MASTER

Modelling evolution of boundaries in growing cells with supply centres

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Abstract

We discuss two models for the growth of cells with supply centres in two dimensions. These supply centres produce material which is distributed to the cell wall. Once the material reaches the wall, the wall expands and the cell grows larger. This type of cell growth is analysed by investigating how the cell wall expands over time. This expansion depends on how material is distributed throughout the cell. The first model we consider assumes that supply material travels in straight lines from the centre to the cell wall. The second model we consider assumes that supply material travels on a random path from the centre to the cell wall. The models are based on research for fungal hyphae, which suppose a unbounded domain. We apply these models in a bounded setting. To describe the evolution of a bounded domain, we analyse how the boundary changes over time. In particular, we derive an evolution equation for a parametrisation function of the boundary. We see that the first model yields a fully nonlinear parabolic differential equation. However the second model does not yield a differential equation, but the resulting equation can be considered fully nonlinear parabolic similar to the first model. We show that both models have a unique short-time solution, applying theory for fully nonlinear problems. Additionally, we show that circular solutions to these models are linearly stable over time. Finally, we show that solutions to the first model satisfy an avoidance principle. This principle states that if one domain is contained in another, then this is preserved over time.

Contents

1 Introduction

The analysis of cell growth has proven to have many real-world applications, most notably in the field of health sciences and biology. These applications include plant architecture [Fourcaud et al., 2008], disease research [Lai and Zou, 2015], and tissue engineering [Croll et al., 2005]. Usually cell growth is mentioned in the same sentence as cell division; as cell grows larger, they can split into multiple [Hanczyc and Szostak, 2004]. After dividing, the individual cells can grow once again and the process repeats. Different type of cells can experience varying types of growth: fungal hyphae grow primarily near the tip [Riquelme and Sánchez-León, 2014], while shells grow accretively, i.e. along the entire boundary [Goriely, 2017, Chapter 1.1.2]. This distinction is illustrated in Figure 1.

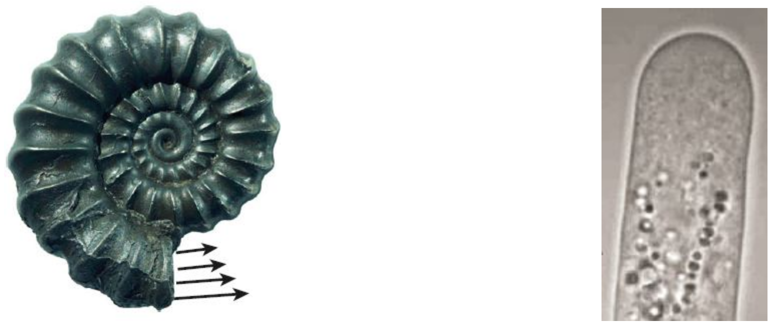

Figure 1: *Examples of accretive growth (left) in shells and tip growth (right) in a hair [Goriely, 2017, Figures 1.2 D, 1.3]. Note that the arrows on the left depict the growth direction.*

However, one aspect of cell growth is consistent: growth can only occur if there is an influx of material. This influx can originate from sources inside or outside of a cell. Examples of these sources include ambient nutrients absorbed by bacterial cells [Shuler et al., 1979] and vesicle supply centres (VSC) inside fungal hyphae [Tindemans et al., 2006]. Bacterial cells transform nutrients outside the cell into cellular material which causes the cell to grow. A VSC produces material inside the hypha near the tip, where the cell growth is most visible. While the supple centre supplies vesicles, it moves in the growth direction, which causes the tipped shape. However, if the supply centre does not initially move, tip growth is be preceded by accretive growth, as illustrated in Figure 2.

Most mathematical models of fungal hyphae focus on tip growth. These models study different distribution methods of vesicles [Nolet, 2020, Tindemans et al., 2006] or the cell structure [de Jong, 2019, Campàs and Mahadevan, 2009]. The simplest cell distribution method is that vesicles travel in straight lines, while it may be more realistic that they do not travel in predefined paths from the centre to the edge. In contrast, the simplest cell structure model is that vesicles at the wall directly expand the boundary, while it may be more realistic that there is a thin viscous membrane which grows and slowly hardens over time. Although these models extensively analyse tip growth, most models do not analyse accretive growth. One example that does incorporate accretive growth, also assumes the domain is spherical [Bartnicki-Garcia et al., 1989]. This paper uses the spherical domain as an initial domain from which tip growth is investigated. A spherical domain is an ideal situation, since natural domains are often not perfectly spherical. However, general domains could be approximated by a sphere. When modelling growth in general domains, the error from the approximation might persist or become larger over time. Therefore, this project will investigate the accretive growth for VSC type models. The tip growth models will first be explained in more detail, and will function as a basis for the accretive model.

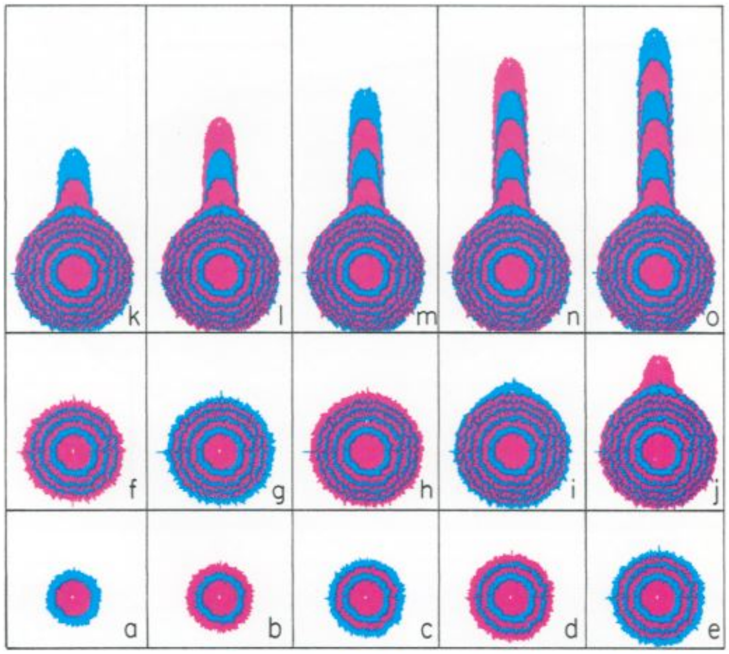

Figure 2: [Bartnicki-Garcia et al., 1989, Figure 2]*Simulation of accretive growth followed by tip growth. The VSC is initially (a-g) in the middle, after which it moves to the top and tip growth occurs (h-o).*

1.1 Previous tip growth models

The models of interest are the Ballistic Ageing Thin viscous Sheet (BATS) model by De Jong [de Jong, 2019] and the ballistic and diffusive VSC model by Nolet [Nolet, 2020]. These models both investigate tip growth in fungal hyphae, but are based on older models, such as found in [Bartnicki-Garcia et al., 1989, Eggen et al., 2011, Campàs and Mahadevan, 2009, Tindemans et al., 2006, Koch, 1982]. An overview of these models may be found in [Keijzer et al., 2010]. One geometrical assumption for tip growth is that the domain is cylindrically symmetric and extends infinitely behind the tip, as illustrated in Figure 3.

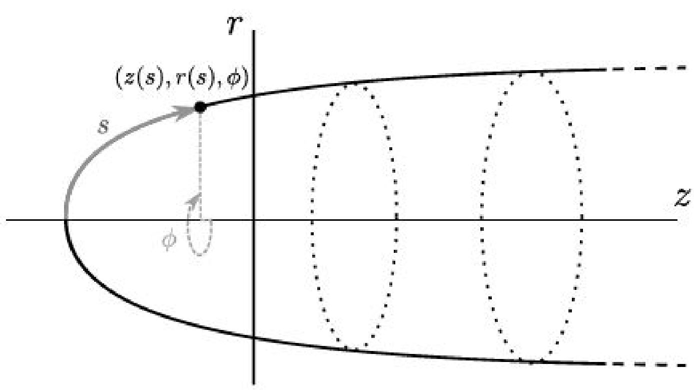

Figure 3: *Mathematical model interpretation of a hypha cell [de Jong, 2019], where the domain extends infinitely in one direction.*

The BATS model as stated by De Jong more accurately describes the biological process of tip growth at the boundary of the cell. It assumes that the cell consists of two parts: the cell and a thin viscous sheet at the boundary. The cell grows through ballistic dispersion of material from the supply centre near the tip. Once the material reaches the boundary, it is absorbed into the wall. This wall slowly hardens over time according to some prescribed viscosity function. While the cell grows, it shows steady tip growth, which is modelled as a travelling wave profile. This profile describes that the cell grows approximately at constant speed and preserves shape.

The model is used for both analytical and numerical results for tip growth. A numerical method is used to compute steady growth solutions. Using a ´topological shooting´ method, existence of solutions to a simplified model is shown. This result is then extended to the original statement, which produces conditions on the existence of steady tip growth solutions. These conditions are verified using a rigorous numerical method, and shows that solutions may be approximated by the topological shooting method.[de Jong, 2019]

The VSC models described by Nolet allows analysis for more types of distribution throughout the cell. Similar to the BATS model, the cell is assumed to have a travelling wave profile. In contrast to the BATS model, the thin viscous sheet is modelled as a flat surface. This simplification eliminates one difficulty so that others may be investigated in more detail. The distribution models focuses on the growth of the boundary, which is described by a flux. The flux can be chosen according to the type of dispersion method, such as ballistic or diffusive. In particular, the ballistic model assumes that vesicles travel in straight lines, while the diffusive model assumes that they travel through diffusive motion, as illustrated in Figure 4.

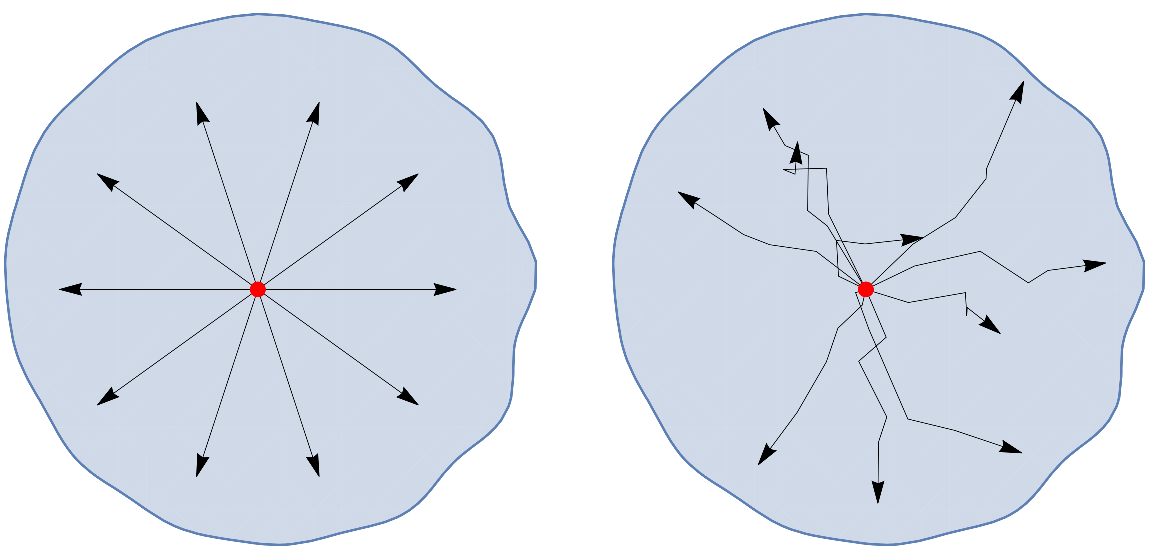

Figure 4: *Visualisation of ballistic (left) and diffusive (right) motion of vesicles from a supply centre (red dot) through a cell.*

Using this model, Nolet proves existence, uniqueness and linear stability of hyphoid solutions to the ballistic model, in addition to asymptotic results in the cylinder and the tip. The diffusive model is shown to have a solution that follows the travelling wave profile. and again asymptotic results away from the tip. Additionally, the diffusive model is analysed through numerical simulations which show the origin of a tip shaped cell, when starting from a spherical domain.

1.2 The accretive VSC models

This project will adapt the models described by Nolet [Nolet, 2020]; the boundary is modelled as a flat surface. However, instead of a domain that extends infinitely in one direction, the domain is assumed to be bounded and star-shaped with respect to the origin. This assumption differentiates the model from the tip growth models as it changes the geometry of the problem. Additionally, the problem is considered in two dimensions. This assumption can be interpreted as a cell between two large slates, similar to a microscope slide or inside a petri dish. Furthermore, vesicle production is assumed to be constant with rate $P > 0$. The setup for both VSC models will be analogous to the models by Nolet.

1.2.1 Mathematical model assumptions

Let $\Omega \subset \mathbb{R}^2$ be a star-shaped domain. Additionally, let $\psi(\theta)$ be such that $\psi(\theta)(\cos(\theta), \sin(\theta))$ is a parametrisation of $\partial\Omega$, where θ is the angle. Then we can write $\Omega = \Omega_\psi$ where

$$\Omega_\psi = \{0\} \cup \{(r, \theta) \mid \theta \in S^1, r \in (0, \psi(\theta))\}, \tag{1.1}$$

and $S^1 = \mathbb{R}\backslash(2\pi\mathbb{Z})$. Since Ω_ψ will change over time, we write $\Omega_\psi = \Omega_\psi(t)$. This time dependency also holds for ψ, so that $\psi = \psi(t, \theta)$. Using the boundary $\partial\Omega_\psi(t)$ we can analyse the evolution of the domain.

Assume that $\Gamma(t) \subset \partial\Omega_\psi(t)$ is a closed curve in the boundary with length $L(t)$. Additionally, we assume that there exists a flux Φ at the boundary which is constant over time. This flux can be used to calculate the change of $L(t)$ over time:

$$\frac{dL}{dt} = \int_{\Gamma(t)} \Phi \cdot n dS, \tag{1.2}$$

where n is the outward unit normal vector. In general, the flux is not defined orthogonal to the boundary, which is why we have to take the inner product with the normal.

Contrastingly, if we assume that $\Gamma(t)$ flows according to some velocity field $v(t)$, we can use the formula for first variation of area [Andrews et al., 2020, Lemma 5.27]. This concept relates the change of the length of $\Gamma(t)$ to its velocity and curvature by

$$\frac{dL}{dt} = \int_{\Gamma(t)} HV_n dS, \tag{1.3}$$

where H is the curvature of $\Gamma(t)$ and V_n the normal velocity of $\Gamma(t)$. Combining (1.3) and (1.2) yields

$$\int_{\Gamma(t)} HV_n - \Phi \cdot n dS = 0. \tag{1.4}$$

Since this equation holds for any arbitrary $\Gamma(t)$, we know that the integrand must vanish:

$$\Phi \cdot n = V_n H. \tag{1.5}$$

We can rewrite this equation by isolating the only time-dependent term V_n to

$$V_n = \frac{\Phi \cdot n}{H}. \tag{1.6}$$

As noted by Nolet, this equation can also describe several popular geometric flows. For example, mean curvature flow $V_n = H$ can be obtained by choosing the flux as $\Phi = H^2 n$. Similarly, inverse curvature flow $V_n = H^{-1}$, such as described in [Huisken and Ilmanen, 2001], can be obtained by $\Phi = n$.

For the ballistic and diffusive model, the quantities V_n, H, n and Φ can be expressed in terms of ψ. The expressions for n, H and V_n are given for both models by

$$n(\psi) = \frac{(\psi_\theta \sin\theta + \psi\cos\theta, -\psi_\theta\cos\theta + \psi\sin\theta)}{\sqrt{\psi_\theta^2 + \psi^2}} \tag{1.7}$$

$$H(\psi) = \frac{\psi^2 + 2\psi_\theta^2 - \psi\psi_{\theta\theta}}{(\psi^2 + \psi_\theta^2)^{3/2}}, \tag{1.8}$$

$$V_n(\psi) = \psi_t \frac{\psi}{\sqrt{\psi^2 + \psi_\theta^2}}. \tag{1.9}$$

The derivations of these quantities are found in Appendix A. The only quantity that remains is the flux, which changes with the choice of the dispersion method; there is a difference between the ballistic flux and the diffusive flux.

Ballistic flux

The ballistic VSC model assumes that vesicles are distributed in straight lines from the centre. Consequently, the flux Φ_B is given by

$$\Phi_B(\mathbf{x}) = \frac{P}{2\pi}\frac{\mathbf{x}}{|\mathbf{x}|^2},$$

for $\mathbf{x} \in \mathbb{R}^2$.

Since we need Φ_B evaluated at $\partial\Omega_\psi$, we can substitute $x = \psi\cos\theta, y = \psi\sin\theta$. This substitution yields

$$\Phi_B(\psi) = \frac{P}{2\pi}\frac{(\cos\theta, \sin\theta)}{\psi}. \tag{1.10}$$

Diffusive flux

The diffusive model assumes that the vesicles move in random paths from the centre to the boundary. This new model solves the unrealistic nature of the ballistic model, as vesicles usually do not travel in straight lines to their destination [Koch, 1994]. The randomness of the individual paths can be eliminated by regarding the concentration or density u of vesicles instead. The density should adhere to a diffusion equation. We assume that P is the total amount of surface area produced by the VSC per unit of time. Additionally, we assume that the diffusion time of vesicle is fast compared to the motion of the cell wall, such that the density is in equilibrium. Finally we assume that vesicles are fully absorbed into the cell wall. The concentration u therefore satisfies the Poisson equation with homogeneous boundary condition:

$$\begin{cases} \Delta u = -P\delta & \text{in } \Omega_\psi, \\ u = 0 & \text{on } \partial\Omega_\psi, \end{cases}, \tag{1.11}$$

where δ the Dirac delta distribution at zero. The normal component of the corresponding diffusive flux Φ_D is equal to the negative of the normal derivative of u since the concentration at the boundary is smaller than the concentration close to the centre. Thus we obtain $\Phi_D \cdot n = -\frac{\partial u}{\partial n}$.

1.3 Overview

This project will discuss the ballistic and diffusive VSC models for accretive growth on bounded domains. In Chapter 2 we will discuss the ballistic model. First we will analyse the simple case of a growing circle. After this analysis, we derive the evolution equation for generic perturbations ψ. This evolution equation will be used to show short time existence and uniqueness. Additionally, we will show that circular domains are stable over time using linear stability. Finally we will use maximum principles to show that solutions to the ballistic model satisfy an avoidance principle.

In Chapter 3 we will discuss the diffusive model. We first redefine the needed components for the evolution equations. As explained before, we use the concentration u of vesicles and (1.11) to define the flux Φ. We will use a diffeomorphism to transform the corresponding differential equation on Ω_ψ to one on Ω_1. The new expression will yield an implicit expression for Φ in terms of ψ. Using similar methods to Chapter 2 we can achieve short time and existence and uniqueness of solutions to the corresponding evolution equation. Moreover, we will show linear stability of circular solutions for the diffusive model. However, since the evolution equation is non-local, we cannot use maximum principles to show an avoidance principles.

2 Ballistic VSC problem

The first model we analyse is the ballistic VSC model. We begin in Section 2.1 by deriving the evolution equation for general perturbations ψ. Then we will analyse the behaviour on circular domains in Section 2.2. In Section 2.3 we continue to analyse the general setting. We will first show short-time existence and uniqueness of solutions to the derived evolution equation. Then we will show long-time linear stability of circular solutions in Section 2.4. Finally we will show that solutions satisfy an avoidance principle.

First we introduce some model assumptions. In particular, we assume that Ω_ψ is non-empty, bounded and strictly convex. These assumptions correspond to the following conditions on ψ:

- There exists a ν such that $\psi, H(\psi) > \nu$.
- ψ is twice differentiable on S^1.

Note that boundedness of ψ follows from continuity since S^1 is bounded.

2.1 The evolution equation

To derive the evolution equation for ψ from (1.6), we can substitute the expressions for Φ, n, H, V_n to obtain

$$V_n(\psi)H(\psi) = \psi_t \frac{\psi}{\sqrt{\psi^2+\psi_\theta^2}} \frac{\psi^2 + 2\psi_\theta^2 - \psi\psi_{\theta\theta}}{(\psi^2+\psi_\theta^2)^{3/2}} = \psi_t \frac{\psi(\psi^2 + 2\psi_\theta^2 - \psi\psi_{\theta\theta})}{(\psi^2+\psi_\theta^2)^2}$$

$$\begin{aligned}\Phi(\psi)\cdot n(\psi) &= \frac{P}{2\pi}\frac{(\cos\theta,\sin\theta)}{\psi}\cdot\frac{(\psi\cos\theta+\psi_\theta\sin\theta, -\psi_\theta\cos\theta+\psi\sin\theta)}{\sqrt{\psi_\theta^2+\psi^2}}\\ &= \frac{P}{2\pi}\frac{1}{\psi}\cdot\frac{\psi\cos^2\theta+\psi_\theta\sin\theta\cos\theta-\psi_\theta\cos\theta\sin\theta+\psi\sin^2\theta}{\sqrt{\psi_\theta^2+\psi^2}}\\ &= \frac{P}{2\pi}\frac{1}{\psi}\frac{\psi}{\sqrt{\psi_\theta^2+\psi^2}} = \frac{P}{2\pi}\frac{1}{\sqrt{\psi_\theta^2+\psi^2}}.\end{aligned}$$

These expressions yield the following evolution equation for ψ:

$$\psi_t = F(\psi) := \frac{P}{2\pi}\frac{(\psi_\theta^2+\psi^2)^{3/2}}{\psi(\psi^2+2\psi_\theta^2-\psi\psi_{\theta\theta})}. \tag{2.1}$$

Observe that F is not a linear operator in ψ. Since there are multiple types of nonlinearity, we will briefly elaborate on this subject. Nonlinear partial differential equations are usually divided into semilinear, quasilinear and fully nonlinear. The types can be found in [Evans, 2010, Page 2]. We see that F is fully nonlinear in ψ, since it depends nonlinearly on the highest derivative $\psi_{\theta\theta}$. Thus we will have to apply techniques for fully nonlinear problems to analyse the evolution equation.

The complete problem statement requires an initial condition for ψ. Adding the condition $\psi(0,\theta) = \psi_{in}(\theta)$ for $\theta \in S^1$, we obtain the complete statement

$$\begin{cases}\partial_t\psi(t,\theta) = F(\psi(t,\theta)), & (t,\theta)\in(0,\infty)\times S^1\\ \psi(0,\theta) = \psi_{in}(\theta), & \theta\in S^1\end{cases} \tag{2.2}$$

Before we analyse the general case, we will first apply the model to circles.

2.2 Simple example: circles

For circular domains, we have that the corresponding ψ_c as defined in 1.2 does not depend on θ. The independence of θ means that we can simplify (2.1) to

$$F(\psi_c) = \frac{P}{2\pi}\frac{(\psi_c^2)^{3/2}}{\psi_c^3} = \frac{P}{2\pi}. \tag{2.3}$$

This simple expression means that the evolution equation for ψ_c becomes

$$\frac{d\psi_c}{dt} = \frac{P}{2\pi},$$

which implies that (2.2) is solved by

$$\psi_c(t) = \psi_s + \frac{P}{2\pi}t, \tag{2.4}$$

where we assume that $\psi_c(0) = \psi_{in}(\theta) \equiv \psi_s$. This solution shows that circular domains stay circular and grow linearly over time.

2.3 General case: local existence and uniqueness

In this section, we will show local existence and uniqueness of solutions to (2.2). We note that the evolution equation is a fully nonlinear parabolic partial differential equation. For fully nonlinear problems, (local) existence and uniqueness of solutions is relatively difficult to show, when compared to other types of problems. For example, Theorem 2.1 gives conditions for which local existence and uniqueness of solutions holds.

Theorem 2.1. *[Lunardi, 1995, Theorem 8.1.1] Let D be a Banach space, with norm $||\cdot||_D$, continuously embedded in X, and $\mathcal{O} \subset D$ be an open set. Let $F : [0,T] \times \mathcal{O} \mapsto X$ a sufficiently smooth function, non-linear function, with $T \in (0,\infty)$. Consider the initial value problems*

$$u'(t) = F(t,u(t)), t > 0, u(0) = u_0, \tag{2.5}$$

for some $u_0 \in \mathcal{O}$.

