$ and $y \in \mathfrak{g}^k$ such that $z = [x, y]$. Since we have $\mathfrak{g}^k \ni \exp(x, 1)(y) = y + [x, y] + g_x(y)x$, we deduce that $z = [x, y]$ is a linear combination of extremal elements in $\mathfrak{g}$ and $\mathfrak{g}^k$, which are themselves linear combinations of elements in $E(\mathfrak{g})$ by the induction hypothesis. Consequently, every $z \in \mathfrak{g}^{k+1}$ can be written as a linear combination of extremal elements. The claim then follows by induction, and so does the lemma. □

Under the conditions specified by the above lemma, it is possible to extend the extremal form of an element in $E(\mathfrak{g})$ to a bilinear form on $\mathfrak{g}$ which will moreover be symmetric. This is characterised by the following proposition.

Proposition 3.1.10. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ generated by its set $E(\mathfrak{g})$ of extremal elements. Then there exists a unique bilinear symmetric form $g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ such that $g(x, y) = g_x(y)$ for every $y \in \mathfrak{g}$ and $x \in E(\mathfrak{g})$ with extremal form $g_x : \mathfrak{g} \to \mathbb{F}$.*

In particular, the form $g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ is associative in the sense that $g(x, [y, z]) = g([x, y], z)$ for all $x, y, z \in E(\mathfrak{g})$.

Proof. For all $x \in E(\mathfrak{g})$, $y \in \mathfrak{g}$ and $\lambda \in \mathbb{F}^*$ we have

$$[\lambda x, [\lambda x, y]] = \lambda^2 [x, [x, y]] = \lambda^2 \cdot 2 g_x(y) x = 2(\lambda g_x(y))(\lambda x),$$

hence $\lambda x \in E(\mathfrak{g})$ with extremal form $g_{\lambda x} : \mathfrak{g} \to \mathbb{F}$ given by $g_{\lambda x}(y) = \lambda g_x(y)$ with $y \in \mathfrak{g}$. Given a basis $\{v_i\}_{i \in I}$ of $\mathfrak{g}$ indexed by some index set I with $v_i \in E(\mathfrak{g})$, $i \in I$, we may write any $x \in E(\mathfrak{g})$ as a linear combination of these basis elements by Lemma 3.1.9, i.e. $x = \sum_{i \in I} \lambda_i v_i$ with $\lambda_i \in \mathbb{F}^*$, $i \in I$, whose extremal form $g_x : \mathfrak{g} \to \mathbb{F}$ we define to be $g_x(y) = \sum_{i \in I} \lambda_i g_{v_i}(y) = \sum_{i \in I} g_{\lambda_i v_i}(y)$ by the above with $y \in \mathfrak{g}$. We show that this extremal form is well-defined. To this extent, suppose that we can also write $x = \sum_{i \in I} \mu_i v_i$ with $\mu_i \in \mathbb{F}$, $i \in I$, such that $\mu_i \neq \lambda_i$ for at least one $i \in I$. By using Corollary 3.1.5, we have for all $y \in E(\mathfrak{g})$ that

$$\sum_{i \in I} g_{\lambda_i v_i}(y) = \sum_{i \in I} g_y(\lambda_i v_i) = g_y\left(\sum_{i \in I} \lambda_i v_i\right) = g_y\left(\sum_{i \in I} \mu_i v_i\right) = \sum_{i \in I} g_y(\mu_i v_i) = \sum_{i \in I} g_{\mu_i v_i}(y).$$

Since $\mathfrak{g} = \langle E(\mathfrak{g})\rangle$, the above is even true if $y \in \mathfrak{g}$, hence the forms $\sum_{i\in I} g_{\lambda_i v_i}$ and $\sum_{i\in I} g_{\mu_i v_i}$ are identical on $\mathfrak{g}$. It follows that g_x is well-defined. Consequently, the form $g : \mathfrak{g}\times\mathfrak{g} \to \mathbb{F}$ given by $g(x, y) = g_x(y)$ with $y \in \mathfrak{g}$ and $x \in E(\mathfrak{g})$ having extremal form $g_x : \mathfrak{g} \to \mathbb{F}$ as given above is well-defined. Linearity of $g_x : \mathfrak{g} \to \mathbb{F}$ implies linearity of $g : \mathfrak{g}\times\mathfrak{g} \to \mathbb{F}$ in its second coordinate, and for $x = \sum_{i\in I} \lambda_i v_i \in E(\mathfrak{g})$ and $y = \sum_{i\in I} v_i \in E(\mathfrak{g})$ with $\lambda_i, \mu_i \in \mathbb{F}$, $i \in I$, we find for all $z \in \mathfrak{g}$ that

$$\begin{aligned} g(\alpha x + \beta y, z) = g_{\alpha x+\beta y}(z) &= \sum_{i\in I}(\alpha\lambda_i + \beta\mu_i)g_{v_i}(z) = \alpha\sum_{i\in I}\lambda_i g_{v_i}(z) + \beta\sum_{i\in I}\mu_i g_{v_i}(z) \\ &= \alpha g_x(z) + \beta g_y(z) = \alpha g(x, z) + \beta g(y, z) \end{aligned}$$

for all $\alpha, \beta \in \mathbb{F}$, showing that $g : \mathfrak{g}\times\mathfrak{g} \to \mathbb{F}$ is also linear in its first coordinate. Thus, the form $g : \mathfrak{g}\times\mathfrak{g} \to \mathbb{F}$ is bilinear. In particular, it is unique because $\mathfrak{g}$ is linearly spanned by $E(\mathfrak{g})$ as a consequence of Lemma 3.1.9. Symmetry of the form follows from Corollary 3.1.5; for all $x, y \in E(\mathfrak{g})$ we have $g(x, y) = g_x(y) = g_y(x) = g(y, x)$, which also holds true if $x, y \in \mathfrak{g}$ by Lemma 3.1.9.

To show associativity of the unique bilinear symmetric form $g : \mathfrak{g}\times\mathfrak{g} \to \mathbb{F}$ defined above, it suffices to do so for $x, y, z \in E(\mathfrak{g})$ by Lemma 3.1.9. It follows immediately from Corollary 3.1.5 that

$$g(x, [y, z]) = g_x([y, z]) = -g_x([z, y]) = g_z([x, y]) = g_{[x,y]}(z) = g([x, y], z)$$

for all $x, y, z \in E(\mathfrak{g})$, thus $g : \mathfrak{g}\times\mathfrak{g} \to \mathbb{F}$ is associative. □

Given that a Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ is generated by its set $E(\mathfrak{g})$ of extremal elements, we call the unique bilinear symmetric form $g : \mathfrak{g}\times\mathfrak{g} \to \mathbb{F}$ as in the above proposition the *extremal form of* $\mathfrak{g}$. Its *radical* is the set $\operatorname{rad}(g) = \{x \in \mathfrak{g} \mid \forall y \in \mathfrak{g} : g(x, y) = g_x(y) = 0\}$. Notice that we have $x \in \operatorname{rad}(g)$ for all sandwich elements $x \in \mathfrak{g}$, since their extremal forms are identically zero.

Recall from Definition 2.1.28 the Killing form of $\mathfrak{g}$, which we have seen is a bilinear symmetric form on $\mathfrak{g}$ satisfying $\kappa(x, [y, z]) = \kappa([x, y], z)$ for all $x, y, z \in \mathfrak{g}$. As was the case for the Killing form, associativity of the extremal form g of $\mathfrak{g}$ implies that $\operatorname{rad}(g) \subseteq \mathfrak{g}$ is an ideal. This property generalises to arbitrary bilinear symmetric and anti-symmetric forms $f : \mathfrak{g}\times\mathfrak{g} \to \mathbb{F}$ on $\mathfrak{g}$ that satisfy $f(x, [y, z]) = f([x, y], z)$ for all $x, y, z \in \mathfrak{g}$.

We finish this section with relating the radical of the extremal form g of $\mathfrak{g}$ to the structure of $\mathfrak{g}$, following Section 9 of [2]. First consider the following lemma.

Lemma 3.1.11. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ with* $\operatorname{char}(\mathbb{F}) \neq 2$ *generated by its set $E(\mathfrak{g})$ of extremal elements and let $g : \mathfrak{g}\times\mathfrak{g} \to \mathbb{F}$ be the extremal form of $\mathfrak{g}$. Then* $\operatorname{rad}(\mathfrak{g}) \subseteq \operatorname{rad}(g)$.

Proof. Let $\mathfrak{i} = \operatorname{rad}(\mathfrak{g})$ be the maximal solvable ideal of $\mathfrak{g}$ and let $x \in \mathfrak{i}$ be arbitrary, which we may assume to be extremal since $\mathfrak{g} = \langle E(\mathfrak{g})\rangle$. We show that $g(x, y) = 0$ for all $y \in \mathfrak{g}$.

Again because $\mathfrak{g} = \langle E(\mathfrak{g})\rangle$, it suffices to do this for all $y \in E(\mathfrak{g})$. We distinguish between two cases.

Assume first that $y \notin \mathfrak{i}$. Then $2g(y,x)y = 2g_y(x)y = [y,[y,x]]$ with $[y,x] \in \mathfrak{i}$ because $x \in \mathfrak{i}$, hence $[y,[y,x]] \in \mathfrak{i}$. But $y \notin \mathfrak{i}$ and $\mathrm{char}(\mathbb{F}) \neq 2$, forcing $g(y,x) = 0$. By symmetry of the extremal form g of $\mathfrak{g}$, then also $g(x,y) = 0$.

Next assume that $y \in \mathfrak{i}$ and suppose towards a contradiction that $g(x,y) \neq 0$. Because $x,y \in E(\mathfrak{g})$, we have $[[y,x],x] = [x,[x,y]] = 2g_x(y)x$ and $[[y,x],y] = -[y,[y,x]] = -2g_y(x)y$, hence $g(x,y) \neq 0$ implies that the triple $\{x,y,[y,x]\}$ is an $\mathfrak{sl}_2$-triple spanning a Lie subalgebra of $\mathfrak{i}$ isomorphic to $\mathfrak{sl}_2(\mathbb{F})$. Since $\mathfrak{i}$ is solvable by assumption, so will $\langle x,y,[x,y]\rangle$ be by Proposition 2.1.19. But we have seen in Example 2.2.2 that $\mathfrak{sl}_2(\mathbb{F})$ is non-solvable because $\mathrm{char}(\mathbb{F}) \neq 2$, which contradicts solvability of $\langle x,y,[x,y]\rangle$. It follows that $g(x,y) = 0$.

We conclude that $g(x,y) = 0$ for all $y \in E(\mathfrak{g})$, and so $x \in \mathrm{rad}(g)$. This concludes the proof. □

Observe that, even though the Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ of $\mathfrak{g}$ is also an associative bilinear symmetric form on $\mathfrak{g}$, the inclusion of radicals is reversed, i.e. $\mathrm{rad}(\kappa) \subseteq \mathrm{rad}(\mathfrak{g})$, which we have established in the first half of the proof of Theorem 2.1.29.

Proposition 3.1.12. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ with* $\mathrm{char}(\mathbb{F}) \neq 2$ *generated by its set $E(\mathfrak{g})$ of extremal elements and let $g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ be the extremal form of $\mathfrak{g}$. Then* $\mathrm{rad}(g) = 0$ *if and only if $\mathfrak{g}$ is a direct sum of simple ideals.*

Proof. Assume first that $\mathfrak{g}$ is a direct sum of simple ideals, i.e. $\mathfrak{g} = \bigoplus_{k=1}^n \mathfrak{g}_k$ where $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ are simple ideals of $\mathfrak{g}$, viewed as Lie subalgebras of $\mathfrak{g}$, with $n \geq 1$ finite. Since this sum is direct, we have $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_i \cap \mathfrak{g}_j = 0$ for all $1 \leq i \neq j \leq n$. Consequently, for all $y \in \mathfrak{g}$ we have $[\mathfrak{g}_i,[\mathfrak{g}_j,y]] \subseteq [\mathfrak{g}_i,\mathfrak{g}_j] = 0$, $1 \leq i \neq j \leq n$, and similarly $[\mathfrak{g}_j,[\mathfrak{g}_i,y]] \subseteq [\mathfrak{g}_j,\mathfrak{g}_i] = 0$. Then for all $x = \sum_{k=1}^n x_k \in E(\mathfrak{g})$ with $x_k \in \mathfrak{g}_k$, $1 \leq k \leq n$, and all $y \in \mathfrak{g}$ we find

$$\sum_{k=1}^n 2g(x,y)x_k = 2g(x,y)x = [x,[x,y]] = \left[\sum_{k=1}^n x_k, \left[\sum_{k=1}^n x_k, y\right]\right] = \sum_{k=1}^n [x_k,[x_k,y]],$$

hence $[x_k,[x_k,y]] = 2g(x,y)x_k$ for all $1 \leq k \leq n$. This shows that $x_k \in E(\mathfrak{g}_k)$ for all $1 \leq k \leq n$, and so $E(\mathfrak{g}_k)$ spans $\mathfrak{g}_k$ linearly since $E(\mathfrak{g})$ spans $\mathfrak{g}$ linearly. In particular, the extremal form of every $\mathfrak{g}_k$, $1 \leq k \leq n$, will be the restriction $g|_{\mathfrak{g}_k}$ of g to $\mathfrak{g}_k$. Now notice that for all $x_i \in E(\mathfrak{g}_i)$ and $y_j \in \mathfrak{g}_j$ with $1 \leq i \neq j \leq n$ we have $2g(x_i,y_j)x_i = [x_i,[x_i,y_j]] = 0$ so that $g(x_i,y_j) = 0$, but since $E(\mathfrak{g}_i)$ spans $\mathfrak{g}_i$ linearly we even have $g(x_i,y_j) = 0$ for all $x_i \in \mathfrak{g}_i$ and $y_i \in \mathfrak{g}_j$. Supposing next that $\mathrm{rad}(g) \neq 0$, we then have $\mathfrak{g}_k \subseteq \mathrm{rad}(g|_{\mathfrak{g}_k})$ for some $1 \leq k \leq n$ because $g(\mathfrak{g}_i,\mathfrak{g}_j) = 0$ for all $1 \leq i \neq j \leq n$ by the above. However, this implies that $g(\mathfrak{g}_k,\mathfrak{g}_k) = g|_{\mathfrak{g}_k} = 0$, hence $(\mathrm{ad}_{\mathfrak{g}_k})^2(y) = [\mathfrak{g}_k,[\mathfrak{g}_k,y]] = 2g(\mathfrak{g}_k,y) = 0$ for all $y \in \mathfrak{g}_k$. It follows that every element in $\mathfrak{g}_k$ is ad-nilpotent, so $\mathfrak{g}_k$ will be nilpotent as well by Theorem 2.1.24, which contradicts simplicity of $\mathfrak{g}_k$. We conclude that $\text{rad}(g|_{\mathfrak{g}_k}) = 0$ for all $1 \leq k \leq n$, and so $\text{rad}(g) = 0$ by the decomposition of $\mathfrak{g}$ as a direct sum of simple ideals.

Now assume that $\text{rad}(g) = 0$. We proceed by induction on $\dim(\mathfrak{g})$, the result being immediate if $\dim(\mathfrak{g}) \leq 1$. Supposing that $\mathfrak{g}$ can be decomposed into a direct sum of simple ideals if $\dim(\mathfrak{g}) \leq k$ for some integer $k \geq 1$, let $\mathfrak{i}$ be a minimal non-zero ideal of $\mathfrak{g}$. Note that any ideal $\mathfrak{i}'$ of $\mathfrak{i}$ will also be an ideal of $\mathfrak{g}$, forcing $\mathfrak{i}' = \mathfrak{i}$ by minimality of $\mathfrak{i}$, which shows simplicity of $\mathfrak{i}$. By Lemma 3.1.11, we have $\text{rad}(\mathfrak{g}) \subseteq \text{rad}(g) = 0$ so that $\mathfrak{g}$ contains no solvable ideals. In particular, $\mathfrak{i}$ cannot be abelian. Now consider the subspace $\mathfrak{i}^\perp = \{x \in \mathfrak{g} \mid \forall y \in \mathfrak{i} : g(x, y) = 0\} \subseteq \mathfrak{g}$, which will also be an ideal of $\mathfrak{g}$; indeed, by associativity of g we have $0 = g(\mathfrak{i}, \mathfrak{i}^\perp) \supseteq g([\mathfrak{i}, \mathfrak{g}], \mathfrak{i}^\perp) = g(\mathfrak{i}, [\mathfrak{i}^\perp, \mathfrak{g}])$ if and only if $[\mathfrak{i}^\perp, \mathfrak{g}] \subseteq \mathfrak{i}^\perp$. If $\mathfrak{i}$ and $\mathfrak{i}^\perp$ intersect non-trivially, then $\mathfrak{i} \subseteq \mathfrak{i} \cap \mathfrak{i}^\perp$ by minimality of $\mathfrak{i}$, forcing equality so that $\mathfrak{i} \subseteq \mathfrak{i}^\perp$, but then also $\mathfrak{i} \subseteq [\mathfrak{i}, \mathfrak{i}] = [\mathfrak{i}, \mathfrak{i}^\perp]$ by minimality of $\mathfrak{i}$ and because $\mathfrak{i}$ is non-abelian, again forcing equality. Again by associativity of g, we obtain $g(\mathfrak{i}, \mathfrak{g}) = g([\mathfrak{i}, \mathfrak{i}^\perp], \mathfrak{g}) = g(\mathfrak{i}, [\mathfrak{i}^\perp, \mathfrak{g}]) \subseteq g(\mathfrak{i}, \mathfrak{i}^\perp) = 0$, implying that $\mathfrak{i} \subseteq \text{rad}(g) = 0$, a contradiction. It follows that $\mathfrak{i} \cap \mathfrak{i}^\perp = 0$, hence $\mathfrak{g} = \mathfrak{i} \oplus \mathfrak{i}^\perp$. Finally, the inclusion $\text{rad}(g|_{\mathfrak{i}^\perp}) \subseteq \text{rad}(g) = 0$ allows us to apply the induction hypothesis to $\mathfrak{i}^\perp$. The proposition then follows. □

An immediate consequence of the above proposition is the following.

Corollary 3.1.13. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ with* $\text{char}(\mathbb{F}) \neq 2$ *generated by its set $E(\mathfrak{g})$ of extremal elements and let $g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ be the extremal form of $\mathfrak{g}$. Then* $\text{rad}(\mathfrak{g}) = \text{rad}(g)$ *if and only if $\mathfrak{g}/\text{rad}(\mathfrak{g})$ is a direct sum of simple ideals.*

Proof. Denote by $g_\mathfrak{i}$ the form on $\mathfrak{g}/\mathfrak{i}$ induced by g, in which $\mathfrak{i}$ is an ideal of $\mathfrak{g}$ contained in $\text{rad}(g)$. Writing $x_\mathfrak{i} = x + \mathfrak{i} \in \mathfrak{g}/\mathfrak{i}$ and $y_\mathfrak{i} = y + \mathfrak{i} \in \mathfrak{g}/\mathfrak{i}$ with $x, y \in \mathfrak{g}$, we get

$$g_\mathfrak{i}(x_\mathfrak{i}, y_\mathfrak{i}) = g_\mathfrak{i}(x + \mathfrak{i}, y + \mathfrak{i}) = g(x, y) + g(x, \mathfrak{i}) + g(\mathfrak{i}, y) + g(\mathfrak{i}, \mathfrak{i}) = g(x, y)$$

because $g(\mathfrak{i}, \mathfrak{g}) = 0$. It follows that $g_\mathfrak{i}$ is well-defined because g is well-defined. By taking $\mathfrak{i} = \text{rad}(\mathfrak{g})$, which satisfies $\text{rad}(\mathfrak{g}) \subseteq \text{rad}(g)$ by Lemma 3.1.11, we obtain the form $g_{\text{rad}(\mathfrak{g})}$ on $\mathfrak{g}/\text{rad}(\mathfrak{g})$ induced by g given by $g_{\text{rad}(\mathfrak{g})}(x_{\text{rad}(\mathfrak{g})}, y_{\text{rad}(\mathfrak{g})}) = g(x, y)$ for all $x_{\text{rad}(\mathfrak{g})} = x + \text{rad}(\mathfrak{g}) \in \mathfrak{g}/\text{rad}(\mathfrak{g})$ and $y_{\text{rad}(\mathfrak{g})} = y + \text{rad}(\mathfrak{g}) \in \mathfrak{g}/\text{rad}(\mathfrak{g})$ with $x, y \in \mathfrak{g}$. But then

$$\begin{aligned} \text{rad}(g_{\text{rad}(\mathfrak{g})}) &= \{x_{\text{rad}(\mathfrak{g})} \in \mathfrak{g}/\text{rad}(\mathfrak{g}) \mid \forall y_{\text{rad}(\mathfrak{g})} \in \mathfrak{g}/\text{rad}(\mathfrak{g}) : g_{\text{rad}(\mathfrak{g})}(x_{\text{rad}(\mathfrak{g})}, y_{\text{rad}(\mathfrak{g})}) = 0\} \\ &= \{x_{\text{rad}(\mathfrak{g})} \in \mathfrak{g}/\text{rad}(\mathfrak{g}) \mid \forall y_{\text{rad}(\mathfrak{g})} \in \mathfrak{g}/\text{rad}(\mathfrak{g}) : g(x, y) = 0\} = \text{rad}(g)/\text{rad}(\mathfrak{g}), \end{aligned}$$

hence $\text{rad}(g_{\text{rad}(\mathfrak{g})}) = 0$ if and only if $\text{rad}(g)/\text{rad}(\mathfrak{g}) = 0$ if and only if $\text{rad}(g) = \text{rad}(\mathfrak{g})$. The corollary now follows by applying Proposition 3.1.12 to $\mathfrak{g}/\text{rad}(\mathfrak{g})$ with extremal form $g_{\text{rad}(\mathfrak{g})}$. □

3.2 Tensors and classical linear Lie algebras

Given a possibly infinite-dimensional vector space V over a field $\mathbb{F}$, the *dual space* of V is the space V^* consisting of all linear maps $\varphi : V \to \mathbb{F}$, possibly infinite-dimensional as well. In case V is finite-dimensional, the spaces V and $(V^*)^*$ are naturally isomorphic with isomorphism $V \to (V^*)^*$ given by $v \mapsto (\varphi \mapsto \varphi(v))$ with $v \in V$ and $\varphi \in V^*$. We additionally have $V \cong V^*$, but here the isomorphism depends on the choice of basis of V; indeed, given a basis $\{v_i\}_{i\in I}$ of V indexed by some index set I, the set $\{\varphi_j\}_{j\in I}$ such that $\varphi_j(v_i) = \delta_{i,j}$ is a basis of V^*.

In this section, we will consider the tensor space $V \otimes W^*$ containing all tensor products of $v \in V$ and $\varphi \in W^*$ with W^* a subspace of V^*, which coincides with V^* if V is finite-dimensional but can be proper otherwise. It is possible to give the vector space $V \otimes W^*$ the structure of a Lie algebra, as shown by the following proposition.

Proposition 3.2.1. *Let V be a vector space over a field $\mathbb{F}$ and let $W^* \subseteq V^*$ be a subspace of the dual space V^* of V. Then equipping $V \otimes W^*$ with the bilinear form $[\cdot,\cdot] : V \otimes W^* \times V \otimes W^* \to V \otimes W^*$ given by*

$$[v \otimes \varphi, w \otimes \psi] = \varphi(w)(v \otimes \psi) - \psi(v)(w \otimes \varphi)$$

with $v, w \in V$ and $\varphi, \psi \in W^*$ turns $V \otimes W^*$ into a Lie algebra over $\mathbb{F}$.

Proof. For all $v \otimes \varphi \in V \otimes W^*$ we have $[v \otimes \varphi, v \otimes \varphi] = \varphi(v)(v \otimes \varphi) - \varphi(v)(v \otimes \varphi) = 0$, hence the form $[\cdot,\cdot]$ is alternating. Moreover, for all $u \otimes \chi, v \otimes \varphi, w \otimes \psi \in V \otimes W^*$, we have

$$\begin{aligned}
&[u \otimes \chi, [v \otimes \varphi, w \otimes \psi]] + [v \otimes \varphi, [w \otimes \psi, u \otimes \chi]] + [w \otimes \psi, [u \otimes \chi, v \otimes \varphi]] \\
&= [u \otimes \chi, \varphi(w)(v \otimes \psi) - \psi(v)(w \otimes \varphi)] + [v \otimes \varphi, \psi(u)(w \otimes \chi) - \chi(w)(u \otimes \psi)] + \\
&\quad [w \otimes \psi, \chi(v)(u \otimes \varphi) - \varphi(u)(v \otimes \chi)] \\
&= \chi(v)\varphi(w)(u \otimes \psi) - \varphi(w)\psi(u)(v \otimes \chi) - \chi(w)\psi(v)(u \otimes \varphi) + \varphi(u)\psi(v)(w \otimes \chi) \\
&\quad + \varphi(w)\psi(u)(v \otimes \chi) - \chi(v)\psi(u)(w \otimes \varphi) - \chi(w)\varphi(u)(v \otimes \psi) + \chi(w)\psi(v)(u \otimes \varphi) \\
&\quad + \chi(v)\psi(u)(w \otimes \varphi) - \chi(v)\varphi(w)(u \otimes \psi) - \varphi(u)\psi(v)(w \otimes \chi) + \chi(w)\varphi(u)(v \otimes \psi) \\
&= (\chi(v)\varphi(w)(u \otimes \psi) - \chi(v)\varphi(w)(u \otimes \psi)) + (\varphi(u)\psi(v)(w \otimes \chi) - \varphi(u)\psi(v)(w \otimes \chi)) + \\
&\quad + (\varphi(w)\psi(u)(v \otimes \chi) - \varphi(w)\psi(u)(v \otimes \chi)) + (\chi(w)\psi(v)(u \otimes \varphi) - \chi(w)\psi(v)(u \otimes \varphi)) \\
&\quad + (\chi(v)\psi(u)(w \otimes \varphi) - \chi(v)\psi(u)(w \otimes \varphi)) + (\chi(w)\varphi(u)(v \otimes \psi) - \chi(w)\varphi(u)(v \otimes \psi)) \\
&= 0
\end{aligned}$$

so that $[\cdot,\cdot]$ satisfies the Jacobi identity. By Definition 2.1.1, the proposition follows. □

We denote by $\mathfrak{g}(V \otimes W^*)$ the vector space $V \otimes W^*$ viewed as a Lie algebra with Lie bracket $[\cdot,\cdot]$ as in the above proposition. Observe that not every element in $V \otimes W^*$ is of the form $v \otimes \varphi$ for some $v \in V$ and $\varphi \in V^*$; for example, the element $[v \otimes \psi, w \otimes \psi]$ is a linear combination of $v \otimes \psi$ and $w \otimes \varphi$ that cannot be written in this form for all $v \otimes \varphi, w \otimes \psi \in V \otimes W^*$. Tensors of the form $v \otimes \varphi$ for some $v \in V$ and $\varphi \in W^*$ are called *pure* tensors. If they additionally satisfy $\varphi(v) = 0$, then they are referred to as *singular pure* tensors. This leads to the following proposition.

Proposition 3.2.2. *Let $\mathfrak{g}(V \otimes W^*)$ be the Lie algebra of $V \otimes W^*$ as in* Proposition 3.2.1 *and let $v \otimes \varphi \in \mathfrak{g}(V \otimes W^*)$. Then $v \otimes \varphi$ is extremal in $\mathfrak{g}(V \otimes W^*)$ if and only if $v \otimes \varphi$ is a singular pure tensor.*

Proof. First suppose that $v \otimes \varphi \in \mathfrak{g}(V \otimes W^*)$ is extremal. Then for all pure tensors $w \otimes \psi \in \mathfrak{g}(V \otimes W^*)$ we have

$$\begin{aligned}\mathbb{F}(v \otimes \varphi) \supseteq [v \otimes \varphi, [v \otimes \varphi, w \otimes \psi]] &= [v \otimes \varphi, \varphi(w)(v \otimes \psi) - \psi(v)(w \otimes \varphi)] \\ &= \varphi(w)[v \otimes \varphi, v \otimes \psi] - \psi(v)[v \otimes \varphi, w \otimes \varphi] \\ &= \varphi(w)(\varphi(v)(v \otimes \psi) - \psi(v)(v \otimes \varphi)) - \psi(v)(\varphi(w)(v \otimes \varphi) - \varphi(v)(w \otimes \varphi)) \\ &= -2\psi(v)\varphi(w)(v \otimes \varphi) + \varphi(v)(\varphi(w)(v \otimes \psi) + \psi(v)(w \otimes \varphi)),\end{aligned}$$

forcing $\varphi(v) = 0$ because the above is true for all pure tensors $w \otimes \psi \in \mathfrak{g}(V \otimes W^*)$. It follows that $v \otimes \varphi$ is a singular pure tensor.

Next, let $v \otimes \varphi \in \mathfrak{g}(V \otimes W^*)$ be a singular pure tensor, i.e. $\varphi(v) = 0$. Since all elements in $\mathfrak{g}(V \otimes W^*)$ are linear combinations of pure tensors, it suffices to check the conditions listed in Definition 3.1.1 for pure tensors. By the above, we have for all pure tensors $w \otimes \psi \in \mathfrak{g}(V \otimes W^*)$ that

$$\begin{aligned}[v \otimes \varphi, [v \otimes \varphi, w \otimes \psi]] &= -2\psi(v)\varphi(w)(v \otimes \varphi) + \varphi(v)(\varphi(w)(v \otimes \psi) + \psi(v)(w \otimes \varphi)) \\ &= -2\psi(v)\varphi(w)(v \otimes \varphi).\end{aligned}$$

We define $g_{v\otimes\varphi} : \mathfrak{g}(V \otimes W^*) \to \mathbb{F}$ to be the linear map given by $g_{v\otimes\varphi}(w \otimes \psi) = -\psi(v)\varphi(w)$. If $\mathrm{char}(\mathbb{F}) \neq 2$, then the proposition follows from Lemma 3.1.3, so assume that $\mathrm{char}(\mathbb{F}) = 2$. For all pure tensors $u \otimes \chi \in \mathfrak{g}(V \otimes W^*)$, we then deduce that

$$\begin{aligned}&[[v \otimes \varphi, w \otimes \psi], [v \otimes \varphi, u \otimes \chi]] \\ &= [\varphi(w)(v \otimes \psi) - \psi(v)(w \otimes \varphi), \varphi(u)(v \otimes \chi) - \chi(v)(u \otimes \varphi)] \\ &= \varphi(w)\varphi(u)[v \otimes \psi, v \otimes \chi] - \varphi(w)\chi(v)[v \otimes \psi, u \otimes \varphi] - \psi(v)\varphi(u)[w \otimes \varphi, v \otimes \chi] \\ &\quad + \psi(v)\chi(v)[w \otimes \varphi, u \otimes \varphi] \\ &= \varphi(w)\varphi(u)(\psi(v)(v \otimes \chi) - \chi(v)(v \otimes \psi)) - \varphi(w)\chi(v)(\psi(u)(v \otimes \varphi) - \varphi(v)(u \otimes \psi)) \\ &\quad - \psi(v)\varphi(u)(\varphi(v)(w \otimes \chi) - \chi(w)(v \otimes \varphi)) + \psi(v)\chi(v)(\varphi(u)(w \otimes \varphi) - \varphi(w)(u \otimes \varphi)) \\ &= (-\varphi(w)\chi(v)\psi(u) + \psi(v)\varphi(u)\chi(w))(v \otimes \varphi) - \varphi(u)\chi(v)(\varphi(w)(v \otimes \psi) - \psi(v)(w \otimes \varphi)) \\ &\quad + \varphi(w)\psi(v)(\varphi(u)(v \otimes \chi) - \chi(v)(u \otimes \varphi)) \\ &= (\psi(u)g_{v\otimes\varphi}(w \otimes \chi) - \chi(w)g_{v\otimes\varphi}(u \otimes \psi))(v \otimes \psi) - \varphi(u)\chi(v)[v \otimes \varphi, w \otimes \psi] \\ &\quad + \varphi(w)\psi(v)[v \otimes \varphi, u \otimes \chi] \\ &= g_{v\otimes\varphi}(\psi(u)(w \otimes \chi) - \chi(w)(u \otimes \psi))(v \otimes \varphi) + g_{v\otimes\varphi}(u \otimes \chi)[v \otimes \varphi, w \otimes \psi] \\ &\quad - g_{v\otimes\varphi}(w \otimes \psi)[v \otimes \psi, u \otimes \chi] \\ &= g_{v\otimes\varphi}([w \otimes \psi, u \otimes \chi])(v \otimes \varphi) + g_{v\otimes\varphi}(u \otimes \chi)[v \otimes \varphi, w \otimes \psi] - g_{v\otimes\varphi}(w \otimes \psi)[v \otimes \psi, u \otimes \chi],\end{aligned}$$

so the first Premet identity holds. Additionally, we have

$$
\begin{aligned}
&[v\otimes\varphi,[w\otimes\psi,[v\otimes\varphi,u\otimes\chi]]]\\
&=[v\otimes\varphi,[w\otimes\psi,\varphi(u)(v\otimes\chi)-\chi(v)(u\otimes\varphi)]\\
&=\varphi(u)[v\otimes\varphi,\psi(v)(w\otimes\chi)-\chi(w)(v\otimes\psi)]-\chi(v)[v\otimes\varphi,\psi(u)(w\otimes\varphi)-\varphi(w)(u\otimes\psi)]\\
&=\varphi(u)\psi(v)[v\otimes\varphi,w\otimes\chi]-\varphi(u)\chi(w)[v\otimes\varphi,v\otimes\psi]-\chi(v)\psi(u)[v\otimes\varphi,w\otimes\varphi]\\
&\quad+\chi(v)\varphi(w)[v\otimes\varphi,u\otimes\psi]\\
&=\varphi(u)\psi(v)(\varphi(w)(v\otimes\chi)-\chi(v)(w\otimes\varphi))-\varphi(u)\chi(w)(\varphi(v)(v\otimes\psi)-\psi(v)(v\otimes\varphi))\\
&\quad-\chi(v)\psi(u)(\varphi(w)(v\otimes\varphi)-\varphi(v)(w\otimes\varphi))+\chi(v)\varphi(w)(\varphi(u)(v\otimes\psi)-\psi(v)(u\otimes\varphi))\\
&=(-\chi(v)\psi(u)\varphi(w)+\varphi(u)\chi(w)\psi(v))(v\otimes\varphi)+\chi(v)\varphi(u)(\varphi(w)(v\otimes\psi)-\psi(v)(w\otimes\varphi))\\
&\quad+\psi(v)\varphi(w)(\varphi(u)(v\otimes\chi)-\chi(v)(u\otimes\varphi))\\
&=(\psi(u)g_{v\otimes\varphi}(w\otimes\chi)-\chi(w)g_{v\otimes\varphi}(u\otimes\psi))(v\otimes\psi)+\varphi(u)\chi(v)[v\otimes\varphi,w\otimes\psi]\\
&\quad+\varphi(w)\psi(v)[v\otimes\varphi,u\otimes\chi]\\
&=g_{v\otimes\varphi}(\psi(u)(w\otimes\chi)-\chi(w)(u\otimes\psi))(v\otimes\varphi)-g_{v\otimes\varphi}(u\otimes\chi)[v\otimes\varphi,w\otimes\psi]\\
&\quad-g_{v\otimes\varphi}(w\otimes\psi)[v\otimes\psi,u\otimes\chi]\\
&=g_{v\otimes\varphi}([w\otimes\psi,u\otimes\chi])(v\otimes\varphi)-g_{v\otimes\varphi}(u\otimes\chi)[v\otimes\varphi,w\otimes\psi]-g_{v\otimes\varphi}(w\otimes\psi)[v\otimes\psi,u\otimes\chi],
\end{aligned}
$$

hence the second Premet identity holds as well. We conclude that all singular pure tensors $v\otimes\varphi\in\mathfrak{g}(V\otimes W^*)$ are extremal by Definition 3.1.1. □

By $E(V\otimes W^*)$ we denote the set of extremal elements of $\mathfrak{g}(V\otimes W^*)$. We have seen in our proof above that every $v\otimes\varphi\in E(V\otimes W^*)$ has extremal form $g_{v\otimes\varphi}(w\otimes\psi)=-\psi(v)\varphi(w)$ with $w\otimes\psi\in\mathfrak{g}(V\otimes W^*)$.

Recall from the previous section the set $\operatorname{Exp}(x)=\{\exp(x,\lambda)\mid\lambda\in\mathbb{F}\}$ with $\exp(x,\lambda)=e^{\lambda\cdot\operatorname{ad}_x}$, in which x is an extremal element of some Lie algebra. For $\mathfrak{g}(V\otimes W^*)$, we obtain the following.

Corollary 3.2.3. *Let $\mathfrak{g}(V\otimes W^*)$ be the Lie algebra of $V\otimes W^*$ as in* Proposition 3.2.1. *Then $\exp(v\otimes\varphi,\lambda)(w\otimes\psi)$ is a pure tensor for all $v\otimes\varphi\in E(V\otimes W^*)$, $w\otimes\psi\in\mathfrak{g}(V\otimes W^*)$ and $\lambda\in\mathbb{F}$.*

In particular, $\exp(v\otimes\varphi,\lambda)(w\otimes\psi)$ is a singular pure tensor if and only if $w\otimes\psi\in E(V\otimes W^*)$.

Proof. Let $v\otimes\varphi\in E(V\otimes W^*)$. Then for all $w\otimes\psi\in\mathfrak{g}(V\otimes W^*)$ and $\lambda\in\mathbb{F}$ we obtain

$$
\begin{aligned}
\exp(v\otimes\varphi,\lambda)(w\otimes\psi)&=w\otimes\psi+\lambda[v\otimes\varphi,w\otimes\psi]+\lambda^2g_{v\otimes\varphi}(w\otimes\psi)(v\otimes\varphi)\\
&=w\otimes\psi+\lambda(\varphi(w)(v\otimes\psi)-\psi(v)(w\otimes\varphi))-\lambda^2\psi(v)\varphi(w)(v\otimes\varphi)\\
&=(w+\lambda\varphi(w)v)\otimes\psi-\lambda\psi(v)(w+\lambda\varphi(w)v)\otimes\varphi\\
&=(w+\lambda\varphi(w)v)\otimes(\psi-\lambda\psi(v)\varphi),
\end{aligned}
$$

therefore $\exp(v \otimes \varphi, \lambda)(w \otimes \psi)$ is a pure tensor. In particular, we have

$$\begin{aligned}(\psi - \lambda\psi(v)\varphi)(w + \lambda\varphi(w)v) &= \psi(w + \lambda\varphi(w)v) - \lambda\psi(v)\varphi(w + \lambda\varphi(w)v) \\ &= \psi(w) + \lambda\psi(v)\varphi(w) - \lambda\psi(v)\varphi(w) - \lambda^2\psi(v)\varphi(w)\varphi(v) \\ &= \psi(w),\end{aligned}$$

so $\exp(v \otimes \varphi, \lambda)(w \otimes \psi)$ is a singular pure tensor if and only if $\psi(w) = 0$ if and only if $w \otimes \psi$ is a singular pure tensor if and only if $w \otimes \psi \in E(V \otimes W^*)$ by Proposition 3.2.2. The corollary now follows. □

Singular pure tensors give rise to a certain type of linear transformations in $\mathfrak{gl}(V)$, which are defined as follows.

Definition 3.2.4 (Infinitesimal transvection & Infinitesimal reflection)**.** *Let V be a vector space on a field $\mathbb{F}$ and let V^* be the dual space of V. The linear map $t_{v,\varphi} : V \to V$ with $v \in V$ and $\varphi \in V^*$ given by $t_{v,\varphi}(w) = \varphi(w)v$ with $w \in V$ is called an **infinitesimal transvection** if $\varphi(v) = 0$. If $\varphi(v) \neq 0$, then $t_{v,\varphi}$ is called an **infinitesimal reflection**.*

Given such an infinitesimal transvection or infinitesimal reflection $t_{v,\varphi}$, we refer to $\langle v \rangle$ as its *center* and $\langle \varphi \rangle$ as its *axis*. Equivalently, the map $t_{v,\varphi}$ is called an infinitesimal transvection if the tensor $v \otimes \varphi \in V \otimes W^*$ is a singular pure tensor, and an infinitesimal reflection otherwise.

If $v \otimes \varphi \in \mathfrak{g}(V \otimes V^*)$, then $t_{v,\varphi}$ is an endomorphism in $\mathfrak{fgl}(V)$. This brings us to the following proposition.

Proposition 3.2.5. *Let $\mathfrak{g}(V \otimes V^*)$ be the Lie algebra of $V \otimes V^*$ as in* Proposition 3.2.1*. Then the linear map $\Phi : \mathfrak{g}(V \otimes V^*) \to \mathfrak{fgl}(V)$ given by $v \otimes \varphi \mapsto t_{v,\varphi}$ with $v \otimes \varphi \in \mathfrak{g}(V \otimes V^*)$ is a Lie algebra homomorphism.*

In particular, Φ is an isomorphism between $\mathfrak{g}(V \otimes V^*)$ and $\mathfrak{fgl}(V)$.

Proof. For all $v \otimes \varphi, w \otimes \psi \in \mathfrak{g}(V \otimes V^*)$ and $u \in V$, we have

$$\begin{aligned}\Phi([v \otimes \varphi, w \otimes \psi])(u) &= \Phi(\varphi(w)(v \otimes \psi) - \psi(v)(w \otimes \varphi))(u) \\ &= \varphi(w)\Phi(v \otimes \psi)(u) - \psi(v)\Phi(w \otimes \varphi)(u) \\ &= \varphi(w)t_{v,\psi}(u) - \psi(v)t_{w,\varphi}(u) = \varphi(w)\psi(u)v - \psi(v)\varphi(u)w \\ &= \psi(u)t_{v,\varphi}(w) - \varphi(u)t_{w,\psi}(v) = t_{v,\varphi}(\psi(u)w) - t_{w,\psi}(\varphi(u)v) \\ &= (t_{v,\varphi}t_{w,\psi})(u) - (t_{w,\psi}t_{v,\varphi})(u) = (t_{v,\varphi}t_{w,\psi} - t_{w,\psi}t_{v,\varphi})(u) \\ &= [t_{v,\varphi}, t_{w,\psi}](u) = [\Phi(v \otimes \varphi), \Phi(w \otimes \psi)](u),\end{aligned}$$

so Φ is a Lie algebra homomorphism by Definition 2.1.6. To show that it is an isomorphism between $\mathfrak{g}(V \otimes V^*)$ and $\mathfrak{fgl}(V)$, we distinguish two cases.

Suppose first that V is finite-dimensional, say $\dim(V) = n \geq 1$, then we have $\mathfrak{fgl}(V) = \mathfrak{gl}(V)$. A basis of $\mathfrak{g}(V \otimes V^*)$ is $\{v_i \otimes \varphi_j\}_{1 \leq i,j \leq n}$ with $\{v_i\}_{1 \leq i \leq n}$ a basis of V and $\{\varphi_j\}_{1 \leq j \leq n}$

a basis of V^* such that $\varphi_j(v_i) = \delta_{i,j}$ for all $1 \leq i, j \leq n$. But then for every $w \in V$ we obtain

$$t_{v_i,\varphi_j}(w) = \varphi_j(w)v_i = \begin{cases} v_i & \text{if } w = v_j, \\ 0 & \text{else} \end{cases}$$

for all $1 \leq i, j \leq n$, hence $\Phi(v_i \otimes \varphi_j)$ may be identified with the matrix $E_{i,j}$, $1 \leq i, j \leq n$, having a one in position (i, j) and zeros elsewhere. We conclude that Φ is a Lie algebra isomorphism between $\mathfrak{g}(V \otimes V^*)$ and $\mathfrak{gl}(V)$, as it maps a basis of $\mathfrak{g}(V \otimes V^*)$ to a basis of $\mathfrak{gl}(V)$.

Suppose next that V is infinite-dimensional. We show injectivity and surjectivity of Φ. Since any element in $\mathfrak{g}(V \otimes V^*)$ can be written as a finite linear combination of pure tensors $\mathfrak{g}(V \otimes V^*)$, we can always identify a finite-dimensional Lie subalgebra $\mathfrak{g}(W \otimes W^*) \subset \mathfrak{g}(V \otimes V^*)$ containing it with $W \subset V$ and $W^* \subset V^*$ finite-dimensional subspaces. Finite-dimensionality of $\mathfrak{g}(W \otimes W^*)$ then implies that the restriction $\Phi|_{\mathfrak{g}(W \otimes W^*)}$ of Φ to $\mathfrak{g}(W \otimes W^*)$ is a Lie algebra isomorphism between $\mathfrak{g}(W \otimes W^*)$ and $\mathfrak{fgl}(W) = \mathfrak{gl}(W)$, hence $\ker(\Phi|_{\mathfrak{g}(W \otimes W^*)}) = \{0\}$. But then necessarily $\ker(\Phi) = \{0\}$ since the above is true for all finite-dimensional subspaces of V and V^*. To show surjectivity of Φ, we prove that $\mathfrak{fgl}(V) = \mathrm{im}(\Phi)$. First consider the inclusion $\mathrm{im}(\Phi) \subseteq \mathfrak{fgl}(V)$. Again because every element in $\mathfrak{g}(V \otimes V^*)$ can be written as a finite linear combination of pure tensors, it suffices to focus only on pure tensors $v \otimes \varphi \in \mathfrak{g}(V \otimes V^*)$. Then $\dim(\Phi(v \otimes \varphi)(V)) = \dim(t_{v,\varphi}(V)) = \dim(\varphi(V))$, but also $\varphi \in V^*$ can be written as a finite linear combination of finitary linear maps in V^* so that $\dim(\varphi(V)) < \infty$, showing the inclusion $\mathrm{im}(\Phi) \subseteq \mathfrak{fgl}(V)$. Conversely, the image of any element in $\mathfrak{fgl}(V)$ is finite-dimensional, hence it is contained in a finite-dimensional Lie subalgebra $\mathfrak{fgl}(W) = \mathfrak{gl}(W) \subset \mathfrak{gl}(V)$ with $W \subset V$ finite-dimensional. But surjectivity of the restriction $\Phi|_{\mathfrak{g}(W \otimes W^*)}$ of Φ to $\mathfrak{g}(W \otimes W^*)$ then implies that $\mathfrak{gl}(W) = \mathrm{im}(\Phi|_{\mathfrak{g}(W \otimes W^*)}) \subseteq \mathrm{im}(\Phi)$. Because the above is true for all finite-dimensional subspaces of V and V^*, this shows the inclusion $\mathfrak{fgl}(V) \subseteq \mathrm{im}(\Phi)$ and forces equality. We conclude that Φ is a Lie algebra isomorphism between $\mathfrak{g}(V \otimes V^*)$ and $\mathfrak{fgl}(V)$. □

As a consequence of the above proposition and Proposition 3.2.2, every linear map $t_{v,\varphi} \in \mathfrak{fgl}(V)$ is extremal if and only if $v \otimes \varphi \in \mathfrak{g}(V \otimes V^*)$ is a singular pure tensor. Specifically, for all infinitesimal transvections $t_{v,\varphi} \in \mathfrak{fgl}(V)$ and $u \in V$, we have

$$\begin{aligned}[] [t_{v,\varphi}, [t_{v,\varphi}, t_{w,\psi}]](u) &= \Phi([v \otimes \varphi, [v \otimes \varphi, w \otimes \psi]])(u) = \Phi(-2\psi(v)\varphi(w)(v \otimes \varphi))(u) \\ &= -2\psi(v)\varphi(w)\Phi(v \otimes \varphi)(u) = -2\psi(v)\varphi(w)t_{v,\varphi}(u) \end{aligned}$$

for all $t_{w,\psi} \in \mathfrak{fgl}(V)$ so that $t_{v,\varphi}$ has extremal form $g_{t_{v,\varphi}}(t_{w,\psi}) = g_{v \otimes \varphi}(w \otimes \psi) = -\psi(v)\varphi(w)$. In addition, we have

$$\begin{aligned} \exp(t_{v,\varphi}, \lambda)(w) &= w + \lambda t_{v,\varphi}(w) + \frac{1}{2}(\lambda t_{v,\varphi})^2(w) = (\mathrm{id}_V + \lambda t_{v,\varphi})(w) + \frac{\lambda^2}{2} t_{v,\varphi}(t_{v,\varphi}(w)) \\ &= (\mathrm{id}_V + \lambda t_{v,\varphi})(w) + \frac{\lambda^2}{2}\varphi(w)\varphi(v)v = (\mathrm{id}_V + \lambda t_{v,\varphi})(w) \end{aligned}$$

for all $w \in V$ and $\lambda \in \mathbb{F}$, and we obtain a group $\operatorname{Exp}(t_{v,\varphi}) = \{\mathrm{id}_V + \lambda t_{v,\varphi} \mid \lambda \in \mathbb{F}, \varphi(v) = 0\}$ called a *transvection group* consisting of elements referred to as *transvections*.

The singular pure tensors $v \otimes \varphi \in \mathfrak{g}(V \otimes V^*)$ generate a Lie subalgebra of $\mathfrak{g}(V \otimes V^*)$, which we will denote by $\mathfrak{g}_0(V \otimes V^*)$. As a corollary of Proposition 3.2.5, we then obtain the following.

Corollary 3.2.6. *Let $\mathfrak{g}(V \otimes V^*)$ be the Lie algebra of $V \otimes V^*$ as in* Proposition 3.2.1 *and let $\mathfrak{g}_0(V \otimes V^*)$ be the Lie subalgebra of $\mathfrak{g}(V \otimes V^*)$ generated by its singular pure tensors. Then $\mathfrak{g}_0(V \otimes V^*) \cong \mathfrak{fsl}(V)$.*

Proof. Assume first that V is finite-dimensional, say $\dim(V) = n \geq 1$. We show that $v \otimes \varphi \in \mathfrak{g}(V \otimes V^*)$ is a singular pure tensor if and only if $t_{v,\varphi} \in \mathfrak{gl}(V)$ is traceless. Let $\{v_i\}_{1 \leq i \leq n}$ and $\{\varphi_j\}_{1 \leq j \leq n}$ be bases of V and V^*, respectively, satisfying $\varphi_j(v_i) = \delta_{i,j}$ for all $1 \leq i, j \leq n$. A basis of $\mathfrak{g}(V \otimes V^*)$ is then given by $\{v_i \otimes \varphi_j\}_{1 \leq i,j \leq n}$, and we have seen in the proof of Proposition 3.2.5 that we may identify $\Phi(v_i \otimes \varphi_j)$ by the matrix $E_{i,j}$, $1 \leq i, j \leq n$, having a one in position (i, j) and zeros elsewhere. Now let $v \otimes \varphi \in \mathfrak{g}(V \otimes V^*)$ be arbitrary and write $v = \sum_{i=1}^{n} \lambda_i v_i \in V$ with $\lambda_i \in \mathbb{F}$, $1 \leq i \leq n$, not all zero and $\varphi = \sum_{j=1}^{n} \mu_j \varphi_j \in V^*$ with $\mu_j \in \mathbb{F}$, $1 \leq j \leq n$, not all zero. On the one hand, we find

$$
\begin{aligned}
t_{v,\varphi} = \Phi(v \otimes \varphi) &= \Phi\left(\left(\sum_{i=1}^{n} \lambda_i v_i\right) \otimes \left(\sum_{j=1}^{n} \mu_j \varphi_j\right)\right) = \Phi\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \mu_j (v_i \otimes \varphi_j)\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \mu_j \Phi(v_i \otimes \varphi_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \mu_j E_{i,j},
\end{aligned}
$$

hence $t_{v,\varphi}$ has trace $\sum_{i=1}^{n} \lambda_i \mu_i$. On the other hand, we have

$$
\varphi(v) = \left(\sum_{j=1}^{n} \mu_j \varphi_j\right)\left(\sum_{i=1}^{n} \lambda_i v_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \mu_j \varphi_j(v_i) = \sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \mu_j \delta_{i,j} = \sum_{i=1}^{n} \lambda_i \mu_i,
$$

from which it immediately follows that $v \otimes \varphi$ is a singular pure tensor if and only if $t_{v,\varphi}$ is traceless. Consequently, $v \otimes \varphi \in \mathfrak{g}_0(V \otimes V^*)$ if and only if $t_{v,\varphi} \in \mathfrak{sl}(V)$, and so the Lie algebra isomorphism $\Phi : \mathfrak{g}(V \otimes V^*) \to \mathfrak{gl}(V)$ given by $v \otimes \varphi \mapsto t_{v,\varphi}$ from Proposition 3.2.5 induces a Lie algebra isomorphism $\Phi_0 : \mathfrak{g}_0(V \otimes V^*) \to \mathfrak{sl}(V)$ given by $v \otimes \varphi \mapsto t_{v,\varphi}$, which settles the finite-dimensional case.

The case of V being infinite-dimensional is proved by using similar arguments as in the proof of Proposition 3.2.5 for V infinite-dimensional restricted to the Lie algebras $\mathfrak{g}_0(V \otimes V^*)$ and $\mathfrak{fsl}(V)$. □

Recall that a basis of $\mathfrak{sl}(V)$ with V finite-dimensional, say $\dim(V) = n \geq 1$, is given by the matrices $E_{i,j}$ with $1 \leq i \neq j \leq n$ and $E_{i,i} - E_{i+1,i+1}$ with $1 \leq i \leq n-1$. For all

$1 \leq i \leq n-1$, we may identify $\Phi(v_i \otimes \varphi_i - v_{i+1} \otimes \varphi_{i+1})$ by the matrix $E_{i,i} - E_{i+1,i+1}$, but $v_i \otimes \varphi_i, \varphi_{i+1} \otimes \varphi_{i+1} \notin \mathfrak{g}_0(V \otimes V^*)$, so we alternatively write

$$v_i \otimes \varphi_i - v_{i+1} \otimes \varphi_{i+1} = (v_i + v_{i+1}) \otimes (\varphi_i - \varphi_{i+1}) + v_i \otimes \varphi_{i+1} - v_{i+1} \otimes \varphi_i,$$

which is a linear combination of singular pure tensors. Thus, a basis of $\mathfrak{g}_0(V \otimes V^*)$ is given by $v_i \otimes \varphi_j$ with $1 \leq i \neq j \leq n$ and $(v_i + v_{i+1}) \otimes (\varphi_i - \varphi_{i+1})$ with $1 \leq i \leq n-1$ satisfying $\varphi_j(v_i) = \delta_{i,j}$ for all $1 \leq i, j \leq n$.

Modulo its center, the Lie algebra $\mathfrak{g}_0(V \otimes V^*)$ is simple because $\mathfrak{sl}(V)$ is simple up to its center. However, in case V is infinite-dimensional, it is possible to obtain proper ideals of $\mathfrak{g}_0(V \otimes V^*)$ because V^* contains proper subspaces. Consider the following definition.

Definition 3.2.7 (Annihilator)**.** *Let V be an infinite-dimensional vector space over a field $\mathbb{F}$ and let W and W^* be subspaces of V and the dual space V^* of V, respectively. The* ***annihilator*** *of W^* in W is the set $\mathrm{Ann}_W(W^*) = \{w \in W \mid \forall \varphi \in W^* : \varphi(w) = 0\}$.*

It is clear that $\mathrm{Ann}_W(W^*) \subset V$ is a finite-dimensional subspace of V for all subspaces $W \subset V$ and $W^* \subset V^*$; indeed, for all $v, w \in \mathrm{Ann}_W(W^*)$ and $\lambda, \mu \in \mathbb{F}$, we obtain $\varphi(\lambda v + \mu w) = \lambda\varphi(v) + \mu\varphi(w) = 0$ for all $\varphi \in W^*$. Observe further that finite-dimensionality of V would imply $W^* = V^*$ for all subspaces $W^* \subseteq V^*$ so that $\mathrm{Ann}_W(V^*) = 0$ for all subspaces $W \subseteq V$. We now propose the following.

Proposition 3.2.8. *Let V be an infinite-dimensional vector space over a field $\mathbb{F}$ with* $\mathrm{char}(\mathbb{F}) \neq 2$ *and let $\mathfrak{g}_0(V \otimes W^*)$ be the Lie subalgebra of $\mathfrak{g}(V \otimes W^*)$ generated by its singular pure tensors with $V \otimes W^*$ as in* Proposition 3.2.1. *Then* $\mathrm{Ann}_V(W^*) = \{0\}$ *if and only if $\mathfrak{g}_0(V \otimes W^*)$ is simple.*

Proof. Suppose first that $\mathrm{Ann}_V(W^*) \neq \{0\}$. Then $U = \mathrm{Ann}_V(W^*)$ is a proper subspace of V. We claim that the subspace $\mathfrak{i} = \mathfrak{g}_0(U \otimes W^*)$ is a proper ideal of $\mathfrak{g}_0(V \otimes W^*)$. Since every element in $\mathfrak{g}_0(V \otimes W^*)$ can be written as a finite linear combination of pure tensors in $\mathfrak{g}_0(V \otimes W^*)$, it suffices to show $[w \otimes \psi, u \otimes \varphi] \in \mathfrak{i}$ for all $u \otimes \varphi \in \mathfrak{g}_0(U \otimes W^*)$ and pure tensors $w \otimes \psi \in \mathfrak{g}_0(V \otimes W^*)$. We find $[w \otimes \psi, u \otimes \varphi] = \psi(u)(w \otimes \varphi) - \varphi(w)(u \otimes \psi) = -\varphi(w)(u \otimes \psi) \in \mathfrak{i}$ because $\psi(u) = 0$ and $\psi \in W^*$. Thus, $\mathfrak{i} \subseteq \mathfrak{g}_0(V \otimes W^*)$ is an ideal that is clearly proper. This shows that $\mathfrak{g}_0(V \otimes W^*)$ is not simple.

Next, suppose that $\mathrm{Ann}_V(W^*) = \{0\}$ and assume that $\mathfrak{i} \subset \mathfrak{g}_0(V \otimes W^*)$ is a proper ideal. We show that $\mathfrak{g}_0(V \otimes W^*) \subseteq \mathfrak{i}$. Since $\mathfrak{g}_0(V \otimes W^*)$ is generated by the singular pure tensors of $\mathfrak{g}(V \otimes W^*)$, any element $0 \neq x \in \mathfrak{i}$ can be written as a finite linear combination of singular pure tensors, i.e. $x = \sum_{i \in I} \lambda_i(v_i \otimes \varphi_i)$ for some finite index set I with $\lambda_i \in \mathbb{F}^*$, $i \in I$, and $0 \neq v_i \otimes \varphi_i \in \mathfrak{g}_0(V \otimes W^*)$, $i \in I$. By Proposition 3.2.2, all singular pure tensors in $\mathfrak{g}(V \otimes W^*)$ are extremal, so for all $0 \neq v \otimes \varphi \in \mathfrak{g}_0(V \otimes W^*)$ we find

$$\mathfrak{i} \ni [v \otimes \varphi, [v \otimes \varphi, x]] = \sum_{i \in I} \lambda_i [v \otimes \varphi, [v \otimes \varphi, v_i \otimes \varphi_i]] = 2(v \otimes \varphi) \sum_{i \in I} \lambda_i g_{v \otimes \varphi}(v_i \otimes \varphi_i)$$
$$= -2(v \otimes \varphi) \sum_{i \in I} \lambda_i \varphi_i(v)\varphi(v_i).$$

If $\varphi(v_i) = 0$ for some $i \in I$, then $v_i \in \mathrm{Ann}_V(W^*) = \{0\}$ because the above is true for all $0 \neq v \otimes \varphi \in \mathfrak{g}_0(V \otimes W^*)$, hence $v_i \otimes \varphi_i = 0 \otimes \varphi_i = 0$, a contradiction. This forces $\varphi_i(v) = 0$ for all $i \in I$ because $\mathrm{char}(\mathbb{F}) \neq 2$ and $\lambda_i \neq 0$ for all $i \in I$. But then necessarily $\varphi(v) = 0$ for all $\varphi \in W^*$ because the above is true for all $0 \neq x \in \mathfrak{i}$ so that $v \in \mathrm{Ann}_V(W^*) = \{0\}$. Consequently, $v \otimes \varphi = 0 \otimes \varphi = 0$, another contradiction. It follows that $v \otimes \varphi \in \mathfrak{i}$, showing the inclusion $\mathfrak{g}_0(V \otimes W^*) \subseteq \mathfrak{i}$ and forcing equality. We conclude that $\mathfrak{g}_0(V \otimes W^*)$ is simple. □

If $\mathrm{char}(\mathbb{F}) \neq 2$, more Lie subalgebras of $\mathfrak{g}(V \otimes V^*)$, with V not necessarily infinite-dimensional, can be obtained by considering sesquilinear spaces (V, f) as discussed in Section 2.2. For a reflexive sesquilinear form f on V and a vector $v \in V$, we define $f_v : V \to \mathbb{F}$ to be the map given by $w \mapsto f(v, w)$ with $w \in V$, and it is readily seen that $f_v \in V^*$ for all $v \in V$.

First, consider a symplectic space (V, f). Since then $f(v, v) = 0$ for all $v \in V$, we deduce that $v \otimes f_v \in \mathfrak{g}_0(V \otimes V^*)$. We will denote the Lie subalgebra of $\mathfrak{g}_0(V \otimes V^*)$ generated by the tensors of the form $v \otimes f_v$ by $\mathfrak{g}_0(V \otimes V^*)_f$, the generators of which we will refer to as *symplectic tensors*. With Φ as in Proposition 3.2.5, we then find for all $u, w \in V$ that

$$\begin{aligned}
0 &= f(v,u)f(v,w) - f(v,u)f(v,w) = f(v,u)f(v,w) + f(u,v)f(v,w) \\
&= f(f(v,u)v, w) + f(u, f(v,w)v) = f(f_v(u)v, w) + f(u, f_v(w)v) \\
&= f(t_{v,f_v}(u), w) + f(u, t_{v,f_v}(w)) = f(\Phi(v \otimes f_v)(u), w) + f(u, \Phi(v \otimes f_v)(w))
\end{aligned}$$

for all $v \otimes f_v \in \mathfrak{g}_0(V \otimes V^*)_f$ so that $\Phi(\mathfrak{g}_0(V \otimes V^*)_f) \subseteq \mathfrak{fsp}(V, f)$ by Lemma 2.2.9. The elements in $\Phi(\mathfrak{g}_0(V \otimes V^*)_f)$ are called *symplectic infinitesimal transvections*. In light of Proposition 3.2.1 and Corollary 3.2.6, we propose the following.

Proposition 3.2.9. *Let $\mathfrak{g}(V \otimes V^*)$ be the Lie algebra of $V \otimes V^*$ as in* Proposition 3.2.1 *and let $\mathfrak{g}_0(V \otimes V^*)_f$ be the Lie subalgebra of $\mathfrak{g}_0(V \otimes V^*)$ generated by its symplectic tensors. Then $\mathfrak{g}_0(V \otimes V^*)_f \cong \mathfrak{fsp}(V, f)$.*

Proof. If V is finite-dimensional, say $\dim(V) = 2n \geq 2$, we proceed in a similar manner as in the proofs of Proposition 3.2.5 and Corollary 3.2.6 by providing a basis of $\mathfrak{g}_0(V \otimes V^*)_f$ that is mapped to a basis of $\mathfrak{sp}(V, f)$ under Φ as defined in Proposition 3.2.5. Let $\{v_i\}_{1 \leq i \leq 2n}$ be a hyperbolic basis of V constituted of the hyperbolic pairs $\{v_i, v_{n+i}\}$ with $1 \leq i \leq n$. Non-degeneracy of f and linear independence of all v_i, $1 \leq i \leq 2n$, then imply linear independence of all f_{v_i}, $1 \leq i \leq 2n$; indeed, if $\sum_{i=1}^{2n} \lambda_i f_{v_i}(w) = 0$ for all $w \in V$ with $\lambda_i \in \mathbb{F}$, $1 \leq i \leq 2n$, then $\sum_{i=1}^{2n} \lambda v_i = 0$ by non-degeneracy of f so that $\lambda_i = 0$ for all $1 \leq i \leq 2n$ by linear independence of all v_i, $1 \leq i \leq 2n$. Then for any $v = \sum_{i=1}^{2n} \lambda_i v_i$ with $\lambda_i \in \mathbb{F}$, $1 \leq i \leq 2n$, not all zero, we have

$$v \otimes f_v = \left(\sum_{i=1}^{2n} \lambda_i v_i \right) \otimes \left(\sum_{j=1}^{2n} \lambda_j f_{v_j} \right) = \sum_{i=1}^{2n} \lambda_i^2 (v_i \otimes f_{v_i}) + \sum_{i=1}^{2n} \sum_{j=i+1}^{2n} \lambda_i \lambda_j (v_i \otimes f_{v_j} + v_j \otimes f_{v_i})$$

so that $\{v_i \otimes f_{v_i}\}_{1\leq i\leq 2n} \cup \{v_i \otimes f_{v_j} + v_j \otimes f_{v_i}\}_{1\leq i<j\leq 2n}$ forms a basis of $\mathfrak{g}_0(V \otimes V^*)_f$ by linear independence of all v_i and f_{v_i}, $1 \leq i \leq 2n$. Now consider the images of these basis elements under Φ. On the one hand, for all $v_i \otimes f_{v_i}$, $1 \leq i \leq 2n$, we obtain for all $w \in V$ that

$$\Phi(v_i \otimes f_{v_i})(w) = f(v_i, w)v_i = \begin{cases} v_i & \text{if } 1 \leq i \leq n \text{ and } w = v_{n+i}, \\ -v_i & \text{if } n+1 \leq i \leq 2n \text{ and } w = v_{i-n}, \\ 0 & \text{else,} \end{cases}$$

hence we may identify $\Phi(v_i \otimes f_{v_i})$ by the matrices $E_{i,n+i}$ for $1 \leq i \leq n$ and $-E_{i,i-n}$ for $n+1 \leq i \leq 2n$. On the other hand, for all $v_i \otimes f_{v_j} + v_j \otimes f_{v_i}$, $1 \leq i < j \leq 2n$, we find for all $w \in V$ that

$$\begin{aligned} &\Phi(v_i \otimes f_{v_j} + v_j \otimes f_{v_i})(w) = f(v_j, w)v_i + f(v_i, w)v_j \\ &= \begin{cases} v_i + v_j & \text{if } 1 \leq i < j \leq n \text{ and } w = v_{n+j} + v_{n+i}, \\ -v_i - v_j & \text{if } n+1 \leq i < j \leq 2n \text{ and } w = v_{j-n} + v_{i-n}, \\ v_i - v_j & \text{if } 1 \leq i \leq n, n+1 \leq j \leq 2n \text{ and } w = v_{j-n} + v_{n+i}, \\ 0 & \text{else,} \end{cases} \end{aligned}$$

therefore we may identify $\Phi(v_i \otimes f_{v_j} + v_j \otimes f_{v_i})$ with the matrices $E_{i,n+j} + E_{j,n+i}$ if $1 \leq i < j \leq n$, $-E_{i,j-n} - E_{j,i-n}$ if $n+1 \leq i < j \leq 2n$, and $E_{i,j-n} - E_{j,n+i}$ if $1 \leq i \leq n$, $n+1 \leq j \leq 2n$. Upon shifting some of the indices and scaling, it is readily seen that Φ maps the basis of $\mathfrak{g}_0(V \otimes V^*)_f$ as given above to the basis of $\mathfrak{sp}(V, f)$ as given in Corollary 2.2.15. It follows that the Lie algebra isomorphism $\Phi : \mathfrak{g}(V \otimes V^*) \to \mathfrak{gl}(V)$ given by $v \otimes \varphi \mapsto t_{v,\varphi}$ from Proposition 3.2.5 induces a Lie algebra isomorphism $\Phi_f : \mathfrak{g}_0(V \otimes V^*)_f \to \mathfrak{sp}(V, f)$ given by $v \otimes \varphi \mapsto t_{v,\varphi}$, and so the finite-dimensional case follows.

