BACHELOR

Travelling wave solutions of a linearized 3D Hele-Shaw flow with kinetic undercooling

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Bachelor Final Project

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Contents

Abstract

This Bachelor Final Project is concerned with a moving boundary problem of Hele-Shaw type in three dimensions. It is a Robin boundary problem for the Laplacian in an exterior domain. It models the flow of a bubble in Hele-Shaw flow with a kinetic undercooling boundary condition as well as electric streamer discharges. We look for travelling-wave solutions and find trivial spherical ones. While discussing the concepts and theories of spherical harmonics, Sobolev spaces and the Fréchet derivative, we turn this problem into a nonlinear operator equation for nontrivial domains that are radial perturbations of the unit sphere. This is done by identifying such a domain with a positive function u, defined on the unit sphere and introducing an operator that solves the Robin boundary problem for the Laplacian. We linearize this operator equation and by using a spherical harmonics expansion we discover the existence of a unique nontrivial traveling wave solution of this linearized problem for each velocity near the trivial one. The smoothness of the domain describing functions u is determined by the order of the Sobolev space on the unit sphere they are in. This order is found to be bounded above by a value which is dependent on a regularizing parameter present in the boundary condition.

1 Introduction

The problem this bachelor final project is concerned with is a so called moving boundary problem. This is a partial differential equation that needs to be solved for an unknown function and on a moving domain which is a priori also unknown. Finding (the boundary of) this domain is part of the desired solution. To be able to find this complete solution set, the PDE is not only accompanied by the usual boundary condition(s), but also by a kinetic condition set at the boundary to describe the movement of this boundary. Examples of processes in nature that are modelled by moving boundary problems are the melting of ice, the growth of tumors and the winning of oil [19].

One of the classes of these moving boundary problems consists of so called Hele-Shaw flows on which our problem is also based. In 1898 Henry Selby Hele-Shaw studied the flow of liquids in a cell that would carry his name, introducing the Hele-Shaw model. He created this cell by placing a fluid layer between two parallel horizontal plates that are so close together that one can view the flow as essentially in two dimensions. By adding one or more driving mechanisms, this fluid layer moves. The pressure p and velocity $\vec{v}$ of this fluid layer are related by Darcy's law; $\vec{v} = -\nabla p$ and because the fluid is assumed to be incompressible we have $0 = \text{div } \vec{v} = -\Delta p$ [19]. Therefore the partial differential equation in Hele-Shaw flows is Laplace's equation and can be accompanied with several boundary conditions. To complete the description of Hele-Shaw flow the kinetic condition is based on the assumption that the boundary moves with the particles. Therefore we have on the boundary that $V_n = \vec{v} \cdot n$, where V_n is the normal velocity of the moving boundary and n denotes the normal vector field [19].

Besides the two-dimensional flow in a Hele-Shaw cell there are also other applications of Hele-Shaw flow. In three dimensions, for example, Hele-Shaw flow describes the flow in porous media, like groundwater flow [19]. In variations of the model it describes the growth of tumors [4] and the process of viscous fingering in two-fluid flow, in which so called fingers emerge in the Hele-Shaw cell when a pressure gradient is applied and the fluids start to move [6, 13]. When this movement occurs, in some cases the less viscous fluid pushes the more viscous fluid away at the interface between the fluids, making it an unstable interface. It is an unstable interface because some fronts push further into the more viscous fluid and the so called fingers arise. The prediction of the width of the fingers is the main focus of such a problem [13].

The problem considered in this thesis is a three dimensional extension of a generalisation of this viscous fingering problem [13]. This generalization is a variation on this two-fluid flow in the sense that we consider a small (air) bubble with unknown shape in a liquid (water), instead of the two fluids next to each other. On the liquid domain Laplace's equation should hold and we have Hele-Shaw's kinetic condition. The model that mirrors the original two dimensional Hele-Shaw cell with such a bubble was already considered in [11, 17]. In this project we expand on these works by considering it in three dimensions, having applications which one would be more likely to encounter in nature. The original modelling setting of a Hele-Shaw cell is however not valid any more in three dimensions, but one can theoretically think about such a cell in three dimensions.

The main example of an encounter with a situation that is modelled by the problem we discuss is electric streamer discharges. Again, up to now research into this has mainly be done in a two dimensional setting to use conformal mapping techniques [6, 13], but a more intuitive three dimensional approach will be used here. Electrical streamers are observed for example as precursors of lightning. Ebert et al. in [6] justify why they are modelled by the problem we discuss; "During the initial 'streamer' phase of spark formation, a weakly ionized region extends in a strong externally applied electric field. As the ionized cloud is electrically conducting, it screens the electric field from its interior by forming a thin surface charge layer. This charged layer moves by electron drift within the local electric field and creates additional ionization, i.e., additional electron–ion–pairs, by collisions of fast electrons with neutral molecules. We here approximate the ionized and hence conducting bulk of the streamer as equipotential. In the non-ionized and hence electrically neutral region outside the streamer, the electric field obeys the Laplace equation. The thin surface charge layer can be approximated as an interface which moves according to the electric field extrapolated from the neutral region onto the interface." When comparing this to the setting of the bubble in the liquid we see that the streamer doubles for the bubble, the exterior of the streamer doubles for the liquid, the electric field, $\vec{E}$ doubles for $\vec{v}$ and the electric potential, Φ, doubles for p. In this setting, the same relation holds between $\vec{E}$ and Φ as between $\vec{v}$ and p; $\vec{E} = -\nabla\Phi$.

The condition that we impose on the boundary is a Robin boundary condition. This means that it is a condition on a linear combination of the function and its (directional) derivative(s). In our case this is a so called kinetic undercooling boundary condition for the pressure p; $p - \gamma\partial_n p = 0$ or for the electric potential Φ; $\Phi - \gamma\partial_n\Phi = 0$ on the boundary. Here n denotes the outer unit vector normal to the boundary of the bubble or streamer and γ is a nonnegative constant. In the streamer model one can consider the boundary between the ionized and non-ionized region to be an interface and γ is an indication on the thickness of this interface. This specific boundary condition results from analysis of the variation of the potential across this interface [6]. The name kinetic undercooling comes from the Stefan problem, a moving boundary problem related to the melting of ice, that is also a variation of the Hele-Shaw model [17]. Furthermore we have an asymptotic condition, that says that the motion of the domain is driven by a velocity field that is uniform like e_3 far away in the bubble case. In the setting of the streamer, this represents the assumption that the electric field becomes homogeneous far away from the streamer. We have the strong asymptotic condition $(1.1)_3$ to guarantee uniqueness of the solution, to see a proof of this fact see the Appendix. Figure 1 displays a two dimensional representation of the streamer problem. All in all, when using the substitution $f = -p$ or $f = -\Phi$, the following summarises the problem to be solved.

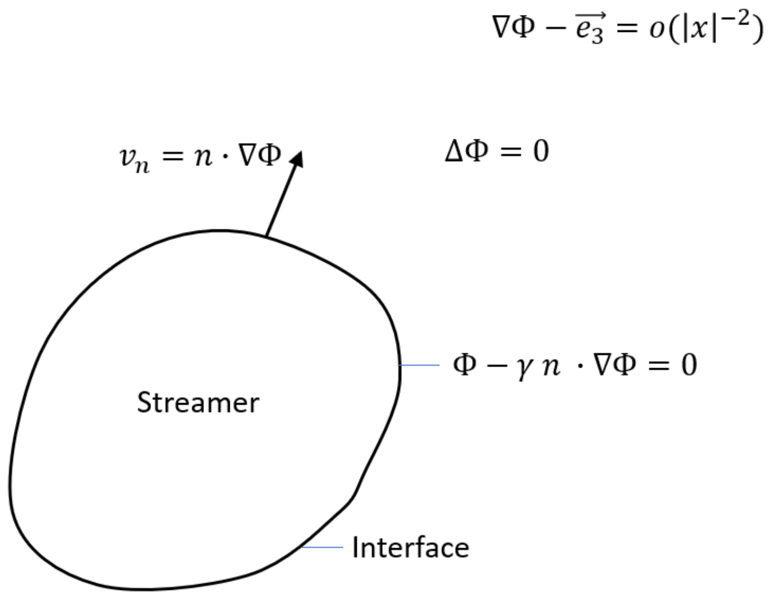


Figure 1: Two dimensional sketch of the 3D streamer moving boundary problem.

We seek a family of bounded moving domains $t \mapsto \Omega(t) \subset \mathbb{R}^3, t \geq 0$ with outer normal $n = n(t)$ and a boundary $\Gamma(t)$ and functions $f = f(\cdot, t)$ defined on $\mathbb{R}^3 \setminus \overline{\Omega(t)}$ such that

$$\left.\begin{aligned} \Delta f &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{\Omega(t)}, \\ f - \gamma \partial_n f &= 0 && \text{on } \Gamma(t), \\ \nabla f - \vec{e_3} &= o(|x|^{-2}) && \text{for } |x| \to \infty, \\ V_n &= \partial_n f. \end{aligned}\right\} \tag{1.1}$$

Here ∂_n denotes the directional derivative of f in the direction of the normal. In other words; $\partial_n f = n \cdot \nabla f$. Moreover, $\vec{e_3}$ denotes the unit vector in the x_3 direction and V_n is the normal velocity of the moving boundary $\Gamma(t)$. We consider a nonnegative constant γ and the initial domain Ω_0 is given.

The structure of the thesis is as follows. In Section 2 we adapt (1.1) to account for removing the inhomogeneous term at infinity and for introducing a moving coordinate system to accompany the traveling-wave solutions we will look for. Moreover, we establish the properties of translational en scaling invariance of this altered system (2.2). Also a trivial solution of spheres moving in $\vec{e_3}$-direction will be determined. In Section 3 an introduction of the concept of Sobolev spaces is given and the spherical harmonics and some of their properties are considered. Then a Sobolev space on the unit sphere in terms of coefficients of a spherical harmonics expansion will be constructed. In Section 4 our problem will be rewritten into an operator equation for an unknown function u which represents radial perturbations of the unit sphere. Section 5 gives an overview on the Fréchet derivative and specific results for the Fréchet derivative of operators defined on Sobolev spaces on the unit sphere. In Section 6 the operator equation is linearized around the trivial solution in terms of the Fréchet derivative. Recurrence relations are established for the spherical harmonics that will be used to get to the main result of solutions of the linearized problem. We conclude that for a small change from the trivial velocity there is a function that describes a unique perturbation of the unit sphere. This function has a spherical harmonics expansion and is such that velocity and function solve the linearized problem. Illustrations of the shape of certain solutions will be displayed and a discussion on the order of the Sobolev space a solution part u can be in and on the problem's invariances is given.

2 Properties and adaption of the problem

In this section we adapt the results from [11] to the 3D setting. Here we discuss some properties of (1.1) and transform it to be able to solve it. Firstly, we observe that the evolution of $\Omega(t)$ over time is volume preserving as we have

$$\frac{d}{dt}\int_{\Omega(t)} dx = \int_{\Gamma(t)} V_n ds = \int_{\Gamma(t)} \partial_n f ds = 0.$$

The verification of this fact can be found in the Appendix.

To remove the inhomogeneous term at infinity we write $f = g + x_3$ and see that

$$\begin{aligned}
\Delta f &= \Delta(g + x_3) = \Delta g + \Delta x_3 = \Delta g, \\
f - \gamma\partial_n f &= g + x_3 - \gamma\partial_n(g + x_3) = g + x_3 - \gamma\partial_n g - \gamma n_3, \\
\nabla f - \vec{e_3} &= \nabla(g + x_3) - \vec{e_3} = \nabla g + \nabla x_3 - \vec{e_3} = \nabla g + \vec{e_3} - \vec{e_3} = \nabla g, \\
V_n &= \partial_n f = \partial_n(g + x_3) = \partial_n g + n \cdot \nabla x_3 = \partial_n g + n_3.
\end{aligned}$$

This results in the new system:

$$\left.\begin{aligned}
\Delta g &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{\Omega(t)}, \\
g - \gamma\partial_n g &= \gamma n_3 - x_3 && \text{on } \Gamma(t), \\
\nabla g &= o(|x|^{-2}) && \text{for } |x| \to \infty, \\
V_n &= \partial_n g + n_3.
\end{aligned}\right\} \tag{2.1}$$

We look for so called "travelling-wave solutions", which are solutions of the type $\Omega(t) = \Omega_0 + \vec{v}t$ where $\vec{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$ is the a priori unknown velocity of the travelling wave. We introduce a corresponding moving coordinate system and find a stationary free boundary problem:

$$\left.\begin{aligned}
\Delta g &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\
g - \gamma\partial_n g &= \gamma n_3 - x_3 && \text{on } \Gamma := \partial\Omega, \\
\nabla g &= o(|x|^{-2}) && \text{for } |x| \to \infty, \\
\partial_n g + (\vec{e_3} - \vec{v}) \cdot n &= 0 && \text{on } \Gamma,
\end{aligned}\right\} \tag{2.2}$$

of which the shape of Ω of the moving domain and of its boundary Γ are a priori unknown. $(2.2)_4$ arises from $(2.1)_4$ as the normal velocity of the moving boundary in the stationary coordinate system (2.1), V_n, corresponds with the velocity of the travelling wave in the normal direction; $\vec{v} \cdot n$. Combining this with $n_3 = \vec{e_3} \cdot n$ on the boundary gives $(2.2)_4$.

2.1 Invariance properties of (2.2)

There exist invariance properties of (2.2) for fixed velocity $\vec{v}$. When (Ω, g, γ) is a solution of (2.2), then we have:

- Translational invariance: For any $a \in \mathbb{R}^3$, $(\Omega + a, T_a g, \gamma)$ is also a solution. Here

  $$\Omega + \{a\} := \{x + a \mid x \in \Omega\},\ T_a g(x) := g(x - a).$$

  This is easily verified. When $\Delta g(x) = 0$ holds for all $x \in \mathbb{R}^3 \setminus \overline{\Omega}$, $\Delta g(x - a) = 0$ holds precisely for those $x \in \mathbb{R}^3 \setminus \overline{\Omega + a}$. As the shape and size of $\Omega + a$ is not different than that of Ω, the normal n is similar and by the same argument as before $(2.2)_2$ and $(2.2)_4$ hold. Obviously $(2.2)_3$ still holds.

- Scaling invariance: For any $R > 0$, $(R\Omega, S_R g, R\gamma)$ is a solution to (2.2), where

$$R\Omega := \{Rx \mid x \in \Omega\}, \ S_R g(x) := Rg(x/R).$$

Proof. We introduce $\tilde{x} = Rx$ and see that

$$\begin{aligned}
\Delta S_R g(\tilde{x}) &= \sum_{i=1}^{3} \frac{\partial^2 S_R g(\tilde{x})}{\partial \tilde{x}_i^2} = \sum_{i=1}^{3} \frac{\partial^2 R g(x)}{\partial \tilde{x}_i^2} \\
&= R \sum_{i=1}^{3} \frac{\partial^2 g(x)}{\partial x_i^2} (\frac{dx_i}{d\tilde{x}_i})^2 + \frac{\partial g(x)}{\partial x_i} \frac{d^2 x_i}{d\tilde{x}_i^2} \\
&= R(\frac{\partial^2 g(x)}{\partial x_1^2} \frac{1}{R^2} + 0 + \frac{\partial^2 g(x)}{\partial x_2^2} \frac{1}{R^2} + 0 + \frac{\partial^2 g(x)}{\partial x_3^2} \frac{1}{R^2} + 0) \\
&= \frac{1}{R} \Delta g(x) = 0.
\end{aligned}$$

This last equation holds for $x \in \mathbb{R}^3 \setminus \overline{\Omega}$ or equivalently for those $\tilde{x} \in \mathbb{R}^3 \setminus \overline{R\Omega}$. We compute the gradient of $S_R g$ with the use of the chain rule:

$$\begin{aligned}
\nabla S_R g(x) &= R \sum_{i=1}^{3} \frac{\partial g(\frac{x_1}{R}, \frac{x_2}{R}, \frac{x_3}{R})}{\partial x_i} \vec{e_i} = R \sum_{i=1}^{3} \frac{\partial g(\hat{x_1}, \hat{x_2}, \hat{x_3})}{\partial \hat{x_i}} \frac{\partial \left(\frac{x_i}{R}\right)}{\partial x_i} \vec{e_i} \\
&= R \sum_{i=1}^{3} \frac{\partial g(\hat{x})}{\partial \hat{x}_i} \frac{1}{R} \vec{e_i} = \nabla g(\hat{x}) = \nabla g(\frac{x}{R}).
\end{aligned}$$

Here we have denoted $\hat{x} = \frac{x}{R}$ and we easily see that $(2.2)_3$ still holds. Moreover, because the shape of region Ω is invariant under scaling, so is its outer normal on the boundary Γ. In other words, the outer normal on $R\Gamma := \partial R\Omega$ at point $\tilde{x}$, $\tilde{n}(\tilde{x})$, equals the corresponding outer normal on Γ at x, i.e. $\tilde{n}(\tilde{x}) = n(x)$. Now we see that

$$\begin{aligned}
S_R g(\tilde{x}) - (R\gamma)\partial_{\tilde{n}(\tilde{x})} S_R g(\tilde{x}) &= Rg(x) - R\gamma n(x) \cdot \nabla g(x) \\
&= R\left(g(x) - \gamma \partial_{n(x)} g(x)\right) \\
&\overset{*}{=} R(\gamma n_3(x) - x_3) \\
&= R\gamma \tilde{n}_3(\tilde{x}) - \tilde{x}_3, \\
\partial_{\tilde{n}(\tilde{x})} S_R g(\tilde{x}) + (\vec{e_3} - \vec{v}) \cdot \tilde{n}(\tilde{x}) &= (\nabla g(x) + \vec{e_3} - \vec{v}) \cdot n(x) \overset{*}{=} 0.
\end{aligned}$$

Here the equations marked with the asterisk hold for $x \in \Gamma$, such that we see that $S_R g$ fulfils $(2.2)_2$ and $(2.2)_4$ for exactly those $\tilde{x} \in R\Gamma = \partial R\Omega$, completing the proof. □

2.2 Trivial solution of traveling unit spheres

Our first result is the existence of a trivial solution of (2.2).

