BACHELOR

The Cramer-Lundberg model and copulas

Jacobs, Tom T.M.M.

Award date:
2021

THE CRAMER-LUNDBERG MODEL AND COPULAS

AUTHOR
T.T.M.M. JACOBS   1360108   APPLIED MATHEMATICS

SUPERVISORS
J. RESING
E.PERRONE

Eindhoven University of Technology
Department of Computer Science and Mathematics

WEDNESDAY 14TH JULY, 2021

EINDHOVEN UNIVERSITY OF TECHNOLOGY

1 Abstract

A standard problem in insurance is the management of the amount of capital to prevent bankruptcy. In ruin theory, the Cramer-Lundberg model is used to describe a claim process at an insurance company. In the Cramer-Lundberg model the inter-arrival time and claim size are independent of each other but this is not realistic. In this thesis we analyze the Cramer-Lundberg model, in the case when there is dependence between inter-arrival time and the claim size. To do so, we use the theory of copulas to introduce and analyze various types of dependence. We derive expressions for the cumulative distribution function and the moment generating function of the bivariate distributions with exponential marginals. The distributions that we will consider are the convex combinations of countermonotonic, independent and comonotonic copula, the Farlie-Gumbel-Morgenstern copula, the Ali-Mikhail-Haq copula, the Moran-Downton distribution, and the Bladt-Nielsen distribution. After these derivations, we analyze the ruin probability via numerical approximations and simulations. We use two numerical approximation methods to find an approximation for the adjustment coefficient: Lagrange-Bürmann inversion and *Findroot* built-in function of *Mathematica* (Newton's method). Next to finding the adjustment coefficient we present a simulation study to analyze the ruin probability and average deficit at ruin.

The results for the methods consist of an analysis of the adjustment coefficient, ruin probability, and average deficit at ruin. We vary the dependence for each distribution such that the exponential marginals and all other parameters are fixed. We will compare the result of each distribution by using the Pearson correlation coefficient. We find that Lagrange-Bürmann inversion does not give a satisfactory result, due to numerical issues that need further research. The *Findroot* method and simulation give satisfactory results. The main finding for our distributions is that a more negative Pearson correlation coefficient increases the ruin probability and the average deficit at the ruin, however, it decreases the adjustment coefficient.

Contents

2 Introduction

An insurance company takes over the risk of its clients in exchange for a premium. Just like its clients the insurance company does not want to go bankrupt. Naturally, an insurance company wants to know how much risk it is exposed to and if the amount of capital it possesses is enough to pay for future claims. An approach for getting insight into this is modeling the amount of capital that the insurance company has.

A model that can be used for this is the Cramer-Lundberg model as described in the book of Kaas et al. [1]. In the Cramer-Lundberg model, an insurance company starts with an initial capital and receives a premium per time unit. Also, the insurance company needs to pay all the claims that arrive. The claim amount is the amount of money the insurer needs to pay when a claim comes in, and the inter-arrival time is the time between the arrival of the previous claim and the current claim. An important risk measure for the Cramer-Lundberg model is the ruin probability, which is the probability that the amount of capital or money the insurance company has, becomes less than zero.

In the Cramer-Lundberg model, the random variables of the claim and the corresponding inter-arrival time are independent, in other words, they do not influence each other. But in reality, there can be dependence between the two. For example, consider an insurance company which has an infinite pile of claims, all those claims need to be checked by an investigator. The investigator checks the claims one at the time. We assume that a longer investigation gives rise to a lower pay out. In other words a longer inter-arrival time corresponds to a smaller claim size. The Bladt-Nielsen distribution is a way to model this process. Another argument for dependence is that claims arrive in groups. In the time between the previous group and the next group, claims are added to the group. Then a larger inter-arrival time means that more claims can arrive during that time, which makes the change of a larger total claim of the group higher. So in this case there would be positive dependence. The Moran-Downton bivariate distribution is a way to model this process.

In this thesis, we investigate what happens when the claim size and inter-arrival time are dependent on each other and how this influences the ruin probability. The main tool that is used to create dependence is the so-called "copula" which means "link" in Latin. A copula links two or more marginal distributions together to create a joint probability distribution. Thus the copula fully describes the dependence between two or more random variables. In our case the copulas link the claim size and inter-arrival distribution together, this results in a bivariate distribution. The book *"An Introduction to Copulas"* by Nelsen is a great start to the subject [2]. There are two reasons why we choose our distributions: they can be easily modeled and/or they have a real-life meaning. Therefore, we also investigate two bivariate distributions which have a real-life interpretation for which we do not know the copula.

The goal of the thesis is to give an overview of possible bivariate distributions that can be used to introduce dependence and investigate how a certain distribution influences the ruin probability. These influences are analyzed with a nummerical approximation and a simulation study.

The thesis will be structured in the following way: In Chapter 3 we describe the Cramer-Lundberg model and copulas and give useful properties that will be used in the thesis. In Chapter 4 bivariate distributions with exponential marginals are introduced and analyzed, this concludes the theoretical part of the thesis. In the simulation part of the thesis, and Chapters 5 and 6, the ruin probability, adjustment coefficient and average deficit at ruin are investigated. We end with a discussion and conclusion in Chapters 7 and 8.

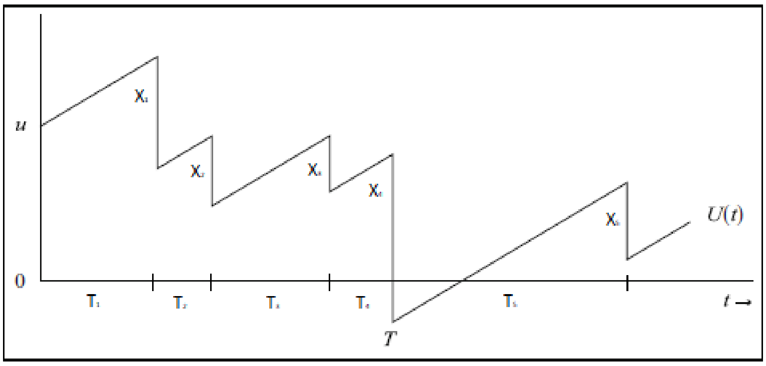


Figure 1: *Illustration of the Cramer-Lundberg model from Kaas et al. [1]*

3 Prerequisites

In this chapter, we introduce definitions of concepts and results that we use in this thesis. This chapter is divided into three subsections, namely insurance, copulas and, dependence and correlation. For the insurance part, most results rely on the book of Kaas et al. [1] and for the copula part, most results rely on the book of Nelsen [2].

3.1 Insurance

In actuarial science, the Cramer-Lundberg model is used to describe the amount of capital an insurance company has at a certain moment in time. We assume that the insurance company starts with an initial capital and gets a premium income per unit of time. The inter-arrival times and the claim sizes follow a certain distribution and are both independent identically distributed or short i.i.d. distributed. The definition of the Cramer-Lundberg is as follows.

Definition 3.1 (Cramer-Lundberg Model)

$$U(t) = u + ct - S(t)\ , t \geq 0, \tag{1}$$

where
$U(t)$ = *the insurer's capital at time t;*
$u = U(0)$ = *the initial capital;*
c = *the (constant) premium income per unit of time;*
$S(t) = X_1 + X_2 + ... + X_{N(t)}$,
$N(t) \sim Poi(\lambda t)$ *with*
$N(t)$ = *the number of claims up to time t, and*
X_i = *the size of the i-th claim, assumed non-negative.*

We refer to the book of Kaas for more details [1]. In (1) we can verify by setting $t = 0$ that the initial capital is indeed u. Also in (1) we can see the units of money earned at time t is ct. A realization of the Cramer-Lundberg model is given in Figure 1, where T_i are the random variables for the inter-arrival and the jumps downward are caused by the claims X_i. Since the assumptions of independence between the claim size and the inter-arrival times $(T_i - T_{i-1})$ is unrealistic, in this thesis we investigate the impact of dependence in this model. To this aim, we introduce another form of the model. In the book of Kaas, there is a second equation that describes the Cramer-Lundberg, which is more convenient for introducing the dependence between the inter-arrival time and the claim size. Therefore, the following form of the model is introduced.

Definition 3.2 *We define the surplus process as follows:*

$$U(n) = u + G_1 + G_2 + ... + G_n\ , n = 0, 1..., \tag{2}$$

where G_i can be seen as the net-profit between the $(i-1)$-th and the i-th claim instant. An additional assumption is that the G_i's are i.i.d..

Now we can use the formulation of the dependence between T_i and X_i for each $i = 1, 2, ...$, by setting $G_i = cT_i - X_i$ such that T_i corresponds to the inter-arrival time between two successive claims, X_i corresponds to the claim size and $c > 0$ is the constant premium income per unit of time. Because of the i.i.d. assumption we can narrow down our investigation to the random vector (T, X) to find the distribution of G. We can use the following result to determine the distribution of G:

Theorem 3.1 *Let (T, X) be a random vector with a known joint probability density function $f_{T,X}(t, x)$ then $cT - X$ has the following probability density function:*

$$f_{cT-X}(z) = \begin{cases} \frac{1}{c}\int_{-z}^{\infty} f_{T,X}(\frac{t+z}{c}, t)dt = \frac{1}{c}\int_{0}^{\infty} f_{T,X}(\frac{t}{c}, t-z)dt \text{ if, } z \leq 0, \\ \frac{1}{c}\int_{0}^{\infty} f_{T,X}(\frac{t+z}{c}, t)dt = \frac{1}{c}\int_{z}^{\infty} f_{T,X}(\frac{t}{c}, t-z)dt \text{ if, } z \geq 0. \end{cases} \tag{3}$$

Theorem 3.1 is helpful if the density of the random vector (T, X) exists. In other words, when the distribution is absolutely continuous.

A core part of the analysis of the Cramer-Lundberg model is to find the ruin probability which is the probability that the insurance company at some point in time has a negative amount of money. For the model described in Definition 3.2 we want to get a closed-form expression for the ruin probability, which is formally defined as follows.

Definition 3.3 (Ruin probability) *Let $U(n)$ be the process described in Definition 3.2. Then the event of ruin can be described as the following random variable:*

$$\tilde{T} = Min\{n : U(n) < 0\}. \tag{4}$$

Now $\tilde{T}$ corresponds to the first instant when the capital of the insurance company goes below 0 (which means the company is ruined). Therefore, the ruin probability can be expressed as:

$$\Psi(u) = \mathbb{P}(\tilde{T} < \infty). \tag{5}$$

To say something about the ruin probability we need to introduce the definition of the adjustment coefficient.

Definition 3.4 (Adjustment coefficient) *The adjustment coefficient $R > 0$ satisfies the following equality:*

$$K_G(-R) = \mathbb{E}[e^{-RG}] = 1. \tag{6}$$

For finding a closed form expression of the ruin probability we need to introduce the deficit at ruin, which is the amount of money the insurance company loses when it gets ruined.

Definition 3.5 (Deficit at ruin) *The deficit at ruin is a random variable with the following distribution:*

$$\mathbb{P}(-U(\tilde{T}) \leq x | \tilde{T} < \infty), \tag{7}$$

where $x \geq 0$.

Finding an explicit expression for the distribution of the deficit at ruin (7) can be hard since the deficit at ruin depends on many factors. Nevertheless, we observe that it is easy to find the distribution if the negative part of a distribution of the random variable $cT - X$ satisfies the memoryless property which we recall below.

Definition 3.6 *A random variable X is said to be memoryless if and only if:*

$$\mathbb{P}(X > x + t | X > x) = \mathbb{P}(X > t) \textit{ for all } x, t \geq 0. \tag{8}$$

If the negative part of the distribution of the random variable $cT - X$ has the memoryless property then the deficit at ruin is equal to the distribution of the negative part. More concretely if:

$$\mathbb{P}(cT - X \leq z | cT - X < 0) \tag{9}$$

has the memoryless property, then:

$$\mathbb{P}(-U(\tilde{T}) \leq z | \tilde{T} < \infty) = \mathbb{P}(X - cT \leq z | cT - X < 0) \text{ for } z \geq 0. \tag{10}$$

To get a better intuition for the deficit at ruin we use Figure 2.

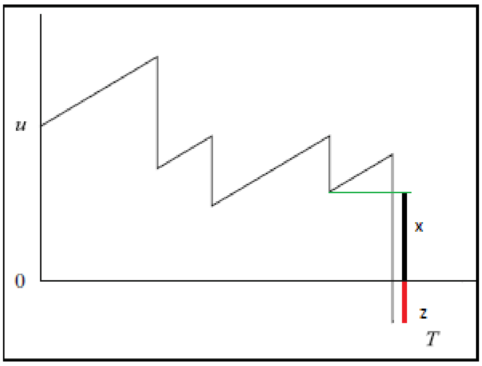


Figure 2: *Deficit at ruin, where the green line indicates that we are in the negative part of the distribution and the red line the deficit at ruin.*

Now we have all the ingredients to determine a closed form expression for the ruin probability, we can give the following theorem.

Theorem 3.2 *Let R be the adjustment coefficient of G as in Definition 3.4 and let $-U(\tilde{T})$ be defined as in Definition 3.5, then the closed expression for the ruin probability is:*

$$\Psi(u) = \frac{e^{-Ru}}{\mathbb{E}[e^{-RU(\tilde{T})} | \tilde{T} < \infty]}. \tag{11}$$

A proof of Theorem 3.2 can be found in Chapter 4.4 in the book of Kaas [1]. Note that the denominator in (11) is always greater than 1, since $-RU(\tilde{T}) > 0$ for all $R > 0$ and by definition of $\tilde{T}$. This means that we can determine an upper bound for the ruin probability as follows:

$$\Psi(u) \leq e^{-Ru}. \tag{12}$$

The upper bound (12) holds for the alternative formulation (2).

We have established that we want to introduce dependence between the inter-arrival time and claim size. In the next section, we will present the tool that will be used for this: copulas.

3.2 Copulas

In this section, we introduce the definition of bivariate copulas and some of the properties that the bivariate copulas possess, by following the book of Nelsen [2]. We also introduce the copula functions that are used in the thesis. The definition of a bivariate copula is as follows.

Definition 3.7 *A two-dimensional copula is a function* $C : [0,1]^2 \to [0,1]$*, satisfying the following properties:*

1. For every u, v *in* $[0,1]$*,*

$$C(u,0) = C(0,v) = 0, \tag{13}$$

and

$$C(u,1) = u \text{ and } C(1,v) = v; \tag{14}$$

2. For every u_1, u_2, v_1, v_2 *in [0,1] such that* $u_1 \le u_2$ *and* $v_1 \le v_2$*,*

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \ge 0. \tag{15}$$

A fundamental theorem that makes copulas useful tools is Sklar's theorem, which links copulas and joint probability distributions. For our purpose we only need the theorem in the case of a bivariate distribution:

Theorem 3.3 (Sklar's theorem for bivariate distributions) *Let H be a joint distribution function with margins F and G. Then there exists a copula C such that for all* t, x *in* $\mathbb{R}$*,*

$$H(t,x) = C(F(t), G(x)). \tag{16}$$

If F and G are continuous, then C is unique; otherwise, C is uniquely determined on $Range(F) \times Range(G)$*. Conversely, if C is a copula and F and G are distribution functions, then the function H defined by (16) is a joint distribution function with margins F and G.*

We only consider exponential marginals, so any copula uniquely defines a joint distribution. Also, from Theorem 3.3 we get the probabilistic interpretation of a copula as a corollary.

Corollary 3.1 *The bivariate copula is a bivariate distribution with uniform marginals on the interval* $[0,1]$*. So the probabilistic definition of a copula C is:*

$$C(u,v) = \mathbb{P}(U < u, V < v), \tag{17}$$

where U and V are random variables with uniform distributions on the interval $[0,1]$*.*

In addition to the probabilistic interpretation of a copula we also need tools to derive the distributions imposed by a copula, this follows as another corollary of Theorem 3.3:

Corollary 3.2 *The density* $h(t,x)$ *of an absolutely continuous bivariate distribution* $H(t,x)$ *with marginal distributions* $F_T(t)$ *and* $F_X(x)$ *can be expressed, by applying the chain rule:*

$$h(t,x) = \frac{d^2 H(t,x)}{dtdx} = \frac{d^2 C(F_T(t), F_X(x))}{dtdx} = c(F_T(t), F_X(x)) f_T(t) f_X(x), \tag{18}$$

where $c(u,v) = \frac{d^2 C(u,v)}{dudv}$*.*

The reason that $F(t,x)$ in Corollary 3.2 needs to be an absolute continuous function is that otherwise, the function is not differentiable everywhere. Corollary 3.2 is useful to get expressions for the probability density functions of absolutely continuous bivariate distributions. We are also interested in linear combinations of copulas. To find the distribution we can use the following theorem:

Theorem 3.4 *Let a copula $C(u,v)$ be a linear combination of two copulas, $C(u,v) = aC_1(u,v) + (1-a)C_2(u,v)$ where $a \in [0,1]$ and $C_1(u,v)$, $C_2(u,v)$ are copulas with the property that the partial derivatives $\frac{\partial C_1}{\partial v}$ and $\frac{\partial C_2}{\partial v}$ or/and $\frac{\partial C_1}{\partial u}$ and $\frac{\partial C_2}{\partial u}$ exist. Then let $F_{(cT-X)_1}$ and $F_{(cT-X)_2}$ be the corresponding distributions for C_1 and C_2 with T and X having continuous marginals F and G. The distribution of $cT-X$ associated with C is equal to:*

$$F_{cT-X} = \begin{cases} F_{(cT-X)_1} \text{ with probability } a, \\ F_{(cT-X)_2} \text{ with probability } 1-a. \end{cases} \tag{19}$$

Now that we have established what a copula is, we introduce special copulas that will be used in the thesis.

3.2.1 The Frechet-Hoeffding copulas and independent copula

Simple examples of copulas are the independent copula Π, the Frechet-Hoeffding lower bound W, and Frechet-Hoeffding upper bound M. The three copulas reflect strong functional relationships and the level of dependence between the two random variables U and V. The independent case corresponds to zero dependence (i.e. independent), the lower bound W corresponds to perfect negative dependence (countermonotonic) and the upper bound M corresponds to perfect positive dependence (comonotonic)[2]. They have the following expressions:

$$W(u,v) = \text{Max}\{u+v-1, 0\}\ , \Pi(u,v) = uv\ , M(u,v) = \text{Min}\{u,v\}. \tag{20}$$

The copulas W and M in (20) are the copula lower bound and upper bound. Also, we can plot (20) to get a better understanding of the three copulas.

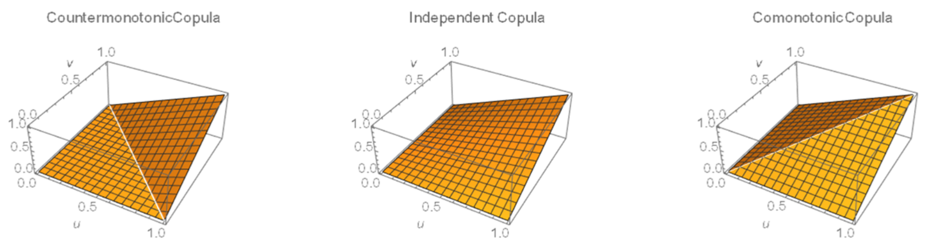


Figure 3: *The copulas W, Π and M.*

We can use Corollary 3.1 to interpret Figure 3.