Assuming that the following conditions are satisfied

(i) $(t,u) \mapsto F(t,u)$ is continuous with respect to (t,u), and it is Fréchet differentiable with respect to u, with derivative $DF(t,u)$.

(ii) For every $t \in [0,T]$ and $v \in \mathcal{O}$, the Fréchet derivative $DF(t,v)$ is sectorial in X, and its graph norm is equivalent to the norm of D.

(iii) There exists $\alpha \in (0,1)$ such that for all $\overline{u} \in \mathcal{O}$ there are $R = R(\overline{u}), L = L(\overline{u}), K = K(\overline{u}) > 0$ verifying

$$||DF(t,v) - DF(t,w)||_{L(D,X)} \le L||v-w||_D$$
$$||F(t,u) - F(s,u)||_X + ||DF(t,u) - DF(s,u)||_{L(D,X)} \le K|t-s|^\alpha,$$

for all $t,s \in [0,T], u,v,w \in B(\overline{u},R) \subset D$.

Fix $\bar{t} \in [0,T], \overline{u} \in \mathcal{O}$ such that $F(\bar{t},\overline{u}) \in \overline{D}$. Then there are $\delta = \delta(\bar{t},\overline{u}) > 0, r = r(\bar{t},\overline{u}) > 0$ such that

(i) For every $t_0 \in [\bar{t}-r,\bar{t}+r] \cap [0,T]$ and $x_0 \in \mathcal{O}$ such that $F(t_0,x_0) \in \overline{D}$ and $||x_0 - \overline{u}||_D \le r$, there is a strict solution $u \in C([t_0,t_0+\delta];D) \cap C^1([t_0,t_0+\delta];X)$ to Equation (2.5) in $[t_0,t_0+\delta]$.

(ii) u belongs to $C_\alpha^\alpha((t_0, t_0+\delta]; D)$, u' belongs to $B_\alpha((t_0, t_0+\delta]; (X, D)_{\alpha,\infty})$, and in addition

$$\lim_{\varepsilon\to 0} \varepsilon^\alpha [u]_{C^\alpha([t_0+\varepsilon, t_0+2\varepsilon]; D)} = 0.$$

Moreover, the solution u is unique and belongs to

$$\bigcup_{0<\beta<1} C_\beta^\beta((t_0, t_0+\delta]; D) \cap C([t_0, t_0+\delta]; D).$$

This theorem shows local existence and uniqueness of solutions for more general evolution equations than parabolic partial differential equations. We will apply this theorem to (2.2) for appropriate choices of $D, \mathcal{O}$ and X.

For D and X, we need a Banach space D that is continuously embedded in X. Additionally, we will require that D is dense in X and both are Banach algebras. Density will ensure that the condition $F(t,u) \in \overline{D}$ in the theorem is automatically satisfied. The Banach algebra will ensure that our definition of F makes sense and prove useful when calculating Fréchet derivatives, as we will see later.

A practical requirement on D is that it involves twice differentiable functions. Examples of these spaces are the Hölder space $C^{2+\gamma}(S^1)$, Sobolev space $W^{2,p}(S^1)$ and little Hölder space $h^{2+\gamma}(S^1)$. We choose $D = h^{2+\gamma}(S^1)$ and $X = h^\gamma(S^1)$, for some fixed $\gamma \in (0,1)$. This space satisfies the additional constraints we set on D. In particular, we know that $h^{2+\gamma}(S^1)$ lies dense in $h^\gamma(S^1)$ where $C^{2+\gamma}(S^1)$ does not lie dense in $C^\gamma(S^1)$. Furthermore, the Hölder spaces are Banach algebras, while Sobolev spaces with $p < \infty$ are not. We note that $C^2(S^1)$ and $C(S^1)$ for D and X are also usable in the ballistic model, however not for the diffusive model. To keep the analysis in the diffusive model similar to the ballistic model, we will choose D and X as the little Hölder spaces.

For $\mathcal{O} \subset D$ we will use the modelling assumptions as stated earlier, in addition to assuming that every $\psi \in \mathcal{O}$ is uniformly bounded in the $2+\gamma$ norm by some fixed $R^* > 0$. Note that these choices for $D, \mathcal{O}$ and X satisfy the previously mentioned constraints. In summary, we have

$$\begin{aligned} D &= h^{2+\gamma}(S^1), \\ \mathcal{O} &= \{\psi \in h^{2+\gamma}(S^1) \mid \nu < \psi, ||\psi||_{2+\gamma} < R^*, H(\psi) > \nu\} \\ X &= h^\gamma(S^1) \end{aligned} \tag{2.6}$$

with $\gamma \in (0,1)$. The spaces $h^\gamma, h^{2+\gamma}$ are endowed with norms $||\cdot||_\gamma, ||\cdot||_{2+\gamma}$ defined by

$$\begin{aligned} ||\psi||_\gamma &= ||\psi||_\infty + [\psi]_\gamma \\ ||\psi||_{2+\gamma} &= ||\psi||_\infty + ||\psi_\theta||_\infty + ||\psi_{\theta\theta}||_\infty + [\psi_{\theta\theta}]_\gamma, \\ [\psi]_\gamma &= \sup_{x,y\in S^1, x\neq y} \left(\frac{|\psi(x)-\psi(y)|}{d_{S^1}(x,y)^\gamma} \right). \end{aligned}$$

We note that the distance $d_{S^1}(x,y)$ on S^1 is different to the distance on $\mathbb{R}$. The main difference is that we choose the shortest distance on S^1, instead of the distance on $\mathbb{R}$. Thus for any $x, y \in S^1$ we have that the distance $d_{S^1}(x,y)$ is given by

$$d_{S^1}(x,y) = \inf_{k\in\mathbb{Z}} |[x] + 2k\pi - [y]|. \tag{2.7}$$

where $[x], [y] \in [0, 2\pi)$ denote the representatives of x, y.

As stated by the theorem, we will use the concept of Fréchet differentiability and sectoriality. We first discuss some preliminaries about these concepts before we continue.

2.3.1 Preliminaries Theorem 2.1

We will first discuss what Fréchet differentiability entails. The definition of a Fréchet derivative is given in Definition 2.2.

Definition 2.2. [Cheney, 2001, Page 115] Let f be a mapping from an open set D in a normed linear space X into a normed linear space Y. Let $x \in D$. If there exists a bounded linear operator $A : X \to Y$ such that

$$\lim_{||h||_X \to 0} \frac{||f(x+h) - f(x) - Ah||_Y}{||h||_X} = 0, \tag{2.8}$$

then f is said to be Fréchet differentiable at x. We write the Fréchet derivative of f at x in direction h as $Df(x)[h]$.

The Fréchet derivative has several properties as described in Lemma 2.3.

Lemma 2.3. *The following statements are true.*

(i) (Linearity) [Cartan, 1971, Proposition 2.3.1]: Let X, Y be normed vector spaces and D an open subset in X. Let $f, g : D \to Y$ be Fréchet differentiable with derivatives Df, Dg. Then $f + g$ is Fréchet differentiable at $x \in D$ with derivative

$$D(f+g)(x)[h] = Df(x)[h] + Dg(x)[h],$$

for $h \in X$.

(ii) (Product rule) [Cartan, 1971, Page 34]: Let X be a normed vector space and Y a Banach algebra and D an open subset in X. Let $f, g : D \to Y$ be Fréchet differentiable with derivatves Df, Dg. Then fg is Fréchet differentiable at $x \in D$ with derivative

$$D(f \cdot g)(x)[h] = Df(x)[h]g(x) + f(x)Dg(x)[h],$$

for $h \in X$.

(iii) (Chain rule) [Cartan, 1971, Theorem 2.2.1]: Let X, Y, Z be normed vector spaces and $D_f \subset X, D_g \subset Y$ open subsets. Let $f : D_f \to D_g, g : D_g \to Z$ be Fréchet differentiable with derivatives Df, Dg. Then $g \circ f : D_f \to Z$ is Fréchet differentiable at $x \in D_f$ with derivative

$$D(g \circ f)(x)[h] = Dg(f(x))[Df(x)[h]],$$

for $h \in X$.

(iv) (Derivative of inverses) [Cartan, 1971, Theorem 2.4.4]: Let X, Y be two Banach spaces. Let $f : Isom(X,Y) \subset \mathcal{L}(X,Y) \to \mathcal{L}(Y,X) : x \mapsto x^{-1}$. Then f is Fréchet differentiable at any $x \in Isom(X,Y)$ with derivative $Df \in \mathcal{L}(\mathcal{L}(X,Y), \mathcal{L}(Y,X))$ given by

$$Df(x)[h] = -x^{-1} \circ h \circ x^{-1},$$

for $h \in L(X,Y)$.

Note that the third and fourth property can be combined to achieve Fréchet differentiability of reciprocals of functions that stay away from 0. Some examples of commonly used functions and their derivatives can be found in Corollary 2.4.

Corollary 2.4. *Consider $f_{1,2,3} : \mathcal{O} \to h^\gamma(S^1)$ defined by*

$$f_1(\psi) = \psi^n,$$
$$f_2(\psi) = \sqrt{\psi},$$
$$f_3(\psi) = \partial_\theta^k \psi,$$

where $n \in \mathbb{N}$ and ∂_θ^k denotes k-th partial derivative of ψ for $k = 1, 2$. Then $f_{1,2,3}$ are Fréchet differentiable at $\psi_0 \in \mathcal{O}$ with derivatives

$$Df_1(\psi_0)[h] = n\psi_0^{n-1} h,$$
$$Df_2(\psi_0)[h] = \frac{1}{2}\psi_0^{-1/2} h,$$
$$Df_3(\psi_0)[h] = \partial_\theta^k h.$$

The proof to this corollary can be found in Appendix B.1. Next we will give the definition of a sectorial operator.

Definition 2.5. [Lunardi, 1995, Definition 2.0.1] Let X be a complex Banach space and $A : D(A) \subset X \to X$ a linear operator. Then A is sectorial if there exist constants $\omega \in \mathbb{R}, \theta \in (\pi/2, \pi), M > 0$ such that

$$\begin{aligned} (i) \quad & \rho(A) \supset S_{\theta,\omega} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}, \\ (ii) \quad & ||R(\lambda, A)||_{L(X)} \leq \frac{M}{|\lambda - \omega|} \ \forall \lambda \in S_{\theta,\omega}. \end{aligned} \tag{2.9}$$

Now we will continue to proving that F satisfies the assumptions for Theorem 2.1. The first property we will show is Fréchet differentiability.

2.3.2 Fréchet differentiability

In Lemma 2.6 we show that F is Fréchet differentiable.

Lemma 2.6. *The function $F(\psi)$ as given in (2.1) is Fréchet differentiable with respect to ψ, with Fréchet derivative $DF(\psi_0)[h]$ at ψ_0 in direction h given by*

$$DF(\psi_0)[h] = a_0(\psi_0)h + a_1(\psi_0)h_\theta + a_2(\psi_0)h_{\theta\theta}. \tag{2.10}$$

where

$$\begin{aligned} a_0(\psi_0) &= \frac{\frac{3P}{2\pi}(\psi_0^2 + \psi_{0,\theta}^2)^{1/2}\psi_0 - F(\psi_0)(3\psi_0^2 + 2\psi_{0,\theta}^2 - 2\psi_0\psi_{0,\theta\theta})}{\psi_0^3 + 2\psi_0\psi_{0,\theta}^2 - \psi_0^2\psi_{0,\theta\theta}}, \\ a_1(\psi_0) &= \frac{\frac{3P}{2\pi}(\psi_0^2 + \psi_{0,\theta}^2)^{1/2}\psi_{0,\theta} - 4F(\psi_0)\psi_{0,\theta}\psi_0}{\psi_0^3 + 2\psi_0\psi_{0,\theta}^2 - \psi_0^2\psi_{0,\theta\theta}}, \\ a_2(\psi_0) &= \frac{F(\psi_0)\psi_0^2}{\psi_0^3 + 2\psi_0\psi_{0,\theta}^2 - \psi_0^2\psi_{0,\theta\theta}}. \end{aligned} \tag{2.11}$$

Proof. Let $\psi_0 \in \mathcal{O}$ and $h \in D$ be fixed. Using the properties in Lemma 2.3, we can establish Fréchet differentiability of F. We will accomplish this by first showing that both the denominator and numerator are Fréchet differentiable, after which we will use a quotient rule to achieve differentiability of F.

First we establish differentiability of the denominator $d_F : \mathcal{O} \to h^\gamma(S^1)$. The denominator is given by

$$d_F(\psi) = \psi^3 + 2\psi\psi_\theta^2 - \psi^2\psi_{\theta\theta}.$$

By Corollary 2.4 we know that powers and partial derivatives of ψ are differentiable. With the sum and product rule, we know that d_F is Fréchet differentiable, with derivative

$$\begin{aligned} D(d_F)(\psi_0)[h] &= 3\psi_0^2 h + 2\psi_{0,\theta}^2 h + 4\psi_{0,\theta}\psi_0 h_\theta - 2\psi_0\psi_{0,\theta\theta} h - \psi_0^2 h_{\theta\theta} \\ &= (3\psi_0^2 + 2\psi_{0,\theta}^2 - 2\psi_0\psi_{0,\theta\theta})h + 4\psi_{0,\theta}\psi_0 h_\theta - \psi_0^2 h_{\theta\theta} \end{aligned}$$

For the numerator $n_F : \mathcal{O} \to h^\gamma(S^1)$, we have

$$n_F(\psi) = \frac{P}{2\pi}(\psi^2 + \psi_\theta^2)^{3/2} = \frac{P}{2\pi}(\psi^2 + \psi_\theta^2)\sqrt{\psi^2 + \psi_\theta^2}.$$

We know that $\psi^2 + \psi_\theta^2$ is Fréchet differentiable. Therefore, we can use the chain rule and product rule to find that n_F is Fréchet differentiable at ψ_0 with derivative

$$\begin{aligned} D(n_F)(\psi_0)[h] &= \frac{3P}{4\pi}(\psi_0^2 + \psi_{0,\theta}^2)^{1/2}(2\psi_{0,\theta}h_\theta + 2\psi_0 h) \\ &= \frac{3P}{2\pi}(\psi_0^2 + \psi_{0,\theta}^2)^{1/2}(\psi_0 h + \psi_{0,\theta}h_\theta). \end{aligned}$$

Next we want to combine these results to achieve differentiability of F. We note that there exists some $\nu^* > 0$ such that $d_F(\psi) \geq \nu^* > 0$ since $\psi \in \mathcal{O}$. Therefore, we can apply the third and fourth property of Lemma 2.3 to obtain that d_F^{-1} is differentiable at ψ_0 with derivative

$$Dd_F^{-1}(\psi_0)[h] = -d_F(\psi_0)^{-1} Dd_F(\psi_0)[h] d_F(\psi_0)^{-1} = -d_F(\psi_0)^{-2} Dd_F(\psi_0)[h].$$

Finally, we can use the product rule to find that F is Fréchet differentiable with derivative

$$\begin{aligned} DF(\psi_0)[h] &= D(n_F/d_F)(\psi_0)[h] \\ &= D(n_F)(\psi_0)[h] d_F(\psi_0)^{-1} - D(d_F)(\psi_0)[h] n_F(\psi_0) d_F(\psi_0)^{-2} \\ &= D(n_F)(\psi_0)[h] d_F(\psi_0)^{-1} - D(d_F)(r)[h] F(\psi_0) d_F(\psi_0)^{-1} \\ &= (D(n_F)(\psi_0)[h] - D(d_F)(\psi_0)[h] F(\psi_0)) d_F(\psi_0)^{-1} \\ &= \frac{1}{\psi_0^3 + 2\psi_0\psi_{0,\theta}^2 - \psi_0^2\psi_{0,\theta\theta}} \left(\frac{3P}{2\pi}(\psi_0^2 + \psi_{0,\theta}^2)^{1/2}(\psi_0 h + \psi_{0,\theta}h_\theta) \right) \\ &\quad - \frac{F(\psi_0)}{\psi_0^3 + 2\psi_0\psi_{0,\theta}^2 - \psi_0^2\psi_{0,\theta\theta}} \left((3\psi_0^2 + 2\psi_{0,\theta}^2 - 2\psi_0\psi_{0,\theta\theta})h + 4\psi_{0,\theta}\psi_0 h_\theta - \psi_0^2 h_{\theta\theta} \right) \\ &= \frac{\frac{3P}{2\pi}(\psi_0^2 + \psi_{0,\theta}^2)^{1/2}\psi_0 - F(\psi_0)(3\psi_0^2 + 2\psi_{0,\theta}^2 - 2\psi_0\psi_{0,\theta\theta})}{\psi_0^3 + 2\psi_0\psi_{0,\theta}^2 - \psi_0^2\psi_{0,\theta\theta}} h \\ &\quad + \frac{\frac{3P}{2\pi}(\psi_0^2 + \psi_{0,\theta}^2)^{1/2}\psi_{0,\theta} - 4F(\psi_0)\psi_{0,\theta}\psi_0}{\psi_0^3 + 2\psi_0\psi_{0,\theta}^2 - \psi_0^2\psi_{0,\theta\theta}} h_\theta \\ &\quad + \frac{F(\psi_0)\psi_0^2}{\psi_0^3 + 2\psi_0\psi_{0,\theta}^2 - \psi_0^2\psi_{0,\theta\theta}} h_{\theta\theta} \\ &= a_0(\psi_0)h + a_1(\psi_0)h_\theta + a_2(\psi_0)h_{\theta\theta}. \end{aligned}$$

Note that $DF(\psi_0)$ is a bounded linear operator, since $a_i(\psi_0) \in h^\gamma(S^1)$ from Lemma B.2, Lemma B.3 and definition of $\mathcal{O}$. $\square$

Lemma 2.6 shows that the first condition for Theorem 2.1 is met. Before we continue to the other conditions, we will show some properties of $a_{0,1,2}$ in Lemma 2.7 which we will need later.

Lemma 2.7. *The following statements about a_0, a_1, a_2 are true.*

(i) There exists a $\xi > 0$ such that $a_2(\psi_0) \geq \xi > 0$ for all $\psi_0 \in \mathcal{O}$;

(ii) The functions $a_0(\psi_0), a_1(\psi_0), a_2(\psi_0)$ are locally Lipschitz continuous with respect to ψ_0;

Proof. **Positivity of a_2**

We can rewrite a_2 so that

$$a_2 = \frac{F(\psi_0)\psi_0^2}{\psi_0^3 + 2\psi_0\psi_{0,\theta}^2 - \psi_0^2\psi_{0,\theta\theta}} = \frac{(\psi_0^2 + \psi_{0,\theta}^2)^{3/2}}{H(\psi_0)^2}.$$

Since ψ_0 is bounded in $h^{2+\gamma}(S^1)$ by R^*, we know that the curvature $H(\psi)$ must be bounded in $h^\gamma(S^1)$ by some $\lambda > 0$ independent of ψ. This bound implies that

$$a_2 \geq \frac{(\psi_0^2 + \psi_{0,\theta}^2)^{3/2}}{\lambda^2} \geq \psi_0^3\lambda^{-2}.$$

Additionally, since $\psi_0 \geq \nu$, we obtain $a_2 \geq \nu^3\lambda^{-2} > 0$.

Local Lipschitz continuity of a_0, a_1, a_2 with respect to ψ_0

Let $\overline{u} \in \mathcal{O}$ and ψ_0, ψ_1 in a ball $B(\overline{u}, R)$ with $R = R(\overline{u})$ such that $B(\overline{u}, R) \subset \mathcal{O}$. We want to show that there exists a constant L_i such that

$$||a_i(\psi_0) - a_i(\psi_1)||_\gamma \leq L_i||\psi_0 - \psi_1||_{2+\gamma}.$$

Since $\psi_{0,1}, H(\psi_{0,1}) > \nu$ and $||\psi_i||_{2+\gamma} < R^*$, we know by Lemma B.2 that there exists a $C_{R^*,\nu,\lambda}$ such that

$$\begin{aligned} ||a_i(\psi_0) - a_i(\psi_1)||_\gamma &= \left|\left|\frac{n_{a_i}(\psi_0)}{\psi_0 H(\psi_0)} - \frac{n_{a_i}(\psi_1)}{\psi_1 H(\psi_1)}\right|\right|_\gamma \\ &\leq C_{R^*,\nu,\lambda}||n_{a_i}(\psi_0) - n_{a_i}(\psi_1)||_\gamma, \end{aligned}$$

where n_{a_i} denote the numerators of a_i. Using the properties of Fréchet differentiation, we obtain that each numerator n_{a_i} is differentiable at $\psi_* \in \mathcal{O}$ with derivatives

$$\begin{aligned} Dn_{a_1}(\psi_*)[h] &= \frac{3P}{2\pi}\psi_*(\psi_*^2 + \psi_{*,\theta}^2)^{-1/2}(\psi_* h + 2\psi_{*,\theta}h_\theta)) + \frac{3P}{2\pi}(\psi_*^2 + \psi_{*,\theta}^2)^{1/2}h \\ &\quad - (a_0(\psi_*)h + a_1(\psi_*)h_\theta + a_2(\psi_*)h_{\theta\theta})(3\psi_*^2 + 2\psi_{*,\theta}^2 - 2\psi_*\psi_{*,\theta\theta}) \\ &\quad - F(\psi_*)(6\psi_* h + 4\psi_{*,\theta}h_\theta - 2\psi_{*,\theta\theta}h - 2\psi_* h_{\theta\theta}) \\ &=: a_{00}(\psi_*)h + a_{01}(\psi_*)h_\theta + a_{02}(\psi_*)h_{\theta\theta}, \\ Dn_{a_2}(\psi_*)[h] &= \frac{3P}{2\pi}\psi_{*,\theta}(\psi_*^2 + \psi_{*,\theta}^2)^{-1/2}(\psi_* h + 2\psi_{*,\theta}h_\theta)) + \frac{3P}{2\pi}(\psi_*^2 + \psi_{*,\theta}^2)^{1/2}h_\theta \\ &\quad - 4\psi_{*,\theta}\psi_*(a_0(\psi_*)h + a_1(\psi_*)h_\theta + a_2(\psi_*)h_{\theta\theta}) \\ &\quad - 4F(\psi_*)(\psi_* h_\theta + \psi_{*,\theta}h), \\ &=: a_{10}(\psi_*)h + a_{11}(\psi_*)h_\theta + a_{12}(\psi_*)h_{\theta\theta} \\ Dn_{a_3}(\psi_*)[h] &= 2\psi_* F(\psi_*)h + \psi_*^2(a_0(\psi_*)h + a_1(\psi_*)h_\theta + a_2(\psi_*)h_{\theta\theta}) \\ &=: a_{20}(\psi_*)h + a_{21}(\psi_*)h_\theta + a_{22}(\psi_*)h_{\theta\theta}. \end{aligned}$$

Note that each coefficient $a_{ij} \in h^\gamma(S^1)$. Additionally, the operator norm of $Dn_{a_i}(\psi_*)$ can be bounded independently of ψ_* since $||\psi_*||_{2+\gamma} < R^*$. Thus we see that all n_{a_i} are locally Lipschitz continuous [Cartan, 1971, Theorem 3.3.2]. Consequently, we have that a_i are locally Lipschitz continuous. □

2.3.3 Sectoriality and equivalence of the graph norm

With Lemma 2.7 we will show that F satisfies the second condition in Lemma 2.10. Sectoriality will be shown in several steps. We will use 2π-periodic continuations $\overline{f}$ of functions f that take arguments from S^1. These periodic continuations are defined such that $\overline{f}$ is 2π-periodic and for all $[x] \in [0, 2\pi)$ we have $\overline{f}([x]) = f(x)$, where $[x]$ is the representant of $x \in S^1$.

First we will show in Lemma 2.8 that the $h^\gamma(\mathbb{R})$ norm of continuations $\overline{f}$ are equal to the $h^\gamma(S^1)$ norm of f. We will use this continuation to translate our problem to the setting of 2π-periodic functions in $h^\gamma(\mathbb{R})$. Then we will show sectoriality of the translated operator in $h^\gamma(\mathbb{R})$ in Lemma 2.9. Finally we will show sectoriality in the original space in Lemma 2.10.

