BACHELOR

Topology-dependent first passage percolation on spatial scale-free networks

Reubsaet, Rick M.H.J.

Award date:
2021

Eindhoven University of Technology

2WH40 - Bachelor Final Project - Topology-dependent First Passage Percolation on Spatial Scale-free Networks

Author:
Rick Reubsaet
r.m.h.j.reubsaet@student.tue.nl
Student number 1346059

Supervisors:
Júlia Komjáthy
Zsolt Bartha

June 3, 2021

1 Abstract

We explore first passage percolation on an infinite geometric inhomogeneous random graph (IGIRG) model. Vertices are distributed in $\mathbb{R}^d$ according to a Poisson process with intensity $0 < \lambda \leq 1$ and vertex fitnesses are distributed according to a power-law with exponent $\tau \in (2, 3)$. Edges are drawn randomly: the probability that vertices u and v are connected is increasing with their fitnesses and decaying with their spatial distance. After the construction of the edges, we equip each edge with a cost or weight: the cost of transmission along an edge consists of an independent random variable L with a cumulative distribution function that is a power-form close to the origin, multiplied by a monomial of the vertex fitnesses and their spatial distance. We look at whether there are infinitely many vertices reachable within finite cost from the origin on a path, which is called explosion, where the cost of the path is the sum of edge-costs on the edges of the path. We characterise whether explosion can happen in the model in terms of the underlying parameters.

2 Introduction

In the current day and age, networks are ubiquitous. People's personal network of acquaintances and family, the internet, the world-wide web, the cellphone network and many other structures can, in their truest essence, be represented as a network of just vertices and edges. The corresponding mathematical object is a *graph*. In many cases, the exact structure of the network is unknown, and only certain properties or features of the network are known. In this case, the real-life network is best modelled by a *random graph*, in which the amount of vertices, their location and the presence of edges is not known beforehand. An interesting class of processes to investigate on such a network are *spreading processes*. Whether it is a meme being shared over the internet or an epidemic spreading across a network of people, comprehending how to influence a spreading process is invaluable. Knowing how to promote the spread may help in promoting an advertisement, while knowing how to contain it can aid in stemming a pandemic.

The purpose of this thesis is to investigate a model for a spreading process on a specific random graph: the infinite geometric inhomogeneous random graph. We will show for what choice of parameters *explosion* occurs, a phenomenon in which the spread happens incredibly fast and infinitely many vertices can be reached in a finite amount of time. We will elaborate on this concept later. This thesis extends results in [25] in a certain new direction. In our model, the transmission time on an edge of the spreading process also depends on the Euclidean length of the edge. While we use similar techniques as in [25], some extensions are non-trivial due to the spatial dependence.

2.a Modelling Spread

To start our discussion, we should first note that there are many types of models for spreading processes. Often, the process is viewed from an epidemiological perspective: a disease spreads through members of a population. The population can be divided in several classes of individuals, depending on the type of disease. The simplest model, the SI model, partitions the population into a group of susceptible individuals and a group of infected individuals. In the beginning, all individuals are susceptible (apart from those who form the 'start' of the epidemic), and they can become infected only once, after which they remain infected forever. This SI model is what we will be using, as it is the most simple model available for our purposes. Many variants of the SI model exist however, such as the Susceptible→Infected→Recovered (SIR), Susceptible→Infected→Susceptible (SIS), Susceptible→Infected→Recovered→Susceptible (SIRS), and Susceptible→Exposed→Infectious→Recovered (SEIR) models, and many more. These models each cater to different types of diseases. In the SIR model, long-term immunity is gained upon recovery, while in the SIS model no immunity is gained and in the SIRS model immunity is only retained for a short period after recovery. In the SEIR model, an incubation period is added (the 'exposed' stage), in which individuals have contracted the disease but are not yet infectious to others.

The next parameter a model must specify is *how* spread occurs. In the introduction, we mentioned that we will be modelling a network, but this is not a necessity. The compartment model, for instance, neglects the underlying network of spread. This model instead chooses to model the population as a fully mixed group of people, where each person can infect another. One of the earliest formulations of such a model was an implementation of the SIR model by McKendrick and Kermack in 1927 [23]. Each 'stage' (S, I or R) is called a compartment and, as before, they partition the population. Typically, these type of models are analysed through ordinary differential equations (ODEs), which is a deterministic approach. Note however, that when using an ODE approach, the population is taken to be continuous; at some points in time the size of the population of a compartment may be non-integer, which detracts from the realism of the model. Extensive literature exists on the compartmental SIR model and for some variants exact analytical results have been derived, such as in [16].

2.b Modelling Real-world Networks

Instead of the compartment model, we will use a graph-based (also called 'agent-based') model for our spreading process. Contrary to the compartment model, our model features a discrete, non-mixed population, i.e, each vertex represents an individual, and spread can only happen via edges between susceptible and infectious vertices. In order to choose an underlying graph model, however, we must first explore the characteristics of the networks we wish to model.

Real-world networks often are scale-free, exhibit clustering [2, 12], have hubs [12, 33] and are small-world [12, 14]. We will briefly elaborate on these properties. Firstly, scale-free means that the vertex degrees (the amount of neighbours a vertex has) follow a power-law distribution: $\mathbb{P}(\deg(v) = k) \asymp k^{1-\tau}$ for some parameter τ, typically in the range $(2, 3)$. In practice, this means that the majority of vertices have relatively few edges, while a minority of them have many neighbours. Such networks in which not all vertices have the same amount of neighbours are called heterogeneous or inhomogeneous.

The concept of clustering relates to the propensity of two vertices to be connected, given the fact that they share a common neighbour. Several measures to make this notion quantitative exist, one of such measures being the global clustering coefficient. This coefficient is defined as the ratio between the total amount of closed triples (triangles) and the total number of connected triples (open and closed). In more general terms, this notion aims to capture group-like structures, in which vertices have many edges to one another, while few edges 'leave' the cluster. Such community structures often arise in social and infrastructure networks [33].

Power-law degree distributions and clustering coefficients are studied in many kinds of network applications. The world-wide web, the internet, the phone call network and neural networks were all found to have (truncated) power-laws as degree distributions, but less obvious examples also include a network of proteins which can be folded into one another, linguistic networks and networks of collaboration or citations [2, 30]. Hubs, also known as influencers or super-spreaders in the context of a spreading process, are simply vertices of large degree. Due to the power-law nature of many real-world networks these are often relatively few in number, but very influential in spreading processes because they have many neighbours [20, 33].

Last is the small-world property. Often used in an everyday-sense by saying "It's a small world, isn't it?", this notion was popularised by an experimental study by Milgram in 1967. This study found that, on average, only 5 intermediary contacts were needed to connect two people in the United States by their network of acquaintances [29]. From a mathematical perspective, small-world means that the expected amount of edges needed to connect two arbitrary vertices is proportional to the logarithm of the amount of vertices in the network.

Several observations have been made in terms of how these features affect spreading in mathematical models which implement them. On the one hand, hubs promote spreading. After the initial infection reaches a hub, it spreads rapidly, causing hubs to potentially be the largest spreaders [12, 33]. Clustering, on the other hand, slows down the spread significantly [20, 33]. Intuitively, the infection has trouble breaking out of the community structures that accompany high clustering, causing 'local' outbreaks instead of 'global' pandemics. The interplay of these features and effects is complex, and we aim to choose a model for our underlying graph which captures all of them in order to observe the rich behaviour one might expect from a real-world network.

2.c Choosing a Graph Model

In the previous section we have examined several desired real-world features that we would like to simulate in our graph model. In this section, we will review several models and results relating to spreading processes, after which we introduce the model we will be using.

Non-geometric network models. Graph models for random networks have been around since the 1960's, having been first introduced by Erdős and Rényi in 1959 [13]. This original model is simple, yet it shows rich behaviour. The main idea is to randomly choose one of the possible graphs which can be formed out of a fixed number n vertices and any configuration of m edges, with equal probability. While this model is small-world, it does not capture the other features of real-world networks we have previously discussed, nor does it reflect any geometric properties.

Next, we will briefly mention Inhomogeneous Random Graphs (IRG). This model is a generalization of the Erdős-Rényi graph: independence of edges is kept, but the edge probabilities are now allowed to be different. A very general format for such models was introduced by Bollobás et al. in [8]. We omit the details due to the complex nature of the format, but the general idea is that every vertex in the graph has a certain type, and edge probabilities are controlled by a kernel. If a specific kernel is chosen, other models may be recovered, such as the Chung-Lu (see e.g. [10]) and Erdős-Rényi models [17].

The configuration model, introduced by Bollobás in [7], is the next model we will discuss. In this model, the degrees of vertices are fixed before the construction of the graph, which in particular allows the vertex degrees to follow any (non-negative, discrete) distribution. Every vertex is assigned as many half-edges as its set degree requires. Then, half-edges are paired randomly, until all are connected. Note that in general, this results in a *multigraph*, a graph in which vertices may have self-loops and multiple edges between pairs of vertices may be present. Naturally, whether this model reflects any of the features mentioned in Section 2.c depends on the choice of distribution for the vertex degrees. Some general results, however, can be found in terms of conditions on the degrees, such as whether an infinite component exists [31].

As a last non-spatial model, we will discuss the preferential attachment model, for which results relating to explosion have already been obtained. This model is interesting, because it is scale-free and contains hubs, features that were shown to occur in real-world networks. Originally introduced by Barabási and Albert, the preferential attachment model differs from the models we have discussed so far in the sense that it is seen as a dynamically growing model [4]. The graph starts with one vertex and no edges. When a new vertex v is added, edges are created between v and m existing vertices, where v is connected to an already-present vertex u with probability $\deg(u)/\sum_j \deg(j)$. Here, $\deg(u)$ is the degree of u, while the denominator sums over all degrees of vertices j present in the graph prior to adding v. As is immediately clear from this attachment rule, new vertices prefer to be attached to high-degree vertices. Moreover, since new vertices keep being connected to high degree vertices, they benefit from a 'rich-get-richer' effect and gain more and more neighbours. This stimulates the formation of hubs and creates a scale-free graph.

Geometric network models. For many real-world networks, incorporating a spatial aspect into the construction of the graph is desirable. In many common applications, the notion of 'distance' influences how likely two vertices are to be directly connected by an edge. For example, in a railway network, big cities close to one another are intuitively much more likely to have a direct route between them than cities which are far apart.

The scale-free percolation model was introduced by Deijfen at al. in [11]. The model features an infinite graph with vertices on $\mathbb{Z}^d$, each assigned an i.i.d. copy of a non-negative weight random variable W. Conditionally on the vertex weights, vertex locations and given parameters $\alpha, \lambda > 0$, the edges are independent and the probability that an edge between vertices u and v is present is $p_{uv} = 1 - \exp(-\lambda W_u W_v / \|x_u - x_v\|^\alpha)$. Parameter λ is a percolation parameter, while α regulates the long-range behaviour of the model. Under certain conditions, if the weight distribution follows a power-law, the degree distribution will also follow a power-law, which was one of the desired features we wished to incorporate. Hubs are also present in this graph model, and it exhibits clustering and small-world behaviour. However, since the model is defined on $\mathbb{Z}^d$, it is not suitable for modelling networks in which vertices are not placed according to this grid-like structure. In terms of explosion, the model's behaviour has been fully characterised for arbitrary edge weights in [19, 26].

Finally we arrive at the model we will use, the infinite geometric inhomogeneous random graph (IGIRG). This model is an extension of the finite geometric inhomogeneous random graph (GIRG) model (introduced in [9]), which in turn is a generalization of the hyperbolic random graph model (see [32]). This model exhibits all the features we desire (scale-free, clustering, hubs and small-world). We leave the specifications of the model to Subsection 3.a, but remark that this model is implemented with vertices distributed according to a Poisson point process in $\mathbb{R}^d$, which is, in some cases, a more realistic approach than the $\mathbb{Z}^d$ grid used by scale-free percolation.

2.d First Passage Percolation Results

In terms of spreading models, one of the first papers on the SI model applied to a graph was by Hammersley and Welsh in 1965 as a means to model flow through a (random) porous medium [15]. In their paper, each edge is allocated an independent and identically distributed random variable representing the cost of that edge. This kind of model is called *first passage percolation*. Since its introduction, the field of first passage percolation has been very active and the principle has been applied to many types of graphs [3].

For the Erdős-Rényi graph with exponentially distributed weights, for instance, several results have been found, such as an analysis of minimal distance and hopcount (number of edges on the shortest path) [6]. Related to this result is the study of first passage percolation on the complete graph: the graph in which all possible edges are present (this can be seen as an Erdős-Rényi graph where each edge is present with probability 1). For this graph, the distribution of the hopcount of the shortest path has also been characterised [18]. Again, this graph model lacks the previously mentioned features that we would like to include (except small-world).

First passage percolation also has been researched on an IRG model. The distribution of the weight of the shortest path between two uniformly chosen vertices in the giant component has been determined. In addition to this, it was shown that the hopcount, if properly normalized, follows a central limit theorem [24].

For the configuration model, multiple first passage percolation results are available. For instance, under power-law assumptions on the provided degrees and some additional conditions, the weight of the shortest path between two uniformly chosen vertices either converges to non-infinite random variables, or diverges to infinity [5]. Another result is the characterisation of the distribution the hopcount converges to [1].

The preferential attachment model has also been analysed in the context of first passage percolation. One result follows from the analysis of three of these preferential attachment models. The typical weighted distance and hopcount were investigated. It is shown that there are two universality classes of weight distributions relevant for these results, the explosive class and the conservative class, which determine the nature of the typical weighted distance and hopcount [22].

Weighted distances in the GIRG graph (with edge cost variables not incorporating the vertex weights) have been studied in [26] and were found to have a connection to the explosion time of the infinite version of the model. For IGIRG, precise conditions for explosion to occur were found. In [25], conditions for explosion were again analysed, but now using edge costs which also incorporated vertex weights. This deviates from first passage percolation, but is more realistic because high-degree vertices should have a higher expected transmission time. This reflects that high-degree vertices have a limited time budget and cannot interact with arbitrarily many neighbours per time unit, as has been observed in e.g. real-world communication networks [30].

2.e Our Contribution

Our contribution will be to expand on the results in [25] by also factoring in the edge length into the edge costs. This models that the distance between two vertices influences the cost of spread. This is relevant in e.g. epidemiological scenarios, where individuals are less likely to infect each other when there is greater distance between them. A recent application of this fact are the social distancing measures implemented to combat the COVID-19 pandemic [34]. In order to provide results for our extension, many of the proofs in [25] had to be (non-trivially) adapted, as they were not originally designed to factor in these edge lengths easily. With the exception of Lemma 5.4 and Lemma 5.5, all relevant lemmas have been revised and are proven for our case.

2.f Notation

We write r.v. for random variable, w.r.t. for with respect to, and i.i.d. for independent and identically distributed. Generally, random variables will be denoted by capital letters (e.g. X) and their cumulative distribution functions (cdfs) by F (so e.g. $F_X(x)$). We say an event A happens almost surely (a.s.) when $\mathbb{P}(A) = 1$. We write $\mathbb{R}^+ := (0, \infty)$ for the domain of positive real numbers. The set $\{1, ..., n\}$ is denoted as $[n]$. The size of a set X is denoted by $|X|$. The floor and ceiling functions of x are denoted by $\lfloor x \rfloor$ and $\lceil x \rceil$ respectively. For any vector $x \in \mathbb{R}^d$, we denote its infinity norm by $\|x\|_\infty := \max_{1 \le i \le d} |x_i|$. The Euclidian norm of x in $\mathbb{R}^d$ is denoted by $\|x\|$. We denote a graph by $G = (\mathcal{V}, \mathcal{E})$, with $\mathcal{V}$ being the vertex set and $\mathcal{E}$ being the edge set. For two vertices u, v, let $u \leftrightarrow v$ denote the event that u and v are connected by an undirected edge (u, v). We denote the *vertex fitness* for a vertex v by W_v. This is also commonly referred to as *vertex weight* in other literature. The meaning of this notion is explained below Assumption 3.2 in Section 3.a.

3 Model Description

We start by discussing the graph model used to model the spreading process. This is a slightly simplified version of the infinite geometric inhomogeneous random graph model (IGIRG) introduced by [9].