Result 2.1. (2.2) *has trivial solutions for $\gamma \geq 0$ given by unit spheres moving in $\vec{e_3}$-direction with constant velocity $\vec{v} = \frac{3}{1+2\gamma}\vec{e_3}$. The solution set is*

$$\Omega = B_1(0) = \{x \in \mathbb{R}^3 \mid |x| = \sqrt{x_1^2 + x_2^2 + x_3^2} < 1\}, \quad \vec{v} = \vec{v_0} = \tfrac{3}{1+2\gamma}\vec{e_3}, \quad g(x) = \tfrac{\gamma-1}{1+2\gamma}\tfrac{x_3}{|x|^3}. \tag{2.3}$$

Proof. We begin by computing the gradient of g:

$$\begin{aligned}\nabla g &= \frac{\partial g}{\partial x_1}\vec{e_1} + \frac{\partial g}{\partial x_2}\vec{e_2} + \frac{\partial g}{\partial x_3}\vec{e_3} \\ &= \frac{\gamma-1}{1+2\gamma}\left(-\frac{3x_1x_3}{|x|^5}\vec{e_1} - \frac{3x_2x_3}{|x|^5}\vec{e_2} + \frac{|x|^3 - 3x_3^2|x|}{|x|^6}\vec{e_3}\right) \\ &= \frac{\gamma-1}{1+2\gamma}\left(-\frac{3x_1x_3}{|x|^5}\vec{e_1} - \frac{3x_2x_3}{|x|^5}\vec{e_2} + \frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^5}\vec{e_3}\right).\end{aligned}$$

From this we see that $\nabla g = o(|x|^{-2})$ for $|x| \to \infty$, making g fulfill $(2.2)_3$. Moreover, on $\Gamma = \partial\Omega = \{x \in \mathbb{R}^3 \mid |x| = 1\} = S^2$ the unit normal pointing outwards is $n = x_1\vec{e_1} + x_2\vec{e_2} + x_3\vec{e_3}$. So on Γ we have (taking into account $|x| = 1$)

$$\begin{aligned}\partial_n g = n \cdot \nabla g &= \frac{\gamma-1}{1+2\gamma}\left(-\frac{3x_1^2x_3}{|x|^5} - \frac{3x_2^2x_3}{|x|^5} + \frac{(x_1^2 + x_2^2 - 2x_3^2)x_3}{|x|^5}\right) \\ &= \frac{\gamma-1}{1+2\gamma}\left(-3x_1^2x_3 - 3x_2^2x_3 + (x_1^2 + x_2^2 - 2x_3^2)x_3\right) \\ &= \frac{\gamma-1}{1+2\gamma}((-2x_1^2 - 2x_2^2 - 2x_3^2)x_3) = -2\frac{\gamma-1}{1+2\gamma}x_3.\end{aligned}$$

This way $(2.2)_2$ and $(2.2)_4$ are easily checked. On Γ we namely have

$$\begin{aligned}g - \gamma\partial_n g &= \frac{\gamma-1}{1+2\gamma}\frac{x_3}{|x|^3} + 2\gamma\frac{\gamma-1}{1+2\gamma}x_3 = \frac{\gamma-1}{1+2\gamma}x_3(1+2\gamma) = (\gamma-1)x_3 = \gamma n_3 - x_3, \\ \partial_n g + (\vec{e_3} - \vec{v}) \cdot n &= -2\frac{\gamma-1}{1+2\gamma}x_3 + ((1 - \frac{3}{1+2\gamma})\vec{e_3}) \cdot (x_1\vec{e_1} + x_2\vec{e_2} + x_3\vec{e_3}) \\ &= \frac{2-2\gamma}{1+2\gamma}x_3 + \frac{2\gamma-2}{1+2\gamma}x_3 = 0.\end{aligned}$$

We are now only left with checking the Laplacian of g:

$$\begin{aligned}\Delta g &= \text{div }(\nabla g) = \text{div }\left(\frac{\gamma-1}{1+2\gamma}\left(-\frac{3x_1x_3}{|x|^5}\vec{e_1} - \frac{3x_2x_3}{|x|^5}\vec{e_2} + \frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^5}\vec{e_3}\right)\right) \\ &= \frac{\gamma-1}{1+2\gamma}\left(-\frac{\partial}{\partial x_1}(\frac{3x_1x_3}{|x|^5}) - \frac{\partial}{\partial x_2}(\frac{3x_2x_3}{|x|^5}) + \frac{\partial}{\partial x_3}(\frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^5})\right) \\ &= \frac{\gamma-1}{1+2\gamma}\left(\frac{x_3(12x_1^2 - 3x_2^2 - 3x_3^2)}{|x|^7} + \frac{x_3(-3x_1^2 + 12x_2^2 - 3x_3^2)}{|x|^7} + \frac{x_3(-9x_1^2 - 9x_2^2 + 6x_3^2)}{|x|^7}\right) = 0.\end{aligned}$$

Note that because of the domain specified for g, we stay away from the origin and therefore never "divide by 0". We see that (2.3) indeed satisfies (2.2). ☐

Result 2.2. *Because of the scaling and translation invariance properties of the problem from Section* 2.1, (2.3) *is part of the family of solutions (for fixed* γ*)*

$$\Omega = B_R(a) = \{x \in \mathbb{R}^3 \mid |x - a| < R\}, \quad \vec{v} = \vec{v_0} = \tfrac{3R}{R+2\gamma}\vec{e_3}, \quad g = R^3\tfrac{\gamma-R}{R+2\gamma}\tfrac{x_3-a_3}{|x-a|^3},\ R > 0, a \in \mathbb{R}^3. \tag{2.4}$$

In our further analysis we exclude the corresponding degrees of freedom by demanding that Ω has volume equal to that of the unit sphere and has the origin as its geometric centre;

$$\int_\Omega d\xi = \frac{4\pi}{3}, \quad \int_\Omega \xi\, d\xi = 0. \tag{2.5}$$

3 Sobolev spaces

In the sequel we make use of so called Sobolev spaces. These function spaces were introduced by the Russian mathematician Sergei L. Sobolev in the late thirties of the last century. Sobolev spaces are vector spaces of functions that are equipped with a norm that is a combination of the L^p-norms of the function and its derivatives up to a given order (see Definition 3.2) [21]. Here we consider derivatives in a weak sense as to complete the space and making it a Banach space. Because we will use Sobolev spaces based on L^2-norms, as is common, we will give a definition of the L^2-space of a general region and for two regions important in this report.

Definition 3.1. *A L^2-space for Ω, denoted with $L^2(\Omega)$, are those functions for which there L^2-norm is finite:*

$$L^2(\Omega) := \left\{ u : \Omega \to \mathbb{C} : ||u||^2_{L^2(\Omega)} = \langle u, u \rangle_{L^2(\Omega)} := \int_\Omega u\overline{u}\, d\Omega = \int_\Omega |u|^2\, d\Omega < \infty \right\}.$$

In particular, the L^2-spaces for the unit circle, $S^1 \subset \mathbb{R}^2$, and unit sphere, $S^2 \subset \mathbb{R}^3$ are defined as follows:

$$L^2(S^1) := \left\{ u : S^1 \to \mathbb{C} : ||u||^2_{L^2(S^1)} = \langle u, u \rangle_{L^2(S^1)} := \int_0^{2\pi} |u(\theta)|^2\, d\theta < \infty \right\},$$
$$L^2(S^2) := \left\{ u : S^2 \to \mathbb{C} : ||u||^2_{L^2(S^2)} = \langle u, u \rangle_{L^2(S^2)} := \int_0^{2\pi} \int_0^{\pi} |u(\theta, \phi)|^2 \sin\theta\, d\theta\, d\phi < \infty \right\}.$$

As we have now (re)familiarized ourselves with (a specific case of) L^p-spaces and -norms, we are in the position to define a Sobolev space for a given region.

Definition 3.2. *[12] Assume that Ω is an open subset of $\mathbb{R}^n$. The Sobolev space $W^{s,p}(\Omega)$, of order $s \in \mathbb{N}$, consists of functions $u \in L^p(\Omega)$ such that for every multi-index α with $|\alpha| \leq s$, the weak derivative $D^\alpha u$ exists and $D^\alpha u \in L^p(\Omega)$. Thus*

$$W^{s,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), |\alpha| \leq s\}.$$

We define its norm by

$$||u||_{W^{s,p}(\Omega)} := \Big(\sum_{|\alpha| \leq s} \int_\Omega |D^\alpha u|^p\, dx \Big)^{1/p} \text{ for } 1 \leq p < \infty,$$
$$\text{and } ||u||_{W^{s,\infty}(\Omega)} := \sum_{|\alpha| \leq s} \operatorname*{ess\,sup}_\Omega |D^\alpha u| \text{ for } p = \infty.$$

As mentioned an important case in this definition is when $p = 2$. This specific Sobolev space has gotten its own notation. The Sobolev space of order s with $p = 2$ is denoted with $H^s(\Omega)$. It has gotten this H, because it makes for a Hilbert space when the following inner product is defined for arbitrary $u, v \in H^s(\Omega)$:

$$\langle u, v \rangle_{H^s(\Omega)} := \sum_{|\alpha| \leq s} \int_\Omega D^\alpha u D^\alpha v\, dx, \text{ such that } ||u||^2_{H^s(\Omega)} = \langle u, u \rangle_{H^s(\Omega)}.$$

This is particularly useful for the Sobolev space of order $s > 0$ defined on the unit circle S^1, $H^s(S^1) = H^s[0, 2\pi]$. Here we can use the coefficients of the Fourier series of any $u \in H^s[0, 2\pi]$. Namely [9]

$$H^s[0, 2\pi] := \{u \in L^2[0, 2\pi] : \sum_{n=0}^{\infty} (1 + n^2)^s |a_k|^2 + \sum_{n=1}^{\infty} (1 + n^2)^s |b_k|^2 < \infty\}, \tag{3.1}$$

$$\langle u, v\rangle_{H^s[0,2\pi]} := \sum_{n=0}^{\infty}(1+n^2)a_n\overline{\alpha_n} + \sum_{n=1}^{\infty}(1+n^2)b_n\overline{\beta_n},$$

where $\{a_n\}$, $\{b_n\}$ and $\{\alpha_n\}$, $\{\beta_n\}$ are the Fourier coefficients of u and v, respectively. Here we have used the convention that $a_0 = \frac{1}{2\pi}\int_0^{2\pi} u(x)\,dx, a_n = \frac{1}{\pi}\int_0^{2\pi} u(x)\cos nx\,dx$ (for $n = 1, 2, 3, ...$) and $b_n = \frac{1}{\pi}\int_0^{2\pi} u(x)\sin nx\,dx$ (for $n = 0, 1, 2, ...$) are the Fourier coefficients of u. It is useful that the Fourier coefficients turn up in this particular definition and the details as to why can be found in [9].

Analogous to what the Fourier series is for functions defined on a circle, there is a counterpart for functions defined on a sphere. This is an expansion with respect to the so called spherical harmonics. These are the homogeneous polynomials in $\mathbb{R}^3$ restricted to the unit sphere, whereas the $\cos nx$ and $\sin nx$ are the homogeneous polynomials in $\mathbb{R}^2$ restricted to the unit circle. In the subsequent sections we will see that we can define a Sobolev space on the unit sphere in terms of a condition on the coefficients of such an expansion in the same way as in (3.1). Now, let us first introduce where we encounter the spherical harmonics.

3.1 Spherical harmonics

The spherical harmonics arise when solving Laplace's equation ($\Delta f = 0$) in spherical coordinates by means of separation of variables [15]. Firstly, we use the convention that
$x_1 = r\sin\theta\cos\phi, x_2 = r\sin\theta\sin\phi, x_3 = r\cos\theta$ with $r \in \mathbb{R}_0^+, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$. Now Laplace's equation becomes

$$0 = \Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} =: \frac{1}{r^2}\left(\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \Delta_{S^2} f\right).$$

Here we denote with Δ_{S^2} the Laplace-Beltrami operator. Because this operator is defined only in terms of of derivatives with respect to the angles and excluding r, one can view this as being on the unit sphere, explaining the S^2 in its definition.
Now separating the r variable only, i.e. assuming $f(r, \theta, \phi) = R(r) \cdot Y(\theta, \phi)$, we get

$$\begin{aligned} Y(\theta,\phi)\cdot\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) &+ R(r)\cdot\Delta_{S^2}Y(\theta,\phi) = 0 \text{ and after division by } R(r)\cdot Y(\theta,\phi); \\ \frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) &= -\frac{1}{Y(\theta,\phi)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y(\theta,\phi)}{\partial\theta}\right) - \frac{1}{Y(\theta,\phi)\sin^2\theta}\frac{\partial^2 Y(\theta,\phi)}{\partial\phi^2}\ (=\lambda) \\ &= -\frac{1}{Y(\theta,\phi)}\Delta_{S^2}Y(\theta,\phi)\,(=\lambda)\,. \end{aligned} \tag{3.2}$$

Both sides of the equation are constant, i.e. equal to some separation constant $\lambda \in \mathbb{Z}$, as the left-hand side is only dependent on r and the right-hand side is only dependent on the angles, θ, ϕ. We now have the following equation for R:

$$\frac{d^2R(r)}{dr^2} + \frac{2}{r}\frac{dR(r)}{dr} - \frac{\lambda}{r^2}R(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) - \frac{\lambda}{r^2}R(r) = 0. \tag{3.3}$$

We now again perform a separation of variables for the angle function Y, i.e. we assume $Y(\theta, \phi) = p(\theta)q(\phi)$. As a result of changing to spherical coordinates this comes with set of conditions for Y, which translates into conditions for p and q. These are that Y is periodic in ϕ with period 2π, which implies q periodic (period 2π) and that Y is regular at the poles. It follows from (3.2), by substituting $Y(\theta,\phi) = p(\theta)q(\phi)$ and multiplying by $-\sin^2\theta$, that

$$\frac{\sin\theta}{p(\theta)q(\phi)}q(\phi)\frac{d}{d\theta}\left(\sin\theta\frac{dp(\theta)}{d\theta}\right) + \frac{1}{p(\theta)q(\phi)}p(\theta)\frac{d^2q(\phi)}{d\phi^2} = -\lambda\sin^2\theta,$$
$$\frac{1}{p(\theta)}\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dp}{d\theta}\right) + \lambda\sin^2\theta = -\frac{1}{q(\phi)}\frac{d^2q}{d\phi^2}\ (=\alpha)\,.$$

Both sides of the last equation are constant (equal to α) because they are only dependent on one of the angles, similarly as before.
The equation for q, keeping in mind that q is periodic, is readily solved. The eigenfunctions for this problem are (constant multiples of) $q(\phi) = e^{im\phi}$ and only for $\alpha = m^2$, with $m \in \mathbb{Z}$, because of periodicity of q. The equation for p, with $\alpha = m^2$ and after dividing by $\sin^2\theta$ is

$$\frac{(d/d\theta)[\sin\theta(dp(\theta)/d\theta]}{\sin\theta} + p(\theta)\cdot[\lambda - \frac{m^2}{\sin\theta^2}] = 0.$$

This equation - with the boundary condition that q is finite at $\theta = 0, \pi$ - changes into the associated Legendre equation after introducing a new variable $s = \cos\theta$. Using the power series method, we find that this equation, combined with the boundary condition that p is finite at $s = \pm 1$, has (any constant times) the associated Legendre polynomials, P_l^m, as eigenfunctions. Here the eigenvalues are $\lambda = l(l+1)$, where l is an integer with $l \geq |m|$. The associated Legendre polynomials are given by

$$P_l^{|m|}(s) = \frac{(-1)^{|m|}}{2^l l!}(1-s^2)^{|m|/2}\frac{d^{l+|m|}}{ds^{l+|m|}}[(s^2-1)^l]. \tag{3.4}$$

This expression shows that the function is nonzero only for the indicated indices corresponding to the eigenvalues; $l \in \mathbb{N}, m \in \mathbb{Z}$ with $|m| \leq l$. If m would be bigger in absolute value than l, then by definition, we would be differentiating a degree $2l$ polynomial more than $2l$ times, resulting in the zero function. Also note that, for odd m's, the Legendre polynomials are not actually polynomials in s. They are polynomials in $\sin\theta$ and $\cos\theta$ after the substitution $s = \cos\theta$ though. This is commonly done, also in this our applications.

We put the eigenfunction-solutions for Y together and get the so called spherical harmonics:

Definition 3.3 (Spherical harmonics)**.** *[15] The functions $Y_l^m : S^2 \to \mathbb{C}$ given by $Y_l^m(\theta,\phi) = P_l^{|m|}(\cos\theta)\, e^{im\phi}$ are the (non-normalized) spherical harmonics of degree l and order m. Here $\theta \in [0,\pi]$ is the colatitude/polar angle/angle with the x_3-axis, so at $\theta = 0$ we are at the north pole and $\phi \in [0, 2\pi]$ is the longtitude/azimuth/angle with the x_1-axis. The functions $P_l^{|m|} : [-1,1] \to \mathbb{R}$ are the aforementioned associated Legendre polynomials. The ranges of the integer indices for nonzero functions are $-l \leq m \leq l, 0 \leq l < \infty$.*

Importantly, because these are the eigenfunctions of the Laplace-Beltrami operator (see rightmost equation of (3.2)) corresponding to the eigenvalue $\lambda = l(l+1)$, all spherical harmonics satisfy the following relation:

$$-\Delta_{S^2}Y_l^m(\theta,\phi) = l(l+1)Y_l^m(\theta,\phi). \tag{3.5}$$

We compute some of these spherical harmonics:

$$\begin{aligned}
Y_0^0(\theta,\phi) &= P_0^0(\cos\theta)e^{i*0*\phi} = \frac{(-1)^0}{2^0 0!}(1-s^2)^{0/2}\frac{d^{0+0}}{ds^{0+0}}[(s^2-1)^0]\Big|_{s=\cos\theta} = 1,\\
Y_1^0(\theta,\phi) &= P_1^0(\cos\theta)e^{i*0*\phi} = \frac{(-1)^0}{2^1 1!}(1-s^2)^{0/2}\frac{d^{1+0}}{ds^{1+0}}[(s^2-1)^1]\Big|_{s=\cos\theta}\\
&= \frac{1}{2}2s\Big|_{s=\cos\theta} = \cos\theta,\\
Y_1^{\pm 1}(\theta,\phi) &= P_1^1(\cos\theta)e^{\pm i\phi} = \frac{(-1)^1}{2^1 1!}(1-s^2)^{1/2}\frac{d^{1+1}}{ds^{1+1}}[(s^2-1)^1]\Big|_{s=\cos\theta}e^{\pm i\phi}\\
&= -\frac{1}{2}(\sin^2\theta)^{1/2}2e^{\pm i\phi} = -\sin\theta e^{\pm i\phi}.
\end{aligned}$$

Explicit spherical harmonics of higher degree and order can be found in tables, such as in [16].

Lemma 3.4. *The spherical harmonics are mutually orthogonal, i.e.* $Y_l^m \perp Y_k^n$ *if and only if* $(m,l) \neq (n,k)$*, with respect to the* L^2*-inner product. Therefore the set of spherical harmonics,* $\{Y_l^m \mid l \in \mathbb{N}_0, m \in \mathbb{Z} : |m| \leq l\}$ *is linearly independent.*

Proof. This follows from the fact that spherical harmonics are eigenfunctions of the Laplace-Beltrami operator. For a worked out computation we let $l,k \in \mathbb{N}\cup\{0\}$ and m,n integers such that $-l \leq m \leq l$ and $-k \leq n \leq k$. Now

$$\begin{aligned}
\langle Y_l^m, Y_k^n\rangle_{L^2(S^2)} &= \int_0^{2\pi}\int_0^{\pi} Y_l^m(\theta,\phi)\overline{Y_k^n(\theta,\phi)}\sin\theta\, d\theta d\phi\\
&= \int_0^{\pi} P_l^{|m|}(\cos\theta)P_k^{|n|}(\cos\theta)\sin\theta\, d\theta \int_0^{2\pi} e^{(m-n)i\phi}\, d\phi\\
&= -2\pi\delta_{mn}\int_1^{-1} P_l^{|m|}(s)P_k^{|n|}(s)\, ds\\
&= 2\pi\delta_{mn}\int_{-1}^{1} P_l^{|m|}(s)P_k^{|m|}(s)\, ds\\
&= 2\pi\delta_{mn}\frac{2}{2l+1}\frac{(l+|m|)!}{(l-|m|)!}\delta_{lk},
\end{aligned}$$

where the orthogonality of the associated Legendre polynomials in the last step follows from the fact that these are eigenfunctions as shown before and can also be found in Section 10.3 of [15]. Eigenfunctions of any Hermitian (differential) operator namely are mutually orthogonal (see Section 1.1 of [18]). For a worked out computation of the last step, see [14] or Section 25.5 of [18].
So we see that $||Y_l^m||_{L^2(S^2)} = \sqrt{\langle Y_l^m, Y_l^m\rangle_{L^2(S^2)}} = \sqrt{\frac{4\pi}{2l+1}\frac{(l+|m|)!}{(l-|m|)!}}$. □

Lemma 3.5. *The set of normalized spherical harmonics form an orthonormal and complete basis for Hilbert space* $L^2_{\mathbb{C}}(S^2)$*. This set is given by*

$$\{\tilde{Y}_l^m \mid \tilde{Y}_l^m := \frac{Y_l^m}{||Y_l^m||} = \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}Y_l^m, l \in \mathbb{N}_0,\, m \in \mathbb{Z} : |m| \leq l\}.$$

Proof. The orthonormality is clear and the completeness is out of the scope of this project. For a justification see Section 9.4 of [20]. □

As a result of this and known Hilbert space theory, we have the following important theorem.