Lemma 2.8. *Let f be a function in $C^\gamma(S^1)$ and $\overline{f}$ be the 2π-periodic continuation. Then $\overline{f} \in C^\gamma(\mathbb{R})$ and*

$$||\overline{f}||_{C^\gamma(\mathbb{R})} = ||f||_{C^\gamma(S^1)}. \tag{2.12}$$

Proof. Let f be a function in $C^\gamma(S^1)$ and $\overline{f}$ its 2π-periodic continuation. We note that by periodicity we obtain

$$||\overline{f}|_{C(\mathbb{R})} = \sup_{x\in\mathbb{R}} |\overline{f}(x)| = \sup_{x\in[0,2\pi)} |\overline{f}(x)| = \sup_{y\in S^1} |f(y)| = ||f||_{C(S^1)}.$$

Next we will show that the Hölder semi-norm are equal. We know that for all $x, y \in S^1$, we have that $d_{S^1}(x,y) \le |[x]-[y]|$. Now let $x, y \in \mathbb{R}$ and $m, n \in \mathbb{Z}$ such that $x - 2m\pi, y - 2n\pi \in [0, 2\pi)$ are representants of $\hat{x}, \hat{y} \in S^1$. Note that $|x - y| \ge d_{S^1}(\hat{x}, \hat{y})$. Then we know that

$$\frac{|\overline{f}(x) - \overline{f}(y)|}{|x-y|^\gamma} = \frac{|\overline{f}(x - 2m\pi) - \overline{f}(y - 2n\pi)|}{|x-y|^\gamma} \le \frac{|f(\hat{x}) - f(\hat{y})|}{d_{S^1}(\hat{x},\hat{y})^\gamma}.$$

Thus we know that $[\overline{f}]_{C^\gamma(\mathbb{R})} \le [f]_{C^\gamma(S^1)}$.

Next we let $x, y \in S^1$ with representants $[x], [y] \in [0, 2\pi)$. If $|[x] - [y]| \le \pi$, we know that $d_{S^1}(x,y) = |[x]-[y]|$ by the definition of the distance on S^1. Thus we know that

$$\frac{|f(x) - f(y)|}{d_{S^1}(x,y)^\gamma} = \frac{|\overline{f}([x]) - \overline{f}([y])|}{|[x]-[y]|^\gamma}$$

If $|[x] - [y]| > \pi$ we know that

$$d_{S^1}(x,y) = |[x] - [y] \pm 2\pi|.$$

Using this result and the periodicity of f we obtain

$$\frac{|f(x) - f(y)|}{d_{S^1}(x,y)^\gamma} = \frac{|\overline{f}([x] \pm 2\pi) - \overline{f}([y])|}{|([x] - [y] \pm 2\pi)|^\gamma} = \frac{|\overline{f}(\overline{x}) - \overline{f}(\overline{y})|}{|\overline{x} - \overline{y}|^\gamma},$$

where $\overline{x} = [x] \pm 2\pi, \overline{y} = [y] \in \mathbb{R}$. Therefore, for every $x, y \in S^1, x \ne y$ there is a pair $\overline{x}, \overline{y} \in [-2\pi, 4\pi), \overline{x} \ne \overline{y}$ with

$$\frac{|f(x) - f(y)|}{d_{S^1}(x,y)^\gamma} = \frac{|\overline{f}(\overline{x}) - \overline{f}(\overline{y})|}{|\overline{x} - \overline{y}|^\gamma}.$$

We can use this result to obtain

$$[f]_{C^\gamma(S^1)} = \sup_{x,y\in S^1, x\ne y} \frac{|f(x) - f(y)|}{|x-y|^\gamma_{S^1}} \le \sup_{\overline{x},\overline{y}\in[-2\pi,4\pi), \overline{x}\ne\overline{y}} \frac{|\overline{f}(\overline{x}) - \overline{f}(\overline{y})|}{|\overline{x} - \overline{y}|^\gamma} \le \sup_{\overline{x},\overline{y}\in\mathbb{R}, \overline{x}\ne\overline{y}} \frac{|\overline{f}(\overline{x}) - \overline{f}(\overline{y})|}{|\overline{x} - \overline{y}|^\gamma} = [\overline{f}]_{C^\gamma(\mathbb{R})}.$$

Consequently, we know that $[f]_{C^\gamma(S^1)} \le [\overline{f}]_{C^\gamma(\mathbb{R})}$. Thus we know that $||\overline{f}||_{C^\gamma(\mathbb{R})} = ||f||_{C^\gamma(S^1)}$, so $\overline{f} \in C^\gamma(\mathbb{R})$. □

We note that this lemma also holds if we replace C^γ with $h^{k+\gamma}$. Next we define the periodic continuation $\overline{DF}(\psi_0)$ of $DF(\psi_0)$ as $\overline{DF}(\psi_0)[h] = \overline{a_2(\psi_0)}h_{\theta\theta} + \overline{a_1(\psi_0)}h_\theta + \overline{a_0(\psi_0)}h$ where $\overline{a_i(\psi_0)}$ are the 2π-periodic continuations of $a_i(\psi_0)$.

Lemma 2.9. *Let DF be the Fréchet derivative of F as defined in* (2.10)*. Then $\overline{DF(\psi_0)}$ is sectorial in $h^\gamma(\mathbb{R})$ for all $\psi_0 \in \mathcal{O}$.*

Proof. Fix $\psi_0 \in \mathcal{O}$. By definition of $\overline{DF(\psi_0)}$, and Lemma 2.8, we can apply Theorem 3.1.14 [Lunardi, 1995] which gives sectoriality of $\overline{DF(\psi_0)}$ in $h^\gamma(\mathbb{R})$. □

Lemma 2.10. *The Fréchet derivative $DF(\psi_0)$ is sectorial in $h^\gamma(S^1)$ for all $\psi_0 \in \mathcal{O}$.*

Proof. Fix $\psi_0 \in \mathcal{O}$. First we will show that the resolvent sets of $\overline{DF(\psi_0)}$ is contained in the resolvent set of $DF(\psi_0)$. Let λ be in the resolvent set $\rho(\overline{DF(\psi_0)})$ and $g \in h^{2+\gamma}(S^1)$. We want to show that there exists a $u \in h^{2+\gamma}(S^1)$ such that $\lambda u - DF(\psi_0)[u] = g$.
We know there exists a $\overline{u} \in h^{2+\gamma}(\mathbb{R})$ such that $\lambda\overline{u} - \overline{DF(\psi_0)}[\overline{u}] = \overline{g}$, where $\overline{g}$ is the periodic continuation of g.

Next we will show that $\overline{u}$ is 2π-periodic so that there exists a $u \in h^{2+\gamma}(S^1)$ which has periodic continuation $\overline{u}$. Since $\overline{g}$ is 2π-periodic, we have $\overline{g}(\theta + 2\pi) = \overline{g}(\theta)$. By linearity of $DF(\psi_0)$, we know that $\lambda I - \overline{DF(\psi_0)}$ is a linear operator, so that

$$\begin{aligned} \lambda\overline{u}(\theta) - \overline{DF(\psi_0)}[\overline{u}(\theta)] = \overline{g}(\theta) = \overline{g}(\theta + 2\pi) = \lambda\overline{u}(\theta + 2\pi) - \overline{DF(\psi_0)}[\overline{u}(\theta + 2\pi))] \\ \Longrightarrow (\lambda I - \overline{DF(\psi_0)})[\overline{u}(\theta)] - \overline{DF(\psi_0)}[\overline{u}(\theta + 2\pi)] = 0 \\ \Longrightarrow (\lambda I - \overline{DF(\psi_0)})[\overline{u}(\theta) - \overline{u}(\theta + 2\pi)] = 0 \\ \Longrightarrow \overline{u}(\theta) - \overline{u}(\theta + 2\pi) = 0, \end{aligned}$$

which means that $\overline{u}$ is 2π-periodic. Thus there exists a $u \in h^{2+\gamma}(S^1)$ such that $\lambda u - DF(\psi_0)[u] = g$. Therefore we know that $\lambda \in \rho(DF(\psi_0))$, which implies that $\overline{DF(\psi_0)} \subset \rho(DF(\psi_0))$. Since $\overline{DF(\psi_0)}$ is sectorial, we know that there exists some sector $S_{\theta^*,\omega} \subset \rho(\overline{DF}) \subset \rho(DF(\psi_0))$, which means that the first condition of sectoriality is satisfied.

Next we investigate the resolvent estimate. Let $\lambda \in S_{\theta^*,\omega}$. Then by Lemma 2.8 we obtain that

$$||(\lambda - DF(\psi_0))^{-1}||_{L(h^\gamma(S^1))} = ||\left(\lambda I - \overline{DF(\psi_0)}\right)^{-1}||_{L(h^\gamma(\mathbb{R}))} \leq \frac{M}{\lambda - \omega},$$

for some $M > 0$. Thus we know that $DF(\psi_0)$ is sectorial in $h^\gamma(S^1)$. □

Lemma 2.11. *The graph norm of $DF(\psi_0)$ and the norm of D are equivalent for all $\psi_0 \in \mathcal{O}$.*

Proof. Let $\psi_0 \in \mathcal{O}$. To show equivalence, we need to show that there exist $C_1, C_2 > 0$ such that

$$C_1||h||_{2+\gamma} \leq ||DF(\psi_0)[h]||_\gamma + ||h||_\gamma \leq C_2||h||_{2+\gamma},$$

for all $h \in D$. Observe that the second inequality follows from boundedness of $DF(\psi_0)$.

For the first inequality, we will first write $h_{\theta\theta}$ in terms of h, h_θ and $DF(\psi_0)[h]$:

$$h_{\theta\theta} = \frac{1}{a_2(\psi_0)}(DF(\psi_0)[h] - a_1(\psi_0)h_\theta - a_0(\psi_0)h) \tag{2.13}$$

Next, we will show that $||h_\theta||_\gamma \le c(||h||_\gamma + ||DF(\psi_0)[h]||_\gamma)$ for some $c > 0$. The proof will use the fact that $||u'||_\infty \le 2||u||_\infty^{1/2}||u''||_\infty^{1/2}$, for all $u \in C^2$[Matioc, 2017, Lemma 3.13]. Additionally, we will use Young's inequality, so that for all $\varepsilon > 0$

$$\begin{aligned}||h_\theta||_\infty &\le 2||h||_\infty^{1/2}||h_{\theta\theta}||_\infty^{1/2}\\ &\le 2(\frac{1}{2\varepsilon}||h||_\infty + \frac{1}{2}\varepsilon||h_{\theta\theta}||_\infty) = \frac{1}{\varepsilon}||h||_\infty + \varepsilon||h_{\theta\theta}||_\infty.\end{aligned}$$

Combining this inequality and (2.13) yields

$$\begin{aligned}||h_\theta||_\infty &\le \tfrac{1}{\varepsilon}||h||_\infty + \varepsilon||h_{\theta\theta}||_\infty\\ &\le \tfrac{1}{\varepsilon}||h||_\infty + \varepsilon||\tfrac{1}{a_2(\psi_0)}||_\infty||DF(\psi_0)[h]||_\infty + \varepsilon||\tfrac{a_1(\psi_0)}{a_2(\psi_0)}||_\infty||h_\theta||_\infty + \varepsilon||\tfrac{a_0(\psi_0)}{a_2(\psi_0)}||_\infty||h||_\infty\\ \implies (1 - \varepsilon||\tfrac{a_1(\psi_0)}{a_2(\psi_0)}||_\infty)||h_\theta||_\infty &\le (\varepsilon + \tfrac{1}{\varepsilon}||\tfrac{a_0(\psi_0)}{a_2(\psi_0)}||_\infty)||h||_\infty + \varepsilon||\tfrac{1}{a_2}||_\infty||DF(\psi_0)[h]||_\infty.\end{aligned}$$

Now we choose $\varepsilon > 0$ such that $\varepsilon||\frac{a_1(\psi_0)}{a_2(\psi_0)}||_\infty < 1$. Consequently, we know $(1 - \varepsilon||\frac{a_1(\psi_0)}{a_2(\psi_0)}||_\infty) > 0$ so that

$$\begin{aligned}||h_\theta||_\infty &\le (1 - \varepsilon||\tfrac{a_1(\psi_0)}{a_2(\psi_0)}||_\infty)^{-1}\left((\varepsilon + \tfrac{1}{\varepsilon}||\tfrac{a_0(\psi_0)}{a_2(\psi_0)}||_\infty)||h||_\infty + \varepsilon||\tfrac{1}{a_2(\psi_0)}||_\infty||DF(\psi_0)[h]||_\infty\right)\\ &= c(||h||_\infty + ||DF(\psi_0)[h]||_\infty),\end{aligned}$$

for some $c = c(\varepsilon, \psi_0) > 0$.

Next we note that there exists a C_γ such that $[h_\theta]_\gamma \le C_\gamma||h_{\theta\theta}||_\infty$, so that

$$\begin{aligned}||h||_{2+\gamma} &= ||h||_\infty + ||h_\theta||_\infty + ||h_{\theta\theta}||_\gamma\\ &= ||h||_\infty + ||h_\theta||_\infty + ||\tfrac{1}{a_2(\psi_0)}(DF(\psi_0)[h] - a_1(\psi_0)h_\theta - a_0(\psi_0)h)||_\gamma\\ &\le ||h||_\gamma + ||h_\theta||_\infty + ||\tfrac{1}{a_2(\psi_0)}||_\gamma||DF(\psi_0)[h]||_\gamma + ||\tfrac{a_1(\psi_0)}{a_2(\psi_0)}||_\gamma||h_\theta||_\gamma + ||\tfrac{a_0(\psi_0)}{a_2(\psi_0)}||_\gamma||h||_\gamma\\ &\le ||h||_\gamma + c(||h||_\infty + ||DF(\psi_0)[h]||_\infty) + ||\tfrac{1}{a_2(\psi_0)}||_\gamma||DF(\psi_0)[h]||_\gamma + ||\tfrac{a_1(\psi_0)}{a_2(\psi_0)}||_\gamma||h_\theta||_\gamma + ||\tfrac{a_0(\psi_0)}{a_2(\psi_0)}||_\gamma||h||_\gamma\\ &\le c_1(||h||_\gamma + ||DF(\psi_0)[h]||_\gamma) + ||\tfrac{a_1(\psi_0)}{a_2(\psi_0)}||_\gamma[h_\theta]_\gamma\\ &\le c_1(||h||_\gamma + ||DF(\psi_0)[h]||_\gamma) + C_\gamma||\tfrac{a_1(\psi_0)}{a_2(\psi_0)}||_\gamma||h_{\theta\theta}||_\infty\\ &\le c_1(||h||_\gamma + ||DF(\psi_0)[h]||_\gamma + ||h_{\theta\theta}||_\infty)\\ &\le c_1\left(||h||_\gamma + ||DF(\psi_0)[h]||_\gamma + ||\tfrac{1}{a_2(\psi_0)}||_\infty||DF(\psi_0)[h]||_\infty + ||\tfrac{a_1(\psi_0)}{a_2(\psi_0)}||_\infty||h_\theta||_\infty + ||\tfrac{a_0(\psi_0)}{a_2(\psi_0)}||_\infty||h||_\infty\right)\\ &\le c_1(||h||_\gamma + ||DF(\psi_0)[h]||_\gamma),\end{aligned}$$

where we note that the coefficient $c_1 > 0$ may differ in each inequality. Therefore, we know that the graph norm of $DF(\psi_0)$ and D are equivalent. □

2.3.4 Local Lipschitz continuity

Finally we will show that F satisfies the third condition in Lemma 2.12.

Lemma 2.12. *For all $\overline{u} \in \mathcal{O}$ there are $R = R(\overline{u}), L = L(\overline{u}) > 0$ such that*

$$||DF(\psi_1) - DF(\psi_0)||_{L(D,X)} \le L||\psi_1 - \psi_0||_{2+\gamma}$$

for all $\psi_{0,1} \in B(\overline{u}, R) \subset D$.

Proof. Fix $\overline{u} \in \mathcal{O}$. The condition we need to show is equivalent to showing

$$||DF(\psi_1)[h] - DF(\psi_0)[h]||_\gamma \leq L||\psi_1 - \psi_0||_{2+\gamma}||h||_{2+\gamma},$$

for all $h \in B(\overline{u}, R)$. To prove this statement, we will use local Lipschitz continuity of a_0, a_1, a_2.

$$\begin{aligned}
||DF(\psi_1)[h] - DF(\psi_0)[h]||_\gamma &= ||a_0(\psi_1)h + a_1(\psi_1)h_\theta + a_2(\psi_1)h_{\theta\theta} - a_0(\psi_0)h - a_1(\psi_0)h_\theta - a_2(\psi_0)h_{\theta\theta}||_\gamma \\
&= ||(a_0(\psi_1) - a_0(\psi_0))h + (a_1(\psi_1) - a_1(\psi_0))h_\theta + (a_2(\psi_1) - a_2(\psi_0))h_{\theta\theta}||_\gamma \\
&\leq ||(a_0(\psi_1) - a_0(\psi_0))h||_\gamma + ||(a_1(\psi_1) - a_1(\psi_0))h_\theta||_\gamma + ||(a_2(\psi_1) - a_2(\psi_0))h_{\theta\theta}||_\gamma \\
&\leq ||(a_0(\psi_1) - a_0(\psi_0))||_\gamma||h||_\infty + ||(a_1(\psi_1) - a_1(\psi_0))||_\gamma||h_\theta||_\infty + ||(a_2(\psi_1) - a_2(\psi_0))||_\gamma||h_{\theta\theta}||_\infty \\
&\leq ||(a_0(\psi_1) - a_0(\psi_0))||_\gamma||h||_{2+\gamma} + ||(a_1(\psi_1) - a_1(\psi_0))||_\gamma||h||_{2+\gamma} + ||(a_2(\psi_1) - a_2(\psi_0))||_\gamma||h||_{2+\gamma} \\
&\leq L||\psi_1 - \psi_0||_{2+\gamma}||h||_{2+\gamma},
\end{aligned}$$

where $L = \max(L_0, L_1, L_2)$. Therefore

$$||DF(\psi_1)[h] - DF(\psi_0)[h]||_{L(D,X)} \leq L||\psi_1 - \psi_0||_{2+\gamma},$$

which was the inequality we needed. □

We can combine Lemmas 2.6, 2.10, and 2.12 to obtain our main theorem for the ballistic model.

Theorem 2.13. *Let $\psi_0 \in \mathcal{O}$ and F defined by (2.1). Then there exist $\delta = \delta(\psi_{in}), r = r(\psi_{in})$ such that for every $t_0 \in [0, r]$ and $\psi_{in} \in \mathcal{O}$ with $||\psi_{in} - \psi_0||_{2+\gamma} \leq r$ there is a strict solution $\psi \in C([0, \delta]; h^{2+\gamma}(S^1)) \cap C^1([0, \delta]; h^\gamma(S^1))$ to*

$$\begin{aligned}
\psi_t(t) &= F(\psi(t)) \quad t \in [0, \delta], \\
\psi(0) &= \psi_{in}.
\end{aligned}$$

In the next section, we will extend this result by comparing the general case to the simple, circular case.

2.4 Long time stability of circles

This section will consist of several parts. First we will introduce a new time-scale to which we will translate the evolution equation. The advantage of the rescaling is that circular solutions are transformed into constant solutions. We will then use the concept of linear stability to analyse the stability of these constant solutions and translate the result back to the original setting.

2.4.1 Introduction of the new time-scale

First we note that for any $\psi \in \mathcal{O}$ and α, such that $\alpha\psi \in \mathcal{O}$, we have that $F(\alpha\psi) = F(\psi)$. With this property in mind, we introduce the new variable $\tilde{\psi} = \frac{\psi}{\psi_c}$, where $\psi_c = \psi_{in} + \frac{P}{2\pi}t$ is the solution to the constant growing circle. This variable transforms circular solutions ψ_c to constant solutions $\tilde{\psi}_c \equiv 1$. Next we will find the evolution equation for $\tilde{\psi}$. Straightforward calculations show that

$$\begin{aligned}
\partial_t\tilde{\psi} &= \partial_t \frac{\psi}{\psi_c} \\
&= \psi_t\psi_c^{-1} - \psi\psi_c^{-2}\frac{P}{2\pi} \\
&= F(\psi)\psi_c^{-1} - \tilde{\psi}\psi_c^{-1}\frac{P}{2\pi} \\
&= F(\tilde{\psi}\psi_c)\psi_c^{-1} - \tilde{\psi}\psi_c^{-1}\frac{P}{2\pi} \\
&= (F(\tilde{\psi}) - \frac{P}{2\pi}\tilde{\psi})\psi_c^{-1}.
\end{aligned}$$

This non-autonomous equation can be made autonomous with the proper time scaling. We introduce $\tilde{t} = g(t)$, with $g'(t) = \psi_c^{-1}$. This definition implies that $g(t) = \frac{2\pi}{P}\ln(\psi_c)$. Substituting this transformation yields

$$\begin{aligned}\psi_c^{-1}\frac{\partial\tilde{\psi}}{\partial\tilde{t}} &= \frac{\partial\tilde{\psi}}{\partial\tilde{t}}\frac{\partial\tilde{t}}{\partial t} \\ &= \frac{\partial\tilde{\psi}}{\partial t} \\ &= \psi_c^{-1}(F(\tilde{\psi}) - \frac{P}{2\pi}\tilde{\psi}).\end{aligned}$$

Thus we see that $\tilde{\psi}(\tilde{t})$ has an autonomous evolution equation, namely

$$\frac{\partial\tilde{\psi}}{\partial\tilde{t}} = F(\tilde{\psi}) - \frac{P}{2\pi}\tilde{\psi} \tag{2.14}$$

Finally, we will translate the variable $\tilde{\psi}$ to $\rho = \tilde{\psi} - 1$ so that 0 corresponds to the trivial solutions. The rescaled equation thus becomes

$$\partial_{\tilde{t}}\rho = F(\rho + 1) - \frac{P}{2\pi}(\rho + 1) = \tilde{F}(\rho). \tag{2.15}$$

2.4.2 Stability of the rescaled problem

We want to analyse perturbations of the stationary solutions to (2.15). Observe that $\tilde{F}(\rho)$ is once again a non-linear parabolic partial differential operator in ρ. However, we can investigate asymptotic behaviour around the equilibrium $\rho = 0$ using linearised stability. The analysis of linear stability uses spectral analysis of the linearised equation around the equilibrium. We will apply a theorem for more general settings to the setting of partial differential equations. In particular, we will use Theorem 2.14.

Theorem 2.14. *[Lunardi, 1995, Theorem 9.1.2] Let $A : D(A) \to X$ be a linear operator satisfying*

- *$A : D(A) \to X$ is sectorial and its graph norm is equivalent to X;*
- *$\sup\{Re\lambda : \lambda \in \sigma(A)\} = -\omega_0 < 0$.*

Let $\mathcal{O}$ be a neighbourhood of the origin in D, and let $G : \mathcal{O} \to X$ be a C^1 function with locally Lipschitz continuous derivative, satisfying

$$G(0) = 0, G'(0) = 0.$$

Fix $\omega \in [0, \omega_0)$. Then there exist $r > 0, M > 0$ such that for each $u_0 \in B(0, r) \subset D$ we have $\tau(u_0) = +\infty$ and

$$||u(t)||_D + ||u'(t)||_\gamma \leq Me^{-\omega t}||u_0||_D, \quad \forall t \geq 0, \tag{2.16}$$

where u satisfies

$$\begin{aligned}u'(t) &= Au(t) + G(u(t)), t \geq 0, \\ u(0) &= u_0.\end{aligned} \tag{2.17}$$

To apply the theorem, we will to define A, G to be

$$A\rho = D\tilde{F}(0)[\rho], G(\rho) = \tilde{F}(\rho) - A\rho. \tag{2.18}$$

Using similar calculation as before, we know that

$$DF(1)[\rho] = \frac{P}{2\pi}\rho_{\theta\theta}$$

which implies that $A\rho = \frac{P}{2\pi}(\rho_{\theta\theta} - \rho)$. Using these new operators, we will prove Theorem 2.15.

Theorem 2.15. *The operators A and G in* (2.18) *satisfy the assumptions of Theorem 2.14 with $D(A) = D, \mathcal{O}$ and X as defined in* (2.6).