For the infinite-dimensional case, we refer to the proof of Proposition 3.2.5, which can be repeated word by word when restricted to the Lie algebras $\mathfrak{g}_0(V \otimes V^*)_f$ and $\mathfrak{fsp}(V, f)$. □

It is also possible to provide a basis of $\mathfrak{g}_0(V \otimes V^*)_f$ consisting entirely of pure tensors. This is done by observing that

$$v_i \otimes f_{v_j} + v_j \otimes f_{v_i} = (v_i + v_j) \otimes f_{v_i+v_j} - v_i \otimes f_{v_i} - v_j \otimes f_{v_j}$$

so that $\{v_i \otimes f_{v_i}\}_{1\leq i\leq 2n} \cup \{(v_i + v_j) \otimes f_{v_i+v_j}\}_{1\leq i<j\leq 2n}$ is also a basis of $\mathfrak{g}_0(V \otimes V^*)_f$.

Now let f be a non-degenerate Hermitian form on V with corresponding involutory anti-automorphism σ and consider the Lie algebra $\mathfrak{g}(V \otimes W^*)_f$ generated by the elements $v \otimes f_v^\sigma \in \mathfrak{g}(V \otimes V^*)$, in which $v \in V$ and $f_v^\sigma : V \to \mathbb{F}$ is the linear map given by $f_v^\sigma(w) = -f(v, w)^\sigma$ with $w \in V$. We will call these generators *unitary tensors*. By Lemma 2.2.7, we may take f to be skew-Hermitian; indeed, if f' is a skew-Hermitian form proportional to f by $\alpha \in \mathbb{F}$, then the elements $v \otimes f_v'^\sigma$ will generate the same Lie algebra

$\mathfrak{g}(V\otimes V^*)_f$ since $v\otimes f_v'^{\sigma} = v\otimes(\alpha f_v)^\sigma = (v\otimes f_v^\sigma)\alpha^\sigma = \alpha^\sigma(v\otimes f_v^\sigma)$. With Φ as in Proposition 3.2.5, we then obtain for all $u, w \in V$ that

$$\begin{aligned} 0 &= f(u,v)f(v,w) - f(u,v)f(v,w) = -f(v,u)^\sigma f(v,w) - f(u, f(v,w)^\sigma v) \\ &= f(-f(v,u)^\sigma v, w) + f(u, -f(v,w)^\sigma v) = f(t_{v,f_v^\sigma}(u), w) + f(u, t_{v,f_v^\sigma}(w)) \\ &= f(\Phi(v\otimes f_v^\sigma)(u), w) + f(u, \Phi(v\otimes f_v^\sigma)(w)) \end{aligned}$$

for all $v\otimes f_v^\sigma \in \mathfrak{g}(V\otimes V^*)_f$, hence $\Phi(\mathfrak{g}(V\otimes V^*)_f) \subseteq \mathfrak{fu}(V,f)$ by Lemma 2.2.9. An element $t_{v,f_v^\sigma} \in \Phi(\mathfrak{g}(V\otimes V^*)_f)$ with v isotropic is called a *unitary infinitesimal transvection*, whereas we refer to it as a *unitary infinitesimal reflection* if v is anisotropic. By recalling that $\mathfrak{fu}(V)$ is a Lie algebra over the subfield $\mathbb{F}_\sigma$ of $\mathbb{F}$ whose elements are fixed by σ, we deduce that a unitary infinitesimal transvection $t_{v,f_v^\sigma} \in \Phi(\mathfrak{g}(V\otimes V^*)_f)$ is extremal if and only if $\mathrm{im}(g_{v,f_v^\sigma}) \subseteq \mathbb{F}_\sigma$, i.e. $g_{t_{v,f_v^\sigma}}(t_{w,f_w^\sigma}) = -f_w^\sigma(v)f_v^\sigma(w) = -f(v,w)f(w,v) \in \mathbb{F}_\sigma$ for all $w \in V$. As before, the Lie algebra $\mathfrak{g}(V\otimes V^*)_f$ generated by the elements $v\otimes f_v^\sigma$ with $v \in V$ is isomorphic to a classical linear Lie algebra. This is characterised by the following proposition.

Proposition 3.2.10. *Let $\mathfrak{g}(V\otimes V^*)$ be the Lie algebra of $V\otimes V^*$ as in* Proposition 3.2.1 *and let $\mathfrak{g}(V\otimes V^*)_f$ be the Lie subalgebra of $\mathfrak{g}(V\otimes V^*)$ generated by its unitary tensors. Then $\mathfrak{g}(V\otimes V^*)_f \cong \mathfrak{fu}(V,f)$.*

Proof. Assume first that V is finite-dimensional, say $\dim(V) = n = 2m+k \geq 1$. Let $\{v_i\}_{1\leq i\leq n}$ be a basis of V such that $\{v_i, v_{m+i}\}$ is a hyperbolic pair for all $1 \leq i \leq m$, $f(v_{2m+i}, v_{2m+i}) = \beta$ with $\beta^\sigma = \beta$ for all $1 \leq i \leq k$ and f evaluates to zero for every other pair of basis elements not constituting a hyperbolic pair. As in the proof of Proposition 3.2.9, all the $f_{v_i}^\sigma$, $1 \leq i \leq n$, are linearly independent because non-degeneracy of f and linear independence of all the v_i, $1 \leq i \leq n$. Then for any $v = \sum_{i=1}^n \lambda_i v_i$ with $\lambda_i \in \mathbb{F}$, $1 \leq i \leq n$, not all zero, we have

$$v\otimes f_v^\sigma = \left(\sum_{i=1}^n \lambda_i v_i\right)\otimes\left(\sum_{j=1}^n \lambda_j^\sigma f_{v_j}^\sigma\right) = \sum_{i=1}^n \lambda_i\lambda_i^\sigma(v_i\otimes f_{v_i}^\sigma) + \sum_{i=1}^n\sum_{j=i+1}^n \gamma_{i,j}(v_i\otimes f_{v_j}^\sigma) + \gamma_{i,j}^\sigma(v_j\otimes f_{v_i}^\sigma),$$

in which $\gamma_{i,j} = \lambda_i\lambda_j^\sigma$ for all $1 \leq i < j \leq n$. Consequently, a basis of $\mathfrak{g}(V\otimes V^*)_f$ is given by

$$\{v_i\otimes f_{v_i}^\sigma\}_{1\leq i\leq n} \cup \{v_i\otimes f_{v_j}^\sigma + v_j\otimes f_{v_i}^\sigma\}_{1\leq i<j\leq n} \cup \{\gamma(v_i\otimes f_{v_j}^\sigma) + \gamma^\sigma(v_j\otimes f_{v_i}^\sigma)\}_{1\leq i<j\leq n}$$

with $\gamma \in \mathbb{F}$ an element not fixed by σ. With Φ as in Proposition 3.2.5, we then find for all $w \in V$ that

$$\Phi(v_i\otimes f_{v_i}^\sigma)(w) = -f(v_i,w)^\sigma v_i = f(w,v_i)v_i = \begin{cases} -v_i & \text{if } 1 \leq i \leq m \text{ and } w = v_{m+i}, \\ v_i & \text{if } m+1 \leq i \leq 2m \text{ and } w = v_{i-m}, \\ \beta v_i & \text{if } 2m+1 \leq i \leq n \text{ and } w = v_i, \\ 0 & \text{else,} \end{cases}$$

hence we may identify $\Phi(v_i \otimes f^\sigma_{v_i})$ by the matrix $-E_{i,m+i}$ if $1 \le i \le m$, $E_{i,i-m}$ if $m+1 \le i \le 2m$ and $\beta E_{i,i}$ if $2m+1 \le i \le n$. Likewise, we have for all $w \in V$ that

$$\Phi(v_i \otimes f_{v_j} + v_j \otimes f_{v_i})(w) = -f(v_j,w)^\sigma v_i - f(v_i,w)^\sigma v_j = f(w,v_j)v_i + f(w,v_i)v_j$$
$$= \begin{cases} -v_i - v_j & \text{if } 1 \le i < j \le m \text{ and } w = v_{m+j} + v_{m+i}, \\ v_i + v_j & \text{if } m+1 \le i < j \le 2m \text{ and } w = v_{j-m} + v_{i=m}, \\ v_i - v_j & \text{if } 1 \le i \le m, m+1 \le j \le 2m \text{ and } w = v_{j-m} + v_{m+i}, \\ \beta v_i + v_j & \text{if } 1 \le i \le m, 2m+1 \le j \le n \text{ and } w = v_j + v_{m+i}, \\ v_i + \beta v_j & \text{if } m+1 \le i \le 2m, 2m+1 \le j \le n \text{ and } w = v_j + v_{i-n}, \\ \beta v_i + \beta v_j & \text{if } 2m+1 \le i < j \le n \text{ and } w = v_j + v_i, \\ 0 & \text{else,} \end{cases}$$

therefore we may identify $\Phi(v_i \otimes f_{v_j} + v_j \otimes f_{v_i})$ by the matrices $-E_{i,m+j} - E_{j,m+i}$ if $1 \le i < j \le m$, $E_{i,j-m} + E_{j,i-m}$ if $m+1 \le i < j \le 2m$, $E_{i,j-m} - E_{j,m+i}$ if $1 \le i \le m$ and $m+1 \le j \le 2m$, $E_{i,j-m} + E_{j,m+i}$ if $1 \le i \le m$ and $m+1 \le j \le 2m$, $\beta E_{i,j} + E_{j,m+i}$ if $1 \le i \le m$ and $2m+1 \le j \le n$, $E_{i,j} + \beta E_{j,i-n}$ if $m+1 \le i \le 2m$ and $2m+1 \le j \le n$, and $\beta E_{i,j} + \beta E_{j,i}$ if $2m+1 \le i < j \le n$. For $\Phi(\gamma(v_i \otimes f_{v_j}) + \gamma^\sigma(v_j \otimes f_{v_i}))$, we obtain similar expressions. Upon shifting some of the indices and scaling, it is readily seen that Φ maps the basis of $\mathfrak{g}(V \otimes V^*)_f$ as given above to the basis of $\mathfrak{u}(V)$ as given in Corollary 2.2.17. It follows that the Lie algebra isomorphism $\Phi : \mathfrak{g}(V \otimes V^*) \to \mathfrak{gl}(V)$ given by $v \otimes \varphi \mapsto t_{v,\varphi}$ from Proposition 3.2.5 induces a Lie algebra isomorphism $\Phi_f : \mathfrak{g}(V \otimes V^*)_f \to \mathfrak{u}(V,f)$ given by $v \otimes \varphi \mapsto t_{v,\varphi}$, which settles the finite-dimensional case.

The infinite-dimensional case follows from arguments similar to those presented in Proposition 3.2.5. □

By writing

$$v_i \otimes f^\sigma_{v_j} + v_j \otimes f^\sigma_{v_i} = (v_i + v_j) \otimes f^\sigma_{v_i+v_j} - v_i \otimes f^\sigma_{v_i} - v_j \otimes f^\sigma_{v_j},$$
$$\gamma(v_i \otimes f^\sigma_{v_j}) + \gamma^\sigma(v_j \otimes f^\sigma_{v_i}) = (\gamma v_i + v_j) \otimes f^\sigma_{\gamma v_i+v_j} - \gamma\gamma^\sigma(v_i \otimes f^\sigma_{v_i}) - v_j \otimes f^\sigma_{v_j},$$

a basis of $\mathfrak{g}(V \otimes V^*)_f$ consisting of pure tensors only is given by

$$\{v_i \otimes f^\sigma_{v_i}\}_{1 \le i \le n} \cup \{(v_i + v_j) \otimes f^\sigma_{v_i+v_j}\}_{1 \le i < j \le n} \cup \{(\gamma v_i + v_j) \otimes f^\sigma_{\gamma v_i+v_j}\}_{1 \le i < j \le n}.$$

Further note that it follows from the above proposition and Corollary 3.2.6 that the unitary tensors $v \otimes f^\sigma_v \in \mathfrak{g}(V \otimes V^*)_f$ with v isotropic generate a Lie subalgebra $\mathfrak{g}_0(V \otimes V^*)_f$ of $\mathfrak{g}(V \otimes V^*)_f$ isomorphic to $\mathfrak{fsu}(V,f)$, provided that the isotropic vectors of V span V linearly.

Lastly, let (V,f) be a symmetric space and consider the Lie algebra $\mathfrak{g}(V \otimes V^*)_f$ generated by the tensors of the form $v \otimes f_w - w \otimes f_v$ with $v, w \in V$, which we will call *symmetric* *tensors*. With Φ as in Proposition 3.2.5, we deduce for all $x, y \in V$ that

$$\begin{aligned}
0 &= f(w,x)f(v,y) - f(w,x)f(v,y) + f(x,v)f(w,y) - f(x,v)f(w,y) \\
&= f(w,x)f(v,y) - f(v,x)f(w,y) + f(x,v)f(w,y) - f(x,w)f(v,y) \\
&= f(f(w,x)v - f(v,x)w, y) + f(x, f(w,y)v - f(v,y)w) \\
&= f(f_w(x)v - f_v(x)w, y) + f(x, f_w(y)v - f_v(y)w) \\
&= f((t_{v,f_w} - t_{w,f_v})(x), y) + f(x, (t_{v,f_w} - t_{w,f_v})(y)) \\
&= f(\Phi(v \otimes f_w - w \otimes f_v)(x), y) + f(x, \Phi(v \otimes f_w - w \otimes f_v)(y))
\end{aligned}$$

for all $v \otimes f_w - w \otimes f_v \in \mathfrak{g}(V \otimes V^*)_f$ so that $\Phi(\mathfrak{g}(V \otimes V^*)_f) \subseteq \mathfrak{fso}(V, f)$ by Lemma 2.2.9. In particular, we obtain the following.

Proposition 3.2.11. *Let $\mathfrak{g}(V \otimes V^*)$ be the Lie algebra of $V \otimes V^*$ as in* Proposition 3.2.1 *and let $\mathfrak{g}(V \otimes V^*)_f$ be the Lie subalgebra of $\mathfrak{g}(V \otimes V^*)$ generated by its symmetric tensors. Then $\mathfrak{g}(V \otimes V^*)_f \cong \mathfrak{fso}(V, f)$.*

Proof. Write $\dim(V) = n \geq 1$ and let $\{v_i\}_{1 \leq i \leq n}$ be a basis of V such that $f(v_i, v_i) = 1$ for all $1 \leq i \leq n$ and f evaluates to zero for every other pair of basis elements. For all $v = \sum_{i=1}^n \lambda_i v_i$ and $w = \sum_{j=1}^n \mu_j v_j$ with $\lambda_i, \mu_j \in \mathbb{F}$, $1 \leq i, j \leq n$, not all zero, we have

$$\begin{aligned}
v \otimes f_w - w \otimes f_v &= \left(\sum_{i=1}^n \lambda_i v_i\right) \otimes \left(\sum_{j=1}^n \mu_j f_{v_j}\right) - \left(\sum_{j=1}^n \mu_j v_j\right) \otimes \left(\sum_{i=1}^n \lambda_i f_{v_i}\right) \\
&= \sum_{i=1}^n \sum_{j=i+1}^n \lambda_i \mu_j (v_i \otimes f_{v_j} - v_j \otimes f_{v_i}),
\end{aligned}$$

hence $\{v_i \otimes f_{v_j} - v_j \otimes f_{v_j}\}_{1 \leq i < j \leq n}$ is a basis of $\mathfrak{g}(V \otimes V^*)_f$ by linear independence of all v_i and f_{v_i}, $1 \leq i \leq n$. With Φ as in Proposition 3.2.5, we then obtain for all $u \in V$ that

$$\begin{aligned}
&\Phi(v_i \otimes f_{v_j} - v_j \otimes f_{v_i})(u) = f(v_j, u)v_i - f(v_i, u)v_j \\
&= \begin{cases} v_i - v_j & \text{if } 1 \leq i < j \leq n \text{ and } u = v_j + v_i, \\ 0 & \text{else,} \end{cases}
\end{aligned}$$

so we may identify $\Phi(v_i \otimes f_{v_j} - v_j \otimes f_{v_i})$ by the matrices $E_{i,j} - E_{j,i}$ for all $1 \leq i < j \leq n$. It is readily seen that Φ maps the basis of $\mathfrak{g}(V \otimes V^*)_f$ as given above to the basis of $\mathfrak{so}(V, f)$ as given in Corollary 2.2.16. Thus, the Lie algebra isomorphism $\Phi : \mathfrak{g}(V \otimes V^*) \to \mathfrak{gl}(V)$ given by $v \otimes \varphi \mapsto t_{v,\varphi}$ from Proposition 3.2.5 induces a Lie algebra isomorphism $\Phi_f : \mathfrak{g}(V \otimes V^*)_f \to \mathfrak{so}(V, f)$ given by $v \otimes \varphi \mapsto t_{v,\varphi}$.

The infinite-dimensional case is settled similar to Proposition 3.2.5. □

We finish this section with a discussion on the extremality of the generators of $\mathfrak{g}(V \otimes V^*)_f$ with (V, f) a symmetric space. First consider the following lemma.

Lemma 3.2.12. *Let $\mathfrak{g}(V \otimes V^*)$ be the Lie algebra of $V \otimes V^*$ as in* Proposition 3.2.1 *and let $\mathfrak{g}(V \otimes V^*)_f$ be the Lie subalgebra of $\mathfrak{g}(V \otimes V^*)$ generated by its symmetric tensors. Then $v \otimes f_w - w \otimes f_v \in \mathfrak{g}(V \otimes V^*)_f$ is extremal if and only if v and w are f-isotropic and orthogonal.*

Proof. Let $v \otimes f_w - w \otimes f_v \in \mathfrak{g}(V \otimes V^*)_f$ be a symmetric tensor. A tedious calculation shows that for all symmetric tensors $x \otimes f_y - y \otimes f_x \in \mathfrak{g}(V \otimes V^*)_f$ we have

$$\begin{aligned}
&[v \otimes f_w - w \otimes f_v, [v \otimes f_w - w \otimes f_v, x \otimes f_y - y \otimes f_x]] \\
&= f(w,x)(f(w,v)(v \otimes f_y) - f(y,v)(v \otimes f_w)) - f(y,v)(f(w,x)(v \otimes f_w) - f(w,v)(x \otimes f_w)) \\
&\quad - f(w,y)(f(w,v)(v \otimes f_x) - f(x,v)(v \otimes f_w)) + f(x,v)(f(w,y)(v \otimes f_w) - f(w,v)(y \otimes f_w)) \\
&\quad - f(v,x)(f(w,w)(v \otimes f_y) - f(y,w)(w \otimes f_w)) + f(y,w)(f(w,x)(v \otimes f_v) - f(v,v)(x \otimes f_w)) \\
&\quad + f(v,y)(f(w,w)(v \otimes f_x) - f(x,v)(w \otimes f_w)) - f(x,w)(f(w,y)(v \otimes f_v) - f(v,v)(y \otimes f_w)) \\
&\quad - f(w,x)(f(v,v)(w \otimes f_y) - f(y,w)(v \otimes f_v)) + f(y,v)(f(v,x)(w \otimes f_w) - f(w,w)(x \otimes f_v)) \\
&\quad + f(w,y)(f(v,v)(w \otimes f_x) - f(x,w)(v \otimes f_v)) - f(x,v)(f(v,y)(w \otimes f_w) - f(w,w)(y \otimes f_v)) \\
&\quad + f(v,x)(f(v,w)(w \otimes f_y) - f(y,w)(w \otimes f_v)) - f(y,w)(f(v,x)(w \otimes f_v) - f(v,w)(x \otimes f_v)) \\
&\quad - f(v,y)(f(v,w)(w \otimes f_x) - f(x,w)(w \otimes f_v)) + f(x,w)(f(v,y)(w \otimes f_v) - f(v,w)(y \otimes f_v)) \\
&= 2(f(v,x)f(w,y) - f(v,y)f(w,x))(v \otimes f_w - w \otimes f_v) \\
&\quad + (f(v,y)f(w,w) - f(v,w)f(w,y))(v \otimes f_x - x \otimes f_v) \\
&\quad + (f(v,w)f(w,x) - f(v,x)f(w,w))(v \otimes f_y - y \otimes f_v) \\
&\quad + (f(v,v)f(w,y) - f(v,w)f(v,y))(w \otimes f_x - x \otimes f_w) \\
&\quad + (f(v,w)f(v,x) - f(v,v)f(w,x))(w \otimes f_y - y \otimes f_w),
\end{aligned}$$

hence $v \otimes f_w - w \otimes f_v$ is extremal if and only if

$$\begin{aligned}
&[v \otimes f_w - w \otimes f_v, [v \otimes f_w - w \otimes f_v, x \otimes f_y - y \otimes f_x]] \\
&= 2(f(v,x)f(w,y) - f(v,y)f(w,x))(v \otimes f_w - w \otimes f_v)
\end{aligned}$$

by Proposition 3.1.3 because $\text{char}(\mathbb{F}) \neq 2$. But the above is true for all $x \otimes f_y - y \otimes f_x \in \mathfrak{g}(V \otimes W^*)_f$, so $v \otimes f_w - w \otimes f_v$ is extremal if and only if $f(v,v) = 0 = f(w,w)$ and $f(v,w) = 0 = f(w,v)$ if and only if v and w are f-isotropic and orthogonal. □

It should be clear from the above lemma that the extremal form of an extremal symmetric tensor $v \otimes f_w - w \otimes f_v \in \mathfrak{g}(V \otimes V^*)_f$ is given by

$$g_{v \otimes f_w - w \otimes f_v}(x \otimes f_y - y \otimes f_x) = f(v,x)f(w,y) - f(v,y)f(w,x)$$

for all $x \otimes f_y - y \otimes f_x \in \mathfrak{g}(V \otimes V^*)_f$. As a consequence of Proposition 3.2.11, the elements $\Phi(v \otimes f_w - w \otimes f_v) = t_{v,f_w} - t_{w,f_v} \in \mathfrak{fso}(V,f)$ will also be extremal whenever $v \otimes f_w - w \otimes f_v$

is extremal, having the same extremal form. Moreover, for all $u \in V$ we have

$$\begin{aligned}(t_{v,f_w} - t_{w,f_v})^2(u) &= (t_{v,f_w} - t_{w,f_v})(f(w,u)v - f(v,u)w) \\ &= f(w,u)(f(w,v)v - f(v,v)w) - f(v,u)(f(w,w))v - f(v,w)w) \\ &= 0,\end{aligned}$$

hence $(e^{\lambda(t_{v,f_w} - t_{w,f_v})})(u) = u + \lambda(t_{v,f_w} + t_{w,f_v})(u) = (\mathrm{id}_V + \lambda(t_{v,f_w} - t_{w,f_v}))(u)$ so that $e^{\lambda(t_{v,f_w} - t_{w,f_v})} = \mathrm{id}_V + \lambda(t_{v,f_w} - t_{w,f_v})$. This gives rise to the following definition.

Definition 3.2.13 (Siegel transformation)**.** *Let V be a possibly infinite-dimensional vector space over a field $\mathbb{F}$ equipped with a non-degenerate symmetric bilinear form $f : V \times V \to \mathbb{F}$. The linear map $\mathcal{S}_{v,w} : V \to V$ given by $\mathcal{S}_{v,w}(u) = u + f(w,u)v - f(v,u)w$ with $u \in V$ is called the **Siegel transformation** of v and w if $v, w \in V$ are two f-isotropic vectors such that $f(v,w) = 0$.*

In other words, Siegel transformations are elements $e^{t_{v,f_w} - t_{w,f_v}}$ with $t_{v,f_w} - t_{w,f_v} \in \mathfrak{fso}(V,f)$ extremal. We can write $t_{v,f_w} - t_{w,f_v} = \mathcal{S}_{v,w} - \mathrm{id}_V$, and the elements $s_{v,w} := t_{v,f_w} - t_{w,f_v}$ with $v, w \in V$ are also referred to as *infinitesimal Siegel transvections* if $v, w \in V$ are both f-isotropic such that $f(v,w) = 0$. The group $\mathrm{Exp}(s_{v,w}) = \{\mathrm{id}_V + \lambda s_{v,w} \mid \lambda \in \mathbb{F}\}$ with $v, w \in V$ both f-isotropic such that $f(v,w) = 0$ is called the *Siegel transvection group.* Note that infinitesimal Siegel transvections exist if and only if f has Witt index at least two.

Infinitesimal Siegel transvections satisfy the following property.

Corollary 3.2.14. *Let V be a possibly infinite-dimensional vector space over a field $\mathbb{F}$ and let $\mathfrak{fso}(V,f)$ be the finitary orthogonal Lie algebra on V for some non-degenerate symmetric bilinear form $f : V \times V \to \mathbb{F}$. Then any two infinitesimal Siegel transvections $s_{v,w}, s_{x,y} \in \mathfrak{fso}(V,f)$ with $v, w, x, y \in V$, if existent, satisfy $\exp(s_{v,w}, \lambda)s_{x,y} = s_{(e^{\lambda s_{v,w}})(x),(e^{\lambda s_{v,w}})(y)}$ for all $\lambda \in \mathbb{F}$.*

Proof. Using Definition 3.2.13, it is readily seen that $s_{v,w}(\lambda u) = \lambda s_{v,w}(u)$ for all $u \in V$ and $s_{\lambda v,w} = \lambda s_{v,w} = s_{v,\lambda w}$ for all $\lambda \in \mathbb{F}$, but also $s_{v+v',w} = s_{v,w} + s_{v',w}$ and $s_{v,w+w'} = s_{v,w} + s_{v,w'}$ for all $v', w' \in V$ such that $f(v',w) = 0 = f(v,w')$, and the same is true upon replacing v and w by x and y, respectively. Since $s_{v,w} \in \mathfrak{fso}(V,f)$, we have $f(s_{v,w}(x),u) = -f(x, s_{v,w}(u))$ and $f(s_{v,w}(y),u) = -f(x, s_{v,w}(u))$ by Lemma 2.2.9. We then find for all $u \in V$ that

$$\begin{aligned}[s_{v,w}, s_{x,y}](u) &= s_{v,w}(s_{x,y}(u)) - s_{x,y}(s_{v,w}(u)) \\ &= s_{v,w}(f(y,u)x - f(x,u)y) - (f(y, s_{v,w}(u))x - f(x, s_{v,w}(u))y) \\ &= (f(y,u)s_{v,w}(x) - f(s_{v,w}(x),u)y) + (f(s_{v,w}(y),u)x - f(x,u)s_{v,w}(y)) \\ &= s_{s_{v,w}(x),y}(u) + s_{x,s_{v,w}(y)}(u) = (s_{s_{v,w}(x),y} + s_{x,s_{v,w}(y)})(u).\end{aligned}$$

Additionally, we have for all $u \in V$ that

$$\begin{aligned}
s_{s_{v,w}(x),s_{v,w}(y)}(u) &= f(s_{v,w}(y),u)s_{v,w}(x) - f(s_{v,w}(x),u)s_{v,w}(y) \\
&= f(f(w,y)v - f(v,y)w, u)(f(w,x)v - f(v,x)w) \\
&\quad - f(f(w,x)v - f(v,x)w, u)(f(w,y)v - f(v,y)w) \\
&= (f(w,y)f(v,u) - f(v,y)f(w,u)(f(w,x)v - f(v,x)w) \\
&\quad - (f(w,x)f(v,u) - f(v,x)f(w,u))(f(w,y)v - f(v,y)w) \\
&= (f(v,x)f(w,u)f(w,y)v - f(w,y)f(v,u)f(v,x)w) \\
&\quad - (f(v,y)f(w,u)f(w,x)v - f(w,x)f(v,u)f(v,y)w) \\
&= f(v,x)f(w,y)(f(w,u)v - f(v,u)w) \\
&\quad - f(v,y)f(w,x)(f(w,u)v - f(v,u)w) \\
&= (f(v,x)f(w,y) - f(v,y)f(w,x))(f(w,u)v - f(v,u)w) \\
&= g_{s_{v,w}}(s_{x,y})s_{v,w}(u).
\end{aligned}$$

By recalling that $e^{\lambda s_{v,w}} = e^{\lambda(t_{v,f_w} - t_{w,f_v})} = \mathrm{id}_V + \lambda(t_{v,f_w} - t_{w,f_v}) = \mathrm{id}_V + \lambda s_{v,w}$ for all $\lambda \in \mathbb{F}$, we finally deduce for all $u \in V$ and $\lambda \in \mathbb{F}$ that

$$\begin{aligned}
\exp(s_{v,w},\lambda)(s_{x,y}(u)) &= s_{x,y}(u) + \lambda[s_{v,w}, s_{x,y}](u) + \lambda^2 g_{s_{v,w}}(s_{x,y})s_{v,w}(u) \\
&= s_{x,y}(u) + \lambda(s_{s_{v,w}(x),y} + s_{x,s_{v,w}(y)}) + \lambda^2 s_{s_{v,w}(x),s_{v,w}(y)} \\
&= s_{x,y}(u) + s_{\lambda s_{v,w}(x),y}(u) + s_{x,\lambda s_{v,w}(y)}(u) + s_{\lambda s_{v,w}(x),\lambda s_{v,w}(y)}(u) \\
&= s_{x+\lambda s_{v,w}(x),y}(u) + s_{x+\lambda s_{v,w}(y),\lambda s_{v,w}(y)}(u) \\
&= s_{x+\lambda s_{v,w}(x),y+\lambda s_{v,w}(y)}(u) = s_{(e^{\lambda s_{v,w}})(x),(e^{\lambda s_{v,w}})(y)}(u),
\end{aligned}$$

therefore $\exp(s_{v,w},\lambda)s_{x,y} = s_{(e^{\lambda s_{v,w}})(x),(e^{\lambda s_{v,w}})(y)}$ for all $\lambda \in \mathbb{F}$, which proves the corollary. $\square$

Chapter 4

Geometry

This chapter is dedicated to an in-depth discussion of point-line geometries, specifically non-degenerate polar spaces and their classification. Additionally, we explore geometries and chamber systems as a means of introducing two other important types of point-line geometries, namely root shadow spaces and root filtration spaces, the latter of which we will discuss in more detail regarding their embeddability in projective spaces.

We mainly follow the theory and notation as presented in [3, 5, 19] throughout this chapter. Some examples given in Section 4.1, Section 4.3 and Section 4.6 are based on or have been taken directly from previous unpublished work of the author [6, 7].

4.1 Basic theory of point-line geometries

We start this chapter with a short discussion of graphs and some related concepts as a means of introducing point-line geometries later on in this section.

Definition 4.1.1 (Graph, Vertex, Edge & Adjacency)**.** *Let V be a possibly infinite set and let $E \subseteq 2^V$ be a set of 2-subsets of V. The pair (V, E) is called a* ***graph****, and the elements in V and E are called* ***vertices*** *and* ***edges****, respectively. Two vertices $x, y \in V$ are said to be* ***adjacent*** *if and only if $\{x, y\} \in E$.*

We will use the symbol $\sim$ to indicate adjacency of two vertices in a graph, i.e. $x \sim y$ if and only if $\{x, y\} \in E$. The *degree* of a vertex $x \in V$ is the cardinality of the set $\{y \in V \mid x \sim y\}$, denoted by $\delta(x)$. A graph (V, E) together with a map $w : E \to \mathbb{R}$ is called a *weighted* graph in which every edge $e \in E$ is given a weight $w(e)$. If $E \subseteq 2^V$ contains (un)ordered pairs, then the graph (V, E) is said to be *(un)directed*. Edges in directed graphs are commonly referred to as *arcs*, in which case we write A instead of E for the set of arcs in a directed graph. The vertices $x \in V$ and $y \in V$ of an arc $(x, y) \in A$ are called its *tail* and *head*, respectively. In directed graphs, we distinguish between the *in-degree* and *out-degree* of a vertex $x \in V$, determined by the number $\delta^-(x)$ or $\delta^+(x)$ of arcs in A such that x is its head or tail, respectively. A *loop* is an edge in E from a vertex in V to itself, and a *multi-edge* is an edge in E between two vertices in V that appears at least twice in E. A unweighted and undirected graph without loops and multi-edges is called a *simple* graph. Henceforth, any graphs mentioned will be simple unless stated otherwise.

Several special types of graphs are the following. A *k-regular* graph, with $k \geq 0$ finite, is a graph (V, E) such that $\delta(x) = k$ for all $x \in V$. If $|V| = n$ and $k = n-1$, we obtain an $(n-1)$-regular graph, also knows as a *complete* graph and satisfying $|E| = \binom{|V|}{2}$. A *bipartite* graph is a graph (V, E) such that $V = V_1 \sqcup V_2$ with $x \not\sim y$ for all $x, y \in V_i$, $i = 1, 2$. Bipartite graphs generalise to *multipartite* graphs, which are graphs (V, E) such that $V = \bigsqcup_{i \in I} V_i$ with $x \not\sim y$ for all $x, y \in V_i$, $i \in I$, where I is some finite index set. A *subgraph* of graph (V, E) is a graph (V', E') such that $V' \subseteq V$ and $E' \subseteq E$. In particular, we call the subgraph (V', E') a *clique* if it is a complete graph. If V' is a subset of V, then the *subgraph induced by* V' of a graph (V, E) is the graph $(V', E \cap (V' \times V'))$. The *complement* of a graph (V, E) is the graph $(V, (V \times V) \setminus E)$, in which loops and multi-edges are usually excluded. The *adjacency* matrix of a graph (V, E) with $V = \{v_1, \ldots, v_n\}$, $n \geq 1$ finite, is the $n \times n$ matrix having a one in position (i, j), $1 \leq i, j \leq n$, if $v_i \sim v_j$ and a zero else.

Example 4.1.2. Figure 1 below shows the graph $\Gamma = (V, E)$ with $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2\}, \{1, 4\}, \{2, 4\}, \{2, 5\}, \{3, 4\}\}$.

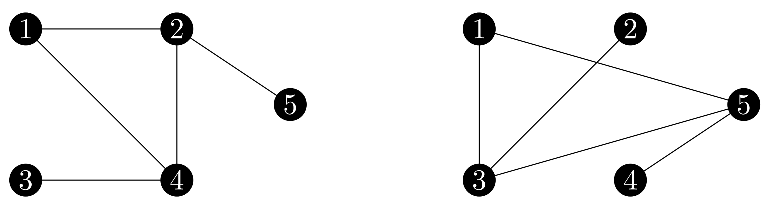


Figure 1: An exemplary graph Γ (left) and its complement $\overline{\Gamma}$ (right).

It is simple and not complete, since $|E| = 5 \neq 10 = \binom{5}{2} = \binom{|V|}{2}$. The vertices $1, 2, 4 \in V$ are pair-wise adjacent, whereas the vertices $1, 3, 5 \in V$ are pair-wise non-adjacent. We have $\delta(2) = \delta(4) = 3$, $\delta(1) = 2$ and $\delta(3) = \delta(5) = 1$, hence Γ is not regular. Its complement is the graph $\overline{\Gamma} = (V, \overline{E})$ with $\overline{E} = \{\{1, 3\}, \{1, 5\}, \{2, 3\}, \{3, 5\}, \{4, 5\}\}$, see also Figure 1 above. The adjacency matrices A_Γ and $A_{\overline{\Gamma}}$ of Γ and $\overline{\Gamma}$, respectively, are given by the 5×5 matrices

$$A_\Gamma = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_{\overline{\Gamma}} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix},$$

where the rows and columns of A_Γ and $A_{\overline{\Gamma}}$ are indexed by the vertices in V in ascending order.

As in Definition 2.1.6, structure-preserving properties between graphs can be described using graph morphisms.

Definition 4.1.3 (Graph morphisms)**.** *Let $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ be two graphs. A* ***homomorphism*** *is a map $\varphi : V \to V'$ such that $\{x, y\} \in E$ implies $\{\varphi(x), \varphi(y)\} \in E'$ for all $x, y \in V$. If Γ is a subgraph of Γ', then φ is called a* ***monomorphism****, and it is called an* ***epimorphism*** *if for every edge $\{x', y'\} \in E'$ there exists an edge $x, y \in E$ such that $\varphi(x) = x'$ and $\varphi(y) = y'$. If φ is bijective maps E bijectively to E', then φ is called an* ***isomorphism****, in which case Γ and Γ' are isomorphic as graphs, denoted by $\Gamma \cong \Gamma'$.*

A graph isomorphism from a graph $\Gamma = (V, E)$ to itself is called a *automorphism*, and the set of all automorphisms of Γ forms a group under ordinary function composition, denoted by $\mathrm{Aut}(\Gamma)$. In particular, $\mathrm{Aut}(\Gamma)$ is a subgroup of the *symmetric group* on V, consisting of all bijections $V \to V$, which are referred to as *permutations*, and denoted by $\mathrm{Sym}(V)$.

Example 4.1.4. Consider the graphs Γ and $\overline{\Gamma}$ from Example 4.1.2, which can both be seen in Figure 1. The map $\varphi : V \to V$ given by $\varphi(1) = 1$, $\varphi(2) = 5$, $\varphi(3) = 2$, $\varphi(4) = 3$ and $\varphi(5) = 4$ establishes a graph morphism from Γ to $\overline{\Gamma}$ because it maps E to $\overline{E}$. In particular, φ is a graph isomorphism between Γ and $\overline{\Gamma}$, since E is mapped bijectively to $\overline{E}$. The map $\psi : V \to V$ given by $\psi(1) = 1$, $\psi(2) = 4$, $\psi(3) = 5$, $\psi(4) = 2$, $\psi(5) = 3$ is a bijective graph morphism from Γ to itself, hence an automorphism of Γ, and we have $\psi \in \mathrm{Aut}(\Gamma) \leq \mathrm{Sym}(V)$. In particular, as $\psi^2 = \mathrm{id}_V$, we deduce that ψ is a permutation on V of order 2.

We may consider graphs as specific types of point-line geometries, which are defined as follows. They lie at the core of this chapter as suggested by its title.

Definition 4.1.5 (Point-line geometry)**.** *Let $\mathcal{P}$ be a possibly infinite set and let $\mathcal{L}$ be a set of subsets of $\mathcal{P}$. If $|\ell| \geq 2$ for every $\ell \in \mathcal{L}$, then $\Gamma = (\mathcal{P}, \mathcal{L})$ is said to be a* ***point-line geometry****.*

Given a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$, the sets $\mathcal{P}$ and $\mathcal{L}$ are oftentimes referred to as the *points*, respectively *lines* of Γ. The point-line geometry Γ is called *thick* if $|\ell| \geq 3$ for all $\ell \in \mathcal{L}$ and *thin* otherwise. For any point $p \in \mathcal{P}$ and line $\ell \in \mathcal{L}$, we say that p *is contained in* ℓ or ℓ *contains* p if $p \in \ell$. Two distinct points $p, p' \in \mathcal{P}$ are said to be *collinear*, denoted by $p \perp p'$, if there exists a line $\ell \in \mathcal{L}$ such that $p, p' \in \ell$. Note that collinearity $\perp$ is a symmetric relation on $\mathcal{P}$, and we will adopt the convention that $p \perp p$ for all $p \in \mathcal{P}$, even though p is not necessarily on some line in $\mathcal{L}$. The set of points collinear to a given point $p \in \mathcal{P}$ is given by $p^\perp = \{p' \in \mathcal{P} \mid p \perp p'\}$. This concept generalises to subsets of points $\mathcal{S} \subseteq \mathcal{P}$; the set of points collinear to every point in $\mathcal{S}$ is given by $\mathcal{S}^\perp = \{p' \in \mathcal{P} \mid \forall p \in \mathcal{S} : p \perp p'\}$. The point-line geometry Γ is said to be a *(partial) linear space* if for all distinct points $p, p' \in \mathcal{P}$ there is (at most) one line $\ell \in \mathcal{L}$ such that $p, p' \in \ell$.

Example 4.1.6. Figure 2 below shows a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ with $\mathcal{P} = \{1,2,3,4,5,6,7\}$ and $\mathcal{L} = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,4,7\},\{3,5,6\}\}$, also knows as the *Fano plane*.

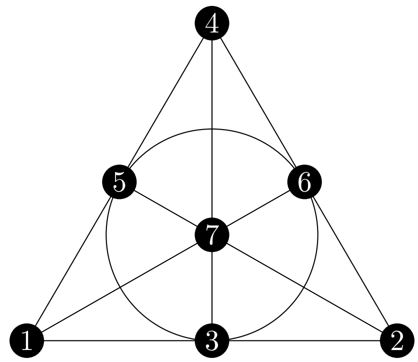


Figure 2: The Fano plane.

Observe that Γ is a thick point-line geometry, since $|\ell| = 3$ for every $\ell \in \mathcal{L}$. We have $\binom{|\mathcal{P}|}{2} = \binom{7}{2} = 21$ pairs of points, and 7 distinct lines in $\mathcal{L}$ each containing 3 such pairs, hence every pair of points uniquely determines a line in $\mathcal{L}$. Equivalently, every pair of points is on a unique line in $\mathcal{L}$, so Γ is a linear space.

A subset of points $\mathcal{S} \subseteq \mathcal{P}$ of a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ is called a *subspace* of Γ if $\ell \subseteq \mathcal{S}$ for every $\ell \in \mathcal{L}$ such that $|\mathcal{S} \cap \ell| \geq 2$. It is called *singular* if $\mathcal{S} \subseteq \mathcal{S}^{\perp}$, i.e. every point $p \in \mathcal{S}$ is collinear with all of $\mathcal{S}$. The *rank* of a singular subspace $\mathcal{S} \subseteq \mathcal{P}$, denoted by $\operatorname{rank}(\mathcal{S})$, is the largest integer $n \geq 0$ such that $\emptyset \subset \mathcal{S}_0 \subset \cdots \subset \mathcal{S}_n \subset \mathcal{S}$ is a chain of non-trivial singular subspaces $\mathcal{S}_i$, $0 \leq i \leq n$, of $\mathcal{S}$ with the convention that $\operatorname{rank}(\mathcal{S}) = \infty$ if no such integer $n \geq 0$ exists. The observation that the intersection of a collection of subspaces again yields a subspace gives rise to the notion of the subspace *generated* by a subset $\mathcal{S} \subseteq \mathcal{P}$, which is the smallest subspace of Γ containing $\mathcal{S}$, denoted by $\langle \mathcal{S} \rangle_\Gamma$ or just $\langle \mathcal{S} \rangle$ if Γ is clear from the context. The *generating rank* of Γ is the cardinality $|\mathcal{S}|$ of the smallest subset $\mathcal{S} \subseteq \mathcal{P}$ that generates $\mathcal{P}$. A *plane* of Γ is a subspace $\langle \ell, \ell' \rangle_\Gamma$ with $\ell, \ell' \in \mathcal{L}$ distinct such that $|\ell \cap \ell'| \geq 1$. A *hyperplane* of Γ is a proper subspace $\mathcal{S} \subset \mathcal{P}$ such that $\mathcal{S} \cap \ell$ is either a single point or ℓ itself for all $\ell \in \mathcal{L}$.

Denoting by E the set of subsets of $\mathcal{P}$ whose elements are collinear points in a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$, we obtain the graph $(\mathcal{P}, E)$ called the *collinearity graph* of Γ. Note that for any singular subspace $\mathcal{S} \subseteq \mathcal{P}$ of Γ the subgraph of the collinearity graph of Γ induced by $\mathcal{S}$ is a complete graph. The *incidence graph* of Γ, on the other hand, is the graph having $\mathcal{P} \sqcup \mathcal{L}$ as its vertex set and in which adjacency is defined by containment. Consequently, the incidence graph of Γ is an undirected bipartite graph with parts $\mathcal{P}$ and $\mathcal{L}$.

Example 4.1.7. Consider again the point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ from Example 4.1.6. The subsets $\{p\}$ and ℓ with $p \in \mathcal{P}$ and $\ell \in \mathcal{L}$ are subspaces of Γ. In particular, they are both singular subspaces of Γ as $\{p\} \subset p^\perp$ and $\ell \subset \ell^\perp$. Any point $p \in \mathcal{P}$ generates the subspace $\langle\{p\}\rangle_\Gamma = \{p\}$, whereas $\langle\{p, p'\}\rangle_\Gamma = \ell$ for any other point $p' \in \mathcal{P}$ with $\ell \in \mathcal{L}$ being the line that contains both p and p'. Adding a third point $p'' \in \mathcal{P}$ not collinear with both p and p' yields $\langle\{p, p', p''\}\rangle_\Gamma = \mathcal{P}$, showing that Γ has generating rank 3. The only hyperplanes of Γ are its lines, and Γ does not contain any proper planes.

Since $p^\perp = \mathcal{P}$ for all $p \in \mathcal{P}$, the collinearity graph of Γ will be a complete graph on $|\mathcal{P}| = 7$ vertices and containing $|E| = \binom{|\mathcal{P}|}{2} = \binom{7}{2} = 21$ edges. On the other hand, as every point $p \in \mathcal{P}$ is on exactly 3 lines and every line $\ell \in \mathcal{L}$ contains exactly 3 points, the incidence graph of Γ will be a 3-regular bipartite graph on $|\mathcal{P} \sqcup \mathcal{L}| = 14$ vertices also containing $|E| = 7 \cdot 3 = 21$ edges.

As a means of describing structure-preserving properties between point-line geometries as we did for graphs, we introduce geometry morphisms.

Definition 4.1.8 (Point-line geometry morphisms)**.** *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ and $\Gamma' = (\mathcal{P}', \mathcal{L}')$ be two point-line geometries. A **homomorphism** is a map $\varphi : \mathcal{P} \to \mathcal{P}'$ such that for all $\ell \in \mathcal{L}$ there exists an $\ell' \in \mathcal{L}'$ satisfying $\ell' = \{\varphi(p) \mid p \in \ell\}$. If Γ is a subspace of Γ', then φ is called a **monomorphism**, and it is called an **epimorphism** if for every line $\ell' \in \mathcal{L}$ there exists a line $\ell \in \mathcal{L}$ such that $\ell' = \{\varphi(p) \mid p \in \ell\}$. If φ is bijective and maps $\mathcal{L}$ bijectively to $\mathcal{L}'$, then φ is called an **isomorphism**, in which case Γ and Γ' are isomorphic as point-line geometries, denoted by $\Gamma \cong \Gamma'$.*

A point-line geometry isomorphism from a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ to itself is called an *automorphism*, and the set of all automorphisms of Γ forms a group under ordinary function composition, denoted by $\mathrm{Aut}(\Gamma)$. In particular, we have $\mathrm{Aut}(\Gamma) \leq \mathrm{Sym}(\mathcal{P})$.

Example 4.1.9. Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be the point-line geometry obtained from Figure 2 by removing point 7 and all lines containing point 7, i.e. $\mathcal{P} = \{1, 2, 3, 4, 5, 6\}$ and $\mathcal{L} = \{\{1, 2, 3\}, \{1, 4, 5\}, \{2, 4, 6\}, \{3, 5, 6\}\}$. Further let $\Gamma' = (\mathcal{P}', \mathcal{L}')$ be the point-line geometry obtained from Figure 2 as follows. We again remove point 7 so that $\mathcal{P}' = \mathcal{P}$, and take $\mathcal{L}' = \bigcup_{p \in \mathcal{P}'}\{q \in \mathcal{P}' \mid p \sim q\}$, where $\sim$ is adjacency in Figure 2 when viewed as a graph. Then $\mathcal{L}' = \{\{1, 3, 5\}, \{2, 3, 6\}, \{4, 5, 6\}, \{2, 3, 4, 5, 6\}, \{1, 3, 4, 5, 6\}, \{1, 2, 3, 5, 6\}\}$. The map $\phi : \mathcal{P}' \to \mathcal{P}$ given by $\phi(1) = 1$, $\phi(2) = 2$, $\phi(3) = 3$, $\phi(4) = 3$, $\phi(5) = 2$ and $\phi(6) = 1$ establishes a homomorphism between Γ and Γ' sending $\mathcal{L}'$ to $\mathcal{L}$. In particular, it is easily verified that all of $\mathcal{L}'$ is sent to $\{1, 2, 3\} \in \mathcal{L}$. This map is not injective and does not map $\mathcal{P}'$ and $\mathcal{L}'$ bijectively to $\mathcal{P}$, respectively $\mathcal{L}$, hence it is not an isomorphism. In particular, $\Gamma \ncong \Gamma'$. The map $\psi : \mathcal{P} \to \mathcal{P}$ given by $\psi(1) = 2$, $\psi(2) = 1$, $\psi(3) = 3$, $\psi(4) = 4$, $\psi(5) = 6$ and $\psi(6) = 5$ is a bijective homomorphism from Γ to itself, therefore $\psi \in \mathrm{Aut}(\Gamma) \leq \mathrm{Sym}(\mathcal{P})$. Specifically, ψ is a permutation on $\mathcal{P}$ of order 2.

Given a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ such that every point is contained in at least two lines, denote by $\mathcal{L}_p$ the set of lines containing point $p \in \mathcal{P}$, i.e. $\mathcal{L}_p = \{\ell \in \mathcal{L} \mid p \in \ell\}$, and write $\mathcal{P}^* = \bigcup_{p\in\mathcal{P}} \mathcal{L}_p$. The pair $(\mathcal{L}, \mathcal{P}^*)$ is called the *dual* of Γ, denoted by Γ^*, and is a point-line geometry by Definition 4.1.5 as $|\mathcal{L}_p| \geq 2$ for all $p \in \mathcal{P}$ by the above. The following then holds.

Proposition 4.1.10. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry such that every point is contained in at least two lines. Then $\Gamma \cong (\Gamma^*)^*$.*

Proof. We have $\Gamma^* = (\mathcal{L}, \mathcal{P}^*)$ with $P^* = \bigcup_{p\in\mathcal{P}} \mathcal{L}_p$, hence $(\Gamma^*)^*$ will be the point-line geometry having point set $\mathcal{P}^*$ and line set $\mathcal{L}^* := \bigcup_{\ell\in\mathcal{L}} \{\mathcal{L}_p \in \mathcal{P}^* \mid \ell \in \mathcal{L}_p\}$. We now claim that the map $\varphi : \Gamma \to (\Gamma^*)^*$ given by $\varphi(p) = \mathcal{L}_p$ establishes an isomorphism of point-line geometries. It is clearly well-defined and surjective on $(\Gamma^*)^*$, hence bijective on the point sets of Γ and $(\Gamma^*)^*$ as $|\mathcal{P}| = |\mathcal{P}^*|$. Furthermore, because $p \in \ell \iff \ell \in \mathcal{L}_p$ for all $p \in \mathcal{P}$ and $\ell \in \mathcal{L}$, we know that φ sends $\{p \in \mathcal{P} \mid p \in \ell\}$ to $\{\mathcal{L}_p \in \mathcal{P}^* \mid \ell \in \mathcal{L}_p\}$ for all $\ell \in \mathcal{L}$, which φ does bijectively in particular as φ is bijective on $\mathcal{P}$ and $\mathcal{P}^*$. But then φ maps $\mathcal{L}$ bijectively to $\mathcal{L}^*$, hence φ is an isomorphism of point-line geometries, and we conclude that $\Gamma \cong (\Gamma^*)^*$. □

Point-line geometries that will be of particular interest to us in later sections are polar spaces. They are defined as follows.

Definition 4.1.11 (Polar space)**.** *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry. If $p^\perp \cap \ell$ is either a single point or ℓ itself for all $p \in \mathcal{P}$ and $\ell \in \mathcal{L}$, then Γ is called a* ***polar space****.*

The *radical* of a polar space $\Gamma = (\mathcal{P}, \mathcal{L})$, denoted by $\mathrm{rad}(\Gamma)$, is the set $\mathcal{P}^\perp$. Note that the inclusion $\mathcal{P}^\perp \subseteq (\mathcal{P}^\perp)^\perp$ shows that $\mathrm{rad}(\Gamma)$ is a singular subspace of Γ. A polar space Γ is called *non-degenerate* if its radical is empty and *degenerate* otherwise. The *rank* of Γ, denoted by $\mathrm{rank}(\Gamma)$, is the largest integer $n \geq 0$ such that $\mathrm{rad}(\Gamma) = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_n$ is a chain of singular subspaces $\mathcal{S}_i$ of Γ, $0 \leq i \leq n$, with the convention that $\mathrm{rank}(\Gamma) = \infty$ if no such integer $n \geq 0$ exists. This concept generalises to partial linear spaces that are not necessarily polar spaces; if $\Gamma = (\mathcal{P}, \mathcal{L})$ is a partial linear space, its *singular rank* is the supremum of the ranks of its singular subspaces.

Given a subspace $\mathcal{S} \subseteq \mathcal{P}$ of a polar space $\Gamma = (\mathcal{P}, \mathcal{L})$, we call $\mathrm{rank}_{\mathrm{nd}}(\mathcal{S}) = \mathrm{rank}(\mathcal{S}) - \mathrm{rank}(\mathrm{rad}(\mathcal{S}))$ with $\mathrm{rad}(\mathcal{S}) = \mathcal{S} \cap \mathcal{S}^\perp$ the *non-degenerate rank* of $\mathcal{S}$. It coincides with the rank of $\mathcal{S}$ if $\mathcal{S}$ is non-degenerate and it is zero if $\mathcal{S}$ is a singular subspace of Γ. Notice that $p \in \mathrm{rad}(\mathcal{S})$ if and only if $p \in \mathcal{S}$ is a point such that $p \in \mathcal{S}^\perp \iff \mathcal{S} \subseteq p^\perp$. In other words, $\mathcal{S}$ is degenerate if and only if there exists a point $p \in \mathcal{S}$ such that $\mathcal{S} \subseteq p^\perp$.

Example 4.1.12. The point-line geometry depicted in Example 4.1.6 is a polar space. Clearly, $p^\perp = \mathcal{P}$ for every $p \in \mathcal{P}$, hence $\mathcal{P}^\perp = \mathcal{P}$ so that Γ is a degenerate polar space.

A generalised quadrangle, denoted by GQ, is a partial linear space $(\mathcal{P}, \mathcal{L})$ such that for all $p \in \mathcal{P}$ and $\ell \in \mathcal{L}$ we have $|\mathcal{L}_p| = t + 1$ and $|\ell| = s + 1$ for some fixed integers $s, t \geq 1$, and with the property that for all $\ell \in \mathcal{L}$ and $p \in \mathcal{P}$ with $p \notin \ell$ there exists a unique line $\ell \neq \ell' \in \mathcal{L}$ and a unique point $p \neq p' \in \mathcal{P}$ such that $p \in \ell'$ and $\ell \cap \ell' = \{p'\}$. A generalised quadrangle $GQ = (\mathcal{P}, \mathcal{L})$ is a polar space; indeed, for every $p \in \mathcal{P}$ and $\ell \in \mathcal{L}$, we have either $p \in \ell$, in which case $p^\perp \cap \ell = \ell$, or $p \notin \ell$, in which case there exists a unique line $\ell \neq \ell' \in \mathcal{L}$ and a unique point $p \neq p' \in \mathcal{P}$ such that $p \in \ell'$ and $\ell \cap \ell' = \{p'\}$, implying that $p^\perp \cap \ell = \ell \cap \ell' = \{p'\}$. Clearly, GQ is non-degenerate, since no point in $\mathcal{P}$ is collinear with every point of GQ. For every $p \in \mathcal{P}$ and $\ell \in \mathcal{L}$, the subsets $\{p\}$ and ℓ of $\mathcal{P}$ are singular subspaces of GQ. In particular, if $p \in \ell$ then the chain of singular subspaces $\mathrm{rad}(GQ) = \emptyset \subset \{p\} \subset \ell$ shows that GQ has rank 2. On the other hand, the line pencil $\mathcal{S}_p = p^\perp$ of any point $p \in \mathcal{P}$ is a hyperplane of GQ, which follows immediately from GQ being a polar space. It is easy to see that $\mathrm{rad}(\mathcal{S}_p) = \bigcap_{\ell \in \mathcal{S}_p} \ell = \{p\}$, hence $\mathrm{rank}(\mathcal{S}_p) = 1$ and $\mathrm{rank}(\mathrm{rad}(\mathcal{S}_p)) = \mathrm{rank}(\{p\}) = 0$ so that $\mathrm{rank}_{\mathrm{nd}}(\mathcal{S}_p) = 1$. The dual $GQ^* = (\mathcal{L}, \mathcal{P}^*)$ of GQ will again be a generalised quadrangle.

We follow up with some properties of non-degenerate polar spaces in particular, exhibited by the following lemma and proposition.

Lemma 4.1.13. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a thick polar space. Then Γ is non-degenerate if and only if $p^\perp \subseteq q^\perp \implies p = q$ for all $p, q \in \mathcal{P}$.*

Proof. Suppose first that Γ is non-degenerate and let $p, q \in \mathcal{P}$ be distinct points such that $p^\perp \subseteq q^\perp$. By non-degeneracy of Γ there exists a point $q' \in \mathcal{P}$ such that $q' \not\perp q$, which moreover satisfies $q' \not\perp p$ as otherwise $q' \in p^\perp \subseteq q^\perp$. Then q' will be collinear with some point r on some line $\ell_{pq} \in \mathcal{L}$ containing p and q. Again by non-degeneracy of Γ, there exists a point $p' \in \mathcal{P}$ not collinear to r. It satisfies $p' \not\perp p$ as otherwise $p' \in p^\perp \subseteq q^\perp$ so that $p' \perp q$, which would force $p' \perp r$. In turn, p' will be collinear to some point $s \in \mathcal{P}$ on some line $\ell_{q'r} \in \mathcal{L}$ containing q' and r. If $s = q'$, then p will be collinear to some point $t \in \mathcal{P}$ on some line $\ell_{p'q'} \in \mathcal{L}$ containing p' and q' different from p' and q', and this point will be collinear to q because $t \in p^\perp \subseteq q^\perp$, forcing $t \perp r$. But then r is collinear to both q' and t, hence also with p', a contradiction. Consequently, $s \neq q'$ and p will be collinear to a point $t \in \mathcal{L}$ different from p' but also from s as otherwise $p \perp r$ would imply $p \perp q'$. As before, this point t will be collinear with both p and q, hence also with r, but then r will be collinear with both s and t, therefore also with p', another contradiction. We conclude that $p = q$.

Assume next that $p^\perp \subseteq q^\perp \implies p = q$ for all $p, q \in \mathcal{P}$ and let $r \in \mathrm{rad}(\Gamma) = \mathcal{P}^\perp$. Then $r^\perp = \mathcal{P}$, hence for all points $s \in \mathcal{P}$ different from r we have $s^\perp \subseteq \mathcal{P} = r^\perp \implies s = r$, a contradiction. This forces $|\mathcal{P}| = 1$, contradicting that Γ is thick, so $\mathrm{rad}(\Gamma) = \emptyset$ and we conclude that Γ is non-degenerate. □

Proposition 4.1.14. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a non-degenerate thick polar space. Then*

(i) *for all non-collinear points $p, q \in \mathcal{P}$ the subset $p^\perp \cap q^\perp \subseteq \mathcal{P}$ is a non-degenerate subspace of Γ,*

(ii) *Γ is a partial linear space,*

(iii) *for all collinear points* $p, q \in \mathcal{P}$ *we have that* $(p^\perp \cap q^\perp)^\perp$ *is the unique line containing both* p *and* q*.*

Proof. For (i), let $p, q \in \mathcal{P}$ such that $p \not\perp q$ and write $\mathcal{S} := p^\perp \cap q^\perp \subseteq \mathcal{P}$. We first show that $\mathcal{S}$ is a subspace of Γ, so let $\ell \in \mathcal{L}$ be a line such that $|\mathcal{S} \cap \ell| \geq 2$. Then both p and q are collinear to two distinct points on ℓ, hence they are both collinear to all points on ℓ, which shows that $\ell \subseteq p^\perp \cap q^\perp = \mathcal{S}$. To show that $\mathcal{S}$ is non-degenerate, suppose towards a contradiction that it is degenerate. Then $\mathrm{rad}(\mathcal{S}) = \mathcal{S} \cap \mathcal{S}^\perp \neq \emptyset$, so there exists a point $r \in \mathcal{S}$ such that every point in $\mathcal{S}$ is collinear with r, which is moreover different from p and q as otherwise $p \perp q$. By non-degeneracy of Γ, there exists a point $s \in \mathcal{P}$ such that $s \perp p$ and $s \not\perp r$; indeed, if $s \perp r$ for all points $s \in \mathcal{P}$ such that $s \perp p$, then $p^\perp \subseteq r^\perp \implies p = r$ by Lemma 4.1.13, a contradiction. Now q is collinear with a point $t \in \mathcal{P}$ on some line $\ell_{ps} \in \mathcal{L}$ containing p and s different from p because $p \not\perp q$, and it is also different from s as otherwise $s \in p^\perp \cap q^\perp = \mathcal{S}$ so that $s \perp r$, a contradiction. However, now $t \in p^\perp \cap q^\perp = \mathcal{S}$, hence $t \perp r$, which forces $s \perp r$ because $p \perp r$, another contradiction. We conclude that $\mathcal{S}$ is non-degenerate.

For (ii), assume towards a contradiction that $p, q \in \mathcal{P}$ are two distinct points both contained in two distinct lines $\ell_{pq}, \ell'_{pq} \in \mathcal{L}$. Let p' be a point on line ℓ_{pq} different from p and q, then by non-degeneracy of Γ there exists a point $r \in \mathcal{P}$ such that $r \perp p'$ and $r \not\perp p$ as otherwise $p'^\perp \subseteq p^\perp \implies p' = p$ by Lemma 4.1.13, a contradiction. Moreover, this point must be collinear to some point $q' \in \mathcal{P}$ on line ℓ'_{pq} different from p; we have $r \not\perp p$ by construction, and if $r \perp q$ then $r \perp p'$ would imply $r \perp p$, a contradiction. Now write $\mathcal{S} := p^\perp \cap r^\perp \subseteq \mathcal{P}$, which is a non-degenerate subspace of Γ by Proposition 4.1.14(i), and let $s \in p'^\perp \cap \mathcal{S}$. Then $s \perp p'$ and $s \perp p$ together imply $s \perp q$, hence $s \perp q'$ so that $p'^\perp \cap \mathcal{S} \subseteq q'^\perp$. Similarly, for any $s \in q'^\perp \cap \mathcal{S}$ we have $s \perp q'$ and $s \perp p$ so that $s \perp q$, therefore $s \perp p'$ and so $q'^\perp \cap \mathcal{S} \subseteq p'^\perp$. But then $p'^\perp \cap \mathcal{S} \subseteq q'^\perp \cap \mathcal{S} \subseteq p'^\perp$ implies $\mathcal{S} \subseteq p'^\perp$, showing that $\mathcal{S}$ is a degenerate subspace of Γ, a contradiction. We conclude that Γ is a partial linear space.