Farlie-Gumbel-Morgenstern and Ali-Mikhail-Haq copula

Besides the lower bound, independent and upper bound copula (20), we will use two other copulas namely the Farlie-Gumbel-Morgenstern (FGM) copula and Ali-Mikhail-Haq (AMH) copula as described in Nelsen [2]. Both copulas can create moderately negative and positive dependence. The FGM copula function has the following form:

$$C_{\text{FGM}}(u,v) = uv(1 + \theta(1-u)(1-v)), \tag{21}$$

where $\theta \in [-1,1]$. To illustrate the copula we make a scatter plot of ten thousand simulations of the FGM copula for some values of the parameter θ:

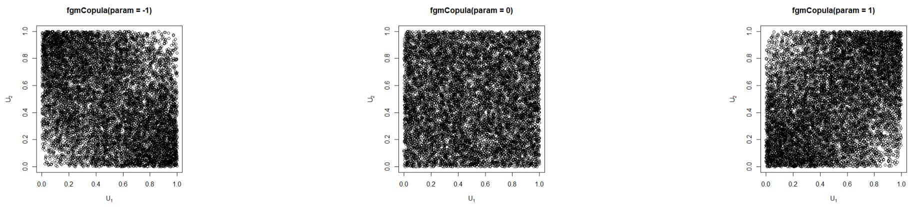


Figure 4: *The FGM copula evaluated at $\theta = -1, 0, 1$.*

The AMH copula function has the following form:

$$C_{\mathrm{AMH}}(u,v) = \frac{uv}{1-\theta(1-u)(1-v)} = uv + uv\sum_{k=1}^{\infty}\theta^k(1-u)^k(1-v)^k, \tag{22}$$

where $\theta \in [-1,1]$. The AMH copula is very similar to the FGM copula:

$$C_{\mathrm{AMH}}(u,v) = C_{\mathrm{FGM}}(u,v) + uv\sum_{k=2}^{\infty}\theta^k(1-u)^k(1-v)^k, \tag{23}$$

this means that the only difference between the two copulas is the second term. Also they are part of the same class of copulas, namely the polynomial copulas. Because they are so closely related, we would expect that they will give similar results for the ruin probability.

3.3 Dependence and correlation

In this section, we will introduce the definition of positive quadrant dependence, negative quadrant dependence, and Pearson's correlation coefficient. The dependence that we want to introduce between random variables can be characterized by certain properties the bivariate distribution possesses. One such characterization is negative/positive quadrant dependence or NQD/PQD as an abbreviation, which has direct implications on the adjustment coefficient from Definition 3.4.

Definition 3.8 *A random vector (T, X) is said to be Negative quadrant dependent if the following inequality holds for all $t, x \in \mathbb{R}$:*

$$\mathbb{P}(T \le t, X \le x) \le \mathbb{P}(T \le t)\mathbb{P}(X \le x), \tag{24}$$

and Positive quadrant dependent if:

$$\mathbb{P}(T \le t, X \le x) \ge \mathbb{P}(T \le t)\mathbb{P}(X \le x). \tag{25}$$

An alternative definition in terms of copulas functions is given in Definition 3.9 below.

Definition 3.9 *A copula function $C(u,v)$ is said to be Negative quadrant dependent if the following inequality holds for all $u, v \in [0,1]$:*

$$C(u,v) \le uv = \Pi(u,v), \tag{26}$$

and Positive quadrant dependent if:

$$C(u,v) \ge uv = \Pi(u,v). \tag{27}$$

From Definition 3.9 we can conclude that the lower bound copula W and, FGM and AMH copula for $\theta \in [-1, 0]$ are NQD and the upper bound copula M and, FGM and AMH copula for $\theta \in [0, 1]$ are PQD. Also, note that the independent copula Π is both NQD and PQD. Now we know which copulas possesses the NQD or and PQD property, we can use this to derive inequalities for the adjustment coefficient.

If a random vector (T, X) is NQD/PQD, by Definition 3.8, then the moment generating function K of the random variable $cT - X$ satisfies the following inequality for all $w \in (-\sigma, 0)$ where σ is the smallest value such that the moment generating functions K and K_I are defined. The moment generating function K_I is the moment generating function of $cT - X$ when the random variables T and X are independent of each other, then if (T, X) is NQD [3]:

$$K(w) = \mathbb{E}[e^{w(cT-X)}] \geq K_I(w), \tag{28}$$

and if (T, X) is PQD:

$$K(w) = \mathbb{E}[e^{w(cT-X)}] \leq K_I(w). \tag{29}$$

This has as a direct consequence that the adjustment coefficient R that corresponds to the random vector (T, X) satisfies the following inequality if (T, X) is NQD:

$$R \leq R_I, \tag{30}$$

and if (T, X) is PQD:

$$R \geq R_I, \tag{31}$$

where R_I is the adjustment coefficient of K_I. From (12) it follows that a pair of random variables that is PQD has a tighter upper bound than a pair of random variables that is NQD for the same initial capital u, premium c and marginal distributions. Now we have a way to check our result for the adjustment coefficient if Definition 3.8 is satisfied. A useful property of the definition of NQD and PQD is that the property is preserved under linear combinations:

Proposition 1 *Let $C_1(u, v)$ and $C_2(u, v)$ be NQD/PQD copulas and $p \in [0, 1]$ then $pC_1(u, v) + (1 - p)C_2(u, v)$ is NQD/PQD.*

Proof:

Let $C_1(u, v)$ and $C_2(u, v)$ be NQD copulas and $p \in [0, 1]$ then:

$$pC_1(u, v) + (1 - p)C_2(u, v) \leq puv + (1 - p)uv = uv. \tag{32}$$

Analogously, let $C_1(u, v)$ and $C_2(u, v)$ be PQD copulas and $p \in [0, 1]$ then:

$$pC_1(u, v) + (1 - p)C_2(u, v) \geq puv + (1 - p)uv = uv. \tag{33}$$

Proposition 1 will be useful when we take linear combinations of copulas to create different dependence structures. Also, recall Section 3.2.1 for copulas W and M. From Albrecher and Teugels [3], similarly to the NQD and PQD property we have that for all copulas $C(u, v)$:

$$W(u, v) \leq C(u, v) \leq M(u, v) \text{ for all } u, v \in [0, 1]. \tag{34}$$

This also gives the same results for the corresponding moment generating function $K(w)$ with fixed marginals:

$$K_W(w) \geq K(w) \geq K_M(w), \tag{35}$$

for those values of $w \in (-\sigma, 0)$, where σ is the smallest value such that the moment generating function are defined. In Eq. (35), K_W is the MGF of the lower bound and K_M the MGF of the upper bound. From Eq. (35) we can conclude the same result as in case of PQD and NQD:

$$R_W \leq R \leq R_M. \tag{36}$$

In addition, we also want to compare various dependent random vectors under the same strength of dependence. For this, we use a correlation measure called Pearson's correlation coefficient:

Definition 3.10 *Let T and X be random variables. Then Pearson's correlation coefficient $\rho_{Pearson}$ is defined as:*

$$\rho_{Pearson} = \frac{\mathbb{E}[TX] - \mathbb{E}[T]\mathbb{E}[X]}{\sqrt{Var(T)}\sqrt{Var(X)}}. \tag{37}$$

Later on, in Chapter 6, Definition 3.10 will be used to compare the results. Also, the following proposition will be helpful when creating bivariate exponential distributions:

Proposition 2 *Let $F_1(t,x)$ and $F_2(t,x)$ be bivariate distributions with the same marginals and Pearson correlation coefficient ρ_1 and ρ_2 then the following bivariate distribution has Pearson correlation coefficient $p\rho_1 + (1-p)\rho_2$:*

$$F(t,x) = \begin{cases} F_1(t,x) \text{ w.p. } p, \\ F_2(t,x) \text{ w.p. } 1-p. \end{cases} \tag{38}$$

The proof of Proposition 2 follows from conditioning on the two events. In the next chapter, we will introduce the bivariate distributions of the random vector (T, X) for which we will find the distributions and the moment generating function for the random variable $cT - X$.

4 Bivariate exponential distributions

In this chapter we introduce some bivariate distributions of the random vector (T, X) for which we will investigate the ruin probability in the next chapter. We will divide the bivariate distributions of our interest into three classes of distributions, namely, the simple copulas, bivariate mixed exponential distribution, and bivariate Erlang distribution. Note that all distributions we investigate have exponential marginals. For the bivariate distributions we will derive analytical expressions for $cT - X$, the moment generating functions and the ruin probability (if they exist). We will see that both the FGM and AMH copula with exponential margins fall in the class of bivariate mixed exponential distributions. Besides these copula models, we will consider two bivariate exponential distributions, namely, the Bladt-Nielsen distribution and the Moran-Downton distribution. The Bladt-Nielsen distribution is in the same class as the FGM and AMH copula. The Moran-Downton distribution is part of the bivariate mixed Erlang distributions and is the main motivation for this class of distributions, because it has an analytical expression for the ruin probability.

4.1 The Frechet-Hoeffding copulas and independent copula with exponential marginals

In this section, the distribution of $cT - X$ will be analyzed under the assumption that the dependence is captured by the independent, Frechet-Hoeffding lower bound and Frechet-Hoeffding upper bound copula with T and X exponential marginals with parameters λ and μ.

Independent Copula

The independent copula $\Pi(u, v) = uv$ is absolutely continuous and therefore the probability density function exist. We can calculate the joint density of T and X, by using Corollary 3.2:

$$f_{T,X}(t, x) = \frac{\partial^2 \Pi(1 - e^{-\lambda t}, 1 - e^{-\mu x})}{\partial t \partial x} = \lambda \mu e^{-\lambda t} e^{-\mu x}, \tag{39}$$

which is exactly the same as the PDF of two independent random variables. Note that the Pearson correlation coefficient for the independent copula with exponential marginals

$$\rho_\Pi = 0, \tag{40}$$

this follows directly from $\mathbb{E}[TX] = \mathbb{E}[T]\mathbb{E}[X]$.

We can use Eq (39) to derive the distribution of $cT - X$ by applying Theorem 3.1, which results in:

$$f_{cT-X}(z) = \begin{cases} \frac{1}{c} \int_{-z}^{\infty} f_{T,X}(\frac{t+z}{c}, t) dt = \frac{1}{c} e^{-\lambda \frac{z}{c}} \int_{-z}^{\infty} \lambda \mu e^{-(\frac{\lambda}{c} + \mu)t} dt = \frac{\lambda \mu}{\lambda + c\mu} e^{\mu z} & \text{if, } z \le 0, \\ \frac{1}{c} \int_{0}^{\infty} f_{T,X}(\frac{t+z}{c}, t) dt = \frac{1}{c} e^{-\lambda \frac{z}{c}} \int_{0}^{\infty} \lambda \mu e^{-(\frac{\lambda}{c} + \mu)t} dt = \frac{\lambda \mu}{\lambda + c\mu} e^{-\frac{\lambda}{c} z} & \text{if, } z \ge 0, \end{cases}. \tag{41}$$

Now that we have the PDF of $cT - X$, we can find the CDF by using the relation between PDF and CDF:

$$F_{cT-X}(z) = \begin{cases} \int_{-\infty}^{z} \frac{\lambda \mu}{\lambda + c\mu} e^{\mu x} dx = \frac{\lambda}{\lambda + c\mu} e^{\mu z} = F_-(z) & \text{if, } z \le 0, \\ F_-(0) + \int_0^z \frac{\lambda \mu}{\lambda + c\mu} e^{-\frac{\lambda}{c} x} dx = \frac{\lambda}{\lambda + c\mu} - \frac{c\mu}{\lambda + c\mu} e^{-\frac{\lambda}{c} z} + \frac{c\mu}{\lambda + c\mu} = 1 - \frac{c\mu}{\lambda + c\mu} e^{-\frac{\lambda}{c} z} & \text{if, } z \ge 0. \end{cases} \tag{42}$$

In terms of random variables, (42) can be represented as:

$$cT - X \sim \begin{cases} c \cdot \exp(\lambda) & \text{w.p.} \frac{c\mu}{\lambda + c\mu}, \\ -\exp(\mu) & \text{w.p.} \frac{\lambda}{\lambda + c\mu}. \end{cases} \tag{43}$$

From (43) we can derive the MGF of $cT - X$ as follows:

$$\begin{gathered} \mathbb{E}[e^{w(cT-X)}] = \frac{c\mu}{\lambda + c\mu}\mathbb{E}[e^{w\exp(\frac{\lambda}{c})}] + \frac{\lambda}{\lambda + c\mu}\mathbb{E}[e^{-w\exp(\mu)}] = \\ \frac{c\mu}{\lambda + c\mu}\frac{\lambda}{\lambda - cw} + \frac{\lambda}{\lambda + c\mu}\frac{\mu}{\mu + w} = \frac{\lambda\mu}{(\lambda - cw)(\mu + w)} \text{ for } w \in (-\mu, \frac{\lambda}{c}). \end{gathered} \tag{44}$$

Now that we have $cT - X$ in terms of random variables (43) and as a MGF (44), we can find the random variable that represents the deficit at ruin and the adjustment coefficient, therefore also the ruin probability. Since the exponential distribution has the memoryless property (see Definition 3.6), the deficit at ruin follows an exponential distribution with parameter μ. The adjustment coefficient can be determined by solving the following equation:

$$\frac{\lambda\mu}{(\lambda + cR)(\mu - R)} = 1 \Rightarrow R = \frac{\lambda - c\mu}{c}. \tag{45}$$

Using Definition 3.3 we get an explicit formula for the ruin probability:

$$\Psi(u) = \frac{\mu - R}{\mu}e^{-Ru} = \frac{\lambda}{\mu c}e^{-(\mu - \frac{\lambda}{c})u}. \tag{46}$$

Frechet-Hoeffding upper bound

For the Frechet-Hoeffding upper bound $M(u, v) = \text{Min}\{u, v\}$, we can not use the same method as for the independent case, since the upper bound is not absolutely continuous, and we can not differentiate and get the joint density. Therefore we use that $T = F_T^{-1}(F_X(x))$ given that $X = x$ (see [3]):

$$\begin{gathered} \mathbb{P}(cT - X < z) = \int_0^\infty \mathbb{P}(cF_T^{-1}(F_X(x)) - x < z)\mu e^{-\mu x}dx = \\ \int_0^\infty \mathbb{P}((\frac{c\mu}{\lambda} - 1)x < z)\mu e^{-\mu x}dx = \int_0^{\frac{z\lambda}{c\mu - \lambda}} \mu e^{-\mu x}dx = 1 - e^{-\frac{\mu\lambda z}{c\mu - \lambda}}, \end{gathered} \tag{47}$$

where $z > 0$. Also, the Pearson correlation coefficient of the Frechet-Hoeffding upper bound copula with exponential marginals becomes:

$$\rho_M = 1, \tag{48}$$

where we also used that $T = F_T^{-1}(F_X(x))$ given that $X = x$, (see [4] for more information).

Now, in terms of random variables, (47) can be represented as:

$$cT - X \sim \exp(\frac{\mu\lambda}{c\mu - \lambda}). \tag{49}$$

From (49) we can derive the MGF of $cT - X$:

$$\mathbb{E}[e^{w(cT-X)}] = \frac{\frac{\mu\lambda}{c\mu-\lambda}}{\frac{\mu\lambda}{c\mu-\lambda} - w} \text{ for } w < \frac{\mu\lambda}{c\mu-\lambda}. \tag{50}$$

We notice that ruin will never occur since the random variable $cT - X$ is non-negative.

Frechet-Hoeffding lower bound

For the Frechet-Hoeffding lower bound we apply the same approach as for the upper bound case, but now we use that $T = F_T^{-1}(1 - F_X(x))$ given $X = x$. In this case we can not find an explicit expression for x in terms of z, λ and μ so easily as for the Frechet-Hoeffding upper bound. Therefore we will adapt the following notation $\mu = \alpha\lambda$ with $\alpha \in \mathbb{R}^+$, (Note that for the implicit expression we take $c = 1$. Also note that we are only interested in $\alpha > 1$, otherwise the ruin probability would be trivial.):

$$\begin{aligned}\mathbb{P}(T - X \leq t) = \int_0^\infty \mathbb{P}(F_T^{-1}(1 - F_X(z)) - z \leq t)\alpha\lambda e^{-\alpha\lambda z}dz = \\ \int_0^\infty \mathbb{P}(-\frac{1}{\lambda}\ln(1 - e^{-\alpha\lambda z}) - z \leq t)\alpha\lambda e^{-\alpha\lambda z}dz.\end{aligned} \tag{51}$$

The inequality $-\frac{1}{\lambda}\ln(1 - e^{-\alpha\lambda z}) - z < t$ is completely deterministic i.e. there is no random variable involved. Therefore, if we evaluate this expression, this gives either one or zero depending on whether the inequality is true or false. We can rewrite the inequality in the following way:

$$\begin{aligned}-\lambda^{-1}\ln(1 - e^{-\alpha\lambda z}) - z \leq t &\Longleftrightarrow \\ \ln(1 - e^{-\alpha\lambda z}) + \lambda z > -\lambda t &\Longleftrightarrow \\ (1 - e^{-\alpha\lambda z})e^{\lambda z} > e^{-\lambda t} &\Longleftrightarrow \\ e^{\alpha\lambda z} - 1 - e^{-\lambda t}e^{(\alpha-1)\lambda z} &> 0.\end{aligned} \tag{52}$$

Setting $e^{\lambda z} = y$ and $e^{-\lambda t} = x$, we can define the function $f(y) = y^\alpha - xy^{\alpha-1} - 1$. We notice that the following properties hold for f:

1. $f(y)$ can be rewritten in the following way $f(y) = y^{\alpha-1}(y - x) - 1$, as a consequence $f(y) \leq -1$ for $y \in [0, x]$.
2. The derivative of $f(y)$ is $f'(y) = \alpha y^{\alpha-1} - (\alpha - 1)xy^{\alpha-2} = y^{\alpha-2}(\alpha y - (\alpha - 1)x)$. Thus for $y > \max\{0, \frac{(\alpha-1)x}{\alpha}\}$ we have that $f'(y) > 0$.
3. $\lim_{y\to\infty} f(y) = \infty$
4. $f(y)$ is continuous.

Observe that for all $\alpha > 0$ we have that $\max\{0, \frac{(\alpha-1)x}{\alpha}\} < x$. We have that $f(y) < 0$ for $y \in [0, x]$ by the first property of the function f and then from the second, third and fourth properties it follows that there is exactly one root $\hat{y}$ such that $f(\hat{y}) = 0$. Therefore, $e^{\lambda z} = \hat{y} \Rightarrow \hat{z} = \frac{1}{\lambda}\ln(\hat{y})$ and by the third property the inequality (52) is always true for $z > \hat{z}$, which means that the distribution is of the following form:

$$\mathbb{P}(T - X \leq t) = \int_{\frac{1}{\lambda}\ln(\hat{y})}^\infty \alpha\lambda e^{-\alpha\lambda z}dz = \frac{1}{\hat{y}^\alpha}. \tag{53}$$

So we have found an implicit expression in case of the Frechet-Hoeffding lower bound distribution. Nevertheless, we can find an explicit expression for the Pearson correlation coefficient:

$$\rho_W = 1 - \frac{\pi^2}{6}, \tag{54}$$

where we also used that $T = F_T^{-1}(1 - F_X(x))$ given that $X = x$. We notice that the value found in (54) is the smallest value of the Pearson correlation coefficient for the bivariate distributions with exponential marginals (for more details, see [4]). Since we only have an implicit expression for the distribution in case of the Frechet-Hoeffding lower bound, the MGF needs to be derived from the beginning:

$$\begin{aligned} \mathbb{E}[e^{w(cT-X)}] &= \int_0^\infty e^{w(ct - \frac{-1}{\mu}\ln(1-e^{-\lambda t}))} \cdot \lambda e^{-\lambda t} dt = \int_0^\infty (1 - e^{-\lambda t})^{\frac{w}{\mu}} \cdot \lambda e^{(wc-\lambda)t} dt \\ &= \int_0^1 (1-x)^{\frac{w}{\mu}} x^{\frac{-wc}{\lambda}} dx = B(1 - \frac{wc}{\lambda}, 1 + \frac{w}{\mu}) \text{ for } w \in (-\mu, \frac{\lambda}{c}), \end{aligned} \tag{55}$$

where we use the substitution $x = e^{-\lambda t}$ and $B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx$ is the beta function.