Proof. The operator A is a linear uniformly elliptic partial differential operator with definition domain $h^{2+\gamma}(S^1)$. Clearly, A is sectorial in $h^{\gamma}(S^1)$ and has graph norm equivalent to the γ-norm. Next we will find the spectrum of A to check whether the second condition is met.

Spectrum of A We need to find all $\lambda \in \mathbb{C}$ such that the operator $\lambda I - A$ is not invertible. Therefore, we will try to solve the equation $\lambda\rho - A\rho = g$ for $\rho \in D$ and arbitrary $g \in X$, and show that $(\lambda I - A)^{-1}$ is a bounded operator for these λ. We note that $h^{\gamma}(S^1) \subset L^2(S^1)$, so we can write ρ and g in term of their Fourier coefficients. By definition for A, we have that

$$\lambda\rho - A\rho = \lambda\rho - \frac{P}{2\pi}\rho_{\theta\theta} + \frac{P}{2\pi}\rho = (\lambda + \frac{P}{2\pi})\rho - \frac{P}{2\pi}\rho_{\theta\theta}.$$

Assume that ρ, g have Fourier coefficients ρ_n and g_n respectively. Then we have that

$$(\lambda + \frac{P}{2\pi})\rho - \frac{P}{2\pi}\rho_{\theta\theta} = \sum_{n\in\mathbb{Z}} (\lambda + \frac{P}{2\pi} - \frac{P}{2\pi}(-n^2))\rho_n \exp(in\theta) = \sum_{n\in\mathbb{Z}} g_n \exp(in\theta).$$

Next we take the inner product with $\exp(im\theta)$ for some $m \in \mathbb{Z}$ on both sides to obtain

$$(\lambda + \frac{P}{2\pi} + \frac{P}{2\pi}m^2)\rho_m = g_m \Longrightarrow \rho_m = \frac{g_m}{\lambda + \frac{P}{2\pi}(1+m^2)},$$

which only holds if $\lambda \neq -\frac{P}{2\pi}(1+n^2)$ for all $n \in \mathbb{Z}$. Next we will to show that ρ as defined here lies in $h^{2+\gamma}(S^1)$. Since $(g_n)_{n\in\mathbb{Z}} \in \ell^2$, we get that $(\rho_n)_{n\in\mathbb{Z}} \in \ell^2$ which implies that ρ is in $L^2(S^1)$. Additionally, for any $\lambda \neq \lambda_m$, we note that $(-\rho_n n^2)_{n\in\mathbb{Z}} \in \ell^2$. Thus we know that ρ is in $W^{2,2}(S^1)$. However, since $W^{2,2}(S^1) \subset h^{\gamma}(S^1)$, we know that $\rho \in h^{\gamma}(S^1)$. Additionally, since $g \in h^{\gamma}(S^1)$, we can also obtain that

$$\rho_{\theta\theta} = \frac{2\pi}{P}A\rho + \rho = \frac{2\pi}{P}(g + \lambda\rho) + \rho \in h^{\gamma},$$

which implies that $\rho \in h^{2+\gamma}$. Therefore the equation $\lambda\rho - A\rho = g$ can be solved in $h^{2+\gamma}$ for any $\lambda \in \mathbb{C}$ unequal to $\lambda_n = -\frac{P}{2\pi}(1+n^2)$ for any $n \geq 0$. Finally, we will show that $(\lambda I - A)^{-1}$ is a bounded operator for any $\lambda \neq \lambda_n$. Let $(\lambda I - A)\rho = g$ for some $g \in h^{\gamma}(S^1), \rho \in h^{2+\gamma}(S^1)$. We can use Morrey's inequality [Evans, 2010, Paragraph 5.6.2] so that there exists some $C > 0$ such that

$$||\rho||_{\gamma} \leq C||\rho||_{W^{1,2}}.$$

By the calculations done earlier we can write ρ in terms of g using Fourier coefficients. Since $(n^2\rho_n)_n \in \ell^2$, we know that there exists some $C_\lambda > 0$ such that

$$||\rho||_{\gamma} \leq C||\rho||_{W^{1,2}} \leq C_\lambda||g||_{L^2} \leq C_\lambda||g||_{\gamma}.$$

Additionally, recall from the proof of Lemma 2.10 that $||\rho_\theta||_\infty \leq ||\rho_{\theta\theta}||_\infty + ||\rho||_\infty$. Thus we know that

$$\begin{aligned}
||\rho||_{2+\gamma} &= ||\rho||_\infty + ||\rho_\theta||_\infty + ||\rho_{\theta\theta}||_\gamma \\
&\leq 2||\rho||_\gamma + 2||\rho_{\theta\theta}||_\gamma \\
&\leq 2||\rho||_\gamma + 2||\frac{2\pi}{P}(g + \lambda\rho) + \rho||_\gamma \\
&\leq (3 + \frac{4\pi}{P}\lambda)||\rho||_\gamma + \frac{4\pi}{P}||g||_\gamma \\
&\leq C_\lambda||g||_\gamma,
\end{aligned}$$

for some constant $C_\lambda > 0$. Therefore we know that $||(\lambda I - A)^{-1} g||_{2+\gamma} \leq ||g||_\gamma$ and thus we know that $(\lambda I - A)^{-1}$ is a bounded linear operator for $\lambda \neq \lambda_n$. This implies that $\lambda \in \rho(A)$ if $\lambda \neq \lambda_n$. Thus A has a discrete spectrum $\sigma(A) \subset \{\lambda_m\}_{m\geq 0}$. Since $\lambda_n \leq -\frac{P}{2\pi}$, we also know that we can choose $\omega_0 = \frac{P}{2\pi}$ so that $\sup\{Re\lambda : \lambda \in \sigma(A)\} \leq -\omega_0$.

Differentiability of G As discussed in Subsection 2.3, we know that G is differentiable with locally Lipschitz continuous derivative. Note that the derivative $G'(0)$ denotes the Fréchet derivative at 0. Additionally, we know that

$$G(0) = F(1) - \frac{P}{2\pi} - DF(1)[0] - 0 = \frac{P}{2\pi} - \frac{P}{2\pi} = 0,$$
$$G'(0) = D\tilde{F}(0) - A = D\tilde{F}(0) - D\tilde{F}(0) = 0,$$

which means that each condition of Theorem 2.14 is satisfied. □

We will now use Theorem 2.15 to show linear stability of circles in the original problem statement in Theorem 2.16.

Theorem 2.16. *Let ψ_c be a circular solution to (2.2) and ψ_{in} close to $\psi_c(0) \in \mathcal{O}$. Then ψ is a solution to (2.2) with initial condition $\psi(0) = \psi_{in}$ for all time $t > 0$ and $\varepsilon \in (0, 1]$ there exists a $M = M(\varepsilon)$ such that*

$$||\psi - \psi_c||_{2+\gamma} \leq M(\psi_c(0) + \frac{P}{2\pi} t)^\varepsilon ||\psi_{in} - \psi_c(0)||_{2+\gamma}.$$

Proof. We will achieve the inequality by using linear stability of the rescaled problem from Theorem (2.15). By this theorem we know that the evolution equation (2.15) can be solved for all time, and $||\rho(\tilde{t})||_{2+\gamma} \leq Me^{-\omega\tilde{t}}||\rho_0||_{2+\gamma}$ for all $t \geq 0$. Note that the initial condition ρ_0 can be obtained by transforming the initial condition ψ_{in}.

$$\rho_0 = \frac{\psi(0)}{\psi_c(0)} - 1.$$

Since we want a result for ψ, we need to substitute the definitions of $\tilde{t}, \tilde{\psi}$ and ρ backwards to obtain

$$\begin{aligned}
||\psi - \psi_c||_{2+\gamma} &= ||\tilde{\psi}\psi_c - \psi_c||_{2+\gamma} \\
&= ||\tilde{\psi} - 1||_{2+\gamma}\psi_c \\
&= ||\rho||_{2+\gamma}\psi_c \\
&\leq Me^{-\omega\tilde{t}(t)}||\rho_0||_{2+\gamma}\psi_c \\
&= Me^{-\omega\frac{2\pi}{P}\ln(\psi_c(t))}\psi_c||\psi_c(0)^{-1}(\psi(0) - \psi_c(0))||_{2+\gamma} \\
&= M\psi_c(0)^{-1}\psi_c^{1-\frac{2\pi}{P}\omega}||\psi(0) - \psi_c(0)||_{2+\gamma} \\
&= M\psi_c(0)^{-1}(\psi_c(0) + \frac{P}{2\pi}t)^{1-\frac{2\pi}{P}\omega}||\psi(0) - \psi_c(0)||_{2+\gamma}
\end{aligned}$$

Since $\omega \in [0, \frac{P}{2\pi})$, we know that for all $\varepsilon \in (0, 1]$, we have an $M = M(\varepsilon, \psi_c(0)) > 0$ such that

$$||\psi - \psi_c||_{2+\gamma} \leq M(\psi_c(0) + \frac{P}{2\pi} t)^\varepsilon ||\psi_{in} - \psi_c(0)||_{2+\gamma}.$$

□

Next we will prove an avoidance principle for solutions to the ballistic model.

2.5 Avoidance principle for solutions of the ballistic model

We have shown that perturbations of circles will grow similar to circles over long time. For circles we know that two circles will never intersect, unless they started with an equal radius. The question for this final part is thus whether perturbations of solutions also satisfy this principle. This principle is often referred to as an avoidance principle. The avoidance principle is a well-known concept for mean curvature flow, which states that two different solutions will not intersect if they did not already intersect at the starting time. The proof for the principle is based on the maximum principle. We will use a similar method to show the principle for the non-linear parabolic partial differential equation of the ballistic VSC model in Theorem 2.17.

Theorem 2.17. *Let $\psi_1(0), \psi_2(0)$ be initial conditions for solutions $\psi_1(t), \psi_2(t)$ of the equation $\psi_t = F(\psi)$, such that $\psi_1(0) < \psi_2(0)$. Then also $\psi_1(t) < \psi_2(t)$ for all $t > 0$, such that $\psi_{1,2}(t)$ exist.*

The proof of this theorem will use a maximum principle, as given in Theorem 2.18.

Theorem 2.18. *[Protter and Weinberger, 1984, Chapter 3, Theorem 2] Let E be a domain and suppose that in $E_{t_1} = \{(x,t) \in E : t \leq t_1\}$ the inequality*

$$L[u] \equiv a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} \geq 0$$

holds, that a and b are bounded functions of x and t, and that L is uniformly parabolic in E_{t_1}, i.e. there exists a $\lambda > 0$ such that $a \geq \nu$. If $u \leq M$ in E_{t_1} and $u(x_1, t_1) = M$, then $u = M$ at every point (x,t) in E_{t_1}, which can be connected with (x_1, t_1) by a horizontal and a vertical line segment, both of which lie in E_{t_1}.

Proof of Theorem 2.18. We will prove by contradiction. Assume there is a minimal time $\tilde{t}$ such that there exists a $\tilde{\theta}$ with $\psi_1(\tilde{t}, \tilde{\theta}) = \psi_2(\tilde{t}, \tilde{\theta})$. Let $w = \psi_2 - \psi_1$. Then

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \frac{\partial \psi_2}{\partial t} - \frac{\partial \psi_1}{\partial t} \\
&= F(\psi_2) - F(\psi_1) \\
&= \int_0^1 DF(s\psi_2 + (1-s)\psi_1)[\psi_2 - \psi_1]ds \\
&= \int_0^1 DF(s\psi_2 + (1-s)\psi_1)ds[\psi_2 - \psi_1] \\
&= \tilde{a}_2 \partial_{\theta\theta} w + \tilde{a}_1 \partial_\theta w + \tilde{a}_0 w,
\end{aligned}$$

where $\tilde{a}_i = \int_0^1 a_i(s\psi_2 + (1-s)\psi_1)ds$. As $a_2 > 0$, we know that $\tilde{a}_2 > 0$, and thus, w satisfies a linear parabolic equation. To apply Theorem 2.18, we will first rewrite the equation.

Let $K > 0$ be such that $\tilde{a}_0 + K \geq 0$ and $v := -e^{Kt} w \leq 0$. Conversely, we obtain that $w = -e^{-Kt} v$. Therefore, we obtain

$$\begin{aligned}
w_t &= \partial_t(-e^{-Kt} v) \\
&= -e^{-Kt} v_t + K e^{-Kt} v \\
&= e^{-Kt}(-v_t + Kv)
\end{aligned}$$

The parabolic equation for w can be rewritten to

$$\begin{aligned}
w_t &= \tilde{a}_2 w_{\theta\theta} + \tilde{a}_1 w_\theta + \tilde{a}_0 w \\
&= \tilde{a}_2 \partial_{\theta\theta}(-e^{-Kt} v) + \tilde{a}_1 \partial_\theta(-e^{-Kt} v) + \tilde{a}_0(-e^{-Kt} v) \\
&= -e^{-Kt}\left(\tilde{a}_2 v_{\theta\theta} + \tilde{a}_1 v_\theta + \tilde{a}_0 v\right).
\end{aligned}$$

Combining these two results yields

$$-v_t + Kv = \tilde{a}_2 v_{\theta\theta} + \tilde{a}_1 v_\theta + \tilde{a}_0 v.$$

Next we reorder the terms so that

$$\tilde{a}_2 v_{\theta\theta} + \tilde{a}_1 v_\theta - v_t = -(\tilde{a}_0 + K)v.$$

Note that $v \leq 0$, $\tilde{a}_0 + K \geq 0$. Therefore, if $w_t = \tilde{a}_2 w_{\theta\theta} + \tilde{a}_1 w_\theta + \tilde{a}_0 w$ then

$$L[v] := \tilde{a}_2 v_{\theta\theta} + \tilde{a}_1 v_\theta - v_t \geq 0.$$

From Lemma 2.7 we know that $a_2 \geq \nu$ and $a_{1,2}$ are bounded. Therefore we obtain that the $\tilde{a}_{1,2}$ must have the same properties. Since $v(\tilde{\theta}, \tilde{t}) = 0$, we can apply Theorem 2.18 with $M = 0$ to obtain that $v \equiv 0$. However, then $w \equiv 0$ must hold, which contradicts the assumption that $\tilde{t}$ is the first time where $w = 0$. Thus, there is no time t and angle θ, such that $\psi_1(t,\theta) = \psi_2(t,\theta)$, which implies that $\psi_2(\theta,t) > \psi_1(\theta,t)$ for all θ, t.

□

3 Diffusive VSC problem

The second model we analyse is the diffusive VSC model. We first define a diffeomorphism between the unit ball and the domain in Section 3.1. This diffeomorphism will be used to transform the diffusion equation for the concentration to one on the unit ball. Using this transformation, we will show local existence and uniqueness to solutions of the evolution equation in Section 3.2. Similar to the previous chapter, we will show this with Theorem 2.1. Finally, we will use linear stability to prove stability of circular solutions in Section 3.3.

For the diffusive model, we will also assume that Ω_ψ is a bounded, non-empty and strictly convex set. Therefore, ψ should satisfy

- There exists a $\nu > 0$ such that $\nu < \psi(\theta), H(\psi(\theta))$ for all $\theta \in S^1$.
- ψ is twice continuously differentiable on S^1.

However, we will later see that ψ has an extra condition when we transform the concentration equation (1.11).

3.1 Deriving the evolution equation

In this section we will derive the evolution equation for the diffusive model. We will first introduce a diffeomorphism between the unit ball Ω_1 and the domain Ω_ψ. This diffeomorphism allows us to rewrite Equation (1.11) to a problem on the unit ball. The rewritten problem will then be used to derive the evolution equation.

3.1.1 Introducing the diffeomorphism

We define the function $z = z(\psi) : \Omega_1 \to \Omega_\psi$, by splitting Ω_1 in two subdomains Ω_a and Ω_d. These subdomains are given by

$$\begin{aligned} \Omega_d &= \{(x,y) \mid x^2 + y^2 < \frac{1}{9}\}, \\ \Omega_a &= \{(r\cos\theta, r\sin\theta) \mid (r,\theta) \in (\tfrac{1}{4}, 1) \times S^1\}, \end{aligned} \tag{3.1}$$

Note that we choose two different coordinate systems for the subdomains. This choice will help to show that $z(\psi)$ will be a global diffeomorphism on Ω_1. For the definition of $z(\psi)$ we suppress the time-dependence of ψ. Now we define $z(\psi)$ as

$$z(\psi) = \begin{cases} z(\psi; x, y) = (x, y) & \text{for } (x,y) \in \Omega_d \\ z(\psi; r\cos\theta, r\sin\theta) = g(r,\theta)(\cos\theta, \sin\theta), & \text{for } (r\cos\theta, r\sin\theta) \in \Omega_a. \end{cases}, \tag{3.2}$$

where we choose χ a smooth, increasing function from $(\frac{1}{4}, 1)$ to $[0,1]$ and $g(r,\theta) = r(1 + \chi(r)(\psi(\theta) - 1))$. In addition, we require that $\chi(r) \equiv 0$ for $r \in (\frac{1}{4}, \frac{1}{3})$ and $\chi(r) \equiv 1$ for $r \in (\frac{1}{2}, 1)$. Furthermore, we introduce $K > 0$ such that $\chi(r) + r\chi'(r) \le K$ for all $r \in (\frac{1}{4}, 1)$. Note that this constant always exists, since χ is smooth on a bounded interval. We can find a lower bound on K using the previously mentioned assumptions for χ. We know that $\chi(r) \ge 0$, and there must exist a $r^* \in (\frac{1}{3}, \frac{1}{2})$ such that $\chi'(r^*) = 6$ by the intermediate value theorem. Additionally, we know that $\chi(r) + r\chi'(r)$ will take its largest value for $r \in (\frac{1}{3}, \frac{1}{2})$. We thus know that $K \ge 0 + \frac{1}{3} \cdot 6 = 2$. This implies that $\psi \ge \frac{1}{2} + \alpha$ for some $\alpha > 0$.

Since $z(\psi)$ is defined on two subdomains that overlap, we need to verify that it is well-defined on the intersection. On Ω_d, we know that $z(\psi; x, y) = (x, y)$ which is the identity map in cartesian coordinates. On $\Omega_a \cap \Omega_d$, we note that the corresponding radius r must be between $\frac{1}{4}$ and $\frac{1}{3}$. However, then we see that $\chi(r) \equiv 0$, so that $z(\psi; r\cos\theta, r\sin\theta) = (r\cos\theta, r\sin\theta)$, which is the identity map in polar coordinates. Thus we see that on the overlap $\Omega_d \cap \Omega_a$, the two definitions for $z(\psi)$ both correspond to the identity map, so $z(\psi)$ is well-defined. Additionally, we note that $z(\psi; x) = 0$ implies that $x = 0 \in \Omega_d$. A visualisation of how $z(\psi)$ transforms Ω_1 is depicted in Figure 5.

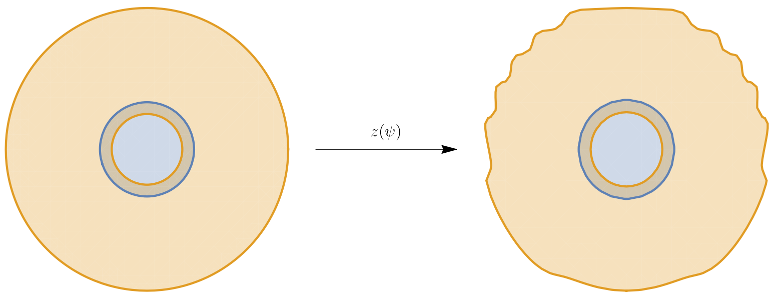

Figure 5: *Demonstration of how Ω_1 is transformed under $z(\psi)$. The subdomains Ω_d, Ω_a, and their images are visualised in blue and orange respectively. Note that the variation of ψ has been embellished to emphasise its perturbative behaviour.*

We will show that z as given in (3.2) is a global diffeomorphism from Ω_1 to Ω_ψ in Lemma 3.1. The proof needs an additional assumption on ψ. In particular we assume that $\psi \geq 1 - \frac{1}{K} + \alpha > 0$, for some $\alpha > 0$.

Lemma 3.1. *Let ψ satisfy the modelling assumptions and $\psi \geq 1 - \frac{1}{K} + \alpha > 0$. Let $z(\psi)$ be given by* (3.2). *Then $z(\psi)$ is a global diffeomorphism from Ω_1 to Ω_ψ.*

Proof. We need to show that $z(\psi)$ is a differentiable bijective map from Ω_1 to Ω_ψ and its inverse $z(\psi)^{-1}$ is differentiable too. We note that the second requirement follows from the inverse function theorem if we can show that the determinant of the Jacobian of $z(\psi)$ stays away from zero. First we will show that $z(\psi)$ is differentiable and the Jacobian has a determinant that stays away from zero. This property would imply that $z(\psi)$ is a local diffeomorphism from Ω_1 to Ω_ψ.

We note that $z(\psi)$ equals the identity map in Ω_d, which is differentiable and the Jacobian has determinant equal to 1. On $\Omega_a \backslash \Omega_d$, we can calculate the partial derivatives of $z(\psi) = (z_1(\psi), z_2(\psi))$ with respect to r and θ. These partial derivatives use the derivatives of $g(r,\theta)$ which are equal to

$$\begin{aligned} \frac{\partial g}{\partial r} &= g_r(r,\theta) = 1 + \chi(r)(\psi(\theta) - 1) + r\chi'(r)(\psi(\theta) - 1), \\ \frac{\partial g}{\partial \theta} &= g_\theta(r,\theta) = r\chi(r)\psi_\theta(\theta). \end{aligned}$$

Therefore the partial derivatives of $z(\psi)$ are given by

$$\begin{aligned} \frac{\partial z_1(\psi)}{\partial r} &= \cos(\theta) g_r(r,\theta), \\ \frac{\partial z_2(\psi)}{\partial r} &= \sin(\theta) g_r(r,\theta), \\ \frac{\partial z_1(\psi)}{\partial \theta} &= -\sin(\theta) g(r,\theta) + \cos(\theta) g_\theta(r,\theta), \\ \frac{\partial z_2(\psi)}{\partial \theta} &= \cos(\theta) g(r,\theta) + \sin(\theta) g_\theta(r,\theta). \end{aligned}$$

The corresponding determinant of the Jacobian is given by

$$\begin{aligned}\det(J) &= \frac{\partial(z_1(\psi))}{\partial r}\frac{\partial z_2(\psi)}{\partial \theta} - \frac{\partial z_2(\psi)}{\partial r}\frac{\partial z_1(\psi)}{\partial \theta} \\ &= \cos(\theta)^2 g_r(r,\theta)g(r,\theta) + \cos(\theta)\sin(\theta)g_r(r,\theta)g_\theta(r,\theta) + \sin(\theta)^2 g_r(r,\theta)g(r,\theta) - \cos(\theta)\sin(\theta)g_r(r,\theta)g_\theta(r,\theta) \\ &= g_r(r,\theta)g(r,\theta).\end{aligned}$$

We note that $g(r,\theta) \geq r\min(1,\psi(\theta)) \geq \frac{1}{4}\min(1, 1-\frac{1}{K}+\alpha) > 0$. Furthermore, since $\chi(r) + r\chi'(r) \in [0, K]$, we know that

$$g_r(r,\theta) = 1 + (\chi(r) + r\chi'(r))(\psi(\theta) - 1) \geq 1 + \min(0, K(-\tfrac{1}{K}+\alpha)) = \min(1, \alpha K) > 0.$$

Therefore, we have that

$$\begin{aligned}\det(J) &\geq \frac{1}{4}\min(1,\alpha K)\min(1,\psi(\theta)) \\ &\geq \frac{1}{4}\min(1,\alpha K, \psi(\theta), \alpha K\psi(\theta)) \\ &\geq \frac{1}{4}\min(1, \alpha K, 1-\frac{1}{K}+\alpha, \alpha(K-1+\alpha K)) > 0.\end{aligned}$$

Next we will show that $z(\psi)$ is a bijective map from Ω_1 to Ω_ψ. By construction of $z(\psi)$ we already know that it is surjective. Thus we will only have to show that it is injective. We will show this in two steps. First we will show that $z(\psi)$ is angle preserving. Then we will show that it is radially monotonically increasing.