3.a Graph Model

Definition 3.1 (Infinite Geometric Inhomogeneous Random Graphs)**.** *Let* $h_\mathrm{I} : \mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}^+ \to [0, 1]$ *be a function, let* $W \ge 1, L \ge 0$ *be random variables, let* $d \ge 0$ *be an integer, and let* $\lambda > 0$. *The infinite graph model* $\mathrm{IGIRG}_{W,L}(\lambda)$ *then is defined as follows. Let* $\mathcal{V}_\lambda$ *be a homogeneous Poisson point process on* $\mathbb{R}^d$ *with intensity* λ*, which forms the positions of the vertices in* $\mathbb{R}^d$. *Let the position of vertex* u *in* $\mathbb{R}^d$ *be denoted* x_u. *For each such vertex* $u \in \mathcal{V}_\lambda$*, draw a (random) fitness* W_u*, which is an i.i.d. copy of* W. *Then, conditioned on* $(x_u, W_u)_{u \in \mathcal{V}_\lambda}$*, edges are present independently with probability*

$$\mathbb{P}(u \leftrightarrow v \text{ in } \mathrm{IGIRG}_{W,L}(\lambda) \mid (x_z, W_z)_{z \in \mathcal{V}_\lambda}) := h_\mathrm{I}(x_u - x_v, W_u, W_v). \tag{3.1}$$

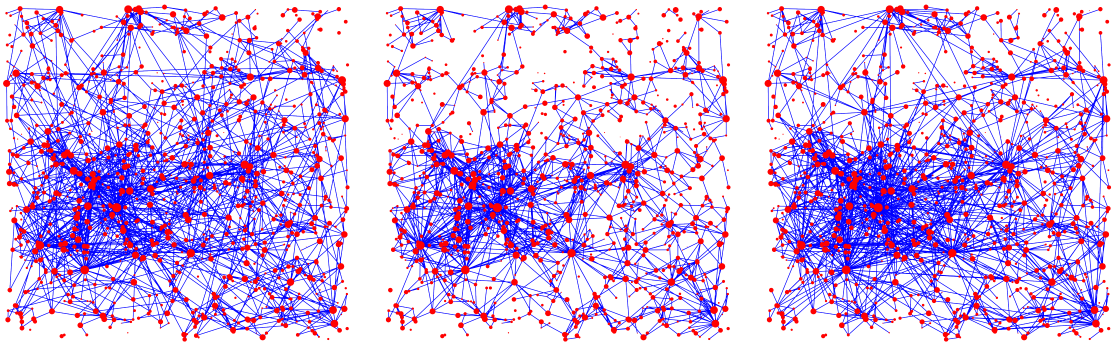
Furthermore, assign to each present edge in the graph an edge weight L_e*, an i.i.d. copy of a random variable* $L \ge 0$. *We write* $(\mathcal{V}_\lambda, \mathcal{E}_\lambda)$ *for the vertex and edge set of the resulting* IGIRG *graph.*

For our specific case, this definition is too general. The edge connectivity function h_I and vertex fitness distribution will be specified by Assumption 3.3 and Assumption 3.2 below. Furthermore, we assume that $0 < \lambda \le 1$. This assumption is not detrimental to the generality of our case, since this corresponds to rescaling the Poisson point process.

Assumption 3.2 (Vertex Fitnesses Obey Power-law)**.** *It is assumed that for all vertices* u*, their fitness* W_u *satisfies*

$$\frac{c_1}{x^{\tau-1}} \le \mathbb{P}(W_u \ge x) \le \frac{c_2}{x^{\tau-1}}, \tag{3.2}$$

for some real $\tau \in (2, 3)$ *and constants* $c_1, c_2 \in \mathbb{R}^+$. *Such a distribution is known as a* power-law *with parameter* τ.



(a) Average degree: 4.022, $\tau = 2.7$, $\alpha = 1.5$ (b) Average degree: 3.930, $\tau = 2.7$, $\alpha = 5$ (c) Average degree: 3.958, $\tau = 2.4$, $\alpha = 5$

Figure 1: Three examples of the GIRG model under our modelling assumptions. Vertices are placed randomly on a unit cube of dimension $d = 2$. Note that the same underlying vertex set is used. Parameters τ and α are varied, while the resulting graph is thinned such that the average degrees are approximately the same. Comparing 1a to 1b, the effect of α can be seen. The lower value of α allows a greater number of long-range edges. Similarly comparing 1b to 1c, the effect of τ can be observed. The smaller choice of τ leads to greater diversity in vertex degrees. In 1c, there are several smaller hubs, as well as some very large hubs. In 1b, there is less variation in hub size. Images are generated using code provided by Joost Jorritsma.

Under Assumption 3.2 and conditioned on its fitness W_u, the expected degree of any vertex u coincides with W_u up to a constant factor and the vertex degrees will follow a power-law with parameter τ (when $\tau > 2$ and $\alpha > 1$ in Assumption 3.3 below) [9, 25]. The parameter τ controls the degree of heterogenity of vertex fitnesses. A smaller choice of τ will yield greater diversity in vertex degrees. We restrain the model to $\tau \in (2, 3)$. The result of taking $\tau > 2$ is that the expectation of W is finite, while the variance is infinite [11]. The case $\tau > 3$ will not be interesting, as in this case explosion is never possible. This was shown for the scale-free percolation model in [19], which in this case behaves similarly enough to the IGIRG model for the results apply to the IGIRG model as well [26].

Assumption 3.3 (Connection Probability)**.** *For any two vertices $u, v \in \mathcal{V}$ located at $x_u, x_v \in \mathbb{R}^d$, it is assumed that the probability that they are connected is given by*

$$\mathbb{P}(u \leftrightarrow v \mid W_u, W_v, x_u, x_v) = p \cdot \min\left(1, \left(\frac{W_u \cdot W_v}{\|x_u - x_v\|^d}\right)^{\alpha}\right), \tag{3.3}$$

for some percolation parameter $p \in (0, 1]$ and long range parameter $\alpha \in (1, \infty)$.

The amount of long-range connections is governed by α, and $\alpha > 1$ is required to avoid infinite vertex degrees. In general, a larger choice of α will result in fewer long-range connections. This is the case because typically, the ratio $W_u W_v / \|x_u - x_v\|^d$ is less than 1 when vertices are far away. An example of a finite version of our model can be seen in Figure 1.

3.b Spreading Model

As mentioned in Subsection 2.a, several spreading models exist. The model we explore is the SI model, in which susceptible individuals (in our case, vertices) can become infected and then remain infected forever. Spread can only happen from an already infected vertex to a direct neighbour. To spread across an edge, the infection incurs a certain cost. We introduce two such cost models later. First, we formally define the notion of cost-distance in the graph.

Definition 3.4 (Distances and Balls)**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph. Let $\mathcal{W} := \{W_u : u \in \mathcal{V}\}$ be the associated set of vertex fitnesses, where $W_u \in [1, \infty)$ for all $u \in \mathcal{V}$. Choose a collection of costs*

$\mathcal{C} := \{C_{(u,v)} : (u,v) \in \mathcal{E}\}$. Then for all walks $\pi = (\pi_0, ..., \pi_k)$ in $\mathcal{G}$, we define the cost of π to be

$$|\pi|_{\mathcal{C}} := \sum_{i=1}^{k} C_{(\pi_{i-1},\pi_i)}. \tag{3.4}$$

For all sets $A, B \subseteq \mathcal{V}$, we define the cost-distance *from A to B by*

$$d_{\mathcal{C}}(A,B) := \inf(\{|\pi|_{\mathcal{C}} : \pi = (\pi_0, ..., \pi_k) \text{ is a path with } \pi_0 \in A \text{ and } \pi_k \in B\} \cup \{\infty\}). \tag{3.5}$$

To abbreviate our notation, we write $d_{\mathcal{C}}(u,v) := d_{\mathcal{C}}(\{u\},\{v\})$. With this distance function in hand, we can now define cost-balls *by*

$$B_{\mathcal{C}}(u,r) := \{v \in \mathcal{V} : d_{\mathcal{C}}(u,v) \leq r\} \text{ for all } u \in \mathcal{V} \text{ and all } r \geq 0. \tag{3.6}$$

Now we can define our cost of spread models. We will study two variations. Both will take the geometric distance into account in terms of cost, but only the second will also factor in vertex fitnesses. Moreover, both also feature an edge weight L, which will be defined first.

Definition 3.5 (Edge Weights)**.** *Let $L_{(u,v)}$ be an i.i.d. copy of the random variable L for each edge $e = (u,v)$ of the graph. Let the cumulative distribution function F_L satisfy*

$$F_L(t) = \mathbb{P}(L \leq t) = \min(1, t^{\beta}), \tag{3.7}$$

for some $\beta \in \mathbb{R}$, $\beta > 0$. The copy $L_{(u,v)}$ of L associated with the edge (u,v) will be referred to as the edge weight *of (u,v).*

Definition 3.6 (Cost of Spread)**.** *For each edge $e = (u,v)$, we specify $C_{(u,v)}$ in Definition 3.4 as*

$$C_{(u,v)} = L_{(u,v)} \cdot \|x_u - x_v\|^{\zeta}, \tag{3.8}$$

for some $\zeta \in \mathbb{R}$ where $\zeta \geq 0$. In this definition, x_u and x_v denote the positions of vertices u and v in $\mathbb{R}^d$ and $L_{(u,v)}$ is the edge weight as defined in Definition 3.5.

Definition 3.7 (Cost of Weighted Spread)**.** *For each edge $e = (u,v)$, we specify $C_{(u,v)}$ in Definition 3.4 as*

$$C^{w}_{(u,v)} := L_{(u,v)} \cdot \|x_u - x_v\|^{\zeta} \cdot W_u^{\mu} \cdot W_v^{\nu}, \tag{3.9}$$

for some $\zeta, \mu, \nu \in \mathbb{R}$ where $\zeta, \mu, \nu \geq 0$. In this definition, x_u and x_v denote the positions of the vertices u and v in $\mathbb{R}^d$ and $L_{(u,v)}$ is the edge weight as defined in Definition 3.5. Random variables W_u and W_v are the fitnesses of the vertices u and v respectively.

Recall that in our case, the distributions of the fitnesses in Definition 3.7 obey Assumption 3.2. Note that our spreading process is not Markovian, since the costs are not memoryless. Having specified the model and its assumptions, we can now formally introduce the concept of explosion under Definition 3.4. This definition is valid for all weighted infinite networks.

Definition 3.8 (Explosion Time and Lengthwise Explosion)**.** *Consider the* $\mathrm{IGIRG}_{W,L}(\lambda)$ *model as described in Definition 3.1 and let $u \in \mathcal{V}_{\lambda}$. Choose a collection of costs $\mathcal{C} := \{C_{(u,v)} : (u,v) \in \mathcal{E}\}$. Define $\sigma_{\mathcal{C}}(u,k) := \inf\{t : |B_{\mathcal{C}}(u,t)| > k \text{ in } \mathrm{IGIRG}_{W,L}(\lambda)\}$. Then we define the* explosion time *of a vertex u (with respect to cost of spread $\mathcal{C}$) as the possibly infinite limit*

$$Y_{\mathcal{C}}(u) := \lim_{k\to\infty} \sigma(u,k). \tag{3.10}$$

The explosion time of the origin, $Y_{\mathcal{C}}(0)$, is defined analogously when we condition on $0 \in \mathcal{V}_{\lambda}$. The $\mathrm{IGIRG}_{W,L}(\lambda)$ *model is called* explosive *(with respect to the collection of costs $\mathcal{C}$) if*

$$\mathbb{P}(Y_{\mathcal{C}}(0) < \infty) > 0. \tag{3.11}$$

Otherwise, we call it conservative*. Furthermore, we call any infinite path π with $|\pi|_{\mathcal{C}} < \infty$ an* explosive path.

In the above definition, $\sigma_{\mathcal{C}}(u,k)$ can be seen as the smallest cost t such that k other vertices are reachable within cost t from u. The explosion time $Y_{\mathcal{C}}(u)$ is the infimum of all costs t such that the ball $B_{\mathcal{C}}(u,t)$ contains *infinitely many vertices*. Hence $Y_{\mathcal{C}}(u)$ is finite if and only if infinitely many vertices are reachable within bounded cost from u. In this thesis, we are mainly interested in *lengthwise explosion*, which is the case where there exists an explosive path from the origin. In other literature (e.g. [25]), one may also find the concept of *sideways explosion*, which is the case in which there is a finite path from the origin to a vertex from which there are infinitely many incident edges of bounded cost. This latter form of explosion is not exclusive with lengthwise explosion. Note that sideways explosion cannot occur in graphs where each vertex has finite degree almost surely (locally finite graphs), and that the model we study is locally finite when $\tau > 2$ and $\alpha > 1$. In this thesis, we will not explore this concept of sideways explosion.

4 Explosion

In this section we will present two theorems (Theorem 4.1 and Theorem 4.2 below) which characterise the constraints on parameters β, ζ and τ and dimension d needed to ensure explosion with respect to the cost models defined in Definition 3.6 and Definition 3.7 respectively. We will show that (lengthwise) explosion occurs by constructing an infinite path with finite total cost. This implies that $\sigma(0,k)$ stays bounded as $k \to \infty$, which ensures that $Y_{\mathcal{C}}(0)$ is finite. The proof for these theorems is adapted from [25]. We use a similar structure and have modified their existing lemmas to suit our model.

Theorem 4.1 (Explosion Theorem)**.** *Let $\alpha \in (1,\infty)$. Consider the model $\mathrm{IGIRG}_{W,L}(\lambda)$ with $0 < \lambda \le 1$. Suppose the vertex fitness distribution satisfies Assumption 3.2, the connection probability satisfies Assumption 3.3 and the cost of spread is as defined in Definition 3.6. Furthermore, let parameters β, ζ and d satisfy*

$$2\beta\frac{\zeta}{d} < 3-\tau. \tag{4.1}$$

Then the model is lengthwise explosive.

Theorem 4.2 (Explosion Theorem for Weighted Cost)**.** *Let $\alpha \in (1,\infty)$. Consider the model $\mathrm{IGIRG}_{W,L}(\lambda)$ with $0 < \lambda \le 1$. Suppose that the vertex fitness distribution satisfies Assumption 3.2, the connection probability satisfies Assumption 3.3 and the cost of spread is as defined in Definition 3.7. Furthermore, let parameters β, ζ, d, μ and ν satisfy*

$$\beta\left(\mu+\nu+2\frac{\zeta}{d}\right) < 3-\tau. \tag{4.2}$$

Then the model is lengthwise explosive.

While we only treat monomials in $\mathcal{C}$, the theorems can be extended for polynomial penalty factors by observing the following. Suppose f_1 and f_2 are monomials in $\mathcal{C}$ and $f = f_1 + f_2$. Then $|\pi|_f = |\pi|_{f_1} + |\pi|_{f_2}$. Hence, ensuring that for an infinite path π all monomials have finite cost shows explosion with respect to the polynomial f.

Before we come to the structure of the proof, we first show an interesting example.

Example 4.3 (Explosion for $\zeta > 1$)**.** *Consider the model $\mathrm{IGIRG}_{W,L}(\lambda)$ with $0 < \lambda \le 1$. Suppose that the vertex fitness distribution satisfies Assumption 3.2, the connection probability satisfies Assumption 3.3 and the cost of spread is as defined in Definition 3.7. Choose $d = 3$, $\zeta = 3/2$, $\tau = 5/2$, $\mu + \nu = 2$ and $\beta < 1/6$. Then the model is lengthwise explosive.*

This example is noteworthy, since it shows that explosion can occur on such a model in $\mathbb{R}^3$, even though we penalise the spread by more than the Euclidian length of the edges (since $\zeta > 1$). Apparently, by choosing τ close to 2 and β small, there are so many cheap edges out of the hubs, that the cost of at least one infinite path is brought down enough to be finite with positive probability. This is only one choice of parameters which shows this possibility, but in general there many possible choices for μ, ν, ζ and d with the same result as long as τ is close to 2 and β is taken small enough.

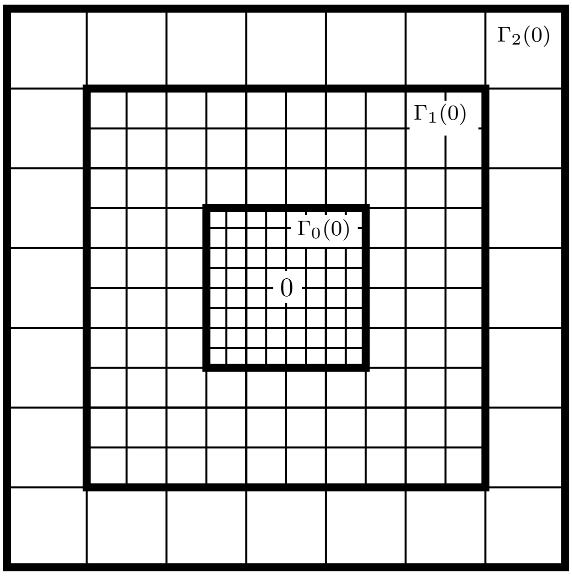


Figure 2: A schematic representation of a boxing system centered around the origin. In most cases, sub-boxes will not perfectly fill each annulus, resulting in some left-over space not illustrated here. The ratio between the annuli and sub-boxes is also not representative. Note that this is only a finite part of the boxing system, as this segment is surrounded by $\Gamma_3(0)$, which in turn is surrounded by $\Gamma_4(0)$, etc. In reality, these nested annuli continue infinitely.