Theorem 3.6. *Every function, $f : S^2 \to \mathbb{C}$, defined on the unit sphere that is square-integrable (is in L^2) can be expanded in a series of normalized spherical harmonics $\tilde{Y}_l^m$.*
i.e. $f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m \tilde{Y}_l^m(\theta,\phi)$, where constants f_l^m can be computed by integral:

$$f_l^m = \langle f, \tilde{Y}_l^m\rangle_{L^2(S^2)} = \int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)\overline{\tilde{Y}_l^m(\theta,\phi)}\sin\theta\, d\theta\, d\phi$$
$$= \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}\int_0^{2\pi}\int_0^{\pi} f(\theta,\phi)P_l^{|m|}(\cos\theta)e^{-mi\phi}\sin\theta\, d\theta\, d\phi.$$

3.2 Sobolev space on the unit sphere

Similarly as in the two dimensional case, where a Sobolev space on the unit circle was defined in terms of a condition on the Fourier coefficients of functions on the circle (see (3.1)), we are now in the position to define a Sobolev space on the unit sphere.

Definition 3.7 A. *The Sobolev space on the unit sphere of even order $s \in \mathbb{N}_0$ (based on the L^2-norm), is defined by $H^s(S^2) = H^s([0,2\pi]\times[0,\pi]) := \{f \in L^2(S^2) : (I - \Delta_{S^2})^{s/2} f \in L^2(S^2)\}$. Here we denote with Δ_{S^2} the Laplace-Beltrami operator as defined before.*

The Laplace-Beltrami operator is the equivalent of the Laplace operator on the sphere and it takes the role of second derivatives. So we want that $H^2(S^2)$ includes up to the "second derivatives", which it does in this definition. Note that $H^0(S^2) = L^2(S^2)$.

We have seen in Theorem 3.6 that every square integrable function defined on the unit sphere can be expanded in a series of spherical harmonics. Assume we have such an expansion, i.e. $f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\langle f, \tilde{Y}_l^m\rangle_{L^2(S^2)}\,\tilde{Y}_l^m(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m\tilde{Y}_l^m(\theta,\phi)$ and consider the differential operator $\mathcal{L} = I - \Delta_{S^2}$. Now, keeping in mind (3.5), one has

$$\mathcal{L}f = \mathcal{L}(\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m\tilde{Y}_l^m) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(1 + l(l+1))f_l^m\tilde{Y}_l^m.$$

More generally, for all $s \in \mathbb{N}_0$

$$\mathcal{L}^s f = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(1 + l(l+1))^s f_l^m\tilde{Y}_l^m.$$

Because of what was previously discussed, in Lemma 3.5 and Theorem 3.6 specifically, with the help of Parseval's identity for Hilbert spaces, a norm in terms of the coefficients of a spherical harmonics expansion can be determined. For $f \in L^2(S^2)$ with spherical harmonics expansion $\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m\tilde{Y}_l^m(\theta,\phi)$, we have

$$||f||^2_{L^2(S^2)} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left|\langle f, Y_l^m\rangle_{L^2(S^2)}\right|^2 = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}|f_l^m|^2.$$

So now that this has been established, we see that

$$||\mathcal{L}^s f||^2_{L^2(S^2)} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}|(1 + l(l+1))^s f_l^m|^2 = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}(1 + l(l+1))^{2s}|f_l^m|^2.$$

We can now extend Definition 3.7 A to include all positive orders in terms of the coefficients of a spherical harmonics expansion in such a way that in coincides with Definition 3.7 A for the even positive orders:

Definition 3.7 B. *For any $f \in L^2(S^2)$, let its spherical harmonics expansion be denoted by $\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m \tilde{Y}_l^m(\theta,\phi)$. The Sobolev space on the unit sphere of order $s > 0$ (based on the L^2-norm) is defined by*

$$H^s(S^2) = \left\{ f \in L^2(S^2) : ||\mathcal{L}^{s/2} f||^2_{L^2(S^2)} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} (1 + l(l+1))^s |f_l^m|^2 < \infty \right\} \tag{3.6}$$

$$= \left\{ f \in L^2(S^2) : ||f||^2_{H^s(S^2)} := \sum_{l=0}^{\infty}\sum_{m=-l}^{l} (1 + l^2)^s |f_l^m|^2 < \infty \right\}. \tag{3.7}$$

The newly defined $H^s(S^2)$-norm was used for ease of computation and to resemble the norm used in the two dimensional case, see (3.1). The equivalence of the $H^s(S^2)$-norm and the $L^2(S^2)$-based norm used in (3.6) needs to be shown. Therefore we show that there exist constants $c_0, c_1 \in \mathbb{R}$ such that for any $f \in L^2(S^2)$, we have that

$$c_0 \cdot ||\mathcal{L}^{s/2} f||^2_{L^2(S^2)} \leq ||f||^2_{H^s(S^2)} \leq c_1 \cdot ||\mathcal{L}^{s/2} f||^2_{L^2(S^2)}.$$

Proof. Let $s > 0$ be arbitrary and let $f \in L^2(S^2)$, with spherical harmonics expansion $\sum_{l=0}^{\infty}\sum_{m=-l}^{l} f_l^m \tilde{Y}_l^m(\theta,\phi)$. We have for all $l \in \mathbb{N}_0$ that $1 + l^2 \leq 1 + l(l+1)$, such that clearly $||f||^2_{H^s(S^2)} \leq ||\mathcal{L}^{s/2} f||^2_{L^2(S^2)}$.
On the other hand we have that $l^2 + 1 \geq l$ for all $l \in \mathbb{N}_0$. From this we have that for all $l \in \mathbb{N}_0$, $2(1 + l^2) \geq 1 + l(l+1)$ and

$$2^s \cdot ||f||^2_{H^s(S^2)} = \sum_{l,m} (2(1 + l^2))^s |f_l^m|^2 \geq \sum_{l,m} (1 + l(l+1))^s |f_l^m|^2 = ||\mathcal{L}^{s/2} f||^2_{L^2(S^2)}.$$

So we see that $2^{-s} \cdot ||\mathcal{L}^{s/2} f||^2_{L^2(S^2)} \leq ||f||^2_{H^s(S^2)}$. So taking $c_0 = 2^{-s}$, $c_1 = 1$ completes the proof. $\square$

This now ensures that for all $f \in L^2(S^2)$ with finite $||\mathcal{L}^{s/2} f||^2_{L^2(S^2)}$, also $||f||^2_{H^s(S^2)} < \infty$ and vice versa. This implies equality between (3.6) and (3.7).

4 Operator problem

In the sequel, let us denote by e_r, e_θ, e_ϕ the spherical unit vectors. Expressed in the Cartesian unit vectors $\vec{e_1}, \vec{e_2}, \vec{e_3}$ they are:

$$\begin{aligned} e_r &= \sin\theta\cos\phi\,\vec{e_1} + \sin\theta\sin\phi\,\vec{e_2} + \cos\theta\,\vec{e_3}, \\ e_\theta &= \cos\theta\cos\phi\,\vec{e_1} + \cos\theta\sin\phi\,\vec{e_2} - \sin\theta\,\vec{e_3}, \\ e_\phi &= -\sin\phi\,\vec{e_1} + \cos\phi\,\vec{e_2}. \end{aligned}$$

Moreover, we will use a shorthand notation for partial derivatives in the sequel; for example the partial derivative of a function f with respect to r will be denoted by $f_r := \frac{\partial f}{\partial r}$.

4.1 Operator equation

Now we have established the Sobolev space on the unit sphere, we turn back to the problem at hand, i.e. (2.2). We will continue with this in the same way as in [11]. We restrict ourselves to domains that are star-shaped with respect to the origin. We want to transform our problem, (2.2), to a non-linear operator problem defined on the unit sphere, S^2. To shorten notations, we will identify functions defined on the unit sphere, e.g. $u(x) = u(x_1, x_2, x_3)$ with functions $u(\theta, \phi) = u(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, $\theta, \phi \in \mathbb{R}$, which are π-periodic in θ and 2π-periodic in ϕ. Now define for positive functions defined on the unit sphere $u : S^2 \to \mathbb{R}^+$

$$\Omega_u := \left\{x \in \mathbb{R}^3 : 0 \le |x| < u(x/|x|)\right\}, \;\; \Gamma_u = \partial\Omega_u. \tag{4.1}$$

We denote with $x(u)(\theta, \phi) = u(\theta, \phi)(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = u(\theta, \phi)\,e_r$, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi]$ a parameterization of Γ_u and the exterior unit normal to Ω_u in the point $x(u)(\theta, \phi) \in \Gamma_u$ with $n(u)(\theta, \phi)$. We have that

$$\begin{aligned} n(u) &= \frac{x(u)_\theta \times x(u)_\phi}{|x(u)_\theta \times x(u)_\phi|} \\ &= \frac{(u_\theta\,e_r) \times (u_\phi\,e_r) + (u\,e_\theta) \times (u_\phi\,e_r) + (u_\theta\,e_r) \times (u\sin\theta\,e_\phi) + (u\,e_\theta) \times (u\sin\theta\,e_\phi)}{|(u_\theta\,e_r) \times (u_\phi\,e_r) + (u\,e_\theta) \times (u_\phi\,e_r) + (u_\theta\,e_r) \times (u\sin\theta\,e_\phi) + (u\,e_\theta) \times (u\sin\theta\,e_\phi)|} \\ &= \frac{-uu_\phi\,e_\phi - uu_\theta\sin\theta\,e_\theta + u^2\sin\theta\,e_r}{\sqrt{(uu_\phi)^2 + (uu_\theta\sin\theta)^2 + (u^2\sin\theta)^2}} \\ &= (u_\phi^2 + \sin^2\theta(u_\theta^2 + u^2))^{-\frac{1}{2}}\left[u\sin\theta\,e_r - u_\theta\sin\theta\,e_\theta - u_\phi\,e_\phi\right]. \end{aligned} \tag{4.2}$$

Let us define the operator $A_\gamma(u)$ which acts on sufficiently smooth $f : S^2 \to \mathbb{R}$ and is given by $A_\gamma(u)f = g \circ x(u)$, where $g : \mathbb{R}^3 \setminus \overline{\Omega_u} \to \mathbb{R}$ is the solution to the exterior Robin problem

$$\left.\begin{aligned} \Delta g &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{\Omega_u}, \\ g - \gamma\partial_{n(u)}g &= f \circ x(u)^{-1} && \text{on } \Gamma_u, \\ \nabla g &= o(|x|^{-2}) && \text{for } |x| \to \infty. \end{aligned}\right\} \tag{4.3}$$

Notice that $A_\gamma(u)$ is a linear operator. This operator is introduced such that it resembles $(2.2)_{1-3}$ for the right function f. Now we can namely turn our problem into an operator problem. We take $f := \gamma n_3 - x_3$, just as in (2.2) and now take g such that $A_\gamma(u)f = g \circ x(u)$. g fulfills $(2.2)_{1-3}$ on $\mathbb{R}^3 \setminus \overline{\Omega_u}$. We also want it to fulfill $(2.2)_4$ on Γ_u and we can rewrite $(4.3)_2$ to obtain

$$\begin{aligned} g &= \gamma\partial_{n(u)}g + f \circ x(u)^{-1} \\ &\overset{(2.2)_4}{=} \gamma(\vec{v} - \vec{e_3}) \cdot (n(u) \circ x(u)^{-1}) + (\gamma n_3 - x_3) \circ x(u)^{-1} \end{aligned}$$

on Γ_u and after composing with $x(u)$, we have $A_\gamma(u)[\gamma n_3(u) - x_3(u)] = \gamma\vec{v}\cdot n(u) - \gamma\vec{e_3}\cdot n(u) + \gamma n_3(u) - x_3(u)$ on S^2. So we have our operator problem;

$$F(u,\vec{v}) := x_3(u) - \gamma\vec{v}\cdot n(u) + A_\gamma(u)[\gamma n_3(u) - x_3(u)] = 0 \text{ on } S^2. \tag{4.4}$$

We can now also rewrite (2.5), by changing to spherical coordinates and considering that $\int_{\Omega_u} \xi\, d\xi = 0$ implies that $\int_{\Omega_u} x_i\, dx_i = 0$ for $i = 1, 2, 3$. We obtain

$$\begin{aligned}&\int_0^\pi \int_0^{2\pi} u^3(\theta,\phi)\sin\theta\, d\phi\, d\theta = 4\pi, \quad \int_0^\pi \int_0^{2\pi} u^4(\theta,\phi)\sin^2\theta\cos\phi\, d\phi\, d\theta = \\ &\int_0^\pi \int_0^{2\pi} u^4(\theta,\phi)\sin^2\theta\sin\phi\, d\phi\, d\theta = \int_0^\pi \int_0^{2\pi} u^4(\theta,\phi)\cos\theta\sin\theta\, d\phi\, d\theta = 0.\end{aligned} \tag{4.5}$$

Now we can represent our problem as looking for an $u \in H^s_+(S^2) := \{v \in H^s(S^2) | v > 0\}$, with a sufficiently high s and corresponding vector $\vec{v} \in \mathbb{R}^3$ that represents the velocity of the travelling wave that solve

$$\begin{cases} F(u,\vec{v}) = 0 \quad \text{ on } S^2, \\ \langle u^3, \mathbf{1}\rangle_{L^2(S^2)} = 4\pi, \\ \langle u^4, \sin\theta\cos\phi\rangle_{L^2(S^2)} = \langle u^4, \sin\theta\sin\phi\rangle_{L^2(S^2)} = \langle u^4, \cos\theta\rangle_{L^2(S^2)} = 0. \end{cases} \tag{4.6}$$

Here we have abused notation by specifying the argument for the cos and sin functions, for clarity. For solving this system ultimately we need the following result:

Result 4.1. *For $u = \mathbf{1}$ on S^2 and $f(\theta,\phi) = \sum_{l=0}^\infty \sum_{m=-l}^{l} f_l^m \tilde{Y}_l^m(\theta,\phi)$ we have that*

$$(A_\gamma(\mathbf{1})f)(\theta,\phi) = f_0^0\tilde{Y}_0^0(\theta,\phi) + \sum_{l=1}^\infty \sum_{m=-l}^{l} \frac{f_l^m}{(1+l)\gamma + 1}\tilde{Y}_l^m(\theta,\phi).$$

Proof. Firstly, $u = \mathbf{1}$ on S^2 implies that $\Omega_1 = B_1(0)$, $\Gamma_1 = S^2$, so $r = 1$ on Γ_1 and also $x(\mathbf{1}) = e_r$, $n(\mathbf{1}) = e_r$. We have seen in Section 3.1, that solving the the Laplace equation in spherical coordinates by means of separation of variables gives rise to the spherical harmonics. We also solve (3.3) to have a full solution of $(4.3)_1$. We solve $r^2R''(r) + 2rR'(r) - \lambda R(r) = 0$, which is a Cauchy-Euler equation. Here we have found when solving the associated Legendre equation that $\lambda = l(l+1)$ with $l \in \mathbb{N}_0$ are the eigenvalues that we consider.

Now, by trying a solution of the form $R(r) = r^a$, we find that for $l \in \mathbb{N}_0$, $R_l(r) = c_l^+ r^l + c_l^- r^{-(l+1)}$. Here $c_l^\pm \in \mathbb{C}$ are constants. We have already found that the other part, concerning the angles, gives rise to constant multiples of spherical harmonics as solution; $Y_l^m(\theta,\phi) = y_l^m\tilde{Y}_l^m(\theta,\phi)$, with $y_l^m \in \mathbb{C}$ constants. So we have that $g(r,\theta,\phi) = \sum_{l=0}^\infty \sum_{m=-l}^{l} R_l(r)Y_l^m(\theta,\phi)$.

To make this g fulfill $(2.2)_3$, we need to compute the partial derivatives of its summands;

$$\begin{aligned}\frac{d}{dr}R_l(r) &= lc_l^+ r^{l-1} - (l+1)c_l^- r^{-(l+2)}, \\ \frac{\partial}{\partial\theta}Y_l^m(\theta,\phi) &= y_l^m\left(\frac{d_l^m}{\sin\theta}\tilde{Y}_{l+1}^m(\theta,\phi) + \frac{e_l^m}{\sin\theta}\tilde{Y}_{l-1}^m(\theta,\phi)\right) =: V_l^m(\theta,\phi), \\ \frac{\partial}{\partial\phi}Y_l^m(\theta,\phi) &= i\, m\, y_l^m\tilde{Y}_l^m(\theta,\phi) =: W_l^m(\theta,\phi).\end{aligned}$$

Here d_l^m and e_l^m are coefficients dependent on m and l only, as computed in (6.8). We see that for $l = 0$ (and therefore also $m = 0$), most terms vanish. We have the derivatives with respect to

θ, ϕ equal to zero and for the derivative with respect to r the term corresponding to the constant, r^0, also equals zero, leaving only $\frac{d}{dr}R_0(r) = -c_0^- r^{-2}$. For arbitrary bigger l, we see that because of the boundedness of the Y, W, V's that the behavior of ∇g is only dependent on the r-part;

$$\begin{aligned}
r^2 \nabla g(r,\theta,\phi) =& r^2 \left(g_r\, e_r + \frac{1}{r} g_\theta\, e_\theta + \frac{1}{r\sin\theta} g_\phi\, e_\phi \right) \\
=& - c_0^-\, e_r + \sum_{l=1}^{\infty} \sum_{m=-l}^{l} (l c_l^+ r^{l+1} - (l+1) c_l^- r^{-l}) Y_l^m(\theta,\phi)\, e_r \\
&+ \sum_{l=1}^{\infty} \sum_{m=-l}^{l} (c_l^+ r^{l+1} + c_l^- r^{-l}) \left(V_l^m(\theta,\phi)\, e_\theta + \frac{W_l^m(\theta,\phi)}{\sin\theta}\, e_\phi \right).
\end{aligned}$$

We need this to go to zero as $r \to \infty$ for $\nabla g = o(|x|^{-2})$ and because r^{-l} does this and r^{l+1} does not (for $l \in \mathbb{N}_+$), we have to have $c_0^- = 0$ and $c_l^+ = 0$ for $l \in \mathbb{N}_+$. We conclude that $g(r,\theta,\phi) = g_0^0 Y_0^0(\theta,\phi) + \sum_{l=1}^{\infty} \sum_{m=-l}^{l} g_l^m r^{-(l+1)} \tilde{Y}_l^m(\theta,\phi)$, where $g_0^0 = c_0^+ y_0^0$ and $g_l^m = c_l^- y_l^m$.

