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Mathematics of the ‘Lights Out’ game

de Nooij, Koen A.P.

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Koen de Nooij (1011650)
Supervisor: Prof. Dr. Hans Cuypers

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1 Introduction

Consider the electronic game 'Lights Out'. The game consists of a 5×5 grid of lights that can either be *on* or *off*. When the game starts, the lights in the grid take on an initial configuration. A move consists of pressing any light in the grid, after which the light itself and all of its adjacent lights will toggle their state. The player is challenged with switching all lights *off* in as few moves as possible. The game has been studied quite well and the effects of moves [1] and conditions for when the game can be solved [2] are known.

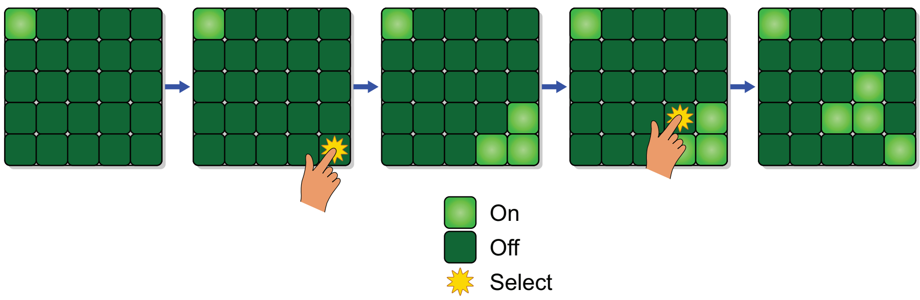

Figure 1: An illustration of the game Lights Out showing two consecutive moves.

1.1 Lit-only σ-game

In this report, we will look at a derived game, oftentimes referred to as the lit-only σ-game. Instead of a grid, the playing board is generalized to an undirected graph of any finite size. Furthermore, the game follows slightly different rules: lights can only be pressed when they are toggled *on*. When a light is pressed, it will only toggle its adjacent lights, not itself. Notice that it is impossible to turn *off* all lights when starting with a configuration that has a light turned *on*. Therefore, we define a new goal for the game: turning all the lights *on*.

With this in mind, one can begin to ask some interesting questions:

- What are the effects of moves?
- How can we go from one configuration to another?
- Can every game be solved?
- What are the consequences of the properties of our graph?

While not all questions can be easily answered, there have been a number of papers looking into different classes of graphs. A topic of particular interest is estimating the minimum light number for a given graph [3][4]. For any initial configuration, one can arrive at a configuration in which the number of lights that are *on* is no greater than this number. In this report however, we will not go into depth about this. Instead we focus on how certain properties of the graph influence the way the game is played. We find a result similar to one found by Yoakun Wu (2009) [5], but through different means.

1.2 Overview

In Section 2 we give an introduction to some important concepts and lay out the necessary definitions to analyze the game as a mathematical group, although it is assumed that the reader is familiar with elementary concepts from linear algebra. In Section 3 this group is better classified by relating it to a well-explored group. Section 4 introduces a geometric tool for 'translating' the conditions under which we can relate the symmetric group to our game. These conditions turn out to be properties of the graph on which we play the game. Section 5 details ways of recognizing these properties and describes an efficient algorithm to do so. Finally, Section 6 is a conclusion of the results.

2 Basic concepts and definitions

Definition 2.1. An (**undirected**) **graph** is a pair $\Gamma = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a finite set whose elements are called **vertices** or **points**, and $\mathcal{E}$ a set of paired vertices, called **edges**. We assume the graph is simple, i.e. does not contain loops.

Definition 2.2. A graph $\Gamma = (\mathcal{V}, \mathcal{E})$ is **connected** if and only if for each two vertices u and v in $\mathcal{V}$, there exists a sequence of edges $\{w_1, w_2\}$, $\{w_2, w_3\}$, $\dots$, $\{w_{n-1}, w_n\}$ in $\mathcal{E}$ such that $u = w_1$ and $v = w_n$.

Definition 2.3. Let $v \in \mathcal{V}$ be a vertex. We refer to the set $\{u \in \mathcal{V} \mid \{u, v\} \in \mathcal{E}\}$ as the **neighbours of** v, or **vertices adjacent to** v. Note that v is never adjacent to itself.

A playing board of the lit-only σ-game can be represented as a finite undirected graph $\Gamma = (\mathcal{V}, \mathcal{E})$ with vertices $\mathcal{V}$ and edges $\mathcal{E}$. Lights in the game form the set of vertices $\mathcal{V}$. The terms light, point and vertex will be used interchangeably. Two lights $u, v \in \mathcal{V}$ are neighbours in the game if and only if there is an edge between them in the graph Γ, i.e. $\{u, v\} \in \mathcal{E}$.

Definition 2.4. The finite field of two elements $\{0, 1\}$ is referred to as $\mathbb{F}_2$. In this field we have $0 + 0 = 0$, $0 + 1 = 1$, but $1 + 1 = 0$. Also $0 \cdot 0 = 0$, $0 \cdot 1 = 0$ and $1 \cdot 1 = 1$.

For reasons that will become evident, we construct a vector space on the set of vertices in the game. Let $V = \mathbb{F}_2^{\mathcal{V}}$ be the vector space over $\mathbb{F}_2$, with lights $\mathcal{V}$ as a basis. A light $v \in \mathcal{V}$ can now be represented as the vector $v \in V$. A set of lights $\{v_1, v_2, \ldots\} \in \mathcal{V}$ can be represented as the vector $v_1 + v_2 + \cdots \in V$.

Definition 2.5. Given a vector space V over a field F, the **dual space** V^* is defined as the set of all linear maps $f : V \to F$. Notice that V^* forms a vector space over F as well.

We can represent the state of a light with the field $\mathbb{F}_2$. We associate 0 with the light being *off* and 1 with the light being *on*. We can encode the state of our game with maps in the dual space V^* of V.

Definition 2.6. For each light $v \in \mathcal{V}$, we define the function $f_v \in V^*$ by

$$f_v(u) = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \neq v \end{cases} \tag{1}$$

Remark. The vector space V^* has the set $\{f_v \mid v \in \mathcal{V}\}$ as a basis.

Notation. If we are working with a indexed set of vertices, i.e. $\{v_1,\ v_2,\ \ldots,\ v_n\}$, sometimes the shorthand notation f_i is used to mean f_{v_i}.

Definition 2.7. We refer to the state of the game on a graph Γ (i.e. which lights are *on* or *off*) as a **configuration** on Γ. Each configuration on Γ can be represented by a vector $f \in V^*$, where:

$$f = \sum_{v \in \mathcal{V}_{on}} f_v \tag{2}$$

where $\mathcal{V}_{on} \subseteq \mathcal{V}$ is the subset of lights that are turned *on*.

We can test the state of a light $v \in \mathcal{V}$ in a configuration $f \in V^*$ by applying the map to it. $f(v) = 1$ if and only if v is *on*.

Definition 2.8. Let $v \in \mathcal{V}$ be a light and $N \subseteq \mathcal{V}$ be the set of lights adjacent to v. We denote the vector in V^* corresponding to the set of lights adjacent to v as A_v. We have $A_v = \sum_{u \in N} f_u$.

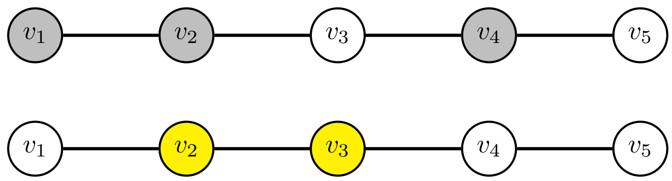

Figure 2: A graph of five points on a line. The top figure shows a subset of points indicated with the grey fill. The bottom figure shows a configuration on the graph, with the yellow fill indicating that a light is *on*.

Example 2.9. *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph on 5 points as is shown in Fig. 2. We represent the subset $\{v_1, v_2, v_4\} \in \mathcal{V}$ of grey points in the top figure as the vector $v_1 + v_2 + v_4 \in V$. The configuration in the bottom figure is represented as the map $f = f_2 + f_3$. Suppose we want to test whether v_3 is turned on. We compute $f(v_3) = f_2(v_3) + f_3(v_3) = 0 + 1 = 1$, meaning it is on.*

Also consider $f(v_2 + v_3) = f(v_2) + f(v_3) = 1 + 1 = 0$, which might go against our initial intuition.

2.1 Moves

With the basic structure in place, we can now define moves in the game. These can be seen as maps taking one configuration to another, depending on which vertex is pressed and the state of that vertex. Pressing a light switches the state of all adjacent lights in the graph.

Definition 2.10. A **move** on vertex $v \in \mathcal{V}$ is the linear map $\mu_v : V^* \to V^*$ defined by

$$\mu_v(f) = f + f(v)A_v \tag{3}$$

This way, only if the vertex v is turned on, $f(v)$ gives 1, adding the vector of adjacent lights to the configuration. Since we are working on $\mathbb{F}_2$, this turns neighbours *on* that were previously *off* and vice versa.

Notation. If we are working with a indexed set of vertices, i.e. $\{v_1,\ v_2,\ \ldots,\ v_n\}$, sometimes the shorthand notation μ_i is used to mean μ_{v_i}.

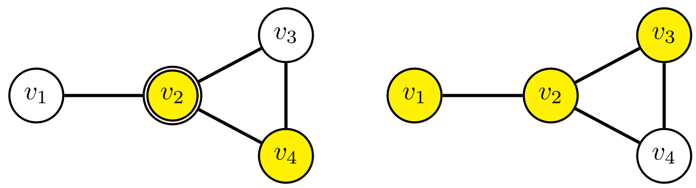

Figure 3: The left figure shows an initial configuration. The right figure shows the same graph after making a move on vertex v_2.

Example 2.11. *Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph as is shown in Fig. 3. We start with an initial configuration $f = f_2 + f_4 \in V$ and make a move μ_2 on vertex v_2.*

$$\mu_2(f) = f + f(v_2)A_{v_2}$$

Note that

$$\begin{aligned} f(v_2) &= [f_2 + f_4](v_2) \\ &= f_2(v_2) + f_4(v_2) \\ &= 1 + 0 = 1 \end{aligned}$$

Also $A_{v_2} = f_1 + f_3 + f_4$. Thus the result becomes

$$\begin{aligned} \mu_2(f) &= f_2 + f_4 + 1 \cdot (f_1 + f_3 + f_4) \\ &= f_1 + f_2 + f_3 \end{aligned}$$

as can be seen in the right figure.

2.2 Groups

Definition 2.12. Let G be a set and $*$ a binary operation on G. A tuple

$$(G,\ * : G \times G \to G,\ \text{inv} : G \to G,\ e \in G)$$

is a **group** if and only if it satisfies the following **group axioms**:

(G1) for all $f, g, h \in G$, $(f * g) * h = f * (g * h)$. We call $*$ the **group operation**.

