MASTER

# Compensation-Based Control for Lossy Communication Networks

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# Abstract

In this thesis we are concerned with the stability analysis and the design of stabilizing compensators for networked control systems (NCSs) that exhibit packet dropouts. We propose a new type of dropout compensator, depending on the local dropout history, for which we provide LMI-based synthesis conditions. The analysis and design framework includes both worst-case bound and stochastic models to describe the packet dropout behavior in both the sensor-to-controller and controller-to-actuator channel. Numerical examples demonstrate the significantly improved robustness with respect to packet dropouts using the proposed dropout compensator, compared to using the zero strategy and the hold strategy.

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# Compensation-Based Control for Lossy Communication Networks

T.M.P. Gommans, W.P.M.H. Heemels, N.W. Bauer, N. van de Wouw

## 1 Introduction

Networked control systems (NCSs) are feedback control systems, in which the communication between spatially distributed components, such as sensors, actuators and controllers, occurs through a shared communication network. Over the last decade, the study of control systems in which communication takes place via a shared network is receiving more and more attention, see, e.g., the overview papers [21,34,37,39] and the recent book [3]. The reason for this interest is that the use of networks offers many advantages for control systems, such as low installation and maintenance costs, reduced system wiring (in the case of wireless networks) and increased flexibility of the system. However, from a control theory point of view, the presence of a communication network also introduces several, possibly destabilizing, effects, such as packet dropout [20,28–30,32,35], time-varying transmission intervals and delays, see e.g., [15,25,27,31,36], and [8,16,22], respectively. In this paper we focus on packet dropouts, which can occur, for instance, if there are transmission failures or message collisions. As packet dropouts are a potential source of instability in NCSs, it is of interest to investigate measures to mitigate the influence of dropouts on the stability and also performance of a NCS.

In the literature several different strategies have been proposed to deal with packet dropouts. These strategies can be categorized into three groups: strategies for dropouts in the sensor-to-controller channel, strategies for dropouts in the controller-to-actuator channel and strategies for dropouts in both the sensor-to-controller and controller-to-actuator channel. For dropouts in the controller-to-actuator channel, typically model-based observers are used to alleviate the effect of dropouts. For dropouts in the controller-to-actuator channel, a solution proposed in [29] was the zero strategy, in which the actuator input is set to zero if a packet is dropped. The hold strategy, in which the actuator holds the last received control input instead of setting it to zero, was used in [28]. Instead of holding the previous control input or setting the control input to zero, dynamical predictive outage compensators were presented in [20]. This approach is related to our approach, but considers only dropouts in the controller-to-actuator channel. An alternative scheme based on sending future predicted control values to the actuator was proposed in [2,4,6]. For packet dropouts in both the controller-to-actuator channel and sensor-to-controller channel, so-called generalized hold functions, which extend the basic hold strategy were studied in [24], where the optimal hold function is found by solving a LQG problem. The approach in [24] is based on a TCP protocol, and requires acknowledgements of successful packet transmissions.

In this paper we provide systematic design methodologies for a novel dropout compensation strategy that minimizes the influence of dropouts on the stability of the NCS. This new compensation strategy applies for NCSs in which both the controller-to-actuator and the sensor-to-controller channel are subject to dropouts, and does not require any acknowledgement of successful transmissions. In modeling the dropout behavior, we consider two distinct approaches: a worst-case bound approach that only requires an upper bound on the maximum number of subsequent dropouts, and a stochastic approach that employs stochastic information on the occurrence of dropouts, given in the form of the well known Bernoulli or Gilbert-Elliott models [14,17]. For both these dropout modeling approaches we design dropout compensators, which act as model-based, closed-loop observers if information is received and as open-loop predictors if a dropout occurs. These compensators, designed for each lossy channel, depend only on a single channel's dropout history, and hence, we require no additional information to be sent over the network. The conditions for the stability analysis and design of the compensators are given in terms of linear matrix inequalities (LMIs) and can therefore be solved efficiently. The effectiveness of the proposed compensation strategy and the design tools will be illustrated through a numerical example. In particular, we will show that the designed compensators outperform the zero strategy and the hold strategy significantly in terms of robustness of the stability with respect to dropouts.

After introducing some notational conventions, the remainder of this paper is organized as follows. In Section 2 we define the NCS setup as studied in this paper and introduce our compensation-based strategy. For reasons of comparison we also define the zero strategy and the hold strategy. Additionally, we define the two dropout models used throughout this paper. In Section 3 and Section 4 we analyse stability and provide synthesis conditions for stabilizing compensator gains for the worst-case bound and stochastic dropout models, respectively. In Section 5 we present numerical results to illustrate the effectiveness of the compensation-based strategy and we present concluding remarks in Section 6. The appendix contains the proof of Theorem 11.