For (iii), let $p, q \in \mathcal{P}$ be collinear and denote by $\ell_{pq} \in \mathcal{L}$ a line containing p and q, which is unique by Proposition 4.1.14(ii). First let $r \in \ell_{pq}$. Since every point in $p^\perp \cap q^\perp$ is collinear with every point on line ℓ_{pq}, in particular with r, we have $r \in (p^\perp \cap q^\perp)^\perp$ so that $\ell_{pq} \subseteq (p^\perp \cap q^\perp)^\perp$. Now let $r \in (p^\perp \cap q^\perp)^\perp$ such that $r \notin \ell_{pq}$, then $r \perp s$ for all $s \in p^\perp \cap q^\perp$, in particular for all $s \in \ell_{pq}$. By non-degeneracy of Γ, there exists a point $t \in \mathcal{P}$ such that $t \perp r$ and $t \not\perp q$ as otherwise $r^\perp \subseteq q^\perp \implies r = q$ by Lemma 4.1.13, contradicting $r \notin \ell_{pq}$. This results in $t \perp x$ for some point $x \in \ell_{pq}$. Since $t \not\perp r$, the subset $\mathcal{S} := q^\perp \cap t^\perp \subseteq \mathcal{P}$ is a non-degenerate subspace of Γ by Proposition 4.1.14(i). Now for all $y \in \mathcal{S}$ such that $y \perp x$, we have $y \perp p$ because $y \perp q$ as implied by $y \in \mathcal{S}$, hence $y \in p^\perp \cap q^\perp \subseteq r^\perp$, showing that $x^\perp \subseteq r^\perp$. But then by Lemma 4.1.13, applied to the non-degenerate thick polar space $\mathcal{S}$, we have $r = x \in \ell_{pq}$, so $(p^\perp \cap q^\perp)^\perp \subseteq \ell_{pq}$. We conclude that $(p^\perp \cap q^\perp)^\perp$ is the unique line containing both p and q. □

Let V be a possibly infinite-dimensional vector space over a division ring $\mathbb{K}$. The *projective space* on V, denoted by $\mathbb{P}(V)$ or $\mathrm{PG}(n,q)$, $n \geq 1$, if V is an $(n+1)$-dimensional vector space over $\mathbb{F}_q$, is the point-line geometry $(\mathcal{P},\mathcal{L})$ whose points are the 1-dimensional subspaces of V and whose lines are the subsets of points contained in some 2-dimensional subspace of V, two of which are said to *intersect* if their intersection is non-trivial. The points and lines of a projective space are oftentimes referred to as its *projective points* and *projective lines*. Note that projective spaces are linear spaces.

Example 4.1.15. Let $V = \mathbb{F}_q^{n+1}$ be the $(n+1)$-dimensional vector space over the finite field $\mathbb{F}_q$ with q some prime power. In this example, we will count the number of points and lines of the projective space $\mathbb{P}(V) = \mathrm{PG}(n,q)$, as well the number of points on a line and the number of lines containing a point. Since V contains $q^{n+1}-1$ non-zero vectors, each of which span a 1-dimensional subspace of V containing $q-1$ non-zero vectors, we have a total of $\frac{q^{n+1}-1}{q-1}$ points in $\mathrm{PG}(n,q)$. To construct a 2-dimensional $\langle u,v\rangle$ subspace of V, we may choose u to be any vector non-zero in V, of which there are $q^{n+1}-1$. Then v is allowed to be any non-zero vector that is not a non-zero scalar multiple of u, hence we have $(q^{n+1}-1)-(q-1) = q^{n+1}-q$ possibilities for v. It remains to count the number of pairs of non-zero vectors that span a given subspace $\langle u,v\rangle$. We may choose any non-zero vector in $\langle u,v\rangle$ as the first vector of such a pair, of which there are q^2-1. The second vector should not be a non-zero scalar multiple of the first vector, giving a total of $(q^2-1)-(q-1) = q^2-q$ vectors. We conclude that $\mathrm{PG}(n,q)$ contains $\frac{(q^{n+1}-1)(q^{n+1}-q)}{(q^2-1)(q^2-q)}$ lines. Analogous to how we determined the number of points of $\mathrm{PG}(n,q)$, we deduce that every line contains $\frac{q^2-1}{q-1} = q+1$ points. Finally, given a point $\langle u\rangle \in \mathcal{P}$, the orthogonal complement $\langle u\rangle^\perp$ of $\langle u\rangle$ is n-dimensional, so we may choose from q^n-1 non-zero vectors $v \in V$ that make $\langle u,v\rangle$ a 2-dimensional subspace of V. This results in $\frac{q^n-1}{q-1}$ such 2-dimensional subspaces, hence every point lies on $\frac{q^n-1}{q-1}$ lines.

A *triangle* in a point-line geometry $\Gamma = (\mathcal{P},\mathcal{L})$ is a triple of pair-wise intersecting lines in $\mathcal{L}$ in three distinct points in $\mathcal{P}$, which are referred to as its *corners*. We have the following lemma.

Lemma 4.1.16. *Let V be a possibly infinite-dimensional vector space over a division ring $\mathbb{K}$ and let $\mathbb{P}(V)$ be the projective space on V. Then any line in $\mathbb{P}(V)$ intersecting two lines of a triangle in two points different from its corners also intersects its third line in a point different from its corners.*

Proof. Let $u,v,w \in V$ such that the lines $\langle u,v\rangle$, $\langle v,w\rangle$ and $\langle u,w\rangle$ in $\mathbb{P}(V)$ form a triangle in $\mathbb{P}(V)$ with corners $\langle u\rangle$, $\langle v\rangle$ and $\langle w\rangle$, and let $x,y \in V$ such that the line $\langle x,y\rangle$ in $\mathbb{P}(V)$ intersects $\langle u,v\rangle$ and $\langle v,w\rangle$ in points different from $\langle u\rangle$, $\langle v\rangle$ and $\langle w\rangle$. Then there exist scalars $\lambda_{uv},\mu_{uv},\lambda_{xy},\mu_{xy} \in \mathbb{K}$ with λ_{uv} and μ_{uv} non-zero such that $\langle \lambda_{uv}u+\mu_{uv}v\rangle = \langle \lambda_{xy}x+\mu_{xy}y\rangle$, and similarly there exist scalars $\lambda_{vw},\mu_{vw},\lambda'_{xy},\mu'_{xy} \in \mathbb{K}$ with λ_{vw} and μ_{vw} non-zero such that $\langle \lambda_{vw} v + \mu_{vw} w \rangle = \langle \lambda'_{xy} x + \mu'_{xy} y \rangle$. But then

$$\begin{aligned}
\langle (\lambda_{vw}\lambda_{xy} - \lambda'_{xy}\mu_{uv})x + (\lambda_{vw}\mu_{xy} - \mu'_{xy}\mu_{uv})y \rangle &= \langle \lambda_{vw}(\lambda_{xy}x + \mu_{xy}y) - (\lambda'_{xy}x + \mu'_{xy}y)\mu_{uv} \rangle \\
&= \langle \lambda_{vw}(\lambda_{uv}u + \mu_{uv}v) - (\lambda_{vw}v + \mu_{vw}w)\mu_{uv} \rangle \\
&= \langle (\lambda_{vw}\lambda_{uv})u + (-\mu_{vw}\mu_{uv})w \rangle,
\end{aligned}$$

which shows that $\langle x, y \rangle$ intersects $\langle u, w \rangle$ in a point different from $\langle u \rangle$ and $\langle w \rangle$ because λ_{uv}, λ_{vw}, μ_{uv} and μ_{vw} are all non-zero. □

This lemma generalises to the *Veblen-Young axiom* in point-line geometries that are not necessarily projective spaces; in a point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$, if a line in $\mathcal{L}$ meets two of the three lines of a triangle in Γ each in a point different from its corners, then it will also meet the third line of the triangle in a point different from its corners.

Since the characterisation of lines via 2-dimensional subspaces is only possible in projective spaces, this axiom will therefore not be satisfied for all point-line geometries. If it is satisfied, however, together with several other conditions, we obtain a projective geometry, defined as follows.

Definition 4.1.17 (Projective geometry)**.** *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry. If Γ is a thick linear space and satisfies the Veblen-Young axiom, then Γ is called a* ***projective geometry****.*

To distinguish between point-line geometries and projective geometries, we will denote the latter by $\mathbb{P}$ instead of Γ. If any two lines in a projective geometry $\mathbb{P} = (\mathcal{P}, \mathcal{L})$ intersect and if it contains three distinct points all of which are not contained in a single line, then $\mathbb{P}$ is called a *projective plane*.

Example 4.1.18. Consider the point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ from Example 4.1.6. It is readily seen that Γ is a thick linear space. Moreover, it satisfies the Veblen-Young axiom; indeed, for any three distinct lines $\{p, x, q\}, \{p, y, r\}, \{q, z, r\} \in \mathcal{L}$ that form a triangle with corners $p, q, r \in \mathcal{P}$, the subset $\{x, y, z\} \subseteq \mathcal{P}$ is a line in Γ that intersects $\{q, z, r\}$ in z, which is not a corner of the triangle. So, Γ is a projective geometry $\mathbb{P}$. In particular, $\mathbb{P}$ is a projective plane as any two distinct lines intersect and since Γ clearly contains three distinct points not contained in the same line. Now let $V = \mathbb{F}_2^3$ and consider its projective space $\mathbb{P}(V) = \mathrm{PG}(2, 2)$. We know from Example 4.1.15 that it contains $\frac{2^3-1}{2-1} = 7$ points, each of them contained in $\frac{2^2-1}{2-1} = 3$ lines, and $\frac{(2^3-1)(2^3-2)}{(2^2-1)(2^2-2)} = 7$ lines, each of them containing $2 + 1 = 3$ points. It is easily verified that $\mathbb{P} \cong \mathrm{PG}(2, 2)$.

Given a projective plane $\Gamma = (\mathcal{P}, \mathcal{L})$, denote as before by $\mathcal{L}_p$ the set of lines containing point $p \in \mathcal{P}$. The *order* of Γ is the integer $n \geq 2$ such that $|\ell| = n + 1$ for all $\ell \in \mathcal{L}$ and $|\mathcal{L}_p| = n + 1$ for all $p \in \mathcal{P}$. For instance, the point-line geometry from Example 4.1.6, which we have seen is a projective plane in the above example, has order 2 and is in fact the smallest projective plane in terms of order. The following proposition shows that all thick projective planes have an order, and that under some conditions they are the only type of point-line geometries that do.

Proposition 4.1.19. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a finite thick linear space and denote by $\mathcal{L}_p$ the set of lines containing point $p \in \mathcal{P}$. Then Γ is a projective plane if and only if there exists an integer $n \geq 2$ such that $|\ell| = n+1$ for all $\ell \in \mathcal{L}$ and $|\mathcal{L}_p| = n+1$ for all $p \in \mathcal{P}$.*

Proof. First suppose that Γ is a projective plane. Then Γ contains three points not on the same line, so there exists a point $p \in \mathcal{P}$ and a line $\ell \in \mathcal{L}$ not containing p. As every line $\ell_p \in \mathcal{L}_p$ intersects ℓ in a different point, then by finiteness and thickness of Γ there exists an integer $n \geq 2$ such that $|\ell| = n+1$ and $|\mathcal{L}_p| = n+1$. In particular, for any point $p' \in \mathcal{P} \setminus \{p\}$ not on ℓ we have $|\mathcal{L}_{p'}| = n+1$. Now let $\ell_p, \ell_p' \in \mathcal{L}_p$, which exist because $|\mathcal{L}_p| = n+1 \geq 3$, and let $p' \in \mathcal{P} \setminus \{p\}$ be a point on ℓ_p' but not on ℓ nor on ℓ_p. Then by the above we have $|\mathcal{L}_{p'}| = n+1$, and because every line $\ell_{p'} \in \mathcal{L}_{p'}$ intersects ℓ_p in a different point we find $|\ell_p| = n+1$, which shows that every line in $\mathcal{L}_p$ also contains $n+1$ points. Consequently, every point on ℓ is contained in $n+1$ lines. But then every line in $\mathcal{L}$ contains $n+1$ points and every point in $\mathcal{P}$ is contained in $n+1$ lines.

Next assume that there exists an integer $n \geq 2$ such that $|\ell| = n+1$ for all $\ell \in \mathcal{L}$ and $|\mathcal{L}_p| = n+1$ for all $p \in \mathcal{P}$. Clearly, Γ contains three points not on the same line. We show that any two lines in $\mathcal{L}$ intersect. Suppose towards a contradiction that there exist two distinct lines $\ell, \ell' \in \mathcal{L}$ that do not intersect, and let $p \in \mathcal{P}$ be a point on ℓ but not on ℓ'. Then for every point $p' \in \mathcal{P}$ on ℓ' there will be a unique line containing both p and p' because Γ is a linear space. But p is contained in as many lines as there are points on ℓ', hence there must be a point on ℓ' that is also contained in ℓ, a contradiction. It follows that any two lines in $\mathcal{L}$ intersect. But then the Veblen-Young axiom is immediate; any line intersecting two lines of a triangle in two points different from its corners must intersect the third line of the triangle, but it cannot do so in its corners because Γ is a linear space, hence their point of intersection must be a point different from the corners of the triangle. We conclude that Γ is a projective plane by Definition 4.1.17. □

In light of our earlier discussion on non-degenerate thick polar spaces, we obtain the following as a corollary to Lemma 4.1.13 and Proposition 4.1.14.

Corollary 4.1.20. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a non-degenerate thick polar space and let $\mathcal{S} \subseteq \mathcal{P}$ be a singular subspace of Γ having line set $\mathcal{L}_\mathcal{S}$. Then $\mathcal{S}$ is a projective geometry.*

Proof. Thickness of $\mathcal{S}$ is inherited from Γ, and the fact that $p \perp q$ for all $p, q \in \mathcal{S}$, together with Proposition 4.1.14(ii), shows that $\mathcal{S}$ is a linear space, so it remains to show that $\mathcal{S}$ satisfies the Veblen-Young axiom according to Definition 4.1.17. To this extent, let $\ell_{pq}, \ell_{qr}, \ell_{pr} \subseteq \mathcal{S}$ be a triangle with corners $p, q, r \in \mathcal{S}$ and suppose that $\ell_{xy} \subseteq \mathcal{S}$ is a line that intersects ℓ_{pr} in a point $x \in \mathcal{S}$ and ℓ_{qr} in a point $y \in \mathcal{S}$ such that neither x nor y is a corner of the triangle. Now let $s \in \mathcal{P}$ be a point such that $s \perp p$ and $s \perp q$. Note that this point must satisfy $s \not\perp r$; indeed, if $s \perp r$ for all points $s \in \mathcal{P}$ such that $s \perp p$ and $s \perp q$, then $p^\perp \cap q^\perp \subseteq r^\perp \implies r \in (p^\perp \cap q^\perp)^\perp = \ell_{pq}$ by Proposition 4.1.14(iii), a contradiction. Consequently, we must have $s \not\perp x$ and $s \not\perp y$, hence there exists a unique point $z \in \ell_{xy}$ such that $s \perp z$. In particular, we have $z \in \mathcal{S}$ as $x, y \in \mathcal{S}$. Additionally, we deduce that $r \perp z$ because $r \perp x$ and $r \perp y$. But then p, q and z are contained in $\mathcal{S}' := r^\perp \cap s^\perp \subseteq \mathcal{P}$, which is a non-degenerate subspace of Γ by Proposition 4.1.14(i) since $r \not\perp s$. Now for any point $t \in \mathcal{S}'$ such that $t \in p^\perp \cap q^\perp$, we have $t \perp x$ because $t \perp p$ and $t \perp r$, but also $t \perp y$ because $t \perp r$ and $t \perp q$, together implying that $t \perp z$ so that $z \in (p^\perp \cap q^\perp)^\perp$. But then non-degeneracy of $\mathcal{S}$ implies that $z \in (p^\perp \cap q^\perp)^\perp = \ell_{pq}$ by Proposition 4.1.14(iii), hence $z \in \ell_{pq} \cap \ell_{xy}$. Since z is different from p, q and r, it follows that $\mathcal{S}$ satisfies the Veblen-Young axiom. We conclude that $\mathcal{S}$ is a projective space. □

A natural question that now arises is whether every projective geometry may be identified by a projective space on some vector space V over some division ring $\mathbb{K}$. We finish this section with a brief investigation of its answer, for which we require the following theorem.

Theorem 4.1.21 (Desargues' theorem)**.** *Let V be a 3-dimensional vector space over a division ring $\mathbb{K}$ and let $\mathbb{P}(V)$ be the projective space on V. Then for all $u, v, w \in V$ such that the lines $\langle u, v\rangle$, $\langle v, w\rangle$, $\langle u, w\rangle$ form a triangle with corners $\langle u\rangle$, $\langle v\rangle$ and $\langle w\rangle$ and any $p \in V$ such that $\langle p\rangle$ is not on the triangle, the points $\langle u, v\rangle \cap \langle y, z\rangle$, $\langle v, w\rangle \cap \langle x, z\rangle$ and $\langle u, w\rangle \cap \langle x, y\rangle$ with $x, y, z \in V$ such that $\langle x\rangle \subset \langle p, w\rangle$, $\langle y\rangle \subset \langle p, u\rangle$ and $\langle z\rangle \subset \langle p, v\rangle$ arbitrary are collinear.*

Proof. We may assume without loss of generality that $u = (1, 0, 0) \in V$, $v = (0, 1, 0) \in V$ and $w = (0, 0, 1) \in V$. Then $\langle u, v\rangle = \{(\lambda, \mu, 0) \in V \mid \lambda, \mu \in \mathbb{K}\}$, $\langle v, w\rangle = \{(0, \lambda, \mu) \in V \mid \lambda, \mu \in \mathbb{K}\}$ and $\langle u, w\rangle = \{(\lambda, 0, \mu) \in V \mid \lambda, \mu \in \mathbb{K}\}$, which together form a triangle with corners $\langle u\rangle$, $\langle v\rangle$ and $\langle w\rangle$. Again without loss of generality, we may take $p = (1, 1, 1) \in V$, whose span in V is readily seen not to be on the triangle. Consequently, we have $x = (1, 1, \gamma) \in V$ for some $\gamma \in \mathbb{K}$, $y = (\alpha, 1, 1)$ for some $\alpha \in \mathbb{K}$ and $z = (1, \beta, 1) \in V$ for some $\beta \in \mathbb{K}$. It follows that

$$\begin{aligned}
\langle x, y\rangle &= \{\lambda(1, 1, \gamma) + \mu(\alpha, 1, 1) \in V \mid \lambda, \mu \in \mathbb{K}\} = \{(\lambda + \mu\alpha, \lambda + \mu, \lambda\gamma + \mu) \in V \mid \lambda, \mu \in \mathbb{K}\},\\
\langle y, z\rangle &= \{\lambda(\alpha, 1, 1) + \mu(1, \beta, 1) \in V \mid \lambda, \mu \in \mathbb{K}\} = \{(\lambda\alpha + \mu, \lambda + \mu\beta, \lambda + \mu) \in V \mid \lambda, \mu \in \mathbb{K}\}\\
\langle x, z\rangle &= \{\lambda(1, 1, \gamma) + \mu(1, \beta, 1) \in V \mid \lambda, \mu \in \mathbb{K}\} = \{(\lambda + \mu, \lambda + \mu\beta, \lambda\gamma + \mu) \in V \mid \lambda, \mu \in \mathbb{K}\},
\end{aligned}$$

hence $\langle x, y\rangle$, $\langle y, z\rangle$ and $\langle x, z\rangle$ intersect $\langle u, w\rangle$, $\langle u, v\rangle$ and $\langle v, w\rangle$, respectively, if and only if $\lambda + \mu = 0$. We obtain the intersection points

$$\begin{aligned}
\langle u, w\rangle \cap \langle x, y\rangle &= \{\lambda(1, 1, \gamma) - \lambda(\alpha, 1, 1) \in V \mid \lambda \in \mathbb{K}\} = \{\lambda(1 - \alpha, 0, \gamma - 1) \in V \mid \lambda \in \mathbb{K}\},\\
\langle u, v\rangle \cap \langle y, z\rangle &= \{\lambda(\alpha, 1, 1) - \lambda(1, \beta, 1) \in V \mid \lambda \in \mathbb{K}\} = \{\lambda(\alpha - 1, 1 - \beta, 0) \in V \mid \lambda \in \mathbb{K}\}\\
\langle v, w\rangle \cap \langle x, z\rangle &= \{\lambda(1, 1, \gamma) - \lambda(1, \beta, 1) \in V \mid \lambda \in \mathbb{K}\} = \{\lambda(0, 1 - \beta, \gamma - 1) \in V \mid \lambda \in \mathbb{K}\},
\end{aligned}$$

and it is readily seen that $(0, 1-\alpha, \gamma-1)$, $(\alpha-1, 1-\beta, 0)$ and $(0, 1-\beta, \gamma-1)$ span a 2-dimensional subspace of V as $(1-\alpha, 0, \gamma-1) + (\alpha-1, 1-\beta, 0) = (0, 1-\beta, \gamma-1)$. We conclude that $\langle u, v\rangle \cap \langle y, z\rangle$, $\langle v, w\rangle \cap \langle x, z\rangle$ and $\langle u, w\rangle \cap \langle x, y\rangle$ are collinear in $\mathbb{P}(V)$. $\square$

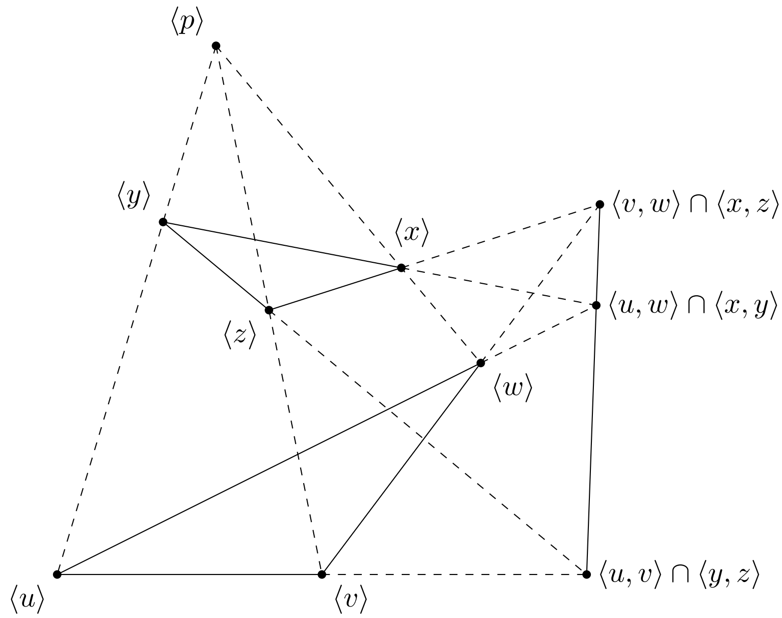


Figure 3: Desargues' configuration.

Figure 3 above provides a pictorial description of the setting in Theorem 4.1.21. Such a setting is also referred to as *Desargues' configuration*.

A projective plane that satisfies Desargues' theorem is called a *Desarguesian plane*. We can now formulate one of the main results on projective spaces that answers our previously posed question and incorporates the above theorem.

Theorem 4.1.22. *Let $\mathbb{P} = (\mathcal{P}, \mathcal{L})$ be a projective geometry containing three non-collinear points. If $\mathbb{P}$ satisfies Desargues' theorem, then $\mathbb{P}$ is isomorphic to the projective space $\mathbb{P}(V)$ on some vector space V over a division ring $\mathbb{K}$.*

Proof. See Theorem 6.2.11 in [3]. $\square$

4.2 Geometries and chamber systems

Recall from the previous section our discussion of graphs and point-line geometries. They can be generalised to *incidence structures*, abstractly defined as triples (P, L, I) in which P and L are sets and $I \subseteq P \times L$ is a binary relation on P and L that describes their mutual structure. Indeed, a graph $\Gamma = (V, E)$ is an incidence structure (V, V, E). Closely related to incidence structures are incidence systems, which are defined as follows.

Definition 4.2.1 (Incidence system)**.** *Let $\mathcal{I}$ be a type set whose element are called types and let X be a set. Further let $* \subseteq X \times X$ be a reflexive and symmetric relation on X and let $\tau : X \to \mathcal{I}$ be a surjective map such that $x * y \implies \tau(x) \neq \tau(y)$ for all distinct $x, y \in X$. Then $\Gamma = (X, *, \tau)$ is called an* ***incidence system*** *over $\mathcal{I}$.*

Given an incidence system $\Gamma = (X, *, \tau)$ over some type set $\mathcal{I}$, we call $*$ the *incidence relation* on X and $\tau : X \to \mathcal{I}$ the *type map* of Γ. As was the case for adjacency in graphs, we will write $x * y$ to indicate incidence of $x, y \in X$. For every $i \in \mathcal{I}$, the elements in $\tau^{-1}(i) \subseteq X$ are called the *i-elements*. The *incidence graph* of Γ is the graph with vertex set X and adjacency defined by $*$, its reflexive property disregarded.

Example 4.2.2. Let $\mathcal{I}$ be a type set and let $\Gamma = (V, E)$ be an undirected multipartite graph such that $V = \bigsqcup_{i \in \mathcal{I}} V_i$ and containing loops from every vertex to itself. Further let $\tau : V \to \mathcal{I}$ be the map given by $\tau(V_i) = i$. Then the triple $(V, \sim, \tau)$ is an incidence structure. Indeed, adjacency $\sim$ in Γ is a reflexive and symmetric relation, and for all $x \in V_i$ and $y \in V_j$, $i, j \in \mathcal{I}$, we have $x \sim y$ only if $i \neq j$, from which it follows that $\tau(x) = i \neq j = \tau(y)$.

Conversely, if $(V, \sim, \tau)$ is an incidence system, define $V_i = \tau^{-1}(i)$ for all $i \in \mathcal{I}$. By setting $V = \bigcup_{i \in \mathcal{I}} V_i$, we obtain an undirected multipartite graph $(V, *)$ containing loops from every vertex to itself. Indeed, if $x \in V_i \cap V_j$ with $i, j \in \mathcal{I}$, then we have $\tau^{-1}(i) = x = \tau^{-1}(j)$ so that $i = j$, showing that $V = \bigsqcup_{i \in \mathcal{I}} V_i$. Moreover, if $x \sim y$ with $x \in V_i$ and $y \in V_j$, $i, j \in \mathcal{I}$, then $\tau(x) = i \neq j = \tau(y)$, hence no two distinct vertices in the same V_i, $i \in \mathcal{I}$, are adjacent.

The type set $\mathcal{I}$ over which an incidence system $\Gamma = (X, *, \tau)$ is defined is commonly referred to as the *type* of Γ. Its cardinality $|\mathcal{I}|$ is the *rank* of Γ, denoted by $\operatorname{rank}(\Gamma)$. The same notions are used for subsets $Y \subseteq X$; they are of type $\tau(Y)$ and have rank $|\tau(Y)|$, denoted by $\operatorname{rank}(Y)$. In addition, we have the notions of *cotype* and *corank*; they are the type, respectively rank of $X \setminus Y$. Whenever we write Y^*, we mean the set $\{x \in X \mid \forall y \in Y : x * y\}$. A definition that is of particular importance is the following.

Definition 4.2.3 (Flag & Chamber)**.** *Let $\Gamma = (X, *, \tau)$ be an incidence system over a type set $\mathcal{I}$. A subset $F \subseteq X$ is called a* ***flag*** *of Γ if $x * y$ for all $x, y \in F$. If moreover $\tau(F) = \mathcal{I}$, i.e. F is of type $\mathcal{I}$, then F is called a* ***chamber****.*

It is clear that any flag of an incidence system $\Gamma = (X, *, \tau)$ over a type set $\mathcal{I}$ can have at most one element of every type in $\mathcal{I}$ by definition of τ, and chambers of an incidence system are subsets $F \subseteq X$ of maximal rank since $\tau(F) = \mathcal{I}$. We further note that F is a flag of Γ if and only if the vertices in F form a clique in the incidence graph of Γ. Consequently, Γ contains a chamber if and only if a maximum clique in the incidence graph of Γ has size $|\mathcal{I}|$, and the number of chambers of Γ is the number of cliques in the incidence graph of Γ of size $|\mathcal{I}|$.

Flags of Γ that are not properly contained in any other flag of Γ are called *maximal* flags. Clearly, chambers are maximal flags, but maximal flags need not be chambers. However, if this is the case, then the incidence system Γ is called a *geometry*. Such a geometry Γ is *firm* or *thick* if every flag of Γ that is not a chamber is contained in at least two, respectively three chambers of Γ. Contrarily, it is *thin* if every flag of type $\mathcal{I} \setminus \{i\}$ for some $i \in \mathcal{I}$ is contained in exactly two chambers of Γ.

Example 4.2.4. Denote by $\mathbb{E}^n$ the n-dimensional Euclidean affine space with $n \geq 2$ finite and let $\mathcal{I} = \{0, \dots, n-1\}$ be a type set. Now consider a polytope in $\mathbb{E}^n$, which is the convex hull of a set of points in $\mathbb{E}^n$, i.e. the smallest convex set in $\mathbb{E}^n$ that contains it. Let X be the set of its i-faces, $i \in \mathcal{I}$. Define $* \subseteq X \times X$ as $x * y$ if and only if $x \subseteq y$ or $y \subseteq x$ for all $x, y \in X$, and define $\tau : X \to \mathcal{I}$ to be the map that sends an element in X to the dimension of the face it represents in the polytope. Then $\Gamma = (X, *, \tau)$ is an incidence system over $\mathcal{I}$ of rank n. In particular, Γ is a geometry over $\mathcal{I}$, which follows from the observation that any i-face of the polytope either contains an $(i-1)$-face, $2 \leq i \leq n-1$, or is contained in an $(i+1)$-face, $1 \leq i \leq n-2$. In other words, any flag of Γ can always be extended to a flag of type $\mathcal{I}$. The geometry Γ is not only firm, but also thin; if $F \subseteq X$ is a flag of type $\mathcal{I} \setminus \{i\}$ for some $i \in \mathcal{I}$, then there are exactly two i-faces in X incident with every element in F, both of which form a chamber together with F.

The *residue* of a flag $F \subseteq X$ of an incidence system $\Gamma = (X, *, \tau)$ over a type set $\mathcal{I}$ is the incidence system $\Gamma_F = (F^* \setminus F, *, \tau)$ over $I \setminus \tau(F)$ with $F^* = \{x \in X \mid \forall y \in F : x * y\}$. Residues have the following properties.

Lemma 4.2.5. *Let $\Gamma = (X, *, \tau)$ be an incidence system over a type set $\mathcal{I}$ and let $\Gamma_F = (F^* \setminus F, *, \tau)$ be the residue of a flag F of Γ. Then*

(i) *F' is a flag of Γ_F if and only if $F \cup F'$ is a flag of Γ,*

(ii) *$(\Gamma_F)_{F'} = \Gamma_{F \cup F'}$ for all flags F' of Γ_F,*

(iii) *Γ_F is a geometry over $\mathcal{I} \setminus \tau(F)$ if Γ is a geometry.*

Proof. For (i), suppose first that F' is a flag of Γ_F. Since $F' \subseteq F^* \setminus F \implies F \cup F' \subseteq F^* \implies F' \subseteq F^*$, we have $x * y$ for all $x \in F'$ and $y \in F$. But we also have $x * y$ for all $x, y \in F$ and $x, y \in F'$ because F and F' are flags, hence $x * y$ for all $x, y \in F \cup F'$, showing that $F \cup F'$ is a flag of Γ. Supposing next that $F \cup F'$ is a flag of Γ, we have $x * y$ for all $x, y \in F'$ so that F' will also be a flag of Γ, and $x * y$ for all $x \in F'$ and $y \in F$ so that $F' \subseteq F^*$. But then $F \cup F' \subseteq F^*$, hence $F' \subseteq F^* \setminus F$ and so F' will be a flag of Γ_F.

For (ii), we have the identities $(F^* \setminus F) \cap (F'^* \setminus F') = (F^* \cap F'^*) \setminus (F \cup F') = (F \cup F')^* \setminus (F \cup F')$ and $(\mathcal{I} \setminus \tau(F)) \setminus \tau(F') = \mathcal{I} \setminus (\tau(F) \cup \tau(F')) = \mathcal{I} \setminus \tau(F \cup F')$. It follows that $(\Gamma_F)_{F'} = \Gamma_{F \cup F'}$ by definition of a residue.

For (iii), let F' be a maximal flag of Γ_F. By Lemma 4.2.5(i), $F \cup F'$ will be a flag of Γ. In particular, because $(\Gamma_F)_{F'} = \Gamma_{F \cup F'}$ by Lemma 4.2.5(ii), it will be a maximal flag of Γ. But Γ is a geometry over $\mathcal{I}$, so $F \cup F'$ will be a chamber and hence of type $\mathcal{I}$. Consequently, F' will be a flag of Γ_F of type $\mathcal{I} \setminus \tau(F)$, showing that it is a chamber. We conclude that Γ_F is a geometry over $\mathcal{I} \setminus \tau(F)$. □

For any incidence system $\Gamma = (X, *, \tau)$ over a type set $\mathcal{I}$, let $X' \subseteq X$ and $\mathcal{I}' \subseteq \mathcal{I}$, and define $*' := *'\big|_{X' \times X'}$ and $\tau' := \tau|_{X'}$. The incidence system $\Gamma' = (X', *', \tau')$ over $\mathcal{I}'$ is called an *incidence subsystem* of Γ, which turns into a *subgeometry* of Γ if Γ' is a geometry. If moreover $\mathcal{I}'$ is chosen such that $\mathcal{I}' = \tau(X')$, then Γ' is called the *incidence subsystem induced by* $\mathcal{I}'$. Examples of incidence subsystems are residues of flags, whereas examples of subgeometries of a polytope in $\mathbb{E}^n$, $n \geq 4$, from Example 4.2.4 are polygons in $\mathbb{E}^2$ and polyhedra in $\mathbb{E}^3$.

A *(simple) chain* in an incidence system $\Gamma = (X, *, \tau)$ over a type set $\mathcal{I}$ is a sequence of (distinct) elements $x_1, \ldots, x_n \in X$, $n \geq 1$ finite, such that for every $1 \leq j \leq n-1$ we have $x_j * x_{j+1}$, and we denote it by γ. If the type of every element in γ is contained in some type subset $\mathcal{J} \subseteq \mathcal{I}$, we also refer to γ as a $\mathcal{J}$-chain. Its *length* is the number of elements it contains, and it is *closed* if $x_1 = x_n$. These concepts generalise to paths, their lengths and circuits, respectively, in its incidence graph. In other words, a (simple) chain in an incidence system Γ corresponds to a (simple) path in the incidence graph of $\mathcal{C}$. For any two elements $x, y \in X$, the *distance* between x and y is the length of a shortest chain starting at x and ending at y or vice versa, denoted by $d(x, y)$. The *diameter* of Γ is the longest shortest chain in $\mathcal{C}$, and Γ is said to be *connected* if its diameter is finite and non-zero. A definition related to connectedness of incidence systems that will be of importance to us later on is the following.

Definition 4.2.6 (Residual connectedness)**.** *Let $\Gamma = (X, *, \tau)$ be an incidence system over a type set $\mathcal{I}$. The incidence system Γ is said to be **residually connected** if the residue $\Gamma_F = (F^* \setminus F, *, \tau)$ of every flag $F \subseteq X$ such that* $\operatorname{rank}(\Gamma_F) \geq 2$ *is connected.*

By noting that the empty set is a flag of any incidence system $\Gamma = (X, *, \tau)$ over some type set $\mathcal{I}$, we deduce that residual connectedness of Γ and $|\mathcal{I}| \geq 2$ together imply that Γ itself is connected. For example, polytopes in $\mathbb{E}^n$, $n \geq 2$ finite, are clearly residually connected. A geometry Γ over a type set $\mathcal{I}$ will be called an *$\mathcal{I}$-geometry* if Γ is firm and residually connected. The following proposition exhibits some properties of residually connected geometries.

Proposition 4.2.7. *Let $\Gamma = (X, *, \tau)$ be a residually connected incidence system over a type set $\mathcal{I}$. Then*

(i) *there exists an $\{i, j\}$-chain between any two elements $x, y \in X$ for any two distinct types $i, j \in \mathcal{I}$ if $\mathcal{I}$ is finite,*

(ii) Γ *is a geometry if every flag* F *of* Γ *such that* $\operatorname{rank}(X \setminus F) = 1$ *is not maximal,*

(iii) *the residue* Γ_F *of every flag* F *of* Γ *is a residually connected geometry if* Γ *is a geometry.*

Proof. For (i), we proceed by induction on $\operatorname{rank}(\Gamma) = |\mathcal{I}| < \infty$, the case $\operatorname{rank}(\Gamma) = 2$ being immediate by connectedness of Γ. Assuming that there is an $\{i, j\}$-chain between any two elements $x, y \in X$ for all $i, j \in \mathcal{I}$ if $\operatorname{rank}(\Gamma) \leq r$ for some $r \geq 2$, consider the case in which $\operatorname{rank}(\Gamma) = r + 1$ and let $x, y \in \Gamma$ and $i, j \in \mathcal{I}$ be arbitrary. By connectedness of Γ, we can always find a chain γ consisting of the elements $x = x_0, \ldots, x_n = y \in X$ with $n \geq 1$ finite. Now suppose that $x_k \in \gamma$, $1 \leq k \leq n-1$, is an element such that $\tau(x_k) \notin \{i, j\}$, which exists as otherwise γ would be an $\{i, j\}$-chain. Since $\{x_k\}$ is a flag of Γ, we obtain the residue $\Gamma_{\{x_k\}}$ all of whose residues of rank at least two are connected by residual connectedness of Γ. But then $\Gamma_{\{x_k\}}$ is residually connected, hence the induction hypothesis applies to $\Gamma_{\{x_k\}}$. In particular, we can find an $\{i, j\}$-chain between x_{k-1} and x_{k+1} inside $\Gamma_{\{x_k\}}$. Repeating this process for every element in γ not having type i or j, the total number of which is finite and decreases after every step, shows that γ is an $\{i, j\}$-chain.

For (ii), let F be a maximal flag of Γ and suppose that $\tau(F) \neq \mathcal{I}$. In particular, we have $\tau(F) \subset \mathcal{I}$ so that $|\mathcal{I} \setminus \tau(F)| \geq 1$. Now for any maximal flag of Γ we have $\operatorname{rank}(X \setminus F) = |\mathcal{I} \setminus \tau(F)| \neq 1$, hence $|\mathcal{I} \setminus \tau(F)| \geq 2$. Consequently, we have $\operatorname{rank}(\Gamma_F) = |\mathcal{I} \setminus \tau(F)| \geq 2$, so the residue Γ_F will be connected. But then Γ_F cannot be empty and so $F^* \setminus F \neq \emptyset$. In turn, $F \cup \{x\}$ will be a flag of Γ that strictly contains F for every $x \in F^* \setminus F$, contradicting maximality of F. We conclude that $\tau(F) = \mathcal{I}$ so that F is a chamber of Γ, showing that Γ is a geometry.

For (iii), let Γ_F be the residue of a flag F of Γ. Since Γ is a geometry, we know that Γ_F will be a geometry over $\mathcal{I} \setminus \tau(F)$ by Lemma 4.2.5(iii). Now for any flag F' of Γ_F, in particular for those having rank at least two, we have $(\Gamma_F)_{F'} = \Gamma_{F \cup F'}$ by Lemma 4.2.5(ii), but $\Gamma_{F \cup F'}$ is connected by residual connectedness of Γ, hence so will $(\Gamma_F)_{F'}$ be. We conclude that Γ_F is residually connected. $\square$

Closely related to chambers of incidence systems, but not equivalent, are chamber systems, which are defined as follows.

Definition 4.2.8 (Chamber system)**.** *Let* $\mathcal{I}$ *be a type set whose elements are called types and let* C *be a set whose elements are referred to as chambers. Define* $\{\sim_i \mid i \in \mathcal{I}\}$ *to be a set of equivalence relations* $\sim_i$ *on* C *indexed by* $i \in \mathcal{I}$*. The pair* $(C, \{\sim_i \mid i \in \mathcal{I}\})$ *is called a* ***chamber system*** *over* $\mathcal{I}$*.*

Given a chamber system $\mathcal{C} = (C, \{\sim_i \mid i \in \mathcal{I}\})$ over a type set $\mathcal{I}$, the *graph of* $\mathcal{C}$ is the graph $(C, \sim)$, in which for all $x, y \in C$ we have $x \sim y$ if and only if $x \sim_i y$ for some $i \in \mathcal{I}$ The pairs $(C, \sim_i)$ with $i \in \mathcal{I}$ also define graphs on C in which two vertices $x, y \in C$ are *i-adjacent* if and only if $x \sim_i y$. In particular, $(C, \sim_i)$ will be a disjoint union of cliques by definition of $\sim_i$. The equivalence classes of $\sim_i$ are called *i-panels*. For any type subset $\mathcal{J} \subseteq \mathcal{I}$, two chambers $x, y \in C$ are said to be *$\mathcal{J}$-adjacent*, denoted by $x \sim_{\mathcal{J}} y$, if there exists a $j \in \mathcal{J}$ such that $x \sim_j y$.

Analogous to incidence systems, the *rank* of a chamber system $\mathcal{C}$ is the cardinality $|\mathcal{I}|$ of the type set $\mathcal{I}$ over which it is defined. The chamber system $\mathcal{C}$ is called *firm*, *thick* or thin if every i-panel of $\mathcal{C}$, $i \in \mathcal{I}$, has size at least two, at least three or exactly two. A *chamber subsystem* of $\mathcal{C}$ is a chamber system $\mathcal{C}' = (C', \{\sim'_{i'} | \ i' \in \mathcal{I}'\})$ with $\mathcal{I}' \subseteq \mathcal{I}$, $C' \subseteq C$ and $\sim'_i \subseteq \sim_i$ for all $i \in \mathcal{I}'$. It is called the *chamber subsystem induced by $\mathcal{I}'$* if we have $\sim'_i = \sim_i|_{C' \times C'}$ for $i \in \mathcal{I}'$ and $\sim'_i = \mathrm{id}_{C' \times C'}$ otherwise.

Example 4.2.9. Consider again a polytope in $\mathbb{E}^n$, $n \geq 2$ finite, as in Example 4.2.4, whose i-faces, $0 \leq i \leq n-1$, we have seen can be used to construct an incidence system Γ over the type set $\mathcal{I} = \{0, \ldots, n-1\}$. The chambers of Γ are the subsets of X consisting of mutually incident i-faces with every $i \in \mathcal{I}$ appearing exactly once. Denote this set of chambers by C, and define for every $i \in \mathcal{I}$ the binary relation $\sim_i \subseteq C \times C$ by $x \sim_i y$ if and only if x and y contain the same i-face, which is clearly an equivalence relation. Then $\mathcal{C} = (C, \{\sim_i | \ i \in \mathcal{I}\})$ is a chamber system. Since every equivalence relation $\sim_i$, $i \in \mathcal{I}$, partitions C into subsets all of whose elements share the same i-face, we deduce that $\mathcal{C}$ is thin if and only if $n = 2$, whereas it will always be thick if $n \geq 3$.

The equivalent of a (simple) chain γ in an incidence system $\Gamma = (X, *, \tau)$ over a type set $\mathcal{I}$ is a *(simple) gallery* in a chamber system $\mathcal{C} = (C, \{\sim_i | \ i \in \mathcal{I}\})$ over $\mathcal{I}$; it is a sequence of (distinct) chambers $x_1, \ldots, x_n \in C$, $n \geq 1$ finite, such that for every $1 \leq j \leq n-1$ we have $x_j \sim_i x_{j+1}$ for some $i \in \mathcal{I}$, and we also denote it by γ. Its *length* is the number of chambers it contains, and it is *closed* if $x_1 = x_n$. These concepts generalise to paths, their lengths and circuits in the graph of $\mathcal{C}$. In other words, a gallery in a chamber system $\mathcal{C}$ corresponds to a path in the graph of $\mathcal{C}$. For any two chambers $x, y \in C$, the *distance* between x and y is the length of a shortest gallery starting at x and ending at y or vice versa, denoted by $d(x, y)$. The *diameter* of $\mathcal{C}$ is the longest shortest gallery in $\mathcal{C}$, and $\mathcal{C}$ is said to be *connected* if its diameter is finite and non-zero. A maximally connected chamber subsystem of $\mathcal{C}$ induced by some type subset $\mathcal{J} \subseteq \mathcal{I}$ is called a *cell*. To emphasise its dependency on $\mathcal{J}$, a cell is also referred to as a *$\mathcal{J}$-cell*. If $\mathcal{J} = \mathcal{I} \setminus \{i\}$ for some $i \in \mathcal{I}$, we refer to a $\mathcal{J}$-cell as an *i-object* of $\mathcal{C}$, denoted by Z_i. If for all type subsets $\mathcal{J} \subseteq \mathcal{I}$ the intersection $\bigcap_{j \in \mathcal{J}} Z_j$ of every collection $\{Z_j \mid j \in \mathcal{J}\}$ of j-objects of $\mathcal{C}$, one for each $j \in \mathcal{J}$, such that any two of them have a non-empty intersection is an $(\mathcal{I} \setminus \mathcal{J})$-cell of $\mathcal{C}$, then $\mathcal{C}$ is said to be *residually connected*.

We provide a definition comparable to Definition 2.1.6, Definition 4.1.3 and Definition 4.1.8 as a means of relating chamber systems to one another.

Definition 4.2.10 (Chamber system morphisms)**.** *Let $\mathcal{C} = (C, \{\sim_i | \ i \in \mathcal{I}\})$ and $\mathcal{C}' = (C', \{\sim'_i | \ i \in \mathcal{I}\})$ be two chamber systems over a type set $\mathcal{I}$. A* ***weak homomorphism*** *between $\mathcal{C}$ and $\mathcal{C}'$ is a map $\varphi : C \to C'$ such that for all $x, y \in C$ and $i \in \mathcal{I}$ we have*

$x \sim_i y$ implies $\varphi(x) \sim_{\sigma(i)} \varphi(y)$ for some permutation $\sigma : \mathcal{I} \to \mathcal{I}$. If moreover φ is bijective and its inverse φ^{-1} is a weak homomorphism, then φ is called a ***correlation****. If $\sigma = id_{\mathcal{I}}$, then φ is called a* ***homomorphism****, and if additionally φ is a correlation then φ is an* ***isomorphism****, in which case $\mathcal{C}$ and $\mathcal{C}'$ are isomorphic as chamber systems, denoted by $\mathcal{C} \cong \mathcal{C}'$.*

Both the set of correlations and the set of isomorphisms from a chamber system $\mathcal{C} = (C, \{\sim_i | \ i \in \mathcal{I}\})$ over a type set $\mathcal{I}$ to itself, referred to as *auto-correlations* and *automorphisms*, respectively, form a group under ordinary function composition, denoted by $\mathrm{Cor}(\mathcal{C})$, respectively $\mathrm{Aut}(\mathcal{C})$. They are related in the sense that $\mathrm{Aut}(\mathcal{C}) \trianglelefteq \mathrm{Cor}(\mathcal{C})$.

Now recall from Example 4.2.9 that a chamber system $\mathcal{C} = (C, \{\sim_i | \ i \in \mathcal{I}\})$ over a type set $\mathcal{I}$ can be obtained from a geometry $\Gamma = (X, *, \tau)$ over $\mathcal{I}$. In general, this is done for an arbitrary incidence system $\Gamma = (X, *, \tau)$ by letting C be its set of chambers and by defining $\sim_i \subseteq C \times C$ as $x \sim_i y$ with $x, y \in C$ if and only if x and y contain the same j-element for every $j \in \mathcal{I} \setminus \{i\}$. Since the relations $\sim_i$, $i \in \mathcal{I}$, are clearly equivalence relations on C, we indeed obtain a chamber system by Definition 4.2.8. To emphasise the dependency of its construction on Γ, we denote the resulting chamber system by $\mathcal{C}(\Gamma)$, and it is firm, thick or thin if Γ is firm, thick or thin, respectively. In light of our discussion on residues of flags of incidence systems, we obtain the following result.

Corollary 4.2.11. *Let $\Gamma = (X, *, \tau)$ be a geometry over a type set $\mathcal{I}$ and let $\mathcal{C}(\Gamma)$ be the chamber system of Γ. Further let F be a flag of Γ and denote by $\mathcal{C}(\Gamma)_F$ the chamber system whose chambers are the chambers of Γ containing F and whose set of equivalence relations is that of $\mathcal{C}(\Gamma)$ restricted to $\mathcal{I} \setminus \tau(F)$. Then $\mathcal{C}(\Gamma_F) \cong \mathcal{C}(\Gamma)_F$.*

Proof. We claim that the map $\varphi : \mathcal{C}(\Gamma_F) \to \mathcal{C}(\Gamma)_F$ given by $\varphi(F') = F \cup F'$ with F' a chamber in $\mathcal{C}(\Gamma_F)$ is an isomorphism of chamber systems. Note that φ indeed maps chambers in $\mathcal{C}(\Gamma_F)$ to chambers in $\mathcal{C}(\Gamma)_F$ by Lemma 4.2.5(i). It is clearly well-defined and injective, and surjectivity follows from the observation that x is a chamber in $\mathcal{C}(\Gamma)_F$ if and only if $x \setminus F$ is a chamber in $\mathcal{C}(\Gamma_F)$, again by Lemma 4.2.5(i), so that $\varphi(x \setminus F) = x$ for all chambers x in $\mathcal{C}(\Gamma)_F$. Now letting F' and F'' be chambers in $\mathcal{C}(\Gamma_F)$ and supposing that $F' \sim_i F''$ for some $i \in \mathcal{I} \setminus \tau(F)$, the identity $F \sim_i F$ immediately shows that $\varphi(F') = (F \cup F') \sim_i (F \cup F'') = \varphi(F'')$. In particular, by Definition 4.2.10, the map φ will be an isomorphism of chamber systems, and we conclude that $\mathcal{C}(\Gamma_F) \cong \mathcal{C}(\Gamma)_F$. □

Before we are able to define buildings, we require particular type sets over which chamber systems are defined. We introduce these type sets by means of Coxeter groups, which are defined as follows.

Definition 4.2.12 (Coxeter group)**.** *Let $\mathcal{I}$ be a type set and let $M = (m_{i,j})_{i,j \in \mathcal{I}}$ be a symmetric matrix with entries over $\mathbb{N} \cup \{\infty\}$ such that $m_{i,i} = 1$ and $m_{i,j} \geq 2$ for all distinct $i, j \in \mathcal{I}$. The* ***Coxeter group*** *of type M is the group $W(M)$ generated by the elements of a set $R = \{r_i \mid i \in \mathcal{I}\}$ satisfying $(r_i r_j)^{m_{i,j}} = 1$ for all $i, j \in \mathcal{I}$.*

The type M of a Coxeter group $W(M)$ is also known as the *Coxeter matrix* or *Coxeter type*, not only applicable to Coxeter groups but also to incidence systems and chamber systems. Together with the ambient set of generators R over which a Coxeter group is defined, the pair $(W(M), R)$ is called a *Coxeter system* of type M, its *rank* being the cardinality $|R|$ of R. The *Coxeter graph* of $W(M)$ is the weighted graph with vertex set R in which two vertices $r_i, r_j \in R$ are connected by an edge of weight $m_{i,j}$ if and only if $m_{i,j} > 2$, and $(W(M), R)$ is called an *irreducible* Coxeter system if and only if its Coxeter graph is connected.

For any type subset $\mathcal{J} \subseteq \mathcal{I}$, the subgroup $W_{\mathcal{J}}(M_{\mathcal{J}})$ of $W(M)$ generated by the elements of $R_{\mathcal{J}} = \{r_j \mid j \in \mathcal{J}\}$ satisfying $(r_i r_j)^{m_{i,j}}$ for all $i, j \in \mathcal{J}$ with $M_{\mathcal{J}}$ being the submatrix of M only containing the rows and column of M indexed by $\mathcal{J}$ is again a Coxeter group whose Coxeter graph is the Coxeter graph of $W(M)$ induced by $R_{\mathcal{J}}$. We obtain a Coxeter system $(W_{\mathcal{J}}(M_{\mathcal{J}}), R_{\mathcal{J}})$ of type $M_{\mathcal{J}}$.

Example 4.2.13. The dihedral group $\mathcal{D}_n$, $n \geq 3$ finite, is the group of order $2n$ consisting of all rotations and reflections on a regular n-gon in $\mathbb{E}^2$ and generated by a reflection and rotation. Now Let $\mathcal{I} = \{1, 2\}$ be a type set and let $R = \{r_1, r_2\}$ be a set of two reflections of a regular n-gon such that their product $r_1 r_2$ yields a rotation of a regular n-gon of order n. Since both r_1 and r_2 are reflections, we clearly have $r_1^2 = r_2^2 = 1$. Their composition $r_1 r_2$ has order n by construction so that $(r_1 r_2)^n = 1$, which additionally implies that $(r_2 r_1)^n = ((r_1 r_2)^{-1})^n = ((r_1 r_2)^n)^{-1} = 1^{-1} = 1$. Then $\mathcal{D}_n$ is a Coxeter group of Coxeter type $M = \left(\begin{smallmatrix} 1 & n \\ n & 1 \end{smallmatrix}\right)$, its Coxeter graph being a single edge with weight n.

A *diagram* over a type set $\mathcal{I}$ is a map D that assigns to every unordered pair $\{i, j\} \subseteq \mathcal{I}$ a class $D(i, j)$ of rank two geometries over $\{i, j\}$. Such a diagram is called a *Coxeter diagram* if there exists a Coxeter matrix $M = (m_{i,j})_{i,j \in \mathcal{I}}$ such that every class $D(i, j)$, $i, j \in \mathcal{I}$, is the class of generalised $m_{i,j}$-gons. Coxeter diagrams can be viewed as weighted graphs with vertex set $\mathcal{I}$, in which two vertices $i, j \in \mathcal{I}$ are connected by an edge of weight $m_{i,j}$ if and only if $D(i, j)$ is the class of generalised $m_{i,j}$-gons. Since clearly $m_{i,j} > 2$ in Coxeter diagrams, they correspond naturally to Coxeter graphs.

The above paragraph, in combination with the construction of chambers systems obtained from geometries, now enables us to establish a connection between chamber systems and Coxeter systems, as characterised by the following proposition.

Proposition 4.2.14. *Let Γ be a geometry over a type set $\mathcal{I}$ and let $M = (m_{i,j})_{i,j \in \mathcal{I}}$ be a Coxeter matrix. If Γ is a thin geometry of Coxeter type M, then the group generated by the set of permutations on the set of chambers of Γ sending a chamber to its unique i-adjacent chamber for every $i \in \mathcal{I}$ is a Coxeter group.*

Proof. Let $\mathcal{C}(\Gamma)$ be the chamber system of Γ. Since Γ is thin, so will $\mathcal{C}(\Gamma)$ be, meaning that every i-panel of $\mathcal{C}(\Gamma)$ has size exactly two for every $i \in \mathcal{I}$. By definition, every equivalence class of $\sim_i$, $i \in \mathcal{I}$, will then have size two so that every chamber in C is i-adjacent to a unique chamber. Now let $\{\sigma_i \mid i \in \mathcal{I}\}$ be the set of permutations on C sending a chamber $x \in C$ to the unique chamber $y \in C$ satisfying $x \sim_i y$. Clearly, we have $\sigma_i^2 = 1$ for all $i \in \mathcal{I}$, since the unique chamber adjacent to the unique chamber adjacent to a chamber $x \in C$ is x itself, showing that $(\sigma_i\sigma_i)^{m_{i,i}} = \sigma_i^2 = 1$ for all $i \in \mathcal{I}$. In addition, for all $i, j \in \mathcal{I}$, any rank two subgeometry of Γ over $\{i, j\}$ will be a generalised $m_{i,j}$-gon because Γ has Coxeter type M, which shows that $\sigma_{ij} = \sigma_i\sigma_j$ has order $m_{i,j}$. Then $(\sigma_i\sigma_j)^{m_{i,j}} = 1$ for all $i, j \in \mathcal{I}$, and so the group generated by $\{\sigma_i \mid i \in \mathcal{I}\}$ is a Coxeter group by Definition 4.2.12. □

Our previous discussion of diagrams D and chamber systems $\mathcal{C} = (C, \{\sim_i \mid i \in \mathcal{I}\})$ obtained from geometries $\Gamma = (X, *, \tau)$ also gives rise to the following definition.

Definition 4.2.15 (Chamber system of type D)**.** *Let $\mathcal{C} = (C, \{\sim_i \mid i \in \mathcal{I}\})$ be a chamber system over a type set $\mathcal{I}$ and let D be a diagram over $\mathcal{I}$. If every $\{i, j\}$-cell is the chamber system of a residually connected geometry Γ belonging to $D(i, j)$ for every type subset $\{i, j\} \subseteq \mathcal{I}$, then $\mathcal{C}$ is said to be a* ***chamber system of type*** *D.*

A similar notion exists for geometries; a *geometry of type* D is a geometry $\Gamma = (X, *, \tau)$ over a type set $\mathcal{I}$ such that for all $\{i, j\} \subseteq \mathcal{I}$ the residue Γ_F of a flag flag F of Γ of type $\{i, j\}$ is isomorphic to a geometry in $D(i, j)$. A geometry Γ of type D with D a Coxeter diagram is also called a geometry of Coxeter type M.

If $\mathcal{C} = (C, \{\sim_i \mid i \in \mathcal{I}\})$ is a chamber system over a type set $\mathcal{I}$ of type D with D a Coxeter diagram, we say that $\mathcal{C}$ is a chamber system of type M. Note that a chamber system of type M will be firm as any generalised $m_{i,j}$-gon is firm. Related to a chamber system of type M is a *Coxeter chamber system* of type $\mathcal{I}$; given a Coxeter system $(W(M), R)$ of type M, it is the chamber system $(W(M), \{\sim_i \mid i \in \mathcal{I}\})$, in which $x \sim_i y$ with $x, y \in W(M)$ if and only if $xr_i = y$ with $r_i \in R$. The chambers $x, y \in W(M)$ are then said to be r_i*-adjacent*, and we denote a Coxeter chamber system by $\mathcal{C}(M)$.

Example 4.2.16. Let $n \geq 2$ be a finite integer and consider a regular n-simplex in $\mathbb{E}^{n+1}$. Since a regular n-simplex is a specific type of polytope in $\mathbb{E}^n$, it is a residually connected geometry Γ over the type set $\mathcal{I} = \{0, \ldots, n-1\}$. It is well-known that the group of symmetries of a regular n-simplex is isomorphic to the symmetric group on $n+1$ elements, which we will denote by Sym_{n+1}. Therefore, we may identify the chambers of a regular n-simplex by the permutations in Sym_{n+1}. Now let $\{\sigma_i \mid i \in \mathcal{I}\}$ be the set of transpositions $\sigma_i = (i+1 \quad i+2) \in \mathrm{Sym}_{n+1}$, $i \in \mathcal{I}$, which generate Sym_{n+1}. We then obtain the chamber system $\mathcal{C}(\Gamma) = (\mathrm{Sym}_{n+1}, \{\sim_i \mid i \in \mathcal{I}\})$, in which $\alpha \sim_i \beta$ with $\alpha, \beta \in \mathrm{Sym}_{n+1}$ if and only if $\alpha\sigma_i = \beta$. Since all σ_i, $i \in \mathcal{I}$, are transpositions, we clearly have $\sigma_i^2 = 1$ for all $i \in \mathcal{I}$. Furthermore, we have for all distinct $i, j \in \mathcal{I}$ that $\sigma_i\sigma_j = (i+1 \quad i+2)(j+1 \quad j+2)$. Assuming w.l.o.g. that $i < j$, we deduce that $\sigma_i\sigma_j$ is a product of disjoint transpositions having order 2 if $i+1 \neq j$, whereas $\sigma_i\sigma_{i+1} = (i+1 \quad i+2)(i+2 \quad i+3) = (i+1 \quad i+2 \quad i+3)$ having order 3, in which composition has been done from right to left. We conclude for all distinct $i, j \in \mathcal{I}$ that $(\sigma_i\sigma_j)^2 = (\sigma_j\sigma_i)^2 = 1$ if $i+1 \neq j$ and $(\sigma_i\sigma_j)^3 = (\sigma_j\sigma_i)^3 = 1$ if

$i+1=j$. This shows that $\mathcal{C}(\Gamma)$ is of Coxeter type M, with M being the symmetric matrix having ones on its diagonals, threes on its off-diagonals and twos everywhere else.

The Coxeter type of the Coxeter group from Example 4.2.13 is I_n, whereas the chamber system from Example 4.2.16 above is of type A_n. The Coxeter systems obtained from both examples will be irreducible. The Coxeter diagrams and their types corresponding to irreducible Coxeter systems can be seen in Table 1 below, its nodes labeled as in Bourbaki [4].

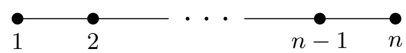
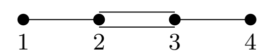
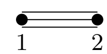
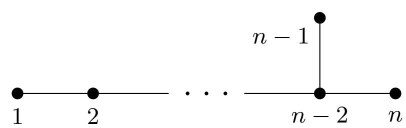
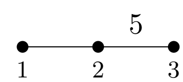
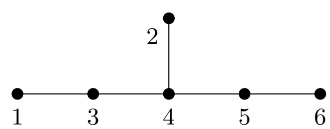
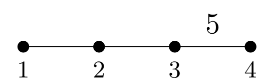
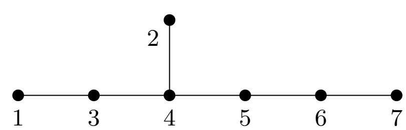
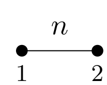
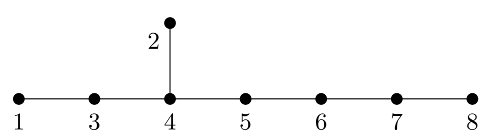
Type	Coxeter diagram	Type	Coxeter diagram
A_n		F_4	
BC_n		G_2	
D_n		H_3	
E_6		H_4	
E_7		I_n	
E_8			

Table 1: An overview of irreducible Coxeter diagrams and their types.

In the above table, the subscript of every type indicates the number of vertices its corresponding Coxeter diagram contains, except for I_n where it indicates the weight of its single edge. Single edges without a specified weight are edges having weight 3, whereas double edges have weight 4 and triple edges have weight 6. Edges having weight greater than 2 but not equal to 3, 4 or 6 have their weight displayed above them.

We return to our discussion of chamber systems $\mathcal{C} = (C, \{\sim_i | \ i \in \mathcal{I}\})$ of Coxeter type M over a type set $\mathcal{I}$. In particular, let $(W(M), R)$ be a Coxeter system of type M and denote by R^* the *free monoid* of R, its elements being all sequences $r_1 \cdots r_n$, $n \geq 0$ finite, of elements $r_1, \ldots, r_n \in R$. Given a simple gallery γ in $\mathcal{C}$ consisting of a sequence of distinct chambers $x_1, \ldots, x_n \in C$, $n \geq 3$ finite, the element $r = r_2 \cdots r_n \in R^*$ is said to be the *type* of γ if x_i is r_i-adjacent to x_{i-1} for every $2 \leq i \leq n$. The type of a simple gallery γ is called *minimal* if $r_2 \cdots r_n \in R^*$ is a *minimal expression* of r, i.e. the number of elements in any other expression of $r \in R^*$ contains at least as many elements as $r_2 \cdots r_n$. This finally enables us to define buildings.

Definition 4.2.17 (Building)**.** *Let $\mathcal{I}$ be a type set and let $\mathcal{C} = (C, \{\sim_i | \ i \in \mathcal{I}\})$ be a chamber system of Coxeter type $M = (m_{i,j})_{i,j \in \mathcal{I}}$. Then $\mathcal{C}$ is called a **building** of type M if every simple closed gallery of minimal type consists of a single chamber in C.*

A building $\mathcal{C}$ of type M is called *spherical* if the Coxeter group $W(M)$ of type M is finite. A building $\mathcal{C}$ is called *firm*, *thick* or *thin* if its underlying chamber system is firm, thick or thin, respectively. It is called *irreducible* if its Coxeter diagram is connected and *reducible* otherwise. Its *rank* is the cardinality $|\mathcal{I}|$ of the type set $\mathcal{I}$. An *apartment* of a building $\mathcal{C}$ of type M is a chamber subsystem of $\mathcal{C}$ isomorphic to the Coxeter chamber system $\mathcal{C}(M)$.

Example 4.2.18. Let $\mathcal{I}$ be a type set and let $\mathcal{C} = (W(M), \{\sim_i | \ i \in \mathcal{I}\})$ be a Coxeter chamber system of type $\mathcal{I}$ with $(W(M), R)$ a Coxeter system of type M. By definition, $\mathcal{C}$ is thin, implying that every chamber in $W(M)$ is i-adjacent to unique chamber in $W(M)$. Consequently, $\mathcal{C}$ cannot contain any simple closed galleries; any simple closed gallery γ consisting of distinct chambers $x_1, \ldots, x_n \in W(M)$, $n \geq 1$ finite, satisfies $x_n = x_1$ and is clearly trivial if $n \leq 2$, but also if $n \geq 3$, the case $n = 3$ being settled by minimality of the type of γ, whereas for $n \geq 4$ we have $x_2 = x_{n-1}$, contradicting simplicity of γ. It follows that any Coxeter chamber system is a thin building.

Recall from Example 4.2.16 the chamber system $\mathcal{C}(\Gamma) = (\mathrm{Sym}_{n+1}, \{\sim_i | \ i \in \mathcal{I}\})$ over type set $\mathcal{I} = \{0, \ldots, n-1\}$ of type A_n obtained from the geometry Γ of a regular n-simplex in $\mathbb{E}^{n+1}$ with Sym_{n+1} the symmetric group on $n+1$ elements, in which two chambers $\alpha, \beta \in \mathrm{Sym}_{n+1}$ are i-adjacent if and only if $\alpha\sigma_i = \beta$ with σ_i being the transposition $(i+1 \quad i+2) \in \mathrm{Sym}_{n+1}$. We have established in Example 4.2.4 that polytopes are thin geometries, hence the regular n-simplex will be a thin geometry. As a consequence, so will $\mathcal{C}(\Gamma)$ be, and we deduce that $\mathcal{C}(\Gamma)$ is a thin building using the same arguments as for Coxeter chamber systems in the previous paragraph. In particular, $\mathcal{C}(\Gamma)$ will be an irreducible thin building of type A_n having rank n.

We finish this section with the following classification of buildings $\mathcal{C}$ of type M under some conditions, which will prove to be useful to us in later chapters.