We know that $z(\psi)$ is the identity map on Ω_d, which is clearly radially monotonically increasing and preserves angles. For Ω_a, we will prove that $\theta_1 = \theta_2$ if $z(\psi; r_1\cos\theta_1, r_1\sin\theta_1) = z(\psi; r_2\cos\theta_2, r_2\sin\theta_2)$. Let $x_1, x_2 \in \Omega_a$ with $x_i = (r_i\cos\theta_i, r_i\sin\theta_i)$ such that $z(\psi; x_1) = z(\psi; x_2)$. Then we know that

$$z_1(\psi; r_1, \theta_1)z_2(\psi; r_2, \theta_2) - z_1(\psi; r_2, \theta_2)z_2(\psi; r_1, \theta_1) = 0.$$

Substituting the expressions for z_1, z_2, we obtain

$$g(r_1,\theta_1)g(r_2,\theta_2)(\cos(\theta_1)\sin(\theta_2) - \cos(\theta_2)\sin(\theta_1)) = 0.$$

We know that $g(r,\theta) > 0$, so we get

$$0 = \cos(\theta_1)\sin(\theta_2) - \cos(\theta_2)\sin(\theta_1) = \sin(\theta_2 - \theta_1).$$

This implies that $\theta_2 = \theta_1$ or $\theta_2 = \theta_1 + \pi$. We will show that the case $\theta_2 = \theta_1 + \pi$ does not yield a valid solution.

Assume that $\theta_2 = \theta_1 + \pi$. We know that

$$g(r_1,\theta_1)\cos(\theta_1) = g(r_2,\theta_2)\cos(\theta_2)$$

Since $g(r,\theta) > 0$, we see that the signs of $\cos(\theta_1)$ and $\cos(\theta_2)$ must match. Similarly, we can obtain that $\sin(\theta_1)$ and $\sin(\theta_2)$ must have matching signs. However, $\cos\theta$ and $\sin\theta$ are never simultaneously equal to zero, thus we have that $\theta_1 = \theta_2$. Therefore, we know that $z(\psi)$ is angle preserving on Ω_1. Additionally, we know that $g_r(r,\theta) > 0$ for any $\theta \in S^1$, which implies that $z(\psi)$ is monotonically increasing in r. Thus we know that $z(\psi)$ is injective. Therefore, we obtain that $z(\psi)$ is a diffeomorphism from Ω_1 to Ω_ψ. $\square$

Next we will use $z(\psi)$ to transform the diffusion equation on Ω_ψ to one on Ω_1.

3.1.2 Transformation of the concentration equation

Recall that the concentration equation is given by (1.11). We split u into $u = -w - \phi$, where

$$\begin{aligned} \Delta\phi &= P\delta \quad \text{in } \Omega_\psi, \\ \Delta w &= 0 \quad \text{in } \Omega_\psi, \\ w &= -\phi \quad \text{on } \partial\Omega_\psi. \end{aligned} \tag{3.3}$$

Note that the corresponding flux can thus be split to

$$\Phi_D = -\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega_\psi} = \left.\frac{\partial \phi}{\partial n}\right|_{\partial\Omega_\psi} + \left.\frac{\partial w}{\partial n}\right|_{\partial\Omega_\psi}.$$

We note that ϕ is a scaled version of the fundamental equation to the Laplace equation. Therefore ϕ is given by

$$\phi(r) = \frac{P}{2\pi}\ln(r). \tag{3.4}$$

We can calculate the gradient of ϕ to be

$$\begin{aligned} \nabla\phi &= (\cos(\theta)\phi_r - \frac{1}{r}\sin(\theta)\phi_\theta, \sin(\theta)\phi_r + \frac{1}{r}\cos(\theta)\phi_\theta) \\ &= (\cos(\theta)\frac{P}{2\pi}\frac{1}{r}, \sin(\theta)\frac{P}{2\pi}\frac{1}{r}) \\ &= \frac{P}{2\pi}\frac{(\cos\theta, \sin\theta)}{r}. \end{aligned}$$

Observe that we can write

$$\left.\frac{\partial\phi}{\partial n}\right|_{\partial\Omega_\psi} = \frac{P}{2\pi}\left.\frac{(\cos\theta,\sin\theta)}{r}\right|_{r=\psi} \cdot n(\psi) = \Phi_B(\psi)\cdot n(\psi).$$

We introduce $c : \Omega_1 \to \mathbb{R}$ such that w is the push-forward of c. Then c satisfies

$$\begin{aligned} \Delta z(\psi)_* c &= 0 \quad \text{in } \Omega_\psi, \\ z(\psi)_* c &= -\phi \quad \text{on } \partial\Omega_\psi. \end{aligned} \tag{3.5}$$

Using the pull-back $z(\psi)^*$ on both equations, we thus know that c satisfies

$$\begin{aligned} L(\psi)[c] &= 0 && \text{in } \Omega_1, \\ c = z(\psi)^*(\phi) &= -\phi\circ z(\psi) = -\frac{P}{2\pi}\ln(\psi) && \text{on } \partial\Omega_1, \end{aligned} \tag{3.6}$$

where $L(\psi) := z(\psi)^* \circ \Delta \circ z(\psi)_*$ is a second order differential operator. In geometric terms, $L(\psi)$ is given by the Laplace-Beltrami operator with respect to the pulled back canonical metric $z(\psi)^* can$. Therefore, we can write $L(\psi)$ using the inverse of the Jacobian of z:

$$L(\psi) = J^{-1}_{z(\psi),ij}\partial_i(J^{-1}_{z(\psi),kj}\partial_k).$$

The inverse Jacobian J_z^{-1} is given by

$$\begin{aligned} J^{-1}_{z(\psi)}(r,\theta) &= \begin{pmatrix} \frac{\partial z_1(\psi)}{\partial r} & \frac{\partial z_1(\psi)}{\partial \theta} \\ \frac{\partial z_2(\psi)}{\partial r} & \frac{\partial z_2(\psi)}{\partial \theta} \end{pmatrix}^{-1} = \frac{1}{\det(J)(r,\theta)}\begin{pmatrix} \frac{\partial z_2(\psi)}{\partial \theta} & -\frac{\partial z_1(\psi)}{\partial \theta} \\ -\frac{\partial z_2(\psi)}{\partial r} & \frac{\partial z_1(\psi)}{\partial r} \end{pmatrix}, \text{ for } (r\cos\theta, r\sin\theta)\in\Omega_a, \\ \det(J)(r,\theta) &= r(1+\chi(r)(\psi(\theta)-1))\big(1+(\chi(r)+r\chi'(r))(\psi(\theta)-1)\big), \text{ for } (r\cos\theta, r\sin\theta)\in\Omega_a, \\ J^{-1}_{z(\psi)}(x,y) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } (x,y)\in\Omega_d. \end{aligned}$$

First we observe that $\det(J)$ stays away from zero, as shown in Lemma 3.1. Secondly we note that $L(\psi)$ is given by the Laplacian when $r \in (\frac{1}{4}, \frac{1}{3})$, thus we see that the operator $L(\psi)$ is well-defined on Ω_1. Furthermore, we can write the operator $L(\psi)$ as

$$L(\psi) = \sum_{i,j} a_{ij}\partial_{ij} + \sum_j b_j \partial_j,$$

where $a_{ij} = \sum_k J^{-1}_{z(\psi),ik} J^{-1}_{z(\psi),jk}$ and $b_j = \sum_{i,k} J^{-1}_{z(\psi),ik} \partial_i J^{-1}_{z(\psi),jk}$. We note that the matrix $(a_{ij})_{ij} = J^{-T}_{z(\psi)} J^{-1}_{z(\psi)}$ is symmetric and positive definite as $J^{-1}_{z(\psi)}$ is invertible. Therefore there exists a $\lambda > 0$ such that

$$a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2$$

for all $\xi \in \mathbb{R}^2$; we have that $L(\psi)$ is a uniformly elliptic operator. Additionally, we can combine $L(\psi)$ and the trace operator Tr in $\mathcal{A}(\psi) = (L(\psi), Tr)$ so that (3.6) becomes

$$\mathcal{A}(\psi)c = (0, -\frac{P}{2\pi}\ln\psi).$$

For now, we derive the expression for the evolution equation. On the later chosen domains, we will be able to prove that $\mathcal{A}(\psi)$ is invertible. With this knowledge, we will be able to write c as

$$c = c(\psi) = \mathcal{A}(\psi)^{-1}(0, -\frac{P}{2\pi}\ln\psi).$$

We use this expression to obtain the evolution equation for ψ.

3.1.3 The evolution equation

The expression for c can be substituted in $\frac{\partial w}{\partial n}$ at $\partial\Omega_\psi$:

$$\begin{aligned}
\left.\frac{\partial w}{\partial n}\right|_{\partial\Omega_\psi} &= \nabla w|_{\partial\Omega_\psi} \cdot n(\psi) \\
&= \nabla(z_* c)|_{\partial\Omega_\psi} \cdot n(\psi) \\
&= z^*\nabla(z_* c)|_{\partial\Omega_1} \cdot n(\psi) \\
&= (z^* \circ \nabla \circ z_*)\mathcal{A}(\psi)^{-1}(0, -\frac{P}{2\pi}\ln\psi)\Big|_{S^1} \cdot n(\psi).
\end{aligned}$$

Using similar reasoning as above, we can express $z^* \circ \nabla \circ z_* f|_{S^1}$ in terms of the Jacobian of z:

$$z^* \circ \nabla \circ z_* f|_{S^1} = J_z^{-T}(\psi)\nabla_a f,$$

where ∇_a is the concatenation of the trace operator to S^1 and the vector of partial derivatives (∂_a, ∂_b) with respect to coordinates v_a, v_b in Ω_1. The derivation of this equality can be found in Appendix C.1. Thus, we obtain the following expression for the diffusive flux:

$$\Phi_D = -J_z^{-T}(\psi)\nabla_a \mathcal{A}(\psi)^{-1}(0, -\frac{P}{2\pi}\ln\psi) \cdot n(\psi) + \Phi_B(\psi)$$

Substituting the diffusive flux in (1.6) we obtain

$$\psi_t \frac{\psi}{\sqrt{\psi^2 + \psi_\theta^2}} = \frac{J_z^{-T}(\psi)\nabla_a \mathcal{A}(\psi)^{-1}(0, -\frac{P}{2\pi}\ln\psi) \cdot n(\psi) + \Phi_B(\psi)}{H(\psi)}.$$

To ease notation, we define $n_0 = n(1) = (\cos\theta, \sin\theta)$. This allows us to rewrite the equation above to

$$\psi_t(n_0 \cdot n(\psi)) = \frac{J_z^{-T}(\psi)\nabla_a \mathcal{A}(\psi)^{-1}(0, -\frac{P}{2\pi}\ln\psi) \cdot n(\psi) + \Phi_B(\psi)}{H(\psi)},$$

which yields the evolution equation

$$\psi_t = F(\psi) := \frac{J_z^{-T}(\psi)\nabla_a \mathcal{A}(\psi)^{-1}(0, -\frac{P}{2\pi}\ln\psi) \cdot n(\psi) + \Phi_B(\psi)}{H(\psi)(n_0 \cdot n(\psi))}. \tag{3.7}$$

Note that we can write $F(\psi)$ as

$$\begin{aligned} F(\psi) &= \frac{J_z^{-T}(\psi)\nabla_a \mathcal{A}(\psi)^{-1}(0, -\frac{P}{2\pi}\ln\psi) \cdot n(\psi)}{H(\psi)(n_0 \cdot n(\psi))} + \frac{\Phi_B(\psi)}{H(\psi)(n_0 \cdot n(\psi))} \\ &= \frac{J_z^{-T}(\psi)\nabla_a \mathcal{A}(\psi)^{-1}(0, -\frac{P}{2\pi}\ln\psi) \cdot n(\psi)}{H(\psi)(n_0 \cdot n(\psi))} + F_B(\psi) \\ &=: F_D(\psi) + F_B(\psi), \end{aligned}$$

where $F_B(\psi)$ is the right hand side function F of the ballistic model given by (2.1). The full problem is given by

$$\begin{aligned} \psi_t(t,\theta) &= F(\psi(t,\theta)) \quad (t,\theta) \in (0,\infty) \times S^1, \\ \psi(0,\theta) &= \psi_{in}(\theta) \quad \theta \in S^1, \end{aligned} \tag{3.8}$$

with F defined as in (3.7). Similar to the ballistic model, we see that F is a fully non-linear. Additionally, we note that $F_D(\psi) = 0$ if $\psi(\theta) \equiv \overline{\psi}$, which implies that the evolution of circular domains in the diffusive model is identical to the ballistic model. Contrasting to the ballistic model, the problem has become non-local due to the diffusive flux, and can thus no longer be represented with a partial differential equation. However, we will still be able to show that F satisfies the conditions for Theorem 2.1.

3.2 Local existence and uniqueness of solutions

In this section, we will show that we have local existence and uniqueness of solutions to (2.2). We choose X, D and $\mathcal{O}$ similar to the ballistic model, namely

$$\begin{aligned} X &= h^{\gamma}(S^1), \\ D &= h^{2+\gamma}(S^1), \\ \mathcal{O} &= \{\psi \in h^{2+\gamma}(S^1) \mid \psi > 1 - \frac{1}{K} + \alpha, H(\psi) > \nu, ||\psi||_{2+\gamma} < R^*\}, \end{aligned} \tag{3.9}$$

for some fixed $\gamma \in (0,1), \alpha, \nu > 0$. Note that the only difference with the ballistic model, is the additional assumption on the lower bound of ψ. Before we show that F satisfies the needed conditions, we will first show some auxiliary results.

We will show that $c(\psi)$ satisfies an a priori estimate in Lemma 3.2. We will use this lemma in Theorem 3.3 to show that for any $\psi \in \mathcal{O}$, $\mathcal{A}(\psi)$ is an isomorphism from $C^{2+\gamma}(\overline{\Omega_1})$ to $C^{\gamma}(\Omega_1) \times C^{2+\gamma}(S^1)$. Note that these results do not hold if we had chosen $D = C^2(S^1)$ and $X = C(S^1)$, which explains why we choose Hölder spaces for D and X. This theorem implies that $\mathcal{A}(\psi)$ is invertible, so that the expression in (3.7) makes sense.

Lemma 3.2. *Let $c \in C^{2+\gamma}(\overline{\Omega_1})$, $\psi \in \mathcal{O}$. Then there exists a $C > 0$ such that*

$$||c||_{2+\gamma} \leq C(||L(\psi)[c]||_{\gamma} + ||c|_{S^1}||_{2+\gamma}). \tag{3.10}$$

Proof. To prove this statement, we will split the solution c into c_a, c_d defined by

$$\begin{aligned} c_d &= \zeta c, \\ c_a &= (1-\zeta)c, \end{aligned} \tag{3.11}$$

where ζ is a smooth, radially decreasing function on Ω_1 such that $\zeta \equiv 1$ on $\Omega_d \backslash \Omega_a$, $\zeta \equiv 0$ on $\Omega_a \backslash \Omega_d$ and

$$\Omega_d \backslash \Omega_a \subset\subset \{\zeta \equiv 1\} \subset \text{supp}(\zeta) \subset\subset \Omega_d.$$

The behaviour of ζ has been visualised in Figure 6. Recall that $L(\psi)$ is uniformly elliptic on both Ω_a

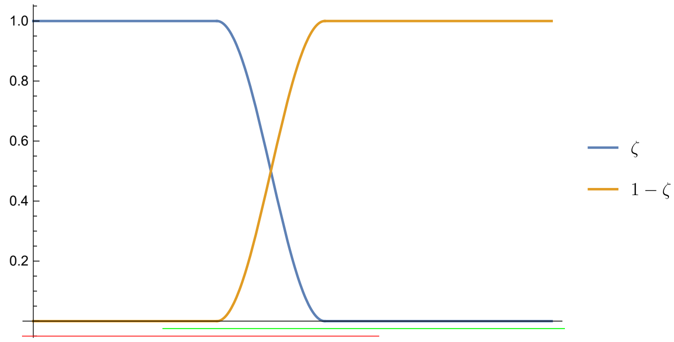

Figure 6: *Visualisation of the behaviour of ζ in Ω_1. The lines at the bottom represent Ω_d and Ω_a in red and green respectively.*

and Ω_d. Additionally we obtain that its coefficients are in $C^\gamma(S^1)$ since $\psi \in \mathcal{O}$. This result follows from the lower bound on the determinant of the Jacobian of $z(\psi)$ we had obtained earlier. Therefore, we have elliptic regularity on $\Omega_{a,d}$ [Gilbarg and Trudinger, 1977, Theorem 6.6]. To make the notation easier, we introduce $||\cdot||_{a,d}$ as a norm on $\Omega_{a,d}$ and $||\cdot||$ as a norm on Ω_1 or S^1. Thus there exist $C_{a,d} > 0$ such that

$$||c_{a,d}||_{a,d,2+\gamma} \leq C_{a,d}(||L(\psi)c_{a,d}||_{a,d,\gamma} + ||c_{a,d}||_{a,d,\infty} + || \, c_{a,d}|_{\partial\Omega_{a,d}} \, ||_{a,d,2+\gamma}), \tag{3.12}$$

where $||\cdot||_{a,d}$ denotes a norm in $\Omega_{a,d}$. The boundary terms in (3.12) can also be expressed in terms of c:

$$\begin{aligned} || \, c_d|_{\partial\Omega_d} \, ||_{d,2+\gamma} &= 0, \\ || \, c_a|_{\partial\Omega_a} \, ||_{a,2+\gamma} &= || \, c|_{S^1} \, ||_{2+\gamma} \end{aligned}$$

Next we observe that we can express $L(\psi)[c_d]$ in terms of $L(\psi)[c]$:

$$\begin{aligned} L(\psi)[\zeta c] &= \sum_{i,j} a_{ij}\partial_{ij}(\zeta c) + \sum_j b_j \partial_j(\zeta c) \\ &= \sum_{i,j} a_{ij}(\partial_{ij} c\zeta + 2\partial_i\zeta\partial_j c + \partial_{ij}\zeta c) + \sum_j b_j(\partial_j c\zeta + \partial_j\zeta c) + \\ &= \zeta\left(\sum_{i,j} a_{ij}\partial_{ij} c + \sum_j b_j\partial_j c\right) + \sum_{i,j} a_{ij}(2\partial_i\zeta\partial_j c + \partial_{ij}\zeta c) + c\sum_j b_j\partial_j\zeta \\ &= \zeta L(\psi)[c] + \sum_{i,j} a_{ij}\partial_i\zeta\partial_j c + (\sum_{i,j} a_{ij}\partial_{ij}\zeta + \sum_j b_j\partial_j\zeta)c. \end{aligned}$$

Thus we can write

$$L(\psi)[c_d] = \zeta L(\psi)[c] + \sum_j b_{d;j}\partial_j c + e_d c, \tag{3.13}$$

$$L(\psi)[c_a] = (1-\zeta)L(\psi)[c] + \sum_j b_{a;j}\partial_j c + e_a c, \tag{3.14}$$

where $b_{a,d;j}$ and $e_{a,d}$ contain derivatives of ζ, and lie in $C^\gamma(\overline{\Omega_1})$ Thus, we can estimate $||c_{a,d}||_{a,d,\infty}$ using [Gilbarg and Trudinger, 1977, Theorem 3.7] by

$$||c_{a,d}||_{a,d,\infty} = \sup_{\Omega_{a,d}} |c_{a,d}| \leq \sup_{\partial\Omega_{a,d}} |c_{a,d}| + \tilde{C}_{a,d} \sup_{\Omega_{a,d}} |L(\psi)c_{a,d}|, \tag{3.15}$$

for some $\tilde{C}_{a,d} > 0$. We will use Lemma C.2 which is an interpolation property of Hölder spaces. Thus for any $\varepsilon_1 > 0$ we know that

$$\begin{aligned}
||c_d||_{d,2+\gamma} &\leq C_d((1+\tilde{C}_d)||L(\psi)[c_d]||_{d,\gamma} + 2||\ c_d|_{\partial\Omega_d}\ ||_{d,2+\gamma}) \\
&\leq C_d(||\zeta L(\psi)[c]||_{d,\gamma} + ||b_{d;i}\partial_i c||_{d,\gamma} + ||e_d c||_{d,\gamma} + 0) \\
&\leq C_d(||L(\psi)[c]||_\gamma + ||\partial_i c||_{d,\gamma} + ||c||_{d,\gamma}) \\
&\overset{\text{Lemma } C.2}{\leq} C_d(||L(\psi)[c]||_\gamma + \varepsilon_1 ||\partial_{ii} c||_{d,\gamma} + C_{\varepsilon_1}||c||_{d,\infty}) \\
&\leq C_d(||L(\psi)[c]||_\gamma + \varepsilon_1||c||_{2+\gamma} + C_{\varepsilon_1}||c||_\infty) \\
&\overset{(3.15)}{\leq} C_d(||L(\psi)[c]||_\gamma + \varepsilon_1||c||_{2+\gamma} + C_{\varepsilon_1}(||\ c|_{S^1}\ ||_\infty + ||L(\psi)c||_\infty)) \\
&\leq C_{\varepsilon_1}(||L(\psi)[c]||_\gamma + ||\ c|_{S^1}\ ||_{2+\gamma}) + C_d\varepsilon_1||c||_{2+\gamma}),
\end{aligned}$$

for some $C_1 = C_1(\varepsilon_1) > 0$ and a different $C_d > 0$ from in (3.12). Furthermore, we note that c satisfies (3.15) on Ω_1, as $L(\psi)$ is elliptic on Ω_1. For c_a we can similarly obtain a $C_2(\varepsilon_2), C_a > 0$ for any fixed $\varepsilon_2 > 0$ such that

$$||c_a||_{a,2+\gamma} \leq C_2(||L(\psi)[c]||_\gamma + ||\ c|_{S^1}\ ||_{2+\gamma}) + C_a\varepsilon_2||c||_{2+\gamma})$$

Combining the inequalities for c_a and c_d, we obtain

$$\begin{aligned}
||c||_{2+\gamma} &= ||(1-\zeta)c + \zeta c||_{2+\gamma} \\
&= ||c_a + c_d||_{C^{2+\gamma}(\Omega_1)} \\
&\leq ||c_a||_{a,2+\gamma} + ||c_d||_{d,2+\gamma} \\
&\leq (C_1 + C_2)(||L(\psi)[c]||_\gamma + ||\ c|_{S^1}\ ||_{2+\gamma}) + (C_d\varepsilon_1 + C_a\varepsilon_2)||c||_{2+\gamma}).
\end{aligned}$$

We choose $\varepsilon_{1,2} = \frac{1}{4C_{d,a}} > 0$ respectively, so that

$$||c||_{2+\gamma} \leq (C_1 + C_2)(||L(\psi)[c]||_\gamma + ||\ c|_{S^1}\ ||_{2+\gamma}) + \frac{1}{2}||c||_{2+\gamma}.$$

Now we set $C = 2(C_1 + C_2)$ to obtain

$$||c||_{2+\gamma} \leq C(||L(\psi)[c]||_\gamma + ||\ c|_{S^1}\ ||_{2+\gamma}).$$

□

Theorem 3.3. *Let $\psi \in \mathcal{O}$. Then the function $\mathcal{A}(\psi) : C^{2+\gamma}(\overline{\Omega_1}) \to C^\gamma(\Omega_1) \times C^{2+\gamma}(S^1)$ is an isomorphism.*

Proof. We can use Lemma 3.2 in combination with [Amann, 1995, Proposition 1.1.1] on $\mathcal{A}$ where we note that $\mathcal{A}$ depends continuously on ψ, so that $\mathcal{A} \in C(\mathcal{O}, \mathcal{L}(C^{2+\gamma}(\overline{\Omega_1}), C^\gamma(\Omega_1) \times C^{2+\gamma}(S^1)))$. From the proposition, we know that $\mathcal{A}(\psi)$ is an isomorphism for all $\psi \in \mathcal{O}$ if there exists a $\psi^* \in \mathcal{O}$ such that $A(\psi^*)$ is an isomorphism. We choose $\psi^* \equiv 1 \in \mathcal{O}$. For this ψ^*, we note that $\mathcal{A}(\psi^*) = (\Delta, Tr)$ which is an isomorphism on the correct space. Therefore, we know that $\mathcal{A}(\psi)$ must be an isomorphism as well for any $\psi \in \mathcal{O}$. □

With the preliminary proofs in place, we will now verify that F satisfies the conditions for Theorem 2.1. Similar to the ballistic model, we will first show Fréchet differentiability.