### 1.1 Nomenclature

The following notational conventions will be used. Let $\mathbb{R}$ and $\mathbb{N}$ denote the field of real numbers and the set of non-negative integers, respectively. We use the notation $\mathbb{R}_{\geq 0}$ to denote the set of non-negative real numbers. For a square matrix $A \in \mathbb{R}^{n \times n}$ we write $A \succ 0$, $A \succeq 0$, $A \prec 0$ and $A \preceq 0$ when $A$ is symmetric and $A$ is positive definite, positive semi-definite, negative definite and negative semi-definite, respectively. For $x \in \mathbb{R}^n$ we denote the Euclidean norm as $\|x\|_2 := \sqrt{x^T x}$. For a matrix $A \in \mathbb{R}^{n \times m}$ we denote its transpose by $A^T$. For the sake of brevity, we sometimes write symmetric matrices of the form $\left[\begin{smallmatrix} A & B^T \\ B & C \end{smallmatrix}\right]$ as $\left[\begin{smallmatrix} A & \star \\ B & C \end{smallmatrix}\right]$. We use $\Xi = \text{diag}(A_1, A_2, \ldots)$ to indicate a block diagonal matrix with matrices $A_1, A_2, \ldots$ on its diagonal. With some abuse of notation, we will use both $(z_0, z_1, \ldots)$ and $\{z_l\}_{l \in \mathbb{N}}$ with $z_l \in \mathbb{R}^n, l \in \mathbb{N}$, to denote a sequence of vectors in $\mathbb{R}^n$. For a bounded sequence $\mathbf{z} := \{z_l\}_{l \in \mathbb{N}}$ with $z_l \in \mathbb{R}^n$, $l \in \mathbb{N}$, let $\|\mathbf{z}\| := \sup\{\|z_l\|_2 | l \in \mathbb{N}\}$. The set of all sequences $\mathbf{z}$ with $\|\mathbf{z}\| < \infty$ is denoted by $\ell_\infty^n$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{K}$-function if it is continuous, strictly increasing and $\gamma(0) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a $\mathcal{KL}$-function if, for each fixed $t \geq 0$, the function $\beta(\cdot, t)$ is a $\mathcal{K}$-function and for each fixed $s \geq 0$, the function $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$. Let $X$ and $Y$ be random variables. We denote by $\mathbb{P}(X = x)$ the probability of the event $X = x$ occurring. The expected value of $X$ is denoted by $\mathbb{E}(X)$. The probability of event $X = x$ occurring, given event $Y = y$ is denoted by $\mathbb{P}(X = x | Y = y)$. The conditional expectation of $X$ given the event $Y = y$ is denoted $\mathbb{E}[X \,|\, Y = y]$.

## 2 Problem Formulation

This section has the following outline: in Section 2.1 we define the NCS and define what is meant by lossy communication links, also we provide different strategies that deal with communication losses. In Section 2.2 we give different models for the lossy communication links. Finally in Section 2.3 we define the problem considered in this paper.

### 2.1 Description of the NCS

In this paper, we consider a NCS consisting of a plant and a controller communicating over a network, see

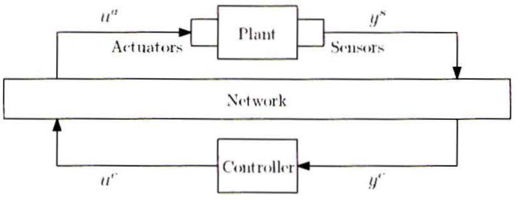



Fig. 1. Scheme of the NCS.

Fig. 1. The plant is given by a discrete-time linear time-invariant system of the form

$$\mathcal{P} : \begin{cases} x_{k+1} &= Ax_k + Bu_k^a, \\ y_k^s &= Cx_k, \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k^a \in \mathbb{R}^m$ is the input to the actuator and $y_k^s \in \mathbb{R}^p$ is the output measured by the sensor, at discrete time $k \in \mathbb{N}$. The controller is given by a discrete-time static output feedback law

$$\mathcal{C} : \quad u_k^c = Ky_k^c, \tag{2}$$

where $y_k^c$ is information of the plant output available at the controller and $u_k^c$ is the desired actuator command computed by the controller, at time $k \in \mathbb{N}$. The reason for introducing both $y^s$ and $y^c$, and both $u^c$ and $u^a$, is the fact that due to a non-ideal communication network $y^s$ and $y^c$ (and $u^c$ and $u^a$) are typically not equal. Therefore, we sometimes call $y^c$ the networked version of $y^s$ and $u^a$ the networked version of $u^c$. In this paper, we are interested in the situation where the differences between $y^s$ and $y^c$, and $u^c$ and $u^a$, are caused by the fact that the network links between the controller and the actuator, and between the sensor and the controller, are lossy, meaning that packet loss can occur. To model packet loss, we introduce the binary variables $\delta_k \in \{0, 1\}$ and $\Delta_k \in \{0, 1\}$, $k \in \mathbb{N}$. In case of a successful transmission in the sensor-to-controller channel at time $k \in \mathbb{N}$, $\Delta_k = 1$, and otherwise $\Delta_k = 0$. Similarly in case of a successful transmission in the controller-to-actuator channel, $\delta_k = 1$, and otherwise $\delta_k = 0$.