Theorem 4.2.19. *Let $\mathcal{I}$ be a type set and let $M = (m_{i,j})_{i,j\in\mathcal{I}}$ be a Coxeter matrix. If $\mathcal{C}$ is an irreducible spherical building of type M, then M occurs in* Table 1.

Proof. Since $\mathcal{C}$ is irreducible and spherical, its underlying Coxeter group will be irreducible and finite. The theorem then follows from Theorem 4.7.3 in [3]. □

4.3 Root systems

We continue our discussion of Coxeter systems and buildings in the setting of Theorem 4.2.19 from the previous section. On the one hand, we will use the geometric representation of Coxeter systems to elaborate on their classification, which we referred to in our proof of Theorem 4.2.19. On the other hand, we will introduce certain types of point-line geometries with which we can identify buildings.

Let $M = (m_{i,j})_{i,j\in\mathcal{I}}$ be a Coxeter matrix over a type set $\mathcal{I}$ and let $W(M)$ be the Coxeter group of Coxeter type M generated by the elements of a set $R = \{r_i \mid i \in \mathcal{I}\}$. A first step towards a geometric representation of Coxeter systems is the following abstract definition.

Definition 4.3.1 (Root system)**.** *Let V be a finite-dimensional real vector space endowed with a positive definite, symmetric bilinear form $\langle\cdot|\cdot\rangle : V \times V \to \mathbb{R}$. A subset $\Phi \subset V$ is called a* ***root system*** *of V if*

(i) $0_V \notin \Phi$ *and* $V = \mathrm{span}(\Phi)$,

(ii) *for all $\alpha \in \Phi$ we have $c\alpha \in \Phi$ if and only if $c = \pm 1$,*

(iii) *for all $\alpha \in \Phi$ we have $\sigma_\alpha(\Phi) = \Phi$, where $\sigma_\alpha : V \to V$ is the reflection on the hyperplane orthogonal to α given by*

$$\sigma_\alpha(\beta) = \beta - \frac{2\langle\alpha|\beta\rangle}{\langle\alpha|\alpha\rangle}\alpha$$

with $\beta \in V$.

The *rank* of a root system $\Phi \subset V$ is the dimension $\dim(V)$ of the ambient vector space V. A root system $\Phi \subset V$ is said to be *irreducible* if there do not exist $\Phi_1, \Phi_2 \subset \Phi$ such that $\Phi = \Phi_1 \cup \Phi_2$ and $\langle\alpha_1|\alpha_2\rangle = 0$ for all $\alpha_1 \in \Phi_1$ and $\alpha_2 \in \Phi_2$. A *base* of a root system Φ is a subset $\Delta \subseteq \Phi$ such that for all $\alpha \in \Phi$ we have $\alpha = \sum_{\delta\in\Delta} \lambda_\delta \delta$ with $\lambda_\delta \in \mathbb{R}$, $\delta \in \Delta$, not all zero and all of them either positive or negative. Consequently, a base of Φ partitions Φ into the subsets $\Phi^\pm = \{\alpha \in \Phi \mid \alpha = \sum_{\delta\in\Delta} \lambda_\delta\delta, \lambda_\delta \in \mathbb{R}_\pm\}$ satisfying $\pm\Phi^+ = \mp\Phi^-$. A root $\alpha \in \Phi$ is called *positive* if $\alpha \in \Phi^+$ and *negative* if $\alpha \in \Phi^-$.

Example 4.3.2. Let $V = \mathbb{R}^2$ be a 2-dimensional vector space over $\mathbb{R}$ with standard basis $\mathcal{B} = \{\alpha_1, \alpha_2\} = \{(1,0)^\top, (0,1)^\top\}$. We endow V with the ordinary dot product

$\langle\cdot|\cdot\rangle : V \times V \to V$ given by $\langle x|y\rangle = x^\top y$, which is positive definite, symmetric and bilinear. Then

$$\begin{aligned}\Phi &= \{\alpha_1, \alpha_2 - \alpha_1, -\alpha_1, \alpha_1 - \alpha_2, \alpha_2, -\alpha_2, \alpha_1 + \alpha_2, -\alpha_1 - \alpha_2\} \\ &= \{(1,0)^\top, (-1,1)^\top, (-1,0)^\top, (1,-1)^\top, (0,1)^\top, (0,-1)^\top, (1,1)^\top, (-1,-1)^\top\} \subset V\end{aligned}$$

is a root system of V of rank 2 with base $\Delta = \{\alpha_1, \alpha_2 - \alpha_1\} = \{(1,0)^\top, (-1,1)^\top\}$ and positive root set $\Phi^+ = \{\alpha_1, \alpha_2 - \alpha_1, \alpha_2, \alpha_1 + \alpha_2\} = \{(1,0)^\top, (-1,1)^\top, (0,1)^\top, (1,1)^\top\}$. Condition Definition 4.3.1(ii) is immediate, and condition Definition 4.3.1(i) follows from the observation that $\mathcal{B} \subset \Phi$. To see that condition Definition 4.3.1(iii) is met, we refer to the geometric visualisation of Φ in Figure 4 below.

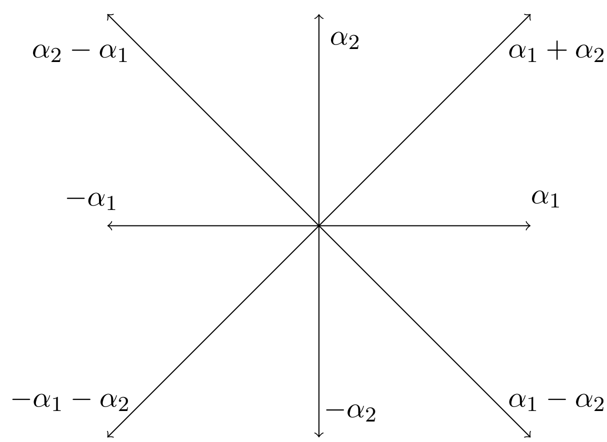


Figure 4: A geometric visualisation of exemplary root system Φ.

Any hyperplane orthogonal to a root $\alpha \in \Phi$ contains two distinct roots different from α that are fixed by σ_α. The remaining roots are all reflected onto each other so that $\sigma_\alpha(\Phi) = \Phi$. Then Φ indeed satisfies condition Definition 4.3.1(iii), hence it is indeed a root system of V.

Now let $\mathcal{B} = \{\alpha_i \mid i \in \mathcal{I}\}$ be a basis of V and define $f_M : V \times V \to \mathbb{R}$ to be the bilinear form given by $f_M(\alpha_i, \alpha_j) = -2\cos(\frac{\pi}{m_{i,j}})$ with $i, j \in \mathcal{I}$. If $m_{i,j} = \infty$, we set $f_M(\alpha_i, \alpha_j) = -2$. The form f_M is clearly a symmetric form as $m_{i,j} = m_{j,i}$ for all $i, j \in \mathcal{I}$, and it is moreover positive definite as $f_M(\alpha_i, \alpha_i) = -2\cos(\frac{\pi}{m_{i,i}}) = -2\cos(\pi) = 2 > 0$ for all $i \in \mathcal{I}$.

For any $\alpha \in V$, define $\sigma_\alpha : V \to V$ to be the reflection on $\alpha^\perp$ given by $\sigma_\alpha(\beta) = \beta - f_M(\alpha, \beta)\alpha$ with $\beta \in V$. We then have the following.

Lemma 4.3.3. *Let $\mathcal{I}$ be a type set and let $(W(M), R)$ be a Coxeter system of Coxeter type $M = (m_{i,j})_{i,j\in\mathcal{I}}$. Further let V be a finite-dimensional real vector space having basis $\{\alpha_i\}_{i\in\mathcal{I}}$ equipped with the positive definite, symmetric bilinear form $f_M : V \times V \to \mathbb{R}$ given* *by* $f_M(\alpha_i,\alpha_j) = -2\cos(\frac{\pi}{m_{i,j}})$ *with* $i,j \in \mathcal{I}$*. Then the group* $\langle \sigma_{\alpha_i} \mid i \in \mathcal{I}\rangle$ *generated by the reflections* $\sigma_{\alpha_i} : V \to V$ *on* $\alpha_i^{\perp}$ *given by* $\sigma_{\alpha_i}(\beta) = \beta - f_M(\alpha_i,\beta)\alpha_i$ *is a Coxeter group of Coxeter type* M*.*

Proof. Denote by Σ the set $\{\sigma_{\alpha_i} \mid i \in \mathcal{I}\}$ of all reflections σ_{α_i} on $\alpha_i^{\perp}$ with $i \in \mathcal{I}$. We show that $\sigma_{\alpha_i}^2 = 1$ for all $i \in \mathcal{I}$, and $(\sigma_{\alpha_i}\sigma_{\alpha_j})^{m_{i,j}} = 1$ for all distinct $i,j \in \mathcal{I}$. The first assertion is immediate from the observation that

$$\begin{aligned}\sigma_{\alpha_i}(\sigma_{\alpha_i}(\beta)) &= \sigma_{\alpha_i}(\beta) - f_M(\alpha_i,\sigma_{\alpha_i}(\beta))\alpha_i = \beta - f_M(\alpha_i,\beta)\alpha_i - f_M(\alpha_i,\beta - f_M(\alpha_i,\beta)\alpha_i)\alpha_i\\ &= \beta - f_M(\alpha_i,\beta)\alpha_i - f_M(\alpha_i,\beta)\alpha_i + f_M(\alpha_i,\alpha_i)f_M(\alpha_i,\beta)\alpha_i\\ &= \beta - 2f_M(\alpha_i,\beta)\alpha_i + 2f_M(\alpha_i,\beta)\alpha_i = \beta\end{aligned}$$

for all $\beta \in V$, implying that σ_{α_i} is an involution for all $i \in \mathcal{I}$. For the second assertion, let $i,j \in \mathcal{I}$ be distinct and consider the subspace $\langle \alpha_i,\alpha_j\rangle \subseteq V$. We first show that the restriction $f_M|_{\langle \alpha_i,\alpha_j\rangle}$ of f_M to $\langle \alpha_i,\alpha_j\rangle$ is positive definite, assuming $m_{i,j} \neq \infty$. The eigenvalues of the Gram matrix

$$\begin{pmatrix} 2 & -2\cos(\frac{\pi}{m_{i,j}})\\ -2\cos(\frac{\pi}{m_{i,j}}) & 2\end{pmatrix}$$

of $f_M|_{\langle \alpha_i,\alpha_j\rangle}$ are the solutions to the equation $(2-\lambda)^2 - 4\cos^2(\frac{\pi}{m_{i,j}}) = 0$, which are readily seen to be $\lambda = 2(1 \mp |\cos(\frac{\pi}{m_{i,j}})|)$. As $|\cos(\frac{\pi}{m_{i,j}})| = 1$ if and only if $m_{i,j} = 1$ or $m_{i,j} = \infty$, both of which cannot occur, we deduce that all eigenvalues are strictly positive so that $f_M|_{\langle \alpha_i,\alpha_j\rangle}$ is positive definite. In particular, it is an inner product on $\langle \alpha_i,\alpha_j\rangle$, and so the angle between $\alpha_i^{\perp}$ and $\alpha_j^{\perp}$ is

$$\cos^{-1}\left(\left|\frac{f_M(\alpha_i,\alpha_j)}{\sqrt{f_M(\alpha_i,\alpha_i)f_M(\alpha_j,\alpha_j)}}\right|\right) = \cos^{-1}\left(\left|\frac{-2\cos(\frac{\pi}{m_{i,j}})}{\sqrt{2\cdot 2}}\right|\right) = \frac{\pi}{m_{i,j}},$$

hence $\sigma_{\alpha_i}\sigma_{\alpha_j}$ is a rotation over an angle of $\frac{2\pi}{m_{i,j}}$, showing that $(\sigma_{\alpha_i}\sigma_{\alpha_j})^{m_{i,j}} = 1$ in $\langle \alpha_i,\alpha_j\rangle$. But $\sigma_{\alpha_i}(\beta) = 0 = \sigma_{\alpha_j}(\beta)$ for all $\beta \in \langle \alpha_i,\alpha_j\rangle^{\perp}$ because $\langle \alpha_i|\beta\rangle = 0 = \langle \alpha_j|\beta\rangle$, so both σ_{α_i} and σ_{α_j} fix $\langle \alpha_i,\alpha_j\rangle^{\perp}$, hence so will their product. It follows that $(\sigma_{\alpha_i}\sigma_{\alpha_j})^{m_{i,j}} = 1$ in the entirety of V. In case $m_{i,j} = \infty$, first observe that

$$\sigma_{\alpha_i}(\alpha_i+\alpha_j) = \alpha_i+\alpha_j - f_M(\alpha_i,\alpha_i+\alpha_j)\alpha_i = \alpha_i+\alpha_j - 2\alpha_i + 2\alpha_i = \alpha_i+\alpha_j$$

and similarly $\sigma_{\alpha_j}(\alpha_i+\alpha_j) = \alpha_i+\alpha_j$. We now claim that $(\sigma_{\alpha_i}\sigma_{\alpha_j})^n = \alpha_i + 2n(\alpha_i+\alpha_j)$ for all integers $n \geq 1$. For $n = 1$, we obtain

$$\begin{aligned}\sigma_{\alpha_i}(\sigma_{\alpha_j}(\alpha_i)) &= \sigma_{\alpha_i}(\alpha_i - f_M(\alpha_j,\alpha_i)\alpha_j) = \sigma_{\alpha_i}(\alpha_i+2\alpha_j)\\ &= \alpha_i + 2\alpha_j - f_M(\alpha_i,\alpha_i+2\alpha_j)\alpha_i = \alpha_i+2\alpha_j-2\alpha_i+4\alpha_i = \alpha_i + 2(\alpha_i+\alpha_j),\end{aligned}$$

hence by linearity of all reflections in Σ we deduce for all $n \geq 2$ that

$$(\sigma_{\alpha_i}\sigma_{\alpha_j})^n(\alpha_i) = (\sigma_{\alpha_i}\sigma_{\alpha_j})^{n-1}(\sigma_{\alpha_i}(\sigma_{\alpha_j}(\alpha_i))) = (\sigma_{\alpha_i}\sigma_{\alpha_j})^{n-1}(\alpha_i + 2(\alpha_i + \alpha_j))$$
$$= (\sigma_{\alpha_i}\sigma_{\alpha_j})^{n-1}(\alpha_i) + (\sigma_{\alpha_i}\sigma_{\alpha_j})^{n-1}(2(\alpha_i + \alpha_j)) = (\sigma_{\alpha_i}\sigma_{\alpha_j})^{n-1}(\alpha_i) + 2(\alpha_i + \alpha_j).$$

But then $(\sigma_{\alpha_i}\sigma_{\alpha_j})^n = \alpha_i + 2n(\alpha_i + \alpha_j)$ by induction, as claimed. We conclude that $\sigma_{\alpha_i}\sigma_{\alpha_j}$ has infinite order, so we may write $(\sigma_{\alpha_i}\sigma_{\alpha_j})^{m_{i,j}} = 1$ as $m_{i,j} = \infty$. It follows that the group generated by the reflections in Σ is a Coxeter group. □

In the setting of the above lemma, an immediate consequence is that a connection can be established between Coxeter systems and real reflections groups obtained from real vector spaces. In particular, upon denoting by $\langle\Sigma\rangle = \langle\sigma_{\alpha_i} \mid i \in \mathcal{I}\rangle$ the group generated by the set $\Sigma = \{\sigma_{\alpha_i} \mid i \in \mathcal{I}\}$, the map sending $r_i \in R$ to $\sigma_{\alpha_i} \in \Sigma$ extends to a surjective group homomorphism $\rho : W(M) \to \langle\Sigma\rangle$ with $\rho(r_i) = \sigma_{\alpha_i}$ for $r_i \in R$, $i \in \mathcal{I}$. This homomorphism is called the *geometric representation* of $W(M)$. An even stronger result due to Tits is the following.

Theorem 4.3.4 (Tits' representation theorem)**.** *Let $\mathcal{I}$ be a type set and let $(W(M), R)$ be a Coxeter system of Coxeter type $M = (m_{i,j})_{i,j\in\mathcal{I}}$. Further let V be a finite-dimensional real vector space with basis $\{\alpha_i\}_{i\in\mathcal{I}}$. Denoting by $\langle\Sigma\rangle$ the group generated by the set of reflections $\Sigma = \{\sigma_{\alpha_i} \mid i \in \mathcal{I}\}$ with σ_{α_i}, $i \in \mathcal{I}$, as in* Lemma 4.3.3*, we have $W(M) \cong \langle\Sigma\rangle$.*

Proof. Lemma 4.3.3 guarantees the existence of a surjective group homomorphism $\rho : W(M) \to \langle\Sigma\rangle$ satisfying $\varphi(r_i) = \sigma_{\alpha_i}$ with $r_i \in R$, $i \in \mathcal{I}$. Injectivity of the geometric representation of $W(M)$ follows from Theorem 4.4.16 in [3]. □

The geometric representation of a Coxeter group gives rise to a root system $\Phi = \bigcup_{w\in W(M)}\{\rho(w)\alpha_i \mid i \in \mathcal{I}\} \subset V$, called the *root system of* $W(M)$. The geometric representation $\rho(W(M))$ of a Coxeter group $W(M)$ is said to be *crystallographic* if the root system Φ of $W(M)$ contains a base $\Delta \subset \Phi$ such that we have for all $\alpha \in \Phi$ that $\alpha = \sum_{\delta\in\Delta} \lambda_\delta \delta$ with $\lambda_\delta \in \mathbb{Z}$, $\delta \in \Delta$, not all zero and all of them either positive or negative. Because the geometric representation of $W(M)$ is an isomorphism by Theorem 4.3.4, we may equivalently say that $W(M)$ is crystallographic. Crystallographic reflections groups are oftentimes referred to as *Weyl groups*, and they have the following property.

Proposition 4.3.5. *Let $\mathcal{I}$ be a type set and let $(W(M), R)$ be a Coxeter system of Coxeter type $M = (m_{i,j})_{i,j\in\mathcal{I}}$. If $W(M)$ is crystallographic, then $m_{i,j} \in \{2, 3, 4, 6\}$.*

Proof. Let $r_i, r_j \in R$ with $i, j \in \mathcal{I}$ distinct. Then $\rho(r_i) = \sigma_{\alpha_i}$ and $\rho(r_j) = \sigma_{\alpha_j}$ with σ_{α_i}, $i \in \mathcal{I}$, as in Lemma 4.3.3. Since the geometric representation of $W(M)$ is crystallographic, the trace of $\rho(r_ir_j) = \sigma_{\alpha_i}\sigma_{\alpha_j}$ must be integral. On the one hand, we have

$$\begin{aligned}\sigma_{\alpha_i}(\sigma_{\alpha_j}(\alpha_i)) &= \sigma_{\alpha_i}(\alpha_i - f_M(\alpha_j,\alpha_i)\alpha_j) = \sigma_{\alpha_i}(\alpha_i + 2\cos(\frac{\pi}{m_{i,j}})\alpha_j)\\ &= \alpha_i + 2\cos(\frac{\pi}{m_{i,j}})\alpha_j - f_M(\alpha_i, \alpha_i + 2\cos(\frac{\pi}{m_{i,j}})\alpha_j)\alpha_i\\ &= -\alpha_i + 2\cos(\frac{\pi}{m_{i,j}})\alpha_j + 4\cos^2(\frac{\pi}{m_{i,j}})\alpha_i\\ &= (4\cos^2(\frac{\pi}{m_{i,j}}) - 1)\alpha_i + 2\cos(\frac{\pi}{m_{i,j}})\alpha_j\end{aligned}$$

and similarly $\sigma_{\alpha_i}(\sigma_{\alpha_j}(\alpha_j)) = 2\cos(\frac{\pi}{m_{i,j}})\alpha_i + (4\cos^2(\frac{\pi}{m_{i,j}}) - 1)\alpha_j$, whereas $\sigma_{\alpha_i}(\sigma_{\alpha_j}(\alpha_k)) = 1$ for all $k \notin \{i,j\}$ because both σ_{α_i} and σ_{α_j} fix $\langle\alpha_i,\alpha_j\rangle^\perp$, which we have established in our proof of Lemma 4.3.3. But then $\sigma_{\alpha_i}\sigma_{\alpha_j}$ has trace $2(4\cos^2(\frac{\pi}{m_{i,j}}) - 1) + (|\mathcal{I}| - 2) = |\mathcal{I}| + 8\cos^2(\frac{\pi}{m_{i,j}}) - 4 \in \mathbb{Z}$, forcing $8\cos^2(\frac{\pi}{m_{i,j}}) \in \mathbb{Z}$. This is the case if and only if $m_{i,j} \in \{2,3,4,6\}$. □

We can represent an irreducible root system Φ of a crystallographic Coxeter group $W(M)$ in a Coxeter graph, as we did for Coxeter groups, as follows. Upon fixing a base $\Delta \subset \Phi$, which satisfies $|\Delta| = \dim(V)$ as Φ has rank $\dim(V)$, we construct a graph having vertex set Δ, conventionally labeled using the numbers 1 up to $|\Delta|$, in which any two vertices $\delta, \delta' \in \Delta$ have

$$\frac{4f_M(\delta,\delta')}{f_M(\delta,\delta)f_M(\delta',\delta')} = f_M(\delta,\delta')^2$$

edges between them. As we have $f_M(\alpha_i,\alpha_j)^2 = 4\cos^2(\frac{\pi}{m_{i,j}}) \in \{0,1,2,3\}$ for all $i,j \in \mathcal{I}$ because $m_{i,j} \in \{2,3,4,6\}$ by the above proposition, the Coxeter graph of $W(M)$ will have at most 3 edges between any two vertices. Notice the correspondence between the number of edges and the numbers $m_{i,j}$: no edge means $m_{i,j} = 2$, one edge means $m_{i,j} = 3$, two edges mean $m_{i,j} = 4$ and three edges mean $m_{i,j} = 6$. This agrees with our definition of Coxeter graphs as given in the previous section. So, we may identify irreducible root systems of crystallographic Coxeter groups by their diagrams and corresponding types as presented in Table 1, excluding those of type H_3, H_4 and I_n as they are correspond to non-crystallographic Coxeter groups.

The number $\sqrt{f_M(\alpha,\alpha)}$ with $\alpha \in \Phi$ is called the *root length* of α, denoted by $|\alpha|$. In the crystallographic geometric representation $\rho(W(M))$ of $W(M)$ whose root system is irreducible, at most two root lengths can occur, and a root having the largest (or smallest) of the two lengths is called *long (or short)*. If exactly one root length occurs, all roots are said to be long by convention. This information can also be included in a Coxeter graph by adding an arrow between two connected vertices pointing from the long root to the short root. Such an extended Coxeter graph is called a *Dynkin diagram*.

Example 4.3.6. Consider the root system $\Phi \subset \mathbb{R}^2$ from Example 4.3.2, which we have seen has base $\Delta = \{\alpha_1, \alpha_2 - \alpha_1\}$. Its Coxeter graph contains the two vertices represented by the roots $\alpha_1, \alpha_2 - \alpha_1 \in \Delta$, connected by a total of

$$\frac{4\langle\alpha_1|\alpha_2 - \alpha_1\rangle}{\langle\alpha_1|\alpha_1\rangle\langle\alpha_2 - \alpha_1|\alpha_2 - \alpha_1\rangle} = \frac{4 \cdot (-1)^2}{1 \cdot 2} = 2$$

edges. We then deduce from Table 1 that Φ is a root system of type B_2. Since $|\alpha_2 - \alpha_1| = \sqrt{2} > 1 = |\alpha_1|$, there is an arrow pointing from vertex $\alpha_2 - \alpha_1$ to vertex α_1 in the Dynkin diagram of Φ.

Let $\mathcal{I}$ be a type set and let $(W(M), R)$ be a Coxeter system of Coxeter type $M = (m_{i,j})_{i,j\in\mathcal{I}}$. The Coxeter diagram of $W(M)$ is said to be of *Weyl type* if it corresponds to a crystallographic Coxeter group, its Weyl type being the type of the Coxeter diagram of $W(M)$. A *root type* is a pair $(M, \mathcal{J})$, in which M is a Weyl type and $\mathcal{J} \subseteq \mathcal{I}$ is a type subset, occurring in one of the columns of Table 2 below.

M	A_n $(n \geq 1)$	BC_n $(n \geq 2)$	D_n $(n \geq 4)$	E_6	E_7	E_8	F_4	G_2
$\mathcal{J}$	$\{1, n\}$	$\{1\}$, $\{2\}$	$\{2\}$	$\{2\}$	$\{1\}$	$\{8\}$	$\{1\}$, $\{4\}$	$\{1\}$, $\{2\}$

Table 2: An overview of all root types.

The nodes corresponding to the elements in a type subset $\mathcal{J}$ of a root type $(M, \mathcal{J})$ are referred to as *root nodes*. They are clearly visible in *affine Coxeter diagrams* as the node(s) to which an extra zeroth node in such a diagram is connected, see Table 3 below. Note the distinction between diagram types $\widetilde{B_n}$ and $\widetilde{C_n}$, and the symmetry in the extended diagrams of type $\widetilde{F_4}$ and $\widetilde{G_2}$ which allows us to also connect the zeroth node to its fourth, respectively second node.

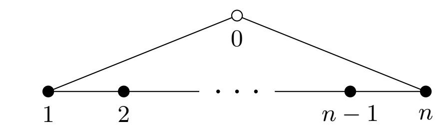
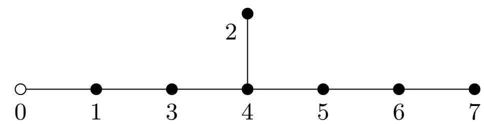
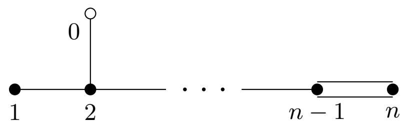
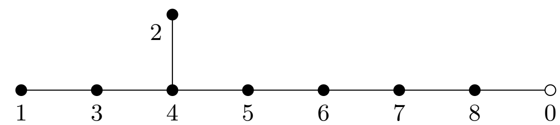
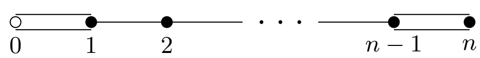
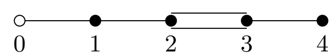
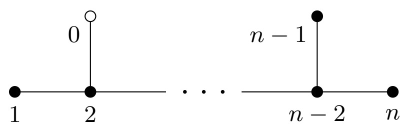
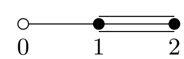
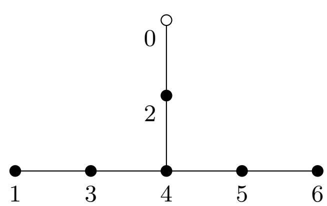
Type	Affine Coxeter diagram	Type	Affine Coxeter diagram
$\widetilde{A}_n$		$\widetilde{E}_7$	
$\widetilde{B}_n$		$\widetilde{E}_8$	
$\widetilde{C}_n$		$\widetilde{F}_4$	
$\widetilde{D}_n$		$\widetilde{G}_2$	
$\widetilde{E}_6$			

Table 3: An overview of irreducible affine Coxeter diagrams and their types.

4.4 Root shadow spaces

We are now almost in a position to introduce root shadow spaces. First, however, we briefly revisit geometries for some additional required theory.

Let $\mathcal{I}$ be a type set and let $\Gamma = (X, *, \tau)$ be an $\mathcal{I}$-geometry over $\mathcal{I}$. Further let $\mathcal{J} \subseteq \mathcal{I}$ be a non-empty type subset of $\mathcal{I}$. Given a flag F of Γ, the *$\mathcal{J}$-shadow of F* is the set of all flags F' of Γ type $\tau(F') = \mathcal{J}$ such that $x * y$ for some $x \in F$ and $y \in F'$, denoted by $\mathrm{Sh}_{\mathcal{J}}(F)$. If F itself has type $\mathcal{I} \setminus \{j\}$ for some $j \in \mathcal{J}$, then the $\mathcal{J}$-shadow $\mathrm{Sh}_{\mathcal{J}}(F)$ of F is called a *j-line*. We have the following definition.

Definition 4.4.1 (Shadow space)**.** *Let $\mathcal{J} \subseteq \mathcal{I}$ be a non-empty type subset of some type set $\mathcal{I}$ and let $\Gamma = (X, *, \tau)$ be an $\mathcal{I}$-geometry over $\mathcal{I}$. Further let $\mathcal{P} = \mathrm{Sh}_{\mathcal{J}}(\emptyset)$ be the set of flags* *of type $\mathcal{J}$ and let $\mathcal{L}$ be the set of all j-lines, $j \in \mathcal{J}$. The pair $(\mathcal{P}, \mathcal{L})$ is called the **shadow space** of Γ on $\mathcal{J}$, denoted by* $\text{ShSp}_{\mathcal{J}}(\Gamma)$.

Given a type subset $\mathcal{J}' \subseteq \mathcal{J}$, the shadow space of Γ on $\mathcal{J}'$, denoted by $\text{ShSp}_{\mathcal{J}}(\Gamma, \mathcal{J}')$, is the shadow space obtained from $\text{ShSp}_{\mathcal{J}}(\Gamma) = (\mathcal{P}, \mathcal{L})$ by removing from $\mathcal{L}$ all j-lines with $j \in \mathcal{J} \setminus \mathcal{J}'$.

Since the set of flags of Γ of type $\mathcal{J}$ incident with some flag of type $\mathcal{I} \setminus \{j\}$, $j \in \mathcal{J}$, is a subset of the set all flags of Γ of type $\mathcal{J}$, we have $\ell \subseteq \mathcal{P}$ for all $\ell \in \mathcal{L}$. Moreover, as Γ is firm, every flag F of type $\mathcal{I} \setminus \{j\}$ for some j is contained in at least two chambers of Γ, each of which will contain a flag of type $\mathcal{J}$ incident with F, implying that $|\ell| \geq 2$ for all $\ell \in \mathcal{L}$. An immediate consequence is then that a shadow space $\text{ShSp}_{\mathcal{J}}(\Gamma) = (\mathcal{P}, \mathcal{L})$ is a point-line geometry by Definition 4.1.5.

Example 4.4.2. Consider a polytope in $\mathbb{E}^n$, $n \geq 2$ finite, which we have seen is an $\mathcal{I}$-geometry Γ over type set $\mathcal{I} = \{0, \ldots, n-1\}$ with $\text{rank}(\Gamma) = n$. Now let $\mathcal{J} = \{0, n-1\} \subset \mathcal{I}$ and let $\text{ShSp}_{\mathcal{J}}(\Gamma) = (\mathcal{P}, \mathcal{L})$ be the shadow space of Γ on $\mathcal{J}$. Its points in $\mathcal{P}$ are all flags of Γ of type $\{0, n-1\}$, i.e. all the incident point-hyperplane pairs (p, H) of Γ. Any 0-line for some flag F of Γ of type $\{1, \ldots, n-1\}$ will contain all incident point-hyperplane pairs (p, H) such that $H \in F$, whereas any $(n-1)$-line for some flag F of Γ of type $\{0, \ldots, n-2\}$ will contain all incident point-hyperplane pairs (p, H) such that $p \in F$. Consequently, two incident points-hyperplane pairs (p, H) and (q, K) are contained in the same 0-line for some flag F of type $\{1, \ldots, n-1\}$ if and only if $H = K$, and they are contained in the same $(n-1)$-line for some flag F of type $\{0, \ldots, n-2\}$ if and only if $p = q$. Note that the point-line geometry $\text{ShSp}_{\mathcal{J}}(\Gamma)$ is a partial linear space because Γ is thin.

Recall from the previous section the construction of a chamber system $\mathcal{C}(\Gamma)$ obtained from an incidence system $\Gamma = (X, *, \tau)$ over some type set $\mathcal{I}$. In order to relate shadow spaces of a geometry $\Gamma = (X, *, \tau)$ over some type set $\mathcal{I}$ to a building $\mathcal{C} = (C, \{\sim_i | \; i \in \mathcal{I}\})$ of Coxeter type $M = (m_{i,j})_{i,j \in \mathcal{I}}$, we will do the opposite and construct an incidence system obtained from a chamber system.

So, let $\mathcal{C} = (C, \{\sim_i | \; i \in \mathcal{I}\})$ be a chamber system over a type set set $\mathcal{I}$. We then let X be the set of all pairs (x, i) with Z_i an i-object of $\mathcal{C}$, define $* \subseteq X \times X$ as $(Z_i, i) * (Z_j, j)$ with $(Z_i, i), (Z_j, j) \in X$ if and only if $Z_i \cap Z_j \neq \emptyset$, and define $\tau : X \to \mathcal{I}$ to be the map given by $\tau((Z_i, i)) = i$ for $(Z_i, i) \in X$. The relation $*$ on X is clearly reflexive and symmetric, and two distinct i-objects Z_i and Z_j of $\mathcal{C}$, $i \in \mathcal{I}$, cannot intersect by definition, forcing $Z_i \cap Z_j = \emptyset$ so that no pairs (Z_i, i) and (Z_j, j) with $x \neq y$ are incident in Γ. Equivalently, any two incident $(Z_i, i), (Z_j, j) \in X$ satisfy $\tau((Z_i, i)) = i \neq j = \tau((Z_j, j))$, hence we indeed obtain an incidence system by Definition 4.2.1, which we will denote by $\Gamma(\mathcal{C})$ to emphasise the dependency of its construction on $\mathcal{C}$. Incidence systems obtained from chambers systems are related to buildings in the following way.

Proposition 4.4.3. *Let $\mathcal{I}$ be a type set and let $\mathcal{C} = (C, \{\sim_i | \; i \in \mathcal{I}\})$ be a building of Coxeter type $M = (m_{i,j})_{i,j \in \mathcal{I}}$. Then $\Gamma(\mathcal{C})$ is an $\mathcal{I}$-geometry of Coxeter type M over $\mathcal{I}$.*

Proof. We prove that $\Gamma(\mathcal{C}) = (X, *, \tau)$ is a firm and residually connected geometry of type M in three steps.

First, we show that $\Gamma(\mathcal{C})$ is a firm geometry. By observing that any flag F of $\Gamma(\mathcal{C})$ is a collection $\{(Z_j, j) \in X \mid j \in \tau(F)\}$ of j-objects Z_j of $\mathcal{C}$, one for each $j \in \tau(F) \subseteq \mathcal{I}$, such that any two of them have a non-empty intersection, we deduce that $Z = \bigcap_{j\in\tau(F)} Z_j$ is an $(\mathcal{I} \setminus \tau(F))$-cell by residual connectedness of $\mathcal{C}$. In particular, Z will be non-empty, so if F is not a chamber then it can be extended to a chamber of $\Gamma(\mathcal{C})$ by adding to it elements $(Z_i, i) \in X$ such that every i-object Z_i, one for each $i \in \mathcal{I} \setminus \tau(F)$, contains some chamber $z \in Z$. It follows that Γ is a geometry, which is moreover firm as $\mathcal{C}$ is a chamber system of type M.

Secondly, we show that $\Gamma(\mathcal{C})$ is residually connected. Let $F = \{(Z_k, k) \in X \mid k \in \tau(F)\}$ be a flag of $\Gamma(\mathcal{C})$ such that its residue $\Gamma(\mathcal{C})_F$ has rank at least two and let $(Z_i, i), (Z_j, j) \in F^* \setminus F$ for an i-object Z_i and j-object Z_j with $i, j \in \mathcal{I} \setminus \tau(F)$. By residual connectedness of $\mathcal{C}$, we have that $Z = \bigcap_{k\in\tau(\mathcal{F})} Z_k$ is a $(\mathcal{I} \setminus \tau(F))$-cell of $\mathcal{C}$, hence any gallery $z_i = z_0, \ldots, z_n = z_j$, $n \geq 1$ finite, with $z_i \in Z_i \cap Z \neq \emptyset$ and $z_j \in Z_j \cap Z \neq \emptyset$, is fully contained in Z. For every $0 \leq m \leq n-1$, we then have $z_m \sim_{k_m} z_{m+1}$ for some $k_m \in \mathcal{I} \setminus \tau(F)$, so the sequence $(Z_i, i), (Z_0, k'_0), \ldots, (Z_{m-1}, k'_{m-1}), (Z_j, j)$, in which for every $0 \leq l \leq m-1$ we have that Z_l is a k'_l-object for some $k'_l \in \mathcal{I} \setminus (\tau(F) \cup \{k_m\})$, is an $(\mathcal{I} \setminus \tau(F))$-chain in $\Gamma(\mathcal{C})_F$ from (Z_i, i) to (Z_j, j) as for every $0 \leq l \leq m-1$ we have $(Z_l, k'_l) * (Z_{l+1}, k'_l)$ because $z_l, z_{l+1} \in Z_l \cap Z_{l+1} \cap Z$. But then $\Gamma(\mathcal{C})_F$ is connected so that $\Gamma(\mathcal{C})$ is residually connected.

Lastly, we show that $\Gamma(\mathcal{C})$ is a geometry of type M. Let F be a flag of $\Gamma(\mathcal{C})$ of type $\{i, j\} \in \mathcal{I}$. Note that the residue $\Gamma(\mathcal{C})_F$ is residually connected by Proposition 4.2.7(iii) as $\Gamma(\mathcal{C})$ is a geometry. The chambers of $\Gamma(\mathcal{C})_F$, which are collections $\{(Z_k, k) \in X \mid k \in \mathcal{I} \setminus \{i, j\}$ of k-objects Z_k of $\mathcal{C}$, one for each $k \in \mathcal{I} \setminus \{i, j\}$, all have type $\mathcal{I} \setminus \{i, j\}$. Therefore, $\bigcap_{k\in\mathcal{I}\setminus\{i,j\}} Z_k$ is an $\{i, j\}$-cell of $\mathcal{C}$ for every chamber $\{(Z_k, k) \in X \mid k \in \mathcal{I} \setminus \{i, j\}\}$ by residual connectedness of $\mathcal{C}$. But $\mathcal{C}$ is a chamber system of type M, hence every $\{i, j\}$-cell is the chamber system $\mathcal{C}(\Gamma)$ of some residually connected geometry Γ belonging to a generalised $m_{i,j}$-gon. In turn, we can identify the collection of all $\{i, j\}$-cells by the chamber system $\mathcal{C}(\Gamma(\mathcal{C})_F)$, which we may because of residual connectedness of $\Gamma(\mathcal{C})_F$, implying that $\Gamma(\mathcal{C})_F$ belongs to a generalised $m_{i,j}$-gon. Since both a generalised $m_{i,j}$-gon and $\Gamma(\mathcal{C})_F$ are residually connected, the two are in fact isomorphic. We conclude that $\Gamma(\mathcal{C})$ is a geometry of type M. □

A consequence of the above proposition that is of particular importance is that shadow spaces are defined for any geometry $\Gamma(\mathcal{C})$ over some type set $\mathcal{I}$ obtained from a building $\mathcal{C} = (C, \{\sim_i \mid i \in \mathcal{I}\})$ of Coxeter type $M = (m_{i,j})_{i,j\in\mathcal{I}}$. This enables us to finally define root shadow spaces.

Definition 4.4.4 (Root shadow space)**.** *Let $\mathcal{I}$ be a type set and let $\mathcal{C} = (C, \{\sim_i \mid i \in \mathcal{I}\})$ be a building of Coxeter type $M = (m_{i,j})_{i,j\in\mathcal{I}}$. The shadow space* $\mathrm{ShSp}_{\mathcal{J}}(\Gamma(\mathcal{C}))$ *of* $\Gamma(\mathcal{C})$ *on some type subset $\mathcal{J} \subseteq \mathcal{I}$ is called a* ***root shadow space*** *if $(M, \mathcal{J})$ is a root type.*

The *type* of a root shadow space $\mathrm{ShSp}_{\mathcal{J}}(\Gamma(\mathcal{C}))$ is its root type $(M, \mathcal{J})$. For the sake of brevity and clarity, we will include the type subset $\mathcal{J}$ as a subscript in the Coxeter type M, e.g. $A_{n,\{1,n\}}$. If $\mathcal{J}$ is a singleton, we will write down only its element instead of $\mathcal{J}$ entirely, e.g. $E_{6,2}$.

Example 4.4.5. Let $\mathcal{I} = \{1, \ldots, n\}$ be a type set and let $\mathcal{C} = (C, \{\sim_i \mid i \in \mathcal{I}\})$ be a building of type A_n, $n \geq 1$ finite. By Proposition 11.1.9(i) in [3], it can be realised as the chamber system $\mathcal{C}(\Gamma)$ of an $\mathcal{I}$-geometry Γ over $\mathcal{I}$ of type A_n whose elements are all i-dimensional subspaces $W \subset V$, $1 \leq i \leq n$, of an $(n+1)$-dimensional vector space V over some division ring $\mathbb{K}$, each having type $\dim(W)$ and in which incidence is defined by inclusion. The shadow space $\mathrm{ShSp}_{\mathcal{J}}(\Gamma(\mathcal{C}(\Gamma)))$ of $\Gamma(\mathcal{C}(\Gamma))$ will be a root shadow space if and only if $\mathcal{J} = \{1, n\}$, in which case $\mathrm{ShSp}_{\mathcal{J}}(\Gamma(\mathcal{C}(\Gamma)))$ will be as described in Example 4.4.2. Any 1-line containing two distinct incidence point-hyperplane pairs (p, H) and (q, H) will consist of all incident point-hyperplane pairs (r, H) such that $r \in \langle p, q \rangle$, and any n-line containing two distinct point-hyperplane pairs (p, H) and (p, K) will consist of all incident point-hyperplane pairs (p, L) such that $H \cap K \subseteq L$.

By Proposition 4.1.14(ii), non-degenerate thick polar spaces are partial linear spaces, and by Lemma 11.4.6 in [3], root shadow spaces are partial linear spaces. A stronger connection between root shadow spaces and non-degenerate polar spaces is established by the following corollary.

Corollary 4.4.6. *Let $\mathcal{I} = \{1, \ldots, n\}$, $n \geq 3$ finite, be a type set and let $\mathcal{C} = (C, \{\sim_i \mid i \in \mathcal{I}\})$ be a building of type B_n. Then the root shadow space of $\Gamma(\mathcal{C})$ of type $BC_{n,1}$ is a non-degenerate polar space of rank n, and the root shadow space of $\Gamma(\mathcal{C})$ of type $BC_{n,2}$ or $D_{n+1,2}$ is the dual of a non-degenerate polar space of rank n.*

Proof. By Proposition 11.1.9(ii) in [3], a building of type B_n, $n \geq 3$ finite, can be realised as the chamber system $\mathcal{C}(\Gamma)$ of an $\mathcal{I}$-geometry Γ of type B_n over $\mathcal{I}$ whose elements are all rank $i-1$ singular subspaces $\mathcal{S} \subset \mathcal{P}$, $1 \leq i \leq n$, of a non-degenerate polar space $(\mathcal{P}, \mathcal{L})$ of rank n, each having type $\mathrm{rank}(\mathcal{S}) + 1$ and in which incidence is defined by inclusion. Its 1-elements are the points in $\mathcal{P}$ and its 2-elements are the lines in $\mathcal{L}$. The root shadow space of $\Gamma(\mathcal{C}(\Gamma))$ of type $BC_{n,1}$ has point set $\mathcal{P}$, and every 1-line $\mathrm{Sh}_1(F)$ for some flag F of $\Gamma(\mathcal{C}(\Gamma))$ of type $\mathcal{I} \setminus \{1\}$ consists of all points in $\mathcal{P}$ incident with the unique 2-element contained in F, i.e. a line in $\mathcal{L}$. It follows that we may identify $BC_{n,1}$ by the non-degenerate polar space $(\mathcal{P}, \mathcal{L})$. The root shadow space of $\Gamma(\mathcal{C}(\Gamma))$ of type $BC_{n,2}$ has point set $\mathcal{L}$, and every 2-line $\mathrm{Sh}_2(F)$ for some flag F of $\Gamma(\mathcal{C}(\Gamma))$ of type $\mathcal{I} \setminus \{2\}$ consists of all lines in $\mathcal{L}$ incident with the unique 3-element contained in F and the unique 1-element contained in F, i.e. the line pencil on this 1-element. So, we may identify $BC_{n,2}$ by the dual of the non-degenerate polar space $(\mathcal{P}, \mathcal{L})$.

Similarly, by Proposition 11.1.10 in [3], a building of D_{n+1}, $n \geq 3$, can be realised as a residually connected chamber system $\mathcal{C}(\Gamma)$ with Γ as above. Although the underlying $\mathcal{I}$-geometry Γ differs from that of a building of type B_n, $n \geq 3$, on the elements of type greater than $n-1$, it coincides on the elements of type $1 \leq i \leq n-1$. Consequently, all that we have said above for the root shadow space of type $BC_{n,2}$ remains true if the root shadow space is of type $D_{n+1,2}$. □

4.5 Root filtration spaces

We further characterise root shadow spaces by introducing root filtration spaces. They will play a vital role in the upcoming chapter, as they establish a connection with buildings and root shadow spaces on the one hand and with Lie algebras and extremal elements on the other hand.

We first introduce some notation that we will adopt in the sequel. For a binary symmetric relation $\mathcal{R} \subseteq S \times S$ on some set S, we denote by $\mathcal{R}(t)$, $t \in S$, the set of all elements $s \in S$ such that $(s,t) \in \mathcal{R}$, i.e. $\mathcal{R}(t) = \{s \in S \mid (s,t) \in \mathcal{R}\}$. Furthermore, we denote by $\mathcal{R}(T)$, $T \subseteq S$, the set of all elements $s \in S$ such that $(s,t) \in \mathcal{R}$ for all $t \in T$, i.e. $\mathcal{R}(T) = \bigcap_{t \in T} \mathcal{R}(t)$. If $T = \{t, t'\} \subseteq S$ or $T = \{t, t', t''\} \subseteq S$, we will also write $\mathcal{R}(T) = \mathcal{R}(t,t')$ or $\mathcal{R}(T) = \mathcal{R}(t,t',t'')$, respectively. If $\{\mathcal{R}_i\}_{i \in \mathcal{I}}$ is a set of relations on S indexed by some index set I ordered by $\leq$, we denote by $\mathcal{R}_{\leq i}$ the subset of relations $\mathcal{R}_j$ with $j \leq i$, i.e. $\mathcal{R}_{\leq i} = \bigcup_{j \leq i} \mathcal{R}_j$.

Definition 4.5.1 (Root filtration space)**.** *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a partial linear space. Then Γ is called a* ***root filtration space*** *if there exists a set $\{\mathcal{P}_i\}_{-2 \leq i \leq 2}$ of disjoint symmetric relations $\mathcal{P}_i \subseteq \mathcal{P} \times \mathcal{P}$ partitioning $\mathcal{P} \times \mathcal{P}$ such that*

(i) *the relation $\mathcal{P}_{-2}$ is equality on $\mathcal{P} \times \mathcal{P}$,*

(ii) *the relation $\mathcal{P}_{-1}$ is collinearity in Γ of distinct points in $\mathcal{P}$,*

(iii) *there exists a map $\varphi : \mathcal{P}_1 \to \mathcal{P}$ satisfying $\mathcal{P}_i(p) \cap \mathcal{P}_j(q) \subseteq \mathcal{P}_{\leq i+j}(\varphi(p,q))$ for all $(p,q) \in \mathcal{P}_1$ and $-2 \leq i, j \leq 2$,*

(iv) *$\mathcal{P}_{\leq 0}(p) \cap \mathcal{P}_{\leq -1}(q) = \emptyset$ for all $(p,q) \in \mathcal{P}_2$,*

(v) *the sets $\mathcal{P}_{\leq -1}(p)$ and $\mathcal{P}_{\leq 0}(p)$ are subspaces of Γ for all $p \in \mathcal{P}$,*

(vi) *the set $\mathcal{P}_{\leq 1}(p)$ is a hyperplane of Γ for all $p \in \mathcal{P}$.*

The set $\{\mathcal{P}_i\}_{-2 \leq i \leq 2}$ of disjoint symmetric relations $\mathcal{P}_i \subseteq \mathcal{P} \times \mathcal{P}$ partitioning $\mathcal{P} \times \mathcal{P}$ of a root filtration space $\Gamma = (\mathcal{P}, \mathcal{L})$ is called the *filtration* of Γ. If additionally $\mathcal{P}_2(p) \neq \emptyset$ for all $p \in \mathcal{P}$ and the collinearity graph of Γ, i.e. the graph $(\mathcal{P}, \mathcal{P}_{-1})$, is connected, then Γ is said to be a *non-degenerate* root filtration space. Two points $p, q \in \mathcal{P}$ such that $(p,q) \in \mathcal{P}_i$ are called *hyperbolic* if $i = 2$, *special* if $i = 1$, *polar* if $i = 0$, *collinear* if $i = -1$ and *commuting* if $i \leq 0$. In case p and q are collinear, we call them *neighbours.*

Applying Definition 4.5.1(iii) to some element $r \in \mathcal{P}_i(p) \cap \mathcal{P}_j(q)$ with $(p,q) \in \mathcal{P}_1$ and $-2 \leq i, j \leq 2$ is referred to as the *filtration around* r. Definition 4.5.1(iv) is oftentimes called the *triangle condition on* p, q *and* r, in which $r \in \mathcal{P}$ is an element satisfying $r \in \mathcal{P}_{\leq 0}(p)$ and $r \notin \mathcal{P}_{\leq -1}(q)$ with $(p,q) \in \mathcal{P}_2$.

Example 4.5.2. As in Example 4.4.5, let $\mathcal{I} = \{1, \dots, n\}$ be a type set and let $\mathcal{C} = (C, \{\sim_i \mid i \in \mathcal{I}\})$ be a building of type A_n, $n \geq 1$ finite. We have seen that the incident point-hyperplane pairs (p, H) of $\Gamma(\mathcal{C}(\Gamma))$ are the points of the root shadow space $\mathrm{ShSp}_{\{1,n\}}(\Gamma(\mathcal{C}(\Gamma))) = (\mathcal{P}, \mathcal{L})$, and two incident point-hyperplane pairs $(p, H), (q, K) \in \mathcal{P}$ are collinear if and only if either $p = q$ or $H = K$. Now let $\{\mathcal{P}_i\}_{-2 \leq i \leq 2}$ be a set of symmetric relations $\mathcal{P}_i \subseteq \mathcal{P} \times \mathcal{P}$ defined by

$$\begin{aligned}
((p,H),(q,K)) \in \mathcal{P}_{-2} &\iff p = q \text{ and } H = K,\\
((p,H),(q,K)) \in \mathcal{P}_{-1} &\iff p = q \text{ or } H = K \text{ and } (p,H) \neq (q,K),\\
((p,H),(q,K)) \in \mathcal{P}_{0} &\iff p \in K \text{ and } q \in H \text{ with } p \neq q \text{ and } H \neq K,\\
((p,H),(q,K)) \in \mathcal{P}_{1} &\iff p \in K \text{ and } q \notin H \text{ or } q \in H \text{ and } p \notin K,\\
((p,H),(q,K)) \in \mathcal{P}_{2} &\iff p \notin K \text{ and } q \notin H
\end{aligned}$$

with $(p, H), (q, H) \in \mathcal{P}$. It is readily seen that these five relations are disjoint and partition $\mathcal{P} \times \mathcal{P}$. It can moreover be shown that these relations satisfy Definition 4.5.1(i)-(vi), that $\mathcal{P}_2((p, H)) \neq \emptyset$ for all $(p, H) \in \mathcal{P}$ and that the graph $(\mathcal{P}, \mathcal{P}_{-1})$ is connected, implying that the root shadow space of $\Gamma(\mathcal{C})$ of type $A_{n,\{1,n\}}$ is a non-degenerate root filtration space.

Now let $\mathcal{C} = (C, \{\sim_i \mid i \in \mathcal{I}\})$ be a building of type B_n or D_{n+1}, $n \geq 3$. We have seen in Corollary 4.4.6 that the root shadow space of $\Gamma(\mathcal{C})$ of type $BC_{n,2}$ or $D_{n+1,2}$ is the dual $(\mathcal{L}, \mathcal{P}^*)$ of a non-degenerate polar space $(\mathcal{P}, \mathcal{L})$ of rank n. The set $\{\mathcal{L}_i\}_{-2 \leq i \leq 2}$ of symmetric relations $\mathcal{L}_i \subseteq \mathcal{L} \times \mathcal{L}$ defined by

$$\begin{aligned}
(\ell, \ell') \in \mathcal{L}_{-2} &\iff \ell = \ell',\\
(\ell, \ell') \in \mathcal{L}_{-1} &\iff \langle \ell, \ell' \rangle \text{ is a singular plane},\\
(\ell, \ell') \in \mathcal{L}_{0} &\iff \langle \ell, \ell' \rangle \text{ is singular but not a plane or } \langle \ell, \ell' \rangle \text{ is a non-singular plane},\\
(\ell, \ell') \in \mathcal{L}_{1} &\iff \langle \ell, \ell'' \rangle \text{ and } \langle \ell', \ell'' \rangle \text{ are singular planes for a unique line } \ell'' \in \mathcal{L},\\
(\ell, \ell') \in \mathcal{L}_{2} &\iff (\ell, \ell') \notin \mathcal{L}_{\leq 1},
\end{aligned}$$

forms a disjoint set of relations that partition $\mathcal{L} \times \mathcal{L}$. In particular, they turn $(\mathcal{L}, \mathcal{P}^*)$ into a root filtration space, which will moreover be non-degenerate.

It is possible to analyse root filtration spaces using elementary graph theory without having to necessarily rely on Definition 4.5.1(i)-(vi). This is characterised by the following proposition, for which we require several properties of the defining relations of a root filtration space that we will not prove here. Instead, we refer to [5] for a thorough overview.

Proposition 4.5.3. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a non-degenerate root filtration space with filtration $\{\mathcal{P}_i\}_{-2\leq i\leq 2}$ and denote by $(\mathcal{P}, \mathcal{P}_{-1})$ the collinearity graph of Γ. Then for all $p, q \in \mathcal{P}$ we have*

(i) $(p, q) \in \mathcal{P}_{-2}$ *if and only if* $p = q$,

(ii) $(p, q) \in \mathcal{P}_{-1}$ *if and only if* p *and* q *are distinct collinear points in* $(\mathcal{P}, \mathcal{P}_{-1})$,

(iii) $(p, q) \in \mathcal{P}_0$ *if and only if* p *and* q *have at least two common neighbours in* $(\mathcal{P}, \mathcal{P}_{-1})$,

(iv) $(p, q) \in \mathcal{P}_1$ *if and only if* p *and* q *have a unique common neighbour in* $(\mathcal{P}, \mathcal{P}_{-1})$,

(v) $(p, q) \in \mathcal{P}_2$ *if and only if* p *and* q *have no common neighbours in* $(\mathcal{P}, \mathcal{P}_{-1})$.

In particular, the distance between any two points in $\mathcal{P}_{-2}$, $\mathcal{P}_{-1}$, $\mathcal{P}_0 \cup \mathcal{P}_1$ or $\mathcal{P}_2$ in $(\mathcal{P}, \mathcal{P}_{-1})$ is 0, 1, 2 *or* 3, *respectively.*

Proof. Both (i) and (ii) immediately follow from Definition 4.5.1(i), respectively Definition 4.5.1(ii).

For (iii), let $(p, q) \in \mathcal{P}_0$. Because Γ is non-degenerate, Lemma 7 of [5] applies, hence there exists a point $r \in \mathcal{P}_{-1}(p, q)$, i.e. r is a common neighbour of p and q. In turn, there exists a point $r' \in \mathcal{P}_{-1}(p, q)$ different from r such that $(r, r') \notin \mathcal{P}_{-1}$ by Lemma 8 of [5]. It follows that both r and r' are common neighbours of p and q $(\mathcal{P}, \mathcal{P}_{-1})$.

For (iv), let $(p, q) \in \mathcal{P}_1$. Then $p \in \mathcal{P}_1(q)$, hence the filtration around $p \in \mathcal{P}_{-2}(p) \cap \mathcal{P}_1(q)$ yields $p \in \mathcal{P}_{\leq -1}(\varphi(p, q))$, in which $\varphi : \mathcal{P}_1 \to \mathcal{P}$ is as in Definition 4.5.1(iii). Similarly, the filtration around $q \in \mathcal{P}_1(p) \cap \mathcal{P}_{-2}(q)$ yields $q \in \mathcal{P}_{\leq -1}(\varphi(p, q))$. Consequently, $\varphi(p, q) \in \mathcal{P}_{\leq -1}(p, q)$. Since $(p, q) \in \mathcal{P}_1$ we have $p \neq q$ and therefore $\varphi(p, q) \notin \mathcal{P}_{-2}(p, q)$ as otherwise $p = \varphi(p, q) = q$. It follows that $\varphi(p, q) \in \mathcal{P}_{-1}(p, q)$. If $r \in \mathcal{P}$ is a point different from $\varphi(p, q)$ such that $r \in \mathcal{P}_{-1}(p, q)$, then again by Definition 4.5.1(iii) we have $r \in \mathcal{P}_{-1}(p) \cap \mathcal{P}_{-1}(q) \subseteq \mathcal{P}_{\leq -2}(\varphi(p, q)) = \mathcal{P}_{-2}(\varphi(p, q)) = \{\varphi(p, q)\}$, so $r = \varphi(p, q)$, a contradiction. We conclude that $\varphi(p, q)$ is the unique neighbour of p and q in $(\mathcal{P}, \mathcal{P}_{-1})$.

For (v), let $(p, q) \in \mathcal{P}_2$ and suppose that $r \in \mathcal{P}_{-1}(p, q)$. Then by Definition 4.5.1(iv), we have $r \in \mathcal{P}_{-1}(p) \cap \mathcal{P}_{-1}(q) \subseteq \mathcal{P}_{\leq 0}(p) \cap \mathcal{P}_{\leq -1}(q) = \emptyset$, a contradiction. Thus, p and q have no common neighbours in $(\mathcal{P}, \mathcal{P}_{-1})$.

For the last assertion, it is clear that any two points in $\mathcal{P}_{-2}$ and $\mathcal{P}_1$ have distance 0 and 1 in $(\mathcal{P}, \mathcal{P}_{-1})$ by Proposition 4.5.3(i) and Proposition 4.5.3(ii), respectively. Any two points in $\mathcal{P}_0 \cup \mathcal{P}_1$ have at least one common neighbour, hence distance 2 in $(\mathcal{P}, \mathcal{P}_{-1})$. Finally, let $(p, q) \in \mathcal{P}_2$ and let $\ell \in \mathcal{L}$ be a line on p. By Definition 4.5.1(vi), the set $\mathcal{P}_{\leq 1}(q)$ is a hyperplane of Γ, hence $|\mathcal{P}_{\leq 1}(q) \cap \ell| = 1$ or $|\mathcal{P}_{\leq 1}(q) \cap \ell| = |\ell|$. If the latter of the two is true, then $(p, q) \in \mathcal{P}_{-1}$, a contradiction. So, we have $(q, r) \in \mathcal{P}_{\leq -1}$ for some point $r \in \ell$. If $(q, r) \in \mathcal{P}_{-2}$ then $q = r \in \ell$ so that $(p, q) \in \mathcal{P}_{-1}$, a contradiction, and if $(q, r) \in \mathcal{P}_{-1}$ then r is a common neighbour of p and q in $(\mathcal{P}, \mathcal{P}_{-1})$, contradicting $(p, q) \in \mathcal{P}_2$ because of Proposition 4.5.3(v). We then must have $(q, r) \in \mathcal{P}_0 \cup \mathcal{P}_1$ so that the distance between q

and r is 2. But $(p, r) \in \mathcal{P}_{-1}$ because $p, r \in \ell$, hence the distance between p and r is 1. It follows that the distance between p and q is 3 in $(\mathcal{P}, \mathcal{P}_{-1})$. $\square$

We finish this section with some important connections between root shadow spaces and root filtration spaces, as mentioned earlier. Both are due to Cohen and Ivanyos [19].

Theorem 4.5.4. *Let $\mathcal{C} = (C, \{\sim_i \mid i \in \mathcal{I}\})$ be a building of Weyl type $M = (m_{i,j})_{i,j \in \mathcal{I}}$ over some type set $\mathcal{I}$. Then the root shadow space of $\Gamma(\mathcal{C})$ is a root filtration space.*

In particular, it is non-degenerate except when the root shadow space of $\Gamma(\mathcal{C})$ is of type $BC_{n,1}$.

Proof. See Corollary 11.6.6 of [3] or Theorem 36 of [19]. $\square$

A partial converse to the above theorem is given by following theorem, which further demonstrates the strength of the connection between root shadow spaces and root filtration spaces.

Theorem 4.5.5. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a non-degenerate thick root filtration space. If the singular rank of Γ is finite, then Γ is isomorphic to a root shadow space of type $A_{n,\{1,n\}}$ $(n \geq 2)$, $BC_{n,2}$ $(n \geq 3)$, $D_{n,2}$ $(n \geq 4)$, $E_{6,2}$, $E_{7,1}$, $E_{8,8}$, $F_{4,1}$ or $G_{2,1}$.*

Proof. See Theorem 11.7.11 of [3] or Theorem 1 of [19]. $\square$

4.6 Classification of polar spaces

In the last paragraph of our proof of Proposition 4.4.3, we made use of the notion of isomorphicity between two geometries over some type set $\mathcal{I}$. In general, incidence systems can be related to one another using the following definition, comparable to Definition 4.2.10 for chamber systems.

Definition 4.6.1 (Incidence system morphisms)**.** *Let $\Gamma = (X, *, \tau)$ and $\Gamma' = (X', *', \tau')$ be two incidence systems over type sets $\mathcal{I}$ and $\mathcal{I}'$, respectively. A* ***weak homomorphism*** *between Γ and Γ' is a map $\varphi : X \to X'$ such that $x * y \implies \varphi(x) *' \varphi(y)$ and $\tau(x) = \tau(y) \iff \tau'(\varphi(x)) = \tau'(\varphi(y))$ for all $x, y \in X$. If moreover φ is bijective and its inverse φ^{-1} is a weak homomorphism, then φ is said to be a* ***correlation****. If $\mathcal{I} = \mathcal{I}'$ and $\tau(x) = \tau'(\varphi(x))$ for all $x \in X$, then φ is called a* ***homomorphism****. If φ is both a correlation and a homomorphism, then it is an* ***isomorphism****, in which case Γ and Γ' are isomorphic as incidence systems, denoted by $\Gamma \cong \Gamma'$.*

An injective homomorphism $\varphi : X \to X'$ between two incidence systems $\Gamma = (X, *, \tau)$ and $\Gamma' = (X', *', \tau')$ is called an *embedding* of Γ in Γ'. An *auto-correlation* is a correlation from an incidence geometry $\Gamma = (X, *, \tau)$ to itself, whereas an *automorphism* is an isomorphism from an incidence geometry to itself. The sets of all auto-correlations and automorphisms of Γ are denoted by $\mathrm{Cor}(\Gamma)$, respectively $\mathrm{Aut}(\Gamma)$, both of which form a group under ordinary function composition. As for chamber systems, they are related in the sense that $\mathrm{Aut}(\Gamma) \trianglelefteq \mathrm{Cor}(\Gamma)$.

Example 4.6.2. Since graphs are polytopes in $\mathbb{E}^2$ over type set $\mathcal{I} = \{0, 1\}$, any graph morphism will be an incidence system morphism. In particular, an incidence system morphism $\varphi : V \to V'$ between two graphs $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ is an embedding if and only if Γ is a subgraph of Γ'. Note that any correlation between Γ and Γ' will automatically be an isomorphism because their type sets coincides and because vertices and edges in Γ are mapped to vertices, respectively edges in Γ' so that $\tau(v) = \tau'(\varphi(v))$ for all $v \in V$.

Let V be a 3-dimensional vector space over $\mathbb{F}_2$. We know from Example 4.4.2 that it gives rise to a geometry that realises a building of type A_2, so we denote it by $\Gamma(A_2)$. Further let Γ be the Fano plane, its vertices labeled as in Example 4.1.6. Define the map $\varphi : \Gamma(A_2) \to \Gamma$ by $(x, y, z) \mapsto 4x + 2y + z$ with $(x, y, z) \in V$. It can easily be verified that φ establishes a homomorphism between $\Gamma(A_2)$ and Γ. In particular, φ is an isomorphism and confirms that the Fano plane is a projective space as established in Example 4.1.18.

Let $\varphi : X \to X$ be an auto-correlation of an incidence system $\Gamma = (X, *, \tau)$ with $\tau(X) = X$. Then φ induces a permutation $\varphi_{\mathcal{I}} : \mathcal{I} \to \mathcal{I}$ by surjectivity of τ. In particular, we obtain a group homomorphism $\psi : \mathrm{Cor}(\Gamma) \to \mathrm{Sym}(\mathcal{I})$ given by $\varphi \mapsto \varphi_{\mathcal{I}}$ with $\ker(\psi) = \mathrm{Aut}(\Gamma)$. If $\varphi_{\mathcal{I}}$ is an involution, then φ is called a *duality*. If moreover φ itself is an involution, then it is called a *polarity*. Examples of dualities and polarities in polytopes in $\mathbb{E}^n$, $n \geq 2$, are rotations and reflections, respectively. We have the following lemma.