Heuristic idea of the proof of Theorem 4.1 and Theorem 4.2. We construct a system of expanding boxes centered around the origin. The kth box has doubly-exponential volume $\exp(MDC^k)$ for some suitably chosen parameters $C, D > 1$ and $M > 0$. The kth annulus formed by these boxes (box k minus box $k-1$) is then packed with approximately $\exp(M(D-1)C^k)$ disjoint sub-boxes, each with volume $\exp(MC^k)$. In each such sub-box, the vertex with maximal fitness is called the *leader* of the sub-box. We construct our infinite path of finite cost greedily as follows. Suppose that we have exposed the kth annulus and have constructed a path reaching some leader vertex c_k in the kth annulus. Then we expose the contents of the $(k+1)$th annulus, and choose the edge (c_k, c_{k+1}) from c_k to a leader vertex c_{k+1} of some sub-box in the $(k+1)$th annulus such that the assigned edge weight $L_{(c_k,c_{k+1})}$ is minimal. In order to show explosion, it suffices to show that the total cost of this path is finite almost surely.

We start by defining the aforementioned boxing system and characterising bounds on the amount of sub-boxes each annulus can have in Lemma 4.5. Afterwards, the leader vertices will be formally defined, and we start to work towards proving Lemma 4.10 and Lemma 4.12 which will serve as the cornerstones of the final proofs of Theorem 4.1 and Theorem 4.2.

Definition 4.4 (Boxes and Annuli)**.** *Given a center vertex $u \in \mathbb{R}^d$ and parameter $k \in \mathbb{N}$, parameters $C, D > 1 \in \mathbb{R}$ and arbitrary parameter $M \in \mathbb{R}$, define a* box *as*

$$\mathrm{Box}_k(u) := \left\{ x \in \mathbb{R}^d : \|x-u\|_\infty \leq \frac{1}{2}\mathrm{e}^{MDC^k/d} \right\}. \tag{4.3}$$

Furthermore, we define the kth annulus *as*

$$\Gamma_k(u) := \mathrm{Box}_k(u) \setminus \mathrm{Box}_{k-1}(u), \ \textit{for } k \geq 1. \tag{4.4}$$

We pack each annulus $\Gamma_k(u)$ with as many disjoint sub-boxes

$$\mathrm{SB}_{k,i}(u) := \left\{ x \in \mathbb{R}^d : \|x-z_{k,i}\|_\infty \leq \frac{1}{2}\mathrm{e}^{MC^k/d} \right\} \tag{4.5}$$

as possible by choosing $z_{k,i}$ accordingly.

Observe that the kth box has radius $\exp(MDC^k/d)/2$ and volume $\exp(MDC^k)$, while the sub-boxes have radius $\exp(MC^k/d)/2$ and volume $\exp(MC^k)$. The centers $z_{k,i}$ of these sub-boxes must be chosen in a way to maximise the amount of sub-boxes in each annulus $\Gamma_k(u)$, but the exact choice of $z_{i,k}$ is not relevant. It must be noted however, that in general the sub-boxes will not perfectly fill up $\Gamma_k(u)$ and some volume will not be covered. Let b_k denote the maximal amount of sub-boxes in annulus $\Gamma_k(u)$ and order the sub-boxes in each annulus arbitrarily from 1 to b_k. We will refer to this setup of packing box-shaped annuli with sub-boxes as a *boxing system centered around* u. An example of a boxing system centered around the origin can be seen in Figure 2.

Lemma 4.5 (Bounds on Amount of Sub-boxes)**.** *Given a boxing system as described in Definition 4.4, the maximal amount of sub-boxes* b_k *in annulus* $\Gamma_k(u)$ *can be bounded by*

$$\frac{1}{2}\mathrm{e}^{M(D-1)C^k} \le b_k \le \mathrm{e}^{M(D-1)C^k}, \tag{4.6}$$

for all sufficiently large $M \ge M(C, D)$.

Proof. The upper bound can be easily seen by examining the ratio of the volume of the box $\mathrm{Box}_k(u)$ (so the volume of the annulus $\Gamma_k(u)$ including the middle) and the volume of a single sub-box. This yields

$$b_k \le \frac{\mathrm{Vol}(\mathrm{Box}_k(u))}{\mathrm{Vol}(\mathrm{SB}_{k,i}(u))} = \frac{\mathrm{e}^{MDC^k}}{\mathrm{e}^{MC^k}} = \mathrm{e}^{M(D-1)C^k}. \tag{4.7}$$

For the lower bound, we investigate the amount of sub-boxes for a particular partial covering of $\Gamma_k(u)$. If we start covering box $\mathrm{Box}_k(u)$, we can fit

$$O_B := \left\lfloor \frac{\mathrm{e}^{MDC^k/d}}{\mathrm{e}^{MC^k/d}} \right\rfloor^d = \left\lfloor \mathrm{e}^{M(D-1)C^k/d} \right\rfloor^d \tag{4.8}$$

sub-boxes in total. The 'inner' box $\mathrm{Box}_{k-1}(u)$ can overlap with at most

$$I_B := \left(\left\lceil \frac{\mathrm{e}^{MDC^{k-1}/d}}{\mathrm{e}^{MDC^k/d}} \right\rceil + 1 \right)^d = \left(\left\lceil \mathrm{e}^{-MDC^{k-1}(C-1)/d} \right\rceil + 1 \right)^d \tag{4.9}$$

of the sub-boxes used to cover $\mathrm{Box}_k(u)$. These particular sub-boxes must therefore be discarded. Using this with the amount of sub-boxes we fitted into the complete box $\mathrm{Box}_k(u)$, we can fit at least $O_B - I_B$ sub-boxes in $\Gamma_k(u)$. For sufficiently large M, we have that

$$O_B - I_B \ge (\mathrm{e}^{M(D-1)C^k/d} - 1)^d - (\mathrm{e}^{-MDC^{k-1}(C-1)/d} + 2)^d \ge \frac{1}{2}\mathrm{e}^{M(D-1)C^k}, \tag{4.10}$$

and thus (4.6) follows. □

We now define the leader vertices of each sub-box and bound their fitnesses from above and from below. Recall that we intend to construct the infinite path of finite cost by greedily adding a leader vertex from the next annulus to the path at each step. In the case of the cost model defined in Definition 3.7, the fitnesses add to the cost of the path, which might cause the total cost to become infinite if we are not careful. Hence we need to bound the fitnesses of the leaders from above. Bounding the fitnesses from below ensures that the leaders have enough neighbours to keep extending the path. Sufficiently many neighbours need to be available to ensure there is a 'cheap enough' option at each step, which keeps the total cost of the path finite. These upper and lower bounds are introduced as the concept of δ-goodness in Definition 4.7 below, and will prove to be crucial to our proofs.

Definition 4.6 (Leader Vertices)**.** *For each sub-box* $\mathrm{SB}_{k,i}(u)$ *containing at least one vertex, the* leader *vertex* $c_{k,i}$ *is defined to be the vertex in the sub-box of maximal fitness. In other words,*

$$c_{k,i} := \arg\max_{v \in \mathrm{SB}_{k,i}} \{W_v\}. \tag{4.11}$$

Definition 4.7 (δ-goodness)**.** *A sub-box* $\mathrm{SB}_{k,i}(u)$ *is called* δ-good *if it has a leader vertex* $c_{k,i}$ *with a fitness* $W_{c_{k,i}}$ *satisfying*

$$\exp\left(MC^k \frac{1-\delta}{\tau-1}\right) \leq W_{c_{k,i}} \leq \exp\left(MC^k \frac{1+\delta}{\tau-1}\right). \tag{4.12}$$

We will also call a leader vertex δ*-good if its fitness satisfies the above constraint.*

We will now present two auxiliary lemmas used in the proof of Lemma 4.10, which will be treated next.

Lemma 4.8 (Markov Bound on Poisson Random Rariable)**.** *Let* X *be a Poisson random variable with mean* μ*. Then it holds that*

$$\mathbb{P}\Big(|X-\mu| \geq \frac{1}{2}\mu\Big) \leq 2\mathrm{e}^{-\mu/15}. \tag{4.13}$$

Proof. The result of this lemma can be obtained by employing Markov's inequality twice. Firstly, we compute (for any $t \in \mathbb{R}^+$ and $\epsilon \in (0,1)$)

$$\mathbb{P}(X \geq (\epsilon+1)\mu) = \mathbb{P}(\mathrm{e}^{tX} \geq \mathrm{e}^{t(\epsilon+1)\mu}) \leq \frac{\mathbb{E}[\mathrm{e}^{tX}]}{\mathrm{e}^{t(\epsilon+1)\mu}}. \tag{4.14}$$

Here the last inequality holds due to Markov's inequality. Note that this expectation can now explicitly be calculated since it corresponds to the generating function of the Poisson random variable X. Therefore,

$$\mathbb{P}(X-\mu \geq \epsilon\mu) = \mathbb{P}(X \geq (\epsilon+1)\mu) \leq \exp(-t(\epsilon+1)\mu - \mu(1-\mathrm{e}^t)) = \exp(-\mu(t(1+\epsilon)+1-\mathrm{e}^t)). \tag{4.15}$$

Similarly,

$$\mathbb{P}(X \leq (1-\epsilon)\mu) = \mathbb{P}(\mathrm{e}^{-tX} \geq \mathrm{e}^{-t(1-\epsilon)\mu}) \leq \frac{\mathbb{E}[\mathrm{e}^{-tX}]}{\mathrm{e}^{-t(1-\epsilon)\mu}}. \tag{4.16}$$

Note that here $\epsilon < 1$ plays a role since we need that $(1-\epsilon)\mu$ is positive in order to employ the Markov bound. Thus we also find

$$\mathbb{P}(X-\mu \leq -\epsilon\mu) = \mathbb{P}(X \leq (1-\epsilon)\mu) \leq \exp(t(1-\epsilon)\mu - \mu(1-\mathrm{e}^{-t})) = \exp(-\mu(t(-1+\epsilon)+1-\mathrm{e}^{-t})). \tag{4.17}$$

Combining the bounds (4.15) and (4.17) and taking $\epsilon = t = 1/2$ yields

$$\mathbb{P}\Big(|X-\mu| \geq \frac{1}{2}\mu\Big) \leq \exp\left(-\mu\Big(\frac{7}{4} - \mathrm{e}^{1/2}\Big)\right) + \exp\left(\mu\Big(-\frac{3}{4} + \mathrm{e}^{-1/2}\Big)\right). \tag{4.18}$$

Observe that $7/4 - \mathrm{e}^{1/2} \approx 0.101 > 0$ and $-3/4 + \mathrm{e}^{-1/2} \approx -0.143 < 0$. Since for Poisson variables $\mu > 0$, we can combine the terms to see

$$\mathbb{P}\Big(|X-\mu| \geq \frac{1}{2}\mu\Big) \leq 2\exp\left(-\mu\Big(\frac{7}{4} - \mathrm{e}^{1/2}\Big)\right) \leq 2\exp\left(-\frac{\mu}{15}\right), \tag{4.19}$$

yielding (4.13). □

A similar results holds for binomial random variables.

Lemma 4.9 (Corollary 2.3 in [21])**.** *Let* X *be a binomial r.v. with mean* μ*. Then for all* $0 < \epsilon \leq 3/2$,

$$\mathbb{P}(|X-\mu| \geq \epsilon\mu) \leq 2e^{-\epsilon^2\mu/3}. \tag{4.20}$$

The following lemma uses the principle of δ-goodness introduced in Definition 4.7 to show that every δ-good leader has many δ-good leader neighbours in the next annulus. This lemma is crucial, as it guarantees the existence of infinite paths and will enable us to greedily construct a low-cost path.

Recall the definitions of leader vertices and δ-goodness from Definition 4.6 and Definition 4.7. The proof of this lemma is adapted from Lemma 4.3 in [25]. We define $N_j(c_{k,i})$ to be the number of δ-good leaders in $\Gamma_j(u)$ that are adjacent to leader $c_{k,i}$ in $\Gamma_k(u)$.

Lemma 4.10 (Fitnesses and Subgraph of Leaders)**.** *Consider* $\mathrm{IGIRG}_{W,L}(\lambda)$ *with parameters* $d \geq 1$, $\tau \in (2,3)$, $\alpha \in (1,\infty)$, *and* $0 < \lambda \leq 1$. *Let* $C, D > 1$ *and* $0 < \delta < 1$ *satisfy*

$$\frac{1-\delta}{\tau-1}(1+C) - DC > 0. \tag{4.21}$$

Consider a boxing system as defined in Definition 4.4. Furthermore, define the events

$$F_k^{(1)} := \Big\{ |i \in [b_k] : \mathrm{SB}_{k,i} \text{ is } \delta\text{-good } | \geq \frac{1}{2} b_k \Big\}, \text{ and} \tag{4.22}$$

$$F_k^{(2)} := \Big\{ \forall i \in [b_k] \text{ such that } \mathrm{SB}_{k,i} \text{ is } \delta\text{-good } : N_{k+1}(c_{k,i}) \geq \frac{p}{8} \mathrm{e}^{M(D-1)C^{k+1}} \Big\}. \tag{4.23}$$

Then there exists an $M_0 > 0$ *such that for all* $M > M_0$

$$\mathbb{P}(\neg \cap_{k\geq 0} (F_k^{(1)} \cap F_k^{(2)})) \leq 3 \exp\Big(- \frac{\lambda}{75} \mathrm{e}^{M \min(D-1,1)/2} \Big). \tag{4.24}$$

Proof. To investigate the probability of $F_k^{(1)}$, we first examine the event that there exists a sub-box with either too many or too few vertices. We denote the amount of vertices in sub-box $\mathrm{SB}_{k,i}$ by $V_{k,i}$. To be exact, we are interested in the probability

$$\mathbb{P}(\neg \mathcal{E}_k^1) := \mathbb{P}\Big(\exists i \leq b_k : V_{k,i} \notin \Big[\frac{1}{2}\lambda \mathrm{e}^{MC^k}, \frac{3}{2}\lambda \mathrm{e}^{MC^k} \Big] \Big). \tag{4.25}$$

Recall that by (4.5), each sub-box has volume $\exp(MC^k)$. Since we study an $\mathrm{IGIRG}_{W,L}$ graph, the vertices are distributed in $\mathbb{R}^d$ according to a Poisson point process with intensity λ. Therefore, the amount of vertices in a sub-box is a Poisson random variable with mean $\lambda \exp(MC^k)$. Thus, we can employ Lemma 4.8. Using a union bound over all sub-boxes and applying the upper bound on b_k from (4.6), this yields

$$\mathbb{P}(\neg \mathcal{E}_k^1) \leq 2 \exp(M(D-1)C^k) \exp\Big(- \frac{\lambda}{15} \mathrm{e}^{MC^k} \Big) \leq \exp\Big(- \frac{\lambda}{20} \mathrm{e}^{MC^k} \Big), \tag{4.26}$$

for sufficiently large M, since the second factor is doubly exponentially small in MC^k while the first is only exponential in MC^k. Now, to derive a bound on the probability of a sub-box not being δ-good, we first look at the error probability of the leader vertex's fitness being too large, conditioned on $\mathcal{E}_k^1$. Note that in this case, by (4.25), we have a deterministic lower bound on $V_{k,i}$. Also recall that since vertices are distributed according to a Poisson point process, the amount of vertices in each sub-box is independent of the amount of vertices in other sub-boxes. Hence we have that

$$\begin{aligned} \mathbb{P}(\max_{v \in \mathrm{SB}_{k,i} \cap \mathcal{V}} W_v \leq y \mid \mathcal{E}_k^1) &\leq (1 - \mathbb{P}(W_v > y))^{\lambda \exp(MC^k)/2} \leq \Big(1 - \frac{c_1}{y^{\tau-1}} \Big)^{\lambda \exp(MC^k)/2} \\ &\leq \exp\Big(- \frac{c_1}{2} y^{-(\tau-1)} \lambda \mathrm{e}^{MC^k} \Big). \end{aligned} \tag{4.27}$$

by Assumption 3.2. Taking y to be the lower bound of δ-goodness, as defined in Definition 4.7, yields

$$\begin{aligned} \mathbb{P}\Big(\max_{v \in \mathrm{SB}_{k,i} \cap \mathcal{V}} W_v \leq \exp\Big(MC^k \frac{1-\delta}{\tau-1} \Big) \Big| \mathcal{E}_k^1 \Big) &\leq \exp\Big(- \frac{c_1}{2} \Big(\exp\Big(MC^k \frac{1-\delta}{\tau-1} \Big) \Big)^{-(\tau-1)} \lambda \mathrm{e}^{MC^k} \Big) \\ &= \exp\Big(- \frac{c_1}{2} \lambda \mathrm{e}^{\delta MC^k} \Big). \end{aligned} \tag{4.28}$$