Using the binary variables $\delta_k$ and $\Delta_k$, $k \in \mathbb{N}$, we can now relate $y^c$ to $y^s$ and $u^a$ to $u^c$. If a transmission over a channel is successful at time $k \in \mathbb{N}$, the networked version of a signal will be equal to the original signal, i.e., $y_k^c = y_k^s$ in case $\delta_k = 1$ and $u_k^a = u_k^c$ in case $\Delta_k = 1$. If however, the transmission fails at time $k$, there are multiple strategies for selecting the values $y_k^c$ and $u_k^a$. In Sections 2.1.1 and 2.1.2 we describe two existing strategies, namely the "zero" strategy and the "hold" strategy, see, e.g., [28,29], while in Section 2.1.3, we will propose a novel "compensation-based" strategy. The latter strategy employs observer-like compensators on both sides of the network to mitigate the effect of packet loss on the stability of the NCS as much as possible.

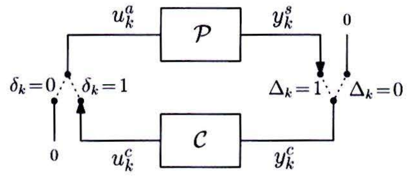



Fig. 2. Scheme of the NCS for "zero" strategy.

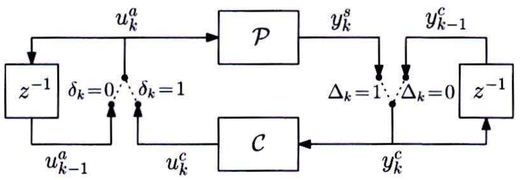



Fig. 3. Scheme of the NCS for "hold" strategy.

#### 2.1.1 *Zero Strategy*

When a transmission fails one can simply set the networked version of the transmitted signal to zero (see Fig. 2). This will be referred to as the "zero" strategy and can be formalized as

$$\begin{aligned} u_k^a &= \delta_k u_k^c, \\ y_k^c &= \Delta_k y_k^s, \end{aligned} \tag{3}$$

for $k \in \mathbb{N}$. This leads to the closed-loop system

$$x_{k+1} = A^z_{\delta_k,\Delta_k} x_k, \tag{4}$$

where

$$A^z_{\delta,\Delta} = A + \delta\Delta BKC \tag{5}$$

for $\delta, \Delta \in \{0,1\}$.

#### 2.1.2 *Hold Strategy*

An alternative to the "zero" strategy is the "hold" strategy, which holds the value of the last successfully transmitted signal (see Fig. 3) in case the current transmission fails. This strategy can be formalized as

$$\begin{aligned} u_k^a &= \delta_k u_k^c + (1-\delta_k)\, u_{k-1}^a, \\ y_k^c &= \Delta_k y_k^s + (1-\Delta_k)\, y_{k-1}^c, \end{aligned} \tag{6}$$

for $k \in \mathbb{N}$. By storing the values of the last successful transmissions in an augmented state $\xi_k^h := [\,x_k^T\ (u_{k-1}^a)^T\ (y_{k-1}^c)^T\,]^T$, we obtain the closed-loop system

$$\xi_{k+1}^h = A^h_{\delta_k,\Delta_k}\xi_k^h \tag{7}$$

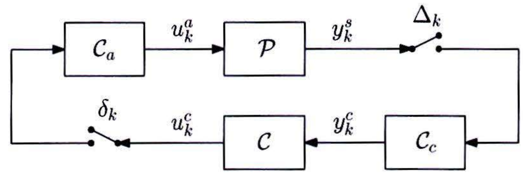



Fig. 4. Scheme of the NCS compensation-based strategy.

with

$$A^h_{\delta,\Delta} = \begin{bmatrix} A+\delta\Delta BKC & (1-\delta)\,B & \delta\,(1-\Delta)\,BK \\ \delta\Delta KC & (1-\delta)\,I_m & \delta\,(1-\Delta)\,K \\ \Delta C & O_{p\times m} & (1-\Delta)\,I_p \end{bmatrix} \tag{8}$$

for $\delta, \Delta \in \{0,1\}$, where $I_p$ and $I_m$ are identity matrices of dimensions $p \times p$ and $m \times m$ respectively, and $O_{p\times m}$ is a zero matrix of dimensions $p \times m$.