Lemma 4.6.3. *Let $\Gamma = (X, *, \tau)$ be the $\mathcal{I}$-geometry over $\mathcal{I} = \{1, \ldots, n\}$, $n \geq 1$, whose elements of type i, $i \in \mathcal{I}$, are the i-dimensional subspaces W of an $(n+1)$-dimensional vector space V over a division ring $\mathbb{K}$ equipped with a non-degenerate reflexive sesquilinear form $f : V \times V \to \mathbb{K}$ and in which incidence is defined by inclusion. Then the map $\pi : \Gamma \to \Gamma$ given by $\pi(W) = W^{\perp}$ is a polarity of Γ.*

Proof. Let $W, W' \subset V$ be non-trivial subspaces. By writing $W^{\perp} = \bigcap_{w \in W} w^{\perp}$ and by noting that non-degeneracy of f implies that $v^{\perp}$ is a hyperplane for all $v \in V$, in particular for all $w \in W$, we deduce that $\dim(W^{\perp}) = \dim(V) - \dim(W)$. Then $\tau(W) = \tau(W') \iff \dim(W) = \dim(W') \iff \dim(V) - \dim(W) = \dim(V) - \dim(W') \iff \dim(W^{\perp}) = \dim(W'^{\perp}) \iff \tau(\pi(W)) = \tau(\pi(W'))$. Moreover, if $W * W'$, which we may assume to mean $W \subseteq W'$, then for all $x \in W'^{\perp}$ we have $f(w, x) = 0$ for all $w \in W'$, in particular for all $w \in W$, hence $f(w, x) = 0$ for all $w \in W$ so that $x \in W^{\perp}$. It follows that $W'^{\perp} \subseteq W^{\perp}$, therefore $W * W' \implies W^{\perp} * W'^{\perp} = \pi(W) * \pi(W')$. This shows that π is a weak homomorphism from Γ to itself. Clearly, it is bijective and its inverse is also a weak homomorphism, turning π into an auto-correlation from Γ to itself.

We then know that $\pi : \Gamma \to \Gamma$ induces the permutation $\pi_{\mathcal{I}} : \mathcal{I} \to \mathcal{I}$ given by $\pi_{\mathcal{I}}(\dim(W)) = \dim(W^{\perp})$. Since $\dim((W^{\perp})^{\perp}) = \dim(V) - \dim(W^{\perp}) = \dim(V) - (\dim(V) -$

$\dim(W)) = \dim(W)$, we have $\pi_{\mathcal{I}}(\pi_{\mathcal{I}}(\dim(W))) = \pi_{\mathcal{I}}(\dim(W^\perp)) = \dim((W^\perp)^\perp) = \dim(W)$ for all non-trivial subspaces $W \subset V$. It follows that $\pi_{\mathcal{I}}$ is an involution. Furthermore, for any $x \in W$ we have $f(x, w) = 0$ for all $w \in W^\perp$. But f is reflexive, hence also $f(w, x) = 0$ so that $x \in (W^\perp)^\perp$. Together with $\dim((W^\perp)^\perp) = \dim(W)$, this implies $(W^\perp)^\perp = W$. In turn, we obtain $\pi(\pi(W)) = \pi(W^\perp) = (W^\perp)^\perp = W$, so π is an involution. We conclude that $\pi : \Gamma \to \Gamma$ is a polarity of Γ. □

Given a projective space $\mathbb{P} = (\mathcal{P}, \mathcal{L})$, we denote by $\Gamma(\mathbb{P})$ the $\mathcal{I}$-geometry over $\mathcal{I} = \{1, \ldots, n\}$ whose elements of type i, $i \in \mathcal{I}$, are the $(i-1)$-dimensional subspaces of $\mathbb{P}$ and in which incidence is defined by inclusion. If $\mathbb{P}$ contains three non-collinear points and satisfies Desargues' theorem, then $\mathbb{P} \cong \mathbb{P}(V)$ for some vector space V over a division ring $\mathbb{K}$ by Theorem 4.1.22, in which case $\Gamma(\mathbb{P})$ is as described in the above lemma. Equipping V with a non-degenerate reflexive sesquilinear form will then yield a polarity $\pi : \Gamma \to \Gamma$. In particular, π maps points of $\mathbb{P}(V)$ to hyperplanes of $\mathbb{P}(V)$, which are subspaces of $\mathbb{P}(V)$ that intersect every line of $\mathbb{P}(V)$ in one or all of its points. Any two points $p, q \in \mathbb{P}(V)$ satisfy $(p^\perp)^\perp = p$ and $(q^\perp)^\perp = q$ so that $p \in q^\perp \implies q = (q^\perp)^\perp \in p^\perp$ and $q \in p^\perp \implies p = (p^\perp)^\perp \in q^\perp$, hence $p \in \pi(q) \iff p \in q^\perp \iff q \in p^\perp \iff q \in \pi(p)$. This leads to the following definition.

Definition 4.6.4 (Quasi-polarity)**.** *Let $\mathbb{P} = (\mathcal{P}, \mathcal{L})$ be a projective space and denote by $\mathcal{H}$ its set of hyperplanes. A map $\pi : \mathcal{P} \to \mathcal{H} \cup \mathcal{P}$ is called a **quasi-polarity** if $p \in \pi(q) \iff q \in \pi(p)$ for all $p, q \in \mathcal{P}$.*

The *kernel* of a quasi-polarity $\pi : \mathcal{P} \to \mathcal{H} \cup \mathcal{P}$ is the set $\ker(\pi) = \{p \in \mathcal{P} \mid \pi(p) = \mathcal{P}\}$, and π is said to be *non-degenerate* if $\ker(\pi) = 0$. In particular, $\ker(\pi) \subseteq \mathcal{P}$ is a subspace of $\mathbb{P}$ if $\mathbb{P}$ contains at least three non-collinear points. To see this, let $p, q \in \ker(\pi)$ and consider a point r on the line through p and q, which is unique because $\mathbb{P}$ is a projective space. Then $r \in \pi(s)$ for all $s \in \mathcal{P}$ as $\pi(p) = \mathcal{P} = \pi(q)$ implies that $s \in \pi(p)$ and $s \in \pi(q)$ for all $s \in \mathcal{P}$ so that $p, q \in \pi(s)$. But then $s \in \pi(r)$ for all $s \in \mathcal{P}$, showing that $\mathcal{P} \subseteq \pi(r)$ and forcing equality, whence $r \in \ker(\pi)$. It follows that the line through p and q is contained in $\ker(\pi)$, thus $\ker(\pi)$ is a subspace of $\mathbb{P}$.

Example 4.6.5. Consider again the projective space $\mathbb{P}(V) = (\mathcal{P}, \mathcal{L})$ on a 3-dimensional vector space V over $\mathbb{F}_2$. As we have seen in Example 4.1.18 and Example 4.1.9, it is isomorphic to the Fano plane, so we may identify $\mathbb{P}(V)$ by Figure 2, its set of hyperplanes being $\mathcal{L}$. Any plane duality $\delta : \mathbb{P}(V) \to \mathbb{P}(V)^*$ on $\mathbb{P}(V)$ sending a point $p \in \mathcal{P}$ to a line $\ell \in \mathcal{L}$ such that $p \in \ell \iff \delta(\ell) \in \delta(p)$ with $\delta(\ell) = \{\delta(p) \in \mathcal{L} \mid p \in \ell\}$ induces a quasi-polarity $\pi : \mathbb{P}(V) \to \mathbb{P}(V)$ on $\mathbb{P}(V)$ sending a point $p \in \mathcal{P}$ to $\delta(p) \in \mathcal{L}$; indeed, as $\delta^2 = \mathrm{id}_{\mathcal{P}}$, we have $p \in \pi(q) \iff \delta(\delta(p)) = p \in \delta(q) \iff q = \delta(\delta(q)) \in \delta(p) \iff q \in \pi(p)$.

Let $\pi : \mathcal{P} \to \mathcal{H} \cup \mathcal{P}$ be a quasi-polarity on a projective space $\mathbb{P}(V) = (\mathcal{P}, \mathcal{L})$ for some vector space V of dimension at least three over a division ring $\mathbb{K}$ with $\mathcal{H}$ the set of hyperplanes of $\mathbb{P}(V)$. Upon endowing V with a non-degenerate sesquilinear form f :

$V \times V \to \mathbb{K}$, we may assume that $\pi(p) = p^\perp$ for all $p \in \mathcal{P}$ as non-degeneracy of f implies that $p^\perp \in \mathcal{H}$ for all $p \in \mathcal{P}$. But then we find for all $p, q \in \mathcal{P}$ that $f(p,q) = 0 \iff q \in p^\perp \iff p \in q^\perp \iff f(q,p) = 0$, turning f into a reflexive form. Conversely, given a non-degenerate reflexive sesquilinear form $f : V \times V \to \mathbb{K}$, we define $\pi : \mathcal{P} \to \mathcal{H} \cup \mathcal{P}$ to be the map given by $\pi(p) = \{q \in \mathcal{P} \mid f(p,q) = 0\} = p^\perp$, which is a hyperplane in $\mathcal{H}$ by non-degeneracy of f. But then $q \in p^\perp \iff f(p,q) = 0 \iff f(q,p) = 0 \iff p \in q^\perp$, which turns π into a quasi-polarity.

The above discussion shows that we may identify any quasi-polarity by a non-degenerate reflexive sesquilinear form and vice versa. This brings us to the following corollary.

Corollary 4.6.6. *Let $\mathbb{P}(V) = (\mathcal{P}, \mathcal{L})$ be a projective space for some vector space V of dimension at least three over a division ring $\mathbb{K}$ and denote by $\mathcal{H}$ its set of hyperplanes. Then any quasi-polarity $\pi : \mathcal{P} \to \mathcal{H} \cup \mathcal{P}$ is a quasi-polarity $\pi_f : \mathcal{P} \to \mathcal{H} \cup \mathcal{P}$ induced by either a symmetric bilinear, alternating bilinear or Hermitian form $f : V \times V \to \mathbb{K}$.*

Proof. By the above, any quasi-polarity $\pi : \mathcal{P} \to \mathcal{H} \cup \mathcal{P}$ of $\mathbb{P}(V)$ gives rise to a non-degenerate reflexive sesquilinear form $f : V \times V \to \mathbb{K}$, and conversely, any non-degenerate reflexive sesquilinear form $f : V \times V \to \mathbb{K}$ gives rise to a quasi-polarity $\pi : \mathcal{P} \to \mathcal{H} \cup \mathcal{P}$ of $\mathbb{P}(V)$. Therefore, any quasi-polarity $\pi : \mathcal{P} \to \mathcal{H} \cup \mathcal{P}$ is a quasi-polarity $\pi_f : \mathcal{P} \to \mathcal{H} \cup \mathcal{P}$ induced by some non-degenerate reflexive sesquilinear form $f : V \times V \to \mathbb{K}$. But every non-degenerate reflexive sesquilinear form $f : V \times V \to \mathbb{K}$ is proportional to a symmetric bilinear form, an anti-symmetric form or a Hermitian form by Theorem 2.2.8. The corollary then follows immediately. □

We construct a point-line geometry from a projective space $\mathbb{P}(V) = (\mathcal{P}, \mathcal{L})$ for some vector space V over a division ring $\mathbb{K}$ using a quasi-polarity $\pi : \mathcal{P} \to \mathcal{H} \cup \mathcal{P}$ with $\mathcal{H}$ the set of hyperplanes of $\mathbb{P}(V)$. It will be the point-line geometry Γ_π having point set $\mathcal{P}_\pi = \{p \in \mathcal{P} \mid p \in \pi(p)\}$, its elements being referred to as *absolute points*, and line set $\mathcal{L}_\pi = \{\ell \in \mathcal{L} \mid \forall p \in \ell : p \in \pi(p)\}$, its elements being referred to as *absolute lines*. We will refer to Γ_π as the *absolute* of π in $\mathbb{P}(V)$.

By the above corollary, $\pi : \mathcal{P} \to \mathcal{H} \cup \mathcal{P}$ is a quasi-polarity $\pi_f : \mathcal{P} \to \mathcal{H} \cup \mathcal{P}$ for some non-degenerate reflexive sesquilinear form $f : V \times V \to \mathbb{K}$. Moreover, any absolute point $p \in \mathcal{P}_\pi$ satisfies $p \in \pi_f(p) \iff f(p,p) = 0$ so that p is an f-isotropic vector in V, and any absolute line $\ell \in \mathcal{L}_\pi$ satisfies $\forall p \in \ell : p \in \pi(p) \iff \forall p \in \ell : f(p,p) = 0$, hence ℓ is a totally f-isotropic subspace of V. We obtain a point-line geometry Γ_f whose points are the f-isotropic vectors of V and whose lines are the 2-dimensional totally f-isotropic subspaces of V. In particular, we have $\Gamma_\pi = \Gamma_f$. This leads to the following.

Lemma 4.6.7. *Let $\mathbb{P}(V) = (\mathcal{P}, \mathcal{L})$ be the projective space of some vector space V of dimension at least three over a division ring $\mathbb{K}$ having at least three non-collinear points and let $f : V \times V \to \mathbb{K}$ be a non-degenerate reflexive sesquilinear form on V. Then the point-line geometry Γ_f whose points are the f-isotropic vectors of V and whose lines are the 2-dimensional totally f-isotropic subspaces of V is a polar space.*

Proof. By Corollary 4.6.6, f induces a polarity $\pi_f : \mathcal{P} \to \mathcal{H} \cup \mathcal{P}$ with $\mathcal{H}$ the set of hyperplanes of $\mathbb{P}(V)$. So, we may equivalently show that the absolute $\Gamma_{\pi_f} = (\mathcal{P}_{\pi_f}, \mathcal{L}_{\pi_f})$ of π_f in $\mathbb{P}(V)$ is a polar space.

Let $p \in \mathcal{P}_\pi$ and $\ell \in \mathcal{L}_\pi$. Then $p \in \pi(p)$, but also $\pi(p) \in \mathcal{H}$ by non-degeneracy of f. Therefore, $|\pi(p) \cap \ell| = 1$ or $|\pi(p) \cap \ell| = |\ell|$. In the former case, p is collinear with a unique point on ℓ, whereas p is collinear with all points of ℓ in the latter case. It follows that Γ_π, hence also Γ_f, is a polar space. □

By Theorem 2.2.8, we may assume the form $f : V \times V \to \mathbb{K}$ to be either symmetric bilinear, alternating bilinear or Hermitian. Up to isomorphism, this limits the possibilities for Γ_f; it provides a first step towards classifying polar spaces. The following definition introduces another type of form on V that is required for this classification. Recall the definition of an admissible pair from Definition 2.2.5.

Definition 4.6.8 (Generalised pseudo-quadratic form)**.** *Let V be a vector space over a division ring $\mathbb{K}$ and let $\mathbb{K}_{\sigma,\epsilon} = \{\lambda - \epsilon\lambda^\sigma \mid \lambda \in \mathbb{K}\}$ with (σ, ϵ) an admissible pair. A map $Q : V \to \mathbb{K}/\mathbb{G}$ with $\mathbb{G}$ a proper additive subgroup of $\mathbb{K}$ containing $\mathbb{K}_{\sigma,\epsilon}$ is called a* ***generalised pseudo-quadratic form*** *if $Q(\lambda v) = \lambda Q(v)\lambda^\sigma$ for all $v \in V$ and $\lambda \in \mathbb{K}$.*

The additive subgroup $\mathbb{G}$ of $\mathbb{K}$ containing $\mathbb{K}_{\sigma,\epsilon}$ satisfies $\lambda\mathbb{G}\lambda^\sigma \subseteq \mathbb{G}$ for all $\lambda \in \mathbb{K}$ and is called a *(σ, ϵ)-form parameter*. If $\mathbb{G} = \{0\}$, the form Q is called a *quadratic form*, whereas it is called a *pseudo-quadratic form* if $\mathbb{G} = \mathbb{K}_{\sigma,\epsilon}$. A generalised pseudo-quadratic form $Q : V \to \mathbb{K}/\mathbb{G}$ induces a (σ, ϵ)-sesquilinear form $f_Q : V \times V \to \mathbb{K}/\mathbb{G}$ satisfying $Q(u+v) = Q(u)+Q(v)+(f(u, v)+\mathbb{G})$ for all $u, v \in V$, referred to as the *sesquilinearisation* of Q. The *radical* of Q is the set $\mathrm{rad}(Q) = Q^{-1}(\mathbb{G}) \cap \mathrm{rad}(f_Q)$, and Q is called *non-degenerate* if $\mathrm{rad}(Q) = \{0\}$ and *degenerate* otherwise. A vector $v \in V$ is said to be *Q-singular* if $Q(v) \in \mathbb{G}$, and a subspace $W \subseteq V$ is said to be *totally Q-singular* if $Q(w) \in \mathbb{G}$ for all $w \in W$. The dimension of a maximal totally Q-singular subspace of V is an invariant, referred to as the *Witt index of Q*.

Example 4.6.9. Let $\mathbb{F}_q$ be the finite field with q a prime power such that $\mathrm{char}(\mathbb{F}_q) = 2$, and let $V = \mathbb{F}_q^{2n}$, with $n \geq 1$ an integer, be a $2n$-dimensional vector space over $\mathbb{F}_q$. Further let (σ, ϵ) be an admissible pair and let $\mathbb{K}_{\sigma,\epsilon} = \{\lambda - \epsilon\lambda^\sigma \mid \lambda \in \mathbb{K}\}$. Endow V with the form $Q : V \to \mathbb{K}/\mathbb{K}_{\sigma,\epsilon}$ given by $Q(v) = v_1v_2 + \cdots + v_{2n-1}v_{2n}$ with $v = (v_1, \ldots, v_{2n})^\top \in V$. By noting that $Q(\lambda v) = \lambda^2 Q(v)$, we take $\sigma = \mathrm{id}_{\mathbb{F}_q}$ so that Q will be a pseudo-quadratic form. An easy calculation shows that $Q(u + v) = Q(u) + Q(v) + u^\top Av$ for all $u, v \in V$, in which A is the block-diagonal matrix with blocks $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ so that $f_Q(u, v) = u^\top Av$ is the sesquilinearisation of Q. It is a symmetric reflexive sesquilinear form since $f(u, v) = f(v, u)$ for all $u, v \in V$, forcing $(\sigma, \epsilon) = (\mathrm{id}_{\mathbb{F}_q}, 1)$ so that $\mathbb{K}_{\sigma,\epsilon} = \{0\}$. We have $f(u, u) = 2Q(u) = 0$ because $\mathrm{char}(\mathbb{F}_q) = 2$, hence every vector in V is f_Q-isotropic. In addition, it is readily seen that f_Q is non-degenerate, so Q will be non-degenerate as well.

A (pseudo-)quadratic form $Q : V \to \mathbb{K}/\mathbb{G}$ gives rise to a point-line geometry $\Gamma_Q = (\mathcal{P}_Q, \mathcal{L}_Q)$ whose points are the Q-singular vectors in V and whose lines are the 2-dimensional totally Q-singular subspaces of V. This point-line geometry is called the *(pseudo-)quadric* of Q in V and has the following properties.

Proposition 4.6.10. *Let V be a vector space of dimension at least three over some division ring $\mathbb{K}$ and assume that the projective space $\mathbb{P}(V)$ of V contains at least three non-collinear points. Further let $Q : V \to \mathbb{K}/\mathbb{G}$ with $\mathbb{G} \supseteq \mathbb{K}_{\sigma,\epsilon} = \{\lambda - \epsilon\lambda^\sigma \mid \lambda \in \mathbb{K}\}$ for some admissible pair (σ, ϵ) be a non-trivial (pseudo-)quadratic form of V. Then the (pseudo-)quadric $\Gamma_Q = (\mathcal{P}_Q, \mathcal{L}_Q)$ of Q in V is a polar space, and it coincides with the absolute $\Gamma_{f_Q} = (\mathcal{P}_{f_Q}, \mathcal{L}_{f_Q})$ of the sesquilinearisation $f_Q : V \times V \to \mathbb{K}/\mathbb{G}$ of Q if $\mathbb{K}_{\sigma,\epsilon} = \mathbb{K}^{\sigma,\epsilon} = \{\lambda \in \mathbb{K} \mid \lambda + \epsilon\lambda^\sigma = 0\}$.*

In particular, Γ_Q is non-degenerate if $\mathcal{P}_Q$ spans $\mathbb{P}(V)$ and if Γ_{f_Q} is non-degenerate.

Proof. As $0 \in \mathbb{K}_{\sigma,\epsilon} \subseteq \mathbb{G}$, it suffices to prove the proposition in case Q is pseudo-quadratic, i.e. $\mathbb{G} = \mathbb{K}_{\sigma,\epsilon}$. For the first assertion, we show that Γ_Q is a subspace of Γ_{f_Q}; since Γ_{f_Q} is a polar space by Lemma 4.6.7, then Γ_Q will be a polar space as well.

First, we show that $\mathcal{P}_Q \subseteq \mathcal{P}_{f_Q}$. So, let $p \in \mathcal{P}_Q$, i.e. $Q(p) \in \mathbb{K}_{\sigma,\epsilon}$, then also $Q(\lambda p) = \lambda Q(p)\lambda^\sigma \in \mathbb{K}_{\sigma,\epsilon}$. By observing that for all $\lambda, \mu \in \mathbb{K}$ such that $\lambda - \epsilon\lambda^\sigma, \mu - \epsilon\mu^\sigma \in \mathbb{K}_{\sigma,\epsilon}$ we have $(\lambda - \epsilon\lambda^\sigma) \pm (\mu - \epsilon\mu^\sigma) = (\lambda \pm \mu) - \epsilon(\lambda \pm \mu)^\sigma \in \mathbb{K}_{\sigma,\epsilon}$, we deduce that $\mathbb{K}_{\sigma,\epsilon}$ is closed under addition and subtraction. Therefore, $\lambda f_Q(p,p)\mu^\sigma = f_Q(\lambda p, \mu p) = Q(\lambda p + \mu p) - Q(\lambda p) - Q(\mu p) + \mathbb{K}_{\sigma,\epsilon} \in \mathbb{K}_{\sigma,\epsilon}$ for all $\lambda, \mu \in \mathbb{K}$. But then $f_Q(p,p) \neq 0$ forces $\mathbb{K}_{\sigma,\epsilon} = \mathbb{K}$ so that $\mathbb{K}/\mathbb{K}_{\sigma,\epsilon} = \{0\}$, contradicting that Q is non-trivial. It follows that $f_Q(p,p) = 0$, hence $p \in \mathcal{P}_{f_Q}$ and so $\mathcal{P}_Q \subseteq \mathcal{P}_{f_Q}$.

Next, we show that $\mathcal{L}_Q \subseteq \mathcal{L}_{f_Q}$, so let $\ell_{pq} \in \mathcal{L}_Q$ be a line on two collinear points $p, q \in \mathcal{P}_Q$. To show that $\ell_{pq} \in \mathcal{L}_{f_Q}$, it suffices to show that $f(p,q) = 0$; indeed, if this is the case then

$$f_Q(\lambda p + \mu q, \lambda p + \mu q) = \lambda f_Q(p,p)\lambda^\sigma + \lambda f_Q(p,q)\mu^\sigma + \mu f_Q(q,p)\lambda^\sigma + \mu f_Q(q,q)\mu^\sigma = 0$$

for all $\lambda, \mu \in \mathbb{K}$, since $p, q \in \mathcal{P}_Q \subseteq \mathcal{P}_{f_Q}$ implies $f_Q(p,p) = 0 = f_Q(q,q)$ and since reflexivity of f_Q by Proposition 2.2.6 implies $f_Q(p,q) = 0 = f_Q(q,p)$. By definition, $Q(\lambda p+\mu q) \in \mathbb{K}_{\sigma,\epsilon}$ for all $\lambda, \mu \in \mathbb{K}$, hence $\lambda f_Q(p,q)\mu^\sigma = f_Q(\lambda p, \mu q) = Q(\lambda p+\mu q) - Q(\lambda p) - Q(\mu q) + \mathbb{K}_{\sigma,\epsilon} \in \mathbb{K}_{\sigma,\epsilon}$ for all $\lambda, \mu \in \mathbb{K}$. As before, we cannot have $f_Q(p,q) \neq 0$ for otherwise Q would be trivial. But then $f_Q(p,q) = 0$ and so $\ell_{pq} \in \mathcal{L}_{f_Q}$ by the above, whence $\mathcal{L}_Q \subseteq \mathcal{L}_{f_Q}$. We conclude that $\Gamma_Q \subseteq \Gamma_{f_Q}$, thus Γ_Q is a polar space.

For the second assertion, it will suffice to prove that $\mathcal{P}_{f_Q} \subseteq \mathcal{P}_Q$, as then automatically $\mathcal{L}_{f_Q} \subseteq \mathcal{L}_Q$; indeed, if $\ell_{pq} \in \mathcal{L}_{f_Q}$, then $f_Q(\lambda p + \mu q, \lambda p + \mu q) = 0$ for all $\lambda, \mu \in \mathbb{K}$ implies $Q(\lambda p+\mu q) \in \mathbb{K}_{\sigma,\epsilon}$ for all $\lambda, \mu \in \mathbb{K}$ and thus $\ell_{pq} \in \mathcal{L}_Q$. So, let $p \in \mathcal{P}_{f_Q}$. Because $f_Q(p,p) = 0$, we have for all $\lambda, \mu \in \mathbb{K}$ that

$$\begin{aligned}\mathbb{K}_{\sigma,\epsilon} \ni Q((\lambda+\mu)p) - Q(\lambda p) - Q(\mu p) &= (\lambda+\mu)Q(p)(\lambda+\mu)^\sigma - \lambda Q(p)\lambda^\sigma - \mu Q(p)\mu^\sigma \\ &= \lambda Q(p)\mu^\sigma + \mu Q(p)\lambda^\sigma,\end{aligned}$$

hence

$$\mathbb{K}_{\sigma,\epsilon} \ni \lambda Q(p)\mu^\sigma + \mu Q(p)\lambda^\sigma - (\mu Q(p)\lambda^\sigma - \epsilon(\mu Q(p)\lambda^\sigma)^\sigma) = \lambda Q(p)\mu^\sigma + \epsilon\lambda^{\sigma^2} Q(p)^\sigma \mu^\sigma$$
$$= \lambda Q(p)\mu^\sigma + \epsilon(\epsilon^{-1}\lambda\epsilon)Q(p)^\sigma\mu^\sigma = \lambda(Q(p) + \epsilon Q(p)^\sigma)\mu^\sigma$$

as $\mathbb{K}_{\sigma,\epsilon}$ is closed under subtraction. But then $Q(p) + \varepsilon Q(p)^\sigma = 0$ for otherwise $\mathbb{K}_{\sigma,\epsilon} = \mathbb{K}$ so that $\mathbb{K}/\mathbb{K}_{\sigma,\epsilon} = \{0\}$, contradicting that Q is non-trivial. It follows that $Q(p) \in \mathbb{K}^{\sigma,\epsilon} = \mathbb{K}_{\sigma,\epsilon}$ and so $p \in \mathcal{P}_Q$.

For the third assertion, assume that $\mathcal{P}_Q$ spans $\mathbb{P}(V)$ and that Γ_{f_Q} is non-degenerate. Now suppose towards a contradiction that $r \in \mathrm{rad}(\Gamma_Q) = \mathcal{P}_Q^\perp$. Then for all $p \in \mathcal{P}_Q$ there is a line $\ell_{pr} \in \mathcal{L}_Q \subseteq \mathcal{L}_{f_Q}$ on p and r, which implies $f_Q(p,r) = 0$ by the above. But $\mathcal{P}_Q$ spans $\mathbb{P}(V)$, hence $\mathcal{P}_Q = \mathcal{P}_{f_Q}$ so that $f_Q(p,r) = 0$ for all $p \in \mathcal{P}_{f_Q}$. But then $r \in \mathrm{rad}(f_Q) = \emptyset$, a contradiction. It follows that $\mathrm{rad}(\Gamma_Q) = \emptyset$ so that Γ_Q is non-degenerate. □

Let $\Gamma = (X, *, \tau)$ be a thick residually connected geometry of type A_n, $n \geq 1$, over type set $\mathcal{I} = \{1, \ldots, n\}$. The *Grassmannian* of Γ of type $i \in \mathcal{I}$ is the shadow space $\mathrm{ShSp}_i(\Gamma)$ on i. Specifically, by Example 4.4.5, the shadow space $\mathrm{ShSp}_1(\Gamma)$ is the projective space $\mathbb{P}(V) = (\mathcal{P}, \mathcal{L})$ of some vector space V, $\dim(V) = n$, over a division ring $\mathbb{K}$, whereas the shadow space $\mathrm{ShSp}_2(\Gamma)$ is the point-line geometry obtained as the dual of $\mathbb{P}(V)$ whose point set is $\mathcal{L}$ and whose lines are the line pencils $\mathcal{L}_p = \{\ell \in \mathcal{L} \mid p \in \ell\}$ with $p \in \mathcal{P}$, all of whose members are contained in a plane of $\mathbb{P}(V)$. Note that such a plane is required to be singular in case $\mathbb{P}(V)$ is a polar space. The shadow space $\mathrm{ShSp}_2(\Gamma)$ is also referred to as the *Grassmannian of lines* of Γ, denoted by $A_{n,2}(\mathbb{K})$ if $\mathrm{ShSp}_2(\Gamma)$ has finite singular rank $n \geq 2$.

The above discussion enables us to define classical polar spaces as a means of classifying polar spaces.

Definition 4.6.11 (Classical polar space)**.** *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a polar space. If Γ is isomorphic to at least one of*

(i) *the Grassmannian of lines $A_{3,2}(\mathbb{K})$ of a thick projective space over some division ring $\mathbb{K}$ of dimension three,*

(ii) *the absolute Γ_f of a Hermitian form $f : V \times V \to \mathbb{K}$ in a vector space V over some division ring $\mathbb{K}$,*

(iii) *the pseudo-quadric Γ_Q of a pseudo-quadratic form $Q : V \to \mathbb{K}/\mathbb{K}_{\sigma,\epsilon}$ with $\mathbb{K}_{\sigma,\epsilon} = \{\lambda - \epsilon\lambda^\sigma \mid \lambda \in \mathbb{K}\}$ for some admissible pair (σ, ϵ) in a vector space V over some division ring $\mathbb{K}$,*

then Γ is said to be a ***classical polar space****.*

We have already seen an example of a classical polar space belonging to Definition 4.6.11(i); the root shadow space of type $BC_{n,2}$ or $D_{n+1,2}$ of the geometry $\Gamma(\mathcal{C})$ of a building $\mathcal{C} = (C, \{\sim_i \mid i \in \mathcal{I}\})$ of type B_n or D_{n+1}, $n \geq 3$, over type set $\mathcal{I} = \{1, \ldots, n\}$ as introduced in Example 4.5.2. Some more examples of classical polar spaces belonging to Definition 4.6.11(ii)-(iii) are given below.

Example 4.6.12. Let V be an $(n+1)$-dimensional vector space over $\mathbb{F}_{q^2}$ with q a prime power and $n \geq 1$ an integer. Define $\sigma : \mathbb{F}_{q^2} \to \mathbb{F}_{q^2}$ to be the map given by $\lambda^\sigma = \lambda^q$ with $\lambda \in \mathbb{F}_{q^2}$, which is an automorphism of $\mathbb{F}_{q^2}$ called the *Frobenius automorphism.* The form $f : V \times V \to \mathbb{F}_{q^2}$ given by $f(u, v) = u^\top v^\sigma$ is readily seen to be a non-degenerate Hermitian form on V. The points of the absolute Γ_f of f in V will be the vectors $v = (v_1, \ldots, v_{n+1})^\top \in V$ satisfying $f(v, v) = v_1^{q+1} + \cdots + v_{n+1}^{q+1} = 0$. We obtain the *Hermitian variety* in $\mathbb{P}(V) = \mathrm{PG}(n, q^2)$, denoted by $\mathrm{H}(n, q^2)$.

Recall from Example 4.6.9 the pseudo-quadratic form $Q : V \to \mathbb{K}$ given by $Q(v) = v_1v_2 + \cdots + v_{2n-1}v_{2n}$ with $v = (v_1, \ldots, v_{2n})^\top \in V$ on a $2n$-dimensional vector space over $\mathbb{F}_q$ with $n \geq 1$. The points of the pseudo-quadric Γ_Q of Q in V will be the vectors $v \in V$ satisfying $Q(v) = 0$. This yields the *hyperbolic quadric* in $\mathbb{P}(V) = \mathrm{PG}(2n-1, q)$, denoted by $\mathrm{Q}^+(2n-1, q)$.

The classification of polar spaces of finite rank is due to Tits [8], Buekenhout and Shult [16] and Veldkamp [12], with extensions to polar spaces of infinite rank due to Buekenhout [31], Johnson [30] and Pasini et al. [22]. We will, however, adapt the version presented in [32] for the sake of notational consistency.

Theorem 4.6.13. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a non-degenerate thick polar space of finite rank at least three. Then Γ is either*

(i) *the pseudo-quadric Γ_Q of a pseudo-quadratic form $Q : V \to \mathbb{K}/\mathbb{K}_{\sigma,\epsilon}$ with $\mathbb{K}_{\sigma,\epsilon} = \{\lambda - \epsilon\lambda^\sigma \mid \lambda \in \mathbb{K}\}$ for some admissible pair (σ, ϵ) in a vector space V over some division ring $\mathbb{K}$, or*

(ii) *the absolute Γ_f of a symplectic form $f : V \times V \to \mathbb{K}$ in a vector space V over a field $\mathbb{F}$ having characteristic different from two, or*

(iii) *the Grassmannian of lines $A_{3,2}(\mathbb{K})$ of a thick projective space over some division ring $\mathbb{K}$ of dimension three, or*

(iv) *the root shadow space of type $E_{7,1}^K$ of a building of type E_7^K with E_7^K the adjoint group of the simple group of exceptional Lie type E_7 corresponding to the Cayley algebra K over some field $\mathbb{F}$.*

Proof. See Theorem 1 of [16]. □

For a classification of the finite classical polar spaces, two of which we have already discussed in Example 4.6.12, we refer to [10].

We finish this section with a classification of polar spaces obtained as the image $\varepsilon(\Gamma)$ of a polar space $\Gamma = (\mathcal{P}, \mathcal{L})$ admitting an embedding $\varepsilon : \Gamma \hookrightarrow \mathbb{P}(V)$ in the projective space $\mathbb{P}(V)$ of some vector space V over a division ring $\mathbb{K}$. To do so, we formalise the concept of embeddings.

Definition 4.6.14 (Projective embedding)**.** *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry and let $\mathbb{P}(V)$ be a projective space for some vector space V over a division ring $\mathbb{K}$. A* ***projective embedding*** *of Γ in $\mathbb{P}(V)$ is an injective map $\varepsilon : \Gamma \hookrightarrow \mathbb{P}(V)$ such that $\varepsilon(\mathcal{P}) = \bigcup_{p\in\mathcal{P}} \varepsilon(p)$ spans $\mathbb{P}(V)$ and $\varepsilon(\mathcal{L}) = \bigcup_{\ell\in\mathcal{L}} \varepsilon(\ell)$ with $\varepsilon(\ell) = \{\varepsilon(p) \mid p \in \ell\}$ is a subset of the set of projective lines of $\mathbb{P}(V)$.*

A point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ that admits an embedding $\varepsilon : \Gamma \hookrightarrow \mathbb{P}(V)$ in the projective space $\mathbb{P}(V)$ of some vector space V over a division ring $\mathbb{K}$ is said to be *embeddable*. If $\varepsilon' : \Gamma \hookrightarrow \mathbb{P}(V')$ is another embedding of Γ in a vector space V' over $\mathbb{K}$, we say that ε and ε' are *(iso)morphic* if there exists a surjective (iso)morphism $\varphi : \mathbb{P}(V) \twoheadrightarrow \mathbb{P}(V')$ such that $\varepsilon' = \varphi \circ \varepsilon$. We have $\ker(\varphi) \cap \varepsilon(\mathcal{P}) = \emptyset$, and the intersection of $\ker(\varphi)$ with $\langle p, q\rangle$ will be trivial for all distinct points $p, q \in \mathcal{P}$. The embedding ε is also said to *cover* ε', which we will denote by $\varepsilon \to \varepsilon'$. If φ is an isomorphism or if $\varepsilon \to \varepsilon' \to \varepsilon$, the embeddings ε and ε' are isomorphic and we write $\varepsilon \cong \varepsilon'$. An embedding $\varepsilon : \Gamma \hookrightarrow \mathbb{P}(V)$ of Γ in $\mathbb{P}(V)$ is called *relatively universal* if $\varepsilon \cong \varepsilon'$ for any embedding $\varepsilon' : \Gamma \hookrightarrow \mathbb{P}(V)$ of Γ in $\mathbb{P}(V)$ such that $\varepsilon' \to \varepsilon$. If moreover $\varepsilon \to \varepsilon'$, then ε is said to be *absolutely universal*. The ε-image $\varepsilon(\mathcal{S})$ of a subspace $\mathcal{S}$ of Γ will be a subspace of $\mathbb{P}(V)$, and conversely, the preimage $\varepsilon^{-1}(\mathcal{S})$ of a subspace $\mathcal{S}$ of $\mathbb{P}(V)$ under ε will be a subspace of Γ.

Example 4.6.15. Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be the point-line geometry with $\mathcal{P} = \{p_1, \dots, p_5\}$ and $\mathcal{L} = \{\ell_1, \ell_2\} = \{\{p_1, p_2, p_3\}, \{p_1, p_4, p_5\}\}$, which is clearly connected. Further let $\mathbb{P}(V) = \mathrm{PG}(2, 2)$ be projective space of a 3-dimensional vector space V over $\mathbb{F}_2$, which we have seen is the Fano plane in Example 4.1.18, with point set $\{p'_1, \dots, p'_7\}$ and line set $\{\ell'_1, \dots, \ell'_7\}$. Under the harmless assumption that $\ell'_1 = \{p'_1, p'_2, p'_3\}$ and $\ell'_2 = \{p'_1, p'_4, p'_5\}$, the map $\varepsilon : \Gamma \hookrightarrow \mathrm{PG}(2, 2)$ given by $\varepsilon(p_i) = p'_i$, $1 \le i \le 5$, will be an embedding of Γ in $\mathrm{PG}(2, 2)$; indeed, the map is clearly injective, the smallest subspace of $\mathrm{PG}(2, 2)$ containing $\varepsilon(\mathcal{P})$ is the entire point set of $\mathrm{PG}(2, 2)$, and $\varepsilon(\mathcal{L}) = \{\ell'_1, \ell'_2\}$ is a subset of the line set of $\mathrm{PG}(2, 2)$. Now let $\varepsilon' : \Gamma \hookrightarrow \mathrm{PG}(2, 2)$ be an embedding of Γ in $\mathrm{PG}(2, 2)$ such that $\varepsilon' \to \varepsilon$. It then follows that $\varepsilon \cong \varepsilon'$ from the observation that the isomorphism $\varphi : \mathrm{PG}(2, 2) \to \mathrm{PG}(2, 2)$ mapping $\varepsilon'(\mathcal{L})$ to $\varepsilon(\mathcal{L})$ yields $\varepsilon = \varphi \circ \varepsilon'$ so that ε is relatively universal in $\mathrm{PG}(2, 2)$. In particular, we have $\varepsilon' = \varphi^{-1} \circ \varepsilon$, hence $\varepsilon \to \varepsilon'$, which shows that ε is absolutely universal in $\mathrm{PG}(2, 2)$. Clearly, the only proper subspaces of Γ are its points $\{p_i\}$, $1 \le i \le 5$, and its lines ℓ_1 and ℓ_2. Their images under ε yield the subspaces $\{p'_i\}$, $1 \le i \le 5$, ℓ'_1 and ℓ'_2 of $\mathrm{PG}(2, 2)$. The converse is true by taking the images under ε^{-1}.

The classification of polar spaces obtained as the image $\varepsilon(\Gamma)$ of a polar space $\Gamma = (\mathcal{P}, \mathcal{L})$ admitting an embedding $\varepsilon : \Gamma \hookrightarrow \mathbb{P}(V)$ in the projective space $\mathbb{P}(V)$ of some vector space V over a division ring $\mathbb{K}$ is due to Tits [8] and relates embedded polar spaces to sesquilinear forms and pseudo-quadratic forms. We present this classification as the following theorem, adopting the notation from Theorem 2.1 of [17].

Theorem 4.6.16. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a polar space of rank $n \geq 2$ admitting a relatively universal embedding $\varepsilon : \Gamma \hookrightarrow \mathbb{P}(V)$ in the projective space $\mathbb{P}(V)$ of some vector space V over a division ring $\mathbb{K}$. Then either $\varepsilon(\Gamma) = \Gamma_Q$ for a non-degenerate pseudo-quadratic form $Q : V \to \mathbb{K}/\mathbb{K}_{\sigma,\epsilon}$ on V with $\mathbb{K}_{\sigma,\epsilon} = \{\lambda - \epsilon\lambda^\sigma \mid \lambda \in \mathbb{K}\}$ for some admissible pair (σ, ϵ), or $\mathrm{char}(\mathbb{K}) \neq 2$ and $\varepsilon(\Gamma) = \Gamma_f$ for a non-degenerate alternating bilinear form $f : V \times V \to \mathbb{K}$ on V.*

In particular, ε will be absolutely universal except if $n = 2$, $\dim(V) = 4$ and either $\varepsilon(\Gamma) = \mathrm{Q}^+(3, q)$ or $\mathbb{K}$ is a quaternion division ring with $\mathbb{K}_{\sigma,\epsilon}$ being 1-dimensional over $Z(\mathbb{K})$ and $\sigma|_{Z(\mathbb{K})} = \mathrm{id}_{Z(\mathbb{K})}$.

Proof. See Corollary 8.7 of [8]. □

4.7 Embeddings of root filtration spaces

We finish our discussion of polar spaces and instead continue with projective embeddings of a non-degenerate root filtration space $\Gamma = (\mathcal{P}, \mathcal{L})$. We start with the following definition.

Definition 4.7.1 (Polarised embedding)**.** *Let $\varepsilon : \Gamma \hookrightarrow \mathbb{P}(V)$ be a projective embedding of an embeddable non-degenerate root filtration space $\Gamma = (\mathcal{P}, \mathcal{L})$ in the projective space $\mathbb{P}(V)$ of some vector space V over a division ring $\mathbb{K}$. The embedding $\varepsilon : \Gamma \hookrightarrow \mathbb{P}(V)$ is said to be* ***polarised*** *if $\varepsilon(p^\perp) = \varepsilon(\mathcal{P}_{\leq 1}(p))$ is contained in a hyperplane of $\mathbb{P}(V)$ for all $p \in \mathcal{P}$.*

Let $\varepsilon : \Gamma \hookrightarrow \mathbb{P}(V)$ be a polarised embedding of a non-degenerate root filtration space $\Gamma = (\mathcal{P}, \mathcal{L})$ with filtration $\{\mathcal{P}_i\}_{-2 \leq i \leq 2}$ in the projective space $\mathbb{P}(V)$ of some vector space V over a division ring $\mathbb{K}$. The *radical* of ε is the set $\mathrm{rad}(\varepsilon) = \bigcap_{p \in \mathcal{P}} \langle \varepsilon(\mathcal{P}_{\leq 1}(p)) \rangle$, i.e. the intersection of the spans of $\varepsilon(\mathcal{P}_{\leq 1}(p))$, $p \in \mathcal{P}$, in $\mathbb{P}(V)$.

Example 4.7.2. Consider again the point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ as described in Example 4.6.15. Using Proposition 4.5.3, we deduce that Γ is a root filtration space with filtration $\{\mathcal{P}_i\}_{-2 \leq i \leq 2}$ only if for all $1 \leq i, j \leq 5$ we have $(p_i, p_j) \in \mathcal{P}_{-2} \iff i = j$, $(p_i, p_j) \in \mathcal{P}_{-1} \iff p_i \perp p_j$, $(p_i, p_j) \in \mathcal{P}_1 \iff p_i \not\perp p_j$ and $\mathcal{P}_0 = \mathcal{P}_2 = \emptyset$. We verify that this set of relations on $\mathcal{P} \times \mathcal{P}$ turns Γ into a root filtration system.

Clearly, $\{\mathcal{P}_i\}_{-2 \leq i \leq 2}$ is set of disjoint relations partitioning $\mathcal{P} \times \mathcal{P}$. Definition 4.5.1(i) and Definition 4.5.1(ii) are immediate, and so is Definition 4.5.1(iv) because $\mathcal{P}_2 = \emptyset$. It is readily seen that $\mathcal{P}_{\leq 1}(p_i) = \mathcal{P}$ for all $1 \leq i \leq 5$, from which Definition 4.5.1(vi) follows. In addition, we have $\mathcal{P}_{\leq -1}(p_1) = \mathcal{P}$, whereas $\mathcal{P}_{\leq -1}(p_2) = \ell_1 = \mathcal{P}_{\leq -1}(p_3)$ and $\mathcal{P}_{\leq -1}(p_4) = \ell_2 = \mathcal{P}_{\leq 1}(p_5)$, from which Definition 4.5.1(v) follows because $\mathcal{P}_0 = \emptyset$. By recalling from our proof of Proposition 4.5.3 that the map $\varphi : \mathcal{P}_1 \to \mathcal{P}$ from Definition 4.5.1(iii) necessarily maps $(p_i, p_j) \in \mathcal{P}_1$, $1 \leq i, j \leq 5$, to their unique neighbour in the collinearity graph $(\mathcal{P}, \mathcal{P}_{-1})$ of Γ, we must have $\varphi(p_i, p_j) = p_1$ for all $1 \leq i, j \leq 5$. Since $\mathcal{P}_{\leq -1}(p_1) = \mathcal{P}$, it remains to check Definition 4.5.1(iii) in case $i + j = -2$, $1 \leq i, j \leq 5$. For $i = 0$ and $j = -2$ or $i = -2$ and $j = 0$, there is nothing to show as $\mathcal{P}_0 = \emptyset$. For $i = -1 = j$, we have $\mathcal{P}_{-1}(p_k) \cap \mathcal{P}_{-1}(p_l) = \{p_1\} = \mathcal{P}_{-2}(p_1) = \mathcal{P}_{\leq -2}(\varphi(p_k, p_l))$ for all $(p_k, p_l) \in \mathcal{P}_1$, $1 \leq k, l \leq 5$, as only p_1 is collinear to both p_k and p_l. Thus, Definition 4.5.1(iii) holds and so Γ is a root filtration space with filtration $\{\mathcal{P}_i\}_{-2 \leq i \leq 2}$.

We now claim that the projective embedding $\varepsilon : \Gamma \hookrightarrow \mathrm{PG}(2, 2)$ given in Example 4.6.15 is not a polarised embedding; indeed, we have established that $\mathcal{P}_{\leq 1}(p_i) = \mathcal{P}$ for all $1 \leq i \leq 5$, hence $\varepsilon(\mathcal{P}_{\leq 1}(p_i)) = \varepsilon(\mathcal{P})$ for all $1 \leq i \leq 5$, and since the span of $\varepsilon(\mathcal{P})$ is $\mathrm{PG}(2, 2)$, which is not a hyperplane, it follows that ε is not a polarised embedding.

We state and prove several properties of embeddable non-degenerate root filtration spaces pertaining to polarised embeddings and their radicals.

Lemma 4.7.3. *Let $\varepsilon : \Gamma \hookrightarrow \mathbb{P}(V)$ be a projective embedding of an embeddable non-degenerate root filtration space $\Gamma = (\mathcal{P}, \mathcal{L})$ with filtration $\{\mathcal{P}_i\}_{-2 \leq i \leq 2}$ in the projective space $\mathbb{P}(V)$ of some vector space V over a division ring $\mathbb{K}$. Further let $\varepsilon' : \Gamma \hookrightarrow \mathbb{P}(V)$ be a polarised embedding of Γ in $\mathbb{P}(V)$ such that $\varepsilon \to \varepsilon'$. Then ε is a polarised embedding and* $\mathrm{rad}(\varepsilon)$ *contains the kernel of the projection of ε to ε'.*

In particular, we have $\varepsilon/\mathrm{rad}(\varepsilon) \cong \varepsilon'$ if and only if $\mathrm{rad}(\varepsilon')$ *is trivial.*

Proof. As $\varepsilon \to \varepsilon'$, there exists a surjective map $\varphi : \mathbb{P}(V) \to \mathbb{P}(V)$ such that $\varepsilon' = \varphi \circ \varepsilon$. Specifically, $\varphi(\varepsilon(\mathcal{P}_{\leq 1}(p))) = \varepsilon'(\mathcal{P}_{\leq 1}(p))$ is contained in a hyperplane of $\mathbb{P}(V)$ for all $p \in \mathcal{P}$ because ε' is a polarised embedding. But φ maps hyperplanes of $\mathbb{P}(V)$ to hyperplanes of $\mathbb{P}(V)$, hence $\varepsilon(\mathcal{P}_{\leq 1}(p)))$ will also be contained in a hyperplane of $\mathbb{P}(V)$ for all $p \in \mathcal{P}$. It follows that ε is a polarised embedding.

To show that $\ker(\varphi) \subseteq \mathrm{rad}(\varepsilon) = \bigcap_{p \in \mathcal{P}} \langle \varepsilon(\mathcal{P}_{\leq 1}(p)) \rangle$, suppose towards a contradiction that this is not the case. Then there exists a point $p \in \mathcal{P}$ such that $\langle \varepsilon(\mathcal{P}_{\leq 1}(p)) \rangle$ does not contain $\ker(\varphi)$. But then necessarily $\mathbb{P}(V) = \varphi(\langle \varepsilon(\mathcal{P}_{\leq 1}(p)) \rangle) = \langle \varphi(\varepsilon(\mathcal{P}_{\leq 1}(p))) \rangle = \langle \varepsilon'(\mathcal{P}_{\leq 1}(p)) \rangle$, contradicting that ε' is polarised. Thus, $\mathrm{rad}(\varepsilon)$ contains $\ker(\varphi)$.

For the final assertion, suppose first that $\mathrm{rad}(\varepsilon')$ is trivial. As φ is surjective, we know that $\varepsilon/\ker(\varphi) \cong \mathrm{im}(\varphi) = \varepsilon'$ by the first isomorphism theorem. By the above, we additionally have $\ker(\varphi) \subseteq \mathrm{rad}(\varepsilon)$, so we need only show that $\mathrm{rad}(\varepsilon) \subseteq \ker(\varphi)$. To this extent, let $r \in \mathrm{rad}(\varepsilon)$, then

$$\varphi(r) \subseteq \varphi\left(\bigcap_{p \in \mathcal{P}} \langle \varepsilon(\mathcal{P}_{\leq -1}(p)) \rangle\right) = \bigcap_{p \in \mathcal{P}} \langle \varphi(\varepsilon(\mathcal{P}_{\leq 1}(p))) \rangle = \bigcap_{p \in \mathcal{P}} \langle \varepsilon'(\mathcal{P}_{\leq 1}(p)) \rangle = \mathrm{rad}(\varepsilon'),$$

hence $\varphi(r)$ is trivial since $\mathrm{rad}(\varepsilon')$ is trivial. Consequently, $r \in \ker(\varphi)$ and so $\mathrm{rad}(\varepsilon) \subseteq \ker(\varphi)$. We conclude that $\ker(\varphi) = \mathrm{rad}(\varepsilon)$ so that $\varepsilon/\mathrm{rad}(\varepsilon) \cong \varepsilon'$. Conversely, if $\varepsilon' \cong$

$\varepsilon/\mathrm{rad}(\varepsilon)$, then by taking radicals on both sides we obtain $\mathrm{rad}(\varepsilon') \cong \mathrm{rad}(\varepsilon/\mathrm{rad}(\varepsilon)) = \mathrm{rad}(\varepsilon)/\mathrm{rad}(\varepsilon) = \emptyset$, hence $\mathrm{rad}(\varepsilon')$ is trivial. □

The last assertion in the above lemma generalises to the first assertion in the proposition below, for which we require Γ to satisfy some additional conditions.

Proposition 4.7.4. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be an embeddable non-degenerate root filtration space admitting an absolutely universal embedding $\varepsilon : \Gamma \hookrightarrow \mathbb{P}(V)$ in the projective space $\mathbb{P}(V)$ of some vector space V over a division ring $\mathbb{K}$. If $\varepsilon' : \Gamma \hookrightarrow \mathbb{P}(V)$ is a polarised embedding of Γ in $\mathbb{P}(V)$, then $\varepsilon/\mathrm{rad}(\varepsilon) \cong \varepsilon'/\mathrm{rad}(\varepsilon')$.*

In particular, we have $\varepsilon'' \to \varepsilon'$ for any polarised embedding $\varepsilon'' : \Gamma \hookrightarrow \mathbb{P}(V)$ of Γ in $\mathbb{P}(V)$ if $\mathrm{rad}(\varepsilon')$ is trivial.

Proof. Since ε is absolutely universal, we have $\varepsilon \to \varepsilon'$, hence there exists a surjective map $\varphi : \mathbb{P}(V) \twoheadrightarrow \mathbb{P}(V)$ such that $\varepsilon' = \varphi \circ \varepsilon$. As in Lemma 4.7.3, we then have $\varepsilon' = \mathrm{im}(\varepsilon') \cong \varepsilon/\ker(\varphi)$ by the first isomorphism theorem. Taking radicals on both sides yields $\mathrm{rad}(\varepsilon') \cong \mathrm{rad}(\varepsilon/\ker(\varphi)) = \mathrm{rad}(\varepsilon)/\ker(\varphi)$. Then by the third isomorphism theorem we obtain $\varepsilon'/\mathrm{rad}(\varepsilon') \cong (\varepsilon/\ker(\varphi))/(\mathrm{rad}(\varepsilon)/\ker(\varphi)) \cong \varepsilon/\mathrm{rad}(\varepsilon)$, as desired.

Next, let ε'' be another polarised embedding and assume that $\mathrm{rad}(\varepsilon')$ is trivial. As ε is an absolutely universal embedding, we have $\varepsilon \to \varepsilon'$ and $\varepsilon \to \varepsilon''$, therefore there exist surjective maps $\varphi' : \mathbb{P}(V) \twoheadrightarrow \mathbb{P}(V)$ and $\varphi'' : \mathbb{P}(V) \twoheadrightarrow \mathbb{P}(V)$ such that $\varepsilon' = \varphi' \circ \varepsilon$ and $\varepsilon'' = \varphi'' \circ \varepsilon$. By Lemma 4.7.3, ε is a polarised embedding and $\mathrm{rad}(\varepsilon)$ contains both $\ker(\varphi')$ and $\ker(\varphi')$. In particular, we have $\varepsilon/\ker(\varphi') \cong \varepsilon' \cong \varepsilon/\mathrm{rad}(\varepsilon)$ by the first isomorphism theorem and again by Lemma 4.7.3 because $\mathrm{rad}(\varepsilon')$ is trivial. But then $\ker(\varphi') = \mathrm{rad}(\varepsilon)$, hence $\ker(\varphi'') \subseteq \mathrm{rad}(\varepsilon) = \ker(\varphi')$. This implies that there exists a surjective map $\varphi : \mathbb{P}(V) \twoheadrightarrow \mathbb{P}(V)$ such that $\varphi' = \varphi \circ \varphi''$. Consequently, we obtain $\varepsilon' = \varphi' \circ \varepsilon = (\varphi \circ \varphi'') \circ \varepsilon = \varphi \circ (\varphi'' \circ \varepsilon) = \varphi \circ \varepsilon''$, which shows that $\varepsilon'' \to \varepsilon'$. □

We finish this section with an important result on root shadow spaces regarding the existence and uniqueness of certain embeddings.

Theorem 4.7.5. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be an embeddable root shadow space of type $BC_{n,2}$ ($n \geq 3$), $D_{n,2}$ ($n \geq 4$), $E_{6,2}$, $E_{7,1}$, $E_{8,8}$ or $F_{4,1}$. Then Γ admits an absolutely universal embedding $\varepsilon : \Gamma \hookrightarrow \mathbb{P}(V)$ in the projective space $\mathbb{P}(V)$ of some vector space V over a division ring $\mathbb{K}$.*

In particular, Γ admits, up to isomorphism, a unique polarised embedding $\varepsilon' : \Gamma \hookrightarrow \mathbb{P}(V)$ in $\mathbb{P}(V)$ such that $\mathrm{rad}(\varepsilon')$ is trivial.

Proof. For the first assertion, see Theorem 2.8 of [13].

For the second assertion, assume that $\varepsilon'' : \Gamma \hookrightarrow \mathbb{P}(V)$ is a second polarised embedding of Γ in $\mathbb{P}(V)$ such that $\mathrm{rad}(\varepsilon'')$ is trivial. By Theorem 4.5.4, Γ is a non-degenerate root filtration space as Γ is not of type $BC_{n,1}$. Since Γ admits an absolutely universal embedding by the first assertion, we may apply Proposition 4.7.4. Therefore, we have $\varepsilon'' \to \varepsilon'$ because $\mathrm{rad}(\varepsilon')$ is trivial, but we also have $\varepsilon' \to \varepsilon''$ because $\mathrm{rad}(\varepsilon'')$ is trivial. It follows that $\varepsilon' \to \varepsilon'' \to \varepsilon'$ so that $\varepsilon' \cong \varepsilon''$. We conclude that ε' is unique up to isomorphism. □

Chapter 5

Lie algebras and geometry

In this chapter, we will combine the theory from the previous chapters to introduce the extremal geometry of a Lie algebra, in particular describe some of its properties as well as demonstrate its significance in terms of determining the Lie algebra it is defined on.

Throughout Section 5.1, we follow the theory and adopt the notation from [5, 13, 15, 23].

5.1 The extremal geometry of a Lie algebra

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ generated by its set $E(\mathfrak{g})$ of extremal elements and define $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ to be the point-line geometry whose points are the spans $\mathbb{F}x$ of an extremal element $x \in E(\mathfrak{g})$ and whose lines are the linear combinations $\mathbb{F}x + \mathbb{F}y$ of two commuting and linearly independent extremal elements $x, y \in E(\mathfrak{g})$ such that $\mathbb{F}x + \mathbb{F}y \subseteq E(\mathfrak{g}) \cup \{0\}$. Note that $\Gamma_{\mathfrak{g}}$ is a partial linear space; indeed, supposing that $z, z' \in E(\mathfrak{g})$ are distinct extremal elements in $\mathfrak{g}$ such that $z, z' \in \ell \cap \ell'$ with $\ell = \mathbb{F}x + \mathbb{F}y \in \mathcal{L}$ and $\ell' = \mathbb{F}x' + \mathbb{F}y' \in \mathcal{L}$, then $\mathbb{F}z, \mathbb{F}z' \subset \mathbb{F}x + \mathbb{F}y$ and $\mathbb{F}z, \mathbb{F}z' \subset \mathbb{F}x' + \mathbb{F}y'$ together imply that x' and y' are both linearly dependent on x and y, forcing $\ell = \ell'$. The following lemma describes a necessary and sufficient condition for two extremal elements to span a line in $\mathcal{L}$.

Lemma 5.1.1. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, generated by its set $E(\mathfrak{g})$ of extremal elements. Then $\mathbb{F}x + \mathbb{F}y \subseteq E(\mathfrak{g}) \cup \{0\}$ with $x, y \in E(\mathfrak{g})$ commuting and linearly independent if and only if $[x, [y, z]] = g_x(z)y + g_y(z)x$ for all $z \in \mathfrak{g}$.*

Proof. Let $x, y \in E(\mathfrak{g})$ such that $\mathbb{F}x \neq \mathbb{F}y$ and $[x, y] = 0$, and suppose first that $\mathbb{F}x + \mathbb{F}y \subseteq E(\mathfrak{g}) \cup \{0\}$. It follows from the Jacobi identity that $[y, [x, z]] = -[z, [y, x]] - [x, [z, y]] = [x, [y, z]]$ for all $z \in \mathfrak{g}$, hence for all $\lambda, \mu \in \mathbb{F}^*$ and $z \in \mathfrak{g}$ we have on the one hand that

$$2g_{\lambda x + \mu y}(z)(\lambda x + \mu y) = 2\lambda^2 g_x(z)x + 2\lambda\mu(g_x(z)y + g_y(z)x) + 2\mu^2 g_y(z)y,$$

whereas on the other hand we have

$$\begin{aligned}2g_{\lambda x+\mu y}(z)(\lambda x+\mu y) &= [\lambda x+\mu y,[\lambda x+\mu y,z]]\\ &= \lambda^2[x,[x,z]]+\lambda\mu([x,[y,z]]+[y,[x,z]])+\mu^2[y,[y,z]]\\ &= 2\lambda^2 g_x(z)x+2\lambda\mu[x,[y,z]]+2\mu^2 g_y(z)y.\end{aligned}$$

But then $[x,[y,z]] = g_x(z)y + g_y(z)x$ for all $z \in \mathfrak{g}$, as desired.

Assuming next that $[x,[y,z]] = g_x(z)y + g_y(z)x$ for all $z \in \mathfrak{g}$, similar calculations show that for all $\lambda, \mu \in \mathbb{F}^*$ we then have

$$\begin{aligned}[\lambda x+\mu y,[\lambda x+\mu y,z]] &= \lambda^2[x,[x,z]]+\lambda\mu([x,[y,z]]+[y,[x,z]])+\mu^2[y,[y,z]]\\ &= 2\lambda^2 g_x(z)x+2\lambda\mu[x,[y,z]]+2\mu^2 g_y(z)y\\ &= 2\lambda^2 g_x(z)x+2\lambda\mu(g_x(z)y+g_y(z)x)+2\mu^2 g_y(z)y\\ &= 2g_{\lambda x}(z)(\lambda x)+2g_{\lambda x}(z)(\mu y)+2g_{\mu y}(z)(\lambda x)+2g_{\mu y}(z)(\mu y)\\ &= 2g_{\lambda x+\mu y}(z)(\lambda x+\mu y)\end{aligned}$$

by Proposition 3.1.10 so that $\lambda x + \mu y \subseteq E(\mathfrak{g})$, from which it follows that $\mathbb{F}x + \mathbb{F}y \subseteq E(\mathfrak{g}) \cup \{0\}$. □

We now define $\{E_i\}_{-2\leq i\leq 2}$ to be the set of relations $E_i \subseteq E(\mathfrak{g}) \times E(\mathfrak{g})$ given by

$$\begin{aligned}(x,y)\in E_{-2} &\iff \mathbb{F}x=\mathbb{F}y,\\ (x,y)\in E_{-1} &\iff \mathbb{F}x+\mathbb{F}y\subseteq E(\mathfrak{g})\cup\{0\} \text{ with } [x,y]=0 \text{ and } x,\, y \text{ linearly independent},\\ (x,y)\in E_{0} &\iff [x,y]=0 \text{ and } (x,y)\notin E_{\leq -1},\\ (x,y)\in E_{1} &\iff [x,y]\neq 0 \text{ and } g_x(y)=0,\\ (x,y)\in E_{2} &\iff g_x(y)\neq 0.\end{aligned}$$

In addition, we define $\{\mathcal{E}_i\}_{-2\leq i\leq 2}$ to be the set of relations $\mathcal{E}_i \subseteq \mathcal{E} \times \mathcal{E}$ given by $(\mathbb{F}x, \mathbb{F}y) \in \mathcal{E}_i \iff (x, y) \in E_i$. This gives rise to the following definition.

Definition 5.1.2 (Extremal geometry)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ generated by its set $E(\mathfrak{g}$ of extremal elements. Then the point-line geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ together with the set $\{\mathcal{E}_i\}_{-2\leq i\leq 2}$ of relations $\mathcal{E}_i \subseteq \mathcal{E} \times \mathcal{E}$, both as described above, form the* ***extremal geometry*** *of $\mathfrak{g}$.*

In the sequel, we will sometimes omit the set of relations $\{\mathcal{E}_i\}_{-2\leq i\leq 2}$ whenever we refer to the extremal geometry of a Lie algebra $\mathfrak{g}$; instead, we will identify its extremal geometry only by the point-line geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ and only emphasise its filtration $\{\mathcal{E}_i\}_{-2\leq i\leq 2}$ if we make explicit use of it.

Under certain conditions, the set of relations $\{\mathcal{E}_i\}_{-2\leq i\leq 2}$ of the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of a Lie algebra $\mathfrak{g}$ turn it into a root filtration space with filtration $\{\mathcal{E}_i\}_{-2\leq i\leq 2}$. This is characterised by the following proposition.

Proposition 5.1.3. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements. Then the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ is a root filtration space with filtration $\{\mathcal{E}_i\}_{-2\leq i\leq 2}$.*

Proof. The relations $\{E_i\}_{-2\leq i\leq 2}$ are clearly disjoint, symmetric by Corollary 3.1.5 and partition $E(\mathfrak{g}) \times E(\mathfrak{g})$, hence the same is true for the relations $\{\mathcal{E}_i\}_{-2\leq i\leq 2}$. Definition 4.5.1(i) and Definition 4.5.1(ii) are immediate from construction of $\Gamma_{\mathfrak{g}}$, $\{E_i\}_{-2\leq i\leq 2}$ and $\{\mathcal{E}_i\}_{-2\leq i\leq 2}$.

For Definition 4.5.1(iii), the map $\varphi : \mathcal{E}_1 \to \mathcal{E}$ must send $(x, y) \in \mathcal{E}_1$ to their unique neighbour in the collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ of $\Gamma_{\mathfrak{g}}$ by Proposition 4.5.3. Therefore, we define $\varphi : \mathcal{E}_1 \to \mathcal{E}$ to be the map given by $\varphi(\mathbb{F}x, \mathbb{F}y) = \mathbb{F}[x, y]$. Note that $[x, y] = (x + [x, y]) - x = (x - [y, x]) - x = \exp(y, -1)x - x = (\exp(y, -1) - 1)x \in E(\mathfrak{g})$, so φ is well-defined. We refer to Theorem 28(iii) of [5] for a proof that Definition 4.5.1(iii) holds.