Next, we compute $\partial_{n(\mathbf{1})} g = n(\mathbf{1}) \cdot \nabla g = e_r \cdot (g_r\, e_r + \frac{1}{r} g_\theta\, e_\theta + \frac{1}{r\sin\theta} g_\phi\, e_\phi) = g_r$ on Γ_1. We can now also incorporate $(4.3)_2$, with $r = 1$, and get

$$\begin{aligned}
0 &= g(1,\theta,\phi) - \gamma \partial_{n(\mathbf{1})} g - f(\theta,\phi) \\
&= g(1,\theta,\phi) - \gamma g_r(1,\theta,\phi) - \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m \tilde{Y}_l^m(\theta,\phi) \\
&= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} g_l^m \tilde{Y}_l^m(\theta,\phi) + \gamma \sum_{l=1}^{\infty} \sum_{m=-l}^{l} (l+1) g_l^m \tilde{Y}_l^m(\theta,\phi) - \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_l^m \tilde{Y}_l^m(\theta,\phi) \\
&= \sum_{l=1}^{\infty} \sum_{m=-l}^{l} ((\gamma l + \gamma + 1) g_l^m - f_l^m) \tilde{Y}_l^m(\theta,\phi) + (g_0^0 - f_0^0) \tilde{Y}_0^0(\theta,\phi).
\end{aligned}$$

The orthogonality of the spherical harmonics gives us that for all $l \in \mathbb{N}_+$ and every integer $|m| \leq l$, $g_l^m = \frac{f_l^m}{\gamma(l+1)+1}$ and that $g_0^0 = f_0^0$, such that indeed $A_\gamma(\mathbf{1})[f](\theta,\phi) = g(1,\theta,\phi) = f_0^0 \tilde{Y}_0^0(\theta,\phi) + \sum_{l=1}^{\infty} \sum_{m=-l}^{l} f_l^m \tilde{Y}_l^m(\theta,\phi)/((1+l)\gamma + 1)$. □

Quite obviously $u = \mathbf{1}, \vec{v} = \vec{v_0}$ is a solution to (4.4), (4.5) according to (2.3), so it nicely solves (4.6). Moreover we see that the volume preservation reappears as

$$\int_{\Gamma_u} F(u,\vec{v})(\theta,\phi) |x(u)_\theta \times x(u)_\phi|\, d\theta\, d\phi = \int_{\Gamma_u} F(u,\vec{v})(\theta,\phi) (u_\phi^2 + \sin^2\theta (u_\theta^2 + u^2))^{\frac{1}{2}}\, d\theta\, d\phi = 0.$$

5 Fréchet derivative

In the sequel we will make use of the so called Fréchet derivative. It is named after the French mathematician Maurice Fréchet (1878-1973) and it is a generalisation of the derivative known for (real-valued) functions for operators on Banach spaces. Commonly the derivative of a real valued function $f : \mathbb{R} \to \mathbb{R}$ at a point x_0 is interpreted as the slope of the tangent line of the graph of f at point x_0. The idea that the Fréchet derivative is based on is that one can see this slope $f'(x_0)$ as a linear mapping, in the way that for x "close to" x_0, $f'(x_0) \approx \frac{f(x)-f(x_0)}{x-x_0}$, which gives us that $f(x) \approx f(x_0) + (x - x_0)f'(x_0)$. By generalizing this idea to operators on Banach spaces, one gets the Fréchet derivative.

This section is based on Section 4 of a similar Bachelor Final Project [17] and follows the same structure while referring to the proofs given in it, as rewriting them here would be superfluous. We will discuss elementary definitions and theorems of the Fréchet derivative, based on [1].

Definition 5.1 (Fréchet derivative)**.** *Let X, Y be Banach spaces and $U \subset X$ be open. The a function $f : U \to Y$ is called Fréchet differentiable at $x_0 \in U$ if there exists a linear mapping $Df(x_0) : X \to Y$ (called the Fréchet derivative at x_0) such that for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever $0 < ||x - x_0||_X < \delta$ we have that*

$$\frac{||f(x) - f(x_0) - Df(x_0)[x - x_0]||_Y}{||x - x_0||_X} < \epsilon. \tag{5.1}$$

Here the norms have a subscript indicating the space on which the norm is defined. This definition is equivalent to saying that $\lim_{||x-x_0||_X \to 0} \frac{||f(x)-f(x_0)-Df(x_0)[x-x_0]||_Y}{||x-x_0||_X} = 0$. Yet another way is saying that $\lim_{||h||_X \to 0} \frac{||f(x_0+h)-f(x_0)-Df(x_0)[h]||_Y}{||h||_X} = 0$. This last definition containing the h will be used in this report and proofs in [17]. Now we see that this derivative really is a (linear) mapping from U to Y that maps an $h \in U \subset X$ to $Df(x_0)[h] = f(x_0 + h) - f(x_0) + o(h)$.

5.1 Theorems on the Fréchet derivative

We begin our discussion of theorems on the Fréchet derivative with the uniqueness of it.

Theorem 5.2 (Uniqueness of Fréchet derivative)**.** *For a given function $f : U \subset X \to Y$, there can be at most one such a linear mapping described in Definition 5.1 at a given point x_0.*

Because of this we can speak of the Fréchet derivative $Df : X \to L(X, Y)$ of a function $f : X \to Y$ that is Fréchet differentiable at all points $x_0 \in X$. Here $L(X, Y)$ denotes the space of bounded linear operators from X to Y.

When both $X = Y = \mathbb{R}$ and we consider the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ at a point $x \in \mathbb{R}$ in the Fréchet way, we have that $Df(x)[h] = f'(x)h$. This can be seen when realising that $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} - f'(x) = 0$. We will use this specific and well known example to illustrate the different theorems we will mention.

Firstly if a function $f \in L(X, Y)$ itself, then it is Fréchet differentiable.

Theorem 5.3 (Fréchet derivative of linear operator)**.** *If $f : X \to Y$ is a bounded linear operator, then f is Fréchet differentiable with Fréchet derivative $Df \equiv f$. In other words, the derivative Df at $x_0 \in X$ is given by $X \ni h \mapsto Df(x_0)[h] = f(h)$.*

So for $f : [a, b] \to \mathbb{R}$ given by $f(x) = \alpha x$, we have that $Df(x)[h] = f(h) = \alpha h = f'(x)h$.

The known rules for differentiating real functions, also hold true for the Fréchet derivative, in a more general sense. To start with the derivative operator is linear.

Theorem 5.4 (Linearity of Fréchet derivative)**.** *Let $f, g : U \subset X \to Y$ be differentiable mappings and α, β be scalars. Now the mapping $\alpha f + \beta g$ is differentiable with Fréchet derivative $\alpha Df + \beta Dg$, such that $D(\alpha f + \beta g)(h) = \alpha Df[h] + \beta Dg[h]$.*

Also, the chain rule also applies in this more general setting.

Theorem 5.5 (Chain rule)**.** *Suppose $f : U \subset X \to V \subset Y$, $g : V \to Z$ are Fréchet differentiable on their domains (U, V open), then $g \circ f$ is differentiable, with derivative $D(g \circ f) = (Dg \circ f) \circ Df$. So the Fréchet derivative at $x \in U$ is given by $Dg(f(x))[h] = Dg(y)[h]|_{y=f(x),h=Df(x)[h]}$.*

For $f, g : \mathbb{R} \to \mathbb{R}$, we know the regular chain rule: $\frac{d}{dx}g(f(x)) = g'(f(x))f'(x)$. This coincides with this theorem as $D(g(y))[h] = g'(y)h$ and $D(f(x))[h] = f'(x)h$, such that $D(g(f(x))[h] = g'(f(x))f'(x)h = \frac{d}{dx}g(f(x))h$.

Before we state the product rule for the Fréchet derivative, we first introduce the concept of a Banach algebra.

Definition 5.6 ((Banach) algebra)**.** *A vector space X over a field F is called an algebra, if there exists an unique product $xy \in X$ for all $x, y \in X$ such that it has the following properties:*

$$\begin{array}{lll} \textit{(Associativity)} & (xy)z = x(yz) & x, y, z \in X, \\ \textit{(Distributivity)} & (x + y)z = xz + yz & x, y, z \in X, \\ & x(y + z) = xy + xz & x, y, z \in X, \\ \textit{(Scalar multiplication)} & \alpha xy = (\alpha x)y = x(\alpha y) & x, y \in X, r \in F. \end{array} \tag{5.2}$$

Here the field F is usually the real or complex numbers. If X also contains an element e, called the identity, such that for all $x \in X$ we have that $xe = ex = x$, it is called an algebra with identity. If X is a Banach space and also an algebra it is called an Banach algebra if there exists a positive constant c such that for all $x, y \in X$, $||xy||_X \leq c||x||_X||y||_X$.

The real numbers are an example of (Banach) algebra (with identity), with the norm being the absolute value. We are now in the position to state the product rule for functions defined on Banach algebra.

Theorem 5.7 (Product rule)**.** *Let $f, g : U \subset X \to Y$ be Fréchet differentiable on the open U and X is Banach algebra then the product $fg : U \to Y$ given by $fg(x) = f(x)g(x)$ is Fréchet differentiable with Fréchet derivative at $x \in U$ is given by*

$$D(fg)(x)[h] = Df(x)[h]g(x) + Dg(x)[h]f(x).$$

These resembles the product rule we know from real functions, i.e. for $f, g : \mathbb{R} \to \mathbb{R}$ we have that $D(fg)(x)[h] = Df(x)[h]g(x) + Dg(x)[h]f(x) = f'(x)hg(x) + g'(x)hf(x) = (fg)'(x)h$. Now we have discussed the differentiation rules for the Fréchet derivative we move on to the Fréchet derivative for functions taking values in a Cartesian product of Banach spaces.

Theorem 5.8 (Cartesian product)**.** *Let $f_i : U \subset X \to Y_i$ ($i = 1, ..., n$), be Fréchet differentiable. Now the function defined as the Cartesian product of these mappings f is Fréchet differentiable. Here $f_1 \times ... \times f_n := f : U \to Y_i \times ... \times Y_n$ is given by $f(x) = (f_1(x), ..., f_n(x))$ and its Fréchet derivative is the Cartesian product of the Fréchet derivatives of the component functions f_i. In other words $Df(x)[h] = (Df_1(x)[h], ..., Df_n(x)[h])$.*

When we consider a function the that has a domain that is the Cartesian product of Banach spaces we encounter the concept of a partial Fréchet derivative.

Definition 5.9 (Partial Fréchet derivative). *Let X, Y, Z be Banach spaces, $f : X \times Y \supset U \to Z$ and $(x_0, y_0) \in U$. Then if the mappings $X \ni x \mapsto f(x, y_0)$ and $Y \ni y \mapsto f(x_0, y)$ are Fréchet differentiable at x_0 and y_0 respectively, then the Fréchet derivative of these mappings are called the partial Fréchet derivatives of f at (x_0, y_0) and are denoted with $D_x f(x_0, y_0) \in L(X, Z)$ and $D_y f(x_0, y_0) \in L(Y, Z)$.*

One can thus interpret a partial Fréchet derivative with respect to one variable, as the Fréchet derivative of the function of which the other variable is fixed, as is also the case in for example functions like $f : \mathbb{R}^2 \to \mathbb{R}$. When keeping this interpretation in mind, one sees that the rules for and the properties of a partial Fréchet derivative equal that of the Fréchet derivative previously discussed. One can naturally extend this definition (and the theorem following) to the case of a function having an $n-$fold Cartesian product as domain, but for the purpose of simplicity and because in this project we only consider functions with domain being a Cartesian product of at most two Banach spaces, we refrain from doing so here. There exist relations between the Fréchet derivative of a function and its partial Fréchet derivatives.

Theorem 5.10. *Let $f : X \times Y \supset U \to Z$ be differentiable with U open. Then the partial derivatives at a point $(x_0, y_0) \in U$ exist and are given by $D_x f(x_0, y_0)[h] = Df(x_0, y_0)[h, 0]$ and $D_y f(x_0, y_0)[k] = Df(x_0, y_0)[0, k]$, where $h \in X$ and $k \in Y$. From this we also see that*

$$Df(x_0, y_0)[h, k] = Df(x_0, y_0)[h, 0] + Df(x_0, y_0)[0, k] = D_x f(x_0, y_0)[h] + D_y f(x_0, y_0)[k].$$

Proof. For a proof we refer to [1] Proposition 2.4.12. □

Now that we have a notion of partial Fréchet differentiation for functions between Banach spaces, we can state an important theorem that we know from multivariable calculus. We again restrict ourselves to only a Cartesian product of two Banach spaces as domain, as it is easily expanded to a product of more spaces.

Theorem 5.11 (Implicit Function Theorem). *Let X, Y, Z be Banach spaces, $U \subset X, V \subset Y$ be open, with $x_0 \in U, y_0 \in V$ and let $f : U \times V \to Z$ be Fréchet differentiable, with $f(x_0, y_0) = 0$. If $D_y f(x_0, y_0) : Y \to Z$ is an isomorphism, then there exist neighbourhoods $U_0 \ni x_0$ and $V_0 \ni y_0$ and a unique Fréchet differentiable map $g : U_0 \to V_0$ that satisfies $f(x, g(x)) = 0$ and $f(x, y) = 0$ if and only if $y = g(x)$ for all $(x, y) \in U_0 \times V_0$.*

Proof. See [7] for a proof. □

5.2 Fréchet derivative for operators on $H^s(S^2)$

We will later use and apply the knowledge of the previous section on the operator F defined in (4.4). As the domain of this operator is the Cartesian product of $\mathbb{R}^3$ and $H^s_+(S^2)$ (for a certain $s > 0$), we need to verify that the latter is a Banach algebra, such that we can also apply the product rule when Fréchet differentiating with respect to $u \in H^s(S^2)$. Proving this fact here explicitly would be too involved and beyond the scope of this project, so we will point out the way along which one would prove this.

The multiplication in this functions space is pointwise multiplication, i.e. for $u, v \in H^s(S^2)$ its multiplication is given by $uv(\theta, \phi) = u(\theta, \phi)v(\theta, \phi)$ for $\theta \in [0, \pi], \phi \in [0, 2\pi]$. It also has an identity element, the **1**-function and a norm discussed in Section 3, especially in (3.7). To help prove that this together is an Banach algebra we need the notion of the cone condition.

Definition 5.12 (Cone condition). *[3] A region Ω satisfies the cone condition if there exists a finite cone C such that each $x \in \Omega$ is the vertex of a finite cone C_x contained in Ω that is congruent to C.*

As we have defined our Sobolev space on the manifold S^2, we can not directly apply Theorem 4.39 of [3] However, after choosing suitable charts we get from this theorem:

Theorem 5.13. *If $s > 1$, there exists a constant K, dependent on s such that for $u, v \in H^s(S^2)$ the product uv, defined pointwise, satisfies*

$$||uv||_{H^s(S^2)} \leq K||u||_{H^s(S^2)}||v||_{H^s(S^2)},$$

making $H^s(S^2)$ a communtative Banach algebra for $s > 1$.

So in the sequel, whenever using the product rule on a function $u \in H^s(S^2)$, we implicitly assume $s > 1$. Now we are in the position to define some specific operators on $H^s(S^2)$ we will encounter in the sequel and give their Fréchet derivatives. Remember that we have denoted with $H^s_+(S^2) = \{u \in H^s(S^2) | u > 0\}$ the positive functions in $H^s(S^2)$.

Theorem 5.14. *Let $A_n : H^s(S^2) \to H^s(S^2)$ be given by*

$$H^s(S^2) \ni u \mapsto A_n(u) = u^n \text{ such that } A_n(u(\theta, \phi)) = (u(\theta, \phi))^n.$$

Also let $R, S : H^s_+(S^2) \to H^s_+(S^2)$ given by

$$\begin{aligned} H^s_+(S^2) \ni u \mapsto R(u) = \frac{1}{u} \text{ such that } R(u(\theta, \phi)) = \frac{1}{u(\theta, \phi)} \\ \text{and } H^s_+(S^2) \ni u \mapsto S(u) = \sqrt{u} \text{ such that } S(u(\theta, \phi)) = \sqrt{u(\theta, \phi)}. \end{aligned}$$

Then the operators A_n, R, S are Fréchet differentiable on their domains with Fréchet derivatives:

$$\begin{aligned} DA_n(u(\theta, \phi))[h] &= nA_{n-1}(u(\theta, \phi))h = n(u(\theta, \phi))^{n-1}h, \\ DR(u(\theta, \phi)[h] &= -\frac{1}{u(\theta, \phi)^2}h, \\ DS(u(\theta, \phi)[h] &= \frac{1}{2\sqrt{u(\theta, \phi)}}h. \end{aligned}$$

Proof. For a proof we refer to [17], as there are no meaningful differences between the two dimensional setting there and the three dimensional setting here, $H^s(S^1)$ versus $H^s(S^2)$. □

6 Linearization

As finding solutions to (4.6) is outside the scope of this project, we linearize this system around the trivial solution with $u = \mathbf{1}, \vec{v} = \vec{v_0}$ and look for solutions to this linearized system close to this trivial solution. This will give us an indication on the solutions of the non-linear problem near the trivial one. One can view this in the following way: assume that the solutions to (4.6) form a manifold M in $H^s_+(S^2) \times \mathbb{R}^3$, such that $M := \{(u, \vec{v}) \in H^s_+(S^2) \times \mathbb{R}^3 : (u, \vec{v}) \text{ solves } (4.6)\}$, now by linearizing around $(\mathbf{1}, \vec{v_0}) \in M$ we look for combinations of u and $\vec{v}$ that are in the tangent space to M at $(\mathbf{1}, \vec{v_0})$, denoted by $T_{(\mathbf{1},\vec{v_0})}M$. This is visually represented in Figure 2 and one can see that in the neighbourhood of the trivial solution the solutions in the tangent space, and thus solutions of the linearized system, approximate solutions in M. Linearizing in this context means taking the Fréchet derivative at $(\vec{v_0}, \mathbf{1})$. We will see how a small change of $\vec{v}$ from $\vec{v_0}$ will influence u and vice versa, while still satisfying the linearized equation and therefore resembling a solution of (4.6).

Taking the Fréchet derivative of (4.4) at $(\mathbf{1}, \vec{v_0})$ according to Theorem 5.10 gives us

$$0 = DF(\mathbf{1}, \vec{v_0}) = D_{\vec{v}}F(\mathbf{1}, \vec{v_0}) + D_uF(\mathbf{1}, \vec{v_0}) \quad \text{on } S^2.$$

Computing the Fréchet derivative of F with respect to $\vec{v}$ is relatively straightforward as F only contains one term with $\vec{v}$. At $(\mathbf{1}, \vec{v_0})$ it is given by:

$$\begin{aligned} D_{\vec{v}}F(\mathbf{1}, \vec{v_0})[k] &= \gamma k \cdot n(u)\big|_{u=\mathbf{1},\vec{v}=\vec{v_0}} \\ &= \gamma(k_1\vec{e_1} + k_2\vec{e_2} + k_3\vec{e_3}) \cdot e_r \\ &= \gamma(k_1 \sin\theta\cos\phi + k_2 \sin\theta\sin\phi + k_3\cos\theta), \end{aligned}$$

with $k_1, k_2, k_3 \in \mathbb{R}$. The Fréchet derivative with respect to u is a bit more involved and will be computed in the subsequent section, where we will denote it by $\hat{L}h := D_uF(\mathbf{1}, \vec{v_0})[h]$.

We see that the conditions in (4.5) are independent of $\vec{v}$, such that only the Fréchet derivative with respect to u plays a part. Computing these derivatives we see that these conditions turn into

$$\langle u^2h, \mathbf{1}\rangle_{L^2(S^2)} = \langle u^3h, \sin\theta\cos\phi\rangle_{L^2(S^2)} = \langle u^3h, \sin\theta\sin\phi\rangle_{L^2(S^2)} = \langle u^3h, \cos\theta\rangle_{L^2(S^2)} = 0. \quad (6.1)$$

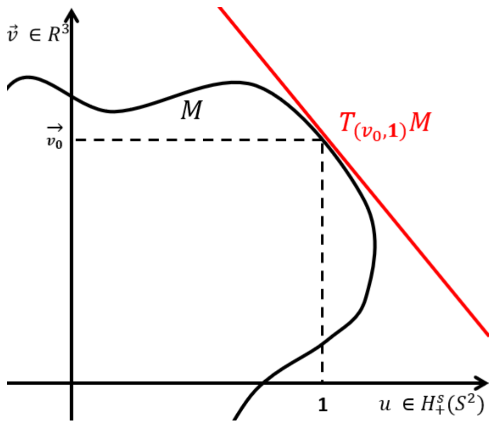


Figure 2: Visualization of the tangent space of M at $(\vec{v_0}, \mathbf{1})$.