The maximum fitness can also be bounded from above. Employing a union bound and using the deterministic upper bound for $V_{k,i}$ given by (4.25), we have that, for all $y > 0$,

$$\mathbb{P}(\max_{v\in \mathrm{SB}_{k,i}\cap \mathcal{V}} W_v > y|\mathcal{E}_k^1) \leq \sum_{v\in \mathrm{SB}_{k,i}\cap \mathcal{V}} \mathbb{P}(W_v > y \mid \mathcal{E}_k^1) \leq \frac{3}{2}\lambda \mathrm{e}^{MC^k} c_2 y^{-(\tau-1)}, \tag{4.29}$$

again by Assumption 3.2. Similarly to above, we now take y to be the upper bound of δ-goodness. This gives the bound

$$\begin{aligned}\mathbb{P}\bigg(\max_{v\in \mathrm{SB}_{k,i}\cap \mathcal{V}} W_v > \exp\bigg(MC^k\frac{1+\delta}{\tau-1}\bigg)\bigg|\mathcal{E}_k^1\bigg) &\leq \frac{3}{2}\lambda \mathrm{e}^{MC^k} c_2\bigg(\exp\bigg(MC^k\frac{1+\delta}{\tau-1}\bigg)\bigg)^{-(\tau-1)} \\ &= \frac{3}{2}\lambda c_2 \mathrm{e}^{-\delta MC^k}.\end{aligned} \tag{4.30}$$

Combining bounds (4.28) and (4.30) yields

$$\mathbb{P}(\mathrm{SB}_{k,i} \text{ is not } \delta\text{-good} \mid \mathcal{E}_k^1) \leq \exp\bigg(-\frac{c_1}{2}\lambda \mathrm{e}^{\delta MC^k}\bigg) + \frac{3}{2}\lambda c_2 \mathrm{e}^{-\delta MC^k} \leq 3\lambda c_2 \mathrm{e}^{-\delta MC^k}, \tag{4.31}$$

for M sufficiently large. Observe that for all k and i, the event of $\mathrm{SB}_{k,i}$ being δ-good depends on the vertex fitnesses within $\mathrm{SB}_{k,i}$ and the amount of vertices only. Both the fitnesses and the amount of vertices are all mutually independent random variables. Therefore, conditioned on $\mathcal{E}_k^1$, the amount of δ-good sub-boxes in $\Gamma_k(u)$ is dominated below by a binomial r.v. with b_k trials and success probability $1-3\lambda c_2\exp(-\delta MC^k)$. By choosing $M \geq \log(12\lambda c_2)/(\delta C^k)$, we can ensure that this success probability is greater than or equal to 3/4. This enforces the inequality

$$\mathbb{P}(\neg F_k^{(1)} \mid \mathcal{E}_k^1) \leq \mathbb{P}\Big(X \leq \frac{1}{2}b_k\Big), \tag{4.32}$$

where $X \sim \mathrm{Bin}(b_k, 3/4)$. This can then be bounded further using a Chernoff bound as given in Lemma 4.9 with $\epsilon = 1/3$. Namely,

$$\mathbb{P}\Big(|X-3b_k/4| \geq b_k/4\Big) \leq 2\mathrm{e}^{-b_k/36} \implies \mathbb{P}\Big(X \leq \frac{1}{2}b_k\Big) \leq 2\mathrm{e}^{-b_k/36}. \tag{4.33}$$

We use this result in the computation of an upper bound for $\mathbb{P}(\neg F_k^{(1)})$ below. We have that

$$\begin{aligned}\mathbb{P}(\neg F_k^{(1)}) &= \mathbb{P}(\neg F_k^{(1)} \mid \neg\mathcal{E}_k^1)\mathbb{P}(\neg\mathcal{E}_k^1) + \mathbb{P}(\neg F_k^{(1)} \mid \mathcal{E}_k^1)\mathbb{P}(\mathcal{E}_k^1) \leq \mathbb{P}(\neg\mathcal{E}_k^1) + 2\mathrm{e}^{-b_k/36} \\ &\leq \exp(-\lambda \mathrm{e}^{MC^k}/20) + 2\exp(-\mathrm{e}^{M(D-1)C^k}/72) \leq 2\exp(-\lambda \mathrm{e}^{\min(D-1,1)MC^k}/74),\end{aligned} \tag{4.34}$$

for sufficiently large M, by (4.26) and the lower bound for b_k of Lemma 4.5. Thus, employing a union bound over k, when M is sufficiently large we have

$$\begin{aligned}\mathbb{P}(\cap_{k\geq 0} F_k^{(1)}) &\geq 1 - 2\sum_{k=0}^{\infty}\exp(-\lambda \mathrm{e}^{\min(D-1,1)MC^k}/74) \\ &\geq 1 - 2\exp(-\lambda \mathrm{e}^{\min(D-1,1)M}/75),\end{aligned} \tag{4.35}$$

since the sum decays faster than a geometric sum and is dominated by its first term.

For $F_k^{(2)}$, we condition on $\cap_{k\geq 0} F_k^{(1)}$. First, we analyse the connection probability between any δ-good leader vertex $c_{k,i}$ in $\Gamma_k(u)$ to any given δ-good leader vertex $c_{k+1,j}$ in $\Gamma_{k+1}(u)$. We denote the fitnesses of $c_{k,i}$ and $c_{k+1,j}$ by w_1 and w_2 respectively, and denote their locations in $\mathbb{R}^d$ by x_1 and x_2. Recall that by Assumption 3.3, the probability that $c_{k,i}$ and $c_{k+1,j}$ are connected satisfies

$$\mathbb{P}(c_{k,i} \leftrightarrow c_{k+1,j} \mid w_1, w_2, x_1, x_2) = p\cdot \min\bigg(1, \bigg(\frac{w_1 w_2}{\|x_1-x_2\|^d}\bigg)^{\alpha}\bigg). \tag{4.36}$$

Recall from Definition 4.4 that because $c_{k,i}$ and $c_{k+1,j}$ both lie in box $\mathrm{Box}_{k+1}(u)$, the distance between them satisfies $\|x_1 - x_2\| \le 2\sqrt{d}\exp(MDC^{k+1}/d)$. Moreover, we also know that both vertices are δ-good, thus they satisfy the fitness constraints from Definition 4.7. Therefore it follows that

$$\begin{aligned}\frac{w_1 w_2}{\|x_1 - x_2\|^d} &\ge \frac{1}{(2\sqrt{d})^d}\exp\left(MC^k\frac{1-\delta}{\tau-1} + MC^{k+1}\frac{1-\delta}{\tau-1} - MDC^{k+1}\right)\\ &= \frac{1}{(2\sqrt{d})^d}\exp\left(MC^k\left(\frac{1-\delta}{\tau-1}(1+C) - DC\right)\right).\end{aligned} \tag{4.37}$$

In the exponent, one can now recognise the condition (4.21). This means that the exponent of the derived lower bound is positive. Therefore, by taking M large enough, we can enforce $w_1 w_2 \ge \|x_1 - x_2\|^d$. In other words, by taking M large enough (since $\alpha > 1$),

$$\mathbb{P}(c_{k,i} \leftrightarrow c_{k+1,j} \mid w_1, w_2, x_1, x_2) = p. \tag{4.38}$$

Thus, conditioned on their fitnesses and locations, edges between δ-good leaders in neighbouring annuli are present independently with a fixed probability p. Let $\mathcal{W} := (x_v, W_v)_{v\in\mathcal{V}}$. Fix a $\mathcal{W}$ which implies $\cap_\ell F_\ell^{(1)}$. Then there are at least $b_{k+1}/2 \ge \exp(M(D-1)C^{k+1})/4$ δ-good leaders in $\Gamma_{k+1}(u)$ by Lemma 4.5. Hence, by (4.38), $N_{k+1}(c_{k,i})$ is dominated below by a binomial random variable with mean $p\exp(M(D-1)C^{k+1})/4$. Employing a Chernoff bound as in Lemma 4.9 with $\epsilon = 1/2$ yields

$$\mathbb{P}\left(N_{k+1}(c_{k,i}) \le p\mathrm{e}^{M(D-1)C^{k+1}}/8 \mid \mathcal{W}\right) \le 2\exp\left(-\frac{p}{48}\mathrm{e}^{M(D-1)C^{k+1}}\right). \tag{4.39}$$

Thus, by using a union bound over all δ-good sub-boxes $i \in [b_k]$ in annulus Γ_k

$$\begin{aligned}\mathbb{P}(\neg F_k^{(2)} \mid \mathcal{W}) &\le 2b_k\exp\left(-\frac{p}{48}\mathrm{e}^{M(D-1)C^{k+1}}\right)\\ &\le 2\exp(M(D-1)C^k)\exp\left(-\frac{p}{48}\mathrm{e}^{M(D-1)C^{k+1}}\right)\\ &\le \exp(-\lambda\mathrm{e}^{3M(D-1)C^{k+1}/4}),\end{aligned} \tag{4.40}$$

for M sufficiently large. Then, again by using a union bound, but this time over all $k \ge 0$,

$$\begin{aligned}\mathbb{P}(\neg\cap_{k\ge 0} F_k^{(2)} \mid \mathcal{W}) &\le \sum_{k=0}^{\infty}\exp(-\lambda\mathrm{e}^{3M(D-1)C^{k+1}/4})\\ &\le \exp(-\lambda\mathrm{e}^{M(D-1)/2}),\end{aligned} \tag{4.41}$$

again for M sufficiently large. Note that this bound holds for any $\mathcal{W}$ as long as $\cap_{k\ge 0}F_k^{(1)}$ holds. Finally, combining (4.35) and (4.41) and using a union bound, we get

$$\begin{aligned}\mathbb{P}(\neg\cap_{k\ge 0}(F_k^{(1)} \cap F_k^{(2)})) &\le \mathbb{P}(\neg\cap_{k\ge 0} F_k^{(1)}) + \mathbb{P}(\neg\cap_{k\ge 0} F_k^{(2)})\\ &\le 2\exp(-\lambda\mathrm{e}^{M\min(D-1,1)}/75) + \exp(-\lambda\mathrm{e}^{M(D-1)/2})\\ &\le 3\exp(-\lambda\mathrm{e}^{M\min(D-1,1)/2}/75),\end{aligned} \tag{4.42}$$

for sufficiently large M. □

With Lemma 4.10 in hand, we are now ready to construct an infinite greedy path under the conditions of this lemma. This greedy path is defined formally below in Definition 4.11. In Lemma 4.12, we then suppose that $\cap_{k\ge 0}(F_k^{(1)} \cap F_k^{(2)})$ occurs (with $F_k^{(1)}$ and $F_k^{(2)}$ as defined in (4.22)), and find an upper bound for the cost of the greedy path. This is only possible due to $\cap_{k\ge 0}(F_k^{(1)} \cap F_k^{(2)})$ occurring, as this guarantees that there are enough δ-good leader vertices to keep extending the path from one annulus to the next, while ensuring that the fitnesses of the vertices on the path are not large enough to drive the total cost to infinity. Lemma 4.12 is a modified version of Claim 4.6 in [25].

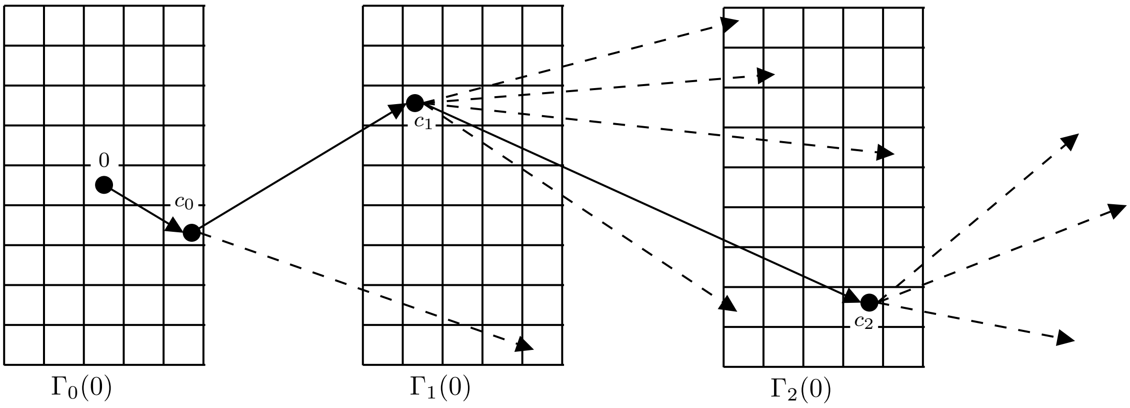


Figure 3: An example of a greedy path created by the approach of Definition 4.11. Annuli are drawn next to each other instead of nested in one another to simplify the picture. The path starts in the origin, after which an arbitrary δ-good leader c_0 in $\Gamma_0(0)$ is selected. Then, the path is greedily extended by choosing the adjacent δ-good leader in the next annulus which minimises $L_{c_k,c_{k+1}}$. There may be multiple options (as indicated by the dotted arrows), but there will always be at least one adjacent δ-good leader from the next annulus since $\cap_{k\geq 0}F_k^{(2)}$ is assumed to occur.

Definition 4.11 (Greedy Path Between δ-good Leaders)**.** *Consider a boxing system centered around $u \in \mathbb{R}^d$ as described in Definition 4.4. Condition on $\cap_{k\geq 0}(F_k^{(1)} \cap F_k^{(2)})$ and let c_0 be any δ-good leader in $\Gamma_0(u)$. This vertex can be greedily extended into an infinite path $\pi^{greedy} = (c_0, c_1, ...)$ as follows. Suppose $c_0, ..., c_k$ are given for some $k \geq 0$ and suppose furthermore that c_k is a δ-good leader. Since $F_k^{(2)}$ occurs, there must be at least one δ-good leader in $\Gamma_{k+1}(u)$ adjacent to c_k. Choose c_{k+1} to be the δ-good leader in $\Gamma_{k+1}(u)$ that minimises $L_{(c_k,c_{k+1})}$.*

An example of a greedy path constructed using Definition 4.11 can be found in Figure 3.

Lemma 4.12 (Cost of a Greedy Path)**.** *Let C, D, δ and M_0 as in Lemma 4.10 and let $(a_1, a_2, ...)$ be any strictly positive infinite sequence. Then for every $M \geq M_0$, with the boxing system as defined in Definition 4.4, the cost of the greedy path starting in a leader c_0 in $\Gamma_0(u)$ w.r.t. cost variables $\mathcal{C}_1$ as defined in Definition 3.6 satisfies*

$$|\pi^{greedy}|_{\mathcal{C}_1} \leq \sum_{k=0}^{\infty} a_k \cdot (2\sqrt{d}\mathrm{e}^{MDC^{k+1}/d})^{\zeta}, \tag{4.43}$$

with probability at least $1 - \sum_{k=0}^{\infty} \exp(-\lceil p\exp(M(D-1)C^{k+1})/8\rceil\, a_k^{\beta})$ conditioned on $\mathcal{V}$, $\{W_v : v \in \mathcal{V}\}$, $\cap_{k\geq 0}(F_k^{(1)} \cap F_k^{(2)})$, and the (unweighted) edge set of the graph. When we instead define the cost variables $\mathcal{C}_2$ as in Definition 3.7, then the error probability remains unchanged and the new bound equals

$$|\pi^{greedy}|_{\mathcal{C}_2} \leq \sum_{k=0}^{\infty} a_k \cdot (2\sqrt{d}\mathrm{e}^{MDC^{k+1}/d})^{\zeta} \left(MC^k \frac{1+\delta}{\tau-1}\right)^{\mu} \left(MC^{k+1} \frac{1+\delta}{\tau-1}\right)^{\nu}. \tag{4.44}$$

Proof. Since $\cap_{k\geq 0}F_k^{(2)}$ occurs, we have a lower bound of the number of δ-good leader neighbours $N_{k+1}(c_k)$. We denote this lower bound by d_k and have $d_k = \lceil p\exp(M(D-1)C^{k+1})/8\rceil$. Leader c_k may be adjacent to more δ-good leaders in $\Gamma_{k+1}(0)$, but they can be ignored here since we are investigating the minimum. Using Definition 3.6, the total cost of π^{greedy} is bounded above by

$$|\pi^{greedy}|_{\mathcal{C}_1} \leq \sum_{k=0}^{\infty} \min(L_{k,1}, ..., L_{k,d_k}) \cdot \|x_{c_k} - x_{c_{k+1}}\|^{\zeta}, \tag{4.45}$$

where $(L_{k,i})_{i\leq d_k}$ are the weights of the first d_k edges from c_k to δ-good leaders in $\Gamma_{k+1}(0)$ (ordered arbitrarily). These are i.i.d. copies of the random variable L, as seen in Definition 3.5. Recall that by (4.4), since c_k and c_{k+1} are in the same box, the length of the edge (c_k, c_{k+1}) can be bounded above by $2\sqrt{d}\exp(MDC^{k+1}/d)$.