For Definition 4.5.1(iv), let $(\mathbb{F}x, \mathbb{F}y) \in \mathcal{E}_2$ and suppose towards towards a contradiction that $\mathcal{E}_{\leq 0}(\mathbb{F}x) \cap \mathcal{E}_{\leq -1}(\mathbb{F}y) \neq \emptyset$. Pick a point $\mathbb{F}z \in \mathcal{E}_{\leq 0}(\mathbb{F}x) \cap \mathcal{E}_{\leq -1}(\mathbb{F}y) \neq \emptyset$. Then $[x, z] = 0 = [y, z]$ and $g_x(z) = 0 = g_y(z)$. If $\mathbb{F}z \in \mathcal{E}_{-2}(\mathbb{F}y)$, then clearly $y = \lambda z$ for some $\lambda \in \mathbb{F}^*$. Since $\mathbb{F}z \in \mathcal{E}_{\leq 0}(\mathbb{F}x)$, it follows that $g_x(y) = g_x(\lambda z) = \lambda g_x(z) = 0$, contradicting that $(\mathbb{F}x, \mathbb{F}y) \in \mathcal{E}_2$. This forces $\mathbb{F}z \in \mathcal{E}_{-1}(\mathbb{F}y)$. But then $y + z \in E(\mathfrak{g})$, hence $[y + z, [y + z, x]] = 2g_{y+z}(x)(y + z)$. Simplifying the left-hand side yields

$$[y + z, [y + z, x]] = [y + z, [y, x] + [z, x]] = [y + z, [y, x]] = [y, [y, x]] + [z, [y, x]] = [y, [y, x]],$$

as $[z, [y, x]] = -[x, [z, y]] - [y, [x, z]] = 0$ by the Jacobi identity. The right-hand side simplifies to

$$2g_{y+z}(x)(y + z) = 2(g_y(x) + g_z(x))(y + z) = 2g_y(x)(y + z) = 2g_y(x)y + 2g_y(x)z,$$

because $g_z(x) = g_x(z) = 0$ by Corollary 3.1.5, so combined we find $2g_y(x)y = 2g_y(x)y + 2g_y(x)z \iff 2g_y(x)z = 0$. Then either $g_x(y) = 0$, contradicting that $(\mathbb{F}x, \mathbb{F}y) \in \mathcal{E}_2$, or $z = 0$, contradicting that $z \in E(\mathfrak{g})$. It follows that $\mathcal{E}_{\leq 0}(\mathbb{F}x) \cap \mathcal{E}_{\leq -1}(\mathbb{F}y) = \emptyset$.

For Definition 4.5.1(v), let $\mathbb{F}z \in \mathcal{E}$. Further let $\ell = \mathbb{F}x + \mathbb{F}y \in \mathcal{L}$ and suppose first that $|\mathcal{E}_{\leq 0}(\mathbb{F}z) \cap \ell| \geq 2$. We may assume w.l.o.g. that $\mathbb{F}x, \mathbb{F}y \in \mathcal{E}_{\leq 0}(\mathbb{F}z) \cap \ell$. Then $[x, z] = 0 = [y, z]$, hence $[\ell, \mathbb{F}z] = [\mathbb{F}x + \mathbb{F}y, \mathbb{F}z] = \mathbb{F}[x, z] + \mathbb{F}[y, z] = 0$. If $\ell = \mathbb{F}x + \mathbb{F}y \not\subset \mathcal{E}_{\leq -1}(\mathbb{F}z)$, then $\ell \subset \mathcal{E}_0(\mathbb{F}z)$, so we have $\ell \subseteq \mathcal{E}_{\leq 0}(\mathbb{F}z)$ regardless, which shows that $\mathcal{E}_{\leq 0}(\mathbb{F}z)$ is a subspace of $\Gamma_{\mathfrak{g}}$. Next, suppose that $|\mathcal{E}_{\leq -1}(\mathbb{F}z) \cap \ell| \geq 2$. As before, we may assume w.l.o.g. that $\mathbb{F}x, \mathbb{F}y \in \mathcal{E}_{\leq 0}(\mathbb{F}z) \cap \ell$, and $[\ell, \mathbb{F}z] = 0$. If $\mathbb{F}x, \mathbb{F}y \in \mathcal{E}_{-2}(\mathbb{F}z)$, then clearly $\mathbb{F}x + \mathbb{F}y \subset \mathcal{E}_{-2}(\mathbb{F}z)$, whereas $\mathbb{F}x + \mathbb{F}y \subset \mathcal{E}_{-1}(\mathbb{F}z)$ if at least one of $\mathbb{F}x$ and $\mathbb{F}y$ is contained in $\mathcal{E}_{-1}(\mathbb{F}z)$. This shows that $\ell \subset \mathcal{E}_{\leq 1}(\mathbb{F}z)$, hence $\mathcal{E}_{\leq -1}(\mathbb{F}z)$ is also a subspace of $\Gamma_{\mathfrak{g}}$.

For Definition 4.5.1(vi), we note that $\mathbb{F}z \in \mathcal{E}_{\leq 1}(\mathbb{F}x)$ if and only if $\mathbb{F}z \notin \mathcal{E}_2(\mathbb{F}x)$ if and only if $g_x(z) = 0$ with $\mathbb{F}x, \mathbb{F}z \in \mathcal{E}$. So, we have $\mathcal{E}_{\leq 1}(\mathbb{F}x) = \{\mathbb{F}y \in \mathcal{E} \mid g_x(y) = 0\}$. Since $\mathfrak{g}$ contains no sandwich elements, the extremal form $g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ of $\mathfrak{g}$ from Proposition 3.1.10 will be non-degenerate so that $\mathcal{E}_{\leq 1}(\mathbb{F}x)$ is a hyperplane for all $\mathbb{F}x \in \mathcal{E}$. □

A continuation of the above proposition is the following theorem due to Cohen and Ivanyos [5], which describes how non-degenerate thick root filtration spaces can be obtained from the extremal geometry of a Lie algebra and how they relate to its Lie subalgebras.

Theorem 5.1.4. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements and let $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ be the extremal geometry of $\mathfrak{g}$ with filtration $\{\mathcal{E}_i\}_{-2\leq i\leq 2}$. Further let $\mathcal{B}_i$, with $i \in I$ for some not necessarily finite index set I, be the connected components of the graph $(\mathcal{E}, \mathcal{E}_2)$ and denote by $\mathfrak{g}_i$ the Lie subalgebra of $\mathfrak{g}$ generated by $\mathcal{B}_i$. Then $\mathcal{B}_i$ is a non-degenerate thick root filtration space or a root filtration space without lines for every $i \in I$. In particular, the Lie subalgebras $\mathfrak{g}_i$, $i \in I$, are ideals of $\mathfrak{g}$ and $\mathfrak{g} = \bigoplus_{i\in I} \mathfrak{g}_i$.*

Proof. See Theorem 28(iii) of [5]. □

A consequence of particular interest is that for all $i \in I$ the Lie subalgebras $\mathfrak{g}_i$ generated by the connected components $\mathcal{B}_i$ of $(\mathcal{E}, \mathcal{E}_2)$ are simple; this follows from the observation that $\mathfrak{g}$ contains no sandwich elements if and only if the extremal form $g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ of $\mathfrak{g}$ is non-degenerate if and only if $\mathfrak{g}$ is a direct sum of simple ideals by Proposition 3.1.12, all of which must necessarily coincide with the Lie subalgebras $\mathfrak{g}_i$, $i \in I$.

We provide an example of an extremal geometry that is a root filtration space. In particular, we will see that it demonstrates how a previously discussed root filtration space can be obtained from a certain Lie algebra.

Example 5.1.5. Let V be an n-dimensional vector space, $n \geq 3$, over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ and let V^* be the dual space of V. Further let $\mathfrak{psl}(V)$ be the projective special Lie algebra of V, which is simple and generated by the transvections $t_{v,\varphi} \in \mathfrak{psl}(V)$ such that $v \otimes \varphi \in V \otimes V^*$ is a singular pure tensors of $V \otimes V^*$ by Corollary 3.2.6. Note that $\mathrm{Ann}_V(V^*) = 0$, ensuring that $\mathfrak{psl}(V)$ does not contain sandwich elements. Then the extremal geometry $\Gamma_{\mathfrak{psl}(V)}$ of $\mathfrak{psl}(V)$ will be a non-degenerate thick root filtration space by Theorem 5.1.4. In fact, it is isomorphic to the root shadow space of type $A_{1,\{1,n\}}$ with filtration $\{\mathcal{P}_i\}_{-2\leq i\leq 2}$ as described in Example 4.5.2. A similar result is proven in [23] if V is infinite-dimensional.

Denote by $\mathbb{P}(\mathfrak{g})$ the projective space of $\mathfrak{g}$, whose points and lines are its 1-dimensional and 2-dimensional subspaces, respectively. Since $\mathfrak{g}$ is generated by its set $E(\mathfrak{g})$ of extremal elements, it will also be linearly spanned by $E(\mathfrak{g})$ by Lemma 3.1.9, hence the points and lines of $\mathbb{P}(\mathfrak{g})$ coincide with the points and lines of the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{g}$. This gives rise to the following definition.

Definition 5.1.6 (Extremal embedding)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements. Further let $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ be the extremal geometry of $\mathfrak{g}$ and let $\mathbb{P}(\mathfrak{g})$ be the projective space of $\mathfrak{g}$. The **extremal embedding** of $\Gamma_{\mathfrak{g}}$ in $\mathbb{P}(\mathfrak{g})$ is the natural embedding $\varepsilon : \Gamma_{\mathfrak{g}} \hookrightarrow \mathbb{P}(\mathfrak{g})$ given by $\varepsilon(\mathbb{F}x) = \mathbb{F}x$ with $\mathbb{F}x \in \mathcal{E}$.*

It should immediately be clear that ε is injective and that $\varepsilon(\mathcal{E})$ spans $\mathbb{P}(\mathfrak{g})$. Moreover, lines $\mathbb{F}x + \mathbb{F}y \in \mathcal{L}$ with $\mathbb{F}x, \mathbb{F}y \in \mathcal{E}$ are clearly mapped to projective lines in $\mathbb{P}(\mathfrak{g})$, so ε is indeed an embedding by Definition 4.6.14. In particular, the extremal embedding of $\Gamma_\mathfrak{g}$ in $\mathbb{P}(\mathfrak{g})$ is a polarised embedding and it is even unique up to isomorphism, as stated by the following proposition.

Proposition 5.1.7. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements and let $\varepsilon : \Gamma_\mathfrak{g} \hookrightarrow \mathbb{P}(\mathfrak{g})$ be the extremal embedding of the extremal geometry $\Gamma_\mathfrak{g} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{g}$ in the projective space $\mathbb{P}(\mathfrak{g})$ of $\mathfrak{g}$. If $\Gamma_\mathfrak{g}$ is isomorphic to a root shadow space of type $BC_{n,2}$ ($n \geq 3$), $D_{n,2}$ ($n \geq 4$), $E_{6,2}$, $E_{7,1}$, $E_{8,8}$ or $F_{4,1}$, then ε is unique up to isomorphism.*

Proof. In light of Theorem 4.7.5, it suffices to show that $\varepsilon : \Gamma_\mathfrak{g} \hookrightarrow \mathbb{P}(V)$ is a polarised embedding whose radical $\mathrm{rad}(\varepsilon)$ is trivial.

So, let $\mathbb{F}x \in \mathcal{E}$. As ε maps hyperplanes of $\Gamma_\mathfrak{g}$ to hyperplanes of $\mathbb{P}(\mathfrak{g})$, it is enough to show that $\mathcal{E}_{\leq 1}(\mathbb{F}x)$ is contained in a hyperplane of $\Gamma_\mathfrak{g}$. But this is immediate from Definition 4.5.1(vi) as $\Gamma_\mathfrak{g}$ is a root filtration space, so ε is indeed a polarised embedding. Since $\mathfrak{g}$ contains no sandwich elements, the radical $\mathrm{rad}(g)$ of the extremal form $g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ of $\mathfrak{g}$ is trivial, hence

$$\begin{aligned}\mathrm{rad}(\varepsilon) = \bigcap_{\mathbb{F}x \in \mathcal{E}} \varepsilon(\mathcal{E}_{\leq 1}(\mathbb{F}x)) &= \bigcap_{\mathbb{F}x \in \mathcal{E}} \{\mathbb{F}y \in \mathcal{E} \mid g_x(y) = 0\} \\ &= \{\mathbb{F}y \in \mathcal{E} \mid \forall \mathbb{F}x \in \mathcal{E} : g(x, y) = 0\} = \mathrm{rad}(g)\end{aligned}$$

implies that $\mathrm{rad}(\varepsilon)$ will be trivial as well. □

We set out to prove that the extremal geometry $\Gamma_\mathfrak{g} = (\mathcal{E}, \mathcal{L})$ of a Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ uniquely determines $\mathfrak{g}$ if $\mathfrak{g}$ contains no sandwich elements and is generated by its extremal elements. First consider the following corollary of the above proposition.

Corollary 5.1.8. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two Lie algebras over a field $\mathbb{F}$ without sandwich elements generated by their sets $E(\mathfrak{g})$ and $E(\mathfrak{g}')$ of extremal elements, and let $\varepsilon : \Gamma_\mathfrak{g} \hookrightarrow \mathbb{P}(\mathfrak{g})$ and $\varepsilon' : \Gamma_{\mathfrak{g}'} \hookrightarrow \mathbb{P}(\mathfrak{g}')$ be the extremal embeddings of their extremal geometries $\Gamma_\mathfrak{g} = (\mathcal{E}, \mathcal{L})$ and $\Gamma_{\mathfrak{g}'} = (\mathcal{E}', \mathcal{L}')$ in their projective spaces $\mathbb{P}(\mathfrak{g})$ and $\mathbb{P}(\mathfrak{g}')$, respectively. If $\Gamma_\mathfrak{g} \cong \Gamma_{\mathfrak{g}'}$, and both $\Gamma_\mathfrak{g}$ and $\Gamma_{\mathfrak{g}'}$ are isomorphic to a root shadow space of type $BC_{n,2}$ ($n \geq 3$), $D_{n,2}$ ($n \geq 4$), $E_{6,2}$, $E_{7,1}$, $E_{8,8}$ or $F_{4,1}$, then $\varepsilon \cong \varepsilon'$.*

Proof. Note that both $\mathrm{rad}(\varepsilon)$ and $\mathrm{rad}(\varepsilon')$ are trivial because the extremal forms $g : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ and $g' : \mathfrak{g}' \times \mathfrak{g}' \to \mathbb{F}$ of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively, are trivial, which we have established in our proof of Proposition 5.1.7. By Theorem 4.7.5, both $\Gamma_\mathfrak{g}$ and $\Gamma_{\mathfrak{g}'}$ admit an absolutely universal embedding, hence we may apply Proposition 4.7.4. Specifically, as $\Gamma_\mathfrak{g} \cong \Gamma_{\mathfrak{g}'}$, we find $\varepsilon' \to \varepsilon$ and $\varepsilon \to \varepsilon'$ so that $\varepsilon \cong \varepsilon'$, as desired. □

So far, we have shown that, up to isomorphism, the extremal embedding of $\Gamma_{\mathfrak{g}}$ in $\mathbb{P}(\mathfrak{g})$ is unique by Proposition 5.1.7 and that it is moreover uniquely determined by $\Gamma_{\mathfrak{g}}$ by the above corollary. It therefore remains to show that the Lie algebra $\mathfrak{g}$ itself is uniquely determined by $\Gamma_{\mathfrak{g}}$, up to isomorphism. This is done by showing that the Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ of $\mathfrak{g}$ is uniquely determined by $\Gamma_{\mathfrak{g}}$.

Proposition 5.1.9. *Let V be a vector space over a field $\mathbb{F}$. Further let $[\cdot,\cdot] : V \times V \to \mathbb{F}$ and $[\cdot,\cdot]' : V \times V \to \mathbb{F}$ be two alternating bilinear forms on V that turn V into the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ such that neither contains sandwich elements and both are generated by their sets $E(\mathfrak{g})$ and $E(\mathfrak{g}')$ of extremal elements, respectively. If the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{g}$ is isomorphic to the extremal geometry $\Gamma_{\mathfrak{g}'} = (\mathcal{E}', \mathcal{L}')$ of $\mathfrak{g}'$, then there exists a $\lambda \in \mathbb{F}^*$ such that $[x,y]' = \lambda[x,y]$ for all $x, y \in V$.*

Proof. See Theorem 5.3.9 of [15]. □

By combining all previous results, we are now in a position to prove that the extremal geometry $\Gamma_{\mathfrak{g}}$ of a Lie algebra $\mathfrak{g}$ uniquely determines $\mathfrak{g}$ up to isomorphism.

Theorem 5.1.10. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements. If the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ is isomorphic to a root shadow space of type $BC_{n,2}$ ($n \geq 3$), $D_{n,2}$ ($n \geq 4$), $E_{6,2}$, $E_{7,1}$, $E_{8,8}$ or $F_{4,1}$, then $\Gamma_{\mathfrak{g}}$ uniquely determines $\mathfrak{g}$ up to isomorphism.*

Proof. Suppose that $\mathfrak{g}'$ is another Lie algebra over $\mathbb{F}$ without sandwich elements generated by its set $E(\mathfrak{g}')$ of extremal elements such that $\Gamma_{\mathfrak{g}} \cong \Gamma_{\mathfrak{g}'}$. Consequently, $\Gamma_{\mathfrak{g}'}$ will also be isomorphic to a root shadow space of type $BC_{n,2}$ ($n \geq 3$), $D_{n,2}$ ($n \geq 4$), $E_{6,2}$, $E_{7,1}$, $E_{8,8}$ or $F_{4,1}$, hence we may apply Corollary 5.1.8, from which we deduce that the extremal embeddings $\varepsilon : \Gamma_{\mathfrak{g}} \hookrightarrow \mathbb{P}(\mathfrak{g})$ and $\varepsilon' : \Gamma_{\mathfrak{g}'} \hookrightarrow \mathbb{P}(V)$ of $\Gamma_{\mathfrak{g}}$ and $\Gamma_{\mathfrak{g}'}$ in the projective spaces $\mathbb{P}(\mathfrak{g})$ and $\mathbb{P}(\mathfrak{g}')$ of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively, are isomorphic. Since also both ε and ε' themselves are, up to isomorphism, unique by Proposition 5.1.7, we may view $\mathfrak{g}'$ as a Lie algebra having the same underlying vector space as $\mathfrak{g}$. Then Proposition 5.1.9 applies, hence $\mathfrak{g}$ is isomorphic to $\mathfrak{g}'$. It follows that $\Gamma_{\mathfrak{g}}$ uniquely determines $\mathfrak{g}$ up to isomorphism. □

Chapter 6

Characterisation of the orthogonal Lie algebras

We introduce local systems of Lie algebras and discuss some of their properties. This is done in preparation for Section 6.2 and 6.3, in which we give our proof of Theorem 1.1.2 in the finite-dimensional case, respectively infinite-dimensional case.

Section 6.1 is based on the theory from [24].

6.1 Local systems of Lie algebras

In this section, we introduce another structure obtained from a Lie algebra $\mathfrak{g}$ that we require for our proof of Theorem 1.1.2. In particular, we will show that this structure also determines $\mathfrak{g}$ up to isomorphism.

Let I be a non-empty index set and define on I a reflexive and transitive binary relation $\preceq$, called a *pre-order*. If for all $i, j \in I$ there exists a $k \in I$ such that $i \preceq k$ and $j \preceq k$, then I is said to be a *directed set*. To emphasise its dependency on the pre-order $\preceq$, we will write $(I, \preceq)$ instead of just I. This gives rise to the following definition.

Definition 6.1.1 (Direct system)**.** *Let $(I, \preceq)$ be a directed partially ordered set and let $(\mathcal{A}, \Phi)$ be a system of algebraic objects $\mathcal{A} = \{\mathcal{A}_i\}_{i \in I}$ and homomorphisms $\Phi = \{\varphi_{i,j} : \mathcal{A}_i \to \mathcal{A}_j\}_{i \preceq j}$. If $\varphi_{i,i} = \mathrm{id}_{\mathcal{A}_i}$ for all $i \in I$ and $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$ for all $i, j, k \in I$ such that $i \preceq j \preceq k$, then $(\mathcal{A}, \Phi)$ is called a* ***direct system*** *over $(I, \preceq)$.*

Let $\mathcal{A}$ be a system of algebraic objects $\{\mathcal{A}_i\}_{i \in I}$ over some directed partially ordered set $(I, \preceq)$. The *disjoint union* of $\mathcal{A}$ is the set of ordered pairs $(a_i, i) \in \mathcal{A}_i \times \{i\}$ with $a_i \in \mathcal{A}_i$ and $i \in I$, denoted by $\bigsqcup_{i \in I} \mathcal{A}_i$. In particular, we have $\bigsqcup_{i \in I} \mathcal{A}_i = \bigcup_{i \in I} (\mathcal{A}_i \times \{i\})$.

Of interest to us is the direct limit of a direct system $(\mathcal{A}, \Phi)$ over a directed set $(I, \preceq)$. In order to define it, we require the following lemma.

Lemma 6.1.2. *Let $(\mathcal{A}, \Phi)$ be a direct system over a directed partially order set $(I, \preceq)$. Define $\sim$ to be the relation on the disjoint union $\bigsqcup_{i\in I} \mathcal{A}_i$ of all algebraic objects $\mathcal{A}_i$, $i \in I$, given by $(a_i, i) \sim (a_j, i)$ with $(a_i, i) \in \mathcal{A}_i \times \{i\}$, $i \in I$, and $(a_j, j) \in \mathcal{A}_j \times \{j\}$, $j \in I$, if and only if there exists a $k \in I$ such that $i \preceq k$, $j \preceq k$ and $\varphi_{i,k}(a_i) = \varphi_{j,k}(a_j)$. Then $\sim$ is an equivalence relation.*

Proof. Reflexivity of $\sim$ follows immediately from the observation that $i \preceq i$ because $\preceq$ is reflexive, and symmetry of $\sim$ is clear from its definition. To show transitivity, let $(a_i, i) \in \mathcal{A}_i \times \{i\}$, $(a_j, j) \in \mathcal{A}_j \times \{j\}$ and $(a_k, k) \in \mathcal{A}_k \times \{k\}$ with $i, j, k \in I$ such that $(a_i, i) \sim (a_j, j)$ and $(a_j, j) \sim (a_k, k)$. Then there exists an $l \in I$ such that $i \preceq l$, $j \preceq l$ and $\varphi_{i,l}(a_i) = \varphi_{j,l}(a_j)$, and there exists an $l' \in I$ such that $j \preceq l'$, $k \preceq l'$ and $\varphi_{j,l'}(a_j) = \varphi_{k,l'}(a_k)$. Because $(I, \preceq)$ is a directed set, there exists an $l'' \in I$ such that $l \preceq l''$ and $l' \preceq l''$. Consequently, we have $i \preceq l \preceq l''$ and $k \preceq l' \preceq l''$, but also $j \preceq l \preceq l''$ and $j \preceq l' \preceq l''$, hence

$$\begin{aligned}
\varphi_{i,l''}(a_i) &= (\varphi_{l,l''} \circ \varphi_{i,l})(a_i) = \varphi_{l,l''}(\varphi_{i,l}(a_i)) = \varphi_{l,l'}(\varphi_{j,l}(a_j)) = (\varphi_{l,l''} \circ \varphi_{j,l})(a_j) \\
&= \varphi_{j,l''}(a_j) = (\varphi_{l',l''} \circ \varphi_{j,l'})(a_j) = \varphi_{l',l''}(\varphi_{j,l'}(a_j)) = \varphi_{l',l''}(\varphi_{k,l'}(a_k)) \\
&= (\varphi_{l',l''} \circ \varphi_{k,l'})(a_k) = \varphi_{k,l''}(a_k),
\end{aligned}$$

from which we conclude that $(a_i, i) \sim (a_k, k)$. □

Let $(\mathcal{A}, \Phi)$ be a direct system over $(I, \preceq)$ for some system of homomorphisms $\Phi = \{\varphi_{i,j} : \mathcal{A}_i \to \mathcal{A}_j\}_{i\preceq j}$ and let $\sim$ be the equivalence relation on $\bigsqcup_{i\in I} \mathcal{A}_i$ as in the above lemma. The *direct limit* of $(\mathcal{A}, \Phi)$ is the algebraic object

$$\bigsqcup_{i\in I} \mathcal{A}_i \Big/ \sim,$$

which we will denote by $\varinjlim \mathcal{A}_i$.

Example 6.1.3. The pair $(\mathbb{N}, \leq)$, in which $\mathbb{N}$ is the set of all natural numbers and $\leq$ is the ordinary order on $\mathbb{N}$, is clearly a directed set. Let V be an n-dimensional vector space over a field $\mathbb{F}$ with $n \in \mathbb{N}$ and consider the special linear Lie algebra $\mathfrak{sl}(V) = \mathfrak{sl}_n(\mathbb{F})$ on V. The map $\varphi_{i,j} : \mathfrak{sl}_i(\mathbb{F}) \to \mathfrak{sl}_j(\mathbb{F})$ with $i, j \in \mathbb{N}$ such that $i \leq j$ given by $\varphi_{i,j}(x) = \mathrm{diag}(x, 0, \ldots, 0)$ with $x \in \mathfrak{sl}_i(\mathbb{F})$ is readily seen to be a Lie algebra homomorphism; indeed, for all $x, y \in \mathfrak{sl}_i(\mathbb{F})$, we have

$$\begin{aligned}
[\varphi_{i,j}(x), \varphi_{i,j}(y)] &= [\mathrm{diag}(x, 0, \ldots, 0), \mathrm{diag}(y, 0, \ldots, 0)] \\
&= \mathrm{diag}(xy, 0, \ldots, 0) - \mathrm{diag}(yx, 0, \ldots, 0) = \varphi_{i,j}([x, y]).
\end{aligned}$$

We claim that the system $(\mathfrak{sl}, \Phi_{\mathfrak{sl}})$, with $\mathfrak{sl} = \{\mathfrak{sl}_n(\mathbb{F})\}_{n\in\mathbb{N}}$ and $\Phi_{\mathfrak{sl}} = \{\varphi_{i,j} : \mathfrak{sl}_i(\mathbb{F}) \to \mathfrak{sl}_j(\mathbb{F})\}_{i\leq j}$ is a direct system over $(\mathbb{N}, \leq)$. We have $\varphi_{i,i}(x) = \mathrm{diag}(x) = x$ for all $i \in \mathbb{N}$ and $x \in \mathfrak{sl}_i(\mathbb{F})$ so that $\varphi_{i,i}$ is the identity on $\mathfrak{sl}_i(\mathbb{F})$, hence it remains to show that $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$ for all $i, j, k \in \mathbb{N}$ such that $i \leq j \leq k$. But this is immediate from the observation that $k-i=(k-j)+(j-i)$ by simply counting the number of zeros appended to an element $x=\mathrm{diag}(x)\in\mathfrak{sl}_i(\mathbb{F})$ by $\varphi_{i,k}$, $\varphi_{j,k}$ and $\varphi_{i,j}$. Using Lemma 6.1.2, it can be verified that the equivalence class $[x]_\sim$ of an element $x\in\mathfrak{sl}_i(\mathbb{F})$ for some $i\in\mathbb{N}$ consists of the elements $\mathrm{diag}(x,0,\dots,0)\in\mathfrak{sl}_j(\mathbb{F})$ with $i\leq j$. Consequently, the choice of any element in $\bigsqcup_{n\in\mathbb{N}}\mathfrak{sl}_n(\mathbb{F})\big/\sim$ is independent of $n\in\mathbb{N}$, therefore the direct limit of $(\mathfrak{sl},\Phi_{\mathfrak{sl}})$ equals

$$\varinjlim\mathfrak{sl}_n(\mathbb{F})=\bigsqcup_{n\in\mathbb{N}}\mathfrak{sl}_n(\mathbb{F})\Big/\sim\;=\bigcup_{n\in\mathbb{N}}(\mathfrak{sl}_n(\mathbb{F})\times\{n\})\Big/\sim\;=\bigcup_{n\in\mathbb{N}}\mathfrak{sl}_n(\mathbb{F}):=\mathfrak{sl}_\infty(\mathbb{F}).$$

In a similar manner, the Lie algebras $\mathfrak{so}_\infty(\mathbb{F})$ and $\mathfrak{sp}_\infty(\mathbb{F})$ can be defined as the direct limit of the direct systems $(\mathfrak{so},\Phi_{\mathfrak{so}})$ and $(\mathfrak{sp},\Phi_{\mathfrak{sp}})$ with $\mathfrak{so}=\{\mathfrak{so}_n(\mathbb{F})\}_{n\in\mathbb{N}}$, $\mathfrak{sp}=\{\mathfrak{sp}_n(\mathbb{F})\}_{n\in\mathbb{N}}$ and both $\Phi_{\mathfrak{so}}$ and $\Phi_{\mathfrak{sp}}$ consisting of the natural embeddings of $\mathfrak{so}_i(\mathbb{F})$ and $\mathfrak{sp}_i(\mathbb{F})$ in $\mathfrak{so}_j(\mathbb{F})$ and $\mathfrak{sp}_j(\mathbb{F})$, respectively, with $i,j\in\mathbb{N}$ such that $i\leq j$.

The above example shows how certain Lie algebras can be defined with the help of direct systems. Conversely, given a possibly infinite-dimensional Lie algebra, it may be possible to extract from it a system of finite-dimensional Lie subalgebras with which a direct system can be constructed. Since a Lie algebra is said to be *locally finite* if all of its finitely generated Lie subalgebras are finite-dimensional, such systems are instead referred to as local systems. Their precise definition is stated below.

Definition 6.1.4 (Local system)**.** *Let $\mathfrak{g}$ be a possibly infinite-dimensional Lie algebra and let I be an index set. A system $\mathfrak{G}=\{\mathfrak{g}_i\}_{i\in I}$ of finite-dimensional Lie subalgebras $\mathfrak{g}_i$ of $\mathfrak{g}$ is called a* ***local system*** *of $\mathfrak{g}$ if $\mathfrak{g}=\bigcup_{i\in I}\mathfrak{g}_i$ and for all $i,j\in I$ there exists a $k\in I$ such that $\mathfrak{g}_i\subseteq\mathfrak{g}_k$ and $\mathfrak{g}_j\subseteq\mathfrak{g}_k$.*

An explicit connection between local systems of Lie algebras, directed sets and direct systems can be made using Example 6.1.3; starting with an infinite dimensional vector space V over a field $\mathbb{F}$ and letting $\mathfrak{sl}_\infty(\mathbb{F}):=\mathfrak{sl}(V)$ be the special linear Lie algebra of V, the system $\mathfrak{G}=\{\mathfrak{sl}_n(\mathbb{F})\}_{n\in\mathbb{N}}$ is readily seen to be a local system of $\mathfrak{sl}_\infty(\mathbb{F})$. In general, local systems, directed sets and direct systems are related in the following way.

Lemma 6.1.5. *Let I be an index set and let $\mathfrak{G}=\{\mathfrak{g}_i\}_{i\in I}$ be a local system of a possibly infinite-dimensional Lie algebra $\mathfrak{g}$. Define $\preceq$ to be the binary relation on I given by $i\preceq j$ with $i,j\in I$ if and only if $\mathfrak{g}_i\subseteq\mathfrak{g}_j$ and define Φ to be the collection of natural embeddings $\varphi_{i,j}:\mathfrak{g}_i\to\mathfrak{g}_j$ of $\mathfrak{g}_i$ in $\mathfrak{g}_j$ with $i,j\in I$ such that $i\preceq j$. Then $(I,\preceq)$ is a directed set, $(\mathfrak{G},\Phi)$ is a direct system and $\mathfrak{g}\cong\varinjlim\mathfrak{g}_i$.*

Proof. Since set inclusion $\subseteq$ is a reflexive and transitive binary relation on $\mathfrak{G}$, it immediately follows that $\preceq$ will be a pre-order on I. Moreover, as $\mathfrak{G}$ is a local system of $\mathfrak{g}$, we have for all $i,j\in I$ that $\mathfrak{g}_i\subseteq\mathfrak{g}_k\iff i\preceq k$ and $\mathfrak{g}_j\subseteq\mathfrak{g}_k\iff j\preceq k$ for some $k\in I$, showing that $(I,\preceq)$ is a directed set.

The natural embedding of a Lie subalgebra $\mathfrak{g}_i$, $i\in I$, of $\mathfrak{g}$ in itself is clearly the identity on $\mathfrak{g}_i$, hence $\varphi_{i,i}=\mathrm{id}_{\mathfrak{g}_i}$. Given three Lie subalgebras $\mathfrak{g}_i$, $\mathfrak{g}_j$ and $\mathfrak{g}_k$ of $\mathfrak{g}$ with $i,j,k\in I$ such that $i \preceq j \preceq k$, embedding $\mathfrak{g}_i$ in $\mathfrak{g}_j$ and then $\mathfrak{g}_j$ in $\mathfrak{g}_k$ is equivalent to embedding $\mathfrak{g}_i$ directly in $\mathfrak{g}_k$, implying that $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$. This shows that $(\mathfrak{G}, \Phi)$ is a direct system.

Let $\mathfrak{g}_i$ and $\mathfrak{g}_j$ be two Lie subalgebras of $\mathfrak{g}$ with $i, j \in I$ and suppose that $x_i \sim x_j$ for two elements $x_i \in \mathfrak{g}_i$ and $x_j \in \mathfrak{g}_j$ with $\sim$ as in Lemma 6.1.2. Then there exists a $k \in I$ such that $\mathfrak{g}_i \subseteq \mathfrak{g}_k$ and $\mathfrak{g}_j \subseteq \mathfrak{g}_k$. In particular, there will be an element $x_k \in \mathfrak{g}_k$ which equals x_i when restricted to $\mathfrak{g}_i$ and x_j when restricted to $\mathfrak{g}_j$. Therefore, as in Example 6.1.3, the choice of any element in $\bigsqcup_{i\in I} \mathfrak{g}_i / \sim$ will be independent of $i \in I$ so that

$$\varinjlim \mathfrak{g}_i = \bigsqcup_{i\in I} \mathfrak{g}_i \Big/ \sim \; = \bigcup_{i\in I} (\mathfrak{g}_i \times \{i\}) \Big/ \sim \; \cong \bigcup_{i\in I} \mathfrak{g}_i = \mathfrak{g},$$

where the last equality follows from $\mathfrak{G}$ being a local system of $\mathfrak{g}$. □

Two Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ over a field $\mathbb{F}$ can be related to one another by means of their local systems $\mathfrak{G} = \{\mathfrak{g}_i\}_{i\in I}$ and $\mathfrak{G}' = \{\mathfrak{g}'_i\}_{i\in I}$, respectively, over some directed set $(I, \preceq)$. Specifically, if for all $i \in I$ the Lie subalgebras $\mathfrak{g}_i$ and $\mathfrak{g}'_i$ of the local systems $\mathfrak{G}$ and $\mathfrak{G}'$, respectively, are isomorphic, it can be shown that the $\mathfrak{g}$ and $\mathfrak{g}'$ themselves are also isomorphic under certain conditions. This is characterised by the following theorem.

Theorem 6.1.6. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two Lie algebras over a field $\mathbb{F}$ and let $\mathfrak{G} = \{\mathfrak{g}_i\}_{i\in I}$ and $\mathfrak{G}' = \{\mathfrak{g}'_i\}_{i\in I}$ be local systems of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively, over some directed set $(I, \preceq)$. If $\Psi = \{\psi_i : \mathfrak{g}_i \to \mathfrak{g}'_i\}_{i\in I}$ is a collection of isomorphisms such that $\psi_i = \psi_j\big|_{\mathfrak{g}_i}$ for all $i, j \in I$ such that $i \preceq j$, then $\mathfrak{g} \cong \mathfrak{g}'$.*

Proof. First note that $\bigsqcup_{i\in I} \mathfrak{g}_i = \bigcup_{i\in I}(\mathfrak{g}_i \times \{i\}) \cong \bigcup_{i\in I}(\mathfrak{g}'_i \times \{i\}) = \bigsqcup_{i\in I} \mathfrak{g}'_i$ because $\mathfrak{g}_i \cong \mathfrak{g}'_i$ under $\psi_i \in \Psi$ for all $i \in I$. Now let $(\mathfrak{G}, \Phi)$ and $(\mathfrak{G}', \Phi')$ be the direct systems of $\mathfrak{g}$ and $\mathfrak{g}'$, respectively, over $(I, \preceq)$ as in Lemma 6.1.5 and consider an element $x \in \mathfrak{g}$ with $\sim$ as in Lemma 6.1.2. As $\mathfrak{G}$ is a local system of $\mathfrak{g}$, we have $\mathfrak{g} = \bigcup_{i\in I} \mathfrak{g}_i$, so $x = x_i \in \mathfrak{g}_i$ for some $i \in I$. The elements in the equivalence class $[x]_\sim = [x_i]_\sim$ will be precisely the elements $x_j \in \mathfrak{g}_j$,$j \in I$, with $i \preceq j$ such that x_j restricted to $\mathfrak{g}_j$ equals x_i. But $\psi_i = \psi_j\big|_{\mathfrak{g}_i}$ whenever $i, j \in I$ such that $i \preceq j$ by assumption, so the elements in $[\psi_i(x_i)]_{\sim'}$ with $\sim'$ as in Lemma 6.1.2 will satisfy the same property as the elements in $[x_i]_\sim$. This shows that the relations $\sim$ and $\sim'$ on $\bigsqcup_{i\in I} \mathfrak{g}_i$ and $\bigsqcup_{i\in I} \mathfrak{g}'_i$, respectively, are equivalent. Finally, as $\mathfrak{g} \cong \varinjlim \mathfrak{g}_i$ and $\mathfrak{g}' \cong \varinjlim \mathfrak{g}'_i$ by Lemma 6.1.5, we obtain

$$\mathfrak{g} \cong \varinjlim \mathfrak{g}_i = \bigsqcup_{i\in I} \mathfrak{g}_i \Big/ \sim \; \cong \bigsqcup_{i\in I} \mathfrak{g}'_i \Big/ \sim' = \varinjlim \mathfrak{g}'_i \cong \mathfrak{g}',$$

proving that $\mathfrak{g} \cong \mathfrak{g}'$. □

Similar to Theorem 5.1.10, we are now able to show that a local system $\mathfrak{G} = (\mathfrak{g}_i)_{i\in I}$ of a Lie algebra $\mathfrak{g}$ over some directed set $(I, \preceq)$ determines $\mathfrak{g}$ uniquely up to isomorphism.

Corollary 6.1.7. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ and let $\mathfrak{G} = \{\mathfrak{g}_i\}_{i \in I}$ be a local system of $\mathfrak{g}$ over some directed set $(I, \preceq)$. Then $\mathfrak{G}$ uniquely determines $\mathfrak{g}$ up to isomorphism.*

Proof. Suppose that $\mathfrak{g}'$ is a second Lie algebra over $\mathbb{F}$ admitting a local system $\mathfrak{G}' = \{\mathfrak{g}'_i\}_{i \in I}$ over $(I, \preceq)$ isomorphic to $\mathfrak{G}$. Then there exist isomorphisms $\psi_i : \mathfrak{g}_i \to \mathfrak{g}'_i$ for all $i \in I$. Now let $i, j \in I$ such that $i \preceq j$ and let $x_i \in \mathfrak{g}_i$ be arbitrary. By Lemma 6.1.5, we have $\mathfrak{g}_i \subseteq \mathfrak{g}_j$ so that $x_i \in \mathfrak{g}_j$. In turn, there exists an element $x_j \in \mathfrak{g}_j$ that equals x_i when restricted to $\mathfrak{g}_i$, hence $\psi_j(x_j) \in \mathfrak{g}'_j$ will equal $\psi_i(x_i) \in \mathfrak{g}'_i$ when restricted to $\mathfrak{g}'_i$. Upon restricting ψ_j to $\mathfrak{g}_i$, which then necessarily also restricts x_j to $\mathfrak{g}_i$, we deduce that $\psi_i(x_i) = \psi_j(x_j) = \psi_j\big|_{\mathfrak{g}_i}(x_i)$. But $x_i \in \mathfrak{g}_i$ was chosen arbitrarily, hence $\psi_i = \psi_j\big|_{\mathfrak{g}_i}$. Now Theorem 6.1.6 applies, from which it follows that $\mathfrak{g}$ is isomorphic to $\mathfrak{g}'$. We conclude that $\mathfrak{G}$ uniquely determines $\mathfrak{g}$ up to isomorphism. □

6.2 Finite-dimensional case

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, without sandwich elements and generated by its set $E(\mathfrak{g})$ of extremal elements whose extremal geometry $\Gamma_g = (\mathcal{E}, \mathcal{L})$ is isomorphic to a root shadow space of type $B_{n,2}$ or $D_{n+1,2}$, $3 \leq n \leq \infty$. Further let $\mathfrak{fso}(V, f)$ be the finitary orthogonal Lie algebra for some finite-dimensional vector space V over $\mathbb{F}$ and some non-degenerate symmetric bilinear form $f : V \times V \to \mathbb{F}$ on V. This section is devoted to proving our first main theorem, Theorem 1.1.2, in the finite-dimensional case. To do so, the following lemmas are necessary.

Lemma 6.2.1. *Let $\mathfrak{fso}(V, f)$ be the orthogonal Lie algebra for some finite-dimensional vector space V over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, and some non-degenerate symmetric bilinear form $f : V \times V \to \mathbb{F}$ on V such that f has Witt index at least two. Then $\mathfrak{fso}(V, f)$ is generated by its set $E(\mathfrak{fso}(V, f))$ of extremal elements and does not contain sandwich elements.*

Proof. Let $f : V \times V \to \mathbb{F}$ be the non-degenerate symmetric bilinear form on V that defines $\mathfrak{fso}(V, f)$ according to Definition 2.2.10. Recall from Section 3.2 the Lie algebra $\mathfrak{g}(V \otimes V^*)_f$ generated by the symmetric tensors $v \otimes f_w - w \otimes f_v \in V \otimes V^*$, in which V^* is the dual space of V. By Proposition 3.2.9, we have $\mathfrak{g}(V \otimes V^*)_f \cong \mathfrak{fso}(V, f)$, showing that $\mathfrak{fso}(V, f)$ is generated by the infinitesimal transvections $t_{v,f_w} - t_{w,f_v} \in \mathfrak{fso}(V, f)$, which exist since f has Witt index at least two. For all $v, w \in V$, such an infinitesimal transvection $t_{v,f_w} - t_{w,f_v}$ is extremal in $\mathfrak{fso}(V, f)$ if and only if $v \otimes f_w - w \otimes f_v$ is extremal in $\mathfrak{g}(V \otimes V^*)_f$ if and only if v and w are f-isotropic and orthogonal by Lemma 3.2.12 if and only if $t_{v,f_w} - t_{w,f_v}$ is an infinitesimal Siegel transvection $s_{v,w} \in \mathfrak{fso}(V, f)$ by Definition 3.2.13. We conclude that the infinitesimal Siegel transvections of $\mathfrak{fso}(V, f)$ not only generate $\mathfrak{fso}(V, f)$, but are also the only extremal elements of $\mathfrak{fso}(V, f)$. Then $\mathfrak{fso}(V, f)$ is generated by its set of extremal elements $E(\mathfrak{fso}(V, f))$.

It remains to show that $\mathfrak{fso}(V,f)$ contains no sandwich elements. As only the infinitesimal Siegel transvections are extremal in $\mathfrak{fso}(V,f)$ by the above, it suffices to show that no infinitesimal Siegel transvection $s_{v,w} \in \mathfrak{fso}(V,f)$ is a sandwich element. So, suppose towards a contradiction that $0 \neq s_{v,w} \in \mathfrak{fso}(V,f)$ is a sandwich element. Then we must have

$$g_{s_{v,w}}(s_{x,y}) = f(v,x)f(w,y) - f(v,y)f(w,x) = f(f(w,y)v - f(v,y)w, x) = 0$$

for all $s_{x,y} \in \mathfrak{fso}(V,f)$. In particular, this is true for all $x \in V$ so that $f(w,y)v - f(v,y)w = 0$ by non-degeneracy of f. But then $s_{v,w}(y) = f(w,y)v - f(v,y)w = 0$ holds for all $y \in V$, hence $s_{v,w} = 0$, a contradiction. It follows that $\mathfrak{fso}(V,f)$ does not contain any sandwich elements. □

Lemma 6.2.2. *The finitary orthogonal Lie algebra $\mathfrak{fso}(V,f)$ for some vector space V, $\dim(V) \neq 2$ and $\dim(V) \neq 4$, over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, and some non-degenerate symmetric bilinear form $f : V \times V \to \mathbb{F}$ such that f has Witt index at least two is a simple Lie algebra.*

Proof. First suppose that V is finite-dimensional, say $\dim(V) = n \geq 1$ with $n \neq 2$ and $n \neq 4$, and denote by $\mathbb{A}_\mathbb{F}$ the algebraic closure of $\mathbb{F}$. It is well-known that $\mathfrak{so}_n(\mathbb{A}_\mathbb{F}, f)$ is simple up to its center if and only if $n \neq 2$ or $n \neq 4$; in the former case, $\mathfrak{so}_2(\mathbb{A}_\mathbb{F}, f)$ is abelian as $\dim(\mathfrak{so}_2(\mathbb{A}_\mathbb{F}, f)) = \binom{2}{2} = 1$ by Corollary 2.2.16, hence $\mathfrak{so}_2(\mathbb{A}_\mathbb{F}, f)$ is not simple by Definition 2.1.13, whereas in the latter case we have $\mathfrak{so}_4(\mathbb{A}_\mathbb{F}, f) \cong \mathfrak{so}_3(\mathbb{A}_\mathbb{F}, f) \times \mathfrak{so}_3(\mathbb{A}_\mathbb{F}, f)$ so that $\mathfrak{so}_3(\mathbb{A}_\mathbb{F}, f) \cong \mathfrak{so}_3(\mathbb{A}_\mathbb{F}, f) \times \{0\}$ is a non-trivial ideal of $\mathfrak{so}_4(\mathbb{A}_\mathbb{F}, f)$, turning $\mathfrak{so}_4(\mathbb{A}_\mathbb{F}, f)$ into a non-simple Lie algebra. But $n \neq 2$ and $n \neq 4$ by assumption, so $\mathfrak{so}_n(\mathbb{A}_\mathbb{F}, f)$ is simple if and only if $Z(\mathfrak{so}_n(\mathbb{A}_\mathbb{F}, f)) = \{0\}$.

By Corollary 2.2.16, $\mathfrak{so}_n(\mathbb{A}_\mathbb{F}, f)$ will have standard basis

$$\{E_{i,j} - E_{j,i}\}_{1 \leq i < \leq n},$$

in which $E_{i,j}$ is the $n \times n$ matrix having a one in position (i,j) and zeros elsewhere. Now let $\varphi \in Z(\mathfrak{so}_n(\mathbb{A}_\mathbb{F}, f))$ and denote by $A_\varphi = (a_{i,j})_{1 \leq i,j \leq n}$ its corresponding matrix. Ordinary matrix multiplication shows that for all $1 \leq i < j \leq n$ we have

$$\begin{aligned}[A_\varphi, E_{i,j} - E_{j,i}] = 0 &\iff A_\varphi(E_{i,j} - E_{j,i}) = (E_{i,j} - E_{j,i})A_\varphi \\ &\iff \sum_{k=1}^{n}(a_{k,i}E_{k,j} - a_{k,j}E_{k,i}) = \sum_{k=1}^{n}(a_{j,k}E_{i,k} - a_{i,k}E_{j,k}),\end{aligned}$$

which yields $a_{i,i} = a_{j,j}$ and $a_{i,j} + a_{j,i} = 0$ for all $1 \leq i < j \leq n$, but also $a_{i,k} = a_{j,k} = 0 = a_{k,i} = a_{k,j}$ for all $k \neq i,j$. Altogether, we deduce that $a_{i,i} = a_{j,j}$ for all $1 \leq i,j \leq n$ and $a_{i,j} = 0$ for all $i \neq j$ so that $A_\varphi = \lambda I_n$ for some $\lambda \in \mathbb{F}^*$, in which I_n is the $n \times n$ identity matrix. But then $\varphi = \lambda \cdot \mathrm{id}_V \notin \mathfrak{so}(\mathbb{A}_\mathbb{F}, f)$, so we conclude that $Z(\mathfrak{so}(\mathbb{A}_\mathbb{F}, f))$ is trivial. It follows that $\mathfrak{so}(\mathbb{A}_{\mathbb{F}}, f)$ is simple. But any ideal of $\mathfrak{so}_n(\mathbb{F}, f) = \mathfrak{so}(V, f)$ is also an ideal of $\mathfrak{so}_n(\mathbb{A}_{\mathbb{F}}, f)$, so $\mathfrak{so}(V, f)$ will be simple as well.

Next, suppose that V is infinite-dimensional and let $\mathfrak{i} \subset \mathfrak{fso}(V, f)$ be a proper ideal. By Lemma 6.2.1, $\mathfrak{fso}(V, f)$ is generated by its infinitesimal Siegel transvections $s_{v,w} \in E(\mathfrak{fso}(V, f))$, which exist since f has Witt index at least two, so any $0 \neq x \in \mathfrak{i}$ can be written as $x = \sum_{i \in I} \lambda_i s_{x_i,y_i}$ for some finite index set I with $\lambda_i \in \mathbb{F}^*$, $i \in I$, and $0 \neq s_{x_i,y_i} \in E(\mathfrak{fso}(V, f))$, $i \in I$. Then for all $0 \neq s_{v,w} \in E(\mathfrak{fso}(V, f))$ we have

$$i \ni [s_{v,w}, [s_{v,w}, x]] = \sum_{i \in I} \lambda_i [s_{v,w}, [s_{v,w}, s_{x_i,y_i}]] = 2 s_{v,w} \sum_{i \in I} \lambda_i g_{s_{v,w}}(s_{x_i,y_i}),$$

but $\lambda_i \neq 0$ for all $i \in I$ and $g_{s_{v,w}}(s_{x_i,y_i}) \neq 0$ for all $s_{v,w} \in E(\mathfrak{fso}(V, f))$ as $\mathfrak{fso}(V, f)$ contains no sandwich elements by Lemma 6.2.1. This forces $s_{v,w} \in \mathfrak{i}$, hence $E(\mathfrak{fso}(V, f)) \subseteq \mathfrak{i}$ and so $\mathfrak{fso}(V, f) = \langle E(\mathfrak{fso}(V, f)) \rangle \subseteq \mathfrak{i}$, from which equality follows. □

Our goal will now be to show that the extremal geometry $\Gamma_{\mathfrak{so}(V,f)} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{so}(V, f)$ is isomorphic to a root shadow space $BC_{n,2}$ or $D_{n+1,2}$, $n \geq 3$; in this case, we may invoke Theorem 5.1.10, from which it will follow that $\mathfrak{g} \cong \mathfrak{so}(V, f)$, as desired.

In Corollary 4.4.6, we have seen how a root shadow space of type $BC_{n,2}$ or $D_{n+1,2}$, $n \geq 3$, can be constructed using the chamber system $\mathcal{C}(\Gamma)$ of an $\mathcal{I}$-geometry Γ of type B_n or D_{n+1} over type set $\mathcal{I} = \{1, \ldots, n\}$. However, this construction is a general one, so we wish to provide a construction that better suits the purposes of this section. To this extent, let V be a vector space having finite dimension at least six over a division ring $\mathbb{K}$ with char($\mathbb{F}$) $\neq 2$. We endow V with a non-degenerate quadratic form $Q : V \to \mathbb{K}$ whose Witt index is at least three. Now consider the sesquilinearisation $f_Q : V \times V \to \mathbb{K}$ of Q, which is uniquely determined by Q as $f_Q(u, v) = Q(u + v) - Q(u) - Q(v)$ for all $v, w \in V$. Following Corollary 4.4.6, $\Gamma_{f_Q} = (\mathcal{P}, \mathcal{L})$ will then be a root shadow space of type $BC_{n,1}$. From it arises a root shadow space of type $BC_{n,2}$ or $D_{n+1,2}$; its point set is $\mathcal{L}$, i.e. all 2-dimensional totally f_Q-isotropic subspaces of V, and its line set is the set of all subsets of $\mathcal{L}$ whose elements are the 2-dimensional totally f_Q-isotropic subspaces of V all of which are contained in some 3-dimensional totally f_Q-isotropic subspace of V and containing some 1-dimensional totally f_Q-isotropic subspace of V.

To relate the root shadow space of type $BC_{n,2}$ or $D_{n+1,2}$ as obtained from Γ_{f_Q} above to the extremal geometry of $\mathfrak{so}(V, f)$, we require the following lemmas.

Lemma 6.2.3. *Let V be a vector space over a division ring $\mathbb{K}$, char($\mathbb{K}$) $\neq 2$, endowed with a non-degenerate sesquilinear form $f : V \times V \to \mathbb{K}$. If $v, w \in V$ are linearly independent, then there exists a $u \in V$ such that $f(v, u) = 0$ and $f(w, u) \neq 0$.*

Proof. Suppose towards a contradiction that $f(v, u) \neq 0$ or $f(w, u) = 0$ for all $u \in V$. On the one hand, for all $u \in V$ such that $f(w, u) \neq 0$, of which at least one exists by non-degeneracy of f, we must have $f(v, u) \neq 0$, hence $f(v - f(v, u) f(w, u)^{-1} w, u) = f(v, u) - f(v, u) f(w, u)^{-1} f(w, u) = 0$. On the other hand, for all $u' \in V$ such that

$f(v,u')=0$ we must have $f(w,u')=0$ so that $f(v-f(v,u)f(w,u)^{-1}w,u')=f(v,u')-f(v,u)f(w,u)^{-1}f(w,u')=0$. But then $v-f(v,u)f(w,u)^{-1}w\in\mathrm{rad}(f)=\{0\}$, contradicting linear independence of v and w. □

Lemma 6.2.4. *Let $\mathfrak{so}(V,f)$ be the orthogonal Lie algebra for some finite-dimensional vector space V over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F})\neq 2$ and symmetric bilinear form $f:V\times V\to\mathbb{F}$ such that f has Witt index at least two. Further let $s_{v,w},s_{v',w'}\in E(\mathfrak{so}(V,f))$ be two linearly independent and commuting infinitesimal Siegel transvections. Then $\mathbb{F}s_{v,w}+\mathbb{F}s_{v',w'}\subseteq E(\mathfrak{so}(V,f))\cup\{0\}$ if and only if either $v=v'$ or $w=w'$.*

Proof. Suppose first that either $v=v'$ or $w=w'$. By properties of infinitesimal Siegel transvections as discussed in our proof of Corollary 3.2.14, we then have either $\mathbb{F}s_{v,w}+\mathbb{F}s_{v,w'}=s_{v,\mathbb{F}w}+s_{v,\mathbb{F}w'}=s_{v,\mathbb{F}w+\mathbb{F}w'}$ or $\mathbb{F}s_{v,w}+\mathbb{F}s_{v',w}=s_{\mathbb{F}v,w}+s_{\mathbb{F}v',w}=s_{\mathbb{F}v+\mathbb{F}v',w}$, both of which are clearly infinitesimal Siegel transvections and hence extremal by Lemma 3.2.12.

Next, suppose that $\mathbb{F}s_{v,w}+\mathbb{F}s_{v',w'}$ is extremal in $\mathfrak{so}(V,f)$. We know from Lemma 5.1.1 that $\mathbb{F}s_{v,w}+\mathbb{F}s_{v',w'}\subseteq E(\mathfrak{so}(V,f))\cup\{0\}$ if and only if $[s_{v,w},[s_{v',w'},z]]=g_{s_{v,w}}(z)s_{v',w'}+g_{s_{v',w'}}(z)s_{v,w}$ for all $z\in\mathfrak{so}(V,f)$. Specifically, we may even assume that z is an infinitesimal Siegel transvection $s_{x,y}\in E(\mathfrak{so}(V,f))$ as $\mathfrak{so}(V,f)$ is generated by $E(\mathfrak{so}(V,f))$ by Lemma 6.2.1, hence linearly spanned by $E(\mathfrak{so}(V,f))$ by Lemma 3.1.9. On the one hand, we have for all $u\in V$ that

$$
\begin{aligned}
&[s_{v,w},[s_{v',w'},s_{x,y}]](u)=[s_{v,w},s_{v',w'}s_{x,y}-s_{x,y}s_{v',w'}](u)\\
&=s_{v,w}(s_{v',w'}s_{x,y}-s_{x,y}s_{v',w'})(u)-(s_{v',w'}s_{x,y}-s_{x,y}s_{v',w'})(s_{v,w}(u))\\
&=s_{v,w}(s_{v',w'}(s_{x,y}(u)))-s_{v,w}(s_{x,y}(s_{v',w'}(u)))-s_{v',w'}(s_{x,y}(s_{v,w}(u)))+s_{x,y}(s_{v',w'}(s_{v,w}(u))),
\end{aligned}
$$

and tedious calculations show that

$$
\begin{aligned}
s_{v,w}(s_{v',w'}(s_{x,y}(u)))&=(f(y,u)f(w',x)-f(x,u)f(w',y))s_{v,w}(v')\\
&\quad-(f(y,u)f(v',x)-f(x,u)f(v',y))s_{v,w}(w'),\\
s_{v,w}(s_{x,y}(s_{v',w'}(u)))&=(f(w',u)f(y,v')-f(v',u)f(y,w'))s_{v,w}(x)\\
&\quad-(f(w',u)f(x,v')-f(v',u)f(x,w'))s_{v,w}(y),\\
s_{v',w'}(s_{x,y}(s_{v,w}(u)))&=(f(w,u)f(y,v)-f(v,u)f(y,w))s_{v',w'}(x)\\
&\quad-(f(w,u)f(x,v)-f(v,u)f(x,w))s_{v',w'}(y),\\
s_{x,y}(s_{v',w'}(s_{v,w}(u)))&=((f(w,u)f(w',v)-f(v,u)f(w',w))s_{x,y}(v')\\
&\quad-(f(w,u)f(v',v)-f(v,u)f(v',w')s_{x,y}(w'),
\end{aligned}
$$

hence $[s_{v,w},[s_{v',w'},s_{x,y}]](u)$ is a linear combination of v, w, v', w', x and y. On the other hand, we have for all $u\in V$ that

$$
\begin{aligned}
g_{s_{v,w}}(s_{x,y})s_{v',w'}(u)+g_{s_{v',w'}}(s_{x,y})s_{v,w}(u)&=(f(v,x)f(w,y)-f(v,y)f(w,x))s_{v',w'}(u)\\
&\quad+(f(v',x)f(w',y)-f(v',y)f(w',x)s_{v,w}(u),
\end{aligned}
$$

which is a linear combination of v, w, v' and w'. Then necessarily $s_{x,y}(s_{v',w'}(s_{v,w}(u)))$ must vanish as this is the only term in $[s_{v,w}, [s_{v',w'}, s_{x,y}]](u)$ that is a linear combination of x and y. This yields

$$\begin{aligned}0 &= ((f(w,u)f(w',v) - f(v,u)f(w',w))s_{x,y}(v') - (f(w,u)f(v',v) - f(v,u)f(v',w')s_{x,y}(w') \\ &= (f(w,u)f(w',v)f(y,v') - f(w,u)f(v',v)f(y,w'))x \\ &\quad - (f(v,u)f(w',w)f(x,v') - f(v,u)f(v,w')f(x,w'))y\end{aligned}$$

for all $x, y, u \in V$. We have $f(w,u) \neq 0$ for some $u \in V$ as otherwise $w = 0$ by non-degeneracy of f, a contradiction. Similarly, we have $f(v,u) \neq 0$ for some $u \in V$, $f(x,w') \neq 0$ for some $x \in V$ and $f(y,w') \neq 0$ for some $y \in V$. But then $f(v',v) = f(v',w) = 0 = f(v,w') = f(w,w')$. In turn, the term $s_{v,w}(s_{v',w'}(s_{x,y}(u)))$ will also vanish, and we moreover obtain the infinitesimal Siegel transvections $s_{v,v'}$, $s_{v,w'}$, $s_{v',w}$ and $s_{w,w'}$. Another tedious calculation now shows that

$$\begin{aligned}&[s_{v,w}, [s_{v',w'}, s_{x,y}]](u) = -s_{v,w}(s_{x,y}(s_{v',w'}(u))) - s_{v',w'}(s_{x,y}(s_{v,w}(u))) \\ &= (f(w',u)f(x,v') - f(v',u)f(x,w'))s_{v,w}(y) - (f(w',u)f(y,v') - f(v',u)f(y,w'))s_{v,w}(x) \\ &\quad + (f(w,u)f(x,v) - f(v,u)f(x,w))s_{v',w'}(y) - (f(w,u)f(y,v) - f(v,u)f(y,w))s_{v',w'}(x) \\ &= f(w',u)f(x,v')f(w,y)v - f(w',u)f(x,v')f(v,y)w - f(v',u)f(x,w')f(w,y)v \\ &\quad + f(v',u)f(x,w')f(v,y)w - f(w',u)f(y,v')f(w,x)v + f(w',u)f(y,v')f(v,x)w \\ &\quad + f(v',u)f(y,w')f(w,x)v - f(v',u)f(y,w')f(v,x)w + f(w,u)f(x,v)f(w',y)v' \\ &\quad - f(w,u)f(x,v)f(v',y)w' - f(v,u)f(x,w)f(w',y)v' + f(v,u)f(x,w)f(v',y)w' \\ &\quad - f(w,u)f(y,v)f(w',x)v' + f(w,u)f(y,v)f(v',x)w' + f(v,u)f(y,w)f(w',x)v' \\ &\quad - f(v,u)f(y,w)f(v',x)w' \\ &= (f(y,w')f(w,x) - f(x,w')f(y,w))s_{v,v'}(u) + (f(x,v')f(w,y) - f(x,w)f(v',y))s_{v,w'}(u) \\ &\quad + (f(x,v)f(w',y) - f(y,v)f(w',x))s_{v',w}(u) + (f(y,v')f(v,x) - f(x,v')f(v,y))s_{w,w'}(u) \\ &= (f(v,x)f(w,y) - f(v,y)f(w,x))s_{v',w'}(u) + (f(v',x)f(w',y) - f(v',y)f(w',x)s_{v,w}(u) \\ &= g_{s_{v,w}}(s_{x,y})s_{v',w'}(u) + g_{s_{v',w'}}(s_{x,y})s_{v,w}(u),\end{aligned}$$

forcing

$$(f(y,w')f(w,x) - f(x,w')f(y,w))s_{v,v'}(u) = 0 = (f(y,v')f(v,x) - f(x,v')f(v,y))s_{w,w'}(u)$$

for all $x, y, u \in V$, or equivalently $g_{s_{w,w'}}(s_{x,y})s_{v,v'}(u) = 0 = g_{s_{v,v'}}(s_{x,y})s_{w,w'}(u)$. If $s_{v,v'} = 0$, then necessarily $v = v'$ and consequently $g_{s_{v,v'}}(s_{x,y}) = 0$. If $g_{s_{w,w'}}(s_{x,y}) = 0$, then we must have $s_{w,w'} = 0$ because $\mathfrak{so}(V, f)$ contains no sandwich elements by Lemma 6.2.1, from which it follows that $w = w'$ so that also $s_{w,w'} = 0$. We conclude that either $v = v'$ or $w = w'$ but not both as otherwise $s_{v,w} = s_{v',w'}$, contradicting their linear independence. □

We are now in a position to establish the desired connection between the extremal geometry $\Gamma_{\mathfrak{so}(V,f)} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{so}(V, f)$ and a root shadow space of type $BC_{n,2}$ or $D_{n+1,2}$, $n \geq 3$.

Proposition 6.2.5. *Let $\mathfrak{so}(V, f)$ be the orthogonal Lie algebra for some vector space V over a field $\mathbb{F}$, $\operatorname{char}(\mathbb{F}) \neq 2$, and some non-degenerate symmetric bilinear form $f : V \times V \to \mathbb{F}$ such that f has Witt index at least three. Then the extremal geometry $\Gamma_{\mathfrak{so}(V,f)} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{so}(V, f)$ is isomorphic to a root shadow space of type $BC_{n,2}$ or $D_{n+1,2}$, $n \geq 3$.*

Proof. By Lemma 6.2.1, $\mathfrak{so}(V, f)$ is generated by the infinitesimal Siegel transvections $s_{v,w} \in V$, all of which are extremal and constitute the set of extremal elements of $\mathfrak{so}(V, f)$ $E(\mathfrak{so}(V, f))$. The points in $\mathcal{E}$ of $\Gamma_{\mathfrak{so}(V,f)} = (\mathcal{E}, \mathcal{L})$ will then be the spans $\mathbb{F}s_{v,w}$ of infinitesimal Siegel transvections $s_{v,w} \in V$, and two infinitesimal Siegel transvections $s_{v,w}, s_{x,y} \in V$ define a line in $\mathcal{L}$ if and only if $s_{v,w}$ and $s_{x,y}$ are commuting and linearly independent infinitesimal Siegel transvections such that $\mathbb{F}s_{v,w} + \mathbb{F}s_{x,y}$ is extremal. Letting $f : V \times V \to \mathbb{F}$ be the non-degenerate symmetric bilinear form on V that defines $\mathfrak{so}(V, f)$ as in Definition 2.2.10, the above construction shows that $\Gamma^*_f = (\mathcal{L}_f, \mathcal{P}^*)$ is a root shadow space of type $BC_{n,2}$ or $D_{n+1,2}$ that arises from the absolute $\Gamma_f = (\mathcal{P}, \mathcal{L}_f)$. Note that $n \geq 3$ since f is assumed to have Witt index at least three. We will prove that the map $\varphi : \Gamma_{\mathfrak{so}(V,f)} \to \Gamma^*_f$ given by $\varphi(s_{v,w}) = \langle v, w\rangle$ with $s_{v,w} \in E(\mathfrak{so}(V, f))$ establishes an isomorphism between $\Gamma_{\mathfrak{so}(V,f)}$ and Γ^*_f.