(G2) e is an **identity element** for $*$, such that for each $g \in G$, $e * g = g = g * e$.

(G3) for each $g \in G$, $\text{inv}(g)$ satisfies $g * \text{inv}(g) = e = \text{inv}(g) * g$. We call $\text{inv}(g)$ the **inverse of** g and inv the **inverse map**.

We will refer to the group as G if the group operation, inverse map and identity element are clear from context.

Definition 2.13. Let S be a set of elements from a group G. We say S **generates** G if any element in G can be expressed as a product of elements in S or inverses of elements in S. So for any $g \in G$, we can write $g = s_1 * s_2 * \cdots * s_n$, where s_i or $\mathrm{inv}(s_i)$ is an element of S for $i = 1 \ldots n$. We say G **is generated by** S, and write $G = \langle S \rangle$.

Definition 2.14. Let Γ be a graph. We define $D_\Gamma = \{\mu_v \mid v \in \mathcal{V}\}$ to be the set of moves on vertices in Γ. Let $\mathcal{M}_\Gamma = \langle D_\Gamma \rangle$ be the **group of moves** generated by D_Γ under function composition. We write $\mathcal{M}$ if the graph on which the moves are defined is clear from context. The group operation of $\mathcal{M}$ is function composition, denoted with $\circ$. The inverse map is denoted with $\cdot^{-1}$ and for a move $\mu \in D_\Gamma$ is simply $\mu^{-1} = \mu$, since making the same move twice does not change the configuration. The identity element is the identity map denoted with id.

For completeness, we show that $\mathcal{M}$ is in fact a group.

Lemma 2.15. $\mathcal{M} = \langle D_\Gamma \rangle$ is a group.

Proof. We check the group axioms (G1)-(G3):

(G1) For all $\mu, \rho, \sigma \in \mathcal{M}$, we have $(\mu \circ \rho) \circ \sigma = \mu \circ (\rho \circ \sigma)$, since function composition is associative.

(G2) $\mathrm{id} \in \mathcal{M}$ is the identity element, since for any $\mu \in \mathcal{M}$, we have $\mathrm{id} \circ \mu = \mu = \mu \circ \mathrm{id}$, by definition of the identity map.

(G3) Remember that each element $\mu \in \mathcal{M}$ can be expressed as a product of elements or inverses of elements in D_Γ. Let $\mu = \rho_1 \circ \rho_2 \circ \cdots \circ \rho_n$ be such a product. Then $\rho_i^{-1} = \rho_i$, and thus

$$\mu \circ \mu^{-1} = (\rho_1 \circ \rho_2 \circ \cdots \circ \rho_n) \circ (\rho_n \circ \rho_{n-1} \circ \cdots \circ \rho_1) = \mathrm{id} = \mu^{-1} \circ \mu$$

□

Notation. Let μ, ρ be elements from $\mathcal{M}$. We leave out the composition operator and write $\rho\mu$ instead of $\rho \circ \mu$ for simplicity.

Now that we have created a group of moves, we can characterize how it interacts with the vector space of configurations V^*.

Definition 2.16. Let G be a group, and let S be a set. We say that G **acts on** S via ρ if ρ is a map

$$\begin{aligned} \rho : G \times S &\to S \\ (g, s) &\mapsto g \cdot s \end{aligned}$$

such that

1. for every $s \in S$, we have that $e \cdot s = s$.
2. for all $s \in S$ and $g, h \in G$, we have that $g \cdot (h \cdot s) = (g * h) \cdot s$.

Definition 2.17. Let G be a group acting on a set S. Let s be an element of S. We define the **orbit** of s as the set

$$\mathrm{Orb}(s) = \{s' \in S \mid g \cdot s = s', \text{ for some } g \in G\}$$

Definition 2.18. Let G be a group acting on a vector space V. A subspace $W \leq V$ is called an **invariant subspace** with respect to G if and only if $g(W) \subseteq W$ for all $g \in G$.

Remark. Note that $\{0\}$ and V are always invariant subspaces of G.

Lemma 2.19. The group of moves $\mathcal{M}$ acts on the vector space of configurations V^* with function application.

Proof. Function application of a move on a configuration is a map $\mathcal{M} \times V^* \to V^*$ defined by $(\mu, f) \mapsto \mu(f)$. We check the necessary properties.

1. for every configuration $f \in V^*$, we have $\mathrm{id}(f) = f$, where $\mathrm{id} \in \mathcal{M}$ is the identity map.
2. for every configuration $f \in V^*$ and group elements $\mu, \rho \in \mathcal{M}$, we have that $\rho(\mu(f)) = (\rho \circ \mu)(f)$ by definition of function composition.

□

Notation. Let G be a group and $g \in G$ an element. We write g^k for some integer k to mean g multiplied k times with itself. We have $g^0 = e$, $g^1 = g$, $g^2 = g * g$, etc.

Definition 2.20. Let G be a group and $g \in G$ an element. The **order** of g is defined as the smallest integer $k > 0$, such that $g^k = e$. We write $|g| = k$. If such k does not exist, we say the order of g is infinite.

A move μ_v on a vertex $v \in \mathcal{V}$ has order 1 or 2, since making a move twice effectively does nothing to the configuration of the game. But the product of different moves, i.e. making one move after the other, can result in elements in $\mathcal{M}$ of different order. In particular we have the following:

Lemma 2.21. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph on which we play the game. For any vertices $u, v \in \mathcal{V}$, and corresponding moves $\mu_u, \mu_v \in \mathcal{M}$, the order of the product of their moves is

$$|\mu_v\mu_u| = \begin{cases} 1 & \text{if } u = v \\ 2 & \text{if } \{u,v\} \notin \mathcal{E} \\ 3 & \text{if } \{u,v\} \in \mathcal{E} \end{cases} \tag{4}$$

Proof. For the first case, we know the order of a move is two, therefore if $u = v$, we have $|\mu_v\mu_u| = |\mu_v^2| = |\mathrm{id}| = 1$.

In the second case, take any $u, v \in \mathcal{V}$, such that $\{u,v\} \notin \mathcal{E}$. For all $f \in V^*$

$$\begin{aligned}
[\mu_v\mu_u](f) &= \mu_v(\mu_u(f)) \\
&= \mu_v(f + f(u)A_u) \\
&= f + f(u)A_u + [f + f(u)A_u](v)A_v \\
&= f + f(u)A_u + f(v)A_v + [f(u)A_u](v)A_v \\
&= f + f(u)A_u + f(v)A_v
\end{aligned}$$

We can make the last step because $\{u,v\} \notin \mathcal{E}$ and therefore $[A_u](v) = 0$ always. Furthermore we make use of the fact that $[A_u](u) = 0$ for any $u \in \mathcal{V}$, giving

$$\begin{aligned}
[(\mu_v\mu_u)^2](f) =&[\mu_v\mu_u](f + f(u)A_u + f(v)A_v) \\
=&f + f(u)A_u + f(v)A_v + \\
&[f + f(u)A_u + f(v)A_v](u)A_u + \\
&[f + f(u)A_u + f(v)A_v](v)A_v \\
=&f + f(u)A_u + f(v)A_v + \\
&f(u)A_u + f(v)A_v \\
=&f
\end{aligned}$$

Therefore $(\mu_v\mu_u)^2 = \mathrm{id}$ and thus $|\mu_v\mu_u| = 2$.

In the last case, when $\{u,v\} \in \mathcal{E}$, we still get $[A_u](u) = 0$ for any $u \in \mathcal{V}$. But now, $[A_u](v) = 1$ for any distinct $u, v \in \mathcal{E}$. This gives

$$\begin{aligned}
[\mu_v\mu_u](f) &= f + f(u)A_u + f(v)A_v + [f(u)A_u](v)A_v \\
&= f + f(u)A_u + f(v)A_v + f(u)A_v
\end{aligned}$$

Now

$$\begin{aligned}
[(\mu_v\mu_u)^2](f) &= [\mu_v\mu_u](f + f(u)A_u + f(v)A_v + f(u)A_v) \\
&= f + f(v)A_u + f(u)A_u + f(v)A_v
\end{aligned}$$

And finally

$$\begin{aligned}[(\mu_v\mu_u)^3](f) &= [\mu_v\mu_u](f + f(v)A_u + f(u)A_u + f(v)A_v) \\ &= f\end{aligned}$$

Therefore $(\mu_v\mu_u)^3 = \text{id}$ and thus $|\mu_v\mu_u| = 3$. □

We have now successfully translated the lit-only σ-game to the language of groups. This way we can reason about properties of the game with results from group theory. We have also explored some of the structure between different elements in $\mathcal{M}$ in Lemma 2.21. To further determine what kind of group $\mathcal{M}$ is, some extra definitions are given.

Definition 2.22. A **group-homomorphism**, or **homomorphism** is a map $f : G \to G'$ where $(G,\ *,\ \text{inv},\ e)$ and $(G',\ *',\ \text{inv}',\ e')$ are groups such that

(H1) $f(e) = e'$.

(H2) for every $a, b \in G$, $f(a * b) = f(a) *' f(b)$.

(H3) for every $a \in G$, $f(\text{inv}(a)) = \text{inv}'(f(a))$.

Definition 2.23. A **group-isomorphism** or **isomorphism** is a bijective homomorphism. If there exists an isomorphism between groups G and G', we write $G \cong G'$.

Definition 2.24. Let $(G,\ *,\ \text{inv},\ e)$ be a group. A subset H of G is called a **subgroup** of G if and only if $(H,\ *|_H,\ \text{inv}|_H,\ e)$ is a group. We denote this by $H \leq G$.

3 Transvections

Definition 3.1. Let W be a vector space over a field F. Let $\tau : W \to W$ be a linear map on W. τ is a **transvection** if and only if τ can be written in the form

$$\tau(x) = x + \alpha(x)w \tag{5}$$

where $w \in W$, $\alpha \in W^*$, $\alpha(w) = 0$ and where W^* denotes the dual space of W.

Lemma 3.2. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph on which we play the game. For any vertex $v \in \mathcal{V}$, we have that the move on v, μ_v, is a transvection.

Proof. The move on vertex v is defined by Eq. (3) to be

$$\mu_v(f) = f + f(v)A_v$$

Using definition Definition 3.1, we set W to be the vector space of configurations V^*. The dual of V^* is $V^{**} = W^*$, the set of all linear maps $\phi : V^* \to \mathbb{F}_2$.

Applying the input configuration $f \in V^*$ to the fixed vertex $v \in V$, can be seen as a linear map $\phi_v : V^* \to \mathbb{F}_2$ defined by $\phi_v(f) = f(v)$ for any $f \in V^*$. We have that $\phi_v \in V^{**}$ and can identify v with ϕ_v. Furthermore, $\phi_v(A_v) = A_v(v) = 0$, since v is not a vertex adjacent to itself.