Now writing $k = \vec{v} - \vec{v_0}$ and $h = u - \mathbf{1}$ the linearization of (4.6) around $(\mathbf{1}, \vec{v_0})$ is

$$\left.\begin{aligned} \gamma(\vec{v} - \vec{v_0}) \cdot (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) + \hat{L}(u - \mathbf{1}) = 0, \\ \langle u - \mathbf{1}, \mathbf{1}\rangle_{L^2(S^2)} = \langle u - \mathbf{1}, \sin\theta\cos\phi\rangle_{L^2(S^2)} = 0, \\ \langle u - \mathbf{1}, \cos\theta\rangle_{L^2(S^2)} = \langle u - \mathbf{1}, \sin\theta\sin\phi\rangle_{L^2(S^2)} = 0. \end{aligned}\right\} \tag{6.2}$$

In the sequel, we will only consider this system and will find solutions, $u \in H^s_+(S^2)$ for sufficiently high s and corresponding $\vec{v} \in \mathbb{R}^3$, to this linearized system only.

6.1 Fréchet derivative of F with respect to u

For the Fréchet derivative of F around $u = \mathbf{1}, \vec{v} = \vec{v_0}$ we have that

$$\hat{L}h := D_u F(\mathbf{1}, \vec{v_0})[h] = D_u(x_3(u) - \gamma\vec{v}\cdot n(u) + A_\gamma(u)[\gamma n_3(u) - x_3(u)])[h]\Big|_{u=\mathbf{1},\vec{v}=\vec{v_0}}.$$

By linearity of the Fréchet derivative (Theorem 5.3), we can compute component-wise; $\hat{L}h = x_3'(u)[h] - \gamma\vec{v}\cdot n'(u)[h] + D_u(A_\gamma(u)[\gamma n_3(u) - x_3(u)])[h]$. Here again the apostrophe $'$ indicates the Fréchet derivative with respect to u. Firstly, as $x(u) = u\, e_r$, we have that

$$x'(u)(\theta,\phi)[h] = x'(\mathbf{1})[h] = h\, e_r \text{ and } x_3'(u)(\theta,\phi)[h] = x_3'(\mathbf{1})(\theta,\phi)[h] = h\cos\theta.$$

The computation of the Fréchet derivative of the unit normal $n(u)$ exploits several properties of this derivative mentioned in Section 5, such as Theorems 5.3, 5.7 and 5.8 and comes down to

$$\begin{aligned} n'(u)(\theta,\phi)[h] = &-(u_\phi^2 + \sin^2\theta(u_\theta^2 + u^2))^{-\frac{3}{2}}(u_\phi h_\phi + \sin^2\theta(u_\theta h_\theta + uh)) \\ &[u\sin\theta\, e_r - u_\theta\sin\theta\, e_\theta - u_\phi\, e_\phi] \\ &+ (u_\phi^2 + \sin^2\theta(u_\theta^2 + u^2))^{-\frac{1}{2}}\left[h\sin\theta\, e_r - h_\theta\sin\theta\, e_\theta - h_\phi\, e_\phi\right]. \end{aligned}$$

With $u = \mathbf{1}$ we have that

$$\begin{aligned} n'(\mathbf{1})(\theta,\phi)[h] &= -h_\theta\, e_\theta - \frac{1}{\sin\theta}h_\phi\, e_\phi \text{ and} \\ n_3'(\mathbf{1})(\theta,\phi)[h] &= h_\theta\sin\theta. \end{aligned}$$

Now because

$$D_u(A_\gamma(u)[\gamma n_3(u) - x_3(u)][h]) = A_\gamma(u)[\gamma n_3'(u)[h] - x_3'(u)[h]] + A_\gamma'(u)\{h\}[\gamma n_3(u) - x_3(u)],$$

we need an expression for $A_\gamma'(u)\{h\}f$ for an arbitrary $f : S^2 \to \mathbb{R}$. We find this in [11], specifically in Lemma 2.1. It states that

$$A_\gamma'(u)\{h\}f = A_\gamma(u)[-\partial_r g\circ x(u))h + \gamma(\partial_r\partial_n g\circ x(u))h + (\nabla g\circ x(u))\cdot n'(u)\{h\})] + \partial_r g\circ x(u))h,$$

where $g = A_\gamma(u)f\circ x(u)^{-1}$. This can be verified by "variation of the domain" of (4.3) as described in [11] and explicitly written out in [17]. A formal proof in a slightly different situation that does not influence the validity of the proof in this case can be found in [10]. In this case, with $u = \mathbf{1}$, we have that $f = \gamma n_3 - x_3 = (\gamma - 1)x_3$ and that $\Omega_1 = B_1(0)$. This corresponds with the system (2.2), such that, according to Result 2.1

$$g = A_\gamma(\mathbf{1})[\gamma n_3 - x_3]\circ x(u)^{-1} = \frac{\gamma - 1}{1 + 2\gamma}\frac{x_3}{|x|^3} = \frac{\gamma - 1}{1 + 2\gamma}\frac{\cos\theta}{r^2}.$$

As we are now working on S^2, things simplify somewhat as $x(\mathbf{1})$ is the identity and $\partial_r = \partial_n$. We compute the necessary (directional) derivatives.

We first compute the gradient of g in spherical coordinates and compute $\nabla g \cdot n'(\mathbf{1})\{h\}$ and $\partial_n g$ on $\Gamma_1 = S^2$ $(r = 1)$ and see that the latter coincides with the computation of Result 2.1.

$$\begin{aligned} \nabla g &= \frac{\gamma - 1}{1 + 2\gamma}\left(-\frac{2\cos\theta}{r^3}\, e_r - \frac{\sin\theta}{r^3}\, e_\theta\right), \\ \partial_r g = \partial_n g = n(\mathbf{1}) \cdot \nabla g\Big|_{r=1} &= -2\frac{\gamma - 1}{1 + 2\gamma}\cos\theta = 2\frac{1-\gamma}{1+2\gamma}x_3, \\ \nabla g \cdot n'(\mathbf{1})\{h\}\Big|_{r=1} &= \frac{\gamma - 1}{1 + 2\gamma}(\sin\theta\, h_\theta). \end{aligned}$$

On S^2, we have that $\partial^2_{n(\mathbf{1})} g = \partial^2_r g\big|_{r=1} = 6\frac{\gamma-1}{1+2\gamma}\cos\theta$. Also when computing in Cartesian coordinates we can use $\partial^2_{n(\mathbf{1})} g = n(\mathbf{1})^T H n(\mathbf{1})$, where H is the Hessian matrix of g. We use that $n(\mathbf{1}) = x_1\, e_1 + x_2\, e_2 + x_3\, e_3$ on S^2. Indeed we have that $\partial^2_{n(\mathbf{1})} g\big|_{|x|=1} = 6\frac{\gamma-1}{1+2\gamma}x_3 = 6\frac{\gamma-1}{1+2\gamma}\cos\theta$.

So we now see that

$$\begin{aligned} A'_\gamma(\mathbf{1})\{h\}[\gamma n_3(\mathbf{1}) - x_3(\mathbf{1})](\theta,\phi) &= A_\gamma(\mathbf{1})\left[-h\partial_n g + \gamma(h\partial_n^2 g + (\nabla g)\cdot n'(\mathbf{1})\{h\})\right] + h\partial_n g \\ &= \frac{\gamma-1}{1+2\gamma}\left(A_\gamma(\mathbf{1})[2h\cos\theta + \gamma(6h\cos\theta + \sin\theta\, h_\theta)] - 2h\cos\theta\right) \\ &= \frac{\gamma-1}{1+2\gamma}\left(A_\gamma(\mathbf{1})[(2+6\gamma)h\cos\theta + \gamma\sin\theta\, h_\theta] - 2h\cos\theta\right). \end{aligned}$$

All together we have that

$$\begin{aligned} \hat{L}h(\theta,\phi) &= h\cos\theta - \gamma\frac{3}{1+2\gamma}\sin\theta\, h_\theta + A_\gamma(\mathbf{1})[\gamma\sin\theta h_\theta - h\cos\theta] \\ &+ \frac{\gamma-1}{1+2\gamma}\left(A_\gamma(\mathbf{1})[(2+6\gamma)h\cos\theta + \gamma\sin\theta\, h_\theta] - 2h\cos\theta\right) \\ &= \frac{1}{1+2\gamma}\Big((1+2\gamma-2(\gamma-1))h\cos\theta - 3\gamma\sin\theta\, h_\theta \\ &+ A_\gamma(\mathbf{1})[(1+2\gamma+\gamma-1)\gamma\sin\theta h_\theta + (-1-2\gamma+(\gamma-1)(2+6\gamma))h\cos\theta]\Big) \\ &= \frac{3}{1+2\gamma}\Big(h\cos\theta - \gamma\sin\theta\, h_\theta + A_\gamma(\mathbf{1})[\gamma^2\sin\theta h_\theta + (-1-2\gamma+2\gamma^2)h\cos\theta]\Big). \qquad (6.3) \end{aligned}$$

6.2 Spherical harmonics expansion

Our goal is to solve (6.2) with the help of (6.3). As the perturbations h of the trivial solution are defined on S^2, we can exploit the properties discussed in Section 3, especially expanding h with respect to the spherical harmonics, i.e. $h = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} h_l^m \tilde{Y}_l^m$.

However we first note that due to conditions imposed (4.5) and linearized in (6.1) the zeroth and first degree of such an expansion vanish. Let us show why this is. When $h \in H^s(S^2)$ satisfies (6.1) at $u = \mathbf{1}$ and has such an expansion, due to the orthogonality/orthonormality (Lemma 3.5) of the spherical harmonics we for example have:

$$
\begin{aligned}
0 &= \langle h, \mathbf{1} \rangle_{L^2(S^2)} \\
&= \left\langle \sum_{l=0}^{\infty} \sum_{m=-l}^{l} h_l^m \tilde{Y}_l^m, 2\sqrt{\pi}\tilde{Y}_0^0 \right\rangle_{L^2(S^2)} \\
&= 2\sqrt{\pi} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} h_l^m \left\langle \tilde{Y}_l^m, \tilde{Y}_0^0 \right\rangle_{L^2(S^2)} \\
&= 2\sqrt{\pi} h_0^0 \implies h_0^0 = 0.
\end{aligned}
$$

Similarly the second and third linearized conditions together in (6.1) at $u = \mathbf{1}$ give that $h_1^1 = h_1^{-1} = 0$ and the fourth condition gives $h_1^0 = 0$.

We conclude that because of the conditions we have imposed to exclude degrees of freedom, that a spherical harmonics expansion of the perturbations h of the trivial solution that solve (6.2) lacks the zeroth and first degree coefficients and therefore is given by

$$
h(\theta, \phi) = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} h_l^m \tilde{Y}_l^m(\theta, \phi). \tag{6.4}
$$

6.3 Recurrence relations spherical harmonics

To be able to solve (6.2) we see that we need expressions for $\cos\theta h$ and $\sin\theta h_\theta$ in (6.3). When h is a spherical harmonic this can be done as there exist recurrence relations for these precisely operators on the associated Legendre polynomials in terms of associated Legendre polynomials of different degrees and by extension the same holds for the spherical harmonics. These recurrence relations are found in tables such as in [2] and proven in [22] with the help of the generating function for the associated Legendre polynomials. Because of the way we have defined these, see (3.4), the generating function for an order m, call it j_m, is given by

$$
j_m(s,t) = (-1)^{|m|}(2|m|-1)!! \frac{(1-s^2)^{\frac{|m|}{2}} t^{|m|}}{(1-2st+t^2)^{|m|+\frac{1}{2}}} = \sum_l P_l^{|m|}(s) t^l.
$$

Here we denote with $n!!$ the double factorial of a natural number n, i.e. the product of the natural numbers smaller or equal to n with the same parity as n.
To get a recurrence relation for the first operator, $\cos\theta Y_l^m$, we follow the approach of [22] which uses the generating function. We differentiate the generating function with respect to t, rewrite to get j_m on one side and equate coefficients of equal powers of t and we get a similar recurrence relation; for $l \in \mathbb{N} \setminus \{0\}$

$$
sP_l^{|m|}(s) = \frac{(l+|m|)P_{l-1}^{|m|}(s) + (l-|m|+1)P_{l+1}^{|m|}(s)}{2l+1}. \tag{6.5}
$$

When we now substitute $s = \cos\theta$, multiply with $e^{im\phi}$ to get the spherical harmonics and normalize, we get the recurrence relation for the normalized spherical harmonics:

$$\begin{aligned}
\cos\theta \tilde{Y}_l^m(\theta,\phi) &= \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}\cos\theta P_l^{|m|}(\cos\theta)e^{im\phi} \\
&= \sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}\frac{(l+|m|)Y_{l-1}^m(\theta,\phi)+(l-|m|+1)Y_{l+1}^m(\theta,\phi)}{2l+1} \\
&= \frac{(l+|m|)\sqrt{\frac{2l+1}{2l-1}\frac{l-|m|}{l+|m|}}\tilde{Y}_{l-1}^m(\theta,\phi)+(l-|m|+1)\sqrt{\frac{2l+1}{2l+3}\frac{l+1+|m|}{l+1-|m|}}\tilde{Y}_{l+1}^m(\theta,\phi)}{2l+1} \\
&= \frac{\sqrt{\frac{(l-|m|)(l+|m|)}{2l-1}}\tilde{Y}_{l-1}^m(\theta,\phi)+\sqrt{\frac{(l+1-|m|)(l+1+|m|)}{2l+3}}\tilde{Y}_{l+1}^m(\theta,\phi)}{\sqrt{2l+1}} \\
&= \frac{1}{\sqrt{2l+1}}\left[\sqrt{\frac{l^2-m^2}{2l-1}}\tilde{Y}_{l-1}^m(\theta,\phi)+\sqrt{\frac{(l+1)^2-m^2}{2l+3}}\tilde{Y}_{l+1}^m(\theta,\phi)\right].
\end{aligned} \tag{6.6}$$

For a worked out derivation of (6.5), see Appendix.

Next, we look at $\partial_\theta h$, for $h = \tilde{Y}_l^m$ a spherical harmonic of degree l and order m. For this we need the derivative of the associated Legendre polynomials, again recursively represented in associated Legendre polynomials (of different degrees or orders). We start by simply applying the product rule and recognizing associated Legendre polynomials;

$$\begin{aligned}
\frac{d}{ds}P_l^{|m|}(s) =& \frac{d}{ds}\left[\frac{(-1)^{|m|}}{2^l l!}(1-s^2)^{|m|/2}\frac{d^{l+|m|}}{ds^{l+|m|}}[(s^2-1)^l]\right] \\
=& -\frac{(-1)^{|m|}}{2^l l!}|m|s(1-s^2)^{|m|/2-1}\frac{d^{l+|m|}}{ds^{l+|m|}}[(s^2-1)^l] \\
&- \frac{1}{(1-s^2)^{1/2}}\frac{(-1)^{|m|+1}}{2^l l!}(1-s^2)^{(|m|+1)/2}\frac{d^{l+|m|+1}}{ds^{l+|m|+1}}[(s^2-1)^l] \\
=& -|m|s(1-s^2)^{-1}P_l^{|m|}(s)-(1-s^2)^{-1/2}P_l^{|m|+1}(s).
\end{aligned}$$

We see that we can write the derivative in terms of the original polynomial. We ultimately want to compute $\sin\theta\partial_\theta P_l^{|m|}(\cos\theta) = -(1-s^2)\frac{d}{ds}P_l^{|m|}(s)\big|_{s=\cos\theta}$ and so we see that we are interested in rewriting $sP_l^{|m|}(s)$ and $(1-s^2)^{1/2}P_l^{|m|+1}(s)$. For the former we can use (6.5) or (6.6) and the following, (6.7), for the latter; for $l \geq 1$:

$$\sqrt{1-s^2}P_l^{|m|+1}(s) = \frac{1}{2l+1}\left[(l-|m|)(l-|m|+1)P_{l+1}^{|m|}(s)-(l+|m|)(l+|m|+1)P_{l-1}^{|m|}(s)\right]. \tag{6.7}$$

For a worked out derivation of this recurrence relation see the Appendix. Now putting everything together we get, for $l \geq 1$,

$$
\begin{aligned}
\sin\theta \frac{d}{d\theta} P_l^{|m|}(\cos\theta) =& -(1-s^2)\frac{d}{ds}P_l^{|m|}(s)\big|_{s=\cos\theta}\\
=& \Big[|m| s P_l^{|m|}(s) + (1-s^2)^{\frac{1}{2}} P_l^{|m|+1}(s)\Big]_{s=\cos\theta}\\
=&\frac{1}{2l+1}\Big[|m|\Big((l+|m|)P_{l-1}^{|m|}(s) + (l-|m|+1))P_{l+1}^{|m|}(s)\Big)\\
&+\Big((l-|m|)(l-|m|+1)P_{l+1}^{|m|}(s) - (l+|m|)(l+|m|+1)P_{l-1}^{|m|}(s)\Big)\Big]_{s=\cos\theta}\\
=&\frac{1}{2l+1}\Big[l(l-|m|+1))P_{l+1}^{|m|}(\cos\theta) - (l+1)(l+|m|)P_{l-1}^{|m|}(\cos\theta)\Big].
\end{aligned}
$$

We multiply with $e^{im\phi}$ and normalize to get the ultimate expression for $\sin\theta h_\theta$ we were looking for, in the same way as done in (6.6), for $l \geq 1$,

$$
\begin{aligned}
\sin\theta h_\theta =& \sin\theta\partial_\theta[\tilde{Y}_l^m(\theta,\phi)] = \sin\theta\frac{\partial}{\partial\theta}\left[\sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}P_l^{|m|}(\cos\theta)e^{im\phi}\right]\\
=&\sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}e^{im\phi}\frac{1}{2l+1}\Big[l(l-|m|+1))P_{l+1}^{|m|}(\cos\theta) - (l+1)(l+|m|)P_{l-1}^{|m|}(\cos\theta)\Big]\\
=&\frac{1}{\sqrt{2l+1}}\left[l\sqrt{\frac{(l+1)^2-m^2}{2l+3}}\tilde{Y}_{l+1}^m(\theta,\phi) - (l+1)\sqrt{\frac{l^2-m^2}{2l-1}}\tilde{Y}_{l-1}^m(\theta,\phi)\right]. \qquad (6.8)
\end{aligned}
$$

Note that in both (6.6) and (6.8) the operators operating on the spherical harmonics are recursively represented by a linear combination of spherical harmonics of one degree higher and of one degree lower, but with constant order. Furthermore, note that the recurrence relations hold for $l \geq 1$. This is no obstacle in our computations however as in our expansion (6.4) we only consider spherical harmonics of degree 2 and larger. Lastly, one should keep in mind that for $|m| = l$, expressions such as $P_l^{|m|+1}$ and $P_{l-1}^{|m|}$ are zero by definition and by extension the same holds for the respective spherical harmonics. One also sees that precisely those terms have a coefficient (containing $l^2 - m^2$) equal to zero in (6.6) and (6.8), such that it is also apparent from their coefficients that precisely those terms do not contribute to the recurrence relation. In the prior computation steps one should keep this fact in mind however.

6.4 Main results and illustrations

Now that we have a way to represent the the operators on the spherical harmonics present in (6.3), we can actually compute $\hat{L}h$ for h being a spherical harmonic.