First, we show that φ is bijective on $\mathcal{E}$ and $\mathcal{L}_f$. Clearly, if $\langle v, w\rangle \subset V$ is a totally f-isotropic subspace, then $f(\lambda v + \mu w, \lambda v + \mu w) = \lambda^2 f(v, v) + 2\lambda\mu f(v, w) + \mu^2 f(w, w) = 0$ for all $\lambda, \mu \in \mathbb{F}$, from which it follows that v and w are f-isotropic and orthogonal so that $s_{v,w}$ is an infinitesimal Siegel transvection by Definition 3.2.13, necessarily extremal by Lemma 6.2.1. Now let $s_{v,w} \in E(\mathfrak{so}(V, f))$ and consider the vectors $s_{v,w}(u) = f(w, u)v - f(v, u)w$ with $u \in V$ arbitrary. Then $s_{v,w} = \langle v, w\rangle$ if and only if for all $\lambda, \mu \in \mathbb{F}$ there exists a $u \in V$ such that $s_{v,w}(u) = \lambda v + \mu w$. So, let $\lambda, \mu \in \mathbb{F}$ be arbitrary. By Lemma 6.2.3, there exists a $u \in V$ such that $f(v, u) = 0$ and $f(w, u) \neq 0$. Similarly, by interchanging v and w, there exists a $u' \in V$ such that $f(w, u') = 0$ and $f(v, u') \neq 0$. Upon scaling, we may assume that $f(w, u) = \lambda$ if $\lambda \neq 0$ and $f(v, u') = -\mu$ if $\mu \neq 0$. But then $s_{v,w}(u + u') = f(w, u + u')v - f(v, u + u')w = f(w, u)v - f(v, u')w = \lambda v + \mu w$ if $\lambda \neq 0$ and $\mu \neq 0$, whereas $s_{v,w}(u) = f(w, u)v - f(v, u)w = \lambda v$ if $\lambda \neq 0$ and $\mu = 0$, $s_{v,w}(u') = f(w, u')v - f(v, u')w = \mu w$ if $\lambda = 0$ and $\mu \neq 0$, and $s_{v,w}(0) = 0$ if $\lambda = 0 = \mu$. Regardless, we have $\varphi(\mathcal{E}) \subseteq \mathcal{L}_f$, forcing equality.

It remains to prove that $\mathcal{L}$ is mapped bijectively to $\mathcal{P}^*$ by φ. On the one hand, any line in $\mathcal{P}^*$ consists of all 2-dimensional totally f-isotropic subspaces of V contained in some 3-dimensional totally f-isotropic subspace $\langle u, v, w\rangle \subset V$, which exists because f has Witt index at least three, and containing w.l.o.g. the 1-dimensional totally f-isotropic subspace $\langle u\rangle \subset V$. These 2-dimensional totally f-isotropic subspaces will then be $\langle \mathbb{F}v + \mathbb{F}w, u\rangle \subset V$. But this yields the infinitesimal Siegel transvection $s_{\mathbb{F}v+\mathbb{F}w,u} = s_{\mathbb{F}v,u} + s_{\mathbb{F}w,u} = \mathbb{F}s_{v,u} + \mathbb{F}s_{w,u}$ by the previous paragraph and by properties of infinitesimal

Siegel transvections that we have established in our proof of Corollary 3.2.14. This shows that $\mathcal{P}^* \subseteq \varphi(\mathcal{L})$. On the other hand, if $s_{v,w}, s_{v',w'} \in E(\mathfrak{so}(V,f))$ are commuting and linearly independent, then $\mathbb{F}s_{v,w} + \mathbb{F}s_{v,w'} \subseteq E(\mathfrak{so}(V,f)) \cup \{0\}$ if and only if either $v = v'$ or $w = w'$ by Lemma 6.2.4. Assuming w.l.o.g. that we are in the latter case, we obtain $\mathbb{F}s_{v,w} + \mathbb{F}s_{v',w} = s_{\mathbb{F}v,w} + s_{\mathbb{F}v',w} = s_{\mathbb{F}v+\mathbb{F}v',w}$, which corresponds to the 2-dimensional totally f-isotropic subspace $\langle \mathbb{F}v+\mathbb{F}v', w\rangle \subset V$ by the previous paragraph. As all such subspaces are contained in the 3-dimensional totally f-isotropic subspace $\langle v, v', w\rangle \subset V$ and contain the 1-dimensional totally f-isotropic subspace $\langle w\rangle \subset V$, we deduce that $\varphi(\mathcal{L}) \subseteq \mathcal{P}^*$. Then φ is an isomorphism between $\Gamma_{\mathfrak{so}(V,f)}$ and Γ_f^* by Definition 4.1.8, so the proposition follows. □

We gather our findings in the following theorem, which settles the first part of our proof of Theorem 1.1.2.

Theorem 6.2.6. *Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements. If the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{g}$ is isomorphic to a root shadow space of type $BC_{n,2}$ or $D_{n+1,2}$, $n \geq 3$, then $\mathfrak{g}$ is isomorphic to the orthogonal Lie algebra $\mathfrak{so}(V,f)$ for some vector space V over $\mathbb{F}$ and some non-degenerate symmetric bilinear form $f : V \times V \to \mathbb{F}$ such that f has Witt index at least three.*

Proof. By Lemma 6.2.1 and Lemma 6.2.2, $\mathfrak{so}(V,f)$ is a simple Lie algebra without sandwich elements generated by its set of extremal elements $E(\mathfrak{so}(V,f))$. By Proposition 6.2.5, we may choose a vector space V over $\mathbb{F}$ such that $\Gamma_{\mathfrak{so}(V,f)}$ is a root shadow space of type $BC_{n,2}$ or $D_{n+1,2}$. But then Theorem 5.1.10 applies, thus $\mathfrak{g} \cong \mathfrak{so}(V,f)$. □

6.3 Infinite-dimensional case

We proceed with the second part of our proof of Theorem 1.1.2. So, let $\mathfrak{fso}(V,f)$ be the finitary orthogonal Lie algebra for some infinite-dimensional vector space V over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, and some non-degenerate symmetric bilinear form $f : V \times V \to \mathbb{F}$ such that f has Witt index at least three. Further let $\mathfrak{g}$ be an infinite-dimensional simple Lie algebra over $\mathbb{F}$ without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements whose extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ is isomorphic to the geometry whose point set is the set of all 2-dimensional totally f-isotropic subspaces of V and whose line set is the set of all subsets of the point set whose elements are the 2-dimensional totally f-isotropic subspaces of V all of which are contained in a fixed 3-dimensional totally f-isotropic subspace of V and containing a fixed 1-dimensional totally f-isotropic subspace of V. We will call this geometry the *orthogonal geometry*, denoted by $\Gamma_{\mathcal{O}}(V,f)$. This time, our goal will be to construct local systems for both $\mathfrak{fso}(V,f)$ and $\mathfrak{g}$ as a means of showing their isomorphicity.

First, we need a directed set that we can use to construct local systems for $\mathfrak{fso}(V,f)$ and $\mathfrak{g}$. Consider first the following lemma.

Lemma 6.3.1. *Let V be a vector space over a division ring $\mathbb{K}$, $\mathrm{char}(\mathbb{K}) \neq 2$, containing isotropic vectors and equipped with a non-degenerate reflexive (σ, ϵ)-sesquilinear form $f : V \times V \to \mathbb{K}$ for some admissible pair (σ, ϵ). Then V is spanned by its f-isotropic vectors if and only if $f(v, v) \in \mathbb{K}_{\sigma,-\epsilon} = \{\lambda + \epsilon\lambda^\sigma \mid \lambda \in \mathbb{K}\}$ for all $v \in V$.*

Proof. Suppose first that V is spanned by its f-isotropic vectors. Then any $v \in V$ can be written as a finite linear combination $\sum_{i\in I} \lambda_i v_i \in V$ of f-isotropic vectors $v_i \in V$, $i \in I$, with $\lambda_i \in \mathbb{K}^*$, $i \in I$, for some finite index set I. Consequently,

$$\begin{aligned} f(v,v) &= \sum_{i,j\in I} f(\lambda_i v_i, \lambda_j v_j) = \sum_{\substack{i,j\in I\\ i\neq j}} f(\lambda_i v_i, \lambda_j v_j) = \sum_{\substack{i,j\in I\\ i<j}} f(\lambda_i v_i, \lambda_j v_j) + \epsilon f(\lambda_i v_i, \lambda_j v_j)^\sigma \\ &= \Bigg(\sum_{\substack{i,j\in I\\ i<j}} f(\lambda_i v_i, \lambda_j v_j)\Bigg) + \epsilon\Bigg(\sum_{\substack{i,j\in I\\ i<j}} f(\lambda v_i, \lambda v_j)\Bigg)^\sigma \in \mathbb{K}_{\sigma,-\epsilon}, \end{aligned}$$

showing that $f(v, v) \in \mathbb{K}_{\sigma,-\epsilon}$ for all $v \in V$.

Next, suppose that $f(v, v) \in \mathbb{K}_{\sigma,-\epsilon}$ for all $v \in V$. Clearly, the span of the isotropic vectors of V is contained in V. Now let $v \in V$ be f-isotropic, which exists by assumption, and let $w \in W$ such that $f(v, w) \neq 0$, which exists by non-degeneracy of f and forces linear independence of v and w. As $f(w, w) \in \mathbb{K}_{\sigma,-\epsilon}$, there exists a $\lambda \in \mathbb{K}$ such that $f(w, w) = \lambda + \epsilon\lambda^\sigma$, hence for all $\mu \in \mathbb{K}$ we obtain

$$\begin{aligned} f(w + \mu v, w + \mu v) &= f(w, w) + f(w, v)\mu^\sigma + \mu f(v, w) \\ &= \lambda + \epsilon\lambda^\sigma + \mu f(v, w) + \epsilon f(v, w)^\sigma \mu^\sigma \\ &= (\lambda + \mu f(v, w)) + \epsilon(\lambda + \mu f(v, w))^\sigma. \end{aligned}$$

Taking $\mu = -\lambda f(v, w)^{-1}$ then yields $f(w + \mu v, w + \mu v) = 0$, so $\langle v, w\rangle \subseteq V$ always contains an f-isotropic vector, which shows that every 2-dimensional subspace of V containing v is generated by its isotropic vectors. But then the span of the isotropic vectors of V contains V, forcing equality. □

Let (V, f) be a symmetric space with V an infinite-dimensional vector space over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, and $f : V \times V \to \mathbb{F}$ a non-degenerate symmetric bilinear form on V such that f has Witt index at least three. Denote by $\mathcal{V}$ the set of finite-dimensional subspace $W \subset V$ such that $f|_W$ is non-degenerate having Witt index at least three. Clearly, inclusion $\subseteq$ defines both a pre-order and a partial order on $\mathcal{V}$. The following lemma shows that $(\mathcal{V}, \subseteq)$ is a directed set.

Lemma 6.3.2. *Let V be an infinite-dimensional vector space over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, equipped with a non-degenerate symmetric bilinear form $f : V \times V \to \mathbb{F}$ and let $\mathcal{V}$ be as defined above. Then for any two finite-dimensional subspaces $U, U' \in \mathcal{V}$ there exists a subspace in $\mathcal{V}$ containing both U and U'.*

Proof. Let $U + U' = \{u + u' \in V \mid u \in U, u' \in U'\}$ be the smallest subspace of V containing both U and U'. Note that $U + U'$ is finite-dimensional; indeed, there are only finitely many vectors $u + u' \in V$ with $u \in U$ and $u' \in U'$ as both U and U' are finite-dimensional by assumption. Setting $X := U + U'$, we show how to extend X to a finite-dimensional subspace $X' \subset V$ such that $\mathrm{rad}(f|_{X'}) = \{0\}$.

If $\mathrm{rad}(f|_X) = \{0\}$, then we are done as clearly $U \subseteq X$ and $U' \subseteq X$. So, assume that $\mathrm{rad}(f|_X) \neq \{0\}$ and let $0 \neq r \in \mathrm{rad}(f|_X)$. We first show that there exists an $0 \neq x \in V \setminus X$ such that $f(x, x) = 0$ and $f(r, x) \neq 0$. Suppose towards the contrary that $f(r, x) = 0$ for all $x \in V \setminus X$ such that $f(x, x) = 0$. Up to proportionality by Theorem 2.2.8, the form f is the sesquilinearisation $f_Q : V \times V \to \mathbb{F}$ of a quadratic form $Q : V \to \mathbb{F}$, which will be non-degenerate as f is non-degenerate. In particular, f_Q uniquely determines Q and we have $Q(v + w) = Q(v) + Q(w) + f(v, w)$ for all $v, w \in V$ so that $f(v, v) = Q(2v) - 2Q(v) = 2Q(v)2^\sigma - 2Q(v) = 2Q(v)$ for all $v \in V$ because f is $(\mathrm{id}_\mathbb{F}, 1)$-sesquilinear. Further note that $\mathbb{F}_{\mathrm{id}_\mathbb{F},-1} = \{\lambda - (-1)\lambda^{\mathrm{id}_\mathbb{F}} \mid \lambda \in \mathbb{F}\} = \{2\lambda \mid \lambda \in \mathbb{F}\}$. As $Q(v) \in \mathbb{F}$ for all $v \in V$, we deduce that $f(v, v) = 2Q(v) \in \mathbb{F}_{\mathrm{id}_\mathbb{F},-1}$. Then Lemma 6.3.1 applies, so $V = \mathrm{span}\{v \in V \mid f(v, v) = 0\}$. But then every $x' \in V \setminus X$ can be written as a linear combination $\sum_{i \in I} \lambda_i x_i$ of f-isotropic vectors $x_i \in V \setminus X$, $i \in I$, with $\lambda_i \in \mathbb{F}^*$, $i \in I$, for some finite index set I. It follows that $f(r, x') = \sum_{i \in I} \lambda_i f(r, x_i) = 0$ for all $x' \in V \setminus X$, but we also have $f(r, y) = 0$ for all $y \in X$ since $r \in \mathrm{rad}(f|_X)$. Combined, this implies that $r \in \mathrm{rad}(f) = \{0\}$, a contradiction. Note that an $x \in V \setminus X$ such that $f(x, x) = 0$ always exists; if not, then all f-isotropic vectors of V are contained in X, but the fact that V is spanned by its f-isotropic vectors then forces X to be V, which is absurd. We conclude that there exists an $x \in V \setminus X$ such that $f(x, x) = 0$ and $f(r, x) \neq 0$. Now consider the subspace $X' := X + \langle x \rangle$, which is clearly finite-dimensional.

We show that $\dim(\mathrm{rad}(f|_{X'})) < \dim(\mathrm{rad}(f|_X))$. As $r \in \mathrm{rad}(f|_X) \setminus \mathrm{rad}(f|_{X'})$, it suffices to show that no vector in $X' \setminus X$ is in $\mathrm{rad}(f|_{X'})$. So, $\dim(\mathrm{rad}(f|_{X'})) < \dim(\mathrm{rad}(f|_X))$ if and only if $y + \lambda x \notin \mathrm{rad}(f|_{X'})$ for all $y \in X' \setminus X$ and $\lambda \in \mathbb{F}$. Supposing for the sake of contradiction that this is not the case, then there exist $y \in X' \setminus X$ and $\lambda \in \mathbb{F}$ such that $f(y + \lambda x, y' + \mu x) = 0$ for all $y' \in X$ and $\mu \in \mathbb{F}$. Note that $\mathrm{rad}(f|_{X'}) \subseteq \mathrm{rad}(f|_X)$ implies not only that $y \neq 0$ as otherwise $x \in \mathrm{rad}(f|_X)$, contradicting $f(x, r) \neq 0$, but also that $\lambda \neq 0$ as otherwise $y \in \mathrm{rad}(f|_X) \subseteq X$, contradicting $y \notin X$. We find $0 = f(y + \lambda x, y' + \mu x) = f(y, x) + \mu f(x, x) = f(y, x)$ by taking $y' = 0$ and $\mu \neq 0$. For $y' = r \in \mathrm{rad}(f|_X) \subseteq X$, we get $0 = f(y + \lambda x, r) = f(y, r) + \lambda f(x, r) = \lambda f(x, r)$, but then $\lambda \neq 0$ implies $f(x, r) = 0$, a contradiction. Thus, the dimension of $\mathrm{rad}(f|_{X'})$ is strictly smaller than the dimension of $\mathrm{rad}(f|_X)$.

Since $\mathrm{rad}(f|_X)$ is finite-dimensional, we can repeat the process described in the previous two paragraphs until we obtain a subspace $W \subset V$ such that $U, U' \subseteq W$ and $\mathrm{rad}(f|_W) = \{0\}$, i.e. $f|_W$ is non-degenerate. As both $f|_U$ and $f|_{U'}$ have Witt index at least three, then $f|_W$ will also have Witt index at least three because $U, U' \subseteq W$. This shows that $W \in \mathcal{V}$ such that $U, U' \subseteq W$, as claimed. □

Let $(\mathcal{V}, \subseteq)$ be the previously defined set of subspaces of V, which is readily seen to be directed by the above lemma. We will construct two systems of Lie subalgebras of $\mathfrak{fso}(V, f)$, respectively $\mathfrak{g}$, over $(\mathcal{V}, \subseteq)$ and show that they are local systems by Definition 6.1.4.

Let $W \in \mathcal{V}$. On the one hand, define $\mathfrak{so}(W, f) = \langle s_{v,w} \in \mathfrak{fso}(V, f) \mid v, w \in W\rangle$ to be the Lie subalgebra of $\mathfrak{fso}(V, f)$ generated by the infinitesimal Siegel transvections in W. On the other hand, denoting by $\Gamma_{\mathfrak{g}(W)} = (\mathcal{E}(W), \mathcal{L}(W))$ the subspace of the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{g}$ whose point set $\mathcal{E}(W)$ is the set of all 2-dimensional totally $f|_W$-isotropic subspaces of W and whose line set $\mathcal{L}(W)$ is the set of all subsets of $\mathcal{E}(W)$ contained in some 3-dimensional $f|_W$-isotropic subspace of W and containing some 1-dimensional totally $f|_W$-isotropic subspace of W, define $\mathfrak{g}(W) = \langle x \mid x \in \mathcal{E}(W)\rangle$ to be the Lie subalgebra of $\mathfrak{g}$ generated by the extremal points of $\Gamma_{\mathfrak{g}(W)}$.

Proposition 6.3.3. *The systems $\mathfrak{O} = \{\mathfrak{so}(W, f)\}_{W\in\mathcal{V}}$ and $\mathfrak{G} = \{\mathfrak{g}(W)\}_{W\in\mathcal{V}}$ with $\mathfrak{so}(W, f)$ and $\mathfrak{g}(W)$, $W \in \mathcal{V}$, as defined above are local systems of $\mathfrak{fso}(V, f)$ and $\mathfrak{g}$, respectively, over the previously defined directed set $(\mathcal{V}, \subseteq)$.*

Proof. First consider the system $\mathfrak{O} = \{\mathfrak{so}(W, f)\}_{W\in\mathcal{V}}$ of finite-dimensional Lie subalgebras of $\mathfrak{fso}(V, f)$. Let $U, U' \in \mathcal{V}$ with corresponding Lie subalgebras $\mathfrak{so}(U, f), \mathfrak{so}(U', f) \subset \mathfrak{fso}(V, f)$. By Proposition 6.3.2, there exists a $W \in \mathcal{V}$ such that $U, U' \subseteq W$, so both $\mathfrak{so}(U, f)$ and $\mathfrak{so}(U', f)$ are contained in $\mathfrak{so}(W, f)$. It remains to show that $\mathfrak{fso}(V, f) = \bigcup_{W\in\mathcal{V}} \mathfrak{so}(W, f)$. Because $\mathfrak{fso}(V, f)$ is generated by its infinitesimal Siegel transvections by Lemma 6.2.1, it is linearly spanned by them as a result of Lemma 3.1.9. So, every element in $\mathfrak{g}$ is a finite linear combination of infinitesimal Siegel transvections, each of them contained in $\mathfrak{so}(W, f)$ for some $W \in \mathcal{V}$, which shows that $\mathfrak{fso}(V, f) \subseteq \bigcup_{W\in\mathcal{V}} \mathfrak{so}(W, f)$ and forces equality. This proves $\mathfrak{O} = \{\mathfrak{so}(W, f)\}_{W\in\mathcal{V}}$ to be a local system of $\mathfrak{fso}(V, f)$ over $(\mathcal{V}, \subseteq)$.

Next, consider the system $\mathfrak{G} = \{\mathfrak{g}(W)\}_{W\in\mathcal{V}}$ of finite-dimensional Lie subalgebras of $\mathfrak{g}$ and let $U, U' \in \mathcal{V}$ with corresponding Lie subalgebras $\mathfrak{g}(U), \mathfrak{g}(U') \subset \mathfrak{g}$. Now Proposition 6.3.2 applies again, so we can find a $W \in \mathcal{V}$ such that $U, U' \subseteq W$. Consequently, we have $\mathcal{E}(U), \mathcal{E}(U') \subseteq \mathcal{E}(W)$, but then the spans of $\mathcal{E}(U)$ and $\mathcal{E}(U')$ will both be contained in the span of $\mathcal{E}(W)$, hence $\mathfrak{g}(U), \mathfrak{g}(U') \subseteq \mathfrak{g}(W)$. The equality $\mathfrak{g} = \bigcup_{W\in\mathcal{V}} \mathfrak{g}(W)$ follows from arguments similar to those given in the previous paragraph; every element in $\mathfrak{g}$ is a finite linear combination of extremal elements, each of them corresponding to an extremal point in $\mathcal{E}(W)$ for some $W \in \mathcal{V}$, hence $\mathfrak{g} \subseteq \bigcup_{W\in\mathcal{V}} \mathfrak{g}(W)$ and equality follows. This shows that $\mathfrak{G} = \{\mathfrak{g}(W)\}_{W\in\mathcal{V}}$ is a local system of $\mathfrak{g}$ over $(\mathcal{V}, \subseteq)$. □

We set out to prove that $\mathfrak{O}$ and $\mathfrak{G}$ are isomorphic as local systems with the aim of applying Theorem 6.1.6. This is done in the following theorem, which settles the infinite-dimensional case.

Theorem 6.3.4. *Let $\mathfrak{g}$ be an infinite-dimensional simple Lie algebra over a field $\mathbb{F}$,* char$(\mathbb{F}) \neq$ 2*, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements. If the extremal* *geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{g}$ is isomorphic to the orthogonal geometry $\Gamma_{\mathcal{O}}(V, f)$ of some vector space V over $\mathbb{F}$ and some non-degenerate symmetric bilinear form $f : V \times V \to \mathbb{F}$ on V such that f has Witt index at least three, then $\mathfrak{g}$ is isomorphic to the finitary orthogonal Lie algebra $\mathfrak{fso}(V, f)$.*

Proof. Let $\mathfrak{O} = \{\mathfrak{so}(W, f)\}_{W \in \mathcal{V}}$ and $\mathfrak{G} = \{\mathfrak{g}(W)\}_{W \in \mathcal{V}}$ be the previously defined systems of finite-dimensional Lie subalgebras of $\mathfrak{fso}(V, f)$ and $\mathfrak{g}$, respectively, over the directed set $(\mathcal{V}, \subseteq)$ obtained from Lemma 6.3.2, which we have seen are both local systems by Proposition 6.3.3.

We first show that $\mathfrak{so}(W, f) \cong \mathfrak{g}(W)$ for all $W \in \mathcal{V}$. So, let $W \in \mathcal{V}$ be arbitrary and consider the Lie subalgebras $\mathfrak{so}(W, f)$ and $\mathfrak{g}(W)$ of $\mathfrak{fso}(V, f)$, respectively $\mathfrak{g}$. As their extremal geometry $\Gamma_{\mathfrak{so}(W,f)}$ and $\Gamma_{\mathfrak{g}(W)}$ are subspaces of the extremal geometries $\Gamma_{\mathfrak{fso}(V,f)}$ and $\Gamma_{\mathfrak{g}}$ of $\mathfrak{fso}(V, f)$ and $\mathfrak{g}$, respectively, we find $\Gamma_{\mathfrak{so}(W,f)}$ and $\Gamma_{\mathfrak{g}(W)}$ to be isomorphic to a root shadow space of type $BC_{m,2}$ or $D_{m+1,2}$, respectively $BC_{m',2}$ or $D_{m'+1,2}$, with $m \geq 3$ and $m' \geq 3$ since $f|_W$ has Witt index at least three. Because W is finite-dimensional, both m and m' must be finite, therefore the polar spaces underlying $\Gamma_{\mathfrak{so}(W,f)}$ and $\Gamma_{\mathfrak{g}(W)}$ arise from two quadratic forms $Q : W \times W \to \mathbb{F}$ and $Q' : W \times W \to \mathbb{F}$ on W. Their sesquilinearisations $f_Q : W \times W \to \mathbb{F}$ and $f'_Q : W \times W \to \mathbb{F}$, which are uniquely determined by Q and Q', respectively, will both be symmetric, i.e. $(\mathrm{id}_{\mathbb{F}}, 1)$-sesquilinear, hence proportional by Theorem 2.2.8 so that $\Gamma_{f_Q} \cong \Gamma_{f'_Q}$. In addition, we have $\mathbb{F}_{\mathrm{id}_{\mathbb{F}},1} = \{\lambda - \lambda \mid \lambda \in \mathbb{F}\} = \{0\} = \{\lambda \in \mathbb{F} \mid 2\lambda = 0\} = \mathbb{F}^{\mathrm{id}_{\mathbb{F}},1}$ since $\mathrm{char}(\mathbb{F}) \neq 2$, so $\Gamma_Q = \Gamma_{f_Q}$ and $\Gamma_{Q'} = \Gamma_{f'_Q}$ by Proposition 4.6.10. But then $\Gamma_Q = \Gamma_{f_Q} \cong \Gamma_{f'_Q} = \Gamma_{Q'}$, forcing $m = m'$ so that $\Gamma_{\mathfrak{so}(W,f)} \cong \Gamma_{\mathfrak{g}(W)}$, hence $\mathfrak{g}(W) \cong \mathfrak{so}(W, f)$ by Theorem 6.2.6 as f has Witt index at least three. In particular, we can find an isomorphism $\varphi_W : \mathfrak{g}(W) \to \mathfrak{so}(W, f)$ that induces the identity on $\Gamma_{\mathfrak{so}(W,f)} \cong \Gamma_{\mathfrak{g}(W)}$.

Denote by $\{\varphi_W : \mathfrak{so}(W, f) \to \mathfrak{g}(W)\}_{W \in \mathcal{V}}$ the collections of isomorphisms as described above. In light of Proposition 6.1.6, it remains show that $\varphi_U = \varphi_W|_U$ for all $U, W \in \mathcal{V}$ such that $U \subseteq W$. Consider the map $\varphi := \varphi_U(\varphi_W|_U)^{-1}$. Note that, as $U \subseteq W$, the map $(\varphi_W|_U)^{-1}$ sends elements in $\mathfrak{so}(W, f)$ contained in $\mathfrak{so}(U, f)$ to elements in $\mathfrak{g}(W)$ contained in $\mathfrak{g}(U)$, whereas φ_U maps elements in $\mathfrak{g}(U)$ to $\mathfrak{so}(U, f)$. As both φ_U and $(\varphi_W|_U)^{-1}$ are Lie algebra isomorphisms, we find φ to be an automorphism of $\mathfrak{so}(U, f)$. Moreover, since both φ_U and φ_W induce the identity on $\Gamma_{\mathfrak{so}(W,f)}$ and $\Gamma_{\mathfrak{so}(U,f)}$, respectively, so will φ on $\Gamma_{\mathfrak{so}(U,f)}$, implying that φ fixes $E(\mathfrak{so}(U, f))$. In turn, φ will fix every line in $\Gamma_{\mathfrak{so}(U,f)}$ up to scalar multiplication, say by some $\lambda \in \mathbb{F}^*$. Writing $\Gamma_{\mathfrak{so}(U,f)} = (\mathcal{E}, \mathcal{L})$, which is a root filtration space with filtration $\{\mathcal{E}_i\}_{-2 \leq i \leq 2}$ by Proposition 5.1.3, the graph $(\mathcal{E}, \mathcal{E}_2)$ is connected by Theorem 5.1.4 since $\mathfrak{so}(U, f)$ is simple, therefore the collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ of $\Gamma_{\mathfrak{so}(U,f)}$ is also connected by Lemma 5(ii) of [5] as $\mathcal{L} \neq \emptyset$. But then φ is a scalar multiplication by λ on the entirety of $\mathfrak{so}(U, f)$. Consequently, for $x, y \in \mathfrak{so}(U, f)$ such that $[x, y] \neq 0$, which exist as otherwise $\mathfrak{so}(U, f)$ would be abelian, specifically not simple, we obtain

$$\lambda[x, y] = \varphi([x, y]) = [\varphi(x), \varphi(y)] = [\lambda x, \lambda y] = \lambda^2[x, y],$$

therefore $\lambda = 1$, forcing φ to be the identity on $\mathfrak{so}(U, f)$. It follows that $\varphi_U = \varphi_W|_U$.

Now Theorem 6.1.6 applies, so we have $\mathfrak{O} \cong \mathfrak{G}$. In particular, the Lie algebras $\mathfrak{fso}(V, f)$ and $\mathfrak{g}$ are isomorphic. $\square$

By combining Theorem 6.2.6 with Theorem 6.3.4, both the finite-dimensional and infinite-dimensional case of Theorem 1.1.2 have been proven. Theorem 1.1.2 then immediately follows.

Chapter 7

Finiteness of the singular rank of the extremal geometry

In Section 7.1, we introduce abstract root subgroups to define point-line geometries related to the extremal geometry of a Lie algebra whose classification we will use to prove Theorem 1.1.4 in Section 7.2.

Section 7.1 and Section 7.2 are based mainly on the theory from [5, 26], with the latter section using [8, 27] for classification purposes.

7.1 The root group geometry of abstract root subgroups

This section serves as an introduction to abstract root subgroups. In particular, we will establish a connection between abstract root subgroups and the extremal geometry of a Lie algebra in preparation for our proof of Theorem 1.1.4. First, however, we briefly turn our attention to some group-theoretical concepts that we require to define abstract root subgroups.

Let G be a group and denote by $G^{\#}$ the set of all elements of G different from its identity element 1. The *conjugate* of an element $g \in G$ by an element $h \in H$ is the element $g^h = h^{-1}gh$, and the *conjugate subgroup* of a subgroup $H \leq G$ by an element $g \in G$ is the subgroup $H^g = \{h^g \mid h \in H\}$. The *commutator* of two elements $g, h \in G$ is the element $[g, h] = g^{-1}h^{-1}gh$, and the *commutator subgroup* of two subgroups $H, K \leq G$ is the subgroup $[H, K] = \{[h, k] \mid h \in H, k \in K\}$. Further recall from Definition 2.1.21 the concept of nilpotency of Lie algebra $\mathfrak{g}$; its group-theoretical equivalent makes use of the *lower central series* of a group G, recursively defined as $G_0 = G$ and $G_i = [G, G_i]$ with $i \geq 0$. In particular, we have $G_{i+1} \trianglelefteq G_i$ for all $i \geq 0$, and G is said to be *nilpotent* if $G_n = 1$ for some $n \geq 0$. The *nilpotency class* of a nilpotent group G is the smallest integer $n \geq 0$ such that $G_n = 1$. Examples of nilpotent groups are abelian groups, but also finite p-groups for some prime number p.

Definition 7.1.1 (Rank one group)**.** *Let X be a group generated by two nilpotent subgroups $A, B \leq X$. If for all $a \in A^{\#}$ there exists some $b \in B^{\#}$ such that $A^b = B^a$ and vice versa, then $X = \langle A, B \rangle$ is called a* ***rank one group****.*

Let $X = \langle A, B \rangle$ be a rank one group for some nilpotent subgroups $A, B \leq X$. The conjugate subgroups A^b and B^a of A by $b \in B^{\#}$, respectively B by $a \in A^{\#}$, are called the *unipotent* subgroups of X. If both A and B are abelian groups, then X is said to be a rank one group with abelian unipotent subgroups, abbreviated AUS. If for all $a \in A^{\#}$ and $b \in B^{\#}$ such that $A^b = B^a$, as in the above definition, we additionally have $a^b = b^{-a} = (b^{-1})^a$, then X is called a *special rank one group*.

Example 7.1.2. Let $\mathbb{K}$ be a division ring and denote by $\mathbb{K}_\sigma = \{\lambda \in \mathbb{K} \mid \lambda^\sigma = \lambda\}$ the subfield of $\mathbb{K}$ consisting of the elements fixed by some anti-automorphism $\sigma : \mathbb{K} \to \mathbb{K}^{\text{opp}}$. We claim that the group $X = \langle A, B \rangle \leq \mathrm{GL}_2(\mathbb{K}_\sigma)$ with $A = \{\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right) \mid \lambda \in \mathbb{K}_\sigma\}$ and $B = \{\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) \mid \lambda \in \mathbb{K}_\sigma\}$ is a special rank one group with AUS.

Write $a_\lambda = \left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right) \in A$ and similarly $b_\lambda = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) \in B$. It is clear that for all $\lambda, \mu \in \mathbb{K}_\sigma$ we have $\lambda + \mu \in \mathbb{K}_\sigma$, hence $a_\lambda a_\mu = a_{\lambda+\mu} = a_\mu a_\lambda$ and $b_\lambda b_\mu = b_{\lambda+\mu} = b_\mu b_\lambda$ shows that both A and B are abelian. As a_0 and b_0 are the identity elements of A and B, respectively, we deduce by the above that $a_\lambda^{-1} = a_{-\lambda}$ and $b_\lambda^{-1} = b_{-\lambda}$ for all $\lambda \in \mathbb{K}_\sigma$. Then for all $a_\lambda \in A^{\#}$ we have $B^{a_\lambda} = \{b_\gamma^{a_\lambda} \mid b_\gamma \in B\} = \{a_\lambda b_\gamma a_{-\lambda} \mid b_\gamma \in B\} = \{\left(\begin{smallmatrix} 1-\gamma\lambda & \gamma \\ -\lambda\gamma\lambda & 1+\lambda\gamma \end{smallmatrix}\right) \mid \gamma \in \mathbb{K}_\sigma\}$, and similarly for all $b_\mu \in B^{\#}$ we have $A^{b_\mu} = \{a_\gamma^{b_\mu} \mid a_\gamma \in A\} = \{b_\mu a_\gamma b_{-\mu} \mid a_\gamma \in A\} = \{\left(\begin{smallmatrix} 1+\mu\gamma & -\mu\gamma\mu \\ \gamma & 1-\gamma\mu \end{smallmatrix}\right) \mid \gamma \in \mathbb{K}_\sigma\}$. Hence, we have for all $a_\lambda \in A^{\#}$ that

$$\begin{aligned} A^{b_{\lambda^{-1}}} &= \{\left(\begin{smallmatrix} 1+\lambda^{-1}\gamma & -\lambda^{-1}\gamma\lambda^{-1} \\ \gamma & 1-\gamma\lambda^{-1} \end{smallmatrix}\right) \mid \gamma \in \mathbb{K}_\sigma\} = \{\left(\begin{smallmatrix} 1+\lambda^{-1}(-\lambda\gamma\lambda) & -\lambda^{-1}(-\lambda\gamma\lambda)\lambda \\ -\lambda\gamma\lambda & 1-(-\lambda\gamma\lambda)\lambda^{-1} \end{smallmatrix}\right) \mid -\lambda\gamma\lambda \in \mathbb{K}_\sigma\} \\ &= \{\left(\begin{smallmatrix} 1-\gamma\lambda & \gamma \\ -\lambda\gamma\lambda & 1+\lambda\gamma \end{smallmatrix}\right) \mid -\lambda\gamma\lambda \in \mathbb{K}_\sigma\} = B^{a_\lambda}, \end{aligned}$$

where the last equality follows from the observation that $-\lambda\gamma\lambda \in \mathbb{K}_\sigma$ if and only if $\gamma \in \mathbb{K}_\sigma$. This shows that $X = \langle A, B \rangle$ is a rank one group. It is moreover a special rank one group, as for all $a_\lambda \in A^{\#}$ and $b_{\lambda^{-1}} \in B^{\#}$ we have

$$\begin{aligned} a_\lambda^{b_{\lambda^{-1}}} = b_{\lambda^{-1}} a_\lambda b_{-\lambda^{-1}} &= \begin{pmatrix} 1 + \lambda^{-1}\lambda & -\lambda^{-1}\lambda\lambda^{-1} \\ \lambda & 1 - \lambda\lambda^{-1} \end{pmatrix} = \begin{pmatrix} 2 & -\lambda^{-1} \\ \lambda & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 - (-\lambda^{-1})\lambda & -\lambda^{-1} \\ -\lambda(-\lambda^{-1})\lambda & 1 + \lambda(-\lambda^{-1}) \end{pmatrix} = a_\lambda b_{-\lambda^{-1}} a_{-\lambda} = (b_{\lambda^{-1}}^{-1})^{a_\lambda} = b_{\lambda^{-1}}^{-a_\lambda}. \end{aligned}$$

Rank one groups play an essential role in sets of abstract root subgroups, which are defined as follows.

Definition 7.1.3 (Abstract root subgroups)**.** *Let $G = \langle \Sigma \rangle$ be a group generated by a set Σ of abelian non-identity subgroups of G. If $\Sigma^g \subseteq \Sigma$ for all $g \in G$ and for all $A, B \in \Sigma$ we either have*

(i) $[A, B] = 1$*, or*

(ii) $X = \langle A, B \rangle$ *is a rank one group, or*

(iii) $Z(\langle A, B \rangle) \geq [A, B] = [a, B] = [A, b] \in \Sigma$ *for all* $a \in A^{\#}$ *and* $b \in B^{\#}$*,*

then Σ is called a set of ***abstract root subgroups*** *of G.*

Let Σ be a set of abstract root subgroups of a group $G = \langle \Sigma \rangle$. If Σ is a conjugacy class in G, we instead refer to Σ as a *class* of abstract root subgroups of G. If for all $A, B \in \Sigma$ such that $X = \langle A, B \rangle$ is a rank one group we have $X \cong (\mathrm{P})\mathrm{SL}_2(\mathbb{F})$ for some commutative field $\mathbb{F}$ with $A, B \leq \mathrm{SL}_2(\mathbb{F})$ unipotent subgroups, then Σ is called a set (or class) of $\mathbb{F}$*-root subgroups* of G. If Definition 7.1.3(iii) never occurs, then Σ is said to be a set (or class) of *abstract transvection groups* of G, or a *degenerate* set (or class) of abstract root subgroups of G. Contrarily, if Definition 7.1.3(i)-(iii) all occur, then Σ is *non-degenerate*.

Example 7.1.4. Similar to the previous example, let $\mathbb{F}$ be a field and consider the subgroups $A = \{\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right) \mid \lambda \in \mathbb{F}\}$ and $B = \{\left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) \mid \lambda \in \mathbb{F}\}$ of $\mathrm{GL}_2(\mathbb{F})$. Then $X = \langle A, B \rangle$ is a rank one group by Example 7.1.2. Writing $\Sigma = \{A, B\}$, then Σ is a set of abstract transvection subgroups of the rank one group $X = \langle A, B \rangle$. We claim that Σ is also a set of $\mathbb{F}$-root subgroups of X. As in Example 7.1.2, write $a_\lambda = \left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right) \in A$ and $b_\lambda = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right) \in B$. First note that $\det(a_\lambda) = 1$ for all $a_\lambda \in A$ and similarly $\det(b_\lambda) = 1$ for all $b_\lambda \in B$, hence both A and B are subgroups of $\mathrm{SL}_2(\mathbb{F})$. Moreover, as $(a_\lambda - I_2)^2 = \left(\begin{smallmatrix} 0 & 0 \\ \lambda & 0 \end{smallmatrix}\right)^2 = 0$ for all $a_\lambda \in A$ and $(b_\lambda - I_2)^3 = \left(\begin{smallmatrix} 0 & \lambda \\ 0 & 0 \end{smallmatrix}\right)^3 = 0$ for all $b_\lambda \in B$, we deduce that both A and B are unipotent subgroups of $\mathrm{SL}_2(\mathbb{F})$. Since the matrices $\left(\begin{smallmatrix} 1 & 0 \\ \lambda & 1 \end{smallmatrix}\right)$, $\lambda \in \mathbb{F}$, and $\left(\begin{smallmatrix} 1 & \lambda' \\ 0 & 1 \end{smallmatrix}\right)$, $\lambda' \in \mathbb{F}$, are known to generate $\mathrm{SL}_2(\mathbb{F})$, we find $X \cong \mathrm{SL}_2(\mathbb{F})$ so that Σ is indeed a set of $\mathbb{F}$-root subgroups of X.

More generally, let V be some finite-dimensional vector space over $\mathbb{F}$ with $\dim(V) = n \geq 3$ and let V^* be the dual space of V. Recall from Section 3.2 the transvection group $\mathrm{Exp}(t_{v,\varphi}) = \{\mathrm{id}_V + \lambda t_{v,\varphi} \mid \lambda \in \mathbb{F}, \varphi(v) = 0\}$, which is a subgroup of $\mathrm{GL}_n(\mathbb{F})$ in its action on $\mathfrak{gl}_n(\mathbb{F})$, obtained from the infinitesimal transvections $t_{v,\varphi} : V \to V$ given by $w \mapsto \varphi(w)v$ with $v \in V$ and $\varphi \in V^*$ such that $\varphi(v) = 0$. Now consider the set $\Sigma = \{\mathrm{Exp}(t_{v,\varphi}) \mid v \in V, \varphi \in V^*, \varphi(v) = 0\} \subseteq \mathrm{SL}_n(\mathbb{F})$. Letting $\mathrm{Exp}(t_{v,\varphi}) \in \Sigma$ be arbitrary, then $\mathrm{Exp}(t_{v,\varphi})$ is isomorphic to the additive group of $\mathbb{F}$ by Corollary 3.1.8, hence abelian, and $\mathrm{Exp}(t_{v,\varphi})^\psi \in \Sigma$ for all $\psi \in \mathrm{SL}_n(\mathbb{F})$ by Corollary 2.1.8. Moreover, $\mathrm{Exp}(t_{v,\varphi})$ is unipotent as $t_{v,\varphi}^2(w) = t_{v,\varphi}(\varphi(w)v) = \varphi(\varphi(w)v)v = \varphi(v)\varphi(w)v = 0$ for all $w \in V$. If additionally $\mathrm{Exp}(t_{w,\psi}) \in \Sigma$, then $\mathrm{Exp}(t_{v,\varphi})$ and $\mathrm{Exp}(t_{w,\psi})$ are as in Definition 7.1.3(i) if and only if $\psi(v) = 0 = \varphi(w)$, Definition 7.1.3(ii) if and only if $\psi(v) \neq 0$ and $\varphi(w) \neq 0$, and Definition 7.1.3(iii) if and only if either $\psi(v) = 0$ and $\varphi(w) \neq 0$ or $\psi(v) \neq 0$ and $\varphi(w) = 0$. Specifically, $\langle \mathrm{Exp}(t_{v,\varphi}), \mathrm{Exp}(t_{w,\psi}) \rangle \cong \mathrm{SL}_2(\mathbb{F})$ if $\psi(v) \neq 0$ and $\varphi(w) \neq 0$, so Σ is a non-degenerate set of $\mathbb{F}$-root subgroups of $\mathrm{SL}_n(\mathbb{F}) = \langle \Sigma \rangle$. See Example II(1.3) of [26] for more details.

As in II§1 of [26], we will fix the following notation throughout the remainder of this section. For a set Σ of abstract root subgroups of a group $G = \langle \Sigma \rangle$ with $\Lambda \subseteq \Sigma$ and $U \leq G$, write

$$\begin{aligned}
N_\Lambda(U) &:= \{A \in \Lambda \mid A \leq N(U)\},\\
C_\Lambda(U) &:= \{A \in \Lambda \mid A \leq C(U)\},\\
\Sigma_A &:= C_\Sigma(A) \setminus \{A\},\\
\Lambda_A &:= \{B \in \Sigma_A \mid \Sigma \cap AB \text{ is a partition of } AB\},\\
\Psi_A &:= \{B \in \Sigma \mid [A, B] \in \Sigma\},\\
\Omega_A &:= \{B \in \Sigma \mid \langle A, B\rangle \text{ is a rank one group}\}.
\end{aligned}$$

With this notation, it is easy to see that for all $A \neq B \in \Sigma$ we have $B \in \Sigma_A$ if and only if A, B are as in Definition 7.1.3(i), $B \in \Omega_A$ if and only if A, B are as in Definition 7.1.3(ii), and $B \in \Psi_A$ if and only if A and B are as in Definition 7.1.3(iii). Consequently, we deduce that

$$\Sigma = \{A\} \sqcup \Sigma_A \sqcup \Psi_A \sqcup \Omega_A$$

for all $A \in \Sigma$.

To relate the extremal elements of a Lie algebra to abstract root subgroups, we first need to establish a connection between abstract root subgroups and point-line geometries. To this end, consider the following definition.

Definition 7.1.5 (Root group geometry)**.** *Let Σ be a non-degenerate class of abstract root subgroups of a group $G = \langle \Sigma \rangle$. The **root group geometry** of Σ is the point-line geometry $\Gamma_\Sigma = (\Sigma, \mathcal{L}_\Sigma)$ having point set Σ and line set $\mathcal{L}_\Sigma = \{AB \cap \Sigma \mid A \in \Sigma, B \in \Lambda_A\}$.*

We are now in a position to establish a connection between abstract root subgroups and Lie algebras. Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements. Recall from Definition 2.1.7 the group $\mathrm{Int}(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$ generated by the automorphisms of the form e^{ad_x} with $x \in \mathfrak{g}$ ad-nilpotent. In particular, we have $\mathrm{Int}(\mathfrak{g}) \trianglelefteq \mathrm{Aut}(\mathfrak{g})$ by Corollary 2.1.8. Since $\mathfrak{g} = \langle E(\mathfrak{g})\rangle$, $\mathfrak{g}$ is linearly spanned by $E(\mathfrak{g})$ as a consequence of Lemma 3.1.9, hence $G := \langle \exp(x, \lambda) \mid x \in E(\mathfrak{g}), \lambda \in \mathbb{F}\rangle \leq \mathrm{Int}(\mathfrak{g})$, in which $\exp(x, \lambda)(y) = y + \lambda[x, y] + \lambda^2 g_x(y)x$ with $y \in \mathfrak{g}$ as introduced in Section 3.1.

Define $\Sigma(\mathfrak{g}) := \{\mathrm{Exp}(x) \mid x \in E(\mathfrak{g})\}$ with $\mathrm{Exp}(x) = \{\exp(x, \lambda) \mid x \in E(\mathfrak{g}), \lambda \in \mathbb{F}\}$. By Corollary 3.1.8, we have $\mathrm{Exp}(x) \leq G$ for all $x \in E(\mathfrak{g})$. Specifically, $\mathrm{Exp}(x)$ is abelian since it is isomorphic to the additive group of $\mathbb{F}$, again by Corollary 3.1.8. We now propose the following.

Proposition 7.1.6. *Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements. Then $\Sigma(\mathfrak{g}) = \{\mathrm{Exp}(x) \mid x \in E(\mathfrak{g})\}$ is a class of abstract root subgroups of $G = \langle \mathrm{Exp}(x) \mid x \in E(\mathfrak{g})\rangle = \langle \Sigma(\mathfrak{g})\rangle$.*

In particular, $\Sigma(\mathfrak{g})$ is a class of $\mathbb{F}$-root subgroups of G.

Proof. By the above, $\Sigma(\mathfrak{g})$ consists of abelian non-identity subgroups of $G = \langle \Sigma(\mathfrak{g}) \rangle$. In addition, as $G \leq \mathrm{Int}(\mathfrak{g}) \trianglelefteq \mathrm{Aut}(\mathfrak{g})$ by Corollary 2.1.8, we have $\mathrm{Exp}(x)^{\varphi} = \mathrm{Exp}(\varphi(x))$ for all $x \in E(\mathfrak{g})$ and $\varphi \in G$. Since any automorphism of $\mathfrak{g}$ must preserve extremality, we find $\varphi(x) \in E(\mathfrak{g})$, hence $\mathrm{Exp}(x)^{\varphi} \in \Sigma(\mathfrak{g})$ so that $\Sigma(\mathfrak{g})^{\varphi} \subseteq \Sigma(\mathfrak{g})$ for all $\varphi \in G$. Now letting $\mathrm{Exp}(x), \mathrm{Exp}(y) \in \Sigma(\mathfrak{g})$ be arbitrary, we check the conditions listed in Definition 7.1.5.

Suppose first that $\mathbb{F}x = \mathbb{F}y$. Then $\mathrm{Exp}(x) = \mathrm{Exp}(y)$, hence $[\mathrm{Exp}(x), \mathrm{Exp}(y)] = [\mathrm{Exp}(x), \mathrm{Exp}(x)] = 1$ because $\mathrm{Exp}(x) \leq G$ is abelian. We are in case Definition 7.1.5(i).

Secondly, assume that x, y are linearly independent such that $[x, y] = 0$ and $\mathbb{F}x + \mathbb{F}y \subseteq E(\mathfrak{g}) \cup \{0\}$. Then also $g_x(y) = 0$ and $[x, [y, z]] = g_x(z)y + g_y(z)x$ for all $z \in E(\mathfrak{g})$ by Lemma 5.1.1. It follows that for all $z \in \mathfrak{g}$ and $\lambda, \mu \in \mathbb{F}$ we have

$$\begin{aligned}
\exp(x,\lambda)\exp(y,\mu)(z) &= \exp(x,\lambda)(z + \mu[y,z] + \mu^2 g_y(z)y) \\
&= z + \mu[y,z] + \mu^2 g_y(z) + \lambda[x, z + \mu[y,z] + \mu^2 g_y(z)y] \\
&\quad + \lambda^2 g_x(z + \mu[y,z] + \mu^2 g_y(z)y)x \\
&= z + [\lambda x + \mu y, z] + \lambda^2 g_x(z)x + \mu^2 g_y(z)y + \lambda\mu[x,[y,z]] \\
&\quad + \lambda^2\mu g_x([y,z]) + \lambda\mu^2 g_y(z)[x,y] + \lambda^2\mu^2 g_x(y)g_y(z)x \\
&= z + [\lambda x + \mu y, z] + \lambda^2 g_x(z)x + \mu^2 g_y(z)y + \lambda\mu(g_x(y)z + g_y(z)x) \\
&= z + [\lambda x + \mu y, z] + g_{\lambda x + \mu y}(z)(\lambda x + \mu y) \\
&= \exp(\lambda x + \mu y, 1)(z),
\end{aligned}$$

where $g_x([y,z]) = g_{[x,y]}(z) = g_0(z) = 0$ by associativity of the extremal form by Proposition 3.1.10. Then $\exp(x,\lambda)\exp(y,\mu) = \exp(\lambda x + \mu y, 1) = \exp(y,\mu)\exp(x,\lambda)$, hence every element in $\mathrm{Exp}(x)$ commutes with every element in $\mathrm{Exp}(y)$ and vice versa so that $[\mathrm{Exp}(x), \mathrm{Exp}(y)] = 1$, again bringing us to case Definition 7.1.5(i).

Thirdly, suppose that x, y are linearly dependent such that $[x, y] = 0$ but $\mathbb{F}x + \mathbb{F}y \not\subseteq E(\mathfrak{g}) \cup \{0\}$. As before, we then also have $g_x(y) = 0$, therefore $\exp(x,\lambda)(y) = y$ for all $\lambda \in \mathbb{F}$. It follows that

$$\begin{aligned}
[\exp(x,\lambda), \exp(y,\mu)] &= \exp(x,-\lambda)\exp(y,-\mu)\exp(x,\lambda)\exp(y,\mu) \\
&= \exp(\exp(x,-\lambda)y, -\mu)\exp(y,\mu) = \exp(y,-\mu)\exp(y,\mu) = 1,
\end{aligned}$$

by Corollary 2.1.8. Consequently, $[\mathrm{Exp}(x), \mathrm{Exp}(y)] = 1$, so we are in case Definition 7.1.5(i) yet again.

Next, assume that x, y are linearly independent such that $[x, y] \neq 0$ but $g_x(y) = 0$. Then $\exp(x,\lambda)(y) = y + \lambda[x,y]$ for all $\lambda \in \mathbb{F}$. Further note that $[x, y] \in E(\mathfrak{g})$; indeed, we have

$$\begin{aligned}
[[x,y],[[x,y],z]] &= 2g_x(y)g_y(z)x - g_x(y)[x,[y,z]] + 2g_x(y)g_x(z)y - g_x(y)[y,[x,z]] \\
&\quad + g_x([y,z])[x,y] - g_y([x,z])[x,y] \\
&= (g_x([y,z]) - g_y([x,z]))[x,y] = (g_x([y,z]) + g_x([y,z]))[x,y] \\
&= 2g_x([y,z])[x,y] = 2g_{[x,y]}(z)[x,y]
\end{aligned}$$

by Corollary 3.1.6(ii), Corollary 3.1.5 and associativity of the extremal form by Proposition 3.1.10. Moreover, by the second Premet identity we have for all $z \in \mathfrak{g}$ that

$$[[x, y], [x, z]] = g_x([y, z])x + g_x(z)[x, y] - g_x(y)[x, z] = g_{[x,y]}(z)x + g_x(z)[x, y],$$

which shows that $\mathbb{F}x + \mathbb{F}[x, y] \in E(\mathfrak{g}) \cup \{0\}$ by Lemma 5.1.1. Furthermore, we then have $\exp(x, \lambda) \exp([x, y], \mu) = \exp(\lambda x + \mu[x, y], 1)$ by the third paragraph, therefore

$$\begin{aligned}
[\exp(x, \lambda), \exp(y, \mu)] &= \exp(x, -\lambda) \exp(y, -\mu) \exp(x, \lambda) \exp(y, \mu) \\
&= \exp(x, -\lambda) \exp(\exp(y, -\mu)x, \lambda) = \exp(x, -\lambda) \exp(x + \mu[x, y], \lambda) \\
&= \exp(x, -\lambda) \exp(x, \lambda) \exp(\mu[x, y], \lambda) = \exp([x, y], \lambda\mu),
\end{aligned}$$

from which it follows that $[\mathrm{Exp}(x), \mathrm{Exp}(y)] = \mathrm{Exp}([x, y]) \in \Sigma(\mathfrak{g})$. By noting that $[x, [x, y]] = 2g_x(y) = 0$ and $[y, [x, y]] = -2g_y(x)y = 0$, we deduce that $\mathrm{Exp}([x, y]) \leq Z(\langle \mathrm{Exp}(x), \mathrm{Exp}(y)\rangle)$. In addition, the identities $\exp(x, \lambda) = \exp(\lambda x, 1)$ and $\exp(y, \mu) = \exp(\mu y, 1)$ for all $\lambda, \mu \in \mathbb{F}^*$ show that $[\mathrm{Exp}(x), \mathrm{Exp}(y)] = [\exp(x, \lambda), \mathrm{Exp}(y)] = [\mathrm{Exp}(x), \exp(y, \mu)]$ for all $\lambda, \mu \in \mathbb{F}^*$, so we are in case Definition 7.1.5(iii).

Finally, suppose that x, y are linearly independent such that $[x, y] \neq 0$ and $g_x(y) \neq 0$. For all $\lambda \in \mathbb{F}^*$ we then have $-\lambda^{-1} g_x(y)^{-1} \in \mathbb{F}^*$, so we find

$$\begin{aligned}
\exp(y, -\lambda^{-1} g_x(y)^{-1})(x) &= x - \lambda^{-1} g_x(y)^{-1}[y, x] + \lambda^{-2} g_x(y)^{-2} g_y(x)y \\
&= x + \lambda^{-1} g_x(y)^{-1}[x, y] + \lambda^{-2} g_x(y)^{-1} y \\
&= \lambda^{-2} g_x(y)^{-1}(y + \lambda[x, y] + \lambda^2 g_x(y)x) \\
&= \lambda^{-2} g_x(y)^{-1} \exp(x, \lambda)(y)
\end{aligned}$$

where $g_x(y) = g_y(x)$ by Corollary 3.1.5. It then follows that from Corollary 2.1.8 that

$$\begin{aligned}
\mathrm{Exp}(y)^{\exp(x,\lambda)} &= \mathrm{Exp}(\exp(x, \lambda)(y)) = \mathrm{Exp}(\lambda^{-2} g_x(y)^{-1} \exp(x, \lambda)(y)) \\
&= \mathrm{Exp}(\exp(y, -\lambda^{-1} g_x(y)^{-1})(x)) = \mathrm{Exp}(x)^{\exp(y, -\lambda^{-1} g_x(y)^{-1})},
\end{aligned}$$

which shows, upon interchanging x and y by symmetry, that $X = \langle \mathrm{Exp}(x), \mathrm{Exp}(y)\rangle$ is a rank one group. We are in case Definition 7.1.5(ii). This shows that $\Sigma(\mathfrak{g})$ is a set of abstract root subgroups of $G = \langle \Sigma_{\mathfrak{g}} \rangle$. Since $G \leq \mathrm{Int}(\mathfrak{g}) \trianglelefteq \mathrm{Aut}(\mathfrak{g})$ by Corollary 2.1.8, we find that $\Sigma(\mathfrak{g})$ is a conjugacy class in G, so $\Sigma(\mathfrak{g})$ is a class of abstract root subgroups of G.

For the final assertion, note that, as in our proof of Lemma 3.1.11, the triple $\{x, y, [x, y]\}$ is easily seen to be an $\mathfrak{sl}_2$-triple if $X = \langle \mathrm{Exp}(x), \mathrm{Exp}(y)\rangle$ with $x, y \in E(\mathfrak{g})$ linearly independent such that $[x, y] \neq 0$ and $g_x(y) \neq 0$ is a rank one group. Moreover, $\mathrm{Exp}(x)$ is a unipotent subgroup of G as for all $z \in \mathfrak{g}$ and $\lambda \in \mathbb{F}$ we have

$$\begin{aligned}
(\exp(x, \lambda) - 1)^3(z) &= (\exp(x, \lambda) - 1)^2(\lambda[x, z] + \lambda^2 g_x(z)x) \\
&= (\exp(x, \lambda) - 1)(\lambda[x, \lambda[x, z] + \lambda^2 g_x(z)x] + \lambda^2 g_x(\lambda[x, z] + \lambda^2 g_x(z)x)x) \\
&= (\exp(x, \lambda) - 1)(\lambda^2[x, [x, z]]) = \lambda[x, \lambda^2 g_x(z)x] + \lambda^2 g_x(\lambda^2 g_x(z)x)x = 0,
\end{aligned}$$

where $g_x([x,z]) = g_{[x,x]}(z) = 0$ because the extremal form is associative by Proposition 3.1.10. Interchanging x and y shows that $\mathrm{Exp}(y)$ is also a unipotent subgroup of G. It now remains to show that $\mathrm{Exp}(x), \mathrm{Exp}(y) \leq \mathrm{SL}_2(\mathbb{F})$ in their action on $\mathfrak{sl}_2(\mathbb{F})$, because then $X \cong \mathrm{SL}_2(\mathbb{F})$ since $\{x, y, [x,y]\}$ is an $\mathfrak{sl}_2$-triple. Equivalently, this amounts to showing that $\det(\exp(x,\lambda)) = 1$ for all $\lambda \in \mathbb{F}$. Using the basis $\{x, y, [x,y]\}$ of $\mathfrak{sl}_2(\mathbb{F})$, we deduce that $\exp(x,\lambda)(x) = x$, $\exp(x,\lambda)(y) = y + \lambda[x,y] + \lambda^2 g_x(y)x$ and $\exp(x,\lambda)([x,y]) = [x,y] + \lambda[x,[x,y]] + \lambda^2 g_x([x,y])x = [x,y] + 2\lambda g_x(y)x$, hence $\exp(x,\lambda)$ has determinant

$$\det(\exp(x,\lambda)) = \begin{vmatrix} 1 & 0 & 0 \\ \lambda^2 g_x(y) & 1 & \lambda \\ 2\lambda g_x(y) & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & \lambda \\ 0 & 1 \end{vmatrix} = 1.$$

It follows that $\mathrm{Exp}(x) \leq \mathrm{SL}_2(\mathbb{F})$, and similarly for $\mathrm{Exp}(y)$. We conclude that $X \cong \mathrm{SL}_2(\mathbb{F})$, thus $\Sigma(\mathfrak{g})$ is a set of $\mathbb{F}$-root subgroups of $G = \langle \Sigma(\mathfrak{g}) \rangle$. □

In the remainder of this chapter, we assume that $\Sigma(\mathfrak{g})$ is a non-degenerate class of $\mathbb{F}$-root subgroups of G, i.e. Definition 7.1.3(i)-(iii) all occur. In addition, we will identify the projective points $\langle x \rangle = \mathbb{F}x$ with $x \in E(\mathfrak{g})$ of the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{g}$ by the extremal elements $x \in E(\mathfrak{g})$ themselves. This identification is harmless as $\mathrm{Exp}(x) = \mathrm{Exp}(\lambda x)$ for all $\lambda \in \mathbb{F}^*$.

Our proof of the above proposition also demonstrates a clear connection between the root group geometry of the non-degenerate class $\Sigma(\mathfrak{g})$ of abstract root subgroups of $G = \langle \Sigma(\mathfrak{g}) \rangle$ and the filtration $\{\mathcal{E}_i\}_{-2 \leq i \leq 2}$ of the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{g}$; for all $x, y \in E(\mathfrak{g})$, we have $(x,y) \in \mathcal{E}_{\leq 0}$ if and only if $\mathrm{Exp}(x), \mathrm{Exp}(y)$ are as in Definition 7.1.5(i), $(x,y) \in \mathcal{E}_1$ if and only if $\mathrm{Exp}(x), \mathrm{Exp}(y)$ are as in Definition 7.1.5(iii), and $(x,y) \in \mathcal{E}_2$ if and only if $\mathrm{Exp}(x), \mathrm{Exp}(y)$ are as in Definition 7.1.5(ii). In particular, we obtain the following.

Corollary 7.1.7. *Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements. Further let $\Sigma(\mathfrak{g})$ be the non-degenerate class of $\mathbb{F}$-root subgroups of $G = \langle \Sigma(\mathfrak{g}) \rangle$ and let $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ be the extremal geometry of $\mathfrak{g}$ with filtration $\{\mathcal{E}_i\}_{-2 \leq i \leq 2}$. Upon identifying every $\mathrm{Exp}(x) \in \Sigma(\mathfrak{g})$ by its defining extremal elements $x \in E(\mathfrak{g})$ through the map $\varphi : \Sigma(\mathfrak{g}) \to E(\mathfrak{g})$ given by $\mathrm{Exp}(x) \mapsto x$, then for all $\mathrm{Exp}(x) \in \Sigma(\mathfrak{g})$ we have $\varphi(\Sigma_{\mathrm{Exp}(x)}) = \mathcal{E}_{-1}(x) \cup \mathcal{E}_0(x)$, $\varphi(\Lambda_{\mathrm{Exp}(x)}) = \mathcal{E}_{-1}(x)$, $\varphi(\Psi_{\mathrm{Exp}(x)}) = \mathcal{E}_1(x)$ and $\varphi(\Omega_{\mathrm{Exp}(x)}) = \mathcal{E}_2(x)$).*

Proof. Let $\mathrm{Exp}(x) \in \Sigma(\mathfrak{g})$ be arbitrary. We have $C_{\Sigma(\mathfrak{g})}(\mathrm{Exp}(x)) = \{\mathrm{Exp}(y) \in \Sigma(\mathfrak{g}) \mid \mathrm{Exp}(y) \leq C(\mathrm{Exp}(x))\} = \mathrm{Exp}(y) \in \Sigma(\mathfrak{g}) \mid [\mathrm{Exp}(x), \mathrm{Exp}(y)] = 1\} = \{y \in E(\mathfrak{g}) \mid [x,y] = 0\}$ by Proposition 7.1.6. Consequently, $\varphi(C_{\Sigma(\mathfrak{g})}(\mathrm{Exp}(x))) = \{y \in E(\mathfrak{g}) \mid y \in \mathcal{E}_{\leq 0}(x)\} = \mathcal{E}_{\leq 0}(x)$, hence $\varphi(\Sigma_{\mathrm{Exp}(x)}) = \varphi(C_{\Sigma(\mathfrak{g})}(\mathrm{Exp}(x)) \setminus \{\mathrm{Exp}(x)\}) = \mathcal{E}_{\leq 0}(x) \setminus \mathcal{E}_2(x) = \mathcal{E}_{-1}(x) \sqcup \mathcal{E}_0(x)$.

Now consider the set $\Lambda_{\mathrm{Exp}(x)}$ consisting of all $\mathrm{Exp}(y) \in \Sigma_{\mathrm{Exp}(x)}$ such that $\Sigma(\mathfrak{g}) \cap \mathrm{Exp}(x)\mathrm{Exp}(y)$ is a partition of $\mathrm{Exp}(x)\mathrm{Exp}(y)$. Then $\varphi(\Lambda_{\mathrm{Exp}(x)})$ is a subset of $\mathcal{E}_{-1}(x) \sqcup \mathcal{E}_0(x)$ by the previous paragraph. On the one hand, if $y \in \mathcal{E}_0(x)$ then $\mathbb{F}x + \mathbb{F}y \not\subseteq E(\mathfrak{g}) \cup \{0\}$, hence

$\Sigma \cap \mathrm{Exp}(x)\mathrm{Exp}(y) = \{\mathrm{Exp}(x), \mathrm{Exp}(y)\}$, which is not a partition of $\mathrm{Exp}(x)\mathrm{Exp}(y)$, showing that $\mathcal{E}_0(x) \not\subseteq \varphi(\Lambda_{\mathrm{Exp}(x)})$. On the other hand, if $y \in \mathcal{E}_{-1}(x)$ then $\mathbb{F}x + \mathbb{F}y \subseteq E(\mathfrak{g}) \cup \{0\}$ so that $\Sigma \cap \mathrm{Exp}(x)\mathrm{Exp}(y) = \{\mathrm{Exp}(\lambda x + \mu y) \mid \lambda, \mu \in \mathbb{F}\}$, which does form a partition of $\mathrm{Exp}(x)\mathrm{Exp}(y)$. We conclude that $\varphi(\Lambda_{\mathrm{Exp}(x)}) = \mathcal{E}_{-1}(x)$.