4.1.1 Convex combinations of Frechet-Hoeffding copulas and independent copula

We can generalize the results that we obtained for the Frechet-Hoeffding copulas and independent copula by making convex combinations of them. For example, let us consider the convex combination of the Frechet-Hoeffding upper bound and independent copula with exponential marginals. This copula takes on the following form:

$$C(u,v) = auv + (1-a)\text{Min}\{u,v\}, \tag{56}$$

where $a \in [0,1]$. We can interpret expression (56) as having the independent copula with probability a and the upper bound copula with probability $1-a$. Also note that (56) is PQD for all $a \in [0,1]$, which follows from the fact that $uv \leq \text{Min}\{u,v\}$ for all $u, v \in [0,1]$. We can now use Theorem 3.4 to get an explicit formula for the distribution. From this explicit formula we get the following expression in terms of random variables:

$$cT - X \sim \begin{cases} \exp(\frac{\lambda}{c}) & \text{w.p. } a\frac{c\mu}{\lambda + c\mu}, \\ -\exp(\mu) & \text{w.p. } a\frac{\lambda}{\lambda+c\mu}, \\ \exp(\frac{\mu\lambda}{c\mu-\lambda}) & \text{w.p. } 1-a. \end{cases} \tag{57}$$

From (57) we can also derive an explicit expression for the ruin probability. The deficit at ruin is an exponential random variable with parameter μ , by the memoryless property (see Definition 3.6). Also, we find that the MGF follows from (57):

$$\mathbb{E}[e^{w(cT-X)}] = a\frac{\mu\lambda}{(\lambda - cw)(\mu + w)} + (1-a)\frac{\frac{\mu\lambda}{c\mu-\lambda}}{\frac{\mu\lambda}{c\mu-\lambda} - w}. \tag{58}$$

In order to find the adjustment coefficient, we solve $\mathbb{E}[e^{w(cT-X)}] = 1$, which is a third order polynomial equation. This can be done by using Mathematica which returns the following solutions:

$$\begin{aligned} w_1 &= 0, \\ w_2 &= \frac{-\lambda^2 + 2c\lambda\mu + ac\lambda\mu - c^2\mu^2 - \sqrt{(-\lambda^2 + 2c\lambda\mu + ac\lambda\mu - c^2\mu^2)^2 + 4c(c\mu-\lambda)(-\lambda^2\mu + c\lambda\mu^2)}}{2c(c\mu-\lambda)}, \\ w_3 &= \frac{-\lambda^2 + 2c\lambda\mu + ac\lambda\mu - c^2\mu^2 + \sqrt{(-\lambda^2 + 2c\lambda\mu + ac\lambda\mu - c^2\mu^2)^2 + 4c(c\mu-\lambda)(-\lambda^2\mu + c\lambda\mu^2)}}{2c(c\mu-\lambda)}, \end{aligned} \tag{59}$$

where the second solution w_2 is the only negative one, since $c\mu - \lambda > 0$. By Definition 3.4, which is the definition of the adjustment coefficient we have that $R = -w_2$:

$$R = -\frac{-\lambda^2 + 2c\lambda\mu + ac\lambda\mu - c^2\mu^2 - \sqrt{(-\lambda^2 + 2c\lambda\mu + ac\lambda\mu - c^2\mu^2)^2 + 4c(c\mu-\lambda)(-\lambda^2\mu + c\lambda\mu^2)}}{2c(c\mu-\lambda)}. \tag{60}$$

Now that we have the adjustment coefficient and the distribution of the deficit at ruin we can determine the formula for the ruin probability, which is:

$$\Psi(u) = \frac{\mu - R}{\mu} e^{-Ru}, \tag{61}$$

where R is the adjustment coefficient reported in (60).

Note that for other convex combinations of Frechet-Hoeffding copulas and independent copula there are no explicit expressions for the distribution and the MGF of $cT - X$ because the Frechet-Hoeffding lower bound has no explicit expression. Hence, there are also no explicit expressions for the ruin probability. Still, we can get implicit expressions for the MGF. Let $a, b \in [0,1]$ such that $1 - a - b \geq 0$ then for the copula:

$$C(u,v) = auv + (1-a-b)\text{Min}\{u,v\} + b\text{Max}\{u+v-1,0\}, \tag{62}$$

we get the MGF, by applying Theorem 3.4 and then using that differentiating and integrating are linear operations:

$$K(w) = a\frac{\lambda\mu}{(\lambda - cw)(\mu + w)} + (1-a-b)\frac{\frac{\mu\lambda}{c\mu-\lambda}}{\frac{\mu\lambda}{c\mu-\lambda} - w} + bB(1 - \frac{wc}{\lambda}, 1 + \frac{w}{\mu}) \text{ for } w \in (-\mu, \frac{\lambda}{c}). \tag{63}$$

In section 5 we will investigate the adjustment coefficient of combinations of the Frechet-Hoeffding copulas and independent copula, for this, we can use (63). Also, we can use Proposition 2 to get the corresponding Pearson correlation coefficient for each combination, since we have the Pearson correlation coefficients of the individual copulas, see (40), (48) and (54).

4.2 Bivariate combination of exponential distributions

In this section we analyze $cT - X$ where the random vector (T, X) follows a bivariate combination of exponential distributions. In particular, we assume that the joint density of the random vector (T, X) is given as follows:

$$f_{T,X}(t,x) = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\lambda_i e^{-\lambda_i t}\mu_j e^{-\mu_j x} \tag{64}$$

where $m \in \mathbb{N}^+$, $c_{i,j} \in \mathbb{R}$ and $\sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j} = 1$ also $\{c_{i,j} : i = 1,2...m, j = 1,2...m\}$ such that $f_{T,X}(t,x) \geq 0$ for all $(t,x) \in \mathbb{R}^2$ and $\lambda_i, \mu_j > 0$ for all $i,j \in \{1,2...,m\}$. Without loss of generality we have that $\lambda_i \neq \lambda_j$ and $\mu_i \neq \mu_j$ for all i and j. We can add additional constraints such that the marginals are exponentially distributed, instead of univariate combinations of exponentials. This will be done in a later stage, since we do not need this assumption yet. In this section we derive an expression for the distribution of $cT - X$ inspired by Cossette et al.[5]. In particular, we investigate under which conditions $cT - X$ has a tractable distribution and we provide examples of bivariate copulas that belong to this class. First we derive an expression for the distribution and MGF of $cT - X$.

4.2.1 The derivation of the distribution of $cT - X$ and its MGF

To get the distribution of $cT - X$, we first derive the density of $cT - X$, which is presented next:

Proposition 3 *The probability density function $f_{cT-X}(z)$ is:*

$$f_{cT-X}(z) = \begin{cases} \sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\frac{\lambda_i\mu_j}{\lambda_i + c\mu_j}e^{\mu_j z}, \ \textit{if } z < 0, \\ \sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\frac{\lambda_i\mu_j}{\lambda_i + c\mu_j}e^{-\frac{\lambda_i}{c}z}, \ \textit{if } z \geq 0. \end{cases} \tag{65}$$

We use the joint density (64), by applying Theorem 3.1 we can find an expression for the density of $cT - X$.
Case 1: $z < 0$

$$\begin{gathered} f_{cT-X}(z) = \frac{1}{c}\int_{-z}^{\infty}\sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\lambda_i e^{-\lambda_i\frac{t+z}{c}}\mu_j e^{-\mu_j t}dt = \\ \frac{1}{c}\sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\int_{-z}^{\infty}\lambda_i e^{-\lambda_i\frac{t+z}{c}}\mu_j e^{-\mu_j t}dt = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\frac{\lambda_i\mu_j}{\lambda_i + c\mu_j}e^{\mu_j z}. \end{gathered} \tag{66}$$

Case 2: $z \geq 0$

$$\begin{gathered} f_{cT-X}(z) = \int_{0}^{\infty}\sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\lambda_i e^{-\lambda_i\frac{t+z}{c}}\mu_j e^{-\mu_j t}dt = \\ \sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\int_{0}^{\infty}\lambda_i e^{-\lambda_i\frac{t+z}{c}}\mu_j e^{-\mu_j t}dt = \sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\frac{\lambda_i\mu_j}{\lambda_i + c\mu_j}e^{-\frac{\lambda_i}{c}z}. \end{gathered} \tag{67}$$

Now that we have an explicit expression for the density of $cT - X$ in Proposition 3 we can calculate the CDF of $cT - X$.

Proposition 4 *The cumulative distribution function $F_{cT-X}(z)$ of $cT - X$ can be expressed as follows:*

$$F_{cT-X}(z) = \begin{cases} 1 - \sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\frac{c\mu_j}{\lambda_i + c\mu_j}(e^{-\frac{\lambda_i}{c}z}), & \textit{if } z \geq 0, \\ \sum_{i=1}^{m}\sum_{j=1}^{m} c_{i,j}\frac{\lambda_i}{\lambda_i + c\mu_j}e^{\mu_j z}, & \textit{if } z < 0. \end{cases} \tag{68}$$

We can prove Proposition 4 by using Proposition 3.

Case 1, $z < 0$:

$$
\begin{aligned}
F_{cT-X}(z) = \int_{-\infty}^{z} \sum_{i=1}^{m} \sum_{j=1}^{m} c_{i,j} \frac{\lambda_i \mu_j}{\lambda_i + c\mu_j} e^{\mu_j x} dx = \\
\sum_{i=1}^{m} \sum_{j=1}^{m} c_{i,j} \int_{-\infty}^{z} \frac{\lambda_i \mu_j}{\lambda_i + c\mu_j} e^{\mu_j x} dx = \\
\sum_{i=1}^{m} \sum_{j=1}^{m} c_{i,j} \frac{\lambda_i}{\lambda_i + c\mu_j} e^{\mu_j z} = F^{-}(z)
\end{aligned}
\tag{69}
$$

Case 2, $z \geq 0$:

$$
\begin{aligned}
F_{cT-X}(z) = F^{-}(0) + \int_{0}^{z} \sum_{i=1}^{m} \sum_{j=1}^{m} c_{i,j} \frac{\lambda_i \mu_j}{\lambda_i + c\mu_j} e^{-\frac{\lambda_i}{c} x} dx = \\
\sum_{i=1}^{m} \sum_{j=1}^{m} c_{i,j} \frac{\lambda_i}{\lambda_i + c\mu_j} + \sum_{i=1}^{m} \sum_{j=1}^{m} c_{i,j} \frac{c\mu_j}{\lambda_i + c\mu_j} (1 - e^{-\frac{\lambda_i}{c} z}) = \\
1 - \sum_{i=1}^{m} \sum_{j=1}^{m} c_{i,j} \frac{c\mu_j}{\lambda_i + c\mu_j} (e^{-\frac{\lambda_i}{c} z})
\end{aligned}
\tag{70}
$$

Realize that we can use Proposition 4 to find the CDF of $cT - X$ for copulas with exponential marginals and with a density that can be expressed as (64), where (T, X) is the corresponding random vector to the copula with exponential marginals.

In a similar way we can derive the MGF of a random vector with the given density as in (64):

$$
\begin{aligned}
\mathbb{E}[e^{-rT-sX}] = \int_{0}^{\infty} \int_{0}^{\infty} e^{-rt-sx} \sum_{i=1}^{m} \sum_{j=1}^{m} c_{i,j} \lambda_i e^{-\lambda_i t} \mu_j e^{-\mu_j x} dt dx = \\
\sum_{i=1}^{m} \sum_{j=1}^{m} \int_{0}^{\infty} \int_{0}^{\infty} e^{-rt-sx} c_{i,j} \lambda_i e^{-\lambda_i t} \mu_j e^{-\mu_j x} dt dx = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{i,j} \frac{\lambda_i}{\lambda_i + r} \frac{\mu_j}{\mu_j + s}
\end{aligned}
\tag{71}
$$

Let $r = -wc$ and $s = w$ then we get:

$$
\mathbb{E}[e^{w(cT-X)}] = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{i,j} \frac{\lambda_i}{\lambda_i - wc} \frac{\mu_j}{\mu_j + w}. \tag{72}
$$

We can now add constraints to assure that the negative part follows a negative exponential distribution such that the deficit at ruin is exponentially distributed. Also, we can use equation (72) to investigate the adjustment coefficient.

4.2.2 The deficit at ruin and exponential marginals

If the negative part of the distribution is exponentially distributed we can only have one $\mu_{j'}$ contributing to the negative part in Proposition 4. Then for every j such that $j \neq j'$ we have

$\sum_{i=1}^{m} c_{i,j} \frac{\lambda_i}{\lambda_i + c\mu_j} = 0$. As a result we have $m-1$ linear constraints, and the distribution takes the following form:

$$F_{cT-X}(x) = \begin{cases} 1 - \sum_{i=1}^{m} \sum_{j=1}^{m} c_{i,j} \frac{c\mu_j}{\lambda_i + c\mu_j} (e^{-\frac{\lambda_i}{c} x}), & \text{if } x \geq 0 \\ \sum_{i=1}^{m} c_{i,j'} \frac{\lambda_i}{\lambda_i + c\mu_{j'}} e^{\mu_{j'} x}, & \text{if } x < 0. \end{cases} \tag{73}$$

We can also add constraints to ensure exponential marginals, this gives us m additional constraints:

$$\sum_{i=1}^{m} c_{i,j} = \begin{cases} 0, & \text{if } j \neq j' \\ 1, & \text{if } j = j'. \end{cases} \tag{74}$$

This follows directly from determining the marginal density and fixing a particular j :

$$f_X(x) = \int_0^\infty \sum_{i=1}^{m} \sum_{j=1}^{m} c_{i,j} \lambda_i e^{-\lambda_i y} \mu_j e^{-\mu_j x} dy = \sum_{i=1}^{m} \sum_{j=1}^{m} c_{i,j} \mu_j e^{-\mu_j x} \tag{75}$$

The same can be done for T, and this results in:

$$\sum_{j=1}^{m} c_{i,j} = \begin{cases} 0, & \text{if } i \neq i' \\ 1, & \text{if } i = i'. \end{cases} \tag{76}$$

Now we have $3m-1$ linear constraints in total and m^2 degrees of freedom. This implies that for $m \geq 3$ there are non-trivial solutions to this system of equations. But note that this is not a guarantee for the existence of these distributions, since the constraint of the positive density for all $t, x \geq 0$ is not taken into account. We will work out the cases for $m = 2$ and $m = 3$ as an illustration. If $m = 2$ we get a set of 5 equations with 4 unknowns:

$$\begin{cases} c_{1,1} + c_{1,2} = 0 \\ c_{2,1} + c_{2,2} = 1 \\ c_{1,1} + c_{2,1} = 0 \\ c_{1,2} + c_{2,2} = 1 \\ c_{1,1} \frac{\lambda_1}{\lambda_1 + c\mu_1} + c_{2,1} \frac{\lambda_2}{\lambda_2 + c\mu_1} = 0 \end{cases} \tag{77}$$

which lead to the following two equalities:

$$\begin{cases} c_{2,1} = c_{1,2} = -c_{1,1} \\ c_{1,1} \frac{\lambda_1}{\lambda_1 + c\mu_1} + c_{2,1} \frac{\lambda_2}{\lambda_2 + c\mu_1} = 0 \end{cases} \quad \rightarrow \lambda_1 = \lambda_2 \tag{78}$$

This contradicts the assumption that $\lambda_1 \neq \lambda_2$.

For $m = 3$ we get 8 equations with 9 unknowns:

$$\begin{cases} c_{1,1}+c_{1,2}+c_{1,3}=0 \\ c_{2,1}+c_{2,2}+c_{2,3}=0 \\ c_{3,1}+c_{3,2}+c_{3,3}=1 \\ c_{1,1}+c_{2,1}+c_{3,1}=0 \\ c_{1,2}+c_{2,2}+c_{3,2}=0 \\ c_{1,3}+c_{2,3}+c_{3,3}=1 \\ \frac{\lambda_1}{\lambda_1+c\mu_1}c_{1,1}+\frac{\lambda_2}{\lambda_2+c\mu_1}c_{2,1}+\frac{\lambda_3}{\lambda_3+c\mu_1}c_{3,1}=0 \\ \frac{\lambda_1}{\lambda_1+c\mu_2}c_{1,2}+\frac{\lambda_2}{\lambda_2+c\mu_2}c_{2,2}+\frac{\lambda_3}{\lambda_3+c\mu_2}c_{3,2}=0 \end{cases} \tag{79}$$

which can be reduced to the following system:

$$\begin{cases} c_{1,1}=(\frac{t_{1,1}}{t_{2,1}}-1)x \\ c_{2,1}=x \\ c_{3,1}=-\frac{t_{1,1}}{t_{2,1}}x \\ c_{1,2}=(\frac{t_{1,2}}{t_{2,2}}-1)y \\ c_{2,2}=y \\ c_{3,2}=-\frac{t_{1,2}}{t_{2,2}}y \\ c_{1,3}=-((\frac{t_{1,1}}{t_{2,1}}-1)x+(\frac{t_{1,2}}{t_{2,2}}-1)y) \\ c_{2,3}=-(x+y) \\ c_{3,3}=\frac{t_{1,1}}{t_{2,1}}x+\frac{t_{1,2}}{t_{2,2}}y+1 \end{cases}, \tag{80}$$

where $t_{i,j}=\frac{\lambda_{i+1}}{\lambda_{i+1}+c\mu_j}-\frac{\lambda_i}{\lambda_i+c\mu_j}$ and $x,y\in\mathbb{R}$ are free variables. Note that the system of equations was a dependent system, since we have 2 degrees of freedom. As we anticipated, in this case it is already not clear if there is a solution that satisfies (64).

4.2.3 FGM copula, AMH copula, and Bladt-Nielsen distribution with exponential marginals

In this section, we derive expressions for the CDF and the MGF for the random variable $cT-X$ when the dependence is created by the FGM copula, AMH copula, and Bladt-Nielsen distribution with exponential marginals. The Bladt-Nielsen distribution with exponential marginals comes from a paper by Bladt and Nielsen [4]. The FGM and AMH copulas functions were introduced in Chapter 3.2. We show that the FGM copula, AMH copula, and Bladt-Nielsen distribution with exponential marginals are part of the class of bivariate combinations of exponentials. Inspired by the paper of Cossette et al. [5], we use Proposition 4 and (71) to derive the explicit expressions for $cT-X$.