Result 6.1. *Let us denote with* $(c_l^m)^+ = \sqrt{\frac{(l+1)^2-m^2}{2l+3}}$ *and* $(c_l^m)^- = \sqrt{\frac{l^2-m^2}{2l-1}}$*, the coefficients present in front of* $\tilde{Y}_{l+1}^m$ *and* $\tilde{Y}_{l-1}^m$ *in the recurrence relations found before respectively. Now we see that for* $l \in \mathbb{N} \setminus \{0,1\}$ *and* $m \in \mathbb{Z}$ *with* $|m| \leq l$*, we have that*

$$
\begin{aligned}
&\hat{L}\tilde{Y}_l^m(\theta,\phi) = \\
&\quad \frac{3}{1+2\gamma}\Big(\cos\theta \tilde{Y}_l^m(\theta,\phi) - \gamma \sin\theta\, \partial_\theta \tilde{Y}_l^m(\theta,\phi) \\
&\qquad + A_\gamma(1)\left[\gamma^2 \sin\theta\, \partial_\theta \tilde{Y}_l^m(\theta,\phi) + (-1-2\gamma+2\gamma^2)\cos\theta \tilde{Y}_l^m(\theta,\phi)\right]\Big) \\
&\overset{(1)}{=} \frac{3}{(1+2\gamma)\sqrt{2l+1}}\Big(\left[(c_l^m)^- \tilde{Y}_{l-1}^m(\theta,\phi) + (c_l^m)^+ \tilde{Y}_{l+1}^m(\theta,\phi)\right] \\
&\qquad - \gamma\left[l(c_l^m)^+ \tilde{Y}_{l+1}^m(\theta,\phi) - (l+1)(c_l^m)^- \tilde{Y}_{l-1}^m(\theta,\phi)\right] \\
&\qquad + A_\gamma(1)\Big[\gamma^2\left[l(c_l^m)^+ \tilde{Y}_{l+1}^m(\theta,\phi) - (l+1)(c_l^m)^- \tilde{Y}_{l-1}^m(\theta,\phi)\right] \\
&\qquad + (-1-2\gamma+2\gamma^2)\left[(c_l^m)^- \tilde{Y}_{l-1}^m(\theta,\phi) + (c_l^m)^+ \tilde{Y}_{l+1}^m(\theta,\phi)\right]\Big]\Big) \\
&\overset{(2)}{=} \frac{3}{(1+2\gamma)\sqrt{2l+1}}\Big((1-\gamma l)(c_l^m)^+ \tilde{Y}_{l+1}^m(\theta,\phi) + (\gamma(l+1)+1)(c_l^m)^- \tilde{Y}_{l-1}^m(\theta,\phi) \\
&\qquad + A_\gamma(1)\Big[-(1+2\gamma+(l-1)\gamma^2)(c_l^m)^- \tilde{Y}_{l-1}^m(\theta,\phi) \\
&\qquad + (-1-2\gamma+(2+l)\gamma^2)(c_l^m)^+ \tilde{Y}_{l+1}^m(\theta,\phi)\Big]\Big) \\
&\overset{(3)}{=} \frac{3}{(1+2\gamma)\sqrt{2l+1}}\Big((1-\gamma l)(c_l^m)^+ \tilde{Y}_{l+1}^m(\theta,\phi) + (\gamma(l+1)+1)(c_l^m)^- \tilde{Y}_{l-1}^m(\theta,\phi) \\
&\qquad - \frac{1+2\gamma+(l-1)\gamma^2}{l\gamma+1}(c_l^m)^- \tilde{Y}_{l-1}^m(\theta,\phi) - \frac{1+2\gamma-(2+l)\gamma^2}{(2+l)\gamma+1}(c_l^m)^+ \tilde{Y}_{l+1}^m(\theta,\phi)\Big) \\
&\overset{(4)}{=} \frac{3\gamma}{(1+2\gamma)\sqrt{2l+1}}\Big(\frac{-1+\gamma+2l+\gamma l^2}{l\gamma+1}(c_l^m)^- \tilde{Y}_{l-1}^m(\theta,\phi) + \frac{\gamma(2-l-l^2)}{(2+l)\gamma+1}(c_l^m)^+ \tilde{Y}_{l+1}^m(\theta,\phi)\Big).
\end{aligned}
$$

In this computation we have filled in the recurrence relations (6.6) and (6.8) in (1), took terms of Y_{l-1}^m and Y_{l+1}^m together in (2), applied Result 4.1 in (3) and again took terms of Y_{l-1}^m and Y_{l+1}^m together, while moving the common factor γ in front of the brackets in (4). Note again that $(c_l^m)^- = 0$ for $|m| = l$. All in all we now have the following result:

Result 6.2. *When the perturbations* h *of the trivial solution are expanded with respect to the spherical harmonics, i.e.* $h = \sum_{l=2}^\infty \sum_{m=-l}^l h_l^m \tilde{Y}_l^m$*, then* $\hat{L}h$ *is given by the following expression:*

$$
\hat{L}h(\theta,\phi) = \frac{3\gamma}{1+2\gamma}\sum_{l=2}^\infty \sum_{m=-l}^{l} a_l^m h_l^m \tilde{Y}_{l-1}^m(\theta,\phi) + b_l^m h_l^m \tilde{Y}_{l+1}^m(\theta,\phi), \tag{6.9}
$$

in which the coefficients a_l^m and b_l^m are given by

$$
\begin{aligned}
a_l^m &= \frac{-1+\gamma+2l+\gamma l^2}{l\gamma+1}\sqrt{\frac{l^2-m^2}{4l^2-1}},\\
b_l^m &= \gamma\frac{2-l-l^2}{(2+l)\gamma+1}\sqrt{\frac{(l+1)^2-m^2}{4(l+1)^2-1}}\\
&= (l-1)\left(\frac{1}{(2+l)\gamma+1}-1\right)\sqrt{\frac{(l+1)^2-m^2}{4(l+1)^2-1}}.
\end{aligned}
$$

Again note that the coefficient in front of a spherical harmonic that would be zero anyway; $a_l^{\pm l}$, equals zero as well. In this way one has to essentially only look at the coefficients.

By grouping terms we can give the Fréchet derivative of F with respect to u at $(\mathbf{1}, \vec{v_0})$ a spherical harmonics representation.

Result 6.3. *When the perturbations h of the trivial solution are expanded with respect to spherical harmonics, i.e. $h = \sum_{l=2}^{\infty}\sum_{m=-l}^{l} h_l^m \tilde{Y}_l^m$, then $\hat{L}h$ also can be expanded with respect to the spherical harmonics, i.e. it is given by $\hat{L}h(\theta,\phi) = \frac{3\gamma}{1+2\gamma}\sum_{l=1}^{\infty}\sum_{m=-l}^{l} k_l^m(h)\tilde{Y}_l^m(\theta,\phi)$. In this case the coefficients $k_l^m(h)$ are dependent on h and are given by*

$$
k_l^m(h) = \begin{cases} a_2^m h_2^m & \text{for } l=1, m=-1,0,1,\\ a_3^m h_3^m & \text{for } l=2, m=-2,...,2,\\ a_{l+1}^m h_{l+1}^m + b_{l-1}^m h_{l-1}^m & \text{for } l\geq 3, |m|<l,\\ a_{l+1}^m h_{l+1}^m & \text{for } l\geq 3, |m|=l. \end{cases} \tag{6.10}
$$

Here we have just collected terms of the same spherical harmonics. For $l = 1, 2$ there is only one term to consider, but for $l \geq 3$ there are two. Note however that for $|m| = l$, $b_{l-1}^m = 0$, such that for these orders this part is absent. With these results at hand we can now get a good view of how a solution to the linearized system (6.2) looks like. The leftmost term in $(6.2)_1$ is a linear combination of the spherical harmonics of degree 1, see (6.13). Therefore we need the following result.

Theorem 6.4. *The equation $\hat{L}h = \sum_{m=-1}^{1}\alpha^m\tilde{Y}_1^m = \alpha^{-1}\tilde{Y}_1^{-1} + \alpha^0\tilde{Y}_1^0 + \alpha^1\tilde{Y}_1^1$, where α^m are constants, has a unique solution given by*

$$
h(\theta,\phi) = \frac{1+2\gamma}{3\gamma}\sum_{l=1}^{\infty}\sum_{m=-1}^{1}\alpha^m c_l^m Y_{2l}^m(\theta,\phi), \tag{6.11}
$$

with the coefficients given by

$$
c_1^m = \frac{1}{a_2^m},\ \ c_l^m = (-1)^{l+1}\frac{b_2^m...b_{2l-2}^m}{a_2^m...a_{2l}^m}\ \ (l\geq 2). \tag{6.12}
$$

Here the coefficients a_l^m and b_l^m are the those from Result 6.2.

Proof. The perturbations h is a spherical harmonics expansion, which we can write in the conventional way as in (6.4), with $h_{2l}^m = \frac{1+2\gamma}{3\gamma}\alpha^m c_l^m$ for $l \in \mathbb{N}+$, $m = -1, 0, 1$ and $h_l^m = 0$

otherwise. Now using this and Result 6.3 we find that

$$
\begin{aligned}
\hat{L}h &\overset{I}{=} \frac{3\gamma}{1+2\gamma}\sum_{l=1}^{\infty}\sum_{m=-l}^{l} k_l^m(h)\tilde{Y}_l^m \\
&\overset{II}{=} \frac{3\gamma}{1+2\gamma}\left(\sum_{m=-1}^{1} a_2^m h_2^m \tilde{Y}_1^m + \sum_{m=-1}^{1} a_3^m h_3^m \tilde{Y}_2^m + \sum_{l=3}^{\infty}\sum_{m=-1}^{1}(a_{l+1}^m h_{l+1}^m + b_{l-1}^m h_{l-1}^m)\tilde{Y}_l^m\right) \\
&\overset{III}{=} \sum_{m=-1}^{1}\alpha^m a_2^m c_1^m \tilde{Y}_1^m + \frac{3\gamma}{1+2\gamma}\left(\sum_{m=-1}^{1} a_3^m \underbrace{h_3^m}_{=0}\tilde{Y}_2^m + \sum_{l=1}^{\infty}\sum_{m=-1}^{1}(a_{2l+2}^m h_{2l+2}^m + b_{2l}^m h_{2l}^m)\tilde{Y}_{2l+1}^m\right) \\
&\overset{IV}{=} \sum_{m=-1}^{1}\alpha^m \tilde{Y}_1^m + \sum_{l=1}^{\infty}\sum_{m=-1}^{1}\alpha^m(a_{2l+2}^m c_{l+1}^m + b_{2l}^m c_l^m)\tilde{Y}_{2l+1}^m \\
&\overset{V}{=} \sum_{m=-1}^{1}\alpha^m \tilde{Y}_1^m + \sum_{l=1}^{\infty}\sum_{m=-1}^{1}\alpha^m(-1)^{l+1}\left(-a_{2l+2}^m\frac{b_2^m\ldots b_{2l}^m}{a_2^m\ldots a_{2l+2}^m} + b_{2l}^m\frac{b_2^m\ldots b_{2l-2}^m}{a_2^m\ldots a_{2l}^m}\right)\tilde{Y}_{2l+1}^m \\
&\overset{VI}{=} \sum_{m=-1}^{1}\alpha^m \tilde{Y}_1^m.
\end{aligned}
$$

Here the following steps were taken in the marked equations:

- I Result 6.3 was applied, here the coefficients $k_l^m(h)$ are the coefficients present in this result.
- II Here the expressions for the coefficients $k_l^m(h)$ were filled in and the case distinction in (6.10) was taken into account by splitting the summation. Realizing that only for $|m| \leq 1$ the coefficients are nonzero, the summation was restricted to include only these.
- III The expressions for h_{2l}^m were filled in and for higher values of l it was recognized that the terms for even l's do not contribute as the h-coefficients are zero then. A change of index was made in the summation to only have an expansion of odd degree spherical harmonics.
- IV The expression for the h-coefficients were filled in, i.e. $h_{2l}^m = \frac{1+2\gamma}{3\gamma}\alpha^m c_l^m$.
- V The expressions for the c-coefficients were filled in according to (6.12).
- VI The coefficients for the higher degree spherical harmonics in the summation are all zero, leaving us with what we wanted to prove.

□

With this last result we can solve the linearized system, (6.2), around $(\mathbf{1}, \vec{v_0})$. Namely;

$$
\begin{aligned}
\hat{L}h &= \alpha^{-1}\tilde{Y}_1^{-1} + \alpha^0\tilde{Y}_1^0 + \alpha^1\tilde{Y}_1^1 \\
&= -\alpha^{-1}\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{-i\phi} + \alpha^0\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta - \alpha^1\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta e^{i\phi} \\
&= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\sin\theta\left(\alpha^{-1}(\cos\phi - i\sin\phi) + \alpha^1(\cos\phi + i\sin\phi)\right) + \alpha^0\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos\theta \\
&= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}(\alpha^{-1}+\alpha^1, i(\alpha^1-\alpha^{-1}), -\sqrt{2}\alpha^0)\cdot(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \qquad (6.13)
\end{aligned}
$$

So now we can relate the velocity of the traveling wave $\vec{v}$ (near $\vec{v_0}$) with the shape of the domain Ω_u via the function u (near $\mathbf{1}$) with Theorem 6.4 in the following way:

$$u(\theta, \phi) = 1 + h(\theta, \phi) = 1 + \frac{1+2\gamma}{3\gamma} \sum_{l=1}^{\infty} \sum_{m=-1}^{1} \alpha^m c_l^m Y_{2l}^m(\theta, \phi).$$

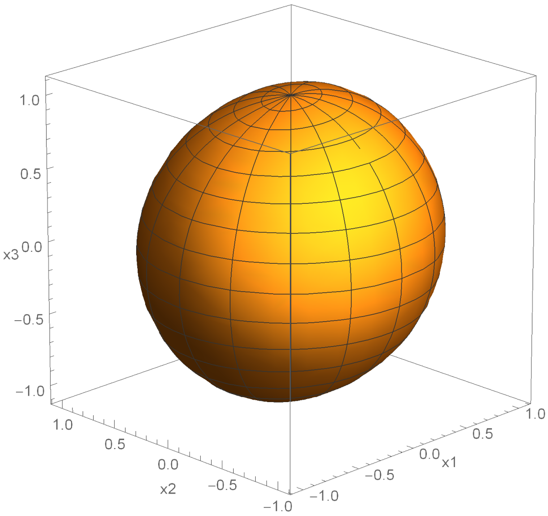
Here the c_l^m are from (6.12) and the α_l^m are chosen in such a way that (6.13) equals the first equation in the linearized system (6.2). In other words, the α_l^m solve the system of equations

$$\frac{1}{2}\sqrt{\frac{3}{2\pi}}(\alpha^{-1} + \alpha^1, i(\alpha^1 - \alpha^{-1}), -\sqrt{2}\alpha^0) = \gamma(\vec{v} - \vec{v_0}). \tag{6.14}$$

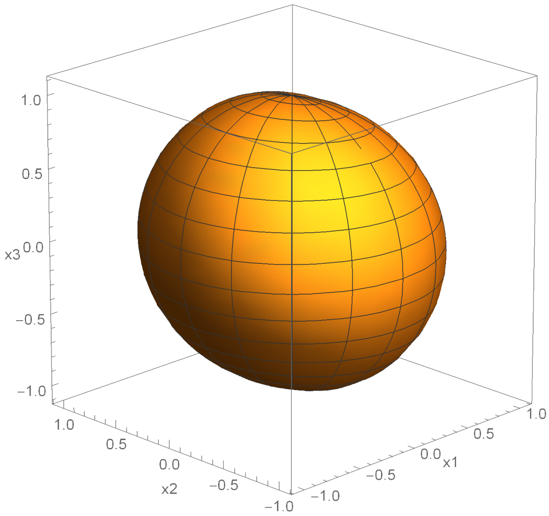
To illustrate this, Figure 3, displays the different shapes of the regions Ω_u that correspond with a small change of the trivial solution (± 0.1 in each of the three Cartesian directions). To get a good insight in the way the shape deforms from the trivial shape, the unit sphere, when perturbing the velocity $\vec{v}$ slightly, the value of γ was fixed at $\gamma = \frac{1}{2}$ and the values of the α-coefficients solving the corresponding system (6.14) are shown in Table 1. One can see that the domains deform from the unit sphere, being lopsided in the direction of the perturbation of the velocity corresponding to the trivial solution (2.3) and in the $\vec{e_3}$-direction. This is to be expected as the velocity $\vec{v}$ now has a component in both directions, as $\vec{v_0}$ is solely in the $\vec{e_3}$-direction. When these two directions coincide, as in Figure 3e and 3f, one only sees this lobsideness as a stretching or compressing of the unit sphere in the $\vec{e_3}$-direction. For $\gamma = \frac{1}{2}$, $|\vec{v_0}| = \frac{3}{2}$ and one sees that making a relatively small change to the velocity changes the shape quite drastically. The Mathematica code written to generate the plots of Figure 3 can be found in the Appendix, there one can see that the function in (6.11) is used. However, the index l only runs to 10 instead to infinity, which would obviously be impossible to implement in Mathematica. The limit 10 was chosen because of the rapid decay of the c_l^m's (see Section 6.5), for higher orders the contribution would be so minor that it would not visible on the plots and drive up the execution time of the script unnecessarily.

$\vec{v} - \vec{v_0}$	α^{-1}	α^0	α^1	$\alpha^{-1}\tilde{Y}_1^{-1} + \alpha^0\tilde{Y}_1^0 + \alpha^1\tilde{Y}_1^1$
$\pm 0.1\vec{e_1}$	$\pm\frac{1}{20}\sqrt{\frac{2\pi}{3}}$	0	$\pm\frac{1}{20}\sqrt{\frac{2\pi}{3}}$	$\mp\frac{1}{20}\sin\theta\cos\phi$
$\pm 0.1\vec{e_2}$	$\pm\frac{i}{20}\sqrt{\frac{2\pi}{3}}$	0	$\mp\frac{i}{20}\sqrt{\frac{2\pi}{3}}$	$\mp\frac{1}{20}\sin\theta\sin\phi$
$\pm 0.1\vec{e_3}$	0	$\mp\frac{1}{10}\sqrt{\frac{\pi}{3}}$	0	$\mp\frac{1}{20}\cos\theta$

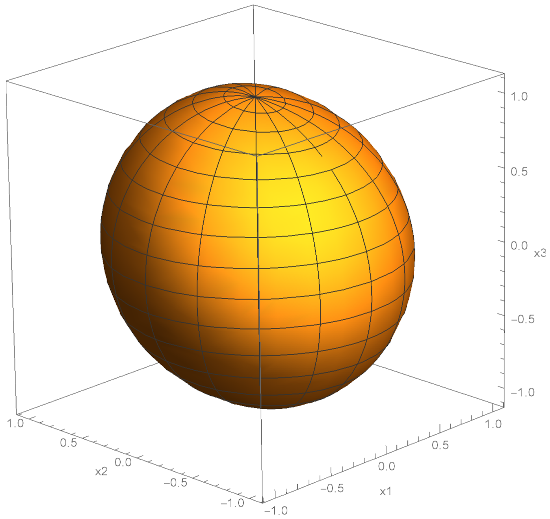
Table 1: Values for the α-coefficients for the visualizations of the solutions shown in Figure 3, with $\gamma = \frac{1}{2}$.