It should be clear that $\varphi(\Psi_{\mathrm{Exp}(x)}) = \varphi(\{\mathrm{Exp}(y) \in \Sigma(\mathfrak{g}) \mid [\mathrm{Exp}(x), \mathrm{Exp}(y)] \in \Sigma(\mathfrak{g})\}) = \mathcal{E}_1(x)$, for $[\mathrm{Exp}(x), \mathrm{Exp}(y)] = \mathrm{Exp}([x, y]) \in \Sigma(\mathfrak{g})$ if and only if x, y are linearly independent such that $[x, y] \neq 0$ and $g_x(y) \neq 0$ by Proposition 7.1.6 if and only if $y \in \mathcal{E}_1(x)$. Similarly, we have $\varphi(\Omega_{\mathrm{Exp}(x)}) = \varphi(\{\mathrm{Exp}(y) \in \Sigma(\mathfrak{g}) \mid \langle \mathrm{Exp}(x), \mathrm{Exp}(y)\rangle$ is a rank one group$\}) = \mathcal{E}_2(x)$, since $\langle \mathrm{Exp}(x), \mathrm{Exp}(y)\rangle$ is a rank one group if and only if x, y are linearly independent such that $[x, y] \neq 0$ and $g_x(y) \neq 0$ by Proposition 7.1.6 if and only if $y \in \mathcal{E}_2(x)$. □

Note that our findings agree with the fact that $\Sigma = \{A\} \sqcup \Sigma_A \sqcup \Psi_A \sqcup \Omega_A$ for any set of abstract root subgroups Σ of a group $G = \langle \Sigma \rangle$ and all $A \in \Sigma$; indeed, using $\{x\} = \mathcal{E}_{-2}(x)$ for any $x \in E(\mathfrak{g})$, by the above corollary we have $E(\mathfrak{g}) = \mathcal{E}_{-2}(x) \sqcup \mathcal{E}_{-1}(x) \sqcup \mathcal{E}_0(x) \sqcup \mathcal{E}_1(x) \sqcup \mathcal{E}_2(x)$ for all $x \in E(\mathfrak{g})$.

The above corollary has an even stronger implication; it shows that there is a one-to-one correspondence between the root group geometry $\Gamma_{\Sigma(\mathfrak{g})} = (\Sigma(\mathfrak{g}), \mathcal{L}_{\Sigma(\mathfrak{g})})$ of the non-degenerate class $\Sigma(\mathfrak{g})$ of abstract root subgroups of $G = \langle \Sigma(\mathfrak{g})\rangle$ and the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{g}$. This correspondence is characterised by the following theorem.

Theorem 7.1.8. *Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements. Further let $\Gamma_{\Sigma(\mathfrak{g})} = (\Sigma(\mathfrak{g}), \mathcal{L}_{\Sigma(\mathfrak{g})})$ be the root group geometry of the non-degenerate class $\Sigma(\mathfrak{g})$ of $\mathbb{F}$-root subgroups of $G = \langle \Sigma(\mathfrak{g})\rangle$ and let $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ be the extremal geometry of $\mathfrak{g}$ with filtration $\{\mathcal{E}_i\}_{-2 \leq i \leq 2}$. Then $\Gamma_{\mathfrak{g}} \cong \Gamma_{\Sigma(\mathfrak{g})}$.*

Proof. We claim that the map $\varphi : \Sigma(\mathfrak{g}) \to E(\mathfrak{g})$ given by $\mathrm{Exp}(x) \mapsto x$ from Corollary 7.1.7 is an isomorphism between $\Gamma_{\Sigma(\mathfrak{g})}$ and $\Gamma_{\mathfrak{g}}$, i.e. we will show that φ is bijective on $\Sigma(\mathfrak{g})$ and $E(\mathfrak{g})$, and maps $\mathcal{L}_{\Sigma(\mathfrak{g})}$ bijectively to $\mathcal{L}$.

The fact that φ is bijective on $\Sigma(\mathfrak{g})$ and $E(\mathfrak{g})$ follows immediately from the observation that every $x \in E(\mathfrak{g})$ corresponds uniquely to $\mathrm{Exp}(x) \in \Sigma(\mathfrak{g})$ and vice versa. So, it remains to show that $\varphi(\mathcal{L}_{\Sigma(\mathfrak{g})}) = \mathcal{L}$.

First let $\ell \in \mathcal{L}$. Then $\ell = \mathbb{F}x + \mathbb{F}y$ for some linearly independent $x, y \in E(\mathfrak{g})$ such that $[x, y] = 0$ and $\mathbb{F}x + \mathbb{F}y \subseteq E(\mathfrak{g}) \cup \{0\}$. Specifically, we have $x \in E(\mathfrak{g})$ and $y \in \mathcal{E}_{-1}(x)$, hence by Corollary 7.1.7 we equivalently have $\mathrm{Exp}(x) \in \Sigma(\mathfrak{g})$ and $\mathrm{Exp}(y) \in \Lambda_{\mathrm{Exp}(x)}$. But then $\ell = \varphi(\{\mathrm{Exp}(\lambda x + \mu y) \mid \lambda, \mu \in \mathbb{F}\}) = \varphi(\mathrm{Exp}(x)\mathrm{Exp}(y) \cap \Sigma(\mathfrak{g})) \in \varphi(\mathcal{L}_{\Sigma(\mathfrak{g})})$, showing that $\mathcal{L} \subseteq \varphi(\mathcal{L}_{\Sigma(\mathfrak{g})})$. Conversely, if $\ell \in \varphi(\mathcal{L}_{\Sigma(\mathfrak{g})})$ then $\ell = \varphi(\mathrm{Exp}(x)\mathrm{Exp}(y) \cap \Sigma(\mathfrak{g}))$ for some $\mathrm{Exp}(x) \in \Sigma(\mathfrak{g})$ and $\mathrm{Exp}(y) \in \Lambda_{\mathrm{Exp}(x)}$. Again by Corollary 7.1.7 we then equivalently have $\ell = \varphi(\{\mathrm{Exp}(\lambda x + \mu y) \mid \lambda, \mu \in \mathbb{F}\})$ with $x \in E(\mathfrak{g})$ and $y \in \mathcal{E}_{-1}(x)$, hence $\ell = \mathbb{F}x + \mathbb{F}y$ with $x, y \in E(\mathfrak{g})$ linearly independent such that $[x, y] = 0$ and $\mathbb{F}x + \mathbb{F}y \subseteq E(\mathfrak{g}) \cup \{0\}$. It follows that $\varphi(\mathcal{L}_{\Sigma(\mathfrak{g})}) \subseteq \mathcal{L}$, forcing equality. We conclude that φ is an isomorphism between $\Gamma_{\mathfrak{g}}$ and $\Gamma_{\Sigma(\mathfrak{g})}$. □

In light of the above theorem and Corollary 7.1.7, in the remainder of this chapter we will identify all $\mathrm{Exp}(x) \in \Sigma(\mathfrak{g})$ by their defining extremal elements $x \in E(\mathfrak{g})$ without explicitly mentioning the isomorphism $\varphi : \Sigma(\mathfrak{g}) \to E(\mathfrak{g})$ given by $\mathrm{Exp}(x) \mapsto x$.

Now let Σ be a set of abstract transvection groups of some group $G = \langle \Sigma \rangle$. Following Notation II(3.13) of [26], we say that Σ *satisfies condition* (H) if for all rank one groups $X = \langle A, B \rangle$ with $A \in \Sigma$ such that $\mathrm{ord}(a) \leq 3$ for all $a \in A^{\#}$ and $B \in \Omega_A$ there exist full unipotent subgroups $A_0 \leq A$ and $B_0 \leq B$ such that $X_0 = \langle A_0, B_0 \rangle \cong \mathrm{(P)SL}_2(\mathbb{F})$ for some commutative field $\mathbb{F}$, $|\mathbb{F}| \geq 4$. Condition (H) enables us to define a point-line geometry on Σ different from the root group geometry of Σ, which we will later see has an important property useful for our purposes. This additional point-line geometry is defined as follows.

Definition 7.1.9 (Abstract transvection geometry)**.** *Let Σ be a set of abstract transvection groups of a group $G = \langle \Sigma \rangle$ that satisfies condition* (H)*. The* ***abstract transvection geometry*** *of Σ is the point-line geometry $\underline{\Gamma}_\Sigma = (\underline{\mathcal{P}}_\Sigma, \underline{\mathcal{L}}_\Sigma)$, in which $\underline{\mathcal{P}}_\Sigma = \{\underline{A} \mid A \in \Sigma\}$ with $\underline{A} := Z(\langle C_\Sigma(A) \rangle) \cap \Sigma$ and $\underline{\mathcal{L}}_\Sigma = \{\ell_{A,B} \mid A, B \in \Sigma\}$ with $\ell_{A,B} := \{\underline{C} \mid C \in \Sigma \text{ such that } \underline{C} \subseteq Z(\langle C_\Sigma(A) \cap C_\Sigma(B) \rangle) \cap \Sigma\}$.*

Under certain conditions, we are able to identify the points, hence also the lines, of the abstract transvection geometry $\underline{\Gamma}_\Sigma = (\underline{\mathcal{P}}_\Sigma, \underline{\mathcal{L}}_\Sigma)$ of a set Σ of abstract transvection groups of a group $G = \langle \Sigma \rangle$ that satisfies condition (H) by the elements in Σ. This is characterised by the following lemma.

Lemma 7.1.10. *Let Σ be a set of abstract root subgroups of a group $G = \langle \Sigma \rangle$ such that $Z(G) = 1$. Define $\mathrm{rad}(G) := \langle N_A \mid A \in \Sigma \rangle$, in which $N_A := N_{\langle \underline{A} \rangle}(\underline{B})$ with $B \in \Omega_A$ arbitrary. Then $\mathrm{rad}(G) = 1$ if and only if $\underline{A} = \{A\}$ for all $A \in \Sigma$.*

Proof. First note that $Z(G) = 1$ implies that $N_{\langle \underline{A} \rangle}(\underline{B})$ is independent of the choice of $B \in \Omega_A$ by Lemma II(4.11) of [26], showing that N_A is well-defined.

Suppose first that $\underline{A} = \{A\}$ for all $A \in \Sigma$. Then $N_A = N_{\langle \underline{A} \rangle}(\underline{B}) = N_A(B) = \{C \in A \mid C \leq N(B)\}$ with $B \in \Omega_A$ arbitrary, hence either $N_A = A$ or $N_A = 1$. If the former is true, then $A \leq N(B)$. Specifically, we have $B^a = B$ for all $a \in A$. But $B \in \Omega_A$ implies that $\langle A, B \rangle$ is a rank one group, so for every $a \in A^{\#}$ there exists a $b \in B^{\#}$ such that $A^b = B^a = B$, implying that $A = B^{b^{-1}} = B$, a contradiction. It follows that $N_A = 1$ for all $A \in \Sigma$ so that $\mathrm{rad}(G) = \langle N_A \mid A \in \Sigma \rangle = 1$.

To prove the converse, we show that $\langle \underline{A} \rangle = AN_A = \{an_A \mid a \in A, n_A \in N_A\}$; as $1 = \mathrm{rad}(G) = \langle N_A \mid A \in \Sigma \rangle$ forces $N_A = 1$ for all $A \in \Sigma$, then we obtain $\langle \underline{A} \rangle = A$ so that $\underline{A} = \{A\}$, as desired. The inclusion $AN_A \subseteq \langle \underline{A} \rangle$ is clear from the observation that $A \in \underline{A} \subseteq \langle \underline{A} \rangle$ and $N_A \subseteq \langle \underline{A} \rangle$, so it remains to show that $\langle \underline{A} \rangle \subseteq AN_A$. To do so, it suffices to show $DN_A = AN_A$ for some $A \neq D \in \underline{A}$. Since $D \in \underline{A}$, we have $\Omega_A = \Omega_D$ by Lemma II(2.22)(3) of [26], hence $B \in \Omega_D$. Consequently, $\langle A, B \rangle$ and $\langle D, B \rangle$ are both rank one groups, so for all $a \in A^{\#}$ there exists a $b \in B^{\#}$ such that $A^b = B^a$, and for all $b \in B^{\#}$ there exists a $d \in D^{\#}$ such that $D^b = B^d$. But then $\underline{B}^a = \underline{A}^b = \underline{B}^d$ as $A, D \in \underline{A}$ so that

$\underline{B}^{ad^{-1}} = \underline{B}$, hence $ad^{-1} \in \langle \underline{A} \rangle$ is an element such that $ad^{-1} \in N(\underline{B})$, i.e. $ad^{-1} \in N_A$. It follows that $dN_A = aN_a$, and since for all $a \in A^{\#}$ we can find such a $d \in D^{\#}$ and vice versa, we find $DN_A = AN_A$. We conclude that $\langle \underline{A} \rangle = AN_A$. □

By combining Proposition 7.1.6 with Corollary 7.1.7 and Lemma 7.1.10, we obtain the following result.

Corollary 7.1.11. *Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements and let $\Sigma(\mathfrak{g})$ be the non-degenerate class of $\mathbb{F}$-root subgroups of $G = \langle \Sigma(\mathfrak{g}) \rangle$. Then $\underline{\mathrm{Exp}(x)} = \{\mathrm{Exp}(x)\}$ for all $\mathrm{Exp}(x) \in \Sigma(\mathfrak{g})$.*

Proof. First observe that the center of G is trivial, i.e. $Z(G) = 1$; indeed, if $\mathrm{Exp}(x) \in \Sigma(\mathfrak{g})$ satisfies $[\mathrm{Exp}(x), \mathrm{Exp}(y)] = 1$ for all $\mathrm{Exp}(y) \in \Sigma(\mathfrak{g})$, then $[x, y] = 0$ for all $y \in E(\mathfrak{g})$ by Proposition 7.1.6. In particular, as $\mathfrak{g} = \langle E(\mathfrak{g}) \rangle$ is linearly spanned by $E(\mathfrak{g})$ by Lemma 3.1.9, we have $[x, y] = 0$ for all $y \in \mathfrak{g}$ so that $x \in Z(\mathfrak{g})$, contradicting that $\mathfrak{g}$ is simple.

Now letting $\mathrm{Exp}(y) \in \Omega_{\mathrm{Exp}(x)}$ be arbitrary, we have $N(\underline{\mathrm{Exp}(y)}) = C(\underline{\mathrm{Exp}(y)})$ because $\underline{\mathrm{Exp}(\lambda y)} = \underline{\mathrm{Exp}(y)}$ for all $\lambda \in \mathbb{F}^*$, therefore $N_{\langle \underline{\mathrm{Exp}(x)} \rangle}(\underline{\mathrm{Exp}(y)}) = \{\mathrm{Exp}(z) \in \langle \underline{\mathrm{Exp}(x)} \rangle \mid \mathrm{Exp}(z) \leq N(\underline{\mathrm{Exp}(y)})\} = \{\mathrm{Exp}(z) \in \langle \underline{\mathrm{Exp}(x)} \rangle \mid \mathrm{Exp}(z) \leq C(\underline{\mathrm{Exp}(y)})\} = \{\mathrm{Exp}(z) \in \langle \underline{\mathrm{Exp}(x)} \rangle \mid [\mathrm{Exp}(z), \underline{\mathrm{Exp}(y)}] = 1\}$. If now $\mathrm{Exp}(z) \in \langle \underline{\mathrm{Exp}(x)} \rangle$, then $\Omega_{\mathrm{Exp}(x)} = \Omega_{\mathrm{Exp}(z)}$ by Lemma II(2.22)(3) of [26] implies that $\mathrm{Exp}(y) \in \Omega_{\mathrm{Exp}(z)}$. But then $y \in \mathcal{E}_2(z)$ by Corollary 7.1.7, specifically $[y, z] \neq 0$. In turn, $[\mathrm{Exp}(z), \mathrm{Exp}(y)] \neq 1$ by Proposition 7.1.6, so then certainly $[\mathrm{Exp}(z), \underline{\mathrm{Exp}(y)}] \neq 1$ as $\mathrm{Exp}(y) \in \underline{\mathrm{Exp}(y)}$. It follows that $N_{\mathrm{Exp}(x)} = 1$ for all $\mathrm{Exp}(x) \in \Sigma(\mathfrak{g})$, thus $\mathrm{rad}(G) = \langle N_{\mathrm{Exp}(x)} \mid \mathrm{Exp}(x) \in \Sigma(\mathfrak{g}) \rangle = 1$. Now Lemma 7.1.10 applies, so we conclude that $\underline{\mathrm{Exp}(x)} = \{\mathrm{Exp}(x)\}$. □

We finish our discussion of abstract root subgroups with an important theorem pertaining to the characterisation of the abstract transvection geometry $\underline{\Gamma}_\Sigma = (\underline{\mathcal{P}}_\Sigma, \underline{\mathcal{L}}_\Sigma)$ of a set Σ of abstract transvection groups of some group $G = \langle \Sigma \rangle$ that satisfies condition (H). This theorem is also due to Cuypers and Meulewaeter in the context of the inner ideal geometry of a Lie algebra [29].

Theorem 7.1.12. *Let Σ be a set of abstract transvection subgroups of a group $G = \langle \Sigma \rangle$ satisfying condition* (H)*. Then the abstract transvection geometry $\underline{\Gamma}_\Sigma = (\underline{\mathcal{P}}_\Sigma, \underline{\mathcal{L}}_\Sigma)$ of Σ is a thick non-degenerate polar space whose rank is at least two but not necessarily finite.*

Proof. See Theorem III(1.4) of [26] or Theorem 6.11(b) of [29]. □

7.2 The singular rank of the extremal geometry of non-classical type

Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry. A pair of point $\{p, q\} \subseteq \mathcal{P}$ is called a *symplectic pair* if $d(p, q) = 2$ and $|p^\perp \cap q^\perp| \geq 2$. A subspace $\mathcal{S} \subseteq \mathcal{P}$ is called *convex* if any point on a shortest path between any two points in $\mathcal{S}$ is contained in $\mathcal{S}$, and 2-*convex* if this is true for any two points in $\mathcal{S}$ at distance two. If $p^\perp \cap \ell$ is either empty, a single point or ℓ itself for all $p \in \mathcal{P}$ and $\ell \in \mathcal{L}$, then Γ is called a *gamma space*. Note that gamma spaces are generalisations of polar spaces, for in polar spaces we only have that $p^\perp \cap \ell$ is either a single point or ℓ itself for all $p \in \mathcal{P}$ and $\ell \in \mathcal{L}$. A *symplecton* is a 2-convex subspace of a gamma space Γ generated by a symplectic pair, i.e. the smallest 2-convex subspace of Γ containing a symplectic pair $\{p, q\} \subseteq \mathcal{P}$.

Of particular interest to us is the following lemma regarding the existence and characterisation of symplecta in root filtration spaces.

Lemma 7.2.1. *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a root filtration space with filtration $\{\mathcal{P}_i\}_{-2 \le i \le 2}$. Then Γ is a gamma space whose symplecta, if existent, are of the form $\mathcal{P}_{\le 0}(\mathcal{P}_{\le 0}(p, q))$ with $(p, q) \in \mathcal{P}_0$.*

Proof. Note that it follows immediately from Definition 4.5.1(v) that Γ is a gamma space; indeed, since $\mathcal{P}_{\le 1}(p) = p^\perp$ is a subspace of Γ for all $p \in \mathcal{P}$, we have $\ell \subseteq \mathcal{P}_{\le -1}(p) = p^\perp$ for all $\ell \in \mathcal{L}$ such that $|\mathcal{P}_{\le -1}(p) \cap \ell| = |p^\perp \cap \ell| \ge 2$, which shows that p is collinear to all points on ℓ. Further note that $\{p, q\} \subseteq \mathcal{P}$ is a symplectic pair if and only if $(p, q) \in \mathcal{P}_0$ by Proposition 4.5.3.

Now let $\mathcal{S} \subseteq \mathcal{P}$ be a symplecton of Γ. Then $\mathcal{S}$ is a 2-convex subspace generated by some symplectic pair $\{p, q\} \subseteq \mathcal{P}$. Consequently, $\mathcal{P}_{\le 0}(p, q)$ is the smallest subspace containing $\{p, q\}$ by Definition 4.5.1(v) and because $(p, q) \in \mathcal{P}_0$ by the above. Now every pair $\{p', q'\} \subseteq \mathcal{P}$ of common neighbours of p and q in $(\mathcal{P}, \mathcal{P}_{-1})$ is contained in $\mathcal{P}_{\le 0}$ since $p, q \in \mathcal{P}_{-1}(p', q')$, hence if $(p', q') \in \mathcal{P}_0$ then for any point $r \in \mathcal{P}_{-1}(p', q')$ different from p and q we have $(p, r) \in \mathcal{P}_{\le 0}$ because $p', q' \in \mathcal{P}_{-1}(p, r)$ and similarly $(q, r) \in \mathcal{P}_{\le 0}$ because $p', q' \in \mathcal{P}_{-1}(q, r)$. But then by 2-convexity of $\mathcal{S}$ we have $\mathcal{S} = \mathcal{P}_{\le 0}(\mathcal{P}_{\le 0}(p, q))$. □

We now introduce two specific types of point-line geometries that we require for our proof of Theorem 1.1.4. The first type is parapolar spaces, following the definition of [18, 19].

Definition 7.2.2 (Parapolar space)**.** *Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a point-line geometry and let $\mathbb{S}$ be a collection of proper symplecta of Γ. If*

(i) *Γ is a thick partial linear gamma space whose collinearity graph is connected,*

(ii) *all symplecta in $\mathbb{S}$ are non-degenerate polar spaces of rank at least two,*

(iii) *for all symplectic pairs $\{p, q\} \subseteq \mathcal{P}$ there exists a unique symplecton in $\mathbb{S}$ containing $\{p, q\}$,*

(iv) *for all $\ell \in \mathcal{L}$ there exists a symplecton in $\mathbb{S}$ containing ℓ,*

then Γ is called a ***parapolar space****.*

A point-line geometry $\Gamma = (\mathcal{P}, \mathcal{L})$ that satisfies Definition 7.2.2(i)-(iii) but not Definition 7.2.2(iv) is called a *weakly parapolar space*. Contrarily, Γ is said to be *strongly parapolar* if Γ is a parapolar space in which no two points have a unique common neighbour in its collinearity graph. If Γ is a parapolar space with a collection $\mathbb{S}$ of symplecta, we say that Γ has *polar rank (at least)* k if all symplecta of Γ in $\mathbb{S}$ have rank (at least) k. n case $\Gamma = (\mathcal{P}, \mathcal{L})$ is both a parapolar space with a collection $\mathbb{S}$ of symplecta and a root filtration space with filtration $\{\mathcal{P}_i\}_{-2 \leq i \leq 2}$, then Γ will be strongly parapolar if and only if $\mathcal{P}_1 = \emptyset$.

Example 7.2.3. Let $\Gamma = (\mathcal{P}, \mathcal{L})$ be a linear space. Then $p^\perp \cap \ell = \ell$ for all $p \in \mathcal{P}$ and $\ell \in \mathcal{L}$ since any two points in $\mathcal{P}$ are collinear, hence clearly Γ satisfies Definition 7.2.2(i). Moreover, because Γ contains no symplectic pairs, the collection $\mathbb{S}$ of symplecta will be empty, in which both Definition 7.2.2(ii) and Definition 7.2.2(iii) are a vacuous truths. As Definition 7.2.2(iv) clearly does not hold, Γ will be a weakly parapolar space.

Let $\mathcal{I} = \{1, 2\}$ be a type set and let $\mathcal{C} = (C, \{\sim \mid i \in \mathcal{I}\})$ be a building of type G_2. From it, we obtain a root shadow space of type $G_{2,1}$, which will be a generalised hexagon by Table 1 as there are three edges between the nodes in the Coxeter diagram corresponding to a Coxeter system of type G_2. A root shadow space of type $G_{2,1}$ has singular rank one, as all of its lines are maximal singular subspaces. Since generalised hexagons are also weakly parapolar spaces with an empty set of symplecta, a root shadow space of type $G_{2,1}$ will therefore be a weakly parapolar space.

Our definition of the second type of point-line geometries follows the one given in [8] and is based on the terminology used by Freudenthal [28].

Definition 7.2.4 (Metasymplectic space)**.** *Let* $\Gamma = (\mathcal{P}, \mathcal{L})$ *be a point-line geometry and let* $\mathbb{S}$ *be a collection of proper symplecta of* Γ*. If*

(i) *the collinearity graph of* Γ *is connected,*

(ii) *all symplecta in* $\mathbb{S}$ *are non-degenerate polar spaces of rank three,*

(iii) *the intersection of any two distinct symplecta in* $\mathbb{S}$ *is either empty, a point in* $\mathcal{P}$*, a line in* $\mathcal{L}$ *or a plane of* Γ*,*

(iv) *for all* $p \in \mathcal{P}$ *the point-line geometry, whose point set is the set of symplecta in* $\mathbb{S}$ *containing* p *and whose line set is the set of subsets of the point set consisting of all symplecta in* $\mathbb{S}$ *containing a fixed plane that contains* p*, is a non-degenerate polar space of rank three,*

then Γ *is called a* ***metasymplectic space****.*

A metasymplectic space $\Gamma = (\mathcal{P}, \mathcal{L})$ is said to be *thick* if all of its symplecta, when viewed as polar spaces, are thick. Note that Definition 7.2.4(i) is superfluous if Γ is a non-degenerate root filtration space, as then the collinearity graph $(\mathcal{P}, \mathcal{P}_{-1})$ of Γ is connected by non-degeneracy of Γ.

We will now turn our attention to proving Theorem 1.1.4. So, let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$, char($\mathbb{F}$) $\neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements whose extremal geometry $\Gamma_\mathfrak{g} = (\mathcal{E}, \mathcal{L})$ with filtration $\{\mathcal{E}_i\}_{-2 \leq i \leq 2}$ is not isomorphic to a root shadow space of classical type. First consider the following proposition, using the notation in III§9 of [26].

Proposition 7.2.5. *Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$,* char($\mathbb{F}$) $\neq 2$*, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements whose extremal geometry $\Gamma_\mathfrak{g} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{g}$ with filtration $\{\mathcal{E}_i\}_{-2 \leq i \leq 2}$ is not isomorphic to a root shadow space of classical type. Then for all $x \in E(\mathfrak{g})$ the subspace $\mathcal{E}_{\leq 0}(x)$ of $\Gamma_\mathfrak{g}$ is maximal and the graph $(\mathcal{E}, \mathcal{E}_2)$ induced by $\Delta_0' := \mathcal{E}_{\leq 0}(x, y)$ is connected for all $y \in \mathcal{E}_2(x)$.*

Proof. First note that $\mathcal{E}_{\leq 0}(x)$ and $\Delta_0' = \mathcal{E}_{\leq 0}(x, y)$ are both subspaces of $\Gamma_\mathfrak{g}$ by Definition 4.5.2(iv) since $\Gamma_\mathfrak{g}$ is a root filtration space by Proposition 5.1.3. In addition, by non-degeneracy of $\Gamma_\mathfrak{g}$ as a result of Theorem 5.1.4 because $\mathfrak{g}$ is simple, we have $\mathcal{E}_2(x) \neq \emptyset$ for all $x \in E(\mathfrak{g})$, so $\Delta_0' \neq \emptyset$. Now let $\Sigma(\mathfrak{g})$ be the non-degenerate class of $\mathbb{F}$-root subgroups of G as defined in the previous section. For $\mathrm{Exp}(x) \in \Sigma(\mathfrak{g})$, set $C_{\mathrm{Exp}(x)} := \langle \Sigma_{\mathrm{Exp}(x)} \rangle$ and define $M_{\mathrm{Exp}(x)} := \mathrm{rad}(C_{\mathrm{Exp}(x)})\underline{\langle \mathrm{Exp}(x) \rangle}$, following the notation in III§1 of [26].

First suppose towards a contradiction that $\mathcal{E}_{\leq 0}(x)$ is not maximal in $\Gamma_\mathfrak{g}$ for some $x \in E(\mathfrak{g})$. Then by Corollary 7.1.7, this equivalently means that $C_{\mathrm{Exp}}(x)$ is not maximal in $\Sigma(\mathfrak{g})$. On the one hand, if $N(\mathrm{Exp}(x))$ is not maximal in G, then $\Gamma_\mathfrak{g}$ is isomorphic to a root shadow space of type $A_{n,\{1,n\}}$ $(n \geq 2)$ by Theorem III(9.3)(2) of [26], a contradiction. On the other hand, if $N(\mathrm{Exp}(x))$ is maximal in G, then either $\Sigma_{\mathrm{Exp}(x)} \subseteq M_{\mathrm{Exp}(x)}$ or $\Sigma_{\mathrm{Exp}(x)} \subseteq M_{\mathrm{Exp}(x)}$. But $\mathrm{Exp}(x) \notin \Sigma_{\mathrm{Exp}(x)}$ whereas $M_{\mathrm{Exp}(x)} = \mathrm{Exp}(x)$ by Corollary 7.1.11, so the former case cannot occur. In the latter case, we find $\Gamma_\mathfrak{g}$ to be isomorphic to a root shadow space of type $BC_{3,2}$ by Theorem III(9.5)(B) of [26], another contradiction. It follows that $\mathcal{E}_{\leq 0}(x)$ must be maximal in $\Gamma_\mathfrak{g}$ for all $x \in E(\mathfrak{g})$.

Next, assume that the graph $(\mathcal{E}, \mathcal{E}_2)$ of $\Delta_0' = \mathcal{E}_{\leq 0}(x, y)$ with $y \in \mathcal{E}_2(x)$ is not connected for some $x \in E(\mathfrak{g})$. By maximality of $C_{\mathrm{Exp}(x)}$ in $\Sigma(\mathfrak{g})$ by the previous paragraph, then Theorem III(9.5)(A) of [26] applies, from which it follows that $\Gamma_\mathfrak{g}$ is isomorphic to a root shadow space of type $BC_{n,2}$ $(n \geq 4)$ or $D_{n,2}$ $(n \geq 4)$, a contradiction. We conclude that the graph $(\mathcal{E}, \mathcal{E}_2)$ of Δ_0' with $y \in \mathcal{E}_2(x)$ is connected for all $x \in E(\mathfrak{g})$. The proposition follows. □

In the remainder of this chapter, we will assume that for all $x \in E(\mathfrak{g})$ the subspace $\mathcal{E}_{\leq 0}(x)$ of $\Gamma_\mathfrak{g}$ is maximal and that the graph $(\mathcal{E}, \mathcal{E}_2)$ induced by $\Delta_0' = \mathcal{E}_{\leq 0}(x, y)$ is connected for all $y \in \mathcal{E}_2(x)$. Note that, as Δ_0' is a subspace of $\Gamma_\mathfrak{g}$, it follows from Proposition 7.1.6 and Corollary 7.1.7 that $\Delta_0 := \{\mathrm{Exp}(z) \in \Sigma(\mathfrak{g}) \mid z \in \Delta_0'\}$ is a set of abstract root subgroups of $\langle \Delta_0 \rangle$. So, it remains to distinguish between Δ_0 being a set of abstract transvection groups or non-degenerate.

Assume first that Δ_0 is a set of abstract transvection groups of $\langle \Delta_0 \rangle$. We will show that, in this case, the extremal geometry $\Gamma_\mathfrak{g} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{g}$ is a thick metasymplectic space.

To do so, we require the following proposition.

Proposition 7.2.6. *Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements and let $\Sigma(\mathfrak{g})$ be the class of abstract root subgroups of $G = \langle \Sigma(\mathfrak{g}) \rangle$. If Δ_0 is a set of abstract transvection groups of $\langle \Delta_0 \rangle$, then the abstract transvection geometry $\underline{\Gamma}_{\Delta_0} = (\underline{\mathcal{P}}_{\Delta_0}, \underline{\mathcal{L}}_{\Delta_0})$ of Δ_0 is a thick non-degenerate polar space of not necessarily finite rank at least two.*

Proof. Since $\Sigma(\mathfrak{g})$ is a class of $\mathbb{F}$-root subgroups of $G = \langle \Sigma(\mathfrak{g}) \rangle$ by Proposition 7.1.6, we have $\langle \mathrm{Exp}(x), \mathrm{Exp}(y) \rangle \cong \mathrm{SL}_2(\mathbb{F})$ for all $\mathrm{Exp}(x) \in \Sigma(\mathfrak{g})$ and $\mathrm{Exp}(y) \in \Omega_{\mathrm{Exp}(x)}$. Consequently, Δ_0 satisfies condition (H) if and only if $|\mathbb{F}| \geq 4$. In particular, condition (H) is a vacuous truth if $|\mathbb{F}| \geq 4$, as $\exp(x, \lambda)^n = \exp(x, n\lambda)$ for all $n \in \mathbb{N}$ and $\lambda \in \mathbb{F}$ by Proposition 3.1.7 implies that $\mathrm{ord}(\exp(x, \lambda)) \geq 4$ for at least one $\lambda \in \mathbb{F}^*$.

Because Δ_0 is a set of abstract transvection groups of $\langle \Delta_0 \rangle$ that satisfies condition (H), the abstract transvection geometry $\underline{\Gamma}_{\Delta_0} = (\underline{\mathcal{P}}_{\Delta_0}, \underline{\mathcal{L}}_{\Delta_0})$ of Δ_0 is well-defined by Definition 7.1.9. Specifically, we may identity the points in $\underline{\mathcal{P}}_{\Delta_0}$ by the elements of Δ_0 and the lines in $\underline{\mathcal{L}}_{\Delta_0}$ by the subsets of Δ_0 all of whose elements commute pair-wise as a consequence of Corollary 7.1.11. But now Theorem 7.1.12 applies, so $\underline{\Gamma}_{\Delta_0}$ is a thick non-degenerate polar space of rank at least two but not necessarily finite.

If $|\mathbb{F}| \leq 3$, the proposition follows Lemma III(7.7)(2) and Lemma III(7.8) of [26]. □

Now define $\Delta_z := \Delta_0 \cap \Lambda_{\mathrm{Exp}(z)}$ with $\mathrm{Exp}(z) \in \Lambda^*_{\mathrm{Exp}(x)} := \Lambda_{\mathrm{Exp}(x)} \cap \Psi_{\mathrm{Exp}(y)}$. By Corollary 7.1.7, we then equivalently have that $\varphi(\Delta_z) = \{z' \in \Delta'_0 \mid z' \in \mathcal{E}_{-1}(z)\} \subseteq \Delta'_0$, in which $z \in \mathcal{E}_{-1}(x) \cap \mathcal{E}_1(y)$. Note that $\mathcal{E}_2(x) \neq \emptyset$ as $\mathcal{E}_{-1} \neq \emptyset$ implies that by simplicity of $\mathfrak{g}$ the extremal geometry $\Gamma_{\mathfrak{g}}$ of $\mathfrak{g}$ will be a non-degenerate root filtration space by Theorem 5.1.4.

For the sake of simplicity, write $\Delta'_z := \varphi(\Delta_z)$. We state some useful properties of subsets of Δ_0 of the form Δ_z with $\mathrm{Exp}(z) \in \Lambda_{\mathrm{Exp}(x)} \cap \Psi_{\mathrm{Exp}(y)}$.

Lemma 7.2.7. *Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements and let $\underline{\Gamma}_{\Delta_0} = (\underline{\mathcal{P}}_{\Delta_0}, \underline{\mathcal{L}}_{\Delta_0})$ be the abstract transvection geometry of Δ_0 with $x \in E(\mathfrak{g})$ and $y \in \mathcal{E}_2(x)$. Then the subset $\Delta_z \subseteq \Delta_0$ with $\mathrm{Exp}(z) \in \Lambda_{\mathrm{Exp}(x)} \cap \Psi_{\mathrm{Exp}(y)}$ is*

(i) *a maximal singular subspace of $\underline{\Gamma}_{\Delta_0}$,*

(ii) *a projective plane of $\underline{\Gamma}_{\Delta_0}$.*

Proof. For (i), first note that $\langle \Delta_z \rangle \cap \Delta_0 = \Delta_z$ by Lemma III(7.2)(2) of [26] implies that Δ_z is a subspace of Δ_0; indeed, the smallest subspace $\langle \Delta_z \rangle$ of Δ_0 containing Δ_z is Δ_z inside Δ_0.

We show next that Δ_z is a singular subspace of Δ_0. Denoting collinearity in $\underline{\Gamma}_{\Delta_0}$ by $\perp_0$, then by Definition 7.1.9 and Corollary 7.1.11 we have $\mathrm{Exp}(p) \perp_0 \mathrm{Exp}(q) \iff [p, q] = 0$ for all $p, q \in \Delta'_0$, for we may identify the points in $\underline{\mathcal{P}}_{\Delta_0}$ by the elements in Δ_0. So, Δ_z is singular if and only if $[p, q] = 0$ for all $p, q \in \Delta'_z$. Supposing now towards a contradiction that $[p, q] \neq 0$ for some $p, q \in \Delta'_z$, then we have $(p, q) \in \mathcal{E}_1 \sqcup \mathcal{E}_2$. By degeneracy of Δ_0, the former case cannot occur, hence $(p, q) \in \mathcal{E}_2$. But then p and q have no common neighbours in the collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ of the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{g}$ by Corollary 4.5.3, which is absurd as $\{p, q\} \subseteq \mathcal{E}_{-1}(z) \iff z \in \mathcal{E}_{-1}(p, q)$, i.e. z is a common neighbour of p and q. We conclude that $(p, q) \notin \mathcal{E}_1 \sqcup \mathcal{E}_2$, therefore $(p, q) \in \mathcal{E}_{\leq 0} \iff [p, q] = 0 \iff \operatorname{Exp}(p) \perp_0 \operatorname{Exp}(q)$.

It remains to show that Δ_z is maximal. Assuming for the sake of contradiction that $\mathcal{S}$ is a subspace of Δ_0 that properly contains Δ_z, let $\operatorname{Exp}(p) \in \mathcal{S} \setminus \Delta_z$ be arbitrary. Then by Corollary 7.1.7 and by connectedness of $(\mathcal{E}, \mathcal{E}_{-1})$, which is a result of non-degeneracy of $\Gamma_{\mathfrak{g}}$ by Theorem 5.1.4 since $\mathfrak{g}$ is simple, there exists a $z' \in \Delta'_z$ such that $p \in \mathcal{E}_{-1}(z')$. Now Lemma III(7.2)(3) of [26] applies, hence $[p, z'] = 0$. But then $[p, \Delta'_z] = 0$ by Lemma III(7.1)(3) of [26], and consequently $p \in \Delta'_z$ by Lemma III(7.2)(1) of [26], which we may invoke by connectivity of the graph $(\mathcal{E}, \mathcal{E}_2)$ induced by Δ'_0. But then we have reached a contradiction, so it follows that Δ_z is maximal, which settles (i).

For (ii), first note that Δ_z is a projective geometry by Corollary 4.1.20 as $\underline{\Gamma}_{\Delta_0}$ is a thick non-degenerate polar space by Proposition 7.2.6 and Δ_z is a singular subspace of $\underline{\Gamma}_{\Delta_0}$ by Lemma 7.2.7(i). So, to show that Δ_z is a projective plane of $\underline{\Gamma}_{\Delta_0}$, it remains to show that Δ_z contains three non-collinear points and that any two lines in Δ_z intersect.

The former property is settled by Lemma III(7.6)(2) of [26]; indeed, it is shown there that $\Delta_z = \operatorname{Exp}(p)^{\perp_0} \cap \ell^{\perp_0}$ for any line $\ell \subseteq \Delta_z$ and point $\operatorname{Exp}(p) \in \Delta_z$ not on ℓ. For the latter property, we again use Corollary 7.1.7 and let $\ell, \ell' \subseteq \Delta'_z$ be two distinct lines. Then by Lemma III(7.6)(1) of [26], there exist distinct $p, q \in \mathcal{E}_{-1}(x, z) \cap \mathcal{E}_1(y)$ such that $\ell = \Delta'_z \cap \Delta'_p$ and $\ell' = \Delta'_z \cap \Delta'_q$. Since x and z are common neighbours of both p and q in $(\mathcal{E}, \mathcal{E}_{-1})$, we cannot have $(p, q) \in \mathcal{E}_1 \sqcup \mathcal{E}_2$ by Proposition 4.5.3, hence $(p, q) \in \mathcal{E}_{\leq 0}$. Now Lemma III(7.3) of [26] applies, which shows that $|\Delta'_p \cap \Delta'_q| \geq 1$. Clearly, we have $(p, q) \notin \mathcal{E}_{-2}$, and if $(p, q) \in \mathcal{E}_{-1}$ then by maximality of $\Delta'_z \cap \Delta'_p$ as a consequence of Lemma 7.2.7(i) we find that q lies on the line through z and p so that $\ell = \Delta'_z \cap \Delta'_p = \Delta'_z \cap \Delta'_q = \ell'$, a contradiction. So, we must have $(p, q) \in \mathcal{E}_0$, hence $|\Delta'_p \cap \Delta'_q| \leq 1$ by Lemma III(7.4) of [26], forcing $|\Delta'_p \cap \Delta'_q| = 1$, i.e. $\Delta'_p \cap \Delta'_q$ contains a single point, say $r \in \Delta'_0$. Now $[r, \ell] = 0 = [r, \ell']$, so for any point $s \in \ell' \setminus \ell \subseteq \Delta'_z \setminus \ell$, which exists by distinctness of ℓ and ℓ', we then have $[r, \ell] = 0 = [r, s]$, implying that $r \in \Delta'_z$, again by Lemma III(7.6)(2) of [26]. It follows that $\{r\} = \Delta'_p \cap \Delta'_q = (\Delta'_z \cap \Delta'_p) \cap (\Delta'_z \cap \Delta'_q) = \ell \cap \ell'$. This shows that Δ_z is a projective plane of Δ_0, as desired. □

It now follows immediately from Proposition 7.2.6 and Lemma 7.2.7 that $\operatorname{rank}(\underline{\Gamma}_{\Delta_0}) = 3$; indeed, as $\operatorname{rank}(\underline{\Gamma}_{\Delta_0}) \geq 2$ by Proposition 7.2.6 and Δ_z is a projective plane of $\underline{\Gamma}_{\Delta_0}$ by Lemma 7.2.7(ii), we have $\operatorname{rank}(\underline{\Gamma}_{\Delta_0}) \geq 3$, but Δ_z being a maximal singular subspace of $\underline{\Gamma}_{\Delta_0}$ by Lemma 7.2.7(i) then forces $\operatorname{rank}(\underline{\Gamma}_{\Delta_0}) = 3$.

The above result is a first step towards establishing that $\Gamma_{\mathfrak{g}}$ is a thick metasymplectic space. Our next step will be to construct a set $\mathbb{S}$ of symplecta of $\Gamma_{\mathfrak{g}}$. As in Notation III(7.9) of [26], for $\text{Exp}(u) \in \Sigma(\mathfrak{g})$ and $\text{Exp}(v) \in \Sigma_{\text{Exp}(u)} \setminus \Lambda_{\text{Exp}(u)}$ define $S(\text{Exp}(u), \text{Exp}(v)) = \text{Exp}(u)\langle \Lambda_{\text{Exp}(u)} \cap \Lambda_{\text{Exp}(v)} \rangle \text{Exp}(v) \cap \Sigma(\mathfrak{g})$ to be the symplecton of the root group geometry $\Gamma_{\Sigma(\mathfrak{g})} = (\Sigma(\mathfrak{g}), \mathcal{L}_{\Sigma(\mathfrak{g})})$ of $\mathfrak{g}$ spanned by $\text{Exp}(u)$ and $\text{Exp}(v)$. By Corollary 7.1.7 and Lemma 7.2.1, we then equivalently obtain the symplecton $S(u,v) = \mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(u,v))$ of the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{g}$ with $(u,v) \in \mathcal{E}_0$ since $\Gamma_{\mathfrak{g}}$ is a root filtration space by Proposition 5.1.3. The following lemma shows that a symplecton is independent of the choice of a symplectic pair contained in it, thereby making them well-defined.

Lemma 7.2.8. *Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$, $\text{char}(\mathbb{F}) \neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements. Then for all $(u,v) \in \mathcal{E}_0$ the symplecton $S(u,v) = \mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(u,v))$ of the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ is the unique symplecton on u and v.*

Proof. To show uniqueness of $S(u,v)$, we prove that $S(u',v') = S(u,v)$ for all $u',v' \in S(u,v)$ such that $(u',v') \in \mathcal{E}_0$. Since $u',v' \in S(u,v) \iff u,v \in S(u',v')$, it suffices to only show the inclusion $S(u',v') \subseteq S(u,v)$ by symmetry.

If $u',v' \notin \mathcal{E}_{-1}(u,v)$, then the lemma follows from Lemma III(7.10)(1) of [26]. So, we may assume that u' and v' are collinear to both u and v. Now let $p \in S(u',v') = \mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(u',v'))$. Since $u,v \in \mathcal{E}_{-1}(u',v')$, we then have $p \in \mathcal{E}_{\leq 0}(u,v)$. If $p \in \mathcal{E}_{\leq -1}(u,v)$, then p lies on a shortest path from u to v in the collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ since $(u,v) \in \mathcal{E}_0$, hence is contained in $S(u,v)$ by 2-convexity, so we may assume w.l.o.g. that $p \in \mathcal{E}_0(u) \cap \mathcal{E}_{\leq 0}(v)$. But then

$$p \in \mathcal{E}_0(\mathcal{E}_{-2}(u)) \cap \mathcal{E}_{\leq 0}(\mathcal{E}_{-2}(v)) \subseteq \mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(u)) \cap \mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(v)) = \mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(u,v)) = S(u,v),$$

showing the inclusion $S(u',v') \subseteq S(u,v)$. ☐

Denote by $\mathbb{S}$ the collection of symplecta $S(u,v) = \mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(u,v))$ of $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ with $(u,v) \in \mathcal{E}_0$. We follow up with some important properties of the symplecta in $\mathbb{S}$.

Corollary 7.2.9. *Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$, $\text{char}(\mathbb{F}) \neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements and let $\mathbb{S}$ be the collection of symplecta of the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{g}$ as defined above. If Δ_0 is a set of abstract transvection groups of $\langle \Delta_0 \rangle$, then*

(i) *every symplecton in $\mathbb{S}$ is a non-degenerate thick polar space of rank three all of whose plane are projective planes over $\mathbb{F}$,*

(ii) *the intersection of any two distinct symplecta in $\mathbb{S}$ is either empty, a point in $\mathcal{E}$, a line in $\mathcal{L}$ or a plane of $\Gamma_{\mathfrak{g}}$.*

Proof. For (i), see Lemma III(7.10)(2) of [26].

For (ii), let $S(u,v), S(u',v') \in \mathbb{S}$ be two distinct symplecta. Now assume towards a contradiction that $S(u,v) \cap S(u',v')$ is not empty, nor a point in $\mathcal{E}$, nor a line in $\mathcal{L}$

nor a plane in $\Gamma_{\mathfrak{g}}$. As all planes of the symplecta in $\mathbb{S}$ are projective planes over $\mathbb{F}$ by Corollary 7.2.9(i), which are moreover maximal singular by Lemma 7.2.7(i)-(ii) because Δ_0 is a set of abstract transvection groups of $\langle\Delta_0\rangle$ by assumption, there exist distinct points $p, q \in S(u,v) \cap S(u',v')$ such that $(p,q) \in \mathcal{E}_0$. But then $S(u,v) = S(p,q) = S(u',v')$ by Lemma 7.2.8, a contradiction. This proves (ii). □

We are now in a position to prove that $\Gamma_{\mathfrak{g}}$ is a thick metasymplectic space.

Theorem 7.2.10. *Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements. If Δ_0 is a set of abstract transvection groups of $\langle\Delta_0\rangle$, then the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ together with the previously defined collection $\mathbb{S}$ of symplecta of $\Gamma_{\mathfrak{g}}$ is a thick metasymplectic space.*

Proof. We check the conditions listed in Definition 7.2.4. By non-degeneracy of $\Gamma_{\mathfrak{g}}$, Definition 7.2.4(i) is obvious. Definition 7.2.4(ii)-(iii) are Corollary 7.2.9(i)-(ii), so it remains to show that Definition 7.2.4(iv) holds.

For arbitrary $\mathrm{Exp}(x) \in \Sigma(\mathfrak{g})$, denote by $\Gamma_{\mathrm{Exp}(x)} = (\mathcal{P}_{\mathrm{Exp}(x)}, \mathcal{L}_{\mathrm{Exp}(x)})$ the point-line geometry whose points in $\mathcal{P}_{\mathrm{Exp}(x)}$ are the symplecta of the form $S(\mathrm{Exp}(x), \mathrm{Exp}(t))$ with $\mathrm{Exp}(t) \in (\Sigma_{\mathrm{Exp}(x)} \setminus \Lambda_{\mathrm{Exp}(x)}) \cap \Delta_0$ and whose lines in $\mathcal{L}_x$ are the sets of symplecta of the form $S(\mathrm{Exp}(x), \mathrm{Exp}(t))$ all of which contain a fixed plane of $\Gamma_{\mathfrak{g}}$ containing $\mathrm{Exp}(x)$. We claim that $\Gamma_{\mathrm{Exp}(x)} \cong \underline{\Gamma}_{\Delta_0}$. In particular, we will show that the map $\varphi : \Gamma_x \to \underline{\Gamma}_{\Delta_0}$ given by $S(\mathrm{Exp}(x), \mathrm{Exp}(t)) \mapsto \mathrm{Exp}(t)$ is an isomorphism. By Corollary 7.1.7, we may equivalently show that the map $S(x,t) \mapsto t$ with $x \in E(\mathfrak{g})$ and $t \in \mathcal{E}_0 \cap \Delta_0'$ is an isomorphism.

It follows readily from uniqueness of $S(x,t)$ by Lemma 7.2.8 that φ maps $S(x,\cdot)$ bijectively to Δ_0', so it remains to show that φ is bijective on $\mathcal{L}_{\mathrm{Exp}(x)}$ and $\underline{\mathcal{L}}_{\Delta_0}$. On the one hand, any line $\ell \in \mathcal{L}_{\mathrm{Exp}(x)}$, which corresponds uniquely to a set of symplecta in $\mathbb{S}$ of the form $S(x,\ell') = \{S(x,x') \in \mathbb{S} \mid x' \in \ell'\}$ for some line $\ell' \subseteq \mathcal{E}_0(x) \cap \Delta_0'$, contains the fixed plane on ℓ' and $x \notin \ell'$, hence corresponds uniquely to the line $\ell' \subseteq \Delta_0'$ so that $\ell \in \underline{\mathcal{L}}_{\Delta_0}$. On the other hand, any line $\ell \in \mathcal{L}_{\Delta_0}$, which corresponds uniquely to a line $\ell' \subseteq \Delta_0'$, gives rise to the fixed plane on ℓ' containing $x \notin \ell'$, hence corresponds uniquely to the set of symplecta $S(x,\ell') = \{S(x,x') \in \mathbb{S} \mid x' \in \ell'\}$ so that $\ell \in \mathcal{L}_{\mathrm{Exp}(x)}$. This shows that $\Gamma_{\mathrm{Exp}(x)} \cong \underline{\Gamma}_{\Delta_0}$, as claimed. Specifically, $\Gamma_{\mathrm{Exp}(x)}$ will be a thick non-degenerate polar space of rank three by Proposition 7.2.6 since Δ_0 is a set of abstract transvection groups of $\langle\Delta_0\rangle$ by assumption, thus Definition 7.2.4(iv) holds.

We conclude that $\Gamma_{\mathfrak{g}}$ together with the collection $\mathbb{S}$ of symplecta of $\Gamma_{\mathfrak{g}}$ is a metasymplectic space, which will moreover be thick as all symplecta in $\mathbb{S}$ are thick by Corollary 7.2.9(i). □

Next, assume that Δ_0 is a non-degenerate set of abstract root subgroups of $\langle\Delta_0\rangle$. In this case, our goal will be to show that the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{g}$ is a parapolar space satisfying two additional conditions.

The first of these additional conditions is that $p^\perp \cap q^\perp$ must be a non-degenerate polar space of rank at least three for all symplectic pairs $\{p,q\} \in \mathcal{E}$, and the second additional condition is that the subset $p^\perp \cap \ell^\perp \subseteq \mathcal{E}$ must either be a single point or a maximal singular subspace of $p^\perp \cap q^\perp$ for all symplectic pairs $\{p, q\} \subseteq \mathcal{E}$ and lines $\ell \in \mathcal{L}$ such that $q \in \ell$ and $p^\perp \cap \ell = \emptyset$. These conditions are the content of the following proposition.

Proposition 7.2.11. *Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$, $\operatorname{char}(\mathbb{F}) \neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements and let $\mathbb{S}$ be the collection of symplecta $S(u, v) = \mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(u, v))$ with $(u, v) \in \mathcal{E}_0$ of the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$. If Δ_0 is a non-degenerate set of abstract root subgroups of $\langle \Delta_0 \rangle$, then*

(i) *every symplecton in $\mathbb{S}$ is a non-degenerate thick polar space of rank at least four,*

(ii) *for all symplectic pairs $\{p, q\} \subseteq \mathcal{E}$ the subset $p^\perp \cap q^\perp \subseteq \mathcal{E}$ is a non-degenerate thick polar space of rank at least three,*

(iii) *for all symplectic pairs $\{p, q\} \subseteq \mathcal{E}$ and lines $\ell \in \mathcal{L}$ such that $q \in \ell$ and $p^\perp \cap \ell = \emptyset$ the subset $p^\perp \cap \ell^\perp$ is either a single point or a maximal singular subspace of $p^\perp \cap q^\perp$.*

Proof. For (i), note first that every symplecton in $\mathbb{S}$ is a non-degenerate thick polar space by Corollary 7.2.9(i). However, since Δ_0 is a non-degenerate set of abstract root subgroups of $\langle \Delta_0 \rangle$, Lemma 7.2.7(i)-(ii) do not apply, hence every symplecton in $\mathbb{S}$ has rank at least three. It is shown in Lemma III(8.8) of [26] that no symplecton in $\mathbb{S}$ can have rank exactly three, from which (i) follows.

For (ii), let $\{p, q\} \in \mathcal{E}$ be a symplectic pair. Then p and q are non-collinear points in $(\mathcal{E}, \mathcal{E}_{-1})$ at distance two having at least two common neighbours, hence $(p, q) \in \mathcal{E}_0$. In particular, p and q then span the symplecton $S(p, q)$ in $\mathbb{S}$. But $S(p, q)$ is a non-degenerate thick polar space of rank at least four by Proposition 7.2.11(i). As clearly $p^\perp \cap q^\perp = \mathcal{E}_{-2}(\mathcal{E}_{\leq -1}(p, q)) \subseteq \mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(p, q)) = S(p, q)$, Proposition 4.1.14 then shows that $p^\perp \cap q^\perp$ is a non-degenerate polar space, which will have rank at least three because $S(p, q)$ has rank at least four and properly contains $p^\perp \cap q^\perp$.

For (iii), let $\{p, q\} \in \mathcal{E}$ be a symplectic pair and let $\ell \in \mathcal{L}$ be a line on q containing no points collinear with p. Then $\emptyset \neq p^\perp \cap \ell^\perp \subseteq p^\perp \cap q^\perp \subseteq S(p, q)$. Because ℓ is spanned by any two points on ℓ, we may fix some $q \neq r \in \ell$ so that $\ell = \mathbb{F}q + \mathbb{F}r$. Since $r \notin \mathcal{E}_{\leq -1}(p)$ as $p^\perp \cap \ell = \emptyset$, then $r \in \mathcal{E}_0(p) \sqcup \mathcal{E}_1(p) \sqcup \mathcal{E}_2(p)$. If $r \in \mathcal{E}_2(p)$, then $p^\perp \cap \ell^\perp \subseteq p^\perp \cap r^\perp = \emptyset$ because p and r have no common neighbours in $(\mathcal{E}, \mathcal{E}_{-1})$ by Proposition 4.5.3, a contradiction. If $r \in \mathcal{E}_1(p)$, then $[p, r] \in E(\mathfrak{g})$ is the unique common neighbour of p and r in $(\mathcal{E}, \mathcal{E}_{-1})$ by Proposition 4.5.3, hence we find $p^\perp \cap \ell^\perp$ to be the single point $[p, r]$. Finally, if $r \in \mathcal{E}_0(p)$, then we find $p^\perp \cap \ell^\perp$ to be a maximal subspace of $p^\perp \cap q^\perp$. In particular, $p^\perp \cap \ell^\perp$ will be a maximal singular subspace of $p^\perp \cap q^\perp$; indeed, supposing for the sake of contradiction that $u, v \in p^\perp \cap \ell^\perp$ are two distinct non-collinear points, then $S(p, q) = S(u, v)$ by Lemma 7.2.8 so that $\ell \subseteq S(u, v) = S(p, q)$, forcing p to be collinear to one or all points on ℓ since $S(p, q)$ is a polar space by Proposition 7.2.11(i), a contradiction. □

Now all that remains is to prove that $\Gamma_{\mathfrak{g}}$ is a parapolar space. To do so, consider first the following lemma.

Lemma 7.2.12. *Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements and let $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ be the extremal geometry of $\mathfrak{g}$. Then*

- (i) *$\Gamma_{\mathfrak{g}}$ is a thick partial linear gamma space whose collinearity graph is connected and has diameter three,*
- (ii) *the graph induced on $p^\perp \cap q^\perp$ is not complete for all collinear $p, q \in \mathcal{E}$ such that $p \perp q$ if Δ_0 is a non-degenerate set of abstract root subgroups of $\langle \Delta_0 \rangle$.*

Proof. For (i), we only need to show that $\Gamma_{\mathfrak{g}}$ is a gamma space; indeed, $\Gamma_{\mathfrak{g}}$ is thick and a partial linear space by construction, the collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ is connected by non-degeneracy of $\Gamma_{\mathfrak{g}}$, and $(\mathcal{E}, \mathcal{E}_{-1})$ has diameter three by Proposition 4.5.3 since the distance between any two points is at most three. To show that $\Gamma_{\mathfrak{g}}$ is a gamma space, we prove that $z^\perp \cap \ell$ is ℓ itself for all $z \in \mathcal{E}$ and $\ell \in \mathcal{L}$ such that $|z^\perp \cap \ell| \geq 2$. W.l.o.g. assume that $|z^\perp \cap \ell| = 2$, say $z^\perp \cap \ell = \{u, v\}$. Since then $\ell = \mathbb{F}u + \mathbb{F}v$, any point $r \in \ell$ is of the form $r = \lambda u + \mu v$ for some $\lambda, \mu \in \mathbb{F}^*$. By Lemma 5.1.1, we have $[z, [u, t]] = g_z(t)u + g_u(t)z$ and $[z, [v, t]] = g_v(t)z + g_z(t)v$ for all $t \in \mathfrak{g}$, hence

$$\begin{aligned}
[z, [r, t]] = [z, [\lambda u + \mu v, t]] &= \lambda[z, u, t]] + \mu[z, [v, t]] = \lambda(g_z(t)u + g_u(t)z) + \mu(g_z(t)v + g_v(t)z) \\
&= g_z(t)(\lambda u) + g_{\lambda u}(t)z + g_z(t)(\mu v) + g_{\mu v}(t)z = g_z(t)(\lambda u + \mu v) + g_{\lambda u + \mu v}(t)z \\
&= g_z(t)r + g_r(t)z,
\end{aligned}$$

for all $t \in \mathfrak{g}$, so $r \perp z$ by Lemma 5.1.1. It follows that $z^\perp \cap \ell$ is ℓ itself, thus $\Gamma_{\mathfrak{g}}$ is a gamma space.

For (ii), let $p, q \in \mathcal{E}$ such that $p \perp q$. Because Theorem III(2.19) of [26] applies by maximality of $\mathcal{E}_{\leq 0}(p)$, there exists an $r \in \mathcal{E}_0(p)$ such that q is a common neighbour of p and r in the collinearity graph $(\mathcal{E}, \mathcal{E}_{-1})$ of $\Gamma_{\mathfrak{g}}$. Since $\{p, r\} \subseteq \mathcal{E}$ is a symplectic pair, the subset $p^\perp \cap r^\perp \subseteq \mathcal{E}$ is a non-degenerate polar space by Proposition 7.2.11(ii), hence the graph induced by $p^\perp \cap r^\perp$ is not complete. Consequently, the graph induced by $p^\perp \cap q^\perp \cap r^\perp$ is not complete, but then neither is the graph induced by $p^\perp \cap q^\perp$. □

Upon combining Proposition 7.2.11 with Lemma 7.2.12, we are now able to prove that $\Gamma_{\mathfrak{g}}$ is a parapolar space.

Theorem 7.2.13. *Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements. If Δ_0 is a non-degenerate set of abstract root subgroups of $\langle \Delta_0 \rangle$, then the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ together with the previously defined collection $\mathbb{S}$ of symplecta of $\Gamma_{\mathfrak{g}}$ is a parapolar space.*

Proof. We check the conditions listed in Definition 7.2.2. Definition 7.2.2(i) is Lemma 7.2.12(i), and Definition 7.2.2(ii) is Proposition 7.2.11(i). Definition 7.2.2(iii) is also immediate, as every symplectic pair $\{p, q\} \subseteq \mathcal{E}$ defines the unique symplecton $S(p, q) = \mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(p, q))$ in $\mathbb{S}$ containing $\{p, q\}$ by Lemma 7.2.8. So, it remains to show that Definition 7.2.2(iv) holds.

To this extent, let $\ell \in \mathcal{L}$ and fix two distinct collinear points $u, v \in \ell$. Then Lemma 7.2.12(ii) applies, so the graph induced by $u^\perp \cap v^\perp$ is not complete. In particular, there exist distinct non-collinear points $u', v' \in u^\perp \cap v^\perp$ which moreover satisfy $(u', v') \in \mathcal{E}_0$ by Proposition 4.5.3 because $u, v \in \mathcal{E}_{-1}(u', v')$. But then $S(u', v')$ is a symplecton in $\mathbb{S}$ such that $u^\perp \cap v^\perp \subseteq \mathcal{E}_{-2}(\mathcal{E}_{-1}(u', v')) \subseteq \mathcal{E}_{\leq 0}(\mathcal{E}_{\leq 0}(u', v')) = S(u', v')$. As clearly $\ell \subseteq u^\perp \cap v^\perp$, we find ℓ to be contained in the symplecton $S(u', v')$, which shows that Definition 7.2.2(iv) holds. It follows that $\Gamma_{\mathfrak{g}}$ is a parapolar space. □

We gather our findings of this section and their implications to finally prove Theorem 1.1.4, for which we will make use of the classification of thick metasymplectic spaces by Tits [8] and parapolar spaces by Cohen and Cooperstein [27].

Theorem 7.2.14. *Let $\mathfrak{g}$ be a simple Lie algebra over a field $\mathbb{F}$, $\mathrm{char}(\mathbb{F}) \neq 2$, without sandwich elements generated by its set $E(\mathfrak{g})$ of extremal elements. If the extremal geometry $\Gamma_{\mathfrak{g}} = (\mathcal{E}, \mathcal{L})$ of $\mathfrak{g}$ is not isomorphic to a root shadow space of classical type, then $\Gamma_{\mathfrak{g}}$ has finite singular rank.*

In particular, $\Gamma_{\mathfrak{g}}$ will have singular rank three, four, five or seven.

Proof. By Proposition 7.2.5, for all $x \in E(\mathfrak{g})$ the subspace $\mathcal{E}_{\leq 0}(x)$ of $\Gamma_{\mathfrak{g}}$ is maximal and the graph $(\mathcal{E}, \mathcal{E}_2)$ induced by $\Delta_0 = \mathcal{E}_{\leq 0}(x, y)$ is connected for all $y \in \mathcal{E}_2(x)$. Since Δ_0 is a set of abstract root subgroups of $\langle \Delta_0 \rangle$, it is either a set of abstract transvections groups or non-degenerate.

On the one hand, if Δ_0 is a set of abstract transvection groups of $\langle \Delta_0 \rangle$, then $\Gamma_{\mathfrak{g}}$ is a thick metasymplectic space by Theorem 7.2.10. But then $\Gamma_{\mathfrak{g}}$ is isomorphic to a root shadow space of type $F_{4,1}$ by Theorem 10.13 of [8]. In particular, $\Gamma_{\mathfrak{g}}$ will have singular rank three since all symplecta in $\Gamma_{\mathfrak{g}}$ have rank three by Corollary 7.2.9(i).

On the other hand, if Δ_0 is a non-degenerate set of abstract root subgroups of $\langle \Delta_0 \rangle$, then $\Gamma_{\mathfrak{g}}$ is a parapolar space by Theorem 7.2.13. By Corollary III(8.10) of [26], $\Gamma_{\mathfrak{g}}$ has singular rank at most ten, hence $\Gamma_{\mathfrak{g}}$ satisfies the hypothesis of Theorem 2 of [27] by Proposition 7.2.11(ii)-(iii). But $\Gamma_{\mathfrak{g}}$ is not isomorphic to a root shadow space of type $BC_{n,1}$ $(n \geq 3)$, which is a non-degenerate polar space of rank n by Corollary 4.4.6, so Theorem 2(i) of [27] cannot occur. It then follows from Theorem 2(ii)-(iv) of [27] that $\Gamma_{\mathfrak{g}}$ is isomorphic to a root shadow space of type $E_{6,2}$, $E_{7,1}$ or $E_{8,8}$. Specifically, $\Gamma_{\mathfrak{g}}$ will have singular rank four, five or seven. □

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