Farlie-Gumbel-Morgenstern copula

For the FGM copula we assume exponential marginals, therefore by Theorem 3.3 we get:

$$\begin{gathered} \mathbb{P}(T\leq t, X\leq x)=C_{\mathrm{FGM}}(F_T(t),F_X(x))= \\ (1-e^{-\lambda t})(1-e^{-\mu x})+\theta(1-e^{-\lambda t})(e^{-\lambda t})(1-e^{-\mu x})(e^{-\mu x}) \end{gathered} \tag{81}$$

where $F_T(t)=1-e^{-\lambda t}$ and $F_X(x)=1-e^{-\mu x}$. The Pearson correlation coefficient of the random vector (T,X) with the FGM copula and exponential marginals becomes:

$$\rho_{\mathrm{FGM}}=\frac{\theta}{4}, \tag{82}$$

see Cossette et al. [5]. Now we can differentiate (81) with respect to t and x to find the joint density function, this gives us:

$$f_{T,X}(t,x) = (1+\theta)\lambda\mu e^{-\lambda t}e^{-\mu x} - 2\theta\lambda\mu e^{-2\lambda t}e^{-\mu x} - 2\theta\lambda\mu e^{-\lambda x}e^{-2\mu x} + 4\theta\lambda\mu e^{-2\lambda t}e^{-2\mu x}, \tag{83}$$

which is clearly of the form (64). Indeed, we have $m = 2$, $c_{1,1} = 1+\theta$, $c_{1,2} = c_{2,1} = -\theta$, $c_{2,2} = \theta$, $\mu_i = i\mu$ and $\lambda_i = i\lambda$ for $i \in \{1,2\}$. Therefore, we can apply Proposition 4 to get the CDF of $cT - X$:

$$F_{cT-X}(z) = \begin{cases} 1 - \left((1+\theta)\frac{c\mu}{\lambda+c\mu}e^{-\frac{\lambda}{c}z} - \theta\frac{c\mu}{2\lambda+c\mu}e^{-2\frac{\lambda}{c}z} - \theta\frac{2c\mu}{\lambda+2c\mu}e^{-\frac{\lambda}{c}z} + \theta\frac{c\mu}{\lambda+c\mu}e^{-2\frac{\lambda}{c}z}\right), & \text{if } z \geq 0 \\ (1+\theta)\frac{\lambda}{\lambda+c\mu}e^{\mu z} - \theta\frac{2\lambda}{2\lambda+c\mu}e^{\mu z} - \theta\frac{\lambda}{\lambda+2c\mu}e^{2\mu z} + \theta\frac{\lambda}{\lambda+c\mu}e^{2\mu z}, & \text{if } z < 0. \end{cases} \tag{84}$$

In addition, we use (72) to derive the MGF $K_{\text{FGM}}(w)$:

$$\begin{aligned} K_{\text{FGM}}(w) = (1+\theta)\frac{\lambda}{\lambda - wc}\frac{\mu}{\mu+w} - \theta\frac{2\lambda}{2\lambda - wc}\frac{\mu}{\mu+w} - \theta\frac{\lambda}{\lambda - wc}\frac{2\mu}{2\mu+s} + \theta\frac{2\lambda}{2\lambda-wc}\frac{2\mu}{2\mu+w} = \\ \frac{\lambda\mu((2\mu+w)(2\lambda-wc) - \theta w^2 c)}{(\lambda-wc)(2\lambda-wc)(\mu+w)(2\mu+w)} \text{ for } w \in (-\mu, \frac{\lambda}{c}). \end{aligned} \tag{85}$$

AMH copula

For the AMH copula, we can follow the same approach as for the FMG copula. The AMH copula can be represented in the following way:

$$C_{\text{AMH}}(u,v) = uv + uv\sum_{k=1}^{\infty}\theta^k(1-u)^k(1-v)^k, \tag{86}$$

where $\theta \in [-1,1]$. Now we can use Theorem 3.3 again to get the bivariate distribution, by plugging in the marginal distributions $F_T(t) = 1 - e^{-\lambda t}$ and $F_X(x) = 1 - e^{-\mu x}$:

$$\begin{aligned} F_{T,X}(t,x) = C_{\text{AMH}}(F_T(t), F_X(x)) = \\ (1-e^{-\lambda t})(1-e^{-\mu x}) + (1-e^{-\lambda t})(1-e^{-\mu x})\sum_{k=1}^{\infty}\theta^k e^{-\lambda t k}e^{-\mu x k} = \\ (1-e^{-\lambda t})(1-e^{-\mu x}) + \sum_{k=1}^{\infty}\theta^k\sum_{i=0}^{1}\sum_{j=0}^{1}(-1)^{i+j}e^{-\lambda t(k+i)}e^{-\mu x(k+j)}. \end{aligned} \tag{87}$$

Similar as for the FGM copula with exponential marginals, Cossette et al. derived an expression for the Pearson correlation coefficient [5]. The Pearson correlation coefficient of the random vector (T,X) with the AMH copula and exponential marginals becomes:

$$\rho_{\text{AMH}} = \sum_{k=1}^{\infty}\theta^k\sum_{i=0}^{1}\sum_{j=0}^{1}(-1)^{i+j}\frac{1}{(k+i)(k+j)}. \tag{88}$$

To get the joint density we can differentiate (87) with respect to t and x:

$$f_{T,X}(t,x) = \lambda\mu e^{-\lambda t}e^{-\mu x} + \sum_{k=1}^{\infty}\theta^k\sum_{i=0}^{1}\sum_{j=0}^{1}(-1)^{i+j}\lambda\mu(k+i)(k+j)e^{-\lambda t(k+i)}e^{-\mu x(k+j)}. \tag{89}$$

Now we can apply Proposition 4 to get the CDF of $cT - X$:

$$F_{cT-X}(z) = \begin{cases} 1 - \frac{c\mu}{\lambda+c\mu} e^{\frac{-\lambda}{c}z} - \sum_{k=1}^{\infty} \theta^k \sum_{i=0}^{1} \sum_{j=0}^{1} (-1)^{i+j} \frac{c\mu(k+j)}{\lambda(k+i)+c\mu(k+j)} e^{-\frac{\lambda(k+i)}{c}z}, & \text{if } z \geq 0, \\ \frac{\lambda}{\lambda+c\mu} e^{\mu z} + \sum_{k=1}^{\infty} \theta^k \sum_{i=0}^{1} \sum_{j=0}^{1} (-1)^{i+j} \frac{\lambda(k+i)}{\lambda(k+i)+c\mu(k+j)} e^{\mu z(k+j)}, & \text{if } z < 0. \end{cases} \tag{90}$$

Note that if we set $\theta = 0$ we indeed get the same distribution as in the independent case. The MGF for the AMH can be found using (72), however this expression is also an infinite sum:

$$K_{\text{AMH}}(w) = \mathbb{E}[e^{w(cT-X)}] = \frac{\lambda\mu}{(\lambda - wc)(\mu + w)} + \sum_{k=1}^{\infty} \theta^k \sum_{i=0}^{1} \sum_{j=0}^{1} (-1)^{i+j} \frac{\lambda(k+i)}{\lambda(k+i) - wc} \frac{\mu(k+j)}{\mu(k+j) + w} \text{ for } w \in (-\mu, \frac{\lambda}{c}). \tag{91}$$

Bladt-Nielsen bivariate distribution

The Bladt-Nielsen distribution has the following density function [4]:

$$f(t,x) = \sum_{l=1}^{n} \sum_{k=1}^{n} c_{l,k} \lambda l e^{-\lambda l t} \mu k e^{-\mu k x} \tag{92}$$

where $c_{l,k} = \frac{(-1)^{l+k-(n+1)}}{n} \binom{n}{l}\binom{n}{k} \sum_{i=n+1-l}^{n} \sum_{j=1}^{k} p_{i,j} (-1)^{-i-j} \binom{l-1}{n-i}\binom{k-1}{n-j}$ with $p_{i,j} = \delta_{i-j}$ or $\delta_{i+j-n-1}$. Here, the delta function is defined in the following way:

$$\delta_t = \begin{cases} 1, \text{ if } t = 0 \\ 0, \text{ else.} \end{cases} \tag{93}$$

Notice that there are other possibilities for $p_{i,j}$ but we are only interested in these two in particular, because they both have a Markov chain representation depicted in Figure 5 and Figure 6:

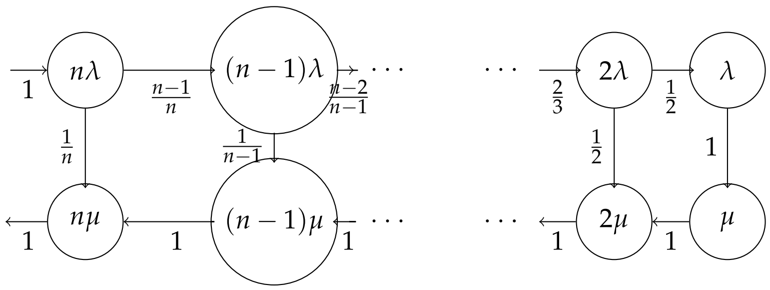


Figure 5: *Markov chain representation of the Bladt-Nielsen distribution with* $\delta_{i+j-(n+1)}$.

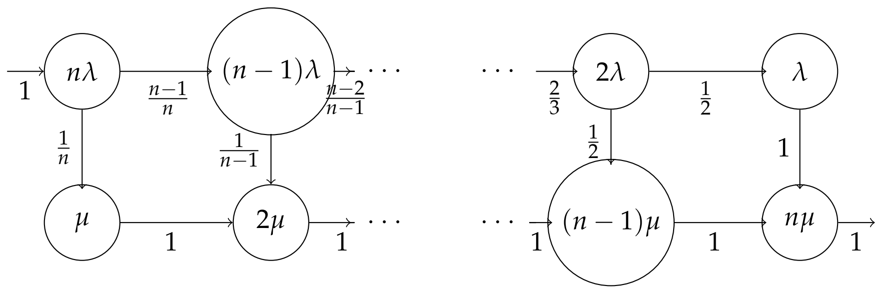


Figure 6: *Markov chain representation of the Bladt-Nielsen distribution with* δ_{i-j}.

In Figures 5 and 6 the first row of circles corresponds to an exponential random variable with parameter λ and the second row of circles corresponds to an exponential random variable with parameter μ and each circle is an exponential random variable with the specified parameter in the circle and the numbers next to or below the arrows correspond to the probability of entering and leaving a circle. In the language of Markov chains a circle is called a state. From Figures 5 and 6 we can see why there is positive or negative correlation in these distributions. In Figure 5 the more states that we travel trough in the first row, the more states we have to travel trough in the second row, therefore there is a positive correlation. In Figure 6 it is exactly the other way around. The corresponding Pearson correlation coefficients are

$$\rho_{\text{i+j-n-1}} = 1 - \frac{1}{n}\sum_{k=1}^{n}\frac{1}{k} \tag{94}$$

$$\rho_{\text{i-j}} = 1 - \sum_{i=1}^{n}\frac{1}{i^2}. \tag{95}$$

Clearly, the Bladt-Nielsen distribution is of the same form as (64). Using Proposition 4 we get:

$$F_{cT-X}(z) = \begin{cases} 1 - \sum_{l=1}^{n}\sum_{k=1}^{n} c_{l,k}\frac{c\mu k}{\lambda l + c\mu k}(e^{-\frac{\lambda}{c}lz}), & \text{if } z \geq 0, \\ \sum_{l=1}^{n}\sum_{k=1}^{n} c_{l,k}\frac{\lambda l}{\lambda l + c\mu k}e^{\mu k z}, & \text{if } z < 0. \end{cases} \tag{96}$$

Also the MGF can be found using (72), but this gives a rather cumbersome formula, since $c_{l,k}$ also depends on $p_{i,j}$:

$$K_{\text{Bladt}}(w) = \sum_{l=1}^{n}\sum_{k=1}^{n} c_{l,k}\frac{\lambda l}{\lambda l - wc}\frac{\mu k}{\mu k + w} \text{ for } w \in (-\mu, \frac{\lambda}{c}), \tag{97}$$

where $c_{l,k}$ is defined as in (92). In the paper by Bladt and Nielsen, the MGF takes on a different expression. In particular,

$$K_{\text{Bladt}}(w) = \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} p_{i,j}\prod_{l=n-i+1}^{n}\frac{l\lambda}{l\lambda - wc}\prod_{k=j}^{n}\frac{k\mu}{k\mu + w} \text{ for } w \in (-\mu, \frac{\lambda}{c}), \tag{98}$$

this can be explained by the Markov chain representation in Figures 5 and 6 when $p_{i,j}$ is $\delta_{i+j-n-1}$ or δ_{i-j}.

4.3 Bivariate mixed Erlang distributions

In this section we will introduce the bivariate mixed Erlang distributions and derive the distribution of $cT - X$. First, we will present the definition of a bivariate mixed Erlang distribution. After this, we will show that the Moran-Downton distribution belongs to this class of distributions. Also, we will show that for this distribution there is a mathematical expression for the ruin probability. The bivariate mixed Erlang distributions can be seen as a generalization of the Moran-Downton distribution.

4.3.1 Generalization of Moran-Downton's bivariate exponential

In this subsection we will investigate the bivariate mixed Erlang distributions, in a similar fashion as the second part of the paper by Cosette et al.[5], which can be used to generalize the bivariate Moran-Downton distribution in the next subsection. Specifically, we are interested in distributions with the following joint density for the random vector (T, X):

$$f_{T,X}(t,x) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} p_{i,j}\frac{t^{i-1}\lambda^i e^{-\lambda t}}{(i-1)!}\frac{x^{j-1}\mu^j e^{-\mu x}}{(j-1)!} \tag{99}$$

where $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} p_{i,j} = 1$.

The MGF of such a distribution is given by:

$$\mathbb{E}[e^{r_1 T + r_2 X}] = \int_0^{\infty}\int_0^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} p_{i,j}\frac{t^{i-1}\lambda^i e^{(-\lambda+r_1)t}}{(i-1)!}\frac{x^{j-1}\mu^j e^{(-\mu+r_2)x}}{(j-1)!}dtdx. \tag{100}$$

Switching the order of integration and summation, we can observe that (100) can be written as a double infinite sum of products of two MGF's of the Erlang distribution with parameters (λ, i) and (μ, j):

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} p_{i,j}\int_0^{\infty}\frac{t^{i-1}\lambda^i e^{(-\lambda+r_1)t}}{(i-1)!}dt\int_0^{\infty}\frac{x^{j-1}\mu^j e^{(-\mu+r_2)x}}{(j-1)!}dx =\\ \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} p_{i,j}\left(\frac{\lambda}{\lambda-r_1}\right)^i\left(\frac{\mu}{\mu-r_2}\right)^j.\end{aligned} \tag{101}$$

Note that in this context λ and μ do not have to correspond to the parameters of the marginal exponential distributions. Now one can set $r_1 = cw$ and $r_2 = -w$ to get the MGF of $cT - X$:

$$\mathbb{E}[e^{w(cT-X)}] = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} p_{i,j}\left(\frac{\lambda}{\lambda-cw}\right)^i\left(\frac{\mu}{\mu-w}\right)^j. \tag{102}$$

4.3.2 Moran-Downton's bivariate exponential distribution

In this section we will present a detailed example of a bivariate distribution named after Moran and Downton. We will show that this distribution has an explicit formula for the adjustment coefficient. We assume T and X as our marginals and we analyze the distribution of $cT - X$. We have the following situation, $T \sim \exp(\lambda)$ and $X \sim \sum_{i=0}^{N(T)} X_i = X_0 + \sum_{i=1}^{N(T)} X_i$ where $N(t)$ is a Poisson process with rate γ and $X_i \sim \exp(\mu)$. In terms of ruin theory, the inter-arrival time is exponentially distributed with parameter λ, and the total claim size can be seen as a random sum of smaller claim sizes. In [6], they showed that this setting is equivalent to the Moran-Downton distribution. The MGF function of the random vector (T, X) has the following expression:

$$\begin{aligned}\mathbb{E}[e^{-sT-\omega X}] = \int_0^\infty e^{-st}\mathbb{E}[e^{-w(X_0+\sum_{i=1}^{N(t)} X_i)}]\lambda e^{-\lambda t}dt = \\ \int_0^\infty \sum_{k=0}^\infty \frac{(\gamma t)^k}{k!}e^{-\gamma t}(\frac{\mu}{\mu+\omega})^{k+1}\lambda e^{-(s+\lambda)x}dt = \int_0^\infty e^{\frac{\gamma\mu t}{\mu+\omega}}\lambda e^{-(s+\lambda+\gamma)t}dt \cdot \frac{\mu}{\mu+\omega} = \\ \frac{\lambda}{s+\lambda+\gamma-\frac{\gamma\mu}{\mu+\omega}} \cdot \frac{\mu}{\mu+\omega} = \frac{\mu\lambda}{(\mu+\omega)(s+\lambda+\gamma)-\gamma\mu}.\end{aligned} \tag{103}$$

The first equality follows from conditioning on T, the second equality follows from conditioning on $N(t)$ and recognizing the MGF of the Erlang distribution, the third equality follows from the definition of the exponential series. We can reparameterize (103) such that the parameters γ and μ vanish. For this reparameterization we introduce the following new parameters $\rho \in [0, 1]$ and $\hat{\mu} > 0$ where ρ is the Pearson correlation coefficient. Now by substituting $\gamma = \frac{\rho\lambda}{1-\rho}$ and $\mu = \frac{\hat{\mu}}{1-\rho}$ into (103) we get:

$$\begin{aligned}\frac{\mu\lambda}{(\mu+\omega)(s+\lambda+\gamma)-\gamma\mu} = \frac{\frac{\hat{\mu}}{1-\rho}\lambda}{(\frac{\hat{\mu}}{1-\rho}+\omega)(s+\lambda+\frac{\rho\lambda}{1-\rho})-\frac{\rho\lambda}{1-\rho}\frac{\hat{\mu}}{1-\rho}} = \\ \frac{\lambda\hat{\mu}}{(s+\lambda)(\omega+\hat{\mu})-\rho\omega s}.\end{aligned} \tag{104}$$

Another interpretation of this distribution would be that (T, X) follows the Moran-Downton distribution, (see [3],[6]). In the investigation of the ruin probability of several distributions that we discussed in chapter 5, we will use (104), Indeed, this parametrization is more convenient as it allows for comparing different adjustment coefficients for the same correlation coefficient. We can set $\omega = w$ and $s = -cw$ in (103) to get the MGF of $cT - X$, which results in the following:

$$\mathbb{E}[e^{w(cT-X)}] = \frac{\mu\lambda}{(\mu+w)(-cw+\lambda+\gamma)-\gamma\mu}. \tag{105}$$

Now we can use partial fraction decomposition and then the inverse Laplace-Stieltjes transform to find the density of the distribution. First, we find the roots of the denominator:

$$x_\pm = \frac{(\lambda+\gamma-c\mu) \pm \sqrt{(\lambda+\gamma-c\mu)^2+4c\mu\lambda}}{2c}, \tag{106}$$

where x_+ corresponds to the positive root and x_- the negative root. Next, we apply partial fraction decomposition:

$$\begin{aligned}\mathbb{E}[e^{w(cT-X)}] = \frac{-\mu\lambda}{x_+-x_-}\frac{1}{w-x_+} + \frac{-\mu\lambda}{x_--x_+}\frac{1}{w-x_-} = \\ \frac{\mu\lambda}{x_+(x_+-x_-)}\frac{x_+}{x_+-w} + \frac{\mu\lambda}{x_-(x_--x_+)}\frac{-x_-}{w-x_-} \text{ for } w \in (x_-, x_+).\end{aligned} \tag{107}$$

Using the definition of the MGF of the exponential random variable and minus an exponential random variable, we get the following expression in terms of MGF's:

$$\mathbb{E}[e^{w(cT-X)}] = \frac{\mu\lambda}{x_+(x_+ - x_-)}\mathbb{E}[e^{w\exp(x_+)}] + \frac{\mu\lambda}{x_-(x_- - x_+)}\mathbb{E}[e^{-w\exp(-x_-)}]. \tag{108}$$

A different interpretation of this would be:

$$cT - X \sim \begin{cases} \exp(x_+) & \text{with probability } \frac{\mu\lambda}{x_+(x_+-x_-)} \\ -\exp(-x_-) & \text{with probability } \frac{\mu\lambda}{x_-(x_--x_+)}. \end{cases} \tag{109}$$