We conclude that μ_v is a transvection. □

Definition 3.3. The **special linear group** $\mathrm{SL}_n(F)$ of degree n over a field F is the set of $n \times n$ matrices with determinant 1.

Lemma 3.4. [6] The matrix representation of any transvection has determinant 1. Furthermore, the set of all transvections on a vector space V of dimension n generates the special linear group $\mathrm{SL}_n(F) \cong \mathrm{SL}(V)$.

Corollary 3.5. Let Γ be a graph with n vertices. The group of moves $\mathcal{M}_\Gamma$ on Γ induces a subgroup of $\mathrm{SL}_n(\mathbb{F}_2) \cong \mathrm{SL}(V^*)$.

Proof. All elements D_Γ generating $\mathcal{M}_\Gamma$ are transvections of degree n and therefore elements of $\mathrm{SL}_n(\mathbb{F}_2)$. Furthermore, $\mathrm{SL}_n(\mathbb{F}_2)$ is closed under function composition (or matrix multiplication really), we have that $\mathcal{M}_\Gamma \subseteq \mathrm{SL}_n(\mathbb{F}_2)$ and thus $\mathcal{M}_\Gamma \leq \mathrm{SL}_n(\mathbb{F}_2)$. □

Lemma 3.6. Let τ be a transvection on a vector space W over $\mathbb{F}_2$. We can uniquely identify τ with the pair (w, α), where $w \in W$ is a point referred to as the **center** and the kernel of $\alpha \in W^*$ a hyperplane in W referred to as the **axis**. We write $\tau = (w, \alpha)$.

Proof. We write τ as $\tau(x) = x + \alpha(x)w$, where $w \in W$, $\alpha \in W^*$ and $\alpha(w) = 0$. Since the image of α is $\{0, 1\}$, it has dimension 1. By the dimension formula we know that $\dim(\ker(\alpha)) = \dim(W) - 1$. This means that $\ker(\alpha)$ forms a hyperplane in W. Notice that $w \in \ker(\alpha)$ always. □

Lemma 3.7. Consider two transvections $\tau = (w, \alpha)$ and $\tau' = (w', \alpha')$. We can have the following situations with corresponding orders of the product.

1. $w = w'$ and $\alpha = \alpha'$, then $|\tau'\tau| = 1$
2. $w = w'$ and $\alpha \neq \alpha'$, then $|\tau'\tau| = 2$
3. $w \neq w'$ and $\alpha = \alpha'$, then $|\tau'\tau| = 2$
4. $w \neq w',\ \alpha'(w) = 0,\ \alpha(w') = 0$ and $\alpha \neq \alpha$, then $|\tau'\tau| = 2$
5. $w \neq w',\ \alpha'(w) = 0,\ \alpha(w') = 1$ and $\alpha \neq \alpha'$, then $|\tau'\tau| = 4$

6. $w \neq w',\ \alpha'(w) = 1,\ \alpha(w') = 0$ and $\alpha \neq \alpha'$, then $|\tau'\tau| = 4$
7. $w \neq w',\ \alpha'(w) = 1,\ \alpha(w') = 1$ and $\alpha \neq \alpha'$, then $|\tau'\tau| = 3$

Proof. We first realize that there is a symmetry between W and its dual space W^*. A point in W^* gives a hyperplane in W and a point in W gives a hyperplane in W^*, following the same reasoning as in the proof of Lemma 3.2. With this, we see that cases (2) and (3) are equivalent. Also (5) and (6) are equivalent.

(1) In this case, the transvections are identical. Applying the same transvection twice results in the identity map, since we are working over $\mathbb{F}_2$. We get $|\tau'\tau| = 1$.

For the following cases, we define $X = \ker(\alpha) \cap \ker(\alpha')$ as a subspace of W. We make use of the fact that X is fixed by both τ and τ', making it invariant. Furthermore, if $\alpha \neq \alpha'$, then X has codimension 2 with W, otherwise it has codimension 1.

(3) Let $\tau = (w, \alpha)$ and $\tau = (w', \alpha)$. We know that X has codimension 1 with W. Since both transvections fix all points in X, we consider only the set $W \backslash X$. Let $p \in (W \backslash X)$. Then by repeatedly applying τ and τ' in alternating order we get one of the following cycles

$$p \overset{\tau}{\longmapsto} p + w \overset{\tau'}{\longmapsto} p + w + w' \overset{\tau}{\longmapsto} p + w' \overset{\tau'}{\longmapsto} p$$
$$p \overset{\tau'}{\longmapsto} p + w' \overset{\tau}{\longmapsto} p + w' + w \overset{\tau'}{\longmapsto} p + w \overset{\tau}{\longmapsto} p$$

From this it becomes clear that $\tau'\tau\tau'\tau = \mathrm{id}$ and $|\tau'\tau| = 2$.

(4) Let $\tau = (w, \alpha)$ and $\tau = (w', \alpha')$. We know X has codimension 2 with W. Furthermore, $\langle w, w' \rangle \subseteq X$. τ acts on any element $p \in (\ker(\alpha') \backslash X)$ with order 2. Similarly, τ' acts with order two on $\ker(\alpha) \backslash X$. Let $M = \{p + p' \mid p \in \ker(\alpha) \text{ and } p' \in \ker(\alpha')\}$ be a hyperplane. Notice that $M \cup \ker(\alpha) \cup \ker(\alpha') = W$. Any $p \in (M \backslash X)$ gives rise to the same cycles as case (3). We conclude that $|\tau'\tau| = 2$.

(5) Let $\tau = (w, \alpha)$ and $\tau = (w', \alpha')$. We know X has codimension 2 with W. Furthermore, $w' \in X$. τ' acts on $\ker(\alpha) \backslash X$ with order 2. Take any point $p \in (\ker(\alpha') \backslash X)$. Applying the transvections in alternating order, starting with τ, we get the following cycle:

$$\begin{aligned} p &\overset{\tau}{\longmapsto} p + w \overset{\tau'}{\longmapsto} p + w + w' \overset{\tau}{\longmapsto} p + w' \overset{\tau'}{\longmapsto} p + w' \\ &\overset{\tau}{\longmapsto} p + w + w' \overset{\tau'}{\longmapsto} p + w \overset{\tau}{\longmapsto} p \overset{\tau'}{\longmapsto} p \end{aligned}$$

Cycles of any $p \in (W \backslash \ker(\alpha))$ are of the same length. We conclude that $|\tau'\tau| = 4$.

(7) Let $\tau = (w, \alpha)$ and $\tau = (w', \alpha')$. Again, X has codimension 2 with W. This time, $w, w', w + w' \notin X$. We have that $\tau(w') = w + w' = \tau'(w)$. Thus τ and τ' transpose w' with $w + w'$ and w with $w + w'$ respectively. The product $\tau'\tau$ is therefore at least of order 3. Notice that $\langle w, w' \rangle$ is of dimension 2 and disjoint from the fixed subspace X. We conclude $|\tau'\tau| = 3$.

□

When we link this back to our game, we know by Lemma 2.21 that moves are transvections and the product of two moves has order 1, 2 or 3.

Indeed, when we apply the same move twice, we have two transvections that are identical, corresponding to case (1).

From Lemma 3.2, we know that a move on a vertex $v \in \mathcal{V}$ can be represented as a tranvection $\mu_v = (A_v, \phi_v)$, where $\phi_v(f) = f(v)$. Note that the axis of this transvection is $\ker(\phi_v) = \langle f_w \mid w \in \mathcal{V}, w \neq v \rangle$. Two moves on vertices $v, w \in \mathcal{V}$ that are not adjacent to each other give a product of order 2. Indeed this corresponds to case (2) if the vertices share the same neighbours, i.e. $A_v = A_w$, or case (4) in any other case. Case (3) does not correspond to a product of two moves in our game, since having identical axes would mean the move is played on the same vertex, but with different adjacent vertices.

The product of two moves on adjacent vertices $v, w \in \mathcal{V}$ corresponds with case (7). We have that their adjacent vertices are not the same, so $A_v \neq A_w$. Furthermore, since v and w are adjacent, we have that $A_w(v) = 1$ and $A_v(w) = 1$. Therefore $\phi_w(v) = 1$ and $\phi_v(w) = 1$.

Cases (5) and (6) cannot occur, since it would mean that the first vertex on which we play a move is adjacent to the second vertex, but the second is not adjacent to the first vertex.

3.1 Groups generated by transvections

We look closer at the group G generated by a set of transvections T on a vector space V over $\mathbb{F}_2$. From Lemma 3.7, we know that for any distinct $d, e \in T$, we have that $|de| = 2,\ 3$ or 4.

Definition 3.8. We define the **diagram** of a set of transvections T to be the graph (T, E), such that for any $\tau, \tau' \in T$ we have $\{\tau, \tau'\} \in E$ if and only if τ and τ' do not commute and thus $|\tau'\tau| > 2$.

For the following results it is assumed that the reader is familiar with the symplectic group, orthogonal group and symmetric group.

Definition 3.9. Let V be a vector space over a field F. We call a mapping $f : V \times V \to F$ a **symplectic** form if and only if

1. f is **bilinear**, i.e. linear in both arguments.
2. $f(v, v) = 0$ for all $v \in V$.

The symplectic form f is called **nondegenerate** if and only if the following is satisfied: Let $v \in V$ be arbitrary, then $f(v, w) = 0$ for all $w \in V$ implies that $w = 0$.

Since we are studying the group of moves generated by a set of transvections, we assume from now on that the order of the product of two transvections is no greater than 3. Let us now define a symplectic form on a set of transvections.

Definition 3.10. Let $\mathbb{F}_2 T$ be a vector space with basis T and let $f : \mathbb{F}_2 T \times \mathbb{F}_2 T \to \mathbb{F}_2$ be a bilinear mapping such that for any $d,\ e \in T$, we have:

$$f(d, e) = \begin{cases} 0 & \text{if } (de)^2 = 1 \\ 1 & \text{if } (de)^3 = 1 \end{cases}$$

Lemma 3.11. f is a symplectic form on $\mathbb{F}_2 T$.

Proof. We check the properties of a symplectic form.

- f is bilinear because it is a linear extension of the definition on elements in T.
- f is alternating, since for any $d \in \mathbb{F}_2 T$ we have $(dd)^2 = 1$ and therefore $f(d, d) = 0$.

□

Notice how the symplectic form f on $\mathbb{F}_2 T$ encodes the diagram of T. We have that two transvections $d, e \in T$ are connected in the diagram if and only if $f(d, e) = 1$.