3.2.1 Fréchet differentiability

We will prove that F is Fréchet differentiable at any $\psi_0 \in \mathcal{O}$ in Lemma 3.4.

Lemma 3.4. *Let $\psi_0 \in \mathcal{O}$. Then F as defined in (3.7) is Fréchet differentiable at ψ_0, and $DF(\psi_0)$ can be written as $A + B$ where A, B are defined by*

$$
\begin{aligned}
A[h] &= a(\psi_0)h_{\theta\theta},\\
B[h] &= \frac{\nabla_a Dc(\psi_0)[h]^T J_z^{-1}(\psi_0)n(\psi_0)}{H(\psi_0)n_0\cdot n(\psi_0)}\\
&\quad - \frac{\nabla_a c(\psi_0)^T J_z^{-1}(\psi_0)DJ_z(\psi_0)[h]J_z^{-1}(\psi_0)n(\psi_0)}{H(\psi_0)n_0\cdot n(\psi_0)}\\
&\quad + \left(\frac{\nabla_a c(\psi_0)^T J_z^{-1}(\psi_0)}{H(\psi_0)n_0\cdot n(\psi_0)} - \frac{F_D(\psi_0)}{n_0\cdot n(\psi_0)}\right)Dn(\psi_0)[h]\\
&\quad + a_1 h_\theta + a_0 h,
\end{aligned}
\tag{3.16}
$$

where

$$
a(\psi_0) = \frac{F_D(\psi_0)\cdot(\psi_0^2+\psi_{0,\theta}^2)^{-3/2}\psi_0}{H(\psi_0)} + a_2
$$

and a_0, a_1, a_2 are given by (2.11).

Proof. To prove Fréchet differentiability, we first note that we only need to show that F_D is Fréchet differentiable, since F_B is by Lemma 2.6. We will obtain differentiability of F_D by showing differentiability of the individual terms in its definition. First we note that H and n are differentiable by the properties discussed in the previous chapter. Their derivatives are given by

$$
\begin{aligned}
DH(\psi_0)[h] &= -\frac{3}{2}(\psi_0^2+2\psi_{0,\theta}^2-\psi_0\psi_{0,\theta\theta})(\psi_0^2+\psi_{0,\theta}^2)^{-5/2}(2\psi_0 h+2\psi_{0,\theta}h_\theta)\\
&\quad + (\psi_0^2+\psi_{0,\theta}^2)^{-3/2}(2\psi_0 h+4\psi_{0,\theta}h_\theta - h\psi_{0,\theta\theta}-\psi_0 h_{\theta\theta})\\
&=: H_0 h + H_1 h_\theta + H_2 h_{\theta\theta},
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
Dn(\psi_0)[h] &= -\frac{1}{2}(\psi_{0,\theta}^2+\psi_0^2)^{-3/2}(2\psi_{0,\theta}h_\theta+2\psi_0 h)(\psi_{0,\theta}\sin\theta+\psi_0\cos\theta, -\psi_{0,\theta}\cos\theta+\psi_0\sin\theta)\\
&\quad + (\psi_{0,\theta}^2+\psi_0^2)^{-1/2}(h_\theta\sin\theta + h\cos\theta, -h_\theta\cos\theta+h\sin\theta)\\
&=: (N_{00}h+N_{01}h_\theta, N_{10}h+N_{11}h_\theta),
\end{aligned}
\tag{3.18}
$$

where we note that each component H_i, N_{ij} are in $h^\gamma(S^1)$. Next we note that the Jacobian J_z has differentiable components, since each partial derivative of z is linear in ψ and ψ_θ. Therefore, it is (componentwise) differentiable with derivative

$$
DJ_z(\psi_0)[h] = \begin{pmatrix}\cos(\theta)h & -\sin(\theta)h+\cos(\theta)h_\theta\\ \sin(\theta)h & \cos(\theta)h+\sin(\theta)h_\theta\end{pmatrix} = \begin{pmatrix}J_{000}h+J_{100}h_\theta & J_{001}h+J_{101}h_\theta\\ J_{010}h+J_{110}h_\theta & J_{011}h+J_{111}h_\theta\end{pmatrix},
\tag{3.19}
$$

where we see that also $J_{ijk} \in h^\gamma(S^1)$. We note that also J_z^{-1} is Fréchet differentiable, since the determinant of J_z stays away from zero. Thus we can apply the inverse rule from Lemma 2.3 so that

$$
DJ_z^{-1}(\psi_0)[h] = -J^{-1}(\psi_0)DJ_z(\psi_0)[h]J^{-1}(\psi_0).
$$

Note that this expression contains only h and h_θ with $h^\gamma(S^1)$ coefficients.

Next we will show Fréchet differentiability of $c(\psi) = \mathcal{A}(\psi)^{-1}(0, -\frac{P}{2\pi}\ln\psi)$. We will not be able to find an explicit expression in terms of the derivatives of h, yet we are able to show that it is not contained in the definition A. First we investigate differentiability of $\mathcal{A}(\psi) = (L(\psi), Tr)$. Clearly, Tr is differentiable with zero derivative. For L, we know that it is differentiable, since J_z^{-1} is differentiable. Thus we can write

$$\begin{aligned}
DL(\psi_0)[h] &= D(J^{-1}_{z,ij}(\psi)\partial_i(J^{-1}_{z,kj}(\psi)\partial_k))(\psi_0)[h] \\
&= D(J^{-1}_{z,ij})(\psi_0)[h]\partial_i(J^{-1}_{z,kj}(\psi)\partial_k) + J^{-1}_{z,ij}(\psi_0)\partial_i(D(J^{-1}_{z,kj})(\psi_0)[h]\partial_k) \\
&= -J^{-2}_{z,ij}(\psi_0)DJ_{z,ij}(\psi_0)[h]\partial_i(J^{-1}_{z,kj}(\psi)\partial_k) - J^{-1}_{z,ij}(\psi_0)\partial_i(J^{-2}_{z,kj}(\psi_0)DJ_{z,kj})(\psi_0)[h]\partial_k) \\
&= -J^{-2}_{z,ij}(\psi_0)DJ_{z,ij}(\psi_0)[h]\partial_i(J^{-1}_{z,kj}(\psi)\partial_k) - J^{-1}_{z,ij}(\psi_0)\partial_i(J^{-2}_{z,kj}(\psi_0)DJ_{z,kj})(\psi_0)[h]\partial_k)
\end{aligned}$$

Using the expression for $DJ_{z,ij}(\psi_0)[h]$ we obtain

$$DL(\psi_0)[h] = -J^{-2}_{z,ij}(\psi_0)(J_{ij0}(\psi_0)h + J_{ij1}(\psi_0)h_\theta)\partial_i(J^{-1}_{z,kj}(\psi_0)\partial_k) - J^{-1}_{z,ij}(\psi_0)\partial_i(J^{-2}_{z,kj}(\psi_0)(J_{kj0}(\psi_0)h + J_{kj1}(\psi_0)h_\theta)\partial_k)$$

Observe that $DL(\psi_0)[h]$ can be written as

$$DL(\psi_0)[h] = hL_0(\psi_0) + h_\theta L_1(\psi_0),$$

where $L_{0,1}(\psi_0)$ are two second order differential operators. Additionally, both have coefficients in $h^\gamma(S^1)$ since $J_{z,ij}(\psi_0), J^{-1}_{z,ij}(\psi_0), J_{ijk}(\psi_0)$ are in $h^\gamma(S^1)$. Therefore we know that $\mathcal{A}(\psi)$ is Fréchet differentiable. Additionally we note that also $\ln\psi$ is Fréchet differentiable, with derivative $D(\ln\psi)(\psi_0)[h] = h\psi_0^{-1}$.

We will use an additional property of Fréchet derivatives as noted in [Cartan, 1971, Proposition 2.5.2]. This proposition states that if $w(x) = f(u(x), v(x))$, and f a continuous bilinear map, and u, v are differentiable at a point, then w is differentiable at that point with derivative $Dw(a)[h] = f(Du[a][h], v(a)) + f(u(a), Dv(a)[h])$. We apply this proposition to $\mathcal{A}(\psi)^{-1}(0, -\frac{P}{2\pi}\ln\psi)$ to obtain

$$\begin{aligned}
D(\mathcal{A}(\psi)^{-1}(0, -\frac{P}{2\pi}(\ln\psi)))(\psi_0)[h] &= D(\mathcal{A}(\psi)^{-1})(0, -\frac{P}{2\pi}(\ln\psi_0)) + \mathcal{A}(\psi_0)^{-1}(0, D(-\frac{P}{2\pi}(\ln\psi))(\psi_0)[h]) \\
&= -\mathcal{A}(\psi_0)^{-1}D\mathcal{A}(\psi_0)[h]\mathcal{A}(\psi_0)^{-1}(0, -\frac{P}{2\pi}(\ln\psi_0)) + \mathcal{A}(\psi_0)^{-1}(0, -\frac{P}{2\pi}\frac{h}{\psi_0}) \\
&= -\mathcal{A}(\psi_0)^{-1}D(L(\psi), Tr)(\psi_0)[h]c(\psi_0) + \mathcal{A}(\psi_0)^{-1}(0, -\frac{P}{2\pi}\frac{h}{\psi_0}) \\
&= -\mathcal{A}(\psi_0)^{-1}(DL(\psi_0)[h]c(\psi_0), 0) + \mathcal{A}(\psi_0)^{-1}(0, -\frac{P}{2\pi}\frac{h}{\psi_0}) \\
&= -\mathcal{A}(\psi_0)^{-1}(DL(\psi_0)[h]c(\psi_0), \frac{P}{2\pi}\frac{h}{\psi_0}).
\end{aligned}$$

In summary, we have that

$$Dc(\psi_0)[h] = -\mathcal{A}(\psi_0)^{-1}(DL(\psi_0)[h]c(\psi_0), \frac{P}{2\pi}\frac{h}{\psi_0}) \tag{3.20}$$

is the Fréchet derivative of c at ψ_0. Next we investigate whether $\nabla_a Dc(\psi_0)[h]$ contains second order derivatives of h.

We know that $DL(\psi_0)[h]$ contains second order differential operators and $c(\psi_0) \in C^{2+\gamma}(\overline{\Omega_1})$, so $DL(\psi_0)[h]c \in C^\gamma(\Omega_1)$. By the properties of $\mathcal{A}$, we consequently see that $Dc(\psi_0)[h]$ lies in $C^{2+\gamma}(\overline{\Omega_1})$. Thus $Dc(\psi_0)[h]$ is twice differentiable, which implies that $\nabla_a Dc(\psi_0)[h]$ cannot have a $h_{\theta\theta}$ term, since h is only twice differentiable in θ. Therefore, we know that $\nabla_a Dc(\psi_0)[h]$ is contained in the definition of $B[h]$.

Finally, we note that $H(\psi)$ and $n_0 \cdot n(\psi)$ stay away from zero for any $\psi \in \mathcal{O}$. Therefore, we can use the properties of the Fréchet derivative to show that F_D is differentiable with derivative at ψ_0 given by

$$\begin{aligned}
DF_D(\psi_0)[h] &= D((\nabla_a c)^T J_z^{-1} n(\psi)(H(\psi)n_0 \cdot n(\psi))^{-1})(\psi_0)[h] \\
&= \frac{D(\nabla_a c)^T(\psi_0)[h] J_{z(\psi_0)}^{-1} n(\psi_0) + (\nabla_a c(\psi_0))^T D J_{z(\psi)}^{-1}(\psi_0)[h] n(\psi_0) + (\nabla_a c(\psi_0))^T J_{z(\psi_0)}^{-1} D(n(\psi))(\psi_0)[h]}{H(\psi_0) n_0 \cdot n(\psi_0)} \\
&- \frac{(\nabla_a c(\psi_0))^T J_{z(\psi_0)}^{-1} n(\psi)}{(H(\psi_0) n_0 \cdot n(\psi_0))^2} D(H(\psi) n_0 \cdot n(\psi))(\psi_0)[h]. \\
&= \frac{(\nabla_a Dc(\psi_0)[h])^T J_{z(\psi_0)}^{-1} n(\psi_0) - (\nabla_a c(\psi_0))^T J_{z(\psi_0)}^{-1} D J_z(\psi_0)[h] J_{z(\psi_0)}^{-1} n(\psi_0) + (\nabla_a c(\psi_0))^T J_{z(\psi_0)}^{-1} D n(\psi_0)[h]}{H(\psi_0) n_0 \cdot n(\psi_0)} \\
&- F_D(\psi_0) \frac{DH(\psi_0)[h] n_0 \cdot n(\psi_0) + H(\psi_0) n_0 \cdot Dn(\psi_0)[h]}{H(\psi_0) n_0 \cdot n(\psi_0)}.
\end{aligned}$$

We see that the individual Fréchet derivatives all have components in $h^\gamma(S^1)$ and have a combination of h, h_θ and $h_{\theta\theta}$ terms. The functions $n(\psi_0), H(\psi_0), \nabla_a c(\psi_0), J_{z(\psi_0)}^{-1}$ are also in $h^\gamma(S^1)$ as established earlier. Now we write $a(\psi_0)$ as

$$a(\psi_0) = (-F_D(\psi_0) \frac{H_2(\psi_0)}{H(\psi_0)} + a_2),$$

so that $DF(\psi_0)$ becomes

$$DF(\psi_0) = A + B,$$

where A, B are given by (3.16). □

Next we will show that $DF(\psi_0)$ is sectorial and its graph norm is equivalent to the norm of D.

3.2.2 Sectoriality and equivalence of the graph norm

To show that F satisfies the second condition for Theorem 2.1, we take a different approach than in the ballistic model. Instead of translating the problem to $h^\gamma(\mathbb{R})$, we will show that A is a sectorial in X and B a perturbation of A in X. Consequently, $A + B = DF(\psi_0)$ is sectorial in X. We will prove sectoriality of A in Lemma 3.5, and the perturbation property of B in Lemma 3.6. Finally we will show that $DF(\psi_0)$ is sectorial in X and its graph norm is equivalent to the norm of D in Lemma 3.7.

Lemma 3.5. *The operator $(A, D(A))$ where $A[h]$ is given by (3.16) and $D(A) = h^{2+\gamma}(S^1)$ is sectorial in X.*

Proof. First we verify that there exists a $\mu > 0$ such that $a \geq \mu$. Even though the sign might not be explicitly computable from the current expression, we can determine the sign from the original problem statement. Recall that $F_D(\psi)$ was given by

$$F_D(\psi) = \frac{\partial_n w|_{\partial\Omega_\psi}}{H(\psi) n_0 \cdot n(\psi)},$$

and w satisfies

$$\begin{aligned} \Delta w &= 0 \quad \text{in } \Omega_\psi, \\ w &= -\phi \text{ on } \partial\Omega_\psi. \end{aligned}$$

Naturally, we know that $\partial_n w(x_0) \geq 0$ for all $x_0 \in \partial\Omega_\psi$ as Δ is an elliptic operator. Therefore we have that $\Phi_D(\psi_0) \cdot n(\psi_0) \geq 0$. Since $H(\psi_0) > \nu$ and $n(\psi_0) \cdot n_0 \geq 0$ we thus obtain that $F_D(\psi_0) \geq 0$. Since $\psi_0^2 + \psi_{0,\theta}^2 \geq 0, \psi_0 \geq 0$ we see that $H_2(\psi_0) = -(\psi_0^2 + \psi_{0,\theta}^2)^{-3/2}\psi_0 \leq 0$. Finally, since $H(\psi_0) > 0$, we know that

$$a(\psi_0) = (-F_D(\psi_0) \frac{H_2}{H(\psi_0)} + a_2) \geq a_2 \geq \xi > 0,$$

where ξ was the coefficient determined by Lemma 2.7. Thus we choose $\mu = \xi > 0$ to obtain $a \geq \mu > 0$.

Since a is strictly positive, we obtain that A is a uniformly elliptic operator. Thus we can use [Lunardi, 1995, Theorem 3.1.14] and similar reasoning as 2.10 to obtain that A is sectorial. □

Lemma 3.6. *The operator $B = DF(\psi_0) - A$ with domain $D(B) = h^{1+\gamma}(S^1)$ has the following property. For all $\varepsilon > 0$ and $h \in D(A)$ there exists a $C_\varepsilon > 0$ such that*

$$||B[h]||_\gamma \leq \varepsilon ||A[h]||_\gamma + C_\varepsilon ||h||_\gamma.$$

Proof. To show this property, let $\varepsilon > 0$ be fixed and $h \in D(A)$. Using the definition of B, we see that B can be written as a sum

$$B[h] = B_1[h] + B_2[h],$$

where

$$\begin{aligned} B_1[h] &= b_0(\psi_0)h + b_1(\psi_0)h_\theta, \\ B_2[h] &= b_2(\psi_0)\nabla_a Dc(\psi_0)[h], \end{aligned}$$

for some appropriate $b_i \in h^\gamma(S^1)$. For B_1 we note that there exists some $C_1 > 0$ such that $||B_1[h]||_\gamma \leq C_1||h||_{1+\gamma}$. Note that C_1 can be chosen independently of ψ_0, since we can estimate the norm of ψ_0 by R^*. By the interpolation property in Lemma C.2, we see that $||B_1[h]||_\gamma \leq C_1(\delta_1||h||_{2+\gamma} + C_{\delta_1}||h||_\infty)$ for all $\delta_1 > 0$.

For B_2 we introduce $\gamma' \in (0, \gamma)$ fixed. Then there exists a $C_2 > 0$ such that

$$||B_2[h]||_\gamma = ||b_2(\psi_0)\nabla_a Dc(\psi_0)[h]||_\gamma \leq C_2||\nabla_a Dc(\psi_0)[h]||_{1+\gamma'}.$$

Recall that ∇_a is the concatenation of the trace to S^1 and the vector of partial derivatives on Ω_1. Since $Dc(\psi_0)$ is a bounded operator on $h^{2+\gamma}(S^1)$, it is also a bounded operator $h^{2+\gamma'}(S^1)$. We can combine these results to obtain a $C_3 > 0$ such that

$$||\nabla_a Dc(\psi_0)[h]||_{1+\gamma'} \leq C_3||h||_{2+\gamma'}.$$

Secondly we apply the interpolation property from Lemma C.2 with $\gamma_1 = 2 + \gamma'$, $\gamma_2 = 2 + \gamma$. Thus we fix $\delta_2 > 0$ so that there exists a $C_{\delta_2} > 0$ with

$$||h||_{2+\gamma'} \leq \delta_2||h||_{2+\gamma} + C_{\delta_2}||h||_\infty.$$

Therefore we obtain that

$$||B_2[h]||_\gamma \leq C_2C_3\delta_2||h||_{2+\gamma} + C_{\delta_2}||h||_\infty.$$

In summary, we have that

$$||B[h]||_\gamma \leq (C_1\delta_1 + C_2C_3\delta_2)||h||_{2+\gamma} + C_{\delta_1,\delta_2}||h||_\infty.$$

Next we use the definition of $||h||_{2+\gamma}$ and apply the inequality $||h_\theta||_\infty \leq \delta_3||h_{\theta\theta}||_\infty + C_{\delta_3}||h||_\infty$ for $\delta_3 = 1$ so that there exists a $C_4 > 0$ such that

$$\begin{aligned} ||h||_{2+\gamma} &= ||h_{\theta\theta}||_\gamma + ||h||_\infty + ||h_\theta||_\infty \\ &\leq ||h_{\theta\theta}||_\gamma + ||h||_\infty + ||h_{\theta\theta}||_\infty + C_4||h||_\infty \\ &\leq 2||1/aA[h]||_\gamma + (1 + C_4)||h||_\infty \\ &\leq C_{a,\mu}||A[h]||_\gamma + (1 + C_4)||h||_\infty, \end{aligned}$$

where $C_{a,\mu}$ comes from Lemma B.2. Thus we can write

$$||B[h]||_\gamma \leq (C_1\delta_1 + C_2C_3\delta_2)C_{a,\mu}||A[h]||_\gamma + C_{\delta_1,\delta_2}||h||_\infty,$$

for some $C_{\delta_1,\delta_2} > 0$. We choose $\delta_1 = \frac{\varepsilon}{2C_1 C_{a,\mu}}, \delta_2 = \frac{\varepsilon}{2C_{a,\mu}C_2C_3}$ so that for all $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that

$$||B[h]||_\gamma \leq \varepsilon||A[h]||_\gamma + C_\varepsilon||h||_\infty.$$

$\square$

Lemma 3.7. *The operator $DF(\psi_0)$ is sectorial in X for any $\psi_0 \in \mathcal{O}$ and its graph norm is equivalent to the norm of D.*

Proof. Fix $\psi_0 \in \mathcal{O}$. Sectoriality of $DF(\psi_0)$ is a direct result of the perturbation property of B with respect to the sectorial operator A from [Engel and Nagel, 2006, Theorem III.2.10]. The proof that the graph norm of $DF(\psi_0)$ is equivalent the norm of D will be similar to the proof for the ballistic model. To show equivalence, we need to show that there exists a $C_1, C_2 > 0$ such that

$$C_1||h||_{2+\gamma} \leq ||h||_X + ||DF(\psi_0)[h]||_\gamma \leq C_2||h||_{2+\gamma}$$

for all $h \in \mathcal{O}$.

The second inequality can be shown using Lemma 3.6. Let $\varepsilon > 0$ be fixed. Then we know that

$$\begin{aligned}
||h||_\gamma + ||DF(\psi_0)[h]||_\gamma &= ||h||_\infty + [h]_X + ||A[h] + B[h]||_\gamma \\
&\leq ||h||_\infty + C||h_\theta||_\infty + ||Ah||_\gamma + ||Bh||_\gamma \\
&\leq C_\varepsilon||h||_\infty + C||h_\theta||_\infty + (1+\varepsilon)||A[h]||_\gamma \\
&\leq C_\varepsilon||h||_\infty + C||h_\theta||_\infty + C_{a,\varepsilon}||h_{\theta\theta}||_\gamma \\
&\leq C_2||h||_{2+\gamma},
\end{aligned}$$

for some $C_2 > 0$. To obtain the first inequality we first show that B is bounded with respect to the graph norm. Let $0 < \varepsilon < 1$ be fixed. Then we know by the perturbation property of B that there exists a $C_\varepsilon > 0$ such that

$$\begin{aligned}
||B[h]||_\gamma &\leq \varepsilon||A[h]||_\gamma + C_\varepsilon||h||_\infty \\
&\leq \varepsilon||DF(\psi_0)[h] - B[h]||_\gamma + C_\varepsilon||h||_\infty \\
&\leq \varepsilon||DF(\psi_0)[h]||_\gamma + \varepsilon||B[h]||_\gamma + C_\varepsilon||h||_\infty \\
\iff (1-\varepsilon)||B[h]||_\gamma &\leq \varepsilon||DF(\psi_0)[h]||_\gamma + C_\varepsilon||h||_\infty \\
\iff ||B[h]||_\gamma &\leq C_\varepsilon(||DF(\psi_0)[h]||_\gamma + ||h||_\gamma).
\end{aligned}$$

Using this inequality, we introduce $\delta > 0$ such that $||h_\theta||_\infty \leq \delta||h_{\theta\theta}||_\infty + C_\delta||h||_\infty$. Thus we know that

$$\begin{aligned}
||h||_{2+\gamma} &= ||h||_\infty + ||h_\theta||_\infty + ||h_{\theta\theta}||_\gamma \\
&\leq c_1||h||_\infty + (\delta+1)||h_{\theta\theta}||_\gamma \\
&\leq c_1(||h||_\infty + ||h_{\theta\theta}||_\gamma) \\
&\leq c_1(||h||_\infty + C_{a,\mu}||Ah||_\gamma) \\
&\leq c_1(||h||_\infty + ||DF(\psi_0)[h]||_\gamma + ||Bh||_\gamma) \\
&\leq c_1(||h||_\gamma + ||DF(\psi_0)[h]||_\gamma + C_\varepsilon(||DF(\psi_0)[h]||_\gamma + ||h||_\gamma)) \\
&\leq c_1(||h||_\gamma + C_\varepsilon||DF(\psi_0)[h]||_\gamma) \\
&\leq c_1(||h||_\gamma + ||DF(\psi_0)[h]||_\gamma),
\end{aligned}$$

for some $c_1 = c_1(a, \mu, \varepsilon, \delta) > 0$. By letting $C_1 = c_1^{-1} > 0$, we also obtain the first inequality. $\square$

3.2.3 Local Lipschitz continuity

The final condition we need to verify is that DF is locally Lipschitz continuous with respect to ψ_0. We will show this condition by showing that F is twice Fréchet differentiable at $\psi_0 \in \mathcal{O}$.