From (109) it follows that the deficit at ruin has distribution $\exp(-x_-)$ by the memoryless property of the exponential distribution. We now get the following expression for the ruin probability:

$$\psi(u) = \frac{e^{-Ru}}{\mathbb{E}(e^{-RU(\tilde{T})}|\tilde{T} < \infty)} = \frac{e^{-Ru}}{\mathbb{E}(e^{R\exp(-x_-)})} = \frac{-x_- - R}{-x_-}e^{-Ru}, \tag{110}$$

where R is the positive solution of $K_{cT-X}(-r) = 1$, which is, $R = \mu - \frac{\lambda+\gamma}{c}$. Note that if we apply the same substitution as in (104), then we get that $R = \frac{c\hat{\mu}-\lambda}{c(1-\rho)}$, which is equivalent to the result presented in Albrecher and Teugels [3]. For the investigation of the ruin probability (110) will be used as a benchmark. Also, we can use that the Moran-Downton bivariate distribution is PQD (Definition 3.8), see Balakrishna Chapter 10.15 [7]. Note that the adjustment coefficient that we found clearly satisfies (31):

$$\frac{c\hat{\mu} - \lambda}{c(1-\rho)} \geq \frac{\lambda - c\hat{\mu}}{c} \Rightarrow \frac{1}{1-\rho} \geq 1 \text{ for } \rho \in [0,1) \tag{111}$$

Now that we know the ruin probability we will show that Moran-Downton bivariate distribution is part of the bivariate mixed Erlang distributions:

$$\begin{aligned} \mathbb{E}[e^{r_1T+r_2X}] = \int_0^\infty \mathbb{E}[e^{r_1t+r_2X}|T=t]\lambda e^{-\lambda t}dt = \int_0^\infty \sum_{n=0}^\infty \mathbb{E}[e^{r_2\sum_{i=0}^n T_i}]\frac{(\gamma t)^n}{n!}e^{-\gamma t}\lambda e^{(-\lambda+r_1)t}dt = \\ \int_0^\infty \sum_{n=0}^\infty (\frac{\mu}{\mu - r_2})^{n+1}\frac{(\gamma t)^n}{n!}e^{-\gamma t}\lambda e^{(-\lambda+r_1)t}dt. \end{aligned} \tag{112}$$

Switching the order of integration and taking the sum, we obtain:

$$\begin{aligned} \sum_{n=0}^\infty (\frac{\mu}{\mu - r_2})^{n+1}\int_0^\infty \frac{(\gamma t)^n}{n!}e^{-\gamma t}\lambda e^{(-\lambda+r_1)t}dt = \sum_{n=0}^\infty (\frac{\mu}{\mu-r_2})^{n+1}\frac{\lambda}{\gamma}(\frac{\gamma}{\lambda+\gamma-r_1})^{n+1} = \\ \sum_{n=0}^\infty (\frac{\mu}{\mu-r_2})^{n+1}\frac{\frac{\lambda}{\gamma}\gamma^{n+1}}{(\lambda+\gamma)^{n+1}}(\frac{\lambda+\gamma}{\lambda+\gamma-r_1})^{n+1}. \end{aligned} \tag{113}$$

The integral can be solved by applying $n+1$ times partial integration. The last step is done by multiplying with $\frac{(\lambda+\gamma)^{n+1}}{(\lambda+\gamma)^{n+1}}$. Since now we have an expression of the form that we described in (101), where

$$p_{i,j} = \begin{cases} \frac{\frac{\lambda}{\gamma}\gamma^{i+1}}{(\lambda+\gamma)^{i+1}}, & \text{if } i = j, \\ 0, & \text{else.} \end{cases} \tag{114}$$

The condition that is set on $p_{i,j}$ also holds:

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} p_{i,j} = \sum_{i=0}^{\infty} \frac{\frac{\lambda}{\gamma}\gamma^{i+1}}{(\lambda+\gamma)^{i+1}} = \frac{\lambda}{\lambda+\gamma}\sum_{i=0}^{\infty}\left(\frac{\gamma}{\lambda+\gamma}\right)^{i} = \frac{\lambda}{\lambda+\gamma}\cdot\frac{1}{1-\frac{\gamma}{\lambda+\gamma}} = 1. \tag{115}$$

5 Model and simulation descriptions

In this section, we will use two different methods to approximate the adjustment coefficient for several distributions: The Lagrange-Bürmann inversion method and a built-in function from Mathematica. Also, we will simulate the ruin process to find an estimate for the ruin probability and average deficit at ruin. We will use the following notations: $K(w)$ is the moment generating function of an arbitrary distribution, R is the adjustment coefficient that corresponds to K, and R_I is the adjustment coefficient for the independent case. We analyze the Moran-Downton distribution, distributions corresponding to the FGM and AMH copulas with exponential marginals, Bladt-Nielsen distribution, and convex combinations of the countermonotonic (Frechet-Hoeffding lower bound), independent and comonotonic (Frechet-Hoeffding upper bound) copula as described in Chapter 4. For all distributions, we assume exponential marginals T and X with parameters λ and μ. We will investigate the adjustment coefficient of the distribution $cT - X$, with $c = 1$. To compare the behavior of the adjustment coefficient of the different distributions, we use the Pearson correlation coefficient ρ as a baseline. We will compare the adjustment coefficients of different distributions with the same Pearson correlation coefficient.

5.1 Lagrange-Bürmann inversion

In this section, we will present the basics to find the adjustment coefficient with the Lagrange-Bürmann inversion theorem. For more details on the theorem, we refer to Schöpf and Supancic [8]. First, we give the general formula for the adjustment coefficient as a direct consequence of the Lagrange-Bürmann inversion theorem and after that, we provide a recursive formula to get an approximation of the adjustment coefficient.

The Lagrange-Bürmann inversion theorem can be used to find an expression for R given that $1 = K(-R)$ in the following way:

$$-R = -R_I + \sum_{n=1}^{\infty} g_n \cdot \frac{(1 - K(-R_I))^n}{n!}, \tag{116}$$

where $g_n = \lim_{w \to -R_I} \frac{d^{n-1}}{dw^{n-1}} \left(\frac{w + R_I}{K(w) - K(-R_I)} \right)^n$. This approach works under the condition that K is analytic on the domain that is being considered and $K'(-R_I) \neq 0$ (this idea comes from [3]). We can use (116) to approximate the adjustment coefficient by computing a finite part of the sum. For this, we use the recursive definition of coefficients g_n which can be found in Schöpf and Supancic [8]. In particular, equation (9) of [8] defines the recursion where $I_n n! = g_n$ for all $n \in \mathbb{N}^+$:

$$I_1(f, z) = \frac{1}{f'(z)} \text{ and } I_n(f, z) = \frac{1}{n} \frac{1}{f'(z)} \cdot \frac{dI_{n-1}(f, z)}{dz}, z \to z_0, \tag{117}$$

where $z_0 \in \mathbb{R}$ such that $f'(z_0) \neq 0$. The recursive relation (117) can be applied to our case, by replacing f with the MGF K of the distribution that we are interested in and z_0 with $-R_I$:

$$I_1(K, z) = \frac{1}{K'(z)} \text{ and } I_n(K, z) = \frac{1}{n} \frac{1}{K'(z)} \cdot \frac{dI_{n-1}(K, z)}{dz}, z \to -R_I. \tag{118}$$

Then, the $N - th$ order approximation is:

$$-R = -R_I + \sum_{n=1}^{N} I_n(K, z)(1 - K(z))^n, z \to -R_I. \tag{119}$$

Schöpf and Supancic also provide the code to implement the Lagrange-Bürmann inversion approximation in *Mathematica*.

5.2 *FindRoot* built-in function *Mathematica*

Mathematica provides several built-in functions that can solve equations algebraically and numerically. In this section we will give an algorithmic description of what the function *FindRoot* does. *FindRoot* has as a default algorithm Newton's method. Such a method requires the choice of a starting point. We have to choose a starting point to ensure convergence to the adjustment coefficient and not some other root. For determining this starting point we can use that the MGF is a continuous convex function. Therefore, there are only two points where the MGF can satisfy the equality $K(w) = 1$. From the definition of the MGF and the assumption that $\mathbb{E}[cT - X] > 0$ we have that $K(0) = 1$ and $K'(0) = \mathbb{E}[cT - X] > 0$. Since the MGF is convex and $K'(0) > 0$, the only other solution must be in the negative domain of the MGF. If we start on the left side of the minimum i.e. the minimum minus some small ϵ, then the only root the method can converge to is the adjustment coefficient. The algorithmic description of Newton's method in our case is:

Algorithm 1 Newton's method

Let $\epsilon_1 = 0.01$, $\epsilon_2 > 0$ the level of accuracy and $M > 0$ be the amount of iterations.
Solve $K'(x) = 0$
Starting point $R_0 = x - \epsilon_1$
while $n < M$ **do**
    $-R_n = -R_{n-1} - \frac{K(-R_{n-1})-1}{K'(-R_{n-1})}$
    n = n+1
    if $|K(-R_{n-1}) - 1| < \epsilon_2$ **then**
        Return R_{n-1}

Notice that we also divide by the derivative of the MGF as in the case of the Lagrange-Bürmann inversion. Nevertheless, there is one important difference; in the case of the inversion we evaluate the derivative on one specific point and rely on it to be not equal to zero, but in the case of Newton's method we have chosen our starting point in such a way that the derivative will never be equal to zero.

5.3 Simulation of the claim process

In this section we describe the simulation procedure of each distribution. First we give a description of how we can simulate the process which is described in Definition 3.2, which is used for all distributions. After this we will zoom in on the individual distributions. The general process can be described in the following way:

Algorithm 2 General process

Let $u > 0$ be the initial capital, $c > 0$ the premium, $\lambda > 0$ the parameter of T and $\mu > 0$ the parameter of X
Let M be the amount of claims that will be simulated, $n = 1$ and $x_0 = u$.
while $n < M$ **do**
    Simulate $cT_n - X_n$
    $x_n = x_{n-1} + cT_n - X_n$
    if $x_n < 0$ **then***Return*$(1, -x_n)$ as a vector, where 1 indicates that ruin occurred and x_n is the deficit at ruin.
Return $(0, 0)$, where the zeros indicate that ruin did not occur and that there is no deficit at ruin.

We can do this multiple times and get an estimate for the ruin probability and average deficit at ruin. Also, in Chapter 4 we got two analytical expressions for the ruin probability and deficit at ruin. One for the convex combination of independent copula and comonotonic copula and one for the Moran-Downton distribution, see Subsections 4.1.1 and 4.3.2. We can use these results to validate our simulation and determine how accurate the simulations are. Now that we have a way to simulate the general process we need a way to simulate the random variable $cT - X$.

FGM and AMH copula

For the two copulas FGM and AMH we use the *copula* package in R, which enables us simulate a random vector (T, X) instantaneously [9]. The outcome of those simulations can be used to compute $cT - X$.

Convex combinations of countermonotonic, independent and comonotonic copula

The convex combinations of countermonotonic, independent and comonotonic copula can also be simulated with the copula package in R, but we will use a different approach. We will make combinations of two copulas: independent and comonotonic, independent and countermonotonic and, countermonotonic and comonotonic. To simulate the combinations we first simulate the inter-arrival time $T \sim \exp(\lambda)$. Then the claim size is either an exponential random variable with parameter μ in the independent case, $F_T^{-1}(F_X(t)) = \frac{\lambda}{\mu}t$ in the comonotonic copula case and $F_T^{-1}(1 - F_X(t)) = \frac{-1}{\mu}\ln(1 - e^{-\lambda t})$ in the countermonotonic copula case, where t is the simulated inter-arrival time. Now we can use a Bernoulli random variable with parameter p to determine which one we will choose. For example, take the independent and comonotonic case: we simulate $T \sim \exp(\lambda)$ than with probability p we simulate an independent claim which follows the random variable $\exp(\mu)$ and with probability $1 - p$ we get the comonotonic part which is completely determined by the inter-arrival time random variable T which gives us $\frac{\lambda}{\mu}T$. This approach for the simulation was also used in Chapter 4.1 to derive expressions for $cT - X$.

Bladt-Nielsen distribution

The two Bladt-Nielsen distributions can be simulated by using their interpretations in terms of random variables. For this, we will use the Markov-chain representation in Figures 5 and 6.

Moran-Downton distribution

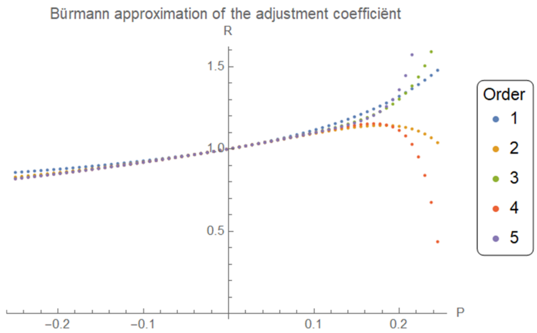
Similar to the Bladt-Nielsen distribution we use the interpretation of the distribution in terms of random variables. In the case of the Moran-Downton distribution we first simulate the inter-arrival time $T \sim \exp(\lambda)$, then a Poisson random variable with parameter γt, where t is the simulated inter-arrival time. The outcome of the Poisson random variable then determines the number of claims that are being simulated. In order to fix the marginal distributions and compare the Moran-Downton distribution, we need to use Eq. (104).

6 Results and interpretation

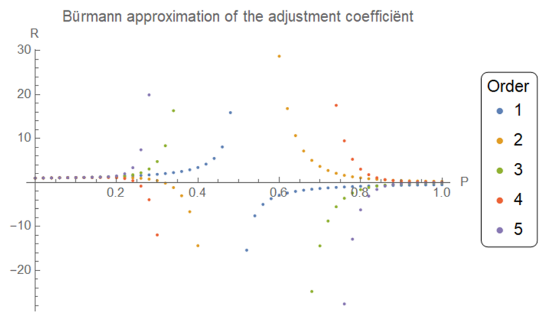
In this section, we present and interpret the results from the investigation of the ruin probability, adjustment coefficient, and average deficit at ruin. We will compare the results by plotting the results as a function of Pearson's correlation coefficient given in Definition 3.10. For each distribution, the corresponding Pearson correlation coefficient in terms of its parameter is in Chapter 4. In addition to this, we use Proposition 2 to determine the Pearson correlation coefficient of the convex combination of Frechet copulas. Note that the Pearson correlation coefficient (88) of the AMH copula is an infinite sum, therefore we use an approximation of the Pearson's correlation coefficient. We assume that $\lambda = 1$, $\mu = 2$, $c = 1$ and $u = 0$. If one of these parameters takes on a different value it will be explicitly stated.

6.1 Lagrange-Bürmann inversion

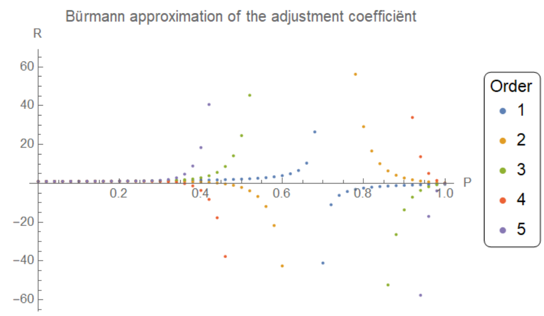
In this part, we present the results from the Lagrange-Bürmann inversion. The results consist of the adjustment coefficient for the FGM copula, Moran-Downton distribution, and the combination between the independent copula and the Frechet upper bound. For all three cases, we provide the first up to fifth-order approximation.



(a) *Adjustment Coefficient of the FGM distribution with Pearson correlation.*



(b) *Adjustment Coefficient of the Moran-Downton distribution.*



(c) *Adjustment Coefficient of the combination of independent and comonotonic distribution.*

Figure 7: *Lagrange-Bürmann inversion approximation of the adjustment coefficient for three families of distributions.*

Looking at Figure 7, we notice that the approach becomes numerically unstable for higher order approximations. To investigate these numerical issues we find the values ρ or parameters for which the derivative $K'(-R_I) = 0$. We will do this for general parameters λ and μ:

$$\begin{aligned}
\rho_{\text{FGM}} &= \frac{1}{4}(1+\frac{2\lambda\mu}{\lambda^2+\mu^2}) \\
\rho_{\text{Moran-Downton}} &= \frac{1}{2} \\
\rho_{\Pi\text{-M}} = p &= \frac{(\lambda^2-\lambda\mu+\mu^2)^2}{\lambda^4-2\mu\lambda^3+4\lambda^2\mu^2-2\lambda\mu^3+\mu^4}.
\end{aligned} \tag{120}$$

where p is the parameter that links the independent copula and Frechet upper bound together $(1-p)\Pi + pM$, see (20). Therefore, from linearity of the Pearson correlation coefficient we get that $\rho_{\Pi\text{-M}} = 0(1-p)+1p = p$. Eq. (120) can be used to determine where the derivative of $K'(-R_I)=0$ in case of the basic parameters $\lambda = 1$ and $\mu = 2$. We get

$$\begin{aligned}
\rho_{\text{FGM}} &= \frac{9}{20} \\
\rho_{\text{Moran-Downton}} &= \frac{1}{2} \\
\rho_{\Pi\text{-M}} &= \frac{9}{13},
\end{aligned} \tag{121}$$

which explains why there is a divergence in the neighbourhood of those points in Figure 7. Every term of the Lagrange-Bürmann inversion has a factor $K'(-R_I)$ in the denominator. Therefore, if this term is close to zero rounding errors make the approximation diverge.

6.2 *FindRoot* built-in function *Mathematica*

In this subsection we present the results of the adjustment coefficient with corresponding Pearson correlation coefficient, by using the built-in *Mathematica* function *FindRoot* (Newton's method).

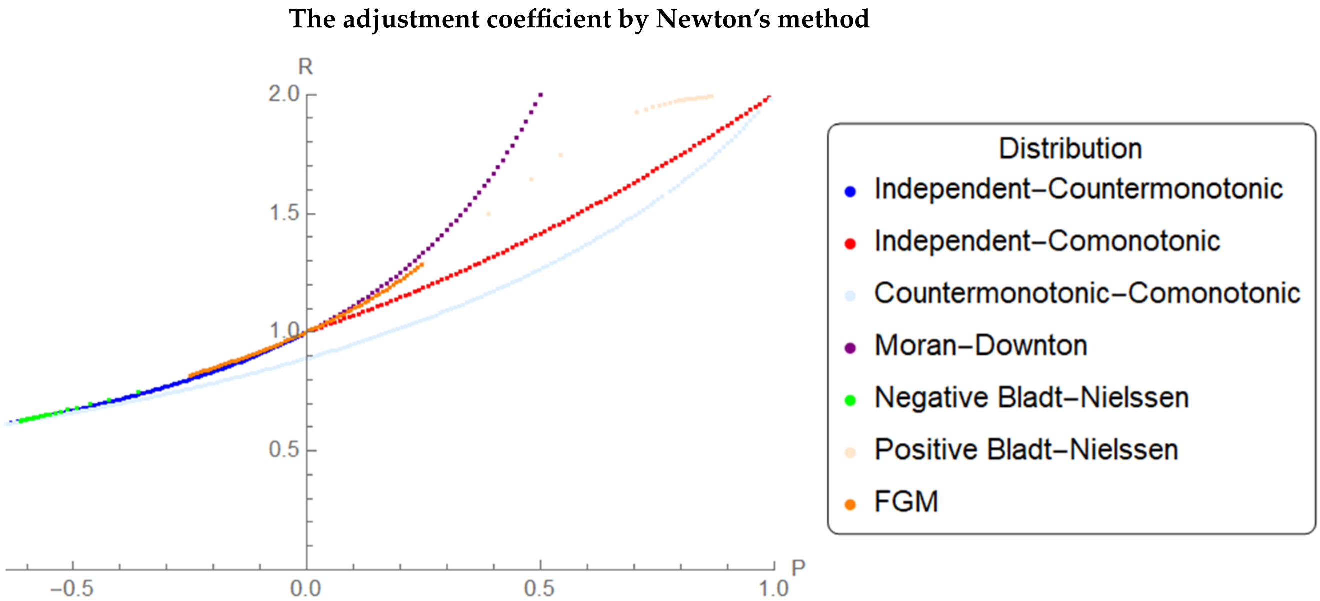


Figure 8: *The adjustment coefficient for all distributions versus the Pearson correlation coefficient.*

In Figure 8 we see that the adjustment coefficient is given for a Pearson correlation coefficient in the interval $[1 - \frac{\pi^2}{6}, 1]$. Notice that for $\rho = 0$ we have that for all distributions the adjustment coefficient is equal to one except the combination between the countermonotonic and comonotonic copula. They coincide because for $\rho = 0$ the distributions are all equal to the independent copula which has $R = \mu - \lambda = 1$, except the combination between the countermonotonic and comonotonic copula. Also, this combination is neither PQD nor NQD, which explains why the adjustment coefficient of the combination can be less than the adjustment coefficient of the independent copula for a positive Pearson correlation coefficient, since PQD (NQD) implies a positive (negative) Pearson correlation coefficient, see Chapter 5 of Nelsen [2].