We turn to examining $\min(L_{k,1}, ..., L_{k,d_k})$ in (4.45). We have that, for i.i.d. $L_{k,i}$,

$$\mathbb{P}\Big(\min_{1\le i\le d_k} L_{k,i} \ge a_k\Big) = (1-\mathbb{P}(L\le a_k))^{d_k} = (1-a_k^{\beta})^{d_k} \le \exp(-d_k a_k^{\beta}), \tag{4.46}$$

where we took $a_k < 1$ (since otherwise $\mathbb{P}(L \le a_k) = 1$ and the upper bound trivially holds) and used d_k as a lower bound for the amount of δ-good neighbours of c_k in $\Gamma_{k+1}(0)$. Filling in d_k, this gives

$$\mathbb{P}\Big(\min_{1\le i\le \lceil p\exp(M(D-1)C^{k+1})/8\rceil} L_{k,i} \ge a_k\Big) \le \exp\Big(-\Big\lceil \frac{p}{8}\mathrm{e}^{M(D-1)C^{k+1}}\Big\rceil a_k^{\beta}\Big). \tag{4.47}$$

Thus, a union bound implies that

$$|\pi^{greedy}|_{\mathcal{C}_1} \le \sum_{k=0}^{\infty} a_k \cdot (2\sqrt{d}\mathrm{e}^{MDC^{k+1}/d})^{\zeta} \tag{4.48}$$

occurs with probability at least $1-\sum_{k=0}^{\infty}\exp(-\lceil p\exp(M(D-1)C^{k+1})/8\rceil a_k^{\beta})$. The case where we use the cost of spread function of Definition 3.7 instead is almost analogous, the only difference being that in the costs, the factors involving vertex fitnesses can be bounded above by the upper bound from δ-goodness, as defined in Definition 4.7. □

The next claim is the final preparation for the proofs of Theorem 4.1 and Theorem 4.2. It specifies the choice of parameters $C, D > 1$ and δ in our boxing system such that they satisfy (4.21) of Lemma 4.10 and ensure that the greedy path we will construct has finite total cost. This claim essentially fulfills the role of Claim 4.4 in [25].

Claim 4.13. *Let $\tau \in (2,3)$, $\mu,\nu \ge 0$, $\beta,\zeta > 0$, $d \ge 1$ and suppose that $\beta(\mu+\nu+2\zeta/d) < 3-\tau$. For $\delta > 0$ sufficiently small, the interval given by*

$$\mathcal{I}_D := \Big(\Big(d+\beta d\Big(\frac{1+\delta}{\tau-1}\Big)\Big(\frac{\mu}{C}+\nu\Big)\Big)(d-\zeta\beta)^{-1},\quad \frac{1-\delta}{\tau-1}\frac{1+C}{C}\Big) \tag{4.49}$$

with $C := 1+\delta$ is non-empty and satisfies $C, D > 1$ for $D \in \mathcal{I}_D$. Moreover, the condition $\beta(\mu+\nu+2\zeta/d) < 3-\tau$ is the mildest condition which guarantees that $\mathcal{I}_D$ is non-empty.

Proof. First, note that $\beta(\mu+\nu+2\zeta/d) < 3-\tau$ implies $\beta\zeta < d$ since $\tau \in (2,3)$. Thus the lower boundary of $\mathcal{I}_D$ is greater than or equal to 1, since d plus some non-negative term is divided by a positive term strictly smaller than d. A quick calculation shows that the upper boundary of $\mathcal{I}_D$ is greater than 1 as $\delta \to 0$ if and only if $C < 1/(\tau-2)$. We will now investigate when the interval given by (4.49) is non-empty and translate this into a requirement for C when $\delta \to 0$. So we require

$$\begin{aligned}\Big(d+\beta d(\tau-1)^{-1}\Big(\frac{\mu}{C}+\nu\Big)\Big)(d-\zeta\beta)^{-1} &< \frac{1}{\tau-1}\Big(1+\frac{1}{C}\Big)\\ \iff C(\beta d\nu+(\tau-1)d+(\zeta\beta-d)) &< (d-\zeta\beta)-\beta d\mu.\end{aligned} \tag{4.50}$$

This cannot directly be converted into an upper bound or lower bound for C, since it is not clear whether $(\beta d\nu+(\tau-1)d+(\zeta\beta-d))$ is positive or not. However, this does signal that the condition for C is linear. So, since C also has to satisfy $C > 1$ and $C < 1/(\tau-2)$, taking C close to one of these two bounds should be optimal.

First we investigate what happens if we maximise the upper boundary of the interval $\mathcal{I}_D$ by choosing C as small as possible. We choose $C := 1+\delta$ in (4.50) and get for $\delta \to 0$ that

$$\frac{d+\beta d(\tau-1)^{-1}(\mu+\nu)}{d-\zeta\beta} < \frac{2}{\tau-1} \iff \beta\Big(\mu+\nu+2\frac{\zeta}{d}\Big) < (3-\tau). \tag{4.51}$$

Hence indeed $\mathcal{I}_D$ is non-empty under the conditions of the claim. What remains is to investigate whether this is the mildest condition which guarantees the results of the claim. Therefore we now investigate the other option for the choice of C.

We can instead take C to be maximal, so $C \to 1/(\tau-2)$, to minimise the lower bound of the interval $\mathcal{I}_D$, then we get

$$\frac{d + \beta d(\tau-1)^{-1}((\tau-2)\mu+\nu)}{d-\zeta\beta} < \frac{1}{\tau-1}\frac{1+(\tau-2)^{-1}}{(\tau-2)^{-1}} = 1. \tag{4.52}$$

Immediately, we can observe that the upper boundary of $\mathcal{I}_D$ gets arbitrarily close to 1 for this choice of C (as $C \to 1/(\tau-2)$). Manipulating the above inequality further yields

$$\frac{\zeta}{d} < -\frac{(\tau-2)\mu+\nu}{\tau-1} \tag{4.53}$$

Note that above, the factor β has dropped out completely and the right-hand side is negative since $\tau \in (2,3)$. Since we require $d, \zeta > 0$, this does not give viable solutions for D. Hence the former case where we choose $C \to 1$ is optimal and the mildest condition for which the claim holds. □

With our preparatory lemmas Lemma 4.10 and Lemma 4.12 ready, we can now provide the proof of Theorem 4.1 and Theorem 4.2. We will first need to ensure that suitable parameters are chosen, such that we can actually use Lemma 4.10 and Lemma 4.12 while retaining a small error probability. Then, we greedily construct an infinite path and show that the cost is finite, which implies lengthwise explosion. The proofs of Theorem 4.1 and Theorem 4.2 will be similar to the proof of Theorem 3.6(ii) in [25], but again adapted to our model.

Proof of Theorem 4.1. Firstly, we investigate which assumptions we need on parameters C, D, ζ and τ to use the lemmas we have prepared so far. Recall that parameters $C, D > 1$ were used to construct the boxing system as defined in Definition 4.4, ζ is involved in the cost of spread in Definition 3.6 and Definition 3.7, and τ is used in the power-law distribution of the vertex fitnesses, as described in Assumption 3.2. First, we need to choose a suitable sequence a_k in Lemma 4.12 such that the error probability is summable. To achieve this, we choose

$$a_k = \left(\frac{1}{\lceil p\mathrm{e}^{M(D-1)C^{k+1}}/8\rceil}\log(k^2)\right)^{1/\beta}. \tag{4.54}$$

Using this a_k, the probability $1-\sum_{k=0}^{\infty}\exp(-\lceil p\exp(M(D-1)C^{k+1}/8\rceil\, a_k^{\beta})$ is still summable. Namely,

$$\sum_{k=0}^{\infty}\exp\left(-\left\lceil\frac{p}{8}\mathrm{e}^{M(D-1)C^{k+1}}\right\rceil a_k^{\beta}\right) \le \sum_{k=0}^{\infty}\frac{1}{k^2} < \infty. \tag{4.55}$$

Thus, by the Borel-Cantelli Lemma, with probability 1, for all k larger than a random, finite k^*, each $\min_{1\le i\le d_k}(L_{k,i})$ will be greater than or equal to a_k. These first k^* terms will only add finite contributions to the sum in (4.43). Below, we will denote these contributions by C_{k^*}.

For now, consider a boxing system where the event $\cap_{k\ge0}(F_k^{(1)} \cap F_k^{(2)})$ occurs. Suppose we aim to construct a greedy path π^{greedy} as described in Definition 4.11. Let c_0 be an arbitrary δ-good leader in $\Gamma_0(0)$ (which exists since $F_0^{(1)}$ occurs). With positive probability, either $c_0 = 0$ or there exists an edge from 0 to c_0. Suppose there is an edge from 0 to c_0 (the case $c_0 = 0$ is identical). We construct the greedy path with initial vertex c_0 and setting $\pi^0 := (0, \pi^{greedy})$. Then, using

Lemma 4.12 and filling in a_k as defined in (4.54), the cost of this path is bounded by

$$\begin{aligned}|\pi^0|_{\mathcal{C}} &\le C_{(0,c_0)} + C_{k^*} + \sum_{k=k^*}^{\infty} a_k (2\sqrt{d} \mathrm{e}^{MDC^{k+1}/d})^{\zeta} \\ &= C_{(0,c_0)} + C_{k^*} + \sum_{k=k^*}^{\infty} \left(\left(\left\lceil \frac{p}{8} \mathrm{e}^{M(D-1)C^{k+1}} \right\rceil \right)^{-1} \log(k^2) \right)^{1/\beta} (2\sqrt{d})^{\zeta} \mathrm{e}^{\zeta MDC^{k+1}/d} \\ &\le C_{(0,c_0)} + C_{k^*} + (2\sqrt{d})^{\zeta} \left(\frac{8}{p} \right)^{1/\beta} \sum_{k=k^*}^{\infty} \log(k^2)^{1/\beta} \exp\left(\zeta MC^{k+1} D/d - \frac{1}{\beta} M(D-1)C^{k+1} \right).\end{aligned} \tag{4.56}$$

For convergence of the sum, we need a negative exponent. This imposes the condition

$$\zeta D/d - \frac{1}{\beta}(D-1) < 0 \iff D > \frac{d}{d - \beta\zeta}. \tag{4.57}$$

Note that this uses $\beta\zeta < d$ since $D, d > 0$. However, we can rewrite (4.1) as $\beta\zeta/d < (3-\tau)/2$, which implies $\beta\zeta < d$, since $\tau \in (2,3)$. Thus, this condition is always satisfied under the assumptions of the theorem. In addition to this condition, we have the condition (4.21) from Lemma 4.10. Combining these bounds for D yields

$$D \in \left(\frac{d}{d-\beta\zeta}, \quad \frac{1-\delta}{(\tau-1)C}(1+C) \right) \quad \text{and} \quad D > 1. \tag{4.58}$$

For valid D to exist, we need to choose a C such that the interval for D given by (4.58) is non-empty. By Claim 4.13 with $\mu = \nu = 0$, valid choices for C and D exist for δ sufficiently small, given that $2\beta\zeta/d < 3 - \tau$. Given that this condition holds and we choose δ, C and D accordingly, we are guaranteed that our original conditions (4.57) and (4.21) are satisfied, which means that the exponent of the exponential in (4.56) is negative and we are allowed to apply Lemma 4.10. Using Lemma 4.10, we can again use the Borel-Cantelli Lemma to show that there exists a boxing system in which $\cap_{k\ge0}(F_k^{(1)} \cap F_k^{(2)})$ occurs, which in turn allows us to apply Lemma 4.12 and the calculations above to show that the cost of the greedy path we constructed is finite.
We choose C, D and δ according to Claim 4.13. The conditions for Lemma 4.10 are now satisfied, and we let M_0 be as in the lemma statement. Define $M_i = M_0 + i$ for all $i > 0$ and construct infinitely many boxing systems around 0 with parameters C, D and M_i. Note that taking $M = M_i$ in (4.24), the right-hand side is summable. By Lemma 4.10 and the Borel-Cantelli Lemma, there exists an i_0 such that for all $i \ge i_0$, the event $\cap_{k\ge0}(F_k^{(1)} \cap F_k^{(2)})$ occurs. Therefore, by our earlier calculations above, the cost of the constructed path π^0 is finite with positive probability. Thus, we have shown that lengthwise explosion occurs with positive probability. □

We can now also prove Theorem 4.2 in a very similar manner.

Proof of Theorem 4.2. We can proceed exactly as in the proof of Theorem 4.1. We choose the same a_k from (4.54) in Lemma 4.12, such that we again get that there is only a finite contribution $C^w_{k^*}$ by terms not satisfying the bound on $\min(L_{k,1}, ..., L_{k,d_k})$. Repeating all of our previous calculations and now using the result (4.44) of Lemma 4.12 instead, the new cost of the greedy path is bounded by

$$|\pi^0|_{\mathcal{C}} \le C^w_{(0,c_0)} + C^w_{k^*} + \sum_{k=k^*}^{\infty} a_k (2\sqrt{d} \mathrm{e}^{MDC^{k+1}/d})^{\zeta} \left(MC^k \frac{1+\delta}{\tau-1} \right)^{\mu} \left(MC^{k+1} \frac{1+\delta}{\tau-1} \right)^{\nu}. \tag{4.59}$$

Again, only the finiteness of the sum is relevant. Filling in our chosen a_k from (4.54), we get

$$\sum_{k=0}^{\infty} \log(k^2)^{1/\beta} \exp\left(-\frac{1}{\beta} M(D-1)C^{k+1} + \frac{1}{d}\zeta MDC^{k+1} + MC^k \frac{1+\delta}{\tau-1}\mu + MC^{k+1}\frac{1+\delta}{\tau-1}\nu \right). \tag{4.60}$$

This sum if finite if and only if the exponent is less than zero. Thus the relevant inequality is

$$\begin{aligned}-\frac{1}{\beta}M(D-1)C^{k+1}+\frac{1}{d}\zeta MDC^{k+1}+MC^{k}\frac{1+\delta}{\tau-1}\mu+MC^{k+1}\frac{1+\delta}{\tau-1}\nu<0\\ \iff -\frac{1}{\beta}(D-1)+\frac{1}{d}\zeta D+\frac{1+\delta}{\tau-1}\left(\frac{1}{C}\mu+\nu\right)<0.\end{aligned} \tag{4.61}$$

Similarly to the proof of Theorem 4.2, we can already see that we again need $\beta\zeta < d$ to have any solutions. Again, this condition is already fulfilled by the theorem's assumptions. Namely, we can rewrite (4.2) as $\beta\zeta/d < (3-\tau)/2 - \mu\beta/2 - \nu\beta/2 < (3-\tau)/2$, which implies $\beta\zeta < d$. Continuing, (4.61) can be manipulated further to find

$$\frac{1}{\beta}+\left(\frac{1+\delta}{\tau-1}\right)\left(\frac{\mu}{C}+\nu\right)<D\left(\frac{1}{\beta}-\frac{\zeta}{d}\right)\iff\left(d+\beta d\left(\frac{1+\delta}{\tau-1}\right)\left(\frac{\mu}{C}+\nu\right)\right)(d-\zeta\beta)^{-1}<D. \tag{4.62}$$

Furthermore, as in Theorem 4.1, we need to satisfy condition (4.21) from Lemma 4.10 as well. This precisely yields that D should be in the interval given by Claim 4.13. Given that $\beta(\mu+\nu+2\zeta/d) < 3-\tau$, we can therefore choose $C, D > 1$ such that $D \in \mathcal{I}_D$. This choice would then allow us to apply Lemma 4.10 while the exponent in (4.60) is negative. Choose C, D and δ according to Claim 4.13. Then we can proceed as in the proof of Theorem 4.1 and conclude that lengthwise explosion occurs with positive probability. □

5 Conservativeness

In this section we will provide the counterpart of Theorems 4.1 and 4.2. We will show that under certain conditions on parameters β, ζ and τ and dimension d, we can prove that the $\mathrm{IGIRG}_{W,L}(\lambda)$ is conservative, i.e, explosion with respect to the cost models defined in Definition 3.6 and Definition 3.7 does not occur. The resulting theorem, Theorem 5.1, applies to both cost definitions. (This can be seen by taking $\mu = \nu = 0$ in the theorem).