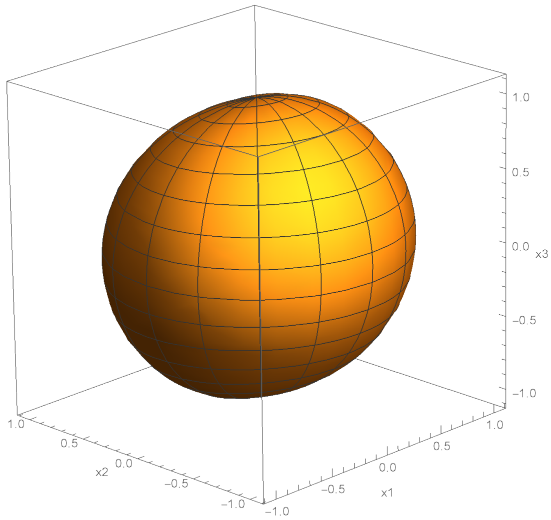
(a) $\vec{v} - \vec{v_0} = -0.1\, e_1$



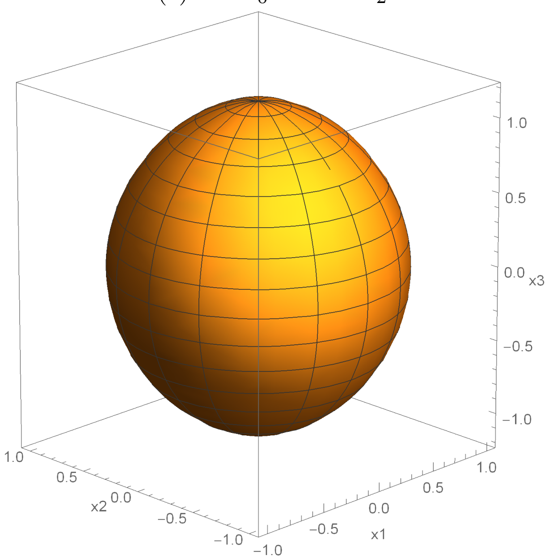
(b) $\vec{v} - \vec{v_0} = 0.1\, e_1$



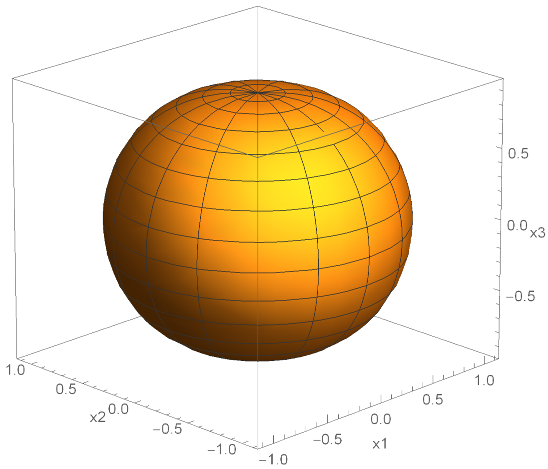
(c) $\vec{v} - \vec{v_0} = -0.1\, e_2$



(d) $\vec{v} - \vec{v_0} = 0.1\, e_2$



(e) $\vec{v} - \vec{v_0} = -0.1\, e_3$



(f) $\vec{v} - \vec{v_0} = 0.1\, e_3$

Figure 3: Domains corresponding to solutions of (6.2), with $\gamma = \frac{1}{2}$ and perturbations of the velocity corresponding to the trivial solution (2.3).

6.5 Smoothness of domain describing function

To assist in computing the c_l^m and with it visualizing the bubble domains, it might be useful to know that for $l \geq 2$ they can be recursively represented in the following way:

$$\begin{aligned} c_{l+1}^m &= (-1)^{l+1+1} \frac{b_2^m ... b_{2l}^m}{a_2^m ... a_{2l+2}^m} \\ &= -(-1)^{l+1} \frac{b_2^m ... b_{2l-2}^m}{a_2^m ... a_{2l}^m} \frac{b_{2l}^m}{a_{2l+2}^m} \\ &= -\frac{b_{2l}^m}{a_{2l+2}^m} c_l^m. \end{aligned}$$

This recursion is also helpful in determining for which value of s the second component of a solution of (6.2), $u(= h + \mathbf{1})$, is in $H_+^s(S^2)$ and with that determine the smoothness of u. This is based on Theorem 6.4 and in this section we will lay out how this is done.

For this we use the gamma function and one of its asymptotic approximations. The gamma function, $\Gamma : \mathbb{C} \setminus \mathbb{Z}_0^- \to \mathbb{C}$, is defined such that $\Gamma(1) = 1$ and $\Gamma(x + 1) = x\Gamma(x)$ for any positive real number x. This factorial behavior is also what we will exploit. First we compute the above recurrence relation for the nonzero entries it has, i.e. for $m = 0, \pm 1$;

$$c_{l+1}^0 = \frac{(l - \frac{1}{2})(l + \frac{1}{2})}{(l + \frac{1+2\gamma+\sqrt{1+\gamma-\gamma^2}}{2\gamma})(l + \frac{1+2\gamma-\sqrt{1+\gamma-\gamma^2}}{2\gamma})} \sqrt{\frac{l + \frac{5}{4}}{l + \frac{1}{4}}} c_l^0, \tag{6.15}$$

$$c_{l+1}^{\pm 1} = \frac{(l + 1)(l - \frac{1}{2})}{4(l + \frac{1+2\gamma+\sqrt{1+\gamma-\gamma^2}}{2\gamma})(l + \frac{1+2\gamma-\sqrt{1+\gamma-\gamma^2}}{2\gamma})} \sqrt{\frac{l(l + 1)(l + \frac{5}{4})}{(l + \frac{1}{4})(l + \frac{1}{2})(l + \frac{3}{2})}} c_l^{\pm 1}.$$

One should realize that we can continuously use this recursion (on c_l^m, c_{l-1}^m, etcetera) until c_{l+1}^m is expressed in terms of c_2^m ($l \geq 2$). We will prove this for $m = 0$;

Proof. We will denote with $D(\gamma)$ a constant dependent on γ that is given

$$\begin{aligned} D(\gamma) &= \frac{\Gamma(2 + \frac{1+2\gamma+\sqrt{1+\gamma-\gamma^2}}{2\gamma})\Gamma(2 + \frac{1+2\gamma-\sqrt{1+\gamma-\gamma^2}}{2\gamma})}{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})} \sqrt{\frac{\Gamma(\frac{9}{4})}{\Gamma(\frac{13}{4})}} \\ &= 8\frac{\Gamma(2 + \frac{1+2\gamma+\sqrt{1+\gamma-\gamma^2}}{2\gamma})\Gamma(2 + \frac{1+2\gamma-\sqrt{1+\gamma-\gamma^2}}{2\gamma})}{3\pi} \sqrt{\frac{\Gamma(\frac{9}{4})}{\Gamma(\frac{13}{4})}} \\ &\approx 0.56589\Gamma(2 + \frac{1 + 2\gamma + \sqrt{1 + \gamma - \gamma^2}}{2\gamma})\Gamma(2 + \frac{1 + 2\gamma - \sqrt{1 + \gamma - \gamma^2}}{2\gamma}) \end{aligned}$$

and prove by induction that

$$c_l^0 = D(\gamma) \frac{\Gamma(l - \frac{1}{2})\Gamma(l + \frac{1}{2})}{\Gamma(l + \frac{1+2\gamma+\sqrt{1+\gamma-\gamma^2}}{2\gamma})\Gamma(l + \frac{1+2\gamma-\sqrt{1+\gamma-\gamma^2}}{2\gamma})} \sqrt{\frac{\Gamma(l + \frac{5}{4})}{\Gamma(l + \frac{1}{4})}} c_2^0 \tag{6.16}$$

for $l \geq 2$. Let us start with the base case $l = 2$. It is clear that when filling in $l = 2$ in (6.16) that all terms with the gamma function cancel with those present in $D(\gamma)$, leaving only c_2^0. Now for the induction step, we assume that (6.16) holds for a certain $n \geq 2$. Then, by applying the recurrence relation (6.15), we have

$$
\begin{aligned}
c^0_{n+1} =& \frac{(n-\frac{1}{2})(n+\frac{1}{2})}{(n+\frac{1+2\gamma+\sqrt{1+\gamma-\gamma^2}}{2\gamma})(n+\frac{1+2\gamma-\sqrt{1+\gamma-\gamma^2}}{2\gamma})}\sqrt{\frac{n+\frac{5}{4}}{n+\frac{1}{4}}}c^0_n \\
\overset{(IH)}{=}& D(\gamma)\frac{(n-\frac{1}{2})\Gamma(n-\frac{1}{2})(n+\frac{1}{2})\Gamma(n+\frac{1}{2})\sqrt{\frac{(n+\frac{5}{4})\Gamma(n+\frac{5}{4})}{(n+\frac{1}{4})\Gamma(n+\frac{1}{4})}}c^2_0}{(n+\frac{1+2\gamma+\sqrt{1+\gamma-\gamma^2}}{2\gamma})\Gamma(n+\frac{1+2\gamma+\sqrt{1+\gamma-\gamma^2}}{2\gamma})(n+\frac{1+2\gamma-\sqrt{1+\gamma-\gamma^2}}{2\gamma})\Gamma(n+\frac{1+2\gamma-\sqrt{1+\gamma-\gamma^2}}{2\gamma})} \\
=& D(\gamma)\frac{\Gamma(n+1-\frac{1}{2})\Gamma(n+1+\frac{1}{2})}{\Gamma(n+1+\frac{1+2\gamma+\sqrt{1+\gamma-\gamma^2}}{2\gamma})\Gamma(n+1+\frac{1+2\gamma-\sqrt{1+\gamma-\gamma^2}}{2\gamma})}\sqrt{\frac{\Gamma(n+1+\frac{5}{4})}{\Gamma(n+1+\frac{1}{4})}}c^0_2.
\end{aligned}
$$

Here we have applied the induction hypothesis in the equality marked with (IH) and then have used the defining property of the gamma function, $x\Gamma(x) = \Gamma(x+1)$, in the consequent equality. We see that (6.16) also holds for $l = n+1$, concluding the proof. $\square$

Now we have represented the coefficients c^0_l as quotients of gamma functions we can use an asymptotic property of the gamma function one can get from well known Sterling's formula. This property says that $\frac{\Gamma(x+\alpha)}{\Gamma(x+\beta)} \sim x^{\alpha-\beta}$ for $x \to \infty$ [8]. With this we can conclude that $c^0_l \sim l^{-(\frac{3}{2}+\frac{1}{\gamma})}$ for $l \to \infty$. As $a^{\pm1}_l, b^{\pm1}_l$ are only slightly different from a^0_l, b^0_l respectively, we see the same asymptotic behavior for $c^{\pm1}_l$ as for c^0_l, i.e. $c^{\pm1}_l \sim l^{-(\frac{3}{2}+\frac{1}{\gamma})}$ for $l \to \infty$. One could prove that in a similar way as done before with

$$
\begin{aligned}
c^{\pm1}_l &= E(\gamma)\frac{\Gamma(l)\Gamma(l-\frac{3}{2})}{4\Gamma(l+\frac{1+\sqrt{1+\gamma-\gamma^2}}{2\gamma})\Gamma(l+\frac{1-\sqrt{1+\gamma-\gamma^2}}{2\gamma})}\sqrt{\frac{\Gamma(l-1)\Gamma(l)\Gamma(l+\frac{1}{4})}{\Gamma(l-\frac{3}{4})\Gamma(l-\frac{1}{2})\Gamma(l+\frac{1}{2})}}c^{\pm1}_2, \quad \text{where} \\
E(\gamma) &= \frac{4\Gamma(2+\frac{1+\sqrt{1+\gamma-\gamma^2}}{2\gamma})\Gamma(2+\frac{1-\sqrt{1+\gamma-\gamma^2}}{2\gamma})}{\Gamma(2)\Gamma(\frac{1}{2})}\sqrt{\frac{\Gamma(\frac{5}{4})\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}{\Gamma(1)\Gamma(2)\Gamma(\frac{9}{4})}}.
\end{aligned}
$$

We conclude that $|c^m_l|^2 \sim l^{-(3+\frac{2}{\gamma})}$ for big enough l. Now to find out for which s, u is in $H^s_+(S^2)$ we need to have $||u||^2_{H^s(S^2)} < \infty$, as per Definition 3.7 B. Because u is a linear combination of only the even spherical harmonics, we are interested in the behavior of $(1+(2l)^2)^s|c^m_l|^2$. Note that (for big l) this quantity is independent of m and that for the convergence of $||u||^2_{H^s(S^2)}$ only these are important. Let us denote with $K(\gamma)$ some finite value that represents the contribution of the lower degree contributions to the norm. Then we have

$$
||u||^2_{H^s(S^2)} = K(\gamma) + \sum_{l=l_0}^{\infty}\sum_{m=-1}^{1}(1+(2l)^2)^s|c^m_l|^2 \sim K(\gamma) + 3\sum_{l=l_0}^{\infty} l^{2s}l^{-(3+\frac{2}{\gamma})},
$$

for some l_0 big enough. For this sum to be convergent we need $2s - 3 - \frac{2}{\gamma} < -1$ and thus we conclude that $u \in H^s_+(S^2)$ for $s < 1 + \frac{1}{\gamma}$. As we have used the product rule prior in the computation, we also have the the condition that $s > 1$. This is based on Theorem 5.13, as for these values $H^s_+(S^2)$ is a Banach algebra. So $s \in (1, 1/\gamma)$ is (formally) the reasonable interval to consider the linearized problem.

6.6 Problem's invariances on level of linearization

As we have now established our result, we will take a look at how the invariances of our problem, as discussed in Section 2.1, translate themselves when changing to the linearized operator problem. These properties should also be present in some way in the linearization. It turns out that they express themselves when computing $\hat{L}\tilde{Y}_l^m$ for degrees not covered in Result 6.1, i.e. for $l = 0, 1$. We namely have that

$$\hat{L}\tilde{Y}_0^0 = \frac{3\gamma^2}{(1+2\gamma)^2\sqrt{\pi}}\cos\theta = \frac{2\sqrt{3}\gamma^2}{(1+2\gamma)^2}\tilde{Y}_1^0, \quad \hat{L}\tilde{Y}_1^{\pm 1} = \hat{L}\tilde{Y}_1^0 = 0.$$

In this section we will point out how the translational invariance expresses itself in the latter equalities on the level of the linearization. We introduce a small translation of our trivial domain; $\Omega_{u_a} = B_1(a)$, where $a \in B_1(0) \subset \mathbb{R}^3$. We can have only a small translation (i.e. $a \in B_1(0)$) to make sure that Ω_{u_a} still is a star-shaped domain with respect to the origin. This is not a problem however, as we will differentiate at $a = 0$ anyway. Here this function $u_a : S^2 \to R$ should describe the domain Ω_{u_a} as in (4.1). Therefore it should satisfy $u_a(\frac{x+a}{|x+a|}) = |x+a|$ and $u_0 = \mathbf{1}$. Then because of the translational invariance we must have that $F(\vec{v_0}, u_a) = 0$ on S^2. As we have linearized this equation around $(\vec{v_0}, \mathbf{1})$, we should take the Fréchet derivative of this equation with respect to a at $a = 0$. We have, by the chain rule (Theorem 5.5), that for all $h \in \mathbb{R}^3$

$$0 = D_a(0)[h] = D_aF(v_0, u_a)\big|_{a=0}[h] = D_uF(v_0, u_a)[D_au_a(x)[h]]\big|_{a=0} = \hat{L}[D_au_a(x)[h]]\big|_{a=0}.$$

To compute this Fréchet derivative with respect to a of $u_a(x)$ at $a = 0$, we first consider the derivative of u_0 with respect to $x \in S^2$, which obviously is zero. This can also be seen in the following way; $D_xu_0(x)[z] = \frac{d}{dt}(u_0 \circ \psi)(t)\big|_{t=0} = \frac{d}{dt}1\big|_{t=0} = 0$. Here ψ is a curve on S^2 with $\psi'(0) = z$. Next we define for any $x_0 \in S^2$ the operator Q_{x_0} given by $Q_{x_0}(a) = \frac{x_0+a}{|x_0+a|}$ such that $u_a(Q_{x_0}(a)) = |x_0 + a|$. Note that $Q_{x_0}(0) = x_0$.

We will differentiate $u_a(\frac{x+a}{|x+a|}) = |x+a|$ with respect to a at $a = 0$. For the left-hand side we use Q and the chain rule; for $h \in \mathbb{R}^3$ we have

$$\begin{aligned} D_au_a(Q_x(a)))\big|_{a=0}[h] &= D_au_a\{h\}(Q_x(a))\big|_{a=0} + D_Qu_a(Q_x(a))[D_aQ_x(a)[h]]\big|_{a=0} \\ &= D_au_a(x)\big|_{a=0}[h] + \underbrace{D_Qu_0(x)}_{=0}[D_aQ_x(a)\big|_{a=0}] \\ &= D_au_a(x)\big|_{a=0}[h]. \end{aligned}$$

Here we have indicated the use of the fact we have just established that the derivative of u_0 with respect to any $x \in S^2$ is zero. As $Q_x(a) \in S^2$, also the highlighted term yields zero.

The right-hand side is quite straightforward as the Frechét derivative of such a finite dimensional is the usual derivative. Particularly, the coordinates of the Jacobian matrix, $\mathbf{J}$, represent it and as the right-hand side is a function $\mathbb{R}^3 \to \mathbb{R}$ it is will just be the gradient.

We have for $x \in S^2$, $h \in \mathbb{R}^3$

$$\begin{aligned}
D_a|x+a|\big|_{a=0}[h] &= D_a((\sum_{i=1}^{3}(x_i+a_i)^2)^{1/2})\big|_{a=0}[h] \\
&= \mathbf{J}((\sum_{i=1}^{3}(x_i+a_i)^2)^{1/2})\big|_{a=0}\cdot h \\
&= \sum_{j=1}^{3}\frac{\partial}{\partial a_j}((\sum_{i=1}^{3}(x_i+a_i)^2)^{1/2})\big|_{a=0}\,h_j \\
&= \sum_{j=1}^{3}(\frac{1}{2}(\sum_{i=1}^{3}(x_i+a_i)^2)^{-1/2}2(x_j+a_j))\big|_{a=0}\,h_j \\
&= \sum_{j=1}^{3}((\sum_{i=1}^{3}x_i^2)^{-1/2}x_j)\,h_j \\
&= \frac{x}{|x|}\cdot h = x\cdot h.
\end{aligned}$$

From this we can conclude that $D_a u_a(x)\big|_{a=0}[h] = x\cdot h$ and ultimately $\hat{L}[x\cdot h]=0$ for all $h\in\mathbb{R}^3$ and $x\in S^2$. With this result we can now see how the translational invariance expresses itself on the linearization level. Let therefore $x=\sin\theta\cos\phi\, e_1+\sin\theta\sin\phi\, e_2+\cos\theta\, e_3\in S^2$ for some $\theta\in[0,\pi],\phi\in[0,2\pi]$. Now

$$\begin{aligned}
\hat{L}[\tilde{Y_1^0}] &= \hat{L}[x\cdot\frac{1}{2}\sqrt{\frac{3}{\pi}}e_3]=0, \\
\hat{L}[\tilde{Y}_1^{\pm1}] &= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\hat{L}[\sin\theta e^{\pm i\phi}] = -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\left(\hat{L}[\sin\theta\cos\phi]\pm i\hat{L}[\sin\theta\sin\phi]\right) \\
&= -\frac{1}{2}\sqrt{\frac{3}{2\pi}}\left(\hat{L}[x\cdot\, e_1]\pm i\hat{L}[x\cdot\, e_2]\right)=0
\end{aligned}$$

and we have the proposed expression of the translational invariance at linearization level. Note that this explanation works for $S^{n-1}\subset\mathbb{R}^n$ for a general $n\in\mathbb{N}_+$, but for the purposes of clear application to our situation, we have only shown it for $n=3$.

A similar approach can be taken to verify that the scaling invariance of our problem translate into $\hat{L}[Y_0^0]=\frac{2\sqrt{3}\gamma^2}{(1+2\gamma)^2}\tilde{Y}_1^0$. We have only detailed the explanation for the translational invariance because this gives the most insight and during this project a lot of time and effort went into verifying this fact. In an earlier stage of this project the translational invariance in $\vec{e_3}$-direction did not show on the linearization level due to an asymptotic condition that was not strong enough (to grant uniformity). After a lot of checking this was rectified and this explanation remains to show the understanding gathered from this different and new way of looking at the problem.

7 Conclusion

All in all, we conclude that the linearized version of our moving boundary problem is uniquely solvable around the trivial solution $(\mathbf{1}, \vec{v_0})$. We have found that for $\vec{v} \in \mathbb{R}^3$ close to $\vec{v_0}$, we can find a bubble/streamer domain described by $u \in H^s_+(S^2)$ for $s < 1 + \frac{1}{\gamma}$, with $\gamma > 0$. No upper bound on the value of γ has been found, while the two dimensional discussion of this (linearized) problem [11] does have a rather tight upper bound and this suggests a similar behavior here. This might result in the chosen value of $\gamma = \frac{1}{2}$ for the visualizations to fall outside of this bound (it would do in the 2D-case). However, this value was chosen such in the visualizations the perturbations of the unit sphere were clearly visible, as this was not the case for lower values.