Lemma 3.8. *The function F is twice Fréchet differentiable at any $\psi_0 \in \mathcal{O}$.*

Proof. We will not give an explicit expression of what the second derivative looks like, since we do not need this to show locally Lipschitz continuity, as can be seen in Corollary 3.9. Let $\psi_0 \in \mathcal{O}$, and recall the definition of $DF(\psi_0)$ from (3.16). We will show that DF is differentiable at ψ_0 with derivative $D^2F(\psi_0)[h_1, h_2]$ for $h_1, h_2 \in D$. Similar to the first derivative, we will show differentiability of DF using properties of the Fréchet derivative.

First we note that DH, DJ_z, Dn and Dc are differentiable by similar arguments for the first derivative. In particular, we have that $D^2J_{z,ij}(\psi_0)[h_1, h_2] = 0$ since $DJ_{z,ij}(\psi_0)$ does not depend on ψ_0. Additionally, the second derivatives of H and n can be written as

$$D^2H(\psi_0)[h_1, h_2] = DH_0(\psi_0)[h_2]h_1 + DH_1(\psi_0)[h_2]h_{1,\theta} + DH_2(\psi_0)[h_2]h_{1,\theta\theta}$$
$$D^2N(\psi_0)[h_1, h_2] = DN_0(\psi_0)[h_2]h_1 + DN_1(\psi_0)[h_2]h_{1,\theta},$$

where the components $H_i, N_i = (N_{i0}, N_{i1})$ are given by (3.17) and (3.18). Similar to before, we obtain that the derivatives of these components can be written as

$$DH_i(\psi_0)[h_2] = H_{i0}(\psi_0)h_2 + H_{i1}(\psi_0)h_{2,\theta} + H_{i2}(\psi_0)h_{2,\theta\theta},$$
$$DN_i(\psi_0)[h_2] = N_{i,0}(\psi_0)h_2 + N_{i,1}(\psi_0)h_{2,\theta},$$

and $H_{ij}(\psi_0), N_{ij,k}(\psi_0) \in h^\gamma(S^1)$. The second derivative of c is equal to

$$\begin{aligned}
D^2c(\psi_0)[h_1, h_2] &= -D(\mathcal{A}(\cdot)^{-1})(\psi_0)[h_2](DL(\psi_0)[h_1]c(\psi_0), \frac{P}{2\pi}\frac{h_1}{\psi_0})) \\
&\quad - \mathcal{A}(\psi_0)^{-1}(D^2L(\psi_0)[h_1, h_2]c(\psi_0) + DL(\psi_0)[h_1]Dc(\psi_0)[h_2], -\frac{P}{2\pi}\frac{h_1h_2}{\psi_0^2}) \\
&= \mathcal{A}(\psi_0)^{-1}D\mathcal{A}(\psi_0)[h_2]\mathcal{A}(\psi_0)^{-1}(DL(\psi_0)[h_1]c(\psi_0), \frac{P}{2\pi}\frac{h_1}{\psi_0})) \\
&\quad - \mathcal{A}(\psi_0)^{-1}(D^2L(\psi_0)[h_1, h_2]c(\psi_0) + DL(\psi_0)[h_1]Dc(\psi_0)[h_2], -\frac{P}{2\pi}\frac{h_1h_2}{\psi_0^2}).
\end{aligned}$$

Using the definition of $Dc(\psi_0)[h_1]$, we obtain

$$\begin{aligned}
D^2c(\psi_0)[h_1, h_2] &= -\mathcal{A}(\psi_0)^{-1}D\mathcal{A}(\psi_0)[h_2]Dc(\psi_0)[h_1] \\
&\quad - \mathcal{A}(\psi_0)^{-1}(D^2L(\psi_0)[h_1, h_2]c(\psi_0) + DL(\psi_0)[h_1]Dc(\psi_0)[h_2], -\frac{P}{2\pi}\frac{h_1h_2}{\psi_0^2}) \\
&= -\mathcal{A}(\psi_0)^{-1}(DL(\psi_0)[h_2]Dc(\psi_0)[h_1], 0) \\
&\quad - \mathcal{A}(\psi_0)^{-1}(D^2L(\psi_0)[h_1, h_2]c(\psi_0) + DL(\psi_0)[h_1]Dc(\psi_0)[h_2], -\frac{P}{2\pi}\frac{h_1h_2}{\psi_0^2}) \\
&= -\mathcal{A}(\psi_0)^{-1}(DL(\psi_0)[h_2]Dc(\psi_0)[h_1] + D^2L(\psi_0)[h_1, h_2]c(\psi_0) + DL(\psi_0)[h_1]Dc(\psi_0)[h_2], -\frac{P}{2\pi}\frac{h_1h_2}{\psi_0^2}).
\end{aligned}$$

Analogously to $DL(\psi_0)[h]$, we have that $D^2L(\psi_0)[h_1, h_2]$ will be a second order differential operator, that can be written as

$$D^2L(\psi_0)[h_1, h_2]c = h_1h_2L_{00}(\psi_0)[c] + h_{1,\theta}h_2L_{10}(\psi_0)[c] + h_1h_{2,\theta}L_{01}(\psi_0)[c] + h_{1,\theta}h_{2,\theta}L_{11}(\psi_0)[c].$$

Thus we have that $D^2L(\psi_0)[h_1,h_2]c(\psi_0) \in C^\gamma(\Omega_1)$. Therefore, we have that

$$(DL(\psi_0)[h_2]Dc(\psi_0)[h_1] + D^2L(\psi_0)[h_1,h_2]c(\psi_0) + DL(\psi_0)[h_1]Dc(\psi_0)[h_2], -\frac{P}{2\pi}\frac{h_1h_2}{\psi_0^2}) \in C^\gamma(\Omega_1)\times C^{2+\gamma}(S^1),$$

which lies in the domain of $\mathcal{A}(\psi_0)^{-1}$, and the definition of the second derivative of c makes sense.

Now we can apply the properties of the Fréchet derivative to achieve that DF is differentiable at ψ_0. This derivative will contain products of first derivatives $Df(\psi_0)[h_1]Dg(\psi_0)[h_2]$ and second derivatives $D^2f(\psi_0)[h_1,h_2]$ and is a bounded, operator in both h_1 and h_2. However, the expression of $D^2F(\psi_0)[h_1,h_2]$ would be too cumbersome to mention. Finally we note that all components of $D^2F(\psi_0)$ depend continuously on ψ_0, which implies that D^2F itself is continuous in ψ_0. □

Corollary 3.9. *For all $\overline{u} \in \mathcal{O}$ there are $R = R(\overline{u}), L = L(\overline{u}) > 0$ such that*

$$||DF(\psi_1) - DF(\psi_0)||_{L(D,X)} \leq L||\psi_1 - \psi_0||_{2+\gamma}, \tag{3.21}$$

for all $\psi_{0,1} \in B(\overline{u}, R) \subset D$.

Proof. This follows from Lemma 3.8, Lemma 2.12 and [Cartan, 1971, Theorem 3.3.2] and continuity of D^2F in ψ_0. □

We can combine Lemmas 3.4, 3.7 and Corollary 3.9 to achieve the main theorem for the diffusive model in Theorem 3.10.

Theorem 3.10. *Let $\psi_0 \in \mathcal{O}$ Then there exist $\delta = \delta(\psi_{in}), r = r(\psi_{in})$ such that for every $t_0 \in [0,r]$ and $\psi_{in} \in \mathcal{O}$ with $||\psi_{in} - \psi_0||_{2+\gamma} \leq r$ there is a strict solution $\psi \in C([0,\delta]; h^{2+\gamma}(S^1)) \cap C^1([0,\delta]; h^\gamma(S^1))$ to*

$$\psi_t(t) = F(\psi(t)) \quad t \in [0,\delta],$$
$$\psi(0) = \psi_{in}.$$

Next, we will investigate linear stability of circular solutions, similarly to the ballistic model.

3.3 Linear stability of circular solutions

In this section we will show that circles are linearly stable under the evolution equation. The proof will be analogous to the proof for the ballistic model. The first property we will show is that $F(\beta\psi) = F(\psi)$ in Lemma 3.11. This property allows the evolution equation for the diffusive model to be transformed analogously to Section 2.4. We will introduce the operators A, G such that the transformed evolution equation satisfies $\rho_t = A\rho + G(\rho)$. Finally we will find the spectrum of A to find an ω_0 and show that A, G satisfy the assumptions of Theorem 2.14.

Lemma 3.11. *Let $\beta, \psi \in \mathcal{O}$ such that $\beta\psi \in \mathcal{O}$. Then $F(\beta\psi) = F(\psi)$.*

Proof. To show that $F(\beta\psi) = F(\psi)$, we first note that F_B already satisfies this property, so we only investigate F_D. For H, n, J_z^{-1} we obtain that

$$H(\beta\psi) = \beta^{-1}H(\psi),$$
$$n(\beta\psi) = n(\psi),$$
$$J_z^{-1}(\beta\psi) = \beta^{-1}J_z^{-1}(\psi).$$

For $\nabla_a c$ we see that $c(\beta\psi)$ has to satisfy the system

$$L(\beta\psi)c = 0 \quad \text{in } \Omega_1,$$
$$c = -\frac{P}{2\pi}\ln(\beta\psi) \quad \text{on } S^1.$$

By definition of L, we have that

$$\begin{aligned} L(\beta\psi) &= J^{-1}_{z,ab}(\beta\psi)\partial_a(J^{-1}_{z,cb})(\beta\psi)\partial_c \\ &= \beta^{-2}J^{-1}_{z,ab}(\psi)\partial_a(J^{-1}_{z,cb})(\psi)\partial_c \\ &= \beta^{-2}L(\psi). \end{aligned}$$

Additionally, since $\ln(\beta\psi) = \ln\beta + \ln\psi$, we see that the function $c(\beta\psi) = c(\psi) - \frac{P}{2\pi}\ln\beta$ solves the system:

$$\begin{aligned} L(\beta\psi)c(\beta\psi) &= \beta^{-2}L(\psi)(c(\psi) - \frac{P}{2\pi}\ln\beta) = 0 \quad \text{in } \Omega_1, \\ c(\beta\psi) &= c(\psi) - \frac{P}{2\pi}\ln\beta = -\frac{P}{2\pi}\ln(\beta\psi) \quad \text{on } S^1. \end{aligned}$$

Therefore, we know that $c(\beta\psi) = \mathcal{A}(\beta\psi)^{-1}(0, -\frac{P}{2\pi}\ln(\beta\psi)) = \mathcal{A}(\psi)^{-1}(0, -\frac{P}{2\pi}\ln\psi) - \frac{P}{2\pi}\ln\beta$ Thus for $F_D(\beta\psi)$ we obtain

$$\begin{aligned} F_D(\beta\psi) &= \frac{\nabla_a c(\beta\psi)^T J_z^{-1}(\beta\psi)n(\beta\psi)}{H(\beta\psi)(n(\beta\psi)\cdot n_0)} \\ &= \frac{\nabla_a (c(\psi) + \ln\beta)^T \beta^{-1} J_z^{-1}(\psi)n(\psi)}{\beta^{-1}H(\psi)(n(\psi)\cdot n_0)} \\ &= \frac{\nabla_a c(\psi)^T J_z^{-1}(\psi)n(\psi)}{H(\psi)(n(\psi)\cdot n_0)} = F_D(\psi), \end{aligned}$$

so $F(\beta\psi) = F(\psi)$ for all $\beta, \psi \in \mathcal{O}$. □

Since this function F also has the scaling property, we will use the same transformation of Section 2.4. Thus we have the evolution equation

$$\rho_{\tilde{t}}(\tilde{t}) = \bar{F}(\rho) = F(\rho+1) - \frac{P}{2\pi}(\rho+1). \tag{3.22}$$

We define the operators A, G as

$$\begin{aligned} A\rho &= D\bar{F}(0)[\rho], \\ G(\rho) &= \bar{F}(\rho) - A\rho, \end{aligned} \tag{3.23}$$

such that $\tilde{F}(\rho) = A\rho + G(\rho)$. Next we will elaborate on $D\bar{F}(0)[\rho] = DF(1)[\rho]$.

3.3.1 Derivative of F at $\psi_0 \equiv 1$

Assume that $\psi_0 \equiv 1$. Then we see that

$$z(\psi_0; r, \theta) = z(1; r, \theta) = (r\cos\theta, r\sin\theta),$$

for $(r\cos\theta, r\sin\theta) \in \Omega_a$. Furthermore, we have that $c(1)$ is given by

$$c(1) = \mathcal{A}(1)^{-1}(0, -\frac{P}{2\pi}\ln 1) = \mathcal{A}(1)^{-1}(0,0) = 0.$$

For $\mathcal{A}(1) = (L(1), Tr)$, we note that $L(1)$ is given by the Laplacian on Ω_1. By definition of $Dc(1)[h]$, we see that it has to satisfy

$$\begin{aligned} \Delta Dc(1)[h] &= 0 \quad \text{in } \Omega_1, \\ Dc(1)[h] &= -\frac{P}{2\pi}h \quad \text{on } S^1. \end{aligned} \tag{3.24}$$

This problem can be solved using Fourier coefficients $h_n = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)\exp(-in\theta)d\theta$. Since we will need the The resulting solution $Dc(1)[h](r,\theta)$ on Ω_a is then given by

$$Dc(1)[h](r,\theta) = -\frac{P}{2\pi}\sum_{n\in\mathbb{Z}} r^{|n|} h_n \exp(in\theta). \tag{3.25}$$

Recall that $\nabla_a Dc(1)[h]$ is given by

$$\nabla_a Dc(1)[h] = (\partial_r Dc(1)[h], \partial_\theta Dc(1)[h])|_{S^1}.$$

We note that derivatives of $Dc(1)[h]$ can be obtained by interchanging the sum and derivatives since h is twice differentiable on S^1. This observation allows us to write

$$\begin{aligned}
\nabla_a Dc(1)[h] &= \begin{pmatrix}\partial_r Dc(1)[h](r,\theta)\\ \partial_\theta Dc(1)[h](r,\theta)\end{pmatrix}\Bigg|_{S^1}\\
&= -\frac{P}{2\pi}\begin{pmatrix}\sum_{n\in\mathbb{Z}} |n| r^{|n|-1} h_n \exp(in\theta)\\ \sum_{n\in\mathbb{Z}} r^{|n|} h_n in \exp(in\theta)\end{pmatrix}\Bigg|_{S^1}\\
&= -\frac{P}{2\pi}\begin{pmatrix}\sum_{n\in\mathbb{Z}} |n| h_n \exp(in\theta)\\ \sum_{n\in\mathbb{Z}} i|n| h_n \exp(in\theta)\end{pmatrix}.
\end{aligned}$$

Additionally, we know that

$$\begin{aligned}
H(1) &= 1,\\
n(1) &= (\cos\theta, \sin\theta) = n_0\\
J_{z(1)}^{-1} &= \begin{pmatrix}\cos(\theta) & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\\
DF_B(1)[h] &= -\frac{P}{2\pi} h_{\theta\theta}
\end{aligned}$$

In summary, we obtain that

$$\begin{aligned}
DF_D(1)[h] &= \frac{1}{1\cdot n_0\cdot n_0} - \frac{P}{2\pi}\left(\sum_{n\in\mathbb{Z}} |n| h_n \exp(in\theta) \quad \sum_{n\in\mathbb{Z}} i|n| h_n \exp(in\theta)\right)\begin{pmatrix}\cos(\theta) & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix} n_0\\
&= -\frac{P}{2\pi}\left(\sum_{n\in\mathbb{Z}} |n| h_n \exp(in\theta)\left(\cos\theta \quad \sin\theta\right) + \sum_{n\in\mathbb{Z}} i|n| h_n \exp(in\theta)\left(-\sin\theta \quad \cos\theta\right)\right) n_0\\
&= -\frac{P}{2\pi}\sum_{n\in\mathbb{Z}} |n| h_n \exp(in\theta).
\end{aligned}$$

so that

$$DF(1)[h] = -\frac{P}{2\pi}\left(h_{\theta\theta} + \sum_{n\in\mathbb{Z}} |n| h_n \exp(in\theta)\right). \tag{3.26}$$

We can thus rewrite A to

$$A\rho = -\frac{P}{2\pi}\left(\sum_{n\in\mathbb{Z}} |n| \rho_n \exp(in\theta) - \rho_{\theta\theta} + \rho\right).$$

3.3.2 Linear stability results

With the operators A and G defined, we will check the assumptions of Theorem 2.14.

Lemma 3.12. *The operators A and G defined in* (3.23) *satisfy the assumptions of Theorem 2.14 for $D(A) = D, \mathcal{O}, X$ as defined in* (3.9).

Proof. The proof for the diffusive model will be analogous to the proof for the ballistic model in Theorem 2.15. We obtain sectoriality of A in X, since $DF(\psi_0)$ is sectorial for all $\psi_0 \in \mathcal{O}$, and in particular for $\psi_0 \equiv 1 \in \mathcal{O}$ and the operator $M : \rho \mapsto \frac{P}{2\pi}\rho$ can be considered a perturbation of $DF(\psi_0)$, so that A is sectorial. Similarly to the ballistic model, we thus obtain that A, G satisfy the first condition of Theorem 2.14. Therefore, we only need to investigate the spectrum of A and find a ω_0 such that $\sup\{Re\ \lambda : \lambda \in \sigma(A)\} = -\omega_0 < 0$. Analogously to the previous chapter, we will try to solve $\lambda\rho - A\rho = g$ for $\lambda \in \mathbb{C}, g \in h^\gamma(S^1)$. We again translate the problem in terms of Fourier coefficients, so that

$$\begin{aligned}
\lambda\rho - A\rho &= \sum_{n\in\mathbb{Z}} \lambda\rho_n \exp(in\theta) + \frac{P}{2\pi}\sum_{n\in\mathbb{Z}}(n^2 + |n| + 1)\rho_n \exp(in\theta) \\
&= \frac{P}{2\pi}\sum_{n\in\mathbb{Z}}(n^2 + |n| + 1 + \frac{2\pi}{P}\lambda)\rho_n \exp(in\theta) = \sum_{n\in\mathbb{Z}} g_n \exp(in\theta) \\
&\Longrightarrow \frac{P}{2\pi}(n^2 + |n| + 1 + \frac{2\pi}{P}\lambda)\rho_n = g_n \\
&\Longrightarrow \rho_n = \frac{g_n}{\frac{P}{2\pi}(n^2 + |n| + 1 + \frac{2\pi}{P}\lambda)},
\end{aligned}$$

which will only yield a solution if $n^2 + |n| + 1 + \frac{2\pi}{P}\lambda \neq 0$, or

$$\lambda \neq \lambda_n := -\frac{P}{2\pi}(n^2 + |n| + 1).$$

Additionally, for any such $\lambda \neq \lambda_n$ we obtain similarly to the ballistic model that

$$||(\lambda I - A)^{-1}g||_{2+\gamma} = ||\rho||_{2+\gamma} \leq C_\lambda||g||_\gamma,$$

for some $C_\lambda > 0$, which implies that $\lambda \neq \sigma(A)$. Thus we have that $\sigma(A) \subset \{\lambda_n | n \in \mathbb{Z}\}$. Since $\lambda_n = \frac{P}{2\pi}(n^2 + |n| + 1) \geq \frac{P}{2\pi}$, we can define $\omega_0 = \frac{P}{2\pi}$ so that $\sup\{Re\lambda : \lambda \in \sigma(A)\} \leq -\omega_0$.

Finally we use Lemma 3.4 and Corollary 3.9 to obtain that G is a C^1 function with locally Lipschitz continuous derivative. Additionally, we have that

$$\begin{aligned}
G(0) &= F(1) - \frac{P}{2\pi} = 0, \\
G'(0) &= DF(1)[0] - \frac{P}{2\pi}0 - A0 = 0,
\end{aligned}$$

so that A, G satisfy the assumptions of Theorem 2.14. □

We will use this lemma to show linear stability of circular solutions to the original evolution equation in the following corollary.

Corollary 3.13. *Let ψ_c, ψ be a solutions to (3.8), where ψ_c is a circular solution, and $\psi(0)$ close to $\psi_c(0)$. Then $\psi(t)$ exists for all time $t > 0$. Additionally, for all $\varepsilon \in (0, 1]$ there exists a $M = M(\varepsilon)$ such that*

$$||\psi - \psi_c||_{2+\gamma} \leq M(\psi_c(0) + \frac{P}{2\pi}t)^\varepsilon||\psi(0) - \psi_c(0)||_{2+\gamma}.$$

Proof. The proof is analogous to the proof of Theorem 2.16. □

This concludes the analysis of the diffusive VSC model. Unfortunately, we are unable to show an avoidance principle similar to the ballistic model. The main problem is that we cannot write the Fréchet derivative of F as a second order partial differential operator. Thus we cannot apply the maximum principle used in Lemma 2.17 to show the avoidance principle. The next chapter will contain a summary and possible future research subjects.

4 Conclusion

4.1 Summary

In this project we have discussed the two dimensional ballistic and diffusive VSC model for the growth of cells. In particular, we were interested in analysing how perturbations of circles would evolve under these models. Circular domains served as a simple example, since the models were identical for circles and showed that circles would grow linearly over time. The models changed when regarding perturbations ψ: the ballistic model only had explicit functions of ψ while the diffusive model contained implicit expressions.

This difference meant that the approach for showing existence to the evolution equation differed too, yet both yielded the same result. The proof for the ballistic model was relatively straight-forward, and the Fréchet derivative could be expressed as a second order differential operator. Contrastingly, the diffusive model did not yield such an expression, yet the gouverning function could be written using the solution for the ballistic model. Additionally, we introduced a diffeomorphism that transformed the differential equation for the concentration function from a changing domain to a fixed domain. However, this diffeomorphism caused the set on which the initial perturbation was defined, $\mathcal{O}$, to decrease. Despite this change, both models allowed short time and long time existence for the evolution equations.

Alongside the existence results, an asymptotics results and a comparison result were achieved. The first result held for both models, which showed that the difference between the solution and an expanding circle could only grow sublinear in time. The second result was exclusive to the ballistic model, which showed that if one initial perturbation was contained in another, then this would hold for all time. This result was achieved using the maximum principle, which cannot be used for the diffusive model. A similar result might however be achievable using other methods.

4.2 Further research options

Next we will discuss some further research topics which can be investigated.

Improving the long time stability result

One additional topic of future research could be to exclude the smallest eigenvalue in the long-time analysis of both models. The current conclusion is that solutions are $\mathcal{O}(t^{\varepsilon})$ close to growing circles for any $\varepsilon \in (0,1]$. This result could be improved if the smallest eigenvalue λ_0 is excluded from the analysis, similar to [Prokert and Vondenhoff, 2009, Chapter 4]. This omission should be achievable by eliminating the corresponding eigenspace E_{λ_0} from the domain of solutions. The eigenspace E_{λ_0} corresponds to the space of constant functions on S^1, i.e. circles, which can be verified using the Fourier coefficients. The solution could then be transformed using an $L^2(S^1)$-orthogonal projection $\mathcal{P}$ to $\{\nu \in h^{2+\gamma}(S^1) \mid (\nu, E_{\lambda_0}) = 0\}$. On this new space, the operator $\mathcal{P} \circ A = \tilde{A}$ has spectrum $\sigma(\tilde{A}) = \sigma(A) \backslash \{\lambda_0\}$. This operator should then give a similar bound to the solutions, except that ε can be smaller than zero. The new bound then implies that the perturbations of circles should vanish as time passes on, and thus the solutions will be similar to circles over time. However, note that this approach is only a sketch based on [Prokert and Vondenhoff, 2009], and should be investigated more thoroughly.