Definition 3.12. The action of a group G on V is called **irreducible** if and only if there is no subspace $W \leq V$ that is invariant under G, except V and $\{0\}$.

Lemma 3.13. Let G be a group generated by transvections on V. Then G is irreducible on V if and only if:

1. The diagram of the transvections in G is connected.
2. The intersection of all axes in G is $\{0\}$.
3. The centers in G span V.

Proof. Suppose the three conditions are met. Let $W \leq V$ be some invariant subspace of the action of G on the vector space V. The intersection of all axes in G is $\{0\}$. That means there is no vector in V, other than 0, that is fixed by all transvections in G. Take any $v \in W$, such that $v \neq 0$, and let $\tau = (w, \alpha) \in G$ be a transvection that does not fix v, i.e. $\alpha(v) \neq 0$. Then $\tau(v) = v + w$ is also an element of W, but $v + (v + w) = w \in W$ as well, since W is a subspace.

Let $\tau' = (w', \alpha') \in G$ be a transvection different from τ. From Lemma 3.7 we know that τ' doesn't fix w if and only if τ and τ' don't commute. Note that τ and τ' don't commute if and only if they are connected in the diagram of the transvections. Suppose this is the case, then $\tau'(w) = w + w'$ and thus $w' \in W$. That means that if a center of a transvection is in W, then the centers of all adjacent transvections are in W as well. Because the diagram is connected, we know that all centers are in W. Since these centers span V, we have $W = V$, meaning the action of G on V is irreducible.

Let G be generated by transvections. Suppose the group is irreducible on V, such that there is no invariant subspace other than $\{0\}$ and V.

1. Let D_1 and D_2 be two components of the diagram of the transvections. Let W_1 and W_2 be the subspaces of V spanned by the centers of transvections in D_1 and D_2 respectively. Any vector $w \in W_1$ is a sum of centers from D_1. We know that transvections from D_1 commute with all transvections in D_2, so all centers in W_1 are fixed by transvections in D_2. Therefore, w is fixed as well. Any transvection in D_1 will either fix w or add a center from W_1 to it, keeping it in W_1. We reach a contradiction, as we conclude that W_1 and W_2 are two invariant subspaces, meaning G would not be irreducible on V.

2. Suppose the intersection of the axes in G contains the vector v. Then v is fixed by all elements of G and thus $\{0, v\}$ would form an invariant subspace, giving a contradiction.

3. Suppose the centers in G span a subspace W of V. Then any vector $w \in W$ is either fixed or is mapped to W by elements of G. We get that W is an invariant subspace of V, making G reducible and thus giving a contradiction.

With that we have proven the statement both ways. □

Assuming the group generated by transvections is irreducible on some vector space, we can further specify it using the following result from McLaughlin.

Theorem 3.14. [7] Let V be a vector space of dimension $n \geq 2$ over $\mathbb{F}_2$ and let G be a subgroup of $\mathrm{SL}(V)$ generated by transvections and irreducible on V.

Then n is even, $n \geq 4$ and G is one of the following subgroups of $\mathrm{Sp}(V)$:

- the symplectic group $\mathrm{Sp}(V)$,
- $n \neq 4$; the orthogonal group $\mathrm{O}^-(V)$ or $\mathrm{O}^+(V)$,
- the symmetric group of degree $n+1$ or $n+2$.

To motivate the third option, we look at the action of the symplectic group on a vector space. Let $\Omega = \{1,\ 2,\ \ldots,\ n\}$. Let $V = \mathbb{F}_2\Omega$ be a vector space with Ω as a basis.

Definition 3.15. Let V be a vector space over a field F, with basis B. Let $v \in V$ be a vector. Then v can be written as

$$v = \sum_{b \in B} x_b \cdot b$$

where x_b is a scalar in F, dependent on b. The **support** of v is the set of elements $b \in B$ such that $x_b \neq 0$, often denoted as $\mathrm{supp}(v)$. The **weight** of v is the cardinality of the support of v.

As an example, consider the vector space V and basis Ω as described above. Let $v = 2+3+4 \in V$. Then the support of v is $\mathrm{supp}(v) = \{2, 3, 4\}$. The weight of v is $|\mathrm{supp}(v)| = 3$.

Consider the **even-weight** subspace $W \leq V$ consisting of vectors of even weight, e.g. $1+2$, or $1+2+5+7$. Notice that W is spanned by $\langle a+b \mid a, b \in \Omega\rangle$.

Lemma 3.16. The even-weight subspace W of V is an invariant subspace of the action of the symmetric group $\mathrm{Sym}(\Omega)$ on V.

Proof. First notice that 0 is in W and that the sum of two vectors with even weight has, again, even weight. Let $w \in W$ be a vector with even weight. All permutations in $\mathrm{Sym}(\Omega)$ keep the weight of vectors constant. An example would be the vector $1+2$ and the permutation $(2\ \ 3)$. Applying the permutation to the vector results in $1+3$, which has the same weight as $1+2$. Thus for all $\sigma \in \mathrm{Sym}(\Omega)$, we have that $\sigma(w) \in W$, making W an invariant subspace. $\square$

It is clear that, because V is of dimension n, the subspace W is of dimension $n-1$. Adding a single vector to W, for example $1 \in V$, would result in V.

Lemma 3.17. If n is odd, the group action restricted to W is irreducible.

Proof. To see this, we first observe that any vector of weight two is in orbit with all other vectors of weight two. That means that any invariant subspace containing a vector of weight two, contains all vectors of weight two. This, in turn, means that this invariant subspace must be W itself, as W is generated by all vectors of weight two.

Now take any even-weight vector $v \in W$. This vector has at most weight $n-1$, meaning there is always an element $a \in \Omega$ that is not in the support of v. Furthermore, let $b \in \operatorname{supp}(v)$ and $\sigma = (a \;\; b) \in \operatorname{Sym}(\Omega)$. Then $v + \sigma(v) = a + b$, which means that any invariant subspace containing v contains a vector of weight 2 and must therefore be W. We conclude that $\operatorname{Sym}(\Omega)$ irreducible on W. $\square$

The argument does not hold in case n is even. We have the vector of weight n, $1 + 2 + \cdots + n$, in W. Any permutation in $\operatorname{Sym}(\Omega)$ on this vector, maps to the same vector. We get that $X = \langle 1 + 2 + \cdots + n \rangle$ forms an invariant subspace of W.

Lemma 3.18. If n is even, the group action on W/X is irreducible.

Proof. For any $v \in W$, we denote $\overline{v} = vX \in W/X$. We have that any invariant subspace containing a vector of weight 2, must be the whole space W/X. Any vector $\overline{w} \in W/X$ has weight $\leq n-2$ or $\overline{w} = \overline{0}$. We can apply the same reasoning as in the proof of Lemma 3.17 and conclude that any invariant subspace containing $\overline{w}$ must be the whole space W/X. $\operatorname{Sym}(\Omega)$ is therefore irreducible on W/X. $\square$

Notice that X has dimension 1 and W has dimension $n-1$. Therefore W/X has dimension $n-2$. This supports the third option of Theorem 3.14. If a symmetric group acts irreducibly on a vector space of dimension m, the group is of degree $m+1$ odd, or $m+2$ even.

Lemma 3.19. For all distinct transvections $d, e \in G$ on some vector space V, we have that $|de| \neq 4$ if and only if for any center $w \in V$ there is at most one axis $H = \ker(\alpha) \in A$ for some $\alpha \in V^*$ such that $(w, \alpha) \in G$.

Proof. Suppose we have two transvections $\tau = (w, \alpha)$, $\tau' = (w', \alpha')$ in G with a product of order 4. From Lemma 3.7, we know that without loss of generality $w \neq w'$, $\alpha \neq \alpha'$, $\alpha'(w) = 0$ and $\alpha(w') = 1$. Now, $(\tau'\tau)^2 = (w, \alpha')$ is also a transvection in G, since $\alpha'(w) = 0$. Therefore, center w has more than one axis in T.

Now suppose that there are two transvections in G with identical centers but different axes, $\tau = (w, \alpha)$, $\tau' = (w, \alpha')$. We get that the product $\tau'\tau$ must be of order two, according to Lemma 3.7. Furthermore, $\tau'\tau = (w, \alpha'')$ such that $\alpha'' \neq \alpha, \alpha'$ and $\ker(\alpha) \cup \ker(\alpha') \cup \ker(\alpha'') = V$. Since all points in V are a center, for each point there must be at least one axis. Take any $p \in V$ such that

$p \notin \ker(\alpha) \cap \ker(\alpha') \cap \ker(\alpha'')$. We know $\sigma = (p, R) \in G$ for some axis R and let's assume without loss of generality that $p \in \ker(\alpha)$. We have two options:

- $w \in R$: in this case $|\tau'\sigma| = 4$ and $|(\tau'\tau)\sigma| = 4$.
- $w \notin R$: in this case $|\tau\sigma| = 4$

□

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph on which we play the game. Let $V = \mathbb{F}_2\mathcal{V}$ be a vector space. We can define a bilinear form on V.

Definition 3.20. Let $B : V \times V \to \mathbb{F}_2$ be a bilinear form defined for all $u, v \in \mathcal{V}$ to be

$$B(u,v) = \begin{cases} 0 & \text{if } \{u,v\} \in \mathcal{E} \\ 1 & \text{else} \end{cases}$$

It is clear that B extends to a symplectic form on V. Also notice how for all $u, v \in \mathcal{V}$, $B(u,v) = f(\mu_u, \mu_v)$, where f is the symplectic form defined in Definition 3.10

Lemma 3.21. B is nondegenerate if and only if Γ is nondegenerate

Proof. We choose an ordered basis for V. Let A be the adjacency matrix of Γ with respect to this basis. We can write $B(u,v) = u^\top Av$. By definition, B is nondegenerate if and only if A is nondegenerate. A is nondegenerate if and only if Γ is nondegenerate. □

We want to find a symplectic form on V^* that is preserved by the group $\mathcal{M}(\Gamma)$. Definition 2.8 of vector A_v for some $v \in \mathcal{V}$ can be extended to a linear map $\theta : V \to V^*$, such that for $v, w \in \mathcal{V}$, $\theta(v + w) = A_v + A_w$. Notice that for all $v, w \in V$, $\theta(v)(w) = B(v,w)$. It is clear that θ is an isomorphism between V and V^* if and only if Γ is nondegenerate.

Definition 3.22. Assuming Γ is nondegenerate, B induces a symplectic form B^* on V^* defined by

$$B^*(x,y) = B(\theta^{-1}(x), \theta^{-1}(y))$$

We have $B^*(A_v, A_w) = 1$ if and only if $\{v,w\} \in \mathcal{E}$. Notice that $B^*(x,y) = x(\theta^{-1}(y))$ for all $x, y \in V^*$.