Theorem 5.1 (Conservativeness Theorem for Weighted Cost)**.** *Consider the model* $\mathrm{IGIRG}_{W,L}(\lambda)$ *with* $0 < \lambda \le 1$. *Suppose that the vertex fitness distribution satisfies Assumption 3.2, the connection probability satisfies Assumption 3.3 and the cost of spread is as defined in Definition 3.7. Furthermore, let parameters* β, ζ, μ *and* ν *satisfy*

$$3-\tau < \beta\left(\mu+\nu+2\frac{\zeta}{d}\right). \tag{5.1}$$

Then the model is conservative. Note that if $d/\zeta \le \beta$*, this condition is trivially satisfied.*

Heuristic idea of the proof of Theorem 5.1. We will rule out sideways explosion and focus on proving that lengthwise explosion does not occur. It is sufficient to show that for some $t_0 < 1$, the probability of having an infinite path starting from 0 with total cost in $[0, t_0]$ is zero. The fact that this is sufficient is not trivial, but we will see that this is the case in Lemma 5.5. We prove that there does not exist any infinite path starting from 0 which only uses edges of cost smaller than t_0 instead, which is a stronger statement. To do this, we employ a path-counting argument in which we count the amount of self-avoiding paths of edges with cost smaller than $t_0 < 1$ emanating from the origin. We show that there exist exponentially decaying upper bounds on the expected number of such paths. Using these bounds, we can then use a suitably chosen t_0 in combination with the Borel-Cantelli Lemma to show that almost surely, no infinite paths only using edges of cost smaller than t_0 exist. This proves that the model is conservative.

First, we will formalise the notion of self-avoiding paths emanating from 0.

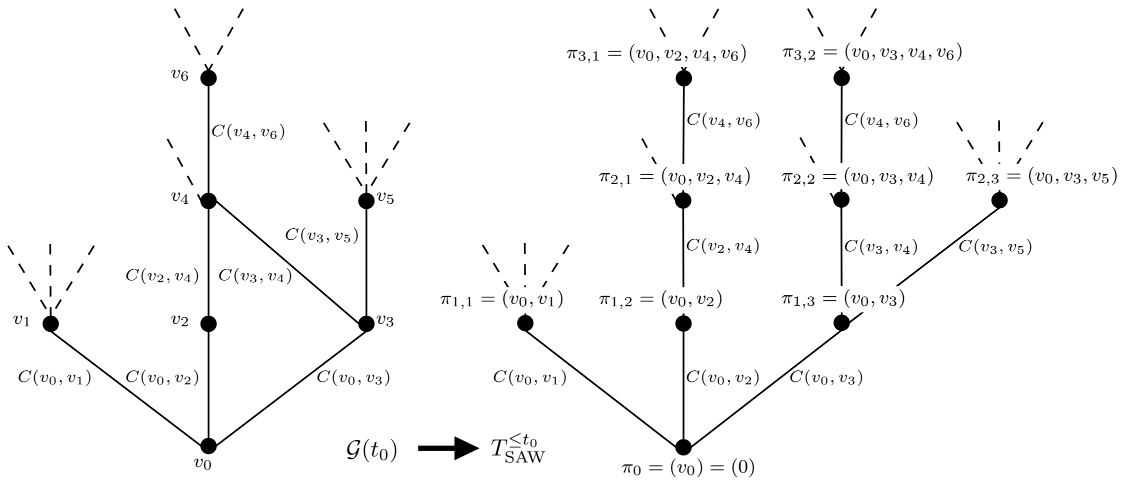


Figure 4: Example of the construction of a self-avoiding walk tree $T_{\text{SAW}}^{\leq t_0}$ as defined in Definition 5.2. On the left is part of the graph $\mathcal{G}(t_0)$. Recall that each edge (u, v) in this graph has cost $C(u, v) \leq t_0$. On the right is the resulting self-avoiding walk tree. Each vertex corresponds to a path in $\mathcal{G}(t_0)$. The paths of vertices at deeper levels of the self-avoiding walk tree are continuations of paths earlier in the tree.

Definition 5.2 (Self-avoiding Walk Tree). *Let the graph* $\text{IGIRG}_{W,L}(\lambda)$ *and a collection of costs* $\mathcal{C} = \{C_{(u,v)} : (u, v) \in \mathcal{E}\}$ *on this graph be given. Let the subgraph* $\mathcal{G}(t_0)$ *of* $\text{IGIRG}_{W,L}(\lambda)$ *be defined by removing all edges of cost greater than* t_0 *from* $\text{IGIRG}_{W,L}(\lambda)$*. We define the self-avoiding walk tree* $T_{\text{SAW}}^{\leq t_0}$ *of* $\mathcal{G}(t_0)$ *as follows. The root of* $T_{\text{SAW}}^{\leq t_0}$ *is the trivial path* $\pi_0 = (v_0)$*, consisting of only the root vertex* $v_0 = 0$*. The direct children of the root of* $T_{\text{SAW}}^{\leq t_0}$ *make up the paths of length* 1 *of the form* $\pi_1 = (v_0, v_1)$*. Here, we set the cost of the edge between* π_0 *and* π_1 *to be* $C_{(v_0,v_1)}$*. In general, every vertex of* $T_{\text{SAW}}^{\leq t_0}$ *is a finite, simple path in* $\mathcal{G}(t_0)$ *emanating from* $v_0 = 0$*. A path* $\pi_k = (v_0, ..., v_k)$ *in the* k*th level of the tree is connected to a path* $\pi'_{k+1} = (v'_0, ..., v'_{k+1})$ *in the* $(k+1)$*th level of the tree if and only if* π'_{k+1} *is a continuation of* π_k*. Thus, if and only if* $v_i = v'_i$ *for all* $i \leq k$*. The cost of the edge between* π_k *and* π_{k+1} *in* $T_{\text{SAW}}^{\leq t_0}$ *is then set to* $C_{(v_k, v'_{k+1})}$*. Note that by construction, the cost-distance of any path* π_k *from* π_0 *within* $T_{\text{SAW}}^{\leq t_0}$ *equals* $|\pi_k|_{\mathcal{C}}$*, the cost of the path itself in* $\mathcal{G}(t_0)$ *and in* $\text{IGIRG}_{W,L}(\lambda)$*.*

An example of a self-avoiding walk tree can be found in Figure 4.

Definition 5.3 (Number of Neighbours and Paths Abiding Cost Constraints). *Let the graph* $\text{IGIRG}_{W,L}(\lambda)$ *and a collection of costs* $\mathcal{C} = \{C_{(u,v)} : (u, v) \in \mathcal{E}\}$ *on this graph be given. For* $u \in \mathcal{V}$ *and* $t \in \mathbb{R}^+$ *we define*

$$N_1^t(u) := |\{(u, v) \in \mathcal{E}_\lambda : C_{(u,v)} \leq t\}|$$

as the number of neighbours of v *such that the edges leading to these neighbours have a cost of at most* t*. More generally, for* $k \geq 1$*, we define* $N_k^{t_0}(0)$ *to be the number of vertices in the* k*th level of* $T_{\text{SAW}}^{\leq t_0}$ *defined in Definition 5.2.*

Observe that $N_k^{t_0}(0)$ corresponds to the number of paths of length k emanating from 0 where each edge has cost less than or equal to t_0. Next, we will introduce Lemma 6.1 from [25], which will help us with ruling out sideways explosion in the final proof. Recall that in sideways explosion, there exists a finite path from the origin to a vertex from which there are infinitely many incident edges of bounded cost. Lemma 5.4 will show that whenever sideways explosion occurs in the $\text{IGIRG}_{W,L}(\lambda)$ model, then the origin itself already has infinitely many incident edges of bounded cost with positive probability. In addition to this, we will also introduce Lemma 6.2 from [25], which shows that if explosion occurs, it happens arbitrarily fast. In-depth proofs for these two lemmas can be found in [25]. Recall that for lengthwise explosion, we defined the explosion time as in Definition 3.8. We argue that for lengthwise explosion, there exists an infinite path $\pi = (\pi_0 = u, \pi_1, \pi_2, ...)$ with total cost $|\pi|_\mathcal{C} < \infty$, i.e,

$$\widetilde{Y}_\mathcal{C}(u) := \inf_{\pi:\pi_0=u,|\pi|=\infty} \{|\pi|_\mathcal{C}\} < \infty. \tag{5.2}$$

We note that in IGIRG the infinite component is unique [28]. Furthermore, a consequence of Lemma 4.10 is that the infinite component exists.

Lemma 5.4 (Lemma 6.1 in [25]). *Consider* $\mathrm{IGIRG}_{W,L}(\lambda)$ *with parameters* $d \geq 1, \tau > 1, \alpha \in (1, \infty]$. *Consider a cost of spread* $\mathcal{C}$ *as described in Definition 3.6 or Definition 3.7 such that for all* $t < \infty$, $\mathbb{P}(N_1^t(0) < \infty) = 1$ *holds. Then sideways explosion almost surely does not happen. Moreover, if* $\tau \in (1, 3)$, *then for any vertex* v *in the infinite component,* $\widetilde{Y}_\mathcal{C}(v) = Y_\mathcal{C}(v)$ *is realised via (at least one) infinite path* $\pi_{opt}(v)$.

Lemma 5.5 (Lemma 6.2 in [25]). *Consider* $\mathrm{IGIRG}_{W,L}(\lambda)$ *with a cost of spread* $\mathcal{C}$ *as described in Definition 3.6 or Definition 3.7, such that explosion occurs with positive probability, but that for all* $t < \infty$, $\mathbb{P}(N_1^t(0) < \infty) = 1$. *Then for all constant* $t_0 > 0$, *with strictly positive probability there is an infinite path from the origin with cost at most* t_0.

The following lemma will be the backbone of our proof for Theorem 5.1 and will employ the path-counting argument we introduced earlier. Lemma 5.7 follows the structure of Lemma 6.3 in [25]. However, because our model differs, the proof had to be adapted. Interestingly, the condition in (5.5) in Lemma 5.7 mirrors condition (4.2) in Theorem 4.2. Furthermore, recall that by Definition 3.5, we have that the edge weights L satisfy $F_L(t) = \min(1, t^\beta)$. Before we get to the lemma, however, we first state the following claim which will aid us in proving part of the lemma.

Claim 5.6 (Finite Expectations of W). *Let the vertex fitness* W *satisfy Assumption 3.2. Moreover, assume that* $W \geq 1$. *Then* $\mathbb{E}[W^\gamma] < \infty$ *for all* $\gamma < \tau - 1$.

Proof. Let $\gamma < \tau - 1$ be given. Then, using the law of the unconscious statistician and integration by parts, we find that

$$\begin{aligned}\mathbb{E}[W^\gamma] &= \int_1^\infty w^\gamma f_W(w)\mathrm{d}w \\ &= [w^\gamma(1 - \mathbb{P}(W \geq w)]_1^\infty - \int_1^\infty \gamma w^{\gamma-1}(1 - \mathbb{P}(W \geq w))\mathrm{d}w.\end{aligned} \tag{5.3}$$

Now we apply Assumption 3.2 to bound $\mathbb{P}(W \geq w)$. For the second term, the sign of γ determines whether c_1 or c_2 should be used to bound $\mathbb{P}(W \geq w)$. Since this is not relevant for the finiteness of the result, denote the optimal choice by c. We then get

$$\begin{aligned}\mathbb{E}[W^\gamma] &\leq [w^\gamma(1 - c_1 w^{1-\tau}]_1^\infty - \int_1^\infty \gamma w^{\gamma-1}(1 - cw^{1-\tau})\mathrm{d}w \\ &= \left[\left(- c_1 + \frac{c\gamma}{\gamma + 1 - \tau}\right) w^{\gamma+1-\tau}\right]_1^\infty,\end{aligned} \tag{5.4}$$

which is finite precisely when $\gamma < \tau - 1$. ☐

With this claim in hand, we are ready to formulate Lemma 5.7.

Lemma 5.7. *Let* $\alpha \in (1, \infty)$. *Consider the model* $\mathrm{IGIRG}_{W,L}(\lambda)$ *with* $0 < \lambda \leq 1$. *Suppose that the vertex fitness distribution is as defined in Assumption 3.2, the connection probability as defined in Assumption 3.3 and the cost of spread* $\mathcal{C}$ *as defined in Definition 3.7. Let*

1. $\mathbb{E}[W^{2-\beta(\mu+\nu+2\zeta/d)}] < \infty$ *and* $\mathbb{E}[W^{1-\beta(\nu+\zeta/d)}] < \infty$ *when* $d/\zeta > \beta$;
2. $\mathbb{E}[W^{-(\mu+\nu)d/\zeta}] < \infty$ *and* $\mathbb{E}[W^{-\nu d/\zeta}] < \infty$ *when* $d/\zeta \leq \beta$.

Let $t_0 < 1$. *Then, for some constant* $C^* > 0$,

a. $\mathbb{E}[N_k^{t_0}(0) \mid W_0 = w_0] \leq (\lambda p C^* t_0^\beta)^k w_0^{1-\beta\mu-\beta\zeta/d} \mathbb{E}[W^{2-\beta(\mu+\nu+2\zeta/d)}]^{k-1} \mathbb{E}[W^{1-\beta(\nu+\zeta/d)}]$ *if* $d/\zeta > \beta$;

b. $\mathbb{E}[N_k^{t_0}(0) \mid W_0 = w_0] \le (\lambda p C^* t_0^{d/\zeta})^k w_0^{-\mu d/\zeta} \mathbb{E}[W^{-(\mu+\nu)d/\zeta}]^{k-1} \mathbb{E}[W^{-\nu d/\zeta}]$ *if* $d/\zeta \le \beta$.

The requirement that the expectations in (1) and (2) are finite is satisfied if

$$3 - \tau < \beta\Big(\mu + \nu + 2\frac{\zeta}{d}\Big). \tag{5.5}$$

Note that if $d/\zeta \le \beta$*, this condition is trivially satisfied.*

Before we start the proof, we first provide an explanation as to why the expectations in the upper bounds are finite, provided that (5.5) holds. By (5.5), we have that $2 - \beta(\mu + \nu - 2\zeta/d) < \tau - 1$. Thus, by Claim 5.6, the expectation $\mathbb{E}[W^{2-\beta(\mu+\nu-2\zeta/d)}]$ is finite. For $\mathbb{E}[W^{1-\beta(\nu-\zeta/d)}]$ to be finite, we need $1 - \beta(\nu + \zeta/d) < \tau - 1$. This is trivially satisfied for $\tau \in (2, 3)$, as it can be rewritten as $2 - \tau < \beta(\nu + \zeta/d)$. So $\mathbb{E}[W^{1-\beta(\nu-\zeta/d)}] < \infty$, again by Claim 5.6. For $\mathbb{E}[W^{-(\mu+\nu)d/\zeta}]$ and $\mathbb{E}[W^{-\nu d/\zeta}]$, it also follows from Claim 5.6 that they are finite.