Notice that the behaviors of the adjustment coefficients for negative correlation are very similar to each other in contrast to the distributions with positive correlation. This difference in behaviour is possible because the adjustment coefficient of the lower bound copula exists, while the comonotonic copula with exponential marginals the adjustment coefficient does not exist. Indeed, the following equation:

$$K_M(w) = \frac{\frac{\mu\lambda}{c\mu-\lambda}}{\frac{\mu\lambda}{c\mu-\lambda} - w} = 1, \tag{122}$$

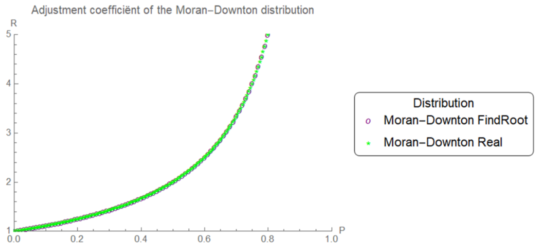
has only the solution $w = 0$. Therefore, Eq. (36) together with the result for the PQD and NQD property explains why the behavior of the adjustment coefficients is possible in Figure 8. Note that we only have a lower bound for the adjustment coefficient, therefore the individual behaviour of the adjustment coefficient for each distribution cannot be explained. In the discussion, we provide additional information for an approach to this.

For the part of Figure 8 with a positive Pearson correlation coefficient, there are more diverse results in contrast to the negative correlated part. We see that the combinations of the independent and comonotonic copula and, the countermonotonic and the comonotonic give the lowest adjustment coefficient for a specific Pearson correlation coefficient. This could be explained by the particular structure their MGFs have and implicitly the underlying distributions.

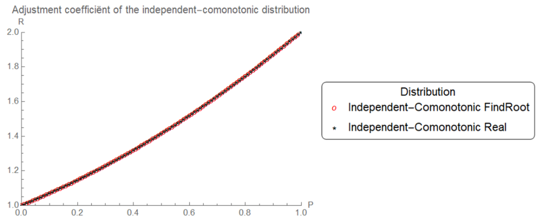
Also, we can observe in Figure 8 that only the adjustment coefficient of the positive Bladt-Nielsen distribution increases less when the Pearson correlation coefficient becomes greater. Heuristically, this can be explained by Figure 5, when we increase n the effect of adding additional states becomes less. Because the probability $\frac{n-1}{n}$ goes to one and the expectation of the additional state is proportional to $\frac{1}{n}$, which implies that adding these states becomes less and lesser of an influence.

Note that the AMH copula is not in Figure 8, because the AMH copula has a similar structure as the FGM copula. Also, the MGF of the AMH copula has an infinite sum representation just as the Pearson correlation coefficient of the AMH copula. Therefore, to use the *Findroot* method we need to approximate the MGF and the Pearson correlation coefficient. But this approximation would be similar to the result of the FGM copula.

In order to check the accuracy we compare the *FindRoot* result with the exact value of the adjustment coefficient in case of the independent-comonotonic copula and Moran-Downton distribution:



(a) *Adjustment coefficient of the Moran-Downton distribution for a specific Pearson correlation coefficient.*



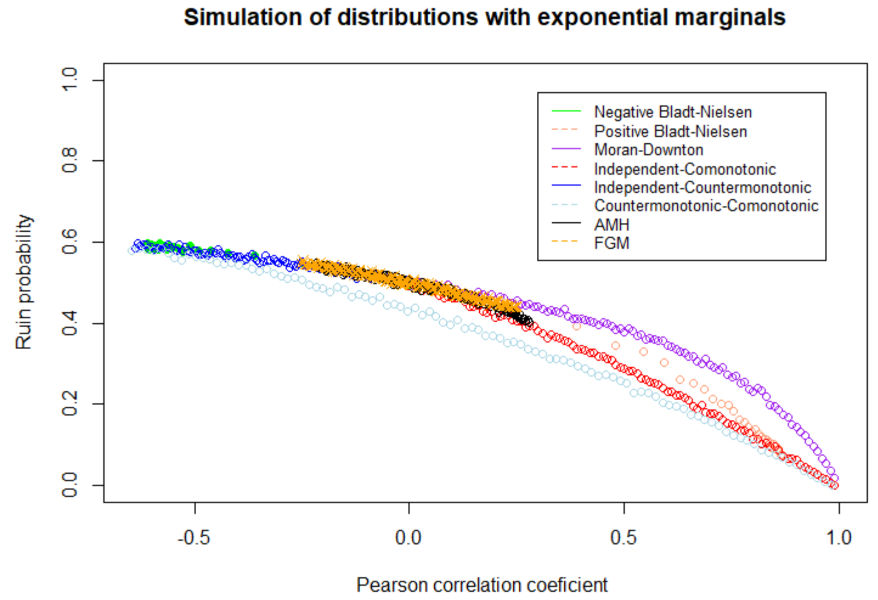
(b) *Adjustment coefficient of the combination independent and comonotonic copula with exponential marginals for a specific Pearson correlation coefficient.*

Figure 9

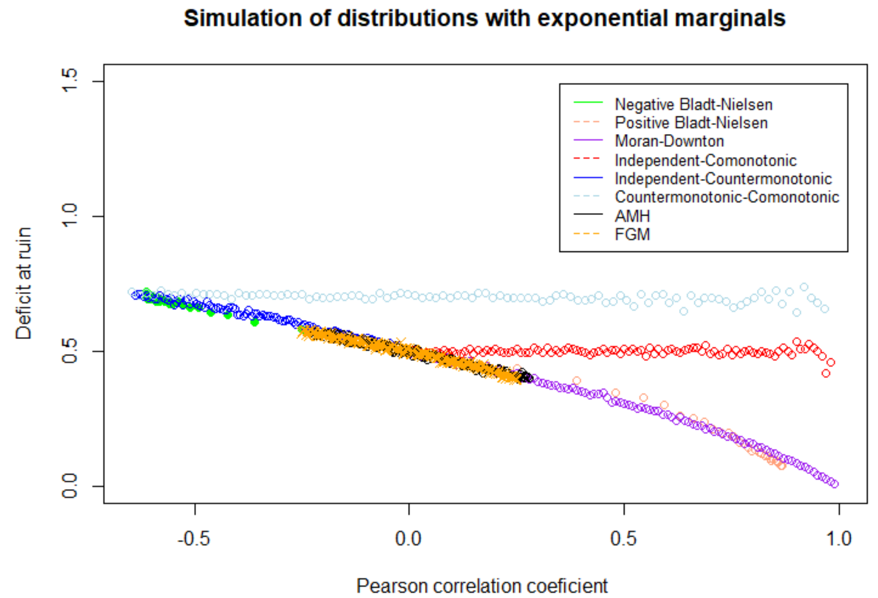
In Figure 9 we see that the *FindRoot* result matches the exact value of the adjustment coefficient.

6.3 Simulation of the claim process

In this section, we present the results of the simulation. The analysis proceeds by simulating the ruin process described in Algorithm 2 with $M = 1000$ claims 10000 times. We first look at the general behavior of the ruin probability and the average deficit at ruin. Then we compare the results from the combination between the independent and comonotonic copula and the Moran-Downton distribution with their explicit formulas. Also, we will compare the FGM and AMH copula, because they have a similar correlation range, see (23). Since the Pearson correlation coefficient of the AMH copula with exponential marginals is an infinite sum (88), we use the approximation $\rho_{\text{AMH}} = \frac{\theta}{4} + \frac{\theta^2}{36}$. The approximation is based on (88), where we only used the terms of the infinite sum up to second order.



(a) Ruin probability of distributions versus Pearson correlation.



(b) Average deficit at ruin versus Pearson correlation

Figure 10: *Comparison of the results for all the distributions.*

In Figure 10 we see that the ruin probability and average deficit at ruin decrease when the Pearson correlation coefficient increases. Another general trend in Figure 10 is that a higher average deficit at ruin seems to correspond to a lower ruin probability for a given correlation. For example, the Moran-Downton distribution has a higher ruin probability than the rest of the distributions for each given correlation but has also a lower average deficit at ruin for each given correlation. Only the positive Bladt-Nielsen distribution seems to have a lower average deficit at ruin for some given correlation.

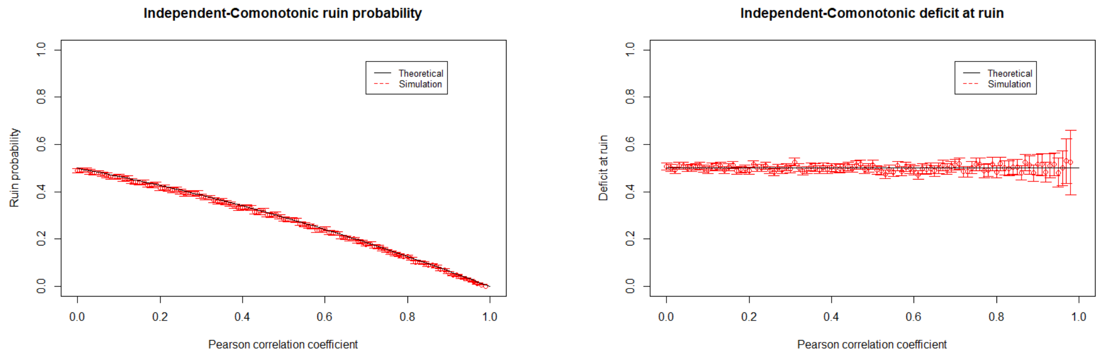
We can use the first order approximation of $\mathbb{E}[e^{-RU(\tilde{T})}|\tilde{T} < \infty] \approx 1 + R\mathbb{E}[-U(\tilde{T})|\tilde{T} < \infty]$ to relate the results of the ruin probability and average deficit at ruin in Figure 10 with the results for the adjustment coefficient in Figure 8:

$$\Psi(0) \approx \frac{1}{1 + R\mathbb{E}[-U(\tilde{T})|\tilde{T} < \infty]}. \tag{123}$$

Note that (123) is also an upper bound for the ruin probability $\Psi(0)$, since all higher-order terms of the approximation of the denominator are positive which makes the ruin probability only lower. For negative Pearson correlation, we see in Figure 8 such adjustment coefficients behave very similarly, and in Figure 10 we see that also for negative Pearson correlation that the average deficit at ruin behaves similarly except in the case of the countermonotonic-comonotonic combination. This also comes back in the results for the ruin probability in Figure 10, where we see that in the case of the countermonotonic-comonotonic case the ruin probability is lower than in the case of the other distributions. The formula given in (123) is a rough estimate. For example, the Bladt-Nielsen distribution and Moran-Downton distribution have a similar average deficit at ruin and the Moran-Downton distribution has a higher adjustment coefficient for each correlation. This implies that the ruin probability of the Moran-Downton distribution should be less than the ruin probability of the Bladt-Nielsen distribution. But this is not the case in Figure 10.

In the simulation of the average deficit at ruin, it seems that independent-comonotonic and countermonotonic-comonotonic have a constant average deficit at ruin despite the change in correlation. For the independent-comonotonic case, this is also predicted by the explicit formula for the deficit at ruin distribution which only depends on the parameter $\mu = 2$. But for the countermonotonic-comonotonic case, we do not have an explicit formula for the deficit at ruin distribution. In this case, ruin only occurs when the countermonotonic part is chosen, since the comonotonic part always gives a positive contribution. An explanation for the constant behavior could be that the Beta function which was used to express the moment generating function of the countermonotonic copula has an alternate expression. Also, note that in both cases the result of the average deficit at ruin seems to decrease in accuracy for higher correlation. This can be explained by the fact that the ruin probability in the simulation tends to zero for a higher Pearson correlation coefficient which implies that there are fewer samples of the deficit at ruin to average over. We will show in the case of the independence-comonotonic case, by computing the 95%-confidence interval, that indeed this effect is due to the smaller sample. Note that for the Bladt-Nielssen and Moran-Downton distribution the average deficit at ruin is more accurate than for the independence-comonotonic and countermonotonic-comonotonic case. This is due to the ruin probability being larger for a higher positive correlation, which implies there are more samples to average over. Also, in the case of the Moran-Downton distribution, the distribution of the deficit at ruin is exponentially distributed with parameter $-x_-$ which goes to infinity for $\rho \to 1$ therefore the variance of the deficit at ruin distribution $\frac{1}{(-x_-)^2} \to 0$ for $\rho \to 1$. This implies that the variance decreases quadratically when $-x_- \to \infty$.

Since we have two distributions that have an explicit expression we will compare the simulation results with them by plotting them together. In the case of the combination between independent and comonotonic copula, we get the result reported in Figure 11.



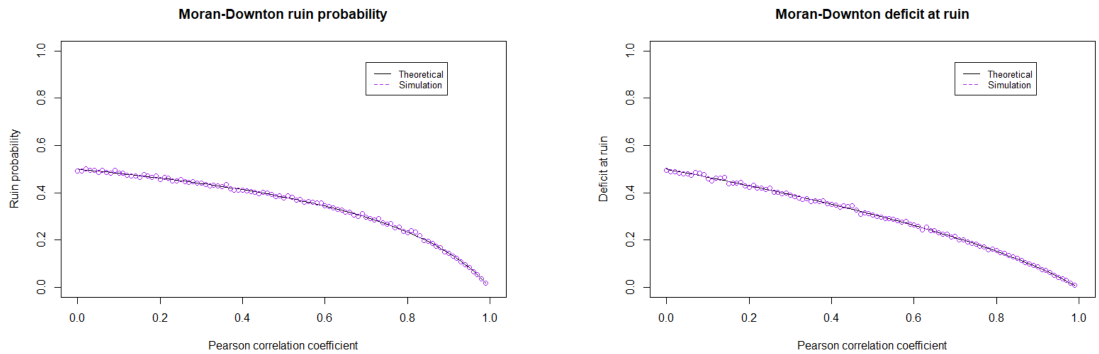
***(a)** Ruin probability of distributions with Pearson correlation.*

***(b)** Average deficit at ruin with Pearson correlation*

***Figure 11:** Comparison of the theoretical results and the simulation results for the indpendent-comonotonic copula with exponential marginals.*

In Figure 11 we see that the simulation matches the theoretical result. Nevertheless, notice that for the average deficit at ruin, for correlation close to one, the simulation gets less accurate. This can be explained by the fact that ruin occurs less often, therefore there are fewer deficits at ruin to average over, which is confirmed by the 95%-confidence interval around the average deficit at ruin.

For the Moran-Downton distribution we get:

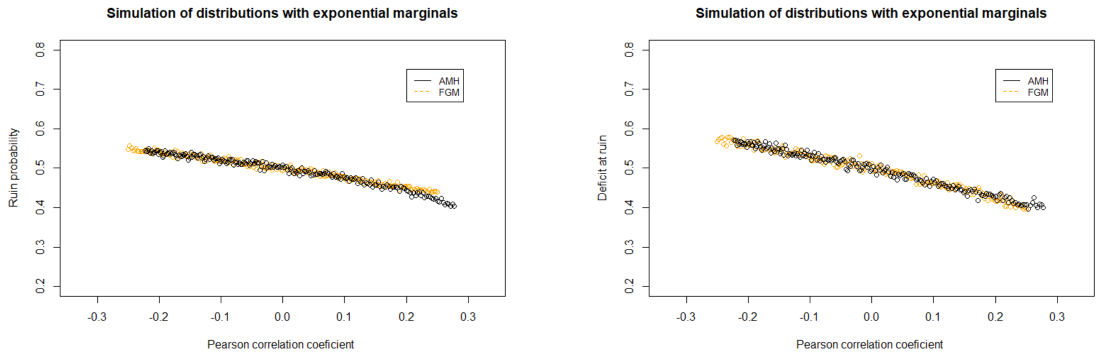


***(a)** Ruin probability of distributions with Pearson correlation.*

***(b)** Average deficit at ruin with Pearson correlation*

***Figure 12:** Comparison of the theoretical results and the simulation results for the Moran-Downton distribution.*

In Figure 12 we see that the simulation matches the theoretical result. We notice that for the average deficit at ruin, for correlation close to one, the simulation stays accurate. This can be explained by the fact that ruin occurs more often, in comparison to the other distributions for similar correlation. Now that we compared the two simulation results of the distributions with the theoretical results, we are interested in the FGM and AMH copula. Because they have a similar structure and the differences were hard to spot in Figure 10, we also created Figure 13.



(a) Ruin probability of distributions with Pearson correlation.

(b) Average deficit at ruin with Pearson correlation

Figure 13: *Comparison of the FGM and AMH copula with exponential marginals.*

In figure 13 we can see that the AMH copula allows a larger positive correlation than the FGM copula. Also, the behavior of the AMH copula is different from the FGM copula for a more positive correlation. The resulting ruin probability decreases more and the average deficit at ruin decreases less. The change cannot be explained by the rough approximation of the Pearson correlation coefficient in the case of the AMH copula, since a better approximation would mean that we rearrange the results with the corresponding Pearson correlation. Nevertheless, the average deficit at ruin becomes constant which implies that a reordering would not change this result. However, a better approximation could explain the sudden decrease in ruin probability for the same reason. To summarize, the change in the ruin probability can be explained with a better approximation but the change of the average deficit at ruin can not be explained.

7 Discussion

In this chapter, we discuss the results of the thesis. We start with the three methods as described in chapter 5 used for the results: Lagrange-Bürmann inversion, *FindRoot* built-in function *Mathematica* and Simulation of the claim process. After this, we discuss the results of the methods.

The first two methods, Lagrange-Bürmann inversion and *FindRoot* built-in function *Mathematica* are used to find the adjustment coefficient. Both methods use the derivatives of the moment generating functions. However, the Lagrange-Bürmann inversion uses higher-order derivatives evaluated at a single point while the *FindRoot* built-in function *Mathematica* uses only the first-order derivative but evaluated at more than one point. Despite that the moment generating function of the lower bound copula with exponential marginals did not have an explicit expression we can use the *Findroot* method. But for the moment generating function of the AMH copula with exponential marginals which is an infinite sum, the method failed. When the lower bound copula was involved (combinations of Frechet-Hoeffding copulas and independent copula) it was possible to use the *FindRoot* function eventhough the MGF of the lower bound copula is an implicit expression. Nevertheless, for the AMH copula this was not possible, because *Mathematica* tries to find a finite expression for the derivative, which is a rather cumbersome expression for the AMH copula. This problem could be solved by computing the derivative manually and giving it as a function to *Mathematica*. In our attempt at using the Lagrange-Bürmann inversion, the methods became numerically unstable around the singularities. This problem could perhaps be resolved by using more accuracy in the computations, nevertheless, the *FindRoot* method does work properly and is more robust. Also, both methods can be used for similar situations, namely, for distributions where we can find the derivatives of the MGF.

The third method is a simulation of the claim process. The software *R* can be used to simulate other copulas than the current copulas in the thesis [10]. For example, the Marshall-Olkin copula could be one of them [7]. For the simulation of the FGM and AMH copulas, we used the *Copula* package from *R*, but for the convex combinations of Frechet-Hoeffding copulas and independent copula, we used a different approach [9]. For the simulation there are two concerns: (1) we can only simulate finitely many claims per run which were 1000 in our case, and (2) the uncertainty of the result of the simulation. The second concern can be dealt with by constructing a confidence interval, to illustrate how certain we are that the simulation results are in this interval. This was done for the independent-comonotonic combination.