Lemma 3.23. The group $\mathcal{M}_\Gamma$ preserves B^*.

Proof. Let $\mu_v \in \mathcal{M}$ be a move on a vertex $v \in \mathcal{V}$. Now for any $x, y \in V^*$, we have that

$$\begin{aligned}
&B^*(\mu_v(x), \mu_v(y)) \\
&= B^*(x + x(v)\theta(v), y + y(v)\theta(v)) \\
&= B^*(x, y) + y(v)B^*(x, \theta(v)) + x(v)B^*(y, \theta(v)) + x(v)y(v)B^*(\theta(v), \theta(v))
\end{aligned}$$

But notice that $B^*(x, \theta(v)) = x(\theta^{-1}(\theta(v))) = x(v)$. Furthermore, we have $B^*(\theta(v), \theta(v)) = 0$. We get

$$\begin{aligned}
&= B^*(x, y) + y(v)x(v) + x(v)y(v) \\
&= B^*(x, y)
\end{aligned}$$

This means that the moves on vertices preserve the symplectic form and thus $\mathcal{M}_\Gamma$ does as well. □

Definition 3.24. Let V be a vector space over a field F. We call a mapping $q : V \to F$ a **quadratic form** on V if and only if it satisfies $q(u + v) = q(u) + q(v) + f(u, v)$ for all $u, v \in V$, where f is a bilinear mapping. A mapping $\rho : V \to V$ is called **orthogonal** if it preserves the quadratic form. That is, $q(\rho(v)) = q(v)$ for all $v \in V$.

Lemma 3.25. Let $\tau(x) = x + \alpha(x)w$ be a transvection. Then τ is an orthogonal mapping with respect to some quadratic form q if and only if $q(w) = 1$ and $f(w, y) = 0$ for all $y \in \ker(\alpha)$, where f is the bilinear component.

Proof. We check the definition for an orthogonal map, making use of the fact that if $x \in \ker(\alpha)$, then $\alpha(x) = 0$, and if $x \notin \ker(\alpha)$, then $f(x, w) = 1$

$$q(\tau(x)) = q(x) + \alpha(x)q(w) + \alpha(x)f(x, w) = q(x)$$

□

With this in mind, we can define a map $Q : V \to \mathbb{F}_2$, such that $Q(v) = 1$ for all $v \in \mathcal{V}$ and $Q(u + v) = Q(u) + Q(v) + B(u, v)$ for all $u, v \in V$. Q extends to a quadratic form on V. When Γ is nondegenerate, Q induces a quadratic form on V^* defined by

$$Q^*(x) = Q(\theta^{-1}(x))$$

for all $x \in V^*$.

Lemma 3.26. The group $\mathcal{M}_\Gamma$ preserves Q^*.

Proof. Let $\mu_v \in \mathcal{M}_\Gamma$ be a move on vertex $v \in \mathcal{V}$. We have $\mu_v(x) = x + x(v)\theta(v)$. The center of this transvection is $\theta(v)$, for which we have $Q^*(\theta(v)) = Q(v) = 1$. Furthermore, for all $y \in V^*$ such that $y(v) = 0$, we have $B^*(y, \theta(v)) = y(v) = 0$. We conclude that the moves on vertices are orthogonal mappings. Hence, $\mathcal{M}_\Gamma$ preserves the quadratic form. $\square$

From Lemma 3.23 and Lemma 3.26 we can conclude that the group $\mathcal{M}_\Gamma$, with Γ nondegenerate, is isomorphic to subgroups of the symplectic group and orthogonal group on V.

4 Partial linear spaces

Definition 4.1. Let $\mathcal{P}$ be a set of points and $\mathcal{L}$ a set of lines, where each line is a subset of $\mathcal{P}$. $\Pi = (\mathcal{P}, \mathcal{L})$ is an incidence structure on the points $\mathcal{P}$ and lines $\mathcal{L}$. Π is a **partial linear space** if the following axioms hold:

1. any line is incident with at least two points
2. any pair of distinct points is incident with at most one line

Definition 4.2. Let x, y be points in a partial linear space $\Pi = (\mathcal{P}, \mathcal{L})$. We say that x and y are **collinear**, and write $x \sim y$, if and only if there is some line $\ell \in \mathcal{L}$, such that $\{x, y\} \subseteq \ell$.

Definition 4.3. $\Pi' = (\mathcal{P}', \mathcal{L}')$ is a **subspace** of $\Pi = (\mathcal{P}, \mathcal{L})$ if and only if:

- $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$, and
- For any pair of points $x, y \in \mathcal{P}'$, we have that x and y are collinear in Π' if and only if they are collinear in Π.

We write $\Pi' \leq \Pi$.

Definition 4.4. Let $\Pi' \leq \Pi$ be a subspace. We call Π' a **clique** of Π if and only if all pairs of points in Π' are collinear.

Definition 4.5. Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a partial linear space. The smallest subspace Π' containing two intersecting lines $\ell_1, \ell_2 \in \mathcal{L}$ is called the **plane** spanned by ℓ_1 and ℓ_2. Let S be the set of subspaces of Π that contain ℓ_1 and ℓ_2:

$$S = \{W \leq \Pi \mid \ell_1, \ell_2 \in \mathcal{L}_W\}$$

The plane spanned by ℓ_1 and ℓ_2 can be constructed as the intersection of all subspaces in S:

$$\Pi' = \bigcap_{W \in S} W$$

4.1 The triangular space

Definition 4.6. Consider some set Ω of cardinality at least 2. We define the **triangular space** $\mathcal{T}(\Omega)$ to be a partial linear space where points are the 2-subsets of Ω and in which there are three points on each line. Two points are collinear if an only if they intersect by exactly one element from Ω. The third point on the line going through two points is the symmetric difference of the two. We observe that the unions of the points on each line form the 3-subsets of Ω.

Theorem 4.7. Let $\Pi = (\mathcal{P}, \mathcal{L})$ be a partial linear space. Π is isomorphic to a triangular space $\mathcal{T}(\Omega)$ for some set Ω if and only if:

1. each line ℓ of $\mathcal{L}$ has three points on it,
2. for each point $p \notin \ell$, p is collinear with none or two points of ℓ. In the latter case, p and ℓ generate a plane isomorphic to a dual affine plane of order 2.
3. for any plane $\Pi' = (\mathcal{P}', \mathcal{L}')$ in Π and a point p such that $p \notin \Pi'$, p is non-collinear to at least one line in $\mathcal{L}'$.

Proof. We first prove that if Π is isomorphic to the triangular space $\mathcal{T}(\Omega)$, then conditions (1), (2) and (3) hold. We use the Fig. 4 for reference.

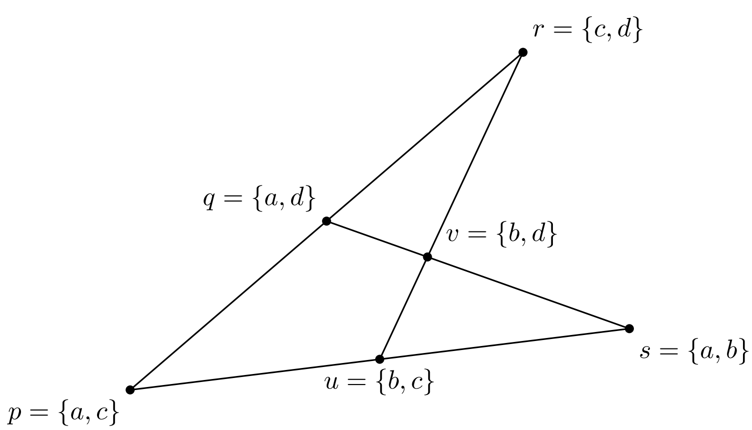

Figure 4: A plane in the triangular space spanned by three points. Each point is labelled with a 2-subset from Ω

1. From the definition of a partial linear space we know that each line in $\mathcal{T}(\Omega)$ connects at least two points with one element in the intersection, say $\{a, b\}$ and $\{a, c\}$. Because the space is triangular, we also know that

the symmetric difference of the two points, namely $\{b, c\}$, also lies on the line. These three points are closed under the symmetric difference, hence each line contains exactly three points.

2. Let $s = \{a, b\}$ be a point and ℓ be a line in $\mathcal{T}(\Omega)$, such that $s \notin \ell$. Let p, q, r be distinct points on ℓ. If s is collinear to p, they intersect in a point $a \in \Omega$. However, since q and r are 2-subsets of a 3-subset of Ω and are distinct, one of them must also contain a. Without loss of generality, we have that $a \in q$ and therefore $s \sim q$ and $s \not\sim r$. We conclude that either s is collinear to no points, or exactly two points of ℓ.

3. Let $\Pi' = (\mathcal{P}', \mathcal{L}')$ be a plane in $\mathcal{T}(\Omega)$. Let $\Omega' = \bigcup \mathcal{P}'$, then $|\Omega'| = 4$. A point p not in Π' contains at most one element from Ω'. Since the four lines in $\mathcal{L}'$ are made up of 3-subsets of Ω', there must be exactly one line which does not have a point containing a. Therefore, p is non-collinear to at least one line in $\mathcal{L}'$.

We now prove the equality the other way around. Assume we have a partial linear space $\Pi = (\mathcal{P}, \mathcal{L})$ for which conditions (1), (2) and (3) hold.

Let p, q and r be points on a line in Π. We define

$$M = M_{p,q} = \{s \in \mathcal{P} \mid s \sim p \text{ and } s \sim q \text{ and } s \neq r\}$$

and claim that M forms a clique.

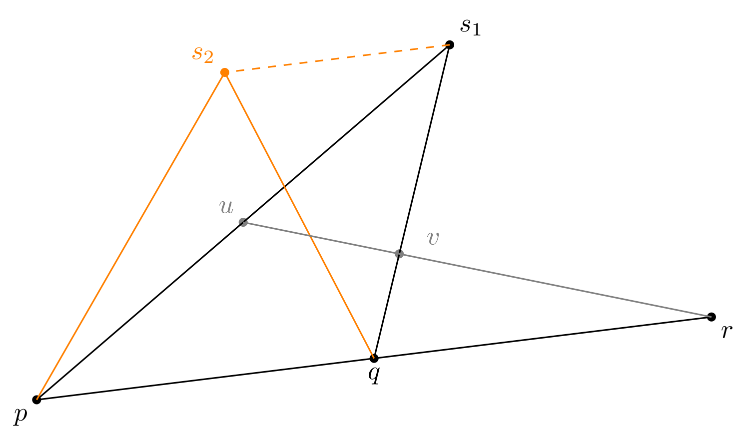

Figure 5: Points collinear with p and q must form a clique.