Transforming the problem to the third dimension

Another interesting topic is to increase the dimension of the problem statement. This project has focused on a two dimensional cell, which raises the question how the models would behave in three dimensions. For example, the trivial solution of a growing circle, would not grow linearly over time but instead would grow in proportion to $\sqrt{t}$. Additionally, the perturbation function ψ would take arguments from the unit ball instead of the circle. Similar changes will be needed for the other components, such as the normal vector and the (mean) curvature. Overall, the approach for the short time existence of the three-dimensional models should look similar to the two-dimensional models.

The approach for the long time existence would likely see more changes needed to the approach. For instance, obtaining the eigenvalues will need to be done on the sphere instead of the circle, which means that the approach using Fourier coefficients can no longer be used. This change can be made through the choice of spherical harmonic coordinates, in which the equation can consequently be solved. More importantly, the scaling property of F might not hold for higher dimensions, which may cause the rescaled equation to be inhomogeneous in time. The comparison result will most likely still hold for the ballistic model.

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A Derivations of the curvature, normal vector and velocity

In this appendix we will derive the expressions for $v(\psi), n(\psi)$ and $H(\psi)$.

A.1 Derivation of the normal vector

For a parametrised curve $\gamma(\theta) = (x(\theta), y(\theta))$, the vector $\frac{d\gamma}{d\theta}$ is tangent to the curve in counterclockwise direction [Goldman, 2005, Section 2.1] . This vector can be rotated $\pi/2$ to the right to obtain the outward normal vector n. The normal vector can thus be written as

$$n = \frac{(y_\theta, -x_\theta)}{\sqrt{y_\theta^2 + x_\theta^2}}. \tag{A.1}$$

The $(x(\theta), y(\theta)) = \psi(\theta)(\cos\theta, \sin\theta)$ to obtain

$$\begin{aligned} x_\theta &= \psi_\theta \cos\theta - \psi \sin\theta \\ y_\theta &= \psi_\theta \sin\theta + \psi \cos\theta \end{aligned}$$

so that

$$\begin{aligned} n(\psi) &= \frac{(\psi_\theta \sin\theta + \psi\cos\theta, -\psi_\theta\cos\theta + \psi\sin\theta)}{\sqrt{\psi_\theta^2\cos^2\theta - 2\psi\psi_\theta\cos\theta\sin\theta + \psi^2\sin^2\theta + \psi_\theta^2\sin^2\theta - 2\psi\psi_\theta\cos\theta\sin\theta + \psi^2\cos^2\theta}} \\ &= \frac{(\psi_\theta \sin\theta + \psi\cos\theta, -\psi_\theta\cos\theta + \psi\sin\theta)}{\sqrt{\psi_\theta^2 + \psi^2}} \end{aligned}$$

A.2 Derivation of the curvature

The curvature H for a parametrised curve $\gamma(\theta) = (x(\theta), y(\theta))$ is given by [Goldman, 2005, Section 2.1]

$$H = \frac{\det(\gamma'\ \gamma'')}{||\gamma'||^3}. \tag{A.2}$$

Note that we need the second derivatives for the curvature. The numerator can be calculated to be

$$\begin{aligned} \det(\gamma'\ \gamma'') &= \det\begin{pmatrix} x_\theta & x_{\theta\theta} \\ y_\theta & y_{\theta\theta} \end{pmatrix} \\ &= x_\theta y_{\theta\theta} - y_\theta x_{\theta\theta}, \end{aligned}$$

so that the curvature becomes

$$H = \frac{x_\theta y_{\theta\theta} - y_\theta x_{\theta\theta}}{(x_\theta^2 + y_\theta)^{3/2}}.$$

Next we substitute the parametrisation $\gamma(\theta) = \psi(\theta)(\cos\theta, \sin\theta)$ to obtain

$$\begin{aligned} x_{\theta\theta} &= \psi_{\theta\theta}\cos\theta - 2\psi_\theta\sin\theta - \psi\cos\theta \\ y_{\theta\theta} &= \psi_{\theta\theta}\sin\theta + 2\psi_\theta\cos\theta - \psi\sin\theta. \end{aligned}$$

so that the curvature equals

$$\begin{aligned} H(\psi) &= \frac{(\psi_\theta\cos\theta - \psi\sin\theta)(\psi_{\theta\theta}\sin\theta + 2\psi_\theta\cos\theta - \psi\sin\theta) - (\psi_\theta\sin\theta + \psi\cos\theta)(\psi_{\theta\theta}\cos\theta - 2\psi_\theta\sin\theta - \psi\cos\theta)}{(\psi^2 + \psi_\theta^2)^{3/2}} \\ &= \frac{2\psi_\theta^2(\cos^2\theta + \sin^2\theta) - \psi\psi_{\theta\theta}(\sin^2\theta + \cos^2\theta) + \psi^2(\sin^2\theta + \cos^2\theta)}{(\psi^2 + \psi_\theta^2)^{3/2}} \\ &= \frac{\psi^2 + 2\psi_\theta^2 - \psi\psi_{\theta\theta}}{(\psi^2 + \psi_\theta^2)^{3/2}}. \end{aligned}$$

A.3 Derivation of the normal velocity

For the velocity of the boundary, we note that this is simply the time derivative of the parametrisation. Therefore the velocity of $\gamma(t,\theta) = \psi(t,\theta)(\cos\theta, \sin\theta)$ is given by $\partial_t\gamma(t,\theta)$. Since we need the normal component V_n of the velocity, we take the inner product with the normal vector:

$$V_n(\psi) = \partial_t\psi(t,\theta)(\cos\theta, \sin\theta) \cdot n(\psi). \tag{A.3}$$

Substituting the expression for $n(\psi)$, we see that

$$\begin{aligned} V_n(\psi) &= \partial_t\psi(\cos\theta, \sin\theta) \cdot (\psi_\theta \sin\theta + \psi\cos\theta, -\psi_\theta\cos\theta + \psi\sin\theta)(\psi^2 + \psi_\theta^2)^{-1/2} \\ &= \partial_t\psi\psi(\cos^2\theta + \sin^2\theta)(\psi^2 + \psi_\theta^2)^{-1/2} \\ &= \partial_t\psi\frac{\psi}{\sqrt{\psi^2 + \psi_\theta^2}}. \end{aligned}$$

B Auxiliary lemmas ballistic model

This appendix contains some auxiliary lemmas with proofs that are used in the ballistic model.

B.1 Fréchet differentiability of several functions

Corollary B.1. *Consider $f_{1,2,3} : \mathcal{O} \to h^\gamma(S^1)$ defined by*

$$\begin{aligned} f_1(\psi) &= \psi^n, \\ f_2(\psi) &= \sqrt{\psi}, \\ f_3(\psi) &= \psi_\theta, \end{aligned}$$

where $n \in \mathbb{N}$. Then $f_{1,2,3}$ are Fréchet differentiable at $\psi_0 \in \mathcal{O}$ with derivatives

$$\begin{aligned} Df_1(\psi_0)[h] &= n\psi_0^{n-1}h, \\ Df_2(\psi_0)[h] &= \frac{1}{2}\psi_0^{-1/2}h, \\ Df_3(\psi_0)[h] &= h_\theta. \end{aligned}$$

Proof. Let $\psi_0 \in \mathcal{O}$ and $h \in h^{2+\gamma}$. The first function we will analyse is ψ^n. We will show differentiability through induction over n.

Base case Consider $f(\psi) = \psi^2$. Then the product rule implies that f is differentiable at ψ_0 with derivative

$$\begin{aligned} Df(\psi_0)[h] &= \psi_0 \cdot D\psi(\psi_0)[h] + D\psi(\psi_0)[h]\psi_0 \\ &= \psi_0 \cdot h + h \cdot \psi_0 = 2\psi_0 h. \end{aligned}$$

Induction step Assume that $f(\psi) = \psi^k$ is Fréchet differentiable for some $k > 2$. Then $g(\psi) = \psi^{k+1} = \psi \cdot \psi^k$ is Fréchet differentiable at ψ_0 with derivative

$$\begin{aligned} Dg(\psi_0)[h] &= h\psi_0^k + \psi_0 D\psi^k(\psi_0)[h] \\ &= h\psi_0^k + \psi_0 \cdot k\psi_0^{k-1}h \\ &= h\psi_0^k + k\psi_0^k h \\ &= (k+1)\psi_0^k h, \end{aligned}$$

which was the induction hypothesis. Therefore $f_1(\psi) = \psi^n$ is Fréchet differentiable at ψ_0 for all $n \in \mathbb{N}$.

The next derivative will be found using the definition of the Fréchet derivative. Clearly we see that $Df_2(\psi_0)$ is a bounded linear operator. Thus we need to show that $||f_2(\psi_0 + h) - f_2(\psi_0) - Df_2(\psi_0)[h]||_\gamma ||h||_{2+\gamma}^{-1} \to 0$ as $||h||_{2+\gamma} \to 0$. We write out the expression to obtain

$$\begin{aligned} f_2(\psi_0 + h) - f_2(\psi_0) - Df_2(\psi_0)[h] &= \sqrt{\psi_0 + h} - \sqrt{\psi_0} - \frac{h}{2\sqrt{\psi_0}} \\ &= \frac{\psi_0 + h - \psi_0}{\sqrt{\psi_0 + h} + \sqrt{\psi_0}} - \frac{h}{2\sqrt{\psi_0}} \\ &= \frac{h\sqrt{\psi_0} - \frac{1}{2}h\sqrt{\psi_0 + h} - \frac{1}{2}h\sqrt{\psi_0}}{(\sqrt{\psi_0})(\sqrt{\psi_0 + h} + \sqrt{\psi_0})} \\ &= \frac{\sqrt{\psi_0 + h} - \sqrt{\psi_0}}{2\sqrt{\psi_0}(\sqrt{\psi_0 + h} + \sqrt{\psi_0})} h. \end{aligned}$$

Therefore we know that

$$\begin{aligned} \lim_{||h||_{2+\gamma} \to 0} \left\| \frac{\sqrt{\psi_0 + h} - \sqrt{\psi_0}}{2\sqrt{\psi_0}(\sqrt{\psi_0 + h} + \sqrt{\psi_0})} h \right\|_\gamma \frac{1}{||h||_{2+\gamma}} &\leq \lim_{||h||_{2+\gamma} \to 0} \left\| \frac{\sqrt{\psi_0 + h} - \sqrt{\psi_0}}{2\sqrt{\psi_0}(\sqrt{\psi_0 + h} + \sqrt{\psi_0})} \right\|_\gamma \\ &= \left\| \frac{\sqrt{\psi_0} - \sqrt{\psi_0}}{2\sqrt{\psi_0}(\sqrt{\psi_0} + \sqrt{\psi_0})} \right\|_\gamma \\ &= \left\| \frac{0}{4\psi_0} \right\|_\gamma = 0. \end{aligned}$$

So we indeed find that $Df_2(\psi_0)[h]$ is the Fréchet derivative of f_2.

For the third function, we note that the partial derivative ∂_θ^k is a linear operator. Clearly $Df_3(\psi_0)$ is a bounded linear operator, thus we find by definition of the Fréchet derivative that f_3 is differentiable at ψ_0 with derivative $Df_3(\psi_0)[h]$. □

B.2 Proof that the quotient of two h^γ functions is still in h^γ

Lemma B.2. *Let $u, v \in X$, $v \geq \mu > 0$. Then there exists a $C = C(||v||_\gamma, \mu) > 0$ such that*

$$||\tfrac{u}{v}||_\gamma \leq C||u||_\gamma \tag{B.1}$$

Proof. Fix u, v with $v \geq \mu > 0$. Then we know that

$$\begin{aligned}
||\frac{u}{v}||_\gamma &= ||\frac{u}{v}||_\infty + [\frac{u}{v}]_\gamma \\
&= \sup_{x \in S^1} \left|\frac{u(x)}{v(x)}\right| + \sup_{x,y \in S^1, x \neq y} \left(\frac{|\frac{u(x)}{v(x)} - \frac{u(y)}{v(y)}|}{|x-y|^\gamma} \right) \\
&\leq \mu^{-1}||u||_\infty + \sup_{x,y \in S^1, x \neq y} \left(\frac{|\frac{u(x)v(y) - u(y)v(x)}{v(x)v(y)}|}{|x-y|^\gamma} \right) \\
&\leq \mu^{-1}||u||_\infty + \sup_{x,y \in S^1, x \neq y} \left(\frac{||v||_\infty |u(x)-u(y)| \frac{1}{|v(x)v(y)|}}{|x-y|^\gamma} \right) \\
&\leq \mu^{-1}||u||_\infty + ||v||_\infty \sup_{x,y \in S^1, x \neq y} \left(\frac{|u(x)-u(y)|\mu^{-2}}{|x-y|^\gamma} \right) \\
&\leq \mu^{-1}||u||_\infty + ||v||_\infty \mu^{-2} \sup_{x,y \in S^1, x \neq y} \left(\frac{|u(x)-u(y)|}{|x-y|^\gamma} \right) \\
&\leq (\mu^{-1} + ||v||_\gamma \mu^{-2})||u||_\gamma =: C||u||_\gamma
\end{aligned}$$

□

B.3 Proof that $\sqrt{u^2 + u_\theta^2} \in h^\gamma$ if $u \in h^{2+\gamma}$.

Lemma B.3. *Let $u \in \mathcal{O}$. Then $\sqrt{u^2 + u_\theta^2} \in X$.*

Proof. For any $u \in \mathcal{O}$ we know that $u > 0$ and continuous, implying there exists a $\lambda > 0$ such that $u \geq \lambda > 0$. Thus we know that $\sqrt{u^2 + u_\theta^2} \geq \lambda$. Using the mean value theorem, we can estimate the Hölder norm $[\sqrt{u^2 + u_\theta^2}]_X$ by

$$\begin{aligned}
[\sqrt{u^2 + u_\theta^2}]_X &= \sup_{x,y \in S^1, x \neq y} \frac{|\sqrt{u^2(y) + u_\theta^2(y)} - \sqrt{u^2(x) - u_\theta^2(x)}|}{|y-x|^\gamma} \\
&\leq \sup_{x,y \in S^1, x \neq y} \frac{1}{|x-y|^\gamma} \sup_{\theta \in S^1} |\frac{\partial}{\partial\theta}\sqrt{u^2(\theta) + u_\theta^2(\theta)}||y-x| \\
&= \sup_{x,y \in S^1, x \neq y} \sup_{\theta \in S^1} |\frac{1}{\sqrt{u^2(\theta) + u_\theta^2(\theta)}}(u(\theta)u_\theta(\theta) + u_\theta(\theta)u_{\theta\theta}(\theta))||y-x|^{1-\gamma} \\
&\leq \sup_{x,y \in S^1, x \neq y} \sup_{\theta \in S^1} |\tfrac{1}{\lambda}(u(\theta)u_\theta(\theta) + u_\theta(\theta)u_{\theta\theta}(\theta))||y-x|^{1-\gamma}.
\end{aligned}$$

Since $u, u_\theta, u_{\theta\theta}$ are bounded functions, we know that there is a $C > 0$ such that $|2u(\theta)u_\theta(\theta) + 2u_\theta(\theta)u_{\theta\theta}(\theta)| \leq C(||u||_\infty + ||u_\theta||_\infty)$. As $x, y \in S^1$ we also know that $|x-y|^{1-\gamma} \leq \mu$ for some $\mu > 0$. Using these estimates, we obtain

$$[\sqrt{u^2 + u_\theta^2}]_X \leq \sup_{x,y \in S^1, x \neq y} \frac{1}{\lambda} C(||u||_\infty + ||u_\theta||_\infty)\mu \tag{B.2}$$

$$= \frac{1}{\lambda} C(||u||_\infty + ||u_\theta||_\infty)\mu \tag{B.3}$$

$$\leq \frac{1}{\lambda} C\mu||u||_D. \tag{B.4}$$

For $||\sqrt{u^2 + u_\theta^2}||_\infty$ we know that

$$|u^2(x) + u_\theta^2(x)|^{1/2} \leq |u(x)| + |u_\theta(x)|,$$

so that

$$||\sqrt{u^2+u_\theta^2}||_\infty \leq ||u||_\infty + ||u_\theta||_\infty \leq ||u||_D.$$

Overall, we obtain that

$$||\sqrt{u^2+u_\theta^2}||_X \leq (\frac{1}{\lambda}C\mu + 1)||u||_D < \infty,$$

which means that the function is in $C^\gamma(S^1)$. For the function to be in $h^\gamma(S^1)$ we see that

$$0 \leq \sup_{x,y\in S^1,|x-y|<\delta} \frac{|\sqrt{u^2(y)+u_\theta^2(y)} - \sqrt{u^2(x)-u_\theta^2(x)}|}{|y-x|^\gamma} \leq \sup_{x,y\in S^1,|x-y|<\delta} C\frac{1}{\lambda}||u||_D|x-y|^{1-\gamma}$$
$$< C\frac{1}{\lambda}||u||_D\delta^{1-\gamma} \to 0 \text{ as } \delta \to 0.$$

By the squeeze theorem we obtain that also

$$\lim_{\delta\to 0} \sup_{x,y\in S^1,|x-y|<\delta} \frac{|\sqrt{u^2(y)+u_\theta^2(y)} - \sqrt{u^2(x)-u_\theta^2(x)}|}{|y-x|^\gamma} = 0,$$

which means that the function is in $h^\gamma(S^1) = X$. □

C Auxiliary lemmas diffusive model

C.1 Derivation of the trace of the transformed gradient on the unit ball

Lemma C.1. *Let $\psi \in \mathcal{O}$ and $f : \Omega_1 \to \Omega_1$ differentiable and $z = z(\psi)$ as defined by Equation (3.2). Then $z^* \circ \nabla \circ z_* f\big|_{S^1}$ can be written as*

$$z^* \circ \nabla \circ z_* f\big|_{S^1} = J_z^{-T}\nabla_a f, \tag{C.1}$$

where $\nabla_a = Tr \circ \nabla_1$, and $\nabla_1 = (\partial_a, \partial_b)$ denotes the vector of derivatives with respect to coordinates in Ω_1.

Proof. Assume that Ω_ψ has coordinates x_1, x_2 and Ω_1 has coordinates v_a, v_b. Additionally, let $\nabla = (\partial_1, \partial_2)$ be the gradient with respect to coordinates in Ω_ψ and $\nabla_1 = (\partial_a, \partial_b)$ where ∂_a is the derivative with respect to v_a in Ω_1. Then we have for every $(v_a, v_b) \in \Omega_1$ a pair $(x_1, x_2) \in \Omega_\psi$ such that $(v_a, v_b) = z^{-1}(x_1, x_2) = (z_a^{-1}(x_1, x_2), z_b^{-1}(x_1, x_2))$. We can use the chain rule to obtain

$$\begin{aligned}\nabla \circ z_* f &= \begin{pmatrix}\partial_1(f(z^{-1}(x_1,x_2)))\\ \partial_2(f(z^{-1}(x_1,x_2)))\end{pmatrix}\\ &= \begin{pmatrix}\partial_a f(z^{-1}(x_1,x_2))\partial_1 z_a^{-1}(x_1,x_2) + \partial_b f(z^{-1}(x_1,x_2))\partial_1 z_b^{-1}(x_1,x_2)\\ \partial_a f(z^{-1}(x_1,x_2))\partial_2 z_a^{-1}(x_1,x_2) + \partial_b f(z^{-1}(x_1,x_2))\partial_2 z_b^{-1}(x_1,x_2)\end{pmatrix}.\end{aligned}$$

Applying the pullback z^* to this expression componentwise yields

$$\begin{aligned}z^* \circ \nabla_1 \circ z_* f &= \begin{pmatrix}z^*(\partial_a f(z^{-1}(x_1,x_2))\partial_1 z_a^{-1}(x_1,x_2) + \partial_b f(z^{-1}(x_1,x_2))\partial_1 z_b^{-1}(x_1,x_2))\\ z^*(\partial_a f(z^{-1}(x_1,x_2))\partial_2 z_a^{-1}(x_1,x_2) + \partial_b f(z^{-1}(x_1,x_2))\partial_2 z_b^{-1}(x_1,x_2))\end{pmatrix}\\ &= \begin{pmatrix}\partial_a f(z^{-1}(z(v_a,v_b)))\partial_1 z_a^{-1}(z(v_a,v_b)) + \partial_b f(z^{-1}(z(v_a,v_b)))\partial_1 z_b^{-1}(z(v_a,v_b))\\ \partial_a f(z^{-1}(z(v_a,v_b)))\partial_2 z_b^{-1}(z(v_a,v_b)) + \partial_b f(z^{-1}(z(v_a,v_b)))\partial_2 z_b^{-1}(z(v_a,v_b))\end{pmatrix}\\ &= \begin{pmatrix}\partial_a f(v_a,v_b)\partial_1 z_a^{-1}(z(v_a,v_b)) + \partial_b f(v_a,v_b)\partial_1 z_b^{-1}(z(v_a,v_b))\\ \partial_a f(v_a,v_b)\partial_2 z_b^{-1}(z(v_a,v_b)) + \partial_b f(v_a,v_b)\partial_2 z_b^{-1}(z(v_a,v_b))\end{pmatrix}\\ &= J_{z^{-1}}^T(z(v_a,v_b))\nabla_1 f(v_a,v_b).\end{aligned}$$

Note that since z is a diffeomorphism, we have that $J_{z^{-1}}(z(v_a, v_b)) = J_z^{-1}(v_a, v_b)$, so that

$$z^* \nabla_1 (z_* c)(v_a, v_b) = J_z^{-T}(v_a, v_b) \nabla_1 f(v_a, v_b).$$

Note that the inverse of the Jacobian evaluated at S^1 is given by

$$J_{z(\psi)}^{-1} = \frac{1}{\psi^2} \begin{pmatrix} \psi \cos(\theta) + \psi_\theta \sin(\theta) & \psi \sin(\theta) - \psi_\theta \cos(\theta) \\ -\psi \sin(\theta) & \psi \cos(\theta) + \psi_\theta \sin(\theta) \end{pmatrix}$$

Thus we obtain that $z^* \circ \nabla z_* f\big|_{S^1}$ is given by

$$z^* \circ \nabla z_* f\big|_{S^1} = J_z^{-T} Tr \circ \nabla_1 f(v_a, v_b) = J_z^{-T} \nabla_a f.$$

□

C.2 Interpolation property of Hölder spaces

Lemma C.2. *Let $\gamma_1 \in (0, \gamma_2)$. Then for all $\varepsilon > 0$ there exists a $C_\varepsilon > 0$ such that*

$$||u||_{\gamma_1} \leq \varepsilon ||u||_{\gamma_2} + C_\varepsilon ||u||_\infty$$

for all $u \in h^{\gamma_1}$.

Proof. First we note that $C^{\gamma_1}(S^1)$ is an interpolation space of class γ_1/γ_2 between $C(S^1)$ and $C^{\gamma_2}(S^1)$.[Lunardi, 1995, Proposition 1.1.3]. Let $u \in h^{\gamma_1}(S^1) \subset C^{\gamma_1}(S^1)$ and $\varepsilon > 0$ fixed. By the interpolation class property, we know that there is a constant c such that

$$||u||_{\gamma_1} \leq c ||u||_{\gamma_2}^{1-\gamma_1/\gamma_2} ||u||_\infty^{\gamma_1/\gamma_2}.$$

Recall that Young's inequality for products states that for all $a, b \geq 0$, we know

$$a^p b^q \leq pa + qb,$$

where we let $p = 1 - \gamma_1/\gamma_2, q = \gamma_1/\gamma_2$. We choose $a = c^{-1} 1/p\varepsilon ||u||_{\gamma_2}, b = c^{p/q} p^{p/q} \varepsilon^{-p/q} ||u||_\infty$ Then we obtain that

$$\begin{aligned} ||u||_{\gamma_1} &\leq c ||u||_{\gamma_2}^p ||u||_\infty^q \\ &= c(1/p\varepsilon ||u||_{\gamma_2})^p (p^{p/q} \varepsilon^{-p/q} ||u||_\infty)^q \\ &= c a^p b^q \\ &\leq cpa + cbq \\ &= c/cp/p\varepsilon ||u||_{\gamma_2} + cqc^{p/q} p^{p/q} \varepsilon^{-p/q} ||u||_\infty \\ &= \varepsilon ||u||_{\gamma_2} + C_\varepsilon ||u||_\infty, \end{aligned}$$

where $C_\varepsilon = qc^{(p+q)/q} p^{p/q} \varepsilon^{-p/q} > 0$. □