In Figure 8 we saw the adjustment coefficient plotted against the Pearson correlation coefficient. We used the PQD and NQD property and, the result of Eq. (36) to explain why the result was possible. Nevertheless, the individual behaviour of the adjustment coefficient for each distribution was not explained. The direct reason for the different behavior is the structure of the MGF. A way to analyze the behaviour of the adjustment coefficient further is to use stochastic orderings such as the PQD and NQD that provide information about the adjustment coefficient.

8 Conclusion

In this thesis, we looked at three sets of bivariate distributions to create dependence between the inter-arrival time and the claim size in the Cramer-Lundberg model. In particular we consider the countermonotonic, independent and comonotonic copula with exponential marginals, bivariate combination of exponential distributions and bivariate mixed Erlang distributions. Then we investigated how this dependence influences the ruin probability, under the assumption of exponential marginals. We used copulas as a tool to construct our bivariate distributions. In addition to the bivariate distributions constructed trough copulas, we analyzed two other bivariate distributions which are standard choices in the field, namely, the Moran-Downton distribution and the Bladt-Nielsen distribution.

We derived expressions for the cumulative distribution function and moment generating function of the difference between premium times inter-arrival time and claim size. For the Moran-Downton distribution and the combination of the independent and upper bound copula, it was also possible to get an explicit expression for the ruin probability. The derivations of the expressions relied on the memoryless property that the exponential distribution possesses. Nevertheless, for the other distributions we could not find explicit expressions for the ruin probability, therefore we resorted to simulation. We used two methods to get an approximation of the adjustment coefficient. The Lagrange-Bürmann method did not yield a satisfactory result in contrast to the *Findroot* method. In our simulation results, we saw that a more negative Pearson correlation corresponds to a higher ruin probability and a higher average deficit at ruin. Also, we showed that the results of each method matched the theoretical result in the case of the Moran-Downton distribution and in the case of a combination of the independent and upper bound copula.

The Cramer-Lundberg model can be used to get insight into a claim process of an insurance company. We introduced dependence into this model to make it more realistic. Nevertheless, other bivariate distributions might be of interest for modeling this process, such as the famous Marshall-Olkin distribution. For the simulation we assumed that we have exponential marginals for the claim size and inter-arrival time and there is no dependence between successive inter-arrival times or between successive claim sizes. These assumptions could be relaxed in further investigations. For example, an insurance company that operates in an area where there are frequent earthquakes or heavy storms would need to use this type of dependence, because when an earthquake or a heavy storm hits, the insurance company would get more claims in a short period. Next to these assumptions on the distributions of the claim sizes and inter-arrival times we made a modeling assumption, by fixing the model parameters. Further research could be done to investigate the impact of these parameters.

References

[1] R. Kaas, M. Goovaerts, J. Dhaene and M. Denuit, *Modern Actuarial Risk Theory*. Springer-Verlag Berlin Heidelberg, 2008.

[2] R. B. Nelsen, *An Introduction to Copulas*. New York, NY: Springer New York, 2006.

[3] H. Albrecher and J. L. Teugels, 'Exponential behavior in the presence of dependence in risk theory,' *Journal of Applied Probability*, vol. 43, no. 1, pp. 257–273, 2006.

[4] M. Bladt and B. F. Nielsen, 'On the construction of bivariate exponential distributions with an arbitrary correlation coefficient,' *Stochastic Models*, vol. 26, no. 2, pp. 295–308, 2010.

[5] H. Cossette, E. Marceau and S. Perreault, 'On two families of bivariate distributions with exponential marginals: Aggregation and capital allocation,' *Insurance: Mathematics and Economics*, vol. 64, pp. 214–224, 2015.

[6] B. Kim and J. Kim, 'Representation of Downton's bivariate exponential random vector and its applications,' *Statistics and Probability Letters*, vol. 81, no. 12, pp. 1743–1750, 2011.

[7] N. Balakrishna and C. D. Lai, *Continuous bivariate distributions*. 2009, pp. 1–684.

[8] H. Schöpf and P. Supancic, 'On Bürmann's Theorem and Its Application to Problems of Linear and Nonlinear Heat Transfer and Diffusion,' *The Mathematica Journal*, vol. 16, 2014.

[9] M. Hofert, I. Kojadinovic, M. Maechler and J. Yan, *Copula: Multivariate Dependence with Copulas. R package version 0.999-13*, Dec. 2012.

[10] RStudio Team, *RStudio: Integrated Development Environment for R*, Boston, 2020.

A Mathematica code Lagrange-Bürmann inversion, *Findroot* method and figures of copulas

```

(* Lagrange-B rmann approach*)
c[f_, p_, 1, y_] :=
 f'[y]/p'[y] (* Starting the recursion for the B rmann approximation*)

 c[f_, p_, n_, y_] :=
 1/n *Together[1/p'[y] D[c[f, p, n - 1, y], y]] (*Recursion step*)

 B rmann[f_, p_, {z_, z0_, m_}] :=
 Module[{n, y},
  f[z0] + Sum[c[f, p, n, y] (1 - p[y])^n, {n, 1, m}] /.
   y -> z0] (* Applying the recursion on an arbitrary function, note \
that 1 in (1-p[y])^n is in the actual case p[z]*)

lambda = 1; (*Initializing the parameters*)
mu = 2;

M = {}
p = Range[-1/3, 1/3, 0.01]; (* p is the parameter of the FGM copula *)
FGM[w_] := lambda*
  mu*((w + 2*mu)*(2*lambda - w) -
      3*p*w^2)/((w + mu)*(w + 2*mu)*(lambda - w)*(2*lambda -
        w))(*MGF of the FGM copula, note:
   specify p in the inverval (-1/3,1/3)*)
For[i = 1, i < 6, i++,
 AppendTo[M,
  B rmann[# &,
   FGM, {R, lambda - mu,
    i}]]] (* Evaluating the B rmann approximation multiple times*)
```

```
G = {}
For[i = 1, i < 6, i++, AppendTo[G, Transpose@{p*3/4, -M[[i]]}]]
Rasterize[
 ListPlot[G,
  PlotLegends ->
   PointLegend[Automatic, {"1", "2", "3", "4", "5"},
    LegendFunction -> "Frame", LegendLabel -> "Order"],
  AxesLabel -> {"P", "R"},
  PlotLabel -> "B rmann␣approximation␣of␣the␣adjustment␣coeffici nt"]]

a = Range[0, 1, 0.02];
Indep[w_] := mu*lambda/((lambda - w) (mu + w))
(* MGF of the independent dist *)
FrechetUp[
  w_] := (mu*lambda/(mu - lambda))/((mu*lambda/(mu - lambda)) -
    w)(* MGF of the Frechet upperbound *)
L[w_] := (1 - a)*Indep[w] +
  a*FrechetUp[
    w](* Linear combination of the independent and the frechet \
upperbound*)
M2 = {}
For[i = 1, i < 6, i++,
 AppendTo[M2,
  B rmann[# &,
   L, {R, lambda - mu,
    i}]]] (* Evaluating the B rmann approximation multiple times*)
G2 = {}
For[i = 1, i < 6, i++, AppendTo[G2, Transpose@{a, -M2[[i]]}]]

Rasterize[
 ListPlot[G2,
  PlotLegends ->
   PointLegend[Automatic, {"1", "2", "3", "4", "5"},
    LegendFunction -> "Frame", LegendLabel -> "Order"],
  AxesLabel -> {"P", "R"},
  PlotLabel -> "B rmann␣approximation␣of␣the␣adjustment␣coeffici nt"]]

p = Range[0, 1, 0.02];
Moran[w_] := (mu*lambda)/((w)^2*(p - 1) + w*(lambda - mu) + lambda*mu)
M3 = {}
For[i = 1, i < 6, i++,
 AppendTo[M3,
  B rmann[# &,
   Moran, {R, lambda - mu,
    i}]]] (* Evaluating the B rmann approximation multiple times*)
G3 = {}
For[i = 1, i < 6, i++, AppendTo[G3, Transpose@{a, -M3[[i]]}]]
Rasterize[
 ListPlot[G3,
  PlotLegends ->
   PointLegend[Automatic, {"1", "2", "3", "4", "5"},
    LegendFunction -> "Frame", LegendLabel -> "Order"],
  AxesLabel -> {"P", "R"},
  PlotLabel -> "B rmann␣approximation␣of␣the␣adjustment␣coeffici nt"]]
```

```
(*Findroot approach*)

FGM[w_, p_] :=
 lambda*mu*((w + 2*mu)*(2*lambda - w) -
     3*p*w^2)/((w + mu)*(w + 2*mu)*(lambda - w)*(2*lambda -
       w))(*MGF of the FGM copula, note: specify p in the inverval \
(-1/3,1/3)*)

Moran[w_,
  p_] := (mu*lambda)/((w)^2*(p - 1) + w*(lambda - mu) + lambda*mu)
(*MGF of the Moran-downton dist, note: specify p, p in the inverval \
(0,1)*)

Indep[w_] := mu*lambda/((lambda - w) (mu + w))
(* MGF of the independent dist *)

BladtNeg[w_, n_] :=
 Sum[Product[l*lambda/(-w + lambda*l), {l, n - i + 1, n}]*
    Product[k*mu/(k*mu + w), {k, i, n}], {i, 1, n}]/n
(* MGF of the bladt-nielsen distribution with delta_{i-j} (negative \
correlation), note: specify n *)

FrechetUp[
  w_] := (mu*lambda/(mu - lambda))/((mu*lambda/(mu - lambda)) -
    w)(* MGF of the Frechet upperbound *)

FrechetLow[w_] :=
 Beta[1 + w/mu, 1 - w/lambda](* MGF of the Frechet lowerbound *)

BladtPos[w_, n_] :=
 Sum[Product[l*lambda/(-w + lambda*l), {l, n - i + 1, n}]*
    Product[k*mu/(k*mu + w), {k, n - i + 1, n}], {i, 1, n}]/n
(* MGF of the bladt-nielsen distribution with delta_{i+j-(n+1)} \
(positive correlation), note: specify n *)

MoranRoots = {};
lambda = 1;
mu = 2;
Q = Range[0, 0.99, 0.01];
Length[Q]
For[i = 1, i < 101, i++,
 Minimum =
  Solve[D[Moran[w, Q[[i]]], w] == 0, w] /. {{(w -> y_)}} -> y ;
 AppendTo[MoranRoots,
  FindRoot[Moran[w, Q[[i]]] - 1 == 0, {w,
     Minimum - 0.01}] /. {(w -> y_)} -> y];

 ]
SetMoran = Transpose@{Q, -MoranRoots};

BladtnegRoots = {};
lambda = 1;
mu = 2;
```

```
n = 30;

For[i = 1, i <= 30, i++,
 Minimum =
  Solve[D[BladtNeg[w, i], w] == 0 && -mu < w < 0,
    w] /. {{(w -> y_)}} -> y ;
 AppendTo[BladtnegRoots,
  FindRoot[BladtNeg[w, i] - 1 == 0, {w,
     Minimum - 0.01}] /. {(w -> y_)} -> y];

 ]
Length[BladtnegRoots]
C6 = {};
For[i = 1, i <= n  , i++;, AppendTo[C6, 1 - Sum[1/n^2, {n, 1, i}]]];
SetBladtneg = Transpose@{C6, -BladtnegRoots};

BladtposRoots = {};
lambda = 1;
mu = 2;
n = 30;
Q = Range[1, n, 1];
Length[Q]
For[i = 1, i <= n, i++,
 Minimum =
  Solve[D[BladtPos[w, i], w] == 0 && -mu < w < 0,
    w] /. {{(w -> y_)}} -> y ;
 AppendTo[BladtposRoots,
  FindRoot[BladtPos[w, i] - 1 == 0, {w,
     Minimum - 0.01}] /. {(w -> y_)} -> y];

 ]
Length[Q]
Length[BladtposRoots]
C6 = {};
For[i = 1, i <= n, i++;, AppendTo[C6, 1 - Sum[1/n, {n, 1, i}]/i]];
SetBladtpos = Transpose@{C6, -BladtposRoots}

FGMRoots = {};
lambda = 1;
mu = 2;
Q = Range[-1/3, 1/3, 0.01];
For[i = 1, i <= Length[Q], i++,
 Minimum =
  Solve[D[FGM[w, Q[[i]]], w] == 0 && -mu < w < 0,
    w] /. {{(w -> y_)}} -> y ;
 AppendTo[FGMRoots,
  FindRoot[FGM[w, Q[[i]]] - 1 == 0, {w,
     Minimum - 0.01}] /. {(w -> y_)} -> y];

 ]
SetFGM = Transpose@{Q*3/4, -FGMRoots}

FreUpRoots = {};
lambda = 1;
```

```
mu = 2;
Q = Range[0, 1,
    0.01];(*Q corresponds to the Pearson correlation by linearity*)
Length[Q];
For[i = 1, i <= Length[Q], i++,
 Minimum =
  Solve[D[Indep[w]*(1 - Q[[i]]) + FrechetUp[w]*Q[[i]], w] ==
      0 && -mu < w < 0, w] /. {{(w -> y_)}} -> y ;
 AppendTo[FreUpRoots,
  FindRoot[Indep[w]*(1 - Q[[i]]) + FrechetUp[w]*Q[[i]] - 1 == 0, {w,
     Minimum - 0.01}] /. {(w -> y_)} -> y];

 ]
SetFreUp = Transpose@{Q, -FreUpRoots};

FreLowRoots = {};
lambda = 1;
mu = 2;
Q = Range[0, 1,
    0.01];(*Q corresponds to the spearman rho correlation by linearity \
of both*)
Length[Q];
For[i = 1, i <= Length[Q], i++,
 Minimum =
  Solve[D[Indep[w]*(1 - Q[[i]]) + FrechetLow[w]*Q[[i]], w] ==
      0 && -mu < w < 0, w] /. {{(w -> y_)}} -> y ;
 AppendTo[FreLowRoots,
  FindRoot[Indep[w]*(1 - Q[[i]]) + FrechetLow[w]*Q[[i]] - 1 == 0, {w,
     Minimum - 0.01}] /. {(w -> y_)} -> y];

 ]
SetFreLowpearson = Transpose@{Q*(1 - Pi^2/6), -FreLowRoots}

UpLowRoots = {};
lambda = 1;
mu = 2;
Q = Range[0, 1,
    0.005];(*Q corresponds to the spearman rho correlation by \
linearity of both*)
Length[Q];
For[i = 1, i <= Length[Q], i++,
 Minimum =
  Solve[D[FrechetLow[w]*(1 - Q[[i]]) + FrechetUp[w]*Q[[i]], w] ==
      0 && -mu < w < 0, w] /. {{(w -> y_)}} -> y ;
 AppendTo[UpLowRoots,
  FindRoot[FrechetLow[w]*(1 - Q[[i]]) + FrechetUp[w]*Q[[i]] - 1 ==
     0, {w, Minimum - 0.01}] /. {(w -> y_)} -> y];

 ]
UpLowRoots
SetUpLowpearson = Transpose@{1 - Pi^2/6 + Q*Pi^2/6, -UpLowRoots}

D2 = List[SetFreLowpearson, SetFreUp, SetUpLowpearson, SetMoran,
    SetBladtneg, SetBladtpos, SetFGM];
```

```
Rasterize[
 ListPlot[D2, PlotRange -> {{1 - Pi^2/6, 1}, {0, 2}},
  PlotLegends ->
   PointLegend[
    Automatic, {"Independent-Countermonotonic",
     "Countermonotonic-Comonotonic", "Countermonotonic-Comonotonic",
     "Moran-Downton", "Negative Bladt-Nielsen",
     "Positive Bladt-Nielsen", "FGM"}, LegendFunction -> "Frame",
    LegendLabel -> "Distribution"],
  AxesLabel -> {"Pearson correlation", "R"},
  PlotLabel -> "Adjustment coeffici nt with negative correlation",
  PlotStyle -> {Blue, Red, LightBlue, Purple, Green, LightOrange,
    Orange }]]


(* Copula pictures*)

(*Frechet Copulas*)
g = Plot3D[u*v, {u, 0, 1}, {v, 0, 1}, AxesLabel -> { u, v},
  PlotLabel -> "Independent Copula"]

g2 = Plot3D[Min[u, v], {u, 0, 1}, {v, 0, 1}, AxesLabel -> { u, v},
  PlotLabel -> "Comonotonic Copula"]

g3 = Plot3D[Max[u + v - 1, 0], {u, 0, 1}, {v, 0, 1},
  AxesLabel -> { u, v} , PlotLabel -> "Countermonotonic Copula"]

Rasterize[
 Grid[{{g3, g, g2}}, ItemSize -> {{20, 20, 20}}, Frame -> False]]

(* Investigating linear combinations of the Frechet copulas*)
Manipulate[
 Plot3D[(1 - p)*Max[u + v - 1, 0] + p*Min[u, v], {u, 0, 1}, {v, 0,
   1}] , {p, 0, 1}]

Manipulate[
 Plot3D[(1 - p)*u*v + p*Min[u, v], {u, 0, 1}, {v, 0,
   1}] , {p, 0, 1}]

 Manipulate[
 Plot3D[(1 - p)*u*v + p*Max[u + v - 1, 0], {u, 0, 1}, {v, 0,
   1}] , {p, 0, 1}]
```