Take any distinct $s_1, s_2 \in M$, now from (2) we have that s_1 and the line $p - r$ form a plane as can be seen in Fig. 5. Suppose s_2 is not collinear with s_1 but is collinear with a point in the plane, then by (3) s_2 must be non-collinear with either line $s_1 - p$ or line $s_1 - q$. But, since $s_2 \in M$, we have that $s_2 \sim p$ and $s_2 \sim q$. Therefore, s_2 must be collinear to s_1 and thus M forms a clique.

Let $\Omega = \{M_{p,q} \mid p, q \in \mathcal{P}\}$ be the collection of cliques in Π. A line ℓ consisting of points $p, q, r \in \mathcal{P}$ generates three distinct cliques $M_{p,q}, M_{p,r}$ and $M_{q,r}$. The points on ℓ are each in exactly two of these. Suppose there is another clique $M_{p,s}$ for some $s \notin \ell$, containing p. Since s is collinear with a point in ℓ, by (2), s must also be collinear with either q or r. Without loss of generality, we say $s \sim q$ and $s \not\sim r$. By definition, we get that $q \in M_{p,s}$ and $s \in M_{p,q}$. Since $M_{p,s}$ is a clique, any point $x \in M_{p,s}$ is collinear to q and thus in $M_{p,q}$, giving $M_{p,s} \subseteq M_{p,q}$. Any point $y \in M_{p,q}$ is collinear to s and thus in $M_{p,s}$, giving $M_{p,q} = M_{p,s}$. We conclude that points can only be in exactly two cliques.

We label the collection of cliques as $\Omega = \{C_1, C_2, \ldots\}$. Each point p in Π, such that $p \in C_i$ and $p \in C_j$, can be mapped to a point $\{C_i, C_j\}$ in $\mathcal{T}(\Omega)$. Two points $p, q \in \Pi$ such that $p \sim q$ are both in the same clique $M_{p,q}$ and therefore their mapped points $\{M_{p,r}, M_{p,q}\}, \{M_{q,r}, M_{p,q}\}$ respectively in $\mathcal{T}(\Omega)$ intersect by exactly one element (clique). The third point r collinear with p, q must be in the cliques $M_{p,r}$ and $M_{p,r}$ and therefore is mapped to the symmetric difference of p and q in $\mathcal{T}(\Omega)$.

We get that Π must be isormorphic to $\mathcal{T}(\Omega)$, completing our proof. □

4.2 Partial linear space on transvections

By Lemma 3.19, we assume that each center has a unique corresponding axis in G. We therefore identify each transvection uniquely by its center from now on and write w instead $\tau = (w, \alpha) \in T$.

Let T be the set of transvections generating G. We have that for any $d,\ e \in T$, either $(de)^2 = 1$ or $(de)^3 = 1$.

Definition 4.8. We define a partial linear space $\Pi_T = (T, \mathcal{L})$, where two transvections $d,\ e \in T$ are collinear if $(de)^3 = 1$. The lines in Π_T are defined as $\mathcal{L} = \{\{d,\ e,\ f\} \subseteq T \mid d \sim e \text{ and where } ded = f\}$.

Let L be the space spanned by the summed lines in Π_T. In other words, $L = \langle d + e + f \mid \{d,\ e,\ f\} \in \mathcal{L}\rangle$. Let $N = \langle p + q \in \mathbb{F}_2 T \mid \forall s \in \mathbb{F}_2 T : s \sim p \Leftrightarrow s \sim q\rangle$ be spanned by the sum of points with identical sets of collinear points.

Definition 4.9. Let f be a symplectic form on some vector space W over a field F. We define the **radical** of f to be the set

$$\mathrm{Rad}(f) = \{v \in W \mid f(v, w) = 0 \text{ for all } w \in W\}$$

We use the symplectic form f as defined in Definition 3.10. Consider the quotient space $V = \mathbb{F}_2T/\mathrm{Rad}(f)$.

Notation. We denote $\overline{d} \in V$ to mean $d + \mathrm{Rad}(f) = \{d + x \mid x \in \mathrm{Rad}(f)\}$ for some $d \in \mathbb{F}_2T$.

Lemma 4.10. $L \subseteq \mathrm{Rad}(f)$ and $N \subseteq \mathrm{Rad}(f)$.

Proof. Take any $\{p, q, r\} \in \mathcal{L}$ and any point $s \in \mathbb{F}_2T$. Without loss of generality, either $f(p, s) = f(q, s) = 1$ and $f(r, s) = 0$, or $f(p, s) = f(q, s) = f(r, s) = 0$. In both cases $f(p + q + r, s) = 0$ and therefore $p + q + r \in \mathrm{Rad}(f)$.

Take any $p + q \in N$. For all $s \in \mathbb{F}_2T$, either $f(p, s) = 1$ or $f(p, s) = 0$. But since p and q have the same collinear points, then also $f(q, s) = 1$ or $f(q, s) = 0$ respectively. In both cases $f(p + q, s) = 0$ and thus $p + q \in \mathrm{Rad}(f)$. ☐

Definition 4.11. Partial linear space Π_T is called *reduced* if and only if $N = \{0\}$, that is, no two points have the same sets of collinear points.

Lemma 4.12. Suppose V contains at least 2 collinear elements. Then V is non-trivial.

Proof. Let $d, e \in \mathbb{F}_2T$, such that d and e are collinear. Suppose V is trivial and thus $\overline{d} = \overline{e}$. Take any r such that $r \sim d$, but $r \not\sim e$, and let $s = drd$. Now $\overline{d} + \overline{r} = \overline{e} + \overline{r}$, but $\overline{d + r} = \overline{s} \neq \overline{e + r}$ since e, r and s are not on a line. Therefore it must be that $\overline{d} \neq \overline{e}$ and thus V is non-trivial. ☐

Remark. It is clear that f on V is nondegenerate. From now on, we work with the assumption that the symplectic form f is nondegenerate unless stated otherwise.

With this form, we know that two intersecting lines in Π_T span a dual affine plane and thus Π_T fulfills conditions (1) and (2) from Theorem 4.7.

For any $p \in T$, we can define the permutation δ_p of T defined as:

$$\delta_p(q) = \begin{cases} q & \text{if } p \not\sim q \\ p + q = r & \text{if } \{p, q, r\} \in \mathcal{L} \end{cases}$$

Lemma 4.13. Let $D = \langle \delta_p \mid p \in T \rangle$. $D \leq \mathrm{Aut}(\Pi_T)$.

Proof. Take $\delta_x \in D$. We show that $p + q + r = 0$ if and only if $\delta_x(p) + \delta_x(q) + \delta_x(r) = 0$.

If $p, q, r \not\sim x$, then by definition $\delta_x(p) + \delta_x(q) + \delta_x(r) = p + q + r = 0$. We assume, without loss of generality, that $p, q \sim x$ and $r \not\sim x$. Now $\delta_x(p) = p + x = p'$,

$\delta_x(q) = q+x = q'$ and $\delta_x(r) = r$. We get $\delta_x(p)+\delta_x(q)+\delta_x(r) = p+x+q+x+r = p+q+r = 0$. $\square$

Theorem 4.14. Assuming that Π_T also fulfills condition (3) from Theorem 4.7 and is thus triangular, then $\langle T \rangle \cong D \cong \mathrm{FSym}(\Omega)$ for some Ω. Where $\mathrm{FSym}(\Omega)$ is the symmetric group generated by permutations with finite support.

Proof. With the assumption, we can construct cliques from the points in Π_T, similarly as in the proof of Theorem 4.7. For any p and q that are collinear, with $r = p+q$, we define:

$$M_{p,q} = \{s \in T \mid s \sim p \text{ and } s \sim q \text{ and } s \neq r\}$$

Consider the collection of cliques $\Omega = \{M_{p,q} \mid p, q \in T \text{ and } p \sim q\}$, labelled as $\Omega = \{C_1, C_2, C_3 \dots\}$. We know that any $p \in T$ is in exactly two cliques, say C_i and C_j. Notice how the permutation δ_p is an involution, i.e. of order 2. Also notice that the support of δ_p is $C_i \cup C_j$. Furthermore, $\delta_p(C_i) = \{\delta_p(x) \mid x \in C_i\} = C_j$ and $\delta_p(C_j) = C_i$. The map $\delta_p \mapsto (C_i\ C_j)$ clearly gives an isomorphism between D and $\mathrm{FSym}(\Omega)$. $\square$

4.3 Consequences for the game

Going back to our game, we want to see under what conditions the group of moves is isomorphic to the symmetric group. We translate the conditions imposed in the previous subsection to the moves on a graph $\Gamma = (\mathcal{V}, \mathcal{E})$. We have already seen that a move μ_v on some vertex $v \in \mathcal{V}$ is a transvection in V^*, defined as

$$\mu_v(f) = f + f(v)A_v.$$

The center of the transvection is A_v. The axis of the transvection is $\ker(f)$, the subspace spanned by all vertices except v. Notice how the axis is a hyperplane uniquely defined by v.

Theorem 4.15. If Γ is connected and nondegenerate, then $\mathcal{M}_\Gamma$ is generated by transvections, irreducible on V^*, and preserves a symplectic form B^* as defined in Definition 3.22.

Proof. We use the conditions in Lemma 3.13. Let $T = \{\mu_v \mid v \in \mathcal{V}\}$ be the transvections that generate $\mathcal{M}_\Gamma$. If Γ is connected, then the diagram of T must be connected as well, since Γ and its diagram are clearly isomorphic.

The axis of the move on some vertex v is the subspace spanned by all vertices except v. All vertices in $\mathcal{V}$ give rise to a move and are therefore excluded at least once in some axis. Thus, the intersection of these axes can only be $\{0\}$.

The centers of T span $W = \langle A_v \mid v \in \mathcal{V} \rangle$, i.e. the row/column space of the adjacency matrix of Γ. If Γ is nondegenerate then its adjacency matrix is invertible and $W = V^*$. $\square$

We assume Γ is connected and nondegenerate. We define the partial linear space $\Pi_T = (T, \mathcal{L})$ on the transvections generating $\mathcal{M}_\Gamma$ the same way as in Definition 4.8, equipped with the same nondegenerate symplectic form.

Theorem 4.16. $\mathcal{M}_\Gamma$ is isomorphic to $\mathrm{Sym}(\Omega)$ for some set Ω if and only if $\Gamma = (\mathcal{V}, \mathcal{E})$ is a line graph of some graph Δ , i.e. $\Gamma = L(\Delta)$, where $\Delta = (\Omega, \mathcal{V})$.

Proof. By Theorem 4.14, we know that if Π_T is triangular then $\mathcal{M}_\Gamma$ is isomorphic to $\mathrm{Sym}(\Omega)$ for some Ω. If $\Pi_T \subseteq \mathcal{T}(\Omega)$, then each vertex in $\mathcal{V}$ is isomorphic to a move on that vertex, which in turn is isomorphic to a pair of elements $\{a, b\} \subseteq \Omega$. For sake of simplicity, we will consider vertices as pairs from Ω.