Not only can further research focus on the nonlinear problem (as done in 2D in [11]), but one can also consider different boundaries conditions such as surface tension or a combination of surface tension and kinetic undercooling. This has been done on the level of the linearization in 2D in [17], with a negative result considering the existence of solutions to the linearized problem, which makes for a similar expectation in 3D.

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Appendix

Uniqueness

To make sure we have a unique solution to (1.1) we need to verify that $\phi \equiv 0$ is only solution to the following system:

$$\left.\begin{array}{rll} \Delta f & = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega(t)}, \\ f - \gamma \partial_n f & = 0 & \text{on } \Gamma(t), \\ \nabla f & = o(|x|^{-2}) & \text{for } |x| \to \infty. \end{array}\right\} \tag{7.1}$$

Proof. Let us fix a $t \geq 0$. Troughout this proof we will make use of a region we denote with W_R. We namely consider balls $B_R(0)$ with R big enough such that $\Gamma(t)$ is completely in the interior. Now W_R is the region enclosed by $\partial B_R(0) \cup \Gamma(t)$. These are compact regions with smooth boundary.

Because f is continuous it is bounded on such a region W_R. Now because of $(7.1)_3$, we have that $|\nabla f\, x^2| < 1$ for all $x \in \mathbb{R}^3 \setminus B_{R_0}(0)$, where $R_0 > 1$ is big enough. We consider the curve K which is the radial line segment connecting the $x \in \mathbb{R}^3$ and $R_0 \frac{x}{|x|} \in \partial B_{R_0}(0)$. We see that

$$f(x) - f(R_0 \frac{x}{|x|}) = \int_K \nabla f \cdot T ds \leq |x - R_0 \frac{x}{|x|}| \max_K |\nabla f| < 1,$$

from which we conclude that f is bounded on $\mathbb{R}^3 \setminus \overline{\Omega(t)}$. Let us say that $|f(x)| \leq M$ on this domain, for $M \in \mathbb{R}_+$. Next, we consider the following integral:

$$\begin{aligned} \int_{W_R} |\nabla f|^2 dx &\overset{(1)}{=} \int_{W_R} \nabla f\, \nabla f dx + \int_{W_R} \Delta f\, f dx \\ &\overset{(2)}{=} \int_{W_R} \operatorname{div}(\nabla f\, f)\, dx \\ &\overset{(3)}{=} -\int_\Gamma f \nabla f \cdot n\, ds + \int_{\partial B_R(0)} f \nabla f \cdot n\, ds \\ &\overset{(4)}{\leq} -\int_\Gamma f \partial_n f\, ds + 4\pi R^2 \max_{\partial B_R(0)} |f \nabla f| \\ &\overset{(5)}{\leq} -\gamma \int_\Gamma \partial_n f \partial_n f\, ds + 4\pi R^2 M \max_{\partial B_R(0)} |\nabla f| \\ &\overset{(6)}{\leq} 4\pi R^2 M \max_{\partial B_R(0)} |\nabla f| \to 0 \text{ as } R = |x| \to \infty. \end{aligned}$$

Here we have added a zero term in (1), as f is harmonic in W_R because of $(7.1)_1$. Then in (2) we have realized that the two integrals together form one. In (3) we applied the divergence theorem. Notice that the normal n points into W_R on $\Gamma(t)$, explaining the minus sign. In (4) we estimate the last integral and in (5) use the boundedness of f. Also we use $(7.1)_2$ on the first integral in (5) and see that this whole term will contribute something nonpositive in (6). We conclude with $(7.1)_3$ that the estimation goes to zero and therefore we have that $\int_{\mathbb{R}^3 \setminus \overline{\Omega}} |\nabla f|^2 dx = 0$ and we need to have $\nabla f = 0$ on $\mathbb{R}^3 \setminus \overline{\Omega}$, implying that f is constant. Now because of $(7.1)_2$ we see we can only have $f \equiv 0$ and this proves the uniqueness of (1.1). □

Volume preservation

Here we provide a proof for the volume preserving property of the evolution of $\Omega(t)$ of (1.1). It is the proof provided in [17] and slightly modified to account for the fact here a three dimensional problem is discussed, as in [17] only two dimensions are considered. First we prove the following lemma:

Lemma 7.1. *Let $\Gamma(t)$ be a smooth, closed surface in $\mathbb{R}^3$ that varies smoothly with t and bounds a domain $\Omega(t)$. Denote with $N(x,t)$ and $V(x,t)$ the unit normal on $\Gamma(t)$ pointing outward and the velocity vector at point x respectively, at time t. Let $f : \mathbb{R}^3 \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a smooth function with function values $f(x,t)$. Then*

$$\frac{d}{dt}\int_{\Omega(t)} f\,dx = \int_{\Omega(t)} \frac{\partial f}{\partial t}\,dx + \int_{\Gamma(t)} fV_n\,ds.$$

Proof. We transform the integral from the time dependent domain $\Omega(t)$ back to a fixed domain Ω_0 with a transformation $x = \eta(x,t)$. Here η should be a solution to the ordinary differential equation $\frac{d\eta}{dt} = V(\eta(x,t),t)$, with initial condition $\eta(x,0) = x$. We denote the Jacobian determinant of the transformation with $det\mathbf{J} = det\mathbf{J}(t)$ and $g(x,t) = f(\eta(x,t),t)$. We have

$$\frac{d}{dt}\int_{\Omega(t)} f(x,t)\,dx = \frac{d}{dt}\int_{\Omega_0} g(x,t)det\mathbf{J}\,dx = \int_{\Omega_0} \frac{\partial g(x,t)}{\partial t}det\mathbf{J} + g(x,t)\frac{d\,det\mathbf{J}}{dt}\,dx$$

We observe that $\frac{d\,det\mathbf{J}}{dt} = \mathbf{J}(V)\cdot\mathbf{J}$, by differentiating the ODE with respect to x. Now we see that the matrix valued function $\mathbf{J}$ satisfies $\frac{d\mathbf{J}}{dt} = A\cdot\mathbf{J}$, for some real matrix $A = A(t)$. Then, according to Liouville's Theorem [5], $\frac{d\,det\mathbf{J}}{dt} = Tr(A)det\mathbf{J}$. So in this case $\frac{d\,det\mathbf{J}}{dt} = Tr(\mathbf{J}(V))det\mathbf{J} = (\nabla\cdot V)det\mathbf{J}$. Now, by the chain rule and transforming back;

$$\begin{aligned}\frac{d}{dt}\int_{\Omega(t)} f(x,t)\,dx &= \int_{\Omega_0}\frac{\partial g(x,t)}{\partial t}det\mathbf{J} + g(x,t)\frac{d\,det\mathbf{J}}{dt}\,dx = \int_{\Omega(t)}\frac{\partial f}{\partial t} + \nabla f\cdot V + f(\nabla\cdot V)\,dx\\ &= \int_{\Omega(t)}\frac{\partial f}{\partial t} + \nabla(fV)\,dx = \int_{\Omega(t)}\frac{\partial f}{\partial t}\,dx + \int_{\Gamma(t)} fV_n\,ds.\end{aligned}$$

Here the last step is due to the divergence theorem and the lemma is proven. □

Now for a fixed t, let R be big enough such that $\overline{\Omega(t)}$ is contained in the ball of radius R, $B_R(0)$. Now $\Gamma(t)$ and $\partial B_R(0)$ form the (piecewise smooth) boundary of a region in $\mathbb{R}^3$, let us call it $W \subset \mathbb{R}^3$. Observe that $W \subset \mathbb{R}^3 \setminus \overline{\Omega(t)}$, such that $\Delta f = 0$ in W. By the divergence theorem we have

$$0 = \int_W \Delta f\,dx = \int_{\partial B_R(0)} \partial_n f\,ds - \int_{\Gamma(t)} \partial_n f\,ds.$$

We see that $\int_{\partial B_R(0)} \partial_n f\,ds = \int_{\Gamma(t)} \partial_n f\,ds$ and that these integrals are independent of R. As $\nabla f - \vec{e_3} = o(R^{-2})$ for $R \to \infty$ we have

$$\begin{aligned}\int_{\partial B_R(0)} \partial_n f\,ds &= \int_{\partial B_R(0)} \nabla f\cdot n\,ds = \int_{\partial B_R(0)} n_3\,ds + \int_{\partial B_R(0)} (\nabla(f - x_3))\cdot n\,ds\\ &= \int_{\partial B_R(0)} (\nabla f - \vec{e_3})\cdot n\,ds\\ &\leq 4\pi R^2 \max|\nabla f - \vec{e_3}| \xrightarrow{R\to\infty} 0.\end{aligned}$$

We conclude that $\int_{\partial B(R)} \partial_n f\,ds = 0$, so when we apply the lemma of this section with $f \equiv 1$ we get

$$\frac{d}{dt}\int_{\Omega(t)} dx = \int_{\Gamma(t)} V_n\,ds = \int_{\Gamma(t)} \partial_n f\,ds = \int_{\partial B_R(0)} \partial_n f\,ds = 0,$$

proving the volume preserving property of the evolution of $\Omega(t)$ of (1.1) over time.

Proofs for recurrence relations associated Legendre polynomials

Because of the somewhat irregular way of defining the associated Legendre polynomials part of the spherical harmonics, sources for recurrence relations for these polynomials are hard to be found. In this part of the Appendix we slightly modify the proofs given in [22] to give justification for using them, without unnecessarily filling the main body of text with proofs that essentially have been done previously. We start with proving the recurrence relation (6.5), which holds for $l \geq 1$ and is given below:

$$sP_l^{|m|}(s) = \frac{(|m|+l)P_{l-1}^{|m|}(s) + (l-|m|+1)P_{l+1}^{|m|}(s)}{2l+1}.$$

Proof. Firstly, for simplicity we will write $n := |m|$. We will differentiate the generating function for the associated Legendre polynomials, j_n, as found before, with respect to t. The generating function is given by

$$j_n(s,t) = (-1)^n(2n-1)!!\frac{(1-s^2)^{\frac{n}{2}}t^n}{(1-2st+t^2)^{n+\frac{1}{2}}} =: \frac{c_n t^n}{(1-2st+t^2)^{n+\frac{1}{2}}} = \sum_{l=0}^{\infty} P_l^n(s)t^l.$$

Note that in comparison to [22], we have an extra factor $(-1)^n$, making our results slightly different. We have defined $c_n := (-1)^n(2n-1)!!(1-s^2)^{\frac{n}{2}}$, as this is independent of t and thus invariant when differentiate with respect to t. We move t^n to the right hand side and differentiate with respect to t, recognise j_n in the left hand side and rewrite:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left[\frac{c_n}{(1-2st+t^2)^{n+\frac{1}{2}}} = \sum_{l=0}^{\infty} P_l^n(s)t^{l-n}\right] \\
\implies & c_n(n+\frac{1}{2})(2s-2t)(1-2st+t^2)^{-(n+\frac{3}{2})} = \sum_{l=0}^{\infty}(l-n)t^{l-n-1}P_l^n(s) \\
\implies & \sum_{l=0}^{\infty}\{(2n+1)(s-t)t^l - [1-2st+t^2](l-n)t^{l-1}\}P_l^n(s) = 0
\end{aligned}$$

Now by shifting indices, we make sure all terms have equal power of t and obtain:

$$\begin{aligned}
&\sum_{l=0}^{\infty}(2n+1)sP_l^n(s)t^l - \sum_{l=1}^{\infty}(2n+1)P_{l-1}^n(s)t^l - \sum_{l=-1}^{\infty}(l+1-n)P_{l+1}^n(s)t^l \\
&+ \sum_{l=0}^{\infty}2(l-n)sP_l^n(s)t^l - \sum_{l=1}^{\infty}(l-1-n)P_{l-1}^n(s)t^l = 0
\end{aligned}$$

And by the identity theorem for power series we now have the following relations:

For $l=-1$ $\quad nP_0^n = 0$, which holds for only option $n=0$

For $l=0$ $\quad sP_0^n(s) + (n-1)P_1^n(s) = 0$, for only option $n=0$ results in $P_1^0(s) = sP_0^0(s) = s$,

For $l \geq 1$ $\quad (2l+1)sP_l^n(s) = (l+n)P_{l-1}^n(s) + (l+1-n)P_{l+1}^n(s)$.

Substituting $n = |m|$ back and the proof is complete. □

Next we prove the recurrence relation (6.7) for $l \geq 1$, which is given by:

$$\sqrt{1-s^2}P_l^{|m|+1}(s) = \frac{1}{2l+1}\left[(l-|m|)(l-|m|+1)P_{l+1}^{|m|}(s) - (l+|m|)(l+|m|+1)P_{l-1}^{|m|}(s)\right].$$

Proof. Therefore we again denote $n := |m|$. We will start with differentiating the generating function j_n with respect to t. As seen before we get

$$\frac{(2n+1)c_n(s-t)}{(1-2st+t^2)^{n+3/2}} = \sum_{l=0}^{\infty}(l-n)t^{l-n-1}P_l^n(s) \implies$$
$$\frac{(2n+1)c_n(s-t)}{(1-2st+t^2)^{1/2}} = (1-2st+t^2)^{n+1}\sum_{l=0}^{\infty}(l-n)t^{l-n-1}P_l^n(s).$$

Now we do another differentiation on both sides with respect to t. For the left hand side we get:

$$\frac{\partial}{\partial t}\left[\frac{(2n+1)c_n(s-t)}{(1-2st+t^2)^{1/2}}\right] = \frac{-(2n+1)c_n(1-s^2)}{(1-2st+t^2)^{3/2}} = \frac{c_{n+1}\sqrt{1-s^2}}{(1-2st+t^2)^{3/2}}.$$

Here we have noticed that $-(2n+1)(1-s^2)c_n = -(2n+1)(1-s^2)(-1)^n(2n-1)!!(1-s^2)^{\frac{n}{2}} = (1-s^2)^{\frac{1}{2}}(-1)^{n+1}(2(n+1)-1)!!(1-s^2)^{\frac{n+1}{2}} = \sqrt{1-s^2}c_{n+1}$. Differentiation on the right hand side yields

$$\frac{\partial}{\partial t}\left[(1-2st+t^2)^{n+1}\sum_{l=0}^{\infty}(l-n)t^{l-n-1}P_l^n(s)\right] =$$
$$t^{-n-1}[1-2st+t^2]^n\sum_{l=0}^{\infty}(l-n)\{(l-n-1)[1-2st+t^2]t^{l-1}+2(n+1)(t-s)t^l\}P_l^n(s).$$

Equating both sides, moving and grouping of terms and recognizing the generating function of $P_l^{n+1}(s)$ gives us

$$\frac{\sqrt{1-s^2}c_{n+1}t^{n+1}}{(1-2st+t^2)^{n+1+\frac{1}{2}}} = \sqrt{1-s^2}\sum_{l=0}^{\infty}P_l^{n+1}(s)t^l =$$
$$\sum_{l=0}^{\infty}(l-n)\{(l-n-1)t^{l-1}-2lst^l+(l+n+1)t^{l+1}\}P_l^n(s)$$
$$\sum_{l=-1}^{\infty}(l+1-n)(l-n)P_{l+1}^n(s)t^l - \sum_{l=0}^{\infty}2l(l-n)sP_l^n(s)t^l + \sum_{l=1}^{\infty}(l-n-1)(l+n)t^lP_{l-1}^n(s).$$

We now compare coefficients of t^l and get as a result of the identity theorem for power series that for $l \geq 1$

$$\begin{aligned}\sqrt{1-s^2}P_l^{n+1}(s) =&(l+1-n)(l-n)P_{l+1}^n(s) - 2l(l-n)sP_l^n(s) + (l-1-n)(l+n)P_{l-1}^n(s)\\ \overset{(6.5)}{=}&(l+1-n)(l-n)P_{l+1}^n(s)\\ &-2l(l-n)\frac{1}{2l+1}[(l+n)P_{l-1}^n(s)+(l+1-n)P_{l+1}^n(s)]\\ &+(l-1-n)(l+n)P_{l-1}^n(s)\\ =&\frac{1}{2l+1}[-(l+n)(l+1+n)P_{l-1}^n(s)+(l-n)(l-n+1)P_{l+1}^n(s)].\end{aligned}$$

Here in the equation marked with (6.5), we used the recurrence relation we found earlier. Substituting $n = |m|$ back gives us the recurrence relation we wished to prove. □

Mathematica script

```
In[1]:= gamma = 1/2;

In[2]:= (*Generating the coefficients in Result 5*)

In[3]:= a[m_, l_] := (-1 + gamma + 2 l + gamma l^2)/(l gamma + 1) * Sqrt[(l^2 - m^2)/(4 * l * l - 1)]

In[4]:= b[m_, l_] := gamma * (2 - l - l^2)/((2 + l) * gamma + 1) * Sqrt[((l + 1)^2 - m^2)/(4 * (l + 1)^2 - 1)]

In[5]:= (*Generating the coefficients in Theorem 6.1*)

In[6]:= c[m_, l_] := (-1)^(l + 1) Product[b[m, 2 j] / a[m, 2 j], {j, 1, l - 1}] / a[m, 2 l]

In[7]:= abs[l_, m_, s_] := Sqrt[((2 l + 1) * (l - Abs[m])!)/((4 * Pi) * (l + Abs[m])!)]
          ((-1)^Abs[m])/(2^l * (l!)) * (1 - s^2)^(Abs[m] / 2) * D[(s^2 - 1)^l, {s, l + Abs[m]}]
          (*Defining the associated Legendre polynomials according to (3.4)*)

In[8]:= Y[m_, l_, theta_, phi_] := abs[l, m, Cos[theta]] *
          ⅇ^(m ⅈ phi) (*Defining the Spherical harmonics according to Definition 3.3*)

In[9]:= h[theta_, phi_, alphamin1_, alpha0_, alphaplus1_] = (1 + 2 gamma)/(3 gamma)
          Sum[alphamin1 * c[-1, l] * Y[-1, 2 l, theta, phi] + alpha0 * c[0, l] * Y[0, 2 l, theta, phi] +
            alphaplus1 * c[1, l] * Y[1, 2 l, theta, phi], {l, 1, 10}];
        (*Defining the h-function, where the alpha coefficients solve the system (6.14)*)

In[10]:= (* Making the plots for different bubble domains
           corresponding to variation of +/- 0.1 in each Cartesian direction,
         name indicates which. Remove semicolon to remove surpressing of
          plot. Two of six plots shown to show procedure*)

In[11]:= Plote2plus = ParametricPlot3D[
           {(h[theta, phi, ⅈ/20 Sqrt[2 Pi/3], 0, -ⅈ/20 Sqrt[2 Pi/3]] + 1) * Sin[theta] Cos[phi],
            (h[theta, phi, ⅈ/20 Sqrt[2 Pi/3], 0, -ⅈ/20 Sqrt[2 Pi/3]] + 1) * Sin[theta] Sin[phi],
            (h[theta, phi, ⅈ/20 Sqrt[2 Pi/3], 0, -ⅈ/20 Sqrt[2 Pi/3]] + 1) * Cos[theta]},
           {theta, 0, Pi}, {phi, 0, 2 Pi}, AxesLabel → {x1, x2, x3}, ViewPoint → {-10, -10, 5}];

In[12]:= Plote3plus = ParametricPlot3D[{(h[theta, phi, 0, -1/10 Sqrt[Pi/3], 0] + 1) * Sin[theta] Cos[phi],
           (h[theta, phi, 0, -1/10 Sqrt[Pi/3], 0] + 1) * Sin[theta] Sin[phi],
           (h[theta, phi, 0, -1/10 Sqrt[2 Pi/3], 0] + 1) * Cos[theta]}, {theta, 0, Pi},
           {phi, 0, 2 Pi}, AxesLabel → {x1, x2, x3}, ViewPoint → {-10, -10, 5}];
```

Figure 4: The Mathematica script used to produce the plots in Figure 3.