B R code for the simulation

```
### Simulation of cramer lundberg model with dependence###

fgm <- fgmCopula(param = 0, dim =2)
FGMmvdc <- mvdc(copula=fgm, margins=c("exp", "exp"),
                 paramMargins=list(list(rate=1),
                                   list(rate=1.1)))
E<- rMvdc(1000,FGMmvdc)
E[,1]-E[,2]
```

```
SimulationRuin<- function(Copula, Marginals, Param, c, u){

  DIST <- mvdc(copula=Copula, margins=Marginals,
                    paramMargins=Param)


  Sim <- rMvdc(1000,DIST)

  T[1] = u

  for(i in 2: 1000){
    T[i] <- T[i-1] + c*Sim[i,1]-Sim[i,2]
    if(T[i]< 0){
      return(c(1,T[i]))
    }
  }

  return(c(0,0))

}

MultipleSimRuin<- function(Copula, Marginals, Param, c, u, n){
  AmountRuins <- 0
  TotalDeficit <- 0
  for(i in 1:n){

  Sim <- SimulationRuin(Copula, Marginals, Param, c, u)
  if(Sim[1] == 1){
    AmountRuins <- AmountRuins + 1
    TotalDeficit <- TotalDeficit + Sim[2]
  }
  }
  DeficitatRuin <- TotalDeficit/AmountRuins
  Ruinprobability <- AmountRuins/n
  R <- log(Ruinprobability)/(-DeficitatRuin-u)
  return(c(Ruinprobability,-DeficitatRuin,R))
}


###FGM###

ResultsFGMRuin <- c()
ResultsFGMDeficit <- c()
ResultsFGMR <- c()
q = 0
for(i in seq(-1,1, by = 0.01)){
  q <- q + 1 #Index counter
  Res <- MultipleSimRuin(fgmCopula(param = i, dim = 2), c("exp","exp"),
      list(list(rate=1), list(rate=2)), 1,0,10000)
  ResultsFGMRuin[q] <- Res[1]
  ResultsFGMDeficit[q] <- Res[2]
  ResultsFGMR[q] <- Res[3]
}
```

```
ResultsFGMRuin
ResultsFGMDeficit
ResultsFGMR

###AMH###

ResultsAMHRuin <- c()
ResultsAMHDeficit <- c()
ResultsAMHR <- c()
q = 0
for(i in seq(-1,1, by = 0.01)){
  q <- q + 1 #Index counter
  Res <- MultipleSimRuin(amhCopula(param = i, dim = 2), c("exp","exp"),
      list(list(rate=1), list(rate=2)), 1,0,10000)
  ResultsAMHRuin[q] <- Res[1]
  ResultsAMHDeficit[q] <- Res[2]
  ResultsAMHR[q] <- Res[3]
}
ResultsAMHRuin
ResultsAMHDeficit
ResultsAMHR <- log(ResultsAMHRuin)/(ResultsAMHDeficit - 1)

###MORAN-DOWNTON###

Moranstep<-function(lambda,muhat,rhoMoran,c){
  gamma <- lambda*rhoMoran/(1-rhoMoran)
  mu <- muhat/(1-rhoMoran)

  interarrival <- rexp(1,lambda)
  amountofclaims <- rpois(1,interarrival*gamma)
  Claim <- rexp(amountofclaims+1,mu)

  return(c*interarrival- sum(Claim))
}

MoranS <-function(lambda,muhat,rhoMoran,c,u,n){
  s <- c(u)
  for(i in 1:n){
  s[i+1] <- s[i] + Moranstep(lambda,muhat,rhoMoran,c)
  }



  FirstNegativeIndex <- which(s < 0)[1]
  FirstNegative <- s[FirstNegativeIndex]
  if(is.na(FirstNegative)){
    return(0)

  }else{
    return(-FirstNegative)

  }
}
```

```
MultiMoranS <- function(lambda,muhat,rhoMoran,c,u,n,m){
  Ruin <- 0
  TotalDeficit <- 0

  for(i in 1:m){
    Sim <- MoranS(lambda,muhat,rhoMoran,c,u,n)

    if(Sim > 0 ){
      TotalDeficit = TotalDeficit + Sim
      Ruin <- Ruin + 1
    }
  }
  DeficitatRuin <- TotalDeficit/Ruin
  Ruinprobability <- Ruin/m
  R<- log(Ruinprobability)/(-DeficitatRuin-u)
  return(c(Ruinprobability,DeficitatRuin,R))
}

ResultsMoranRuin <- c()
ResultsMoranDeficit <- c()
ResultsMoranR <- c()
q = 0
for(i in seq(0,0.99, by = 0.01)){
  q <- q + 1 #Index counter
  Res <- MultiMoranS(1,2,i,1,0,1000,10000)
  print(Res[1])
  ResultsMoranRuin[q] <- Res[1]
  print(ResultsMoranRuin)
  ResultsMoranDeficit[q] <- Res[2]
  ResultsMoranR[q] <- Res[3]
}

### Bladt ###
#Positive correlation

BladtNielsenPosCor <- function(lambda, mu, n, c){
  Total <- 0

  for(i in 1:n){
    Chain <- rexp(2,n-i+1) #Simulating the claim and the interarrival
        together
    Total <- Total + c*Chain[1]/lambda - Chain[2]/mu #scaling propety of
        the exponential random variable
    proceed <- rbinom(1,1,(n-i)/(n-i+1))

    if(proceed == 0){
      return(Total)

    }
  }
  return(Total)
}

BladtNielsenSimP <- function(lambda,mu,n,c, u,m){
```

```
  Run <- c(u)

  for(i in 1:m){
    Run[i+1] = Run[i] + BladtNielsenPosCor(lambda,mu,n,c)
    if(Run[i+1] < 0){
      return(-Run[i+1])
    }

  }
  return(0)
}

MultiBladtSimP <- function(lambda,mu,n,c,u,m,k){
  Ruin <- 0
  Deficit <- 0

  for(i in 1:k){
    Res <- BladtNielsenSimP(lambda,mu,n,c,u,m)
    if(Res > 0){
      Ruin <- append(Ruin,1)
      Deficit <- append(Deficit,Res)
    }
    else{
      Ruin <- append(Ruin,0)
      Deficit <- append(Deficit,0)}
  }
  return(list(Ruin,Deficit))
}

DeficitBladtpos <- c()
RuinBladtpos <- c()
DeficitBladtposSD <- c()
RuinBladtposSD <- c()
j = 0
for(i in 1:30){
  Res <- MultiBladtSimP(1,2,i,1,0,1000,10000)
  RuinBladtpos[j] <- mean(unlist(Res[1]))
  DeficitBladtpos[j] <- mean(unlist(lapply(Res[2], function(x)x[x !=0]))
      )
  RuinBladtposSD[j] <- sd(unlist(Res[1]))
  DeficitBladtposSD[j] <- sd(unlist(lapply(Res[2], function(x)x[x !=0]))
      )

  j <- j +1
}

posCor <- c()
for( i in 1:30){
  posCor[i] <- 1- 1/i*sum(1/1:i)
}

#Negative correlation

BladtNielsenNegCor<- function(lambda,mu,n,c){
```

```
  Total <- 0
  for(i in 1:n){
  Chain <- rexp(1,(n-i+1)*lambda)
  Total <- Total + Chain #Updating the total interarrival time
  proceed <- rbinom(1,1,1/(n-i+1)) #Determines if we go to the claims

  if(proceed == 1){
    Claims <- rexp(n-i+1,1)#simulating the remaining claims

    Weights <- c() #scaling all claims with the corresponding 1/integer
    for(j in 1:n-i+1){
      Weights[j] <- 1/(n-j+1)
    }

    Total <- c*Total -sum(Weights*Claims)/mu #scaling all claims with
        parameter mu
    return(Total)
  }
  }
}

BladtNielsenSimN <- function(lambda,mu,n,c,u,m){
  Run <- c(u)

  for(i in 1:m){
    Run[i+1] = Run[i] + BladtNielsenNegCor(lambda,mu,n,c)
    if(Run[i+1] < 0){
      return(-Run[i+1])
    }

  }
  return(0)
}

MultiBladtSimN <- function(lambda,mu,n,c,u,m,k){
  Ruin <- 0
  Deficit <- 0

  for(i in 1:k){
   Res <- BladtNielsenSimN(lambda,mu,n,c,u,m)
   if(Res > 0){
     Ruin <- append(Ruin,1)
     Deficit <- append(Deficit,Res)
   }
   else{
   Ruin <- append(Ruin,0)
   Deficit <- append(Deficit,0)}
  }
  return(list(Ruin,Deficit))
}

DeficitBladtneg <- c()
RuinBladtneg <- c()
DeficitBladtnegSD <- c()
```

```
RuinBladtnegSD <- c()
j=0
for(i in 1:30){
  Res <- MultiBladtSimN(1,2,i,1,0,1000,10000)
  RuinBladtneg[j] <- mean(unlist(Res[1]))
  DeficitBladtneg[j] <- mean(unlist(lapply(Res[2], function(x)x[x !=0]))
      )
  RuinBladtnegSD[j] <- sd(unlist(Res[1]))
  DeficitBladtnegSD[j] <- sd(unlist(lapply(Res[2], function(x)x[x !=0]))
      )

  j <- j +1
}

negCor <- c()
for( i in 1:30){
  negCor[i] <- 1- sum(1/(1:i)^2)
}

###combination Independent-Comonotonic###

IndepComon<- function(lambda,mu,a,c,u,n){
  Arrivalrate <- rexp(n,lambda)
  IndepClaim <- rexp(n,mu)
  ComonClaim <- lambda/mu*Arrivalrate
  v <- rbinom(n,1,a)
  #print(Arrivalrate)
  #print(IndepClaim*a + (1-a)*ComonClaim)
  steps <- c*Arrivalrate - (IndepClaim*(1-v) + v*ComonClaim)
  Run <- c(u,steps)
  CumRun <- cumsum(Run)

  indexRuin <- min(which(CumRun<0))
  if(indexRuin== Inf){
    return(c(0,0))
  }else{
    return(c(1,CumRun[indexRuin]))
  }
}


MultiIndepComon<- function(lambda,mu,a,c,u,n,m){
  AmountRuin <- c()
  AmountDeficit <- c()
  for(i in 1:m){
    Run <- IndepComon(lambda,mu,a,c,u,n)
    AmountRuin <- append(AmountRuin, Run[1],after = length(AmountRuin))
    AmountDeficit <- append(AmountDeficit,-Run[2],after = length(
        AmountDeficit))

  }

  return(list(AmountRuin,AmountDeficit))
}
```

```

lambda <- 1
mu <- 2
c <- 1
u <- 0
a <- seq(0,1, by = 0.01)

IndepComonRuin <- c()
IndepComonDeficit <- c()
IndepComonRuinSD <- c()
IndepComonDeficitSD <- c()
IndepComonDeficitlength <- c()

j = 0
for(i in a){
  Res <- MultiIndepComon(lambda,mu,i,c,u,1000,10000)
  IndepComonRuin[j] <- mean(unlist(Res[1]))
  IndepComonDeficit[j] <- mean(unlist(lapply(Res[2], function(x)x[x !=
      0])))
  IndepComonRuinSD[j] <- sd(unlist(Res[1]))
  IndepComonDeficitSD[j] <- sd(unlist(lapply(Res[2], function(x)x[x !=
      0])))
  IndepComonDeficitlength[j] <- length(unlist(lapply(Res[2], function(x)
      x[x !=0])))

  j <- j +1

}

###combination Countermonotonic-Independent###

IndepCountermon<- function(lambda,mu,a,c,u,n){
  Arrivalrate <- rexp(n,lambda)
  IndepClaim <- rexp(n,mu)
  CounterClaim <- -1/mu*log(1-exp(-lambda*Arrivalrate))
  v = rbinom(n,1,a)
  #print(Arrivalrate)
  #print(IndepClaim*a + (1-a)*CounterClaim)
  steps <- c*Arrivalrate - (IndepClaim*(1-v) + v*CounterClaim)
  Run <- c(u,steps)
  CumRun <- cumsum(Run)
  indexRuin <- min(which(CumRun<0))
  if(indexRuin== Inf){
    return(c(0,0))
  }else{
    return(c(1,CumRun[indexRuin]))
  }
}

MultiIndepCounter<- function(lambda,mu,a,c,u,n,m){
  AmountRuin <- c()
  AmountDeficit <- c()
  for(i in 1:m){
    Run <- IndepCountermon(lambda,mu,a,c,u,n)
```

```
    AmountRuin <-append(AmountRuin,Run[1])
    AmountDeficit <- append(AmountDeficit,-Run[2])
  }

  return(list(AmountRuin,AmountDeficit))
}

IndepCounterRuin <- c()
IndepCounterDeficit <- c()
IndepCounterRuinSD <- c()
IndepCounterDeficitSD<- c()
j = 0
for(i in a){
  Res <- MultiIndepCounter(lambda,mu,i,c,u,1000,10000)
  IndepCounterRuin[j] <- mean(unlist(Res[1]))
  IndepCounterDeficit[j] <- mean(unlist(lapply(Res[2], function(x)x[x !=
      0])))
  IndepCounterRuinSD[j] <- sd(unlist(Res[1]))
  IndepCounterDeficitSD[j] <- sd(unlist(lapply(Res[2], function(x)x[x !=
      0])))

  j <- j +1

}

###combination Countermonotonic-Comonotonic###

CounterComon<- function(lambda,mu,a,c,u,n){
  Arrivalrate <- rexp(n,lambda)
  CounterClaim <- -1/mu*log(1-exp(-lambda*Arrivalrate))
  ComonClaim <- lambda/mu*Arrivalrate
  v = rbinom(n,1,a)
  #print(Arrivalrate)
  #print(IndepClaim*a + (1-a)*ComonClaim)
  steps <- c*Arrivalrate - (CounterClaim*(1-v) + v*ComonClaim)
  Run <- c(u,steps)
  CumRun <- cumsum(Run)
  indexRuin <- min(which(CumRun<0))
  if(indexRuin== Inf){
    return(c(0,0))
  }else{
    return(c(1,CumRun[indexRuin]))
  }
}
CounterComon(1,2,1/2,1,0,100)
MultiCounterComon<- function(lambda,mu,a,c,u,n,m){
  AmountRuin <- c()
  AmountDeficit <- c()
  for(i in 1:m){
    Run <- CounterComon(lambda,mu,a,c,u,n)
    AmountRuin <- append(AmountRuin,Run[1])
    AmountDeficit <- append(AmountDeficit, -Run[2])
  }

```

```
  return(list(AmountRuin, AmountDeficit))
}
MultiCounterComon(1,2,1/2,1,0,100,100)

CounterComonRuin <- c()
CounterComonDeficit <- c()
CounterComonRuinSD <- c()
CounterComonDeficitSD <- c()

j = 0
for(i in a){
  Res <- MultiCounterComon(lambda,mu,i,c,u,1000,10000)
  CounterComonRuin[j] <- mean(unlist(Res[1]))
  CounterComonDeficit[j] <- mean(unlist(lapply(Res[2], function(x)x[x !=
      0])))
  CounterComonRuinSD[j] <- sd(unlist(Res[1]))
  CounterComonDeficitSD[j] <- sd(unlist(lapply(Res[2], function(x)x[x !=
      0])))

  j <- j +1

}

###Figures###

###Ruin probability of all distributions###
plot(negCor, RuinBladtneg, xlim = c(1-pi^2/6,1), ylim = c(0,1),pch = 16,
     col = "Green", xlab = "Pearson correlation coeficient ", ylab = "
    Ruin probability", main = "Simulation of distributions with
    exponential marginals")
points(posCor, RuinBladtpos, col = "lightsalmon1")
points(seq(0,0.99, by=0.01), ResultsMoranRuin, col = "Purple")
points(a[0:100], IndepComonRuin, col = "Red")
points(a[0:100]*(1-pi^2/6),IndepCounterRuin, col = "blue")
points(a[0:100]+(1-a[0:100])*(1-pi^2/6), CounterComonRuin,col = "light
    blue")
points(seq(-1,1, by=0.01)/4+ seq(-1,1, by=0.01)^2/36, ResultsAMHRuin,
    col ="black" )
points(seq(-1,1, by=0.01)/4, ResultsFGMRuin, col= "orange",pch = 4)
legend(0.3,0.97, legend=c("Negative Bladt-Nielsen", "Positive Bladt-
    Nielsen", "Moran-Downton","Independent-Comonotonic","Independent-
    Countermonotonic","Countermonotonic-Comonotonic","AMH","FGM"), col =
     c("Green","lightsalmon1", "Purple","Red","blue","light blue","black
    ","orange"), lty=1:2, cex=0.8)

###Average deficit at ruin of all distributions###
plot(negCor, DeficitBladtneg, xlim = c(1-pi^2/6,1), ylim = c(0,1.5),pch
    = 16, col = "Green", xlab = "Pearson correlation coeficient ", ylab
    = "Deficit at ruin", main = "Simulation of distributions with
    exponential marginals")
points(posCor, RuinBladtpos, col = "lightsalmon1")
points(seq(0,0.99, by=0.01), ResultsMoranDeficit, col = "Purple")
points(a[0:100], IndepComonDeficit, col = "Red")
points(a[0:100]*(1-pi^2/6),IndepCounterDeficit, col = "blue")
```

```r
points(a[0:100]+(1-a[0:100])*(1-pi^2/6), CounterComonDeficit,col = "light blue")
points(seq(-1,1, by=0.01)/4+ seq(-1,1, by=0.01)^2/36, ResultsAMHDeficit, col ="black" )
points(seq(-1,1, by=0.01)/4, ResultsFGMDeficit, col= "orange",pch = 4)
legend(0.35,1.49, legend=c("Negative Bladt-Nielsen", "Positive Bladt-Nielsen", "Moran-Downton","Independent-Comonotonic","Independent-Countermonotonic","Countermonotonic-Comonotonic","AMH","FGM"), col = c("Green","lightsalmon1", "Purple","Red","blue","light blue", "black", "orange"), lty=1:2, cex=0.8)

###FGM versus AMH###
plot(seq(-1,1, by=0.01)/4, ResultsFGMDeficit, col= "orange", xlim = c(-1/3,1/3), ylim = c(0.2,0.8), xlab = "Pearson correlation coeficient ", ylab = "Deficit at ruin", main = "Simulation of distributions with exponential marginals")
points(seq(-1,1, by=0.01)/4+ seq(-1,1, by=0.01)^2/36, ResultsAMHDeficit, col ="black" )
legend(0.2,0.75, legend=c("AMH","FGM"), col = c("black","orange"), lty=1:2, cex=0.8)

plot(seq(-1,1, by=0.01)/4, ResultsFGMRuin, col= "orange", xlim = c(-1/3,1/3), ylim = c(0.2,0.8), xlab = "Pearson correlation coeficient ", ylab = "Ruin probability", main = "Simulation of distributions with exponential marginals")
points(seq(-1,1, by=0.01)/4+ seq(-1,1, by=0.01)^2/36, ResultsAMHRuin, col ="black" )
legend(0.2,0.75, legend=c("AMH","FGM"), col = c("black","orange"), lty=1:2, cex=0.8)

###Moran-Downton distribution theoretical versus simulation###
curve((((1+x/(1-x)-2/(1-x))-sqrt((1+x/(1-x)-2/(1-x))^2+8/(1-x)))/(-2)-1/(1-x))/(((1+x/(1-x)-2/(1-x))-sqrt((1+x/(1-x)-2/(1-x))^2+8/(1-x)))/(-2)), from=0, to=1, xlab="Pearson correlation coefficient", ylab="Ruin probability",xlim = c(0,1), ylim = c(0,1),main = "Moran-Downton ruin probability")
points(seq(0,0.99, by=0.01), ResultsMoranRuin, col = "Purple")
legend(0.7,0.95, legend = c("Theoretical","Simulation"), col = c("black","purple"), lty=1:2, cex=0.8)

curve(1/(((1+x/(1-x)-2/(1-x))-sqrt((1+x/(1-x)-2/(1-x))^2+8/(1-x)))/(-2)), from=0, to=1, xlab="Pearson correlation coefficient", ylab="Deficit at ruin",xlim = c(0,1), ylim = c(0,1),main = "Moran-Downton deficit at ruin")
points(seq(0,0.99, by=0.01), ResultsMoranDeficit, col = "Purple")
legend(0.7,0.95, legend = c("Theoretical","Simulation"), col = c("black","purple"), lty=1:2, cex=0.8)

###combination of Independent-Comonotonic###
error <- qnorm(0.975)*IndepComonRuinSD/sqrt(10000)
left <- IndepComonRuin-error
right <- IndepComonRuin+error
plotCI(a[1:100], IndepComonRuin, ui=right, li=left, xlab="Pearson correlation coefficient", ylab="Ruin probability", main = "
```

```
    Independent-Comonotonic ruin probability", col = 'red',ylim = c(0,1)
    ,xlim = c(0,1) )
curve((2+1/2*(1-2*x - sqrt(9-4*x+4*x^2)))/(2), from=0, to=1,col = "black
    ",add = TRUE)
legend(0.7,0.95, legend = c("Theoretical","Simulation"), col = c("black"
    ,"red"), lty=1:2, cex=0.8)

error <- qnorm(0.975)*IndepComonDeficitSD/sqrt(IndepComonDeficitlength)
left <- IndepComonDeficit-error
right <- IndepComonDeficit+error
plotCI(a[1:100], IndepComonDeficit, ui=right, li=left, xlab="Pearson 
    correlation coefficient", ylab="Deficit at ruin" , col = 'red', ylim
     = c(0,1),xlim = c(0,1), main = "Independent-Comonotonic deficit at 
    ruin")
points(c(0, 1),c(1/2,1/2),  type = "l", col = 'black')
legend(0.7,0.95, legend = c("Theoretical","Simulation"), col = c("black"
    ,"red"), lty=1:2, cex=0.8)

###Distribution plots of the fgm copula###
par(pty="s") #Makes the plot squares
plot(fgmCopula(param=-1), n = 10000)
plot(fgmCopula(param=0), n = 10000)
plot(fgmCopula(param=1), n = 10000)
```