Proof of Lemma 5.7. First, we introduce the shorthand notation $n_k^{t_0}(0, w_0) := \mathbb{E}[N_k^{t_0}(0) \mid W_0 = w_0]$. Furthermore, let $\mathcal{V}_\lambda^{(k)} := \{(v_i)_{1\le i\le k} \in \mathcal{V}_\lambda \mid v_i \neq v_j \text{ for } 1 \le i < j \le k\}$ be the set of all k-tuples of distinct vertices of the Poisson point process $\mathcal{V}_\lambda$. Expanding $N_k^{t_0}(0)$ by using its definition and summing over all k-tuples gives

$$\begin{aligned} n_k^{t_0}(0, w_0) &= \mathbb{E}\Bigg[\sum_{(v_i)_{i\le k}\in\mathcal{V}_\lambda^{(k)}} \mathbb{1}\{\forall i \le k : v_i \leftrightarrow v_{i-1}, C^w_{(v_{i-1},v_i)} \in [0, t_0]\}\Bigg] \\ &= \mathbb{E}\Bigg[\sum_{(v_i)_{i\le k}\in\mathcal{V}_\lambda^{(k)}} \int_{(w_i)_{i\le k}} \mathbb{1}\{\forall i \le k : W_{v_i} \in [w_i, w_i + \mathrm{d}w_i], v_i \leftrightarrow v_{i-1}, C^w_{(v_{i-1},v_i)} \in [0, t_0]\}\Bigg], \end{aligned} \tag{5.6}$$

by the law of total probability (integrating over the value of the fitnesses), where $\mathrm{d}w_i$ is considered to be infinitesimal. The event in the indicator can be expressed as

$$\mathcal{E}_1(v_1, ..., v_k) \cap \mathcal{E}_2(v_1, ..., v_k) \cap \mathcal{E}_3(v_1, ..., v_k),$$

where, for any distinct fixed points $v_1, ..., v_k$ in $\mathbb{R}^d$,

$$\begin{aligned} \mathcal{E}_1 &:= \{W_{v_i} \in [w_i, w_i + \mathrm{d}w_i] \quad \forall i \in [k]\}; \\ \mathcal{E}_2 &:= \{v_i \leftrightarrow v_{i-1} \quad \forall i \in [k]\}; \\ \mathcal{E}_3 &:= \{C^w_{(v_{i-1},v_i)} \in [0, t_0] \quad \forall i \in [k]\}. \end{aligned} \tag{5.7}$$

To keep the notation in this proof more compact, we define $\|u - v\| := \|x_u - x_v\|$ for two vertices u and v, located at x_u and x_v in $\mathbb{R}^d$ respectively. As $\mathrm{d}w_1, ..., \mathrm{d}w_k \to 0$ and using the definitions of the connection probability in (3.3) from Assumption 3.3 and the cost of spread as in Definition 3.7, we have that, due to the fitnesses being i.i.d,

$$\begin{aligned} \mathbb{P}(\mathcal{E}_1(v_1, ..., v_k)) &\to \prod_{i=1}^k F_W(\mathrm{d}w_i), \\ \mathbb{P}(\mathcal{E}_2(v_1, ..., v_k) \mid \mathcal{E}_1(v_1, ..., v_k)) &\to \prod_{i=1}^k p \cdot \min\Bigg(1, \Big(\frac{w_{i-1}w_i}{\|v_{i-1} - v_i\|^d}\Big)^\alpha\Bigg) \\ \mathbb{P}(\mathcal{E}_3(v_1, ..., v_k) \mid \mathcal{E}_1(v_1, ..., v_k) \cap \mathcal{E}_2(v_1, ..., v_k)) &\to \prod_{i=1}^k F_L\Bigg(\frac{t_0}{\|v_{i-1} - v_i\|^\zeta w_{i-1}^\mu w_i^\nu}\Bigg), \end{aligned} \tag{5.8}$$

where $F_W(\mathrm{d}w_k)$ denotes the Lebesgue-Stieltjes integral with respect to the cdf F_W. Hence we can continue and calculate

$$\begin{aligned} n_k^{t_0}(0,w_0) &\le \mathbb{E}\Bigg[\sum_{(v_i)_{i\le k}\in\mathcal{V}_\lambda^{(k)}}\int_{(w_i)_{i\le k}}\prod_{i=1}^k\left(p\min\left(1,\left(\frac{w_{i-1}w_i}{\|v_{i-1}-v_i\|^d}\right)^\alpha\right)F_L(t^*)F_W(\mathrm{d}w_i)\right)\Bigg] \\ &= \int_{(w_i)_{i\le k}}\mathbb{E}\Bigg[\sum_{(v_i)_{i\le k}\in\mathcal{V}_\lambda^{(k)}}\prod_{i=1}^k p\min\left(1,\left(\frac{w_{i-1}w_i}{\|v_{i-1}-v_i\|^d}\right)^\alpha\right)F_L(t^*)\Bigg]\prod_{i=1}^k F_W(\mathrm{d}w_i). \end{aligned} \tag{5.9}$$

where $t^* := t_0\|v_{i-1}-v_i\|^{-\zeta}w_{i-1}^{-\mu}w_i^{-\nu}$. Observe that the sum within the expectation is over distinct vertices in $\mathbb{R}^d$ which are part of the Poisson point process $\mathcal{V}_\lambda$. We now apply Campbell's formula (see e.g. [27]) to rewrite the expectation as

$$\begin{aligned} T_1 &:= \mathbb{E}\Bigg[\sum_{(v_i)_{i\le k}\in\mathcal{V}_\lambda^{(k)}}\prod_{i=1}^k p\min\left(1,\left(\frac{w_{i-1}w_i}{\|v_{i-1}-v_i\|^d}\right)^\alpha\right)F_L\left(\frac{t_0}{\|v_{i-1}-v_i\|^\zeta w_{i-1}^\mu w_i^\nu}\right)\Bigg] \\ &= \int_{(v_i)_{i\le k}}\prod_{i=1}^k p\min\left(1,\left(\frac{w_{i-1}w_i}{\|v_{i-1}-v_i\|^d}\right)^\alpha\right)F_L\left(\frac{t_0}{\|v_{i-1}-v_i\|^\zeta w_{i-1}^\mu w_i^\nu}\right)\mathrm{d}M_k, \end{aligned} \tag{5.10}$$

where M_k denotes the kth factorial moment measure of the point process. Let the standard measure of the point process, the Lebesgue measure on $\mathbb{R}^d$, be denoted ν_L. Then the term in the integral is non-negative and M_k is dominated from above by $\lambda^k\nu_L^k$. Hence we have that

$$\begin{aligned} T_1 &\le \int_{(v_i)_{i\le k}}\prod_{i=1}^k p\min\left(1,\left(\frac{w_{i-1}w_i}{\|v_{i-1}-v_i\|^d}\right)^\alpha\right)F_L\left(\frac{t_0}{\|v_{i-1}-v_i\|^\zeta w_{i-1}^\mu w_i^\nu}\right)\lambda^k\mathrm{d}\nu_L^k \\ &= \lambda^k p^k\prod_{i=1}^k\left(\int_{v_i\in\mathbb{R}^d}\min\left(1,\left(\frac{w_iw_{i-1}}{\|v_i\|^d}\right)^\alpha\right)\min\left(1,\frac{t_0^\beta}{\|v_i\|^{\beta\zeta}w_{i-1}^{\beta\mu}w_i^{\beta\nu}}\right)\right)\mathrm{d}\nu_L. \end{aligned} \tag{5.11}$$

Above, we have used the translation invariance of the Lebesgue measure and the definition of F_L given in Definition 3.5 to obtain the second step. Observe that w_i and w_{i-1} are constants within T_1. Hence the ith factor T_{1i} in the above product can be computed by applying a case distinction.

$$\begin{aligned} T_{1i} &:= \int_{v_i\in\mathbb{R}^d}\min\left(1,\left(\frac{w_iw_{i-1}}{\|v_i\|^d}\right)^\alpha\right)\min\left(1,\frac{t_0^\beta}{\|v_i\|^{\beta\zeta}w_{i-1}^{\beta\mu}w_i^{\beta\nu}}\right)\mathrm{d}\nu_L \\ &\le \int_{\|v_i\|<\min((w_i,w_{i-1})^{1/d},t_0^{1/\zeta}w_{i-1}^{-\mu/\zeta}w_i^{-\nu/\zeta})}1\mathrm{d}\nu_L \\ &+ \int_{\|v_i\|>\max((w_iw_{i-1})^{1/d},t_0^{1/\zeta}w_{i-1}^{-\mu/\zeta}w_i^{-\nu/\zeta})}\frac{w_i^\alpha w_{i-1}^\alpha}{\|v_i\|^{d\alpha}}\cdot\frac{t_0^\beta}{\|v_i\|^{\beta\zeta}w_{i-1}^{\beta\mu}w_i^{\beta\nu}}\mathrm{d}\nu_L \\ &+ \int_{(w_iw_{i-1})^{1/d}\le\|v_i\|\le t_0^{1/\zeta}w_{i-1}^{-\mu/\zeta}w_i^{-\nu/\zeta}}\frac{w_i^\alpha w_{i-1}^\alpha}{\|v_i\|^{d\alpha}}\mathrm{d}\nu_L\cdot\mathbb{1}\left\{t_0^{1/\zeta}w_{i-1}^{-\mu/\zeta}w_i^{-\nu/\zeta}>(w_iw_{i-1})^{1/d}\right\} \\ &+ \int_{t_0^{1/\zeta}w_{i-1}^{-\mu/\zeta}w_i^{-\nu/\zeta}\le\|v_i\|\le(w_iw_{i-1})^{1/d}}\frac{t_0^\beta}{\|v_i\|^{\beta\zeta}w_{i-1}^{\beta\mu}w_i^{\beta\nu}}\mathrm{d}\nu_L\cdot\mathbb{1}\left\{t_0^{1/\zeta}w_{i-1}^{-\mu/\zeta}w_i^{-\nu/\zeta}<(w_iw_{i-1})^{1/d}\right\}. \end{aligned} \tag{5.12}$$

Note that the indicators, maximum and minimum can be handled by taking t_0 small enough, since we know $w_i\ge 1$ for all i by definition of the $\mathrm{IGIRG}_{W,L}(\lambda)$ model in Definition 3.1. Therefore, taking t_0 small enough drops the third integral in the equation for T_{1i} above. Using polar coordinates and denoting the volume of the unit ball in $\mathbb{R}^d$ by V_d, we get

$$\begin{aligned} T_{1i} \le\, & V_d(t_0^{1/\zeta}w_{i-1}^{-\mu/\zeta}w_i^{-\nu/\zeta})^d \\ & + \int_{r>(w_iw_{i-1})^{1/d}} w_{i-1}^{\alpha-\beta\mu}w_i^{\alpha-\beta\nu}t_0^\beta r^{-\beta\zeta-d\alpha}r^{d-1}\mathrm{d}r \\ & + \int_{t_0^{1/\zeta}w_{i-1}^{-\mu/\zeta}w_i^{-\nu/\zeta}\le r\le(w_iw_{i-1})^{1/d}} t_0^\beta w_{i-1}^{-\beta\mu}w_i^{-\beta\nu}r^{-\beta\zeta}r^{d-1}\mathrm{d}r. \end{aligned} \tag{5.13}$$

This then results in

$$\begin{aligned} T_{1i} \le V_d(t_0^{1/\zeta}w_{i-1}^{-\mu/\zeta}w_i^{-\nu/\zeta})^d + \left[\frac{w_{i-1}^{\alpha-\beta\mu}w_i^{\alpha-\beta\nu}t_0^\beta}{-\beta\zeta-d(\alpha-1)}r^{-\beta\zeta-d(\alpha-1)}\right]_{r=(w_iw_{i-1})^{1/d}}^{\infty} \\ + \left[\frac{t_0^\beta w_{i-1}^{-\beta\mu}w_i^{-\beta\nu}}{d-\beta\zeta}r^{-\beta\zeta+d}\right]_{r=t_0^{1/\zeta}w_{i-1}^{-\mu/\zeta}w_i^{-\nu/\zeta}}^{(w_iw_{i-1})^{1/d}}. \end{aligned} \tag{5.14}$$

Note that the infinite boundary in the second term does not result in an infinite term since $\alpha > 1$. After some simplifications, this then finally reduces to

$$\begin{aligned} T_{1i} \le V_d t_0^{d/\zeta}w_{i-1}^{-\mu d/\zeta}w_i^{-\nu d/\zeta} + \frac{t_0^\beta}{\beta\zeta+d(\alpha-1)}w_{i-1}^{-\beta\mu-\beta\zeta/d+1}w_i^{-\beta\nu-\beta\zeta/d+1} \\ + \frac{t_0^\beta}{d-\beta\zeta}w_{i-1}^{-\beta\mu-\beta\zeta/d+1}w_i^{-\beta\nu-\beta\zeta/d+1} - \frac{t_0^{d/\zeta}}{d-\beta\zeta}w_{i-1}^{-\mu d/\zeta}w_i^{-\nu d/\zeta}. \end{aligned} \tag{5.15}$$

We now use the assumption $t_0 \le 1$ to get

$$T_{1i} \le C^* t_0^{\min(d/\zeta,\beta)}w_{i-1}^{\max(-\mu d/\zeta,-\beta\mu-\beta\zeta/d+1)}w_i^{\max(-\nu d/\zeta,-\beta\nu-\beta\zeta/d+1)}, \tag{5.16}$$

for some constant $C^* > 0$. Thus, we need a case distinction on whether $\beta < d/\zeta$ or not. (The case distinction is the same for the maxima and the minimum). First assume that $\beta < d/\zeta$. Then we can use this upper bound for T_{1i} from (5.16) in the upper bound for T_1 we derived in (5.11) to get

$$\begin{aligned} T_1 &\le \lambda^k p^k \prod_{i=1}^k C^* t_0^\beta w_{i-1}^{-\beta\mu-\beta\zeta/d+1}w_i^{-\beta\nu-\beta\zeta/d+1} \\ &= (\lambda p C^* t_0^\beta)^k w_0^{-\beta\mu-\beta\zeta/d+1}\left(\prod_{i=1}^{k-1} w_i^{-\beta(\mu+\nu)-2\beta\zeta/d+2}\right)w_k^{-\beta\nu-\beta\zeta/d+1}. \end{aligned} \tag{5.17}$$

So then, using this upper bound for T_1 in (5.9), we get

$$\begin{aligned} n_k^{t_0}(0,w_0) &\le \int_{(w_i)_{i\le k}} (\lambda p C^* t_0^\beta)^k w_0^{-\beta\mu-\beta\zeta/d+1}\left(\prod_{i=1}^{k-1} w_i^{-\beta(\mu+\nu)-2\beta\zeta/d+2}\right)w_k^{-\beta\nu-\beta\zeta/d+1}\prod_{i=1}^k F_W(\mathrm{d}w_i) \\ &= (\lambda p C^* t_0^\beta)^k w_0^{1-\beta\mu-\beta\zeta/d}\mathbb{E}[W^{2-\beta(\mu+\nu+2\zeta/d)}]^{k-1}\mathbb{E}[W^{1-\beta(\nu+\zeta/d)}]. \end{aligned} \tag{5.18}$$

Hence we have proven the lemma for the case $d/\zeta > \beta$. We now explore the case where $d/\zeta \le \beta$. In this case using the upper bound for T_{1i} from (5.16) in the upper bound for T_1 in (5.11) gives

$$\begin{aligned} T_1 &\le \lambda^k p^k \prod_{i=1}^k C^* t_0^{d/\zeta}w_{i-1}^{-\mu d/\zeta}w_i^{-\nu d/\zeta} \\ &= (\lambda p C^* t_0^{d/\zeta})^k w_0^{-\mu d/\zeta}\prod_{i=1}^{k-1} w_i^{-(\mu+\nu)d/\zeta}\cdot w_k^{-\nu d/\zeta}. \end{aligned} \tag{5.19}$$

Here we used that $\max(-\mu d/\zeta, -\beta\mu - \beta\zeta/d + 1) = -\mu d\zeta$ since $1 - \beta\zeta/d \leq 0$. Again using this upper bound for T_1 in (5.9) gives

$$\begin{aligned} n_k^{t_0}(0, w_0) &\leq \int_{(w_i)_{i\leq k}} (\lambda p C^* t_0^{d/\zeta})^k w_0^{-\mu d/\zeta} \prod_{i=1}^{k-1} w_i^{-(\mu+\nu)d/\zeta} \cdot w_k^{-\nu d/\zeta} \prod_{i=1}^{k} F_W(\mathrm{d}w_i) \\ &= (\lambda p C^* t_0^{d/\zeta})^k w_0^{-\mu d/\zeta} \mathbb{E}[W^{-(\mu+\nu)d/\zeta}]^{k-1} \mathbb{E}[W^{-\nu d/\zeta}], \end{aligned} \tag{5.20}$$

which shows that the lemma also holds for the case $d/\zeta \leq \beta$, completing the proof. □

The next lemma will show that under the same conditions as in Lemma 5.7, the origin will almost surely not have infinitely many incident edges of bounded cost. This, in combination with Lemma 5.4, will rule out sideways explosion in the proof of Theorem 5.1.

Lemma 5.8. *Let $\alpha \in (1, \infty)$. Consider $\mathrm{IGIRG}_{W,L}(\lambda)$ with $0 < \lambda \leq 1$. Suppose that the vertex fitness distribution is as defined in Assumption 3.2, the connection probability as defined in Assumption 3.3 and the cost of spread $\mathcal{C}$ as defined in Definition 3.7. Then we have that $\mathbb{P}(N_1^t(0) < \infty) = 1$ for all $t \geq 0$ when*

$$3 - \tau < \beta\Big(\mu + \nu + 2\frac{\zeta}{d}\Big). \tag{5.21}$$

Note that if $d/\zeta \leq \beta$, this condition is trivially satisfied.