We can now construct a graph $\Delta = (\Omega, \mathcal{V})$, where $\mathcal{V}$ are the pairs in Ω representing vertices in Γ and edges in Δ. We check the property of a line graph: $\{u, v\} \in \mathcal{E}$ if and only if $u \cap v \neq \emptyset$. Indeed two vertices $u, v \in \mathcal{V}$ are connected in Γ and in Π_T if and only if their respective pairs in Ω, say $\{a, b\}$ and $\{c, d\}$, have an element in common. Therefore, $\Gamma = L(\Delta)$.

Assume $\Gamma = L(\Delta) = (\mathcal{V}, \mathcal{E})$ for some $\Delta = (\Omega, \mathcal{V})$. We can map a move μ_v on any vertex $v \in \mathcal{V}$ such that $v = \{a, b\} \subseteq \Omega$ to the transposition $(a \;\; b) \in \mathrm{Sym}(\Omega)$ giving an isomorphism between $\mathcal{M}(\Gamma)$ and $\mathrm{Sym}(\Omega)$. It is clear that the partial linear space on the pairs from Ω is triangular. $\square$

Definition 4.17. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a line graph of some graph Δ , i.e. $\Gamma = L(\Delta)$. We call Δ the **root graph** of Γ.

4.4 Degeneracy

In the previous sections, we have worked under the assumption that the graph on which the game is played is nondegenerate. We now know that if this graph is a line graph, then the group of moves is isomorphic to the symmetric group of some degree. Let us now look at cases in which the graph is degenerate.

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a connected graph that is not degenerate. We consider the standard symplectic form B on $V = \mathbb{F}_2\mathcal{V}$ as defined in Definition 3.20. Let $\overline{V} = V/\mathrm{Rad}(B)$ be a quotient space of V. Let $\overline{B}$ be the induced nondegenerate symplectic form on $\overline{V}$.

For any $x \in \mathcal{V}$ we define the transvection $\tau_x : V \to V$ by

$$\tau_x(y) = y + B(x, y)x$$

For any $x \in V$, we write $\overline{x}$ to denote $x+\mathrm{Rad}(B) \in \overline{V}$. The transvection τ_x leaves $\mathrm{Rad}(B)$ invariant and therefore acts on $\overline{V}$ as well by the induced transvection

$$\tau_{\overline{x}}(\overline{y}) = \overline{y} + \overline{B}(\overline{x}, \overline{y})\overline{x}$$

Lemma 4.18. [8] Let $G = \langle \tau_x \mid x \in \mathcal{V} \rangle$ be a group generated by transvections. Let $\overline{G} = \langle \tau_{\overline{x}} \mid x \in \mathcal{V} \rangle$ be the induced group acting on $\overline{V}$. Then $\overline{G}$ is an irreducible subgroup of $\mathrm{GL}(\overline{V})$.

Notice that if $x, y \in \mathcal{V}$ have the same neighbours, i.e. $B(x, z) = B(y, z)$ for all $z \in V$, then also $B(x + y, z) = B(x, z) + B(y, z) = 0$ for all $z \in V$. This means that $x + y \in \mathrm{Rad}(B)$ and thus $\overline{x} = x + \mathrm{Rad}(B) = y + \mathrm{Rad}(B) = \overline{y}$. Therefore the induced transvections of x and y are also identical, i.e. $\tau_{\overline{x}} = \tau_{\overline{y}}$. If $x, y \in \mathcal{V}$ do not have the same neighbours, then $\overline{x} \neq \overline{y}$ and thus $\tau_{\overline{x}} \neq \tau_{\overline{y}}$.

Definition 4.19. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a connected graph. For any x, y we write $x \approx y$ if and only if x and y have the same neighbours in Γ. It's clear that $\approx$ is an equivalence relation and we denote the equivalence class of x with $\widetilde{x}$. Now consider the graph $\widetilde{\Gamma} = (\widetilde{\mathcal{V}}, \widetilde{\mathcal{E}})$, where $\widetilde{\mathcal{V}} = \mathcal{V}/\approx$ and $\widetilde{\mathcal{E}} = \{\{\widetilde{u}, \widetilde{v}\} \mid \{u, v\} \in \mathcal{E}\}$. We say Γ is **reduced** if and only if $\widetilde{\Gamma} = \Gamma$.

Again, notice that if $x, y \in \mathcal{V}$ have the same neighbours, then $\widetilde{x} = \widetilde{y}$ and thus these points are identified in $\widetilde{\Gamma}$. If x and y do not have the same neighbours, then $\widetilde{x} \neq \widetilde{y}$ and are therefore distinct points in $\widetilde{\Gamma}$. We can conclude that the transvections in the induced group $\overline{G}$ are in one-to-one correspondence with vertices of the reduced graph $\widetilde{\Gamma}$.

Lemma 4.20. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a connected graph. Let $V = \mathbb{F}_2\mathcal{V}$ and let $G = \langle \tau_x \mid x \in \mathcal{V} \rangle$ be a group generated by transvections on V. Then $\overline{G} \cong \mathrm{Sym}(\Omega)$ for some set Ω if and only if $\widetilde{\Gamma}$ is a line graph.

Proof. Since there is a one-to-one correspondence between transvections in $\overline{G}$ and vertices in $\widetilde{\Gamma}$, we can apply Lemma 4.18 and Theorem 4.14 to come to a similar conclusion as Theorem 4.16. However, $\widetilde{\Gamma}$ is not necessarily nondegenerate and can therefore contain *too many* points, due to hidden relations. It is important to note that unlike in the proof of Theorem 4.16, the isomorphism between $\overline{G}$ and $\mathrm{Sym}(\Omega)$ cannot be directly be constructed from the root graph of $\widetilde{\Gamma}$. □

Let $\mathcal{M}_\Gamma$ be the group of moves on Γ. Note how there is an isomorphism mapping the transvection $\mu_v \in \mathcal{M}_\Gamma$, the move on vertex $v \in \mathcal{V}$, to $\tau_v \in G$. Therefore $\mathcal{M}_\Gamma \cong G$. We also know that $\mathcal{M}_\Gamma$ preserves some symplectic form $B^* : V^* \times V^* \to \mathbb{F}_2$. Consider the induced group $\overline{\mathcal{M}_\Gamma}$ on $\overline{V^*} = V^*/\mathrm{Rad}(B^*)$ which is irreducible. We have $\overline{\mathcal{M}_\Gamma} \cong \overline{G}$.

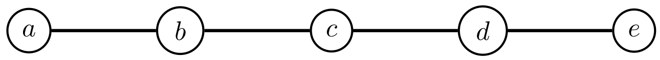

Figure 6: A graph of five points on a line.

Example 4.21. *Consider the graph $\Gamma = (\mathcal{V}, \mathcal{E})$ as shown in Fig. 6. It's clear that Γ is already reduced. The standard symplectic form B on $V = \mathbb{F}_2\mathcal{V}$ is degenerate. In fact, $Rad(B) = \langle a+c+e \rangle$, meaning that $\dim \overline{V} = 4$. Let $G = \langle \tau_x \mid x \in \mathcal{V} \rangle$ be the group generated by transvections on V. Let $\overline{V} = V/Rad(B)$ and let $\overline{G}$ be the induced group acting irreducibly on $\overline{V}$. We have $\overline{x} \neq \overline{y}$ and thus $\tau_{\overline{x}} \neq \tau_{\overline{y}}$ for all $x, y \in \mathcal{V}$. But $\overline{a} = \overline{c+e}$ and thus $\tau_{\overline{a}} = \tau_{\overline{c+e}} = \tau_{\overline{c}}\tau_{\overline{e}}\tau_{\overline{c}}$. Γ is a line graph and thus $\overline{G} \cong S_n$ for some $n \in \mathbb{N}$. We find the following isomorphism, which is also shown in Fig. 7.*

$$\tau_{\overline{a}} \mapsto (1\ \ 2),\ \tau_{\overline{b}} \mapsto (2\ \ 4),\ \tau_{\overline{c}} \mapsto (2\ \ 3),\ \tau_{\overline{d}} \mapsto (3\ \ 5),\ \tau_{\overline{e}} \mapsto (1\ \ 3)$$

From this, we know that the group of moves, $\mathcal{M}_\Gamma$, induces a group $\overline{\mathcal{M}_\Gamma}$ that acts irreducibly on a 2-dimensional vector space and is isomorphic to S_5.

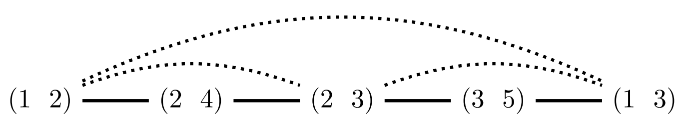

Figure 7: A visualization of the the isomorphism between $\overline{G}$ and S_5. The dotted lines indicate the hidden relation between $\tau_{\overline{a}}$, $\tau_{\overline{c}}$ and $\tau_{\overline{e}}$ in Γ.

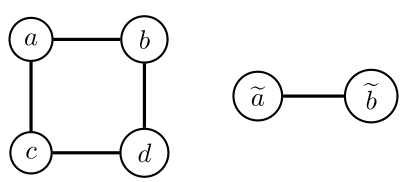

Figure 8: On the left, a graph Γ of four points in a square. On the right, the reduced graph $\widetilde{\Gamma}$.

Example 4.22. *Consider the graph $\Gamma = (\mathcal{V}, \mathcal{E})$ as shown in Fig. 8. In this case $Rad(B) = \langle a+d,\ b+c \rangle$, meaning that $\dim \overline{V} = 2$. Now, $\overline{a} = \overline{d}$ and $\overline{b} = \overline{c}$ and thus $\tau_{\overline{a}} = \tau_{\overline{d}}$ and $\tau_{\overline{b}} = \tau_{\overline{c}}$. As noted earlier, there is a one-to-one correspondence between the transvections in $\overline{G}$ and the vertices in the reduced graph $\widetilde{\Gamma}$, which* *is visualized in Fig. 8 as well. Since $\widetilde{\Gamma}$ is a line graph then $\overline{G} \cong S_n$ for some $n \in \mathbb{N}$. We find the following isomorphism.*

$$\tau_{\overline{a}} \mapsto (1 \;\; 2),\;\; \tau_{\overline{b}} \mapsto (2 \;\; 3),$$

From this, we know that the group of moves, $\mathcal{M}_\Gamma$, induces a group $\overline{\mathcal{M}_\Gamma}$ that acts irreducibly on a 4-dimensional vector space and is isomorphic to S_3.