Proof. We show that the statement is true by showing that $\mathbb{E}[N_1^t(0)] < \infty$. This is most easily done by modifying the proof of Lemma 5.7. The main difference is that we cannot choose $t \leq 1$ and we have that $k = 1$. Again we denote $n_1^t(0, w_0) = \mathbb{E}[N_1^t(0) \mid W_0 = w_0]$. We can again perform the same calculations from (5.6) to (5.9) to see

$$n_1^t(0, w_0) \leq \int_{w_1} \mathbb{E}\left[\sum_{v_1 \in \mathcal{V}} p \min\left(1, \left(\frac{w_0 w_1}{\|v_1\|^d}\right)^\alpha\right) F_L\left(\frac{t}{\|v_1\|^\zeta w_0^\mu w_1^\nu}\right)\right] F_W(\mathrm{d}w_1). \tag{5.22}$$

Again defining T_1 as the inner expectation, applying Campbell's formula and filling in F_L as given in Definition 3.5, we see

$$T_1 \leq \lambda p \int_{v_1 \in \mathbb{R}^d} \min\left(1, \left(\frac{w_0 w_1}{\|v_1\|^d}\right)^\alpha\right) \min\left(1, \frac{t^\beta}{\|v_1\|^{\zeta\beta} w_0^{\mu\beta} w_1^{\nu\beta}}\right) \mathrm{d}\nu_L, \tag{5.23}$$

where the standard measure of the point process, the Lebesgue measure on $\mathbb{R}^d$, again is denoted by ν_L. We can follow the same steps as in (5.12) to (5.16) (now not assuming that $t \leq 1$) to get that

$$T_1 \leq \lambda p C^* \min(t^{d/\zeta}, t^\beta) w_0^{\max(-\mu d/\zeta, 1-\beta\mu-\beta\zeta/d)} w_1^{\max(-\nu d/\zeta, 1-\beta\nu-\beta\zeta/d)}, \tag{5.24}$$

for some constant $C^* > 0$. We again need a case distinction on whether $\beta \leq d/\zeta$ or not. Start by assuming that $\beta \leq d/\zeta$. Then using the upper bound for T_1 given in (5.24) in (5.22) yields

$$\begin{aligned} n_1^t(0, w_0) &\leq \int_{w_1} \lambda p C^* \min(t^{d/\zeta}, t^\beta) w_0^{1-\beta\mu-\beta\zeta/d} w_1^{1-\beta\nu-\beta\zeta/d} F_W(\mathrm{d}w_1) \\ &= \lambda p C^* \min(t^{d/\zeta}, t^\beta) w_0^{1-\beta\mu-\beta\zeta/d} \mathbb{E}[W^{1-\beta(\nu+\zeta/d)}]. \end{aligned} \tag{5.25}$$

In Lemma 5.7 it was proven that the expectation in (5.25) is finite when (5.21) holds, proving the Lemma for the case $\beta \leq d/\zeta$.

For the case $\beta > d/\zeta$, using the upper bound for T_1 given in (5.24) in (5.22) gives

$$\begin{aligned} n_1^t(0, w_0) &\leq \int_{w_1} \lambda p C^* \min(t^{d/\zeta}, t^\beta) w_0^{-\mu d/\zeta} w_1^{-\nu d/\zeta} F_W(\mathrm{d}w_1) \\ &= \lambda p C^* \min(t^{d/\zeta}, t^\beta) w_0^{-\mu d/\zeta} \mathbb{E}[W^{-\nu d/\zeta}]. \end{aligned} \tag{5.26}$$

Again, in Lemma 5.7 it was proven that the expectation in (5.26) is finite when (5.21) holds. Thus we have shown that the lemma holds for the case $\beta > d/\zeta$ as well, completing the proof. □

We are now ready to prove Theorem 5.1. First, Lemma 5.4 and Lemma 5.8 will be used to rule out sideways explosion. We note that if lengthwise explosion occurs, it will occur arbitrarily fast, as was shown in Lemma 5.5. We will prove that for some $t_0 < 1$, the probability of having an infinite path emanating from 0 with total cost in $[0, t_0]$ is zero. This will be done by showing that there exist no infinite paths starting at 0 using edges of cost at most t_0 by using the path-counting argument from Lemma 5.7. Using the exponentially decaying bounds this lemma provides for paths of length k, we can then bound the probability of such paths existing with Markov's inequality. Using this bound on the probability, the Borel-Cantelli Lemma will then show the model is conservative.

Proof of Theorem 5.1. By Lemma 5.8, for all $t \geq 0$, we have that $\mathbb{P}(N_1^t(0) < \infty)$. Then by Lemma 5.4, sideways explosion almost surely does not happen, and since $\tau \in (2, 3)$, the explosion time is realised via at least one infinite path for any vertex in the infinite component. Furthermore, by Lemma 5.5, if the model is lengthwise explosive, for all $t_0 > 0$, there exists an infinite path with total cost at most t_0 with strictly positive probability. In order to show that a model is conservative, it is sufficient to show that for some $t_0 < 1$, the probability of having an infinite path with total cost in $[0, t_0]$ is zero. We will instead show a stronger statement. Namely, we will show that there is no infinite path π starting from 0 that uses only edges (u, v) of cost $C^w_{(u,v)} \leq t_0$. We show this by proving that almost surely, there is no infinite path in $\mathcal{G}(t_0)$, where $\mathcal{G}(t_0)$ is defined as in Definition 5.2. Recall that by Definition 5.3, that $N_k^{t_0}(0)$ counts the number of k-edge paths in $\mathcal{G}(t_0)$ emanating from $v_0 = 0$. We apply Lemma 5.7 and note that we need to apply either the bound given by (a) or the bound given by (b), depending on whether $d/\zeta > \beta$ or not.

First we handle the case where $d/\zeta > \beta$ and the bound given in (a) applies. Choose

$$t_0 := (2\lambda p C^{**}\mathbb{E}[W^{2-\beta(\mu+\nu+2\zeta/d)}])^{-1/\beta} \text{ with } C^{**} := \max(C^*, 1.1/(2\lambda p \mathbb{E}[W^{2-\beta(\mu+\nu+2\zeta/d)}])), \tag{5.27}$$

such that the bound in (a) implies that $\mathbb{E}[N_k^{t_0}(0) \mid W_0 = w_0]$ decays exponentially in k. We use C^{**} rather than C^* to ensure that $t_0 < 1$. By Markov's inequality we have that $\mathbb{P}(N_k^{t_0} \geq 1 \mid W_0 = w_0) \leq \mathbb{E}[N_k^{t_0}(0) \mid W_0 = w_0]$. Therefore, $\sum_{k\geq 1} \mathbb{P}(N_k^{t_0}(0) \geq 1 \mid W_0 = w_0) \leq C \sum_{k \geq 1} 2^{-k} < \infty$ for some constant $C > 0$. By the Borel-Cantelli Lemma, almost surely, there exists a k_0 such that for all $k \geq k_0$, we have that $N_k^{t_0}(0) < 1$. Consequently, almost surely, $N_k^{t_0}(0) = 0$ for all $k \geq k_0$ and there cannot exist an infinite path in $\mathcal{G}(t_0)$. Thus, the model is conservative.

For the case where $d/\zeta \leq \beta$, the proof is analogous except for the choice of t_0. We instead choose $t_0 := (2\lambda p C^{**}\mathbb{E}[W^{-(\mu+\nu)d/\zeta}])^{-d/\zeta}$ with $C^{**} := \max(C^*, 1.1/(2\lambda p \mathbb{E}[W^{-(\mu+\nu)d/\zeta}]))$ and proceed as above to conclude that the model is conservative in this case as well. □

6 Conclusion

6.a Results

The goal of this thesis was to introduce a specific model of the IGIRG graph and extend the results presented by Komjáthy et al. in [25] by adding a multiplicative edge length term to the cost of spread. To achieve this, we modified several of the proofs in [25] to prove our main results. For $\tau \in (2, 3)$, the parameter space is fully covered by Theorem 4.2 and Theorem 5.1. As already mentioned, for $\tau > 3$, explosion cannot occur [19, 26]. The original paper of Komjáthy et al. covered the case $\tau \in (1, 2]$ as well. Recall that in this case $\mathbb{E}[W]$ is infinite. For the conservative case, we can extend our results by assuming that $\beta(\nu + \zeta/d) > 2 - \tau$. Under this assumption, the results of Theorem 5.1 apply and the model is conservative for $\tau \in (1, 2]$. The extra assumption is needed to be able to apply Claim 5.6 to ensure that the expectations in Lemma 5.7 are finite. Extending Theorem 4.2 to $\tau \in (1, 2]$ is a topic of future research. We also did not provide a full analysis of when sideways explosion occurs. In our case, the explosive argument was based on lengthwise explosion only, so what the necessary conditions for sideways explosion are remains an open question.

6.b Different Graph Model

The cost of spread we defined can also be researched on different graph models. A limitation of the IGIRG model is that it is static, i.e, after the vertices and edges have been generated, they no longer change. In some real-world networks such as a social network, this is not desirable: we may be introduced to new individuals, as well as fade out of the life of others. More advanced graph models, dependent on time, could better represent these networks. Another possible limitation of the IGIRG model is that spatial distance influences the existence of *every* edge. Although the IGIRG model contains some long range edges, the probability of these edges being present is still influenced by their geometric distance. It could be a reasonable assumption that for a small fraction of possible edges, the connection probability is not influenced by distance. Coming back to the social network example, this could reflect that sometimes people meet one another through online interactions only. Hence distance between the two such pairs of people was not a factor in getting into contact with one other.

6.c Spread and Cost Extensions

In this thesis, the cost function was restricted to a polynomial of the fitnesses and edge length in the explosive case (Theorem 4.2), and the cost function was restricted to a monomial in the conservative case (5.1). This may not be suited for all modelling purposes; one could imagine that in e.g. physics settings, there might be applications in which sine or cosine terms play a role. Thus, extending the analysis to more general cost functions is desirable.

The spreading model itself could be extended as well. A straightforward extension would be to consider the case in which infected vertices can recover (SIS model) or introduce new vertex classes, such as in the SIR model. More options for different vertex classes can be thought of. For instance, when viewing the spreading process from an epidemiological perspective, vertices could be assigned a class, 'careless', 'normal' or 'careful'. The rate of spread between two vertices can then also depend on the classes of the individuals. The spreading model can also be extended without adding vertex classes. For instance, the cost of spread could be accelerated when many vertices in a fixed radius of an uninfected vertex are already infected. Yet another option is introducing some kind of 'global factors': the rate of spread could be dependent on how many vertices are already infected in total. The possible extensions of the spreading model are endless.

6.d Beyond the Model

A more general future study could investigate how well the IGIRG model and our defined cost of spread reflect real-life networks. It has not been compared to real-life data yet, so whether real-world networks exhibit phenomena similar to explosion under the conditions of our theorems remains uncertain. Whether such phenomena occur when our derived conditions are met is not trivial. Most real-life networks are very large, but, in contrast to our model, ultimately finite.

References

[1] E. Adriaans and J. Komjáthy. "Weighted distances in scale-free configuration models". In: *Journal of Statistical Physics* 173.3 (2018), pp. 1082–1109.

[2] R. Albert and A.-L. Barabási. "Statistical mechanics of complex networks". In: *Reviews of Modern Physics* 74.1 (2002), p. 47.

[3] A. Auffinger, M. Damron, and J. Hanson. "50 years of first passage percolation". In: *arXiv preprint arXiv:1511.03262* (2015).

[4] A.-L. Barabási and R. Albert. "Emergence of scaling in random networks". In: *Science* 286.5439 (1999), pp. 509–512.

[5] E. Baroni, R. v. d. Hofstad, and J. Komjáthy. "Nonuniversality of weighted random graphs with infinite variance degree". In: *Journal of Applied Probability* (2017), pp. 146–164.

[6] S. Bhamidi, R. v. d. Hofstad, and G. Hooghiemstra. "First passage percolation on the Erdős–Rényi random graph". In: *Combinatorics, Probability and Computing* 20.5 (2011), pp. 683–707.

[7] B. Bollobás. "A probabilistic proof of an asymptotic formula for the number of labelled regular graphs". In: *European Journal of Combinatorics* 1.4 (1980), pp. 311–316.

[8] B. Bollobás, S. Janson, and O. Riordan. "The phase transition in inhomogeneous random graphs". In: *Random Structures & Algorithms* 31.1 (2007), pp. 3–122.

[9] K. Bringmann, R. Keusch, and J. Lengler. "Geometric inhomogeneous random graphs". In: *Theoretical Computer Science* 760 (2019), pp. 35–54.

[10] F. Chung and L. Lu. "The average distances in random graphs with given expected degrees". In: *Proceedings of the National Academy of Sciences* 99.25 (2002), pp. 15879–15882.

[11] M. Deijfen, R. v. d. Hofstad, and G. Hooghiemstra. "Scale-free percolation". In: *Annales de l'IHP Probabilités et statistiques*. Vol. 49. 3. 2013, pp. 817–838.

[12] S. N. Dorogovtsev, A. V. Goltsev, and J. F. Mendes. "Critical phenomena in complex networks". In: *Reviews of Modern Physics* 80.4 (2008), p. 1275.

[13] P. Erdős and A. Rényi. "On Random Graphs. I". In: *Publicationes Mathematicae* 6 (1959), pp. 290–297.

[14] C. Giuraniuc et al. "Criticality on networks with topology-dependent interactions". In: *Physical Review E* 74.3 (2006), p. 036108.

[15] J. M. Hammersley and D. J. Welsh. "First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory". In: *Bernoulli 1713, Bayes 1763, Laplace 1813*. Springer, 1965, pp. 61–110.

[16] T. Harko, F. S. Lobo, and M. Mak. "Exact analytical solutions of the Susceptible-Infected-Recovered (SIR) epidemic model and of the SIR model with equal death and birth rates". In: *Applied Mathematics and Computation* 236 (2014), pp. 184–194.

[17] R. v. d. Hofstad. *Random graphs and complex networks*. Vol. 1. Cambridge university press, 2016.

[18] R. v. d. Hofstad, G. Hooghiemstra, and P. v. Mieghem. "First-passage percolation on the random graph". In: *Probability in the Engineering and Informational Sciences* 15.2 (2001), pp. 225–237.

[19] R. v. d. Hofstad and J. Komjathy. "Explosion and distances in scale-free percolation". In: *arXiv preprint arXiv:1706.02597* (2017).

[20] V. Isham, J. Kaczmarska, and M. Nekovee. "Spread of information and infection on finite random networks". In: *Physical Review E* 83.4 (2011), p. 046128.

[21] S. Janson, T. Luczak, and A. Rucinski. *Random graphs*. Vol. 45. John Wiley & Sons, 2011.

[22] J. Jorritsma and J. Komjáthy. "Weighted distances in scale-free preferential attachment models". In: *Random Structures & Algorithms* 57.3 (2020), pp. 823–859.

[23] W. O. Kermack and A. G. McKendrick. "A contribution to the mathematical theory of epidemics". In: *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character* 115.772 (1927), pp. 700–721.

[24] I. Kolossváry and J. Komjáthy. "First passage percolation on inhomogeneous random graphs". In: *Advances in Applied Probability* 47.2 (2015), pp. 589–610.

[25] J. Komjáthy, J. Lapinskas, and J. Lengler. "Stopping explosion by penalising transmission to hubs in scale-free spatial random graphs". In: *arXiv preprint arXiv:2004.01149* (2020).

[26] J. Komjáthy and B. Lodewijks. "Explosion in weighted hyperbolic random graphs and geometric inhomogeneous random graphs". In: *Stochastic Processes and their Applications* 130.3 (2020), pp. 1309–1367.

[27] T. Mattfeldt. "Stochastic geometry and its applications". In: *Journal of Microscopy* 183.3 (1996), pp. 257–257.

[28] R. Meester and R. Roy. *Continuum percolation*. Vol. 119. Cambridge University Press, 1996.

[29] S. Milgram. "The small world problem". In: *Psychology Today* 2.1 (1967), pp. 60–67.

[30] G. Miritello et al. "Time as a limited resource: Communication strategy in mobile phone networks". In: *Social Networks* 35.1 (2013), pp. 89–95.

[31] M. Molloy et al. "A critical point for random graphs with a given degree sequence". In: *The Structure and Dynamics of Networks*. Princeton University Press, 2011, pp. 240–258.

[32] F. Papadopoulos et al. "Greedy forwarding in dynamic scale-free networks embedded in hyperbolic metric spaces". In: *2010 Proceedings IEEE INFOCOM*. IEEE. 2010, pp. 1–9.

[33] R. Pastor-Satorras et al. "Epidemic processes in complex networks". In: *Reviews of Modern Physics* 87.3 (2015), p. 925.

[34] T. P. B. Thu, P. N. H. Ngoc, N. M. Hai, et al. "Effect of the social distancing measures on the spread of COVID-19 in 10 highly infected countries". In: *Science of The Total Environment* 742 (2020), p. 140430.