5 Line graphs

Posing the condition that our game must be played on a line graph that is connected and nondegenerate ensures that the group of moves is isomorphic to the symmetric group. What is left is outlining a way to check this condition. We look closer at triangular partial linear spaces. Considering a plane generated by two lines, condition (3) from Theorem 4.7 states that a point outside of the plane must be noncollinear with at least a line in the plane.

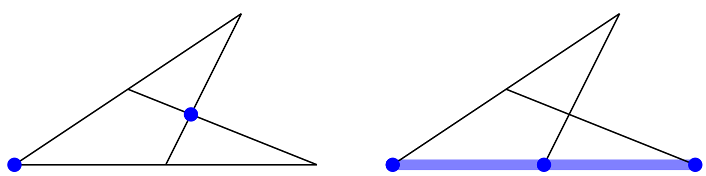

Figure 9: The only two possible situations (up to isomorphism) for an outside point and a symplectic plane to commute. The points marked blue are noncollinear with the outside point. The situation on the right fulfills condition (3) from Theorem 4.7.

If condition (3) is not fulfilled, the only other possibility is that the point outside of the plane is noncollinear with two points that are noncollinear with each other, as can be seen on the left of Fig. 9. Suppose four vertices generated this situation, one of which is the outside point, two of which are the two points noncollinear with it and each other, and the last being an arbitrary point on the plane. The arbitrary vertex is connected to the three vertices giving rise to a claw $K_{1,3}$. By excluding this situation from our graph, we can prevent the partial linear space from breaking condition (3) and thus forms a condition for the graph to be a line graph. In the next section, line graphs will be further characterized.

5.1 Beineke

Definition 5.1. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph. Let $\mathcal{U} \subseteq \mathcal{V}$ be a subset of points. The graph $\Gamma_\mathcal{U} = (\mathcal{U}, \mathcal{E}_\mathcal{U})$, such that for all $u, v \in \mathcal{U}$ we have $\{u, v\} \in \mathcal{E} \Leftrightarrow$

$\{u, v\} \in \mathcal{E}_\mathcal{U}$, is an *induced subgraph*.

There are multiple characterizations of line graphs. One of these was found by Beineke in 1970, giving a set of forbidden induced subgraphs. Part of his result is the following.

Theorem 5.2. [9] A graph Γ is the line graph of some graph if and only if none of the nine graphs in Fig. 10 is an induced subgraph of Γ.

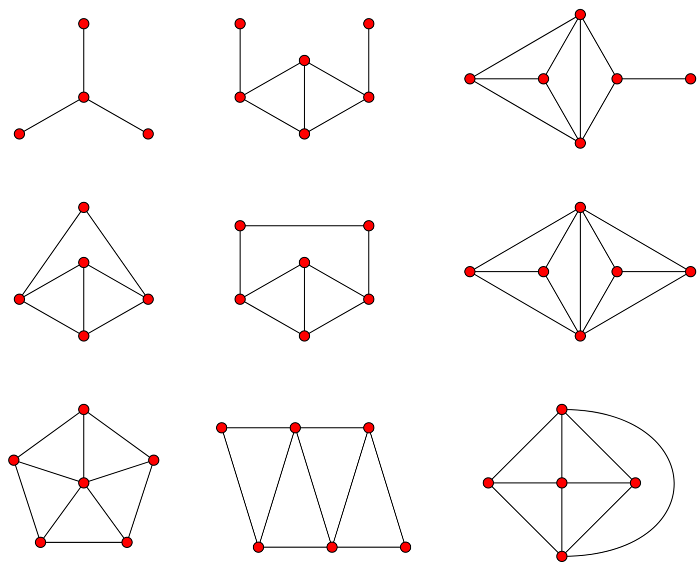

Figure 10: The nine forbidden subgraphs as described by Beineke.

When tasked with deciding whether $\Gamma = (\mathcal{V}, \mathcal{E})$ is a line graph, we can naively apply the results of Beineke. Take any six points of the graph $\mathcal{U} \subseteq \mathcal{V}$ and let $\Gamma_\mathcal{U}$ be the induced subgraph. Although not straightforward, it can be checked whether $\Gamma_\mathcal{U}$ is isomorphic to any forbidden graph on six points, or has a forbidden graph on five or four points as an induced subgraph. We will refer to the procedure as $\textsc{IsForbiddenSubgraph}(\Gamma_\mathcal{U})$. Due to the bounded number of graphs on six points, this procedure takes time $\mathrm{O}(1)$. For a graph on $|\mathcal{V}|$ points we need to call $\textsc{IsForbiddenSubgraph}(\Gamma_\mathcal{U})$ at most $\binom{|\mathcal{V}|}{6}$ times to determine whether Γ is a line graph.

This results in a time $\mathrm{O}\left(\binom{|\mathcal{V}|}{6}\right) = \mathrm{O}(|\mathcal{V}|^6)$ for the algorithm. Although the running time is polynomial, it can be done much quicker.

5.2 Line graph recognition

Degiorgi & Simon [10] have described an algorithm and data structure that is able to dynamically maintain an input graph $\Gamma = (\mathcal{V}, \mathcal{E})$ and, when Γ is a line graph, its root graph $\Delta = (\Omega, \mathcal{V})$, such that $\Gamma = L(\Delta)$. It does so with total running time $O(|\mathcal{E}|)$. The algorithm makes use of the following theorem.

Theorem 5.3. [11] Let Γ and Γ' be connected graphs with isomorphic line graphs. Then G and G' are isomorphic unless one is K_3 and the other is $K_{1,3}$.

Where K_3 is the fully connected graph on three points. $K_{1,3}$ is a claw on four points, where one point is connected to the other three.

Definition 5.4. Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a graph and $\Gamma' = (\mathcal{V}', \mathcal{E}')$ a subgraph. Let $v \in \mathcal{V}$ be a point in Γ such that $v \notin \mathcal{V}'$. Then

$$\Gamma' + v = (\mathcal{V}' \cup \{v\}, \mathcal{E}' \cup \{\{v, w\} \mid \{v, w\} \in \mathcal{E} \text{ and } w \in \mathcal{V}'\})$$

Let $e = \{v, w\}$ be an edge for some endpoints v, w that don't necessarily need to be in $\mathcal{V}$. Then

$$\Gamma + e = (\mathcal{V} \cup e, \mathcal{E} \cup \{e\})$$

Given an input graph Γ, the algorithm builds an input graph Γ' with root graph Δ' by adding vertices e from Γ to it, one by one. This means that if $\Gamma' + e$ is a line graph, then the root graph Δ' has an edge added to it. In other words, if $\Gamma' = L(\Delta')$ and $\Gamma' + e$ is a line graph, then $\Gamma' + e = L(\Delta' + e)$ for some endpoints of e.

The method used to determine the endpoints of e such that $\Delta' + e$ is the root graph of $\Gamma' + e$ only works if Δ' has five or more vertices. Therefore, if Δ' contains four or less points, the root graph $\Delta' + e$ of $\Gamma' + v$ is determined by exhaustive search in a procedure named $\textsc{SmallRoot}(\Gamma' + v)$. The new root graph contains at most 5 points and at most 6 edges. Since there is a bounded number of graphs to check, we get that the procedure $\textsc{SmallRoot}$ takes constant time $O(1)$.

For our game, we work under the assumption that the graph on which the game is played is connected. This simplifies the possibilities in the described algorithm. To determine the root of $\Gamma' + e$ a subgraph $X(e)$ of Δ' is constructed.

Definition 5.5. Let $E_X \subseteq \mathcal{V}'$ be the points in Γ' adjacent to the new vertex e. Then E_X is a subset of the vertices in Δ'. $X(e)$ is defined as the induced subgraph of Δ' such that $X(e) = (V_X, E_X)$.

What is left is determining a placement of e in $X(e)$ to get the new root graph. To do this, we define the following concept.

Definition 5.6. Let T be a subset of V_X. T is an **anchor** if it fulfills:

A1. The subgraph of $E(x)$ induced by T contains no edges.

A2. Every edge in E_X has one endpoint in T.

A3. Every edge not in E_X has no endpoint in T

We assume $\Gamma' = (\mathcal{V}', \mathcal{E}')$ and $\Delta' = (\mathcal{U}', \mathcal{V}')$ are connected, such that $\Gamma' = L(\Delta')$ and $|\mathcal{U}'| \geq 5$. Without going too in-depth, we say an anchor T of $X(e)$ is **'correct'** for e under some conditions specified in the paper. In this case two possibilities are left.

- $|T| = \{x, y\}$ for some $x, y \in \mathcal{U}'$. In this case we set $e = \{x, y\}$.
- $|T| = \{x\}$ for some $x \in \mathcal{U}'$. We now set $e = \{x, y\}$ with $y \notin \mathcal{U}'$ some new point.

We can now describe the step of termining the root of $\Gamma' + e$ with the following two cases.

1. $X(e)$ is connected. The unique correct anchor T for e is computed. If it does not exist, then $\Gamma' + e$ is not a line graph. If it does, the placement of e in Δ' is determined as described above. We have $\Delta' + e$ as the new root graph.
2. $X(e)$ is not connected. There are two connected components X_1 and X_2 of $X(e)$. Now we either get an anchor T with elements in at most one of the components, or an anchor T with one element in each component. In both cases, we use the placement of e as described above. $\Delta' + e$ is the new root graph. If no anchor exists, then $\Gamma' + e$ is not a line graph.

The algorithm assumes Γ' and Δ' are represented as adjacency lists. By choosing an efficient data structure it can be ensured that insertion in the graphs takes time $O(1)$. The algorithm works in steps, each of which consists of inserting an edge in the input graph. For each inserted edge, only the neighbourhood of the edge $(X(e))$ needs to be analyzed to determine a new root graph, which is essentially constant. This makes the running time of the algorithm linear.

6 Conclusion

We have analyzed the lit-only σ-game by looking at the group of moves, $\mathcal{M}_\Gamma$, on a certain graph Γ. We were interested in the cases where this group is isomorphic to the symmetric group S_n for a certain n. To determine this, we first made the observation that a move μ_v on a vertex $v \in \mathcal{V}$ is a transvection. By looking at groups generated by transvections and determining their structure with the use of partial linear spaces, we were able to compare their structure to that of the symmetric group. To be more specific, we determined that a group generated by transvections is isomorphic to a symmetric group if and only if the partial linear space on these transvections is *triangular*. With this, we were able to conclude that the group of moves on a nondegenerate graph Γ is isomorphic to a symmetric group if and only if Γ is a line graph. Furthermore, if Γ is degenerate, then its reduced graph $\widetilde{\Gamma}$ is a line graph if and only if $\mathcal{M}_\Gamma$ induces a group acting irreducibly on some vector space and isomorphic to the symmetric group. After that, we outlined an efficient method for determining whether or not a given graph is a line graph, concluding the analysis of the game in this report.

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