#### 2.1.3 *Compensation-Based Strategy*

In addition to the basic and existing strategies mentioned above, in this paper we also propose a new compensation-based strategy consisting of two packet loss compensators situated before the controller and the actuator, denoted by $\mathcal{C}_c$ and $\mathcal{C}_a$, respectively (see Fig. 4). The main idea behind the functioning of the compensator is that if a packet arrives, the compensator just forwards the packet and, additionally, acts as a model-based closed-loop observer, i.e., the received signal information is also used to innovate the compensator's estimate of the state of the plant. In case of a packet drop, the compensator acts as an open-loop predictor and, additionally, forwards its best prediction of $y_k^c$ or $u_k^a$, based on its estimate of the plant state. To formalize this idea, we propose to give the compensators $\mathcal{C}_c$ and $\mathcal{C}_a$ the following structures:

$$\mathcal{C}_c : \begin{cases} x_{k+1}^c = Ax_k^c + Bu_k^c + \Delta_k L^c_{j_{k-1}}(y_k^s - Cx_k^c) \\ y_k^c = \begin{cases} Cx_k\,(=y_k^s) & \text{if} \quad \Delta_k = 1 \\ Cx_k^c & \text{if} \quad \Delta_k = 0\,, \end{cases} \end{cases} \tag{9}$$

$$\mathcal{C}_a : \begin{cases} x_{k+1}^a = Ax_k^a + Bu_k^a + \delta_k L^a_{i_{k-1}}(u_k^c - KCx_k^a) \\ u_k^a = \begin{cases} Ky_k^c\,(=u_k^c) & \text{if} \quad \delta_k = 1 \\ KCx_k^a & \text{if} \quad \delta_k = 0\,. \end{cases} \end{cases} \tag{10}$$

In (10), we use the fact that the compensator $\mathcal{C}_a$ is collocated with the actuators and, hence, has access to the true implemented control signal $u_k^a$, which is beneficial for the closed-loop observer design. This is not the case for compensator $\mathcal{C}_c$, which is collocated with the controller $\mathcal{C}$, and, consequently, can only employ the controller output $u_k^c$ at time $k \in \mathbb{N}$. Note that $u_k^c$ is typically not equal to the true control signal $u_k^a$ that is implemented at the actuators at time $k \in \mathbb{N}$. This complicates the closed-loop observer design considerably (see also Remark 1 below). The gains $L_{j_{k-1}}^c$ and $L_{i_{k-1}}^a$ are designed to improve the robustness of the stability of the NCS in the presence of dropouts. Note that in (9) and (10) these gains are only effective (i.e. innovation is applied) at time $k \in \mathbb{N}$, if a packet is received, i.e., $\Delta_k = 1$ or $\delta_k = 1$. Moreover, these compensator gains depend on the counters $i_{k-1}$ and $j_{k-1}$, which are the number of successive dropouts that occurred just before and including time $k-1$, in the controller-to-actuator and sensor-to-controller channel, respectively. More specifically, these cumulative dropout counters are defined as

$$\begin{aligned} i_k &:= \min\{l^a \in \mathbb{N} \mid \delta_{k-l^a} = 1,\ k - l^a \geq -1\}, \\ j_k &:= \min\{l^c \in \mathbb{N} \mid \Delta_{k-l^c} = 1,\ k - l^c \geq -1\}, \end{aligned} \tag{11}$$

for $k \in \mathbb{N}$, where we set $\delta_{-1} := 1$ and $\Delta_{-1} := 1$.

To obtain a closed-loop model for the control system including these compensators, we denote the estimation errors at time $k \in \mathbb{N}$ corresponding to the compensators $\mathcal{C}_c$ and $\mathcal{C}_a$ as $e_k^c := x_k - x_k^c$ and $e_k^a := x_k - x_k^a$, respectively, and define the augmented state $\xi_k^{cb} := [x_k^T\ (e_k^a)^T\ (e_k^c)^T]^T$. The closed-loop dynamics for the compensation-based strategy can then be given by

$$\xi_{k+1}^{cb} = A_{\delta_k,\Delta_k,i_{k-1},j_{k-1}}^{cb} \xi_k^{cb}, \tag{12}$$

with $A_{\delta,\Delta,i,j}^{cb}$ as in (13), for $\delta \in \{0,1\}$, $\Delta \in \{0,1\}$, $i \in \mathbb{N}$ and $j \in \mathbb{N}$. For ease of notation, we define

$$\mu_k := (\delta_k, \Delta_k, i_{k-1}, j_{k-1}) \quad \text{and} \quad \mu := (\delta, \Delta, i, j) \tag{15}$$

collecting the parameters on which $A_{\delta,\Delta,i,j}^{cb}$ in (13) depends. This allows a compact representation of (12), i.e.,

$$\xi_{k+1}^{cb} = A_{\mu_k}^{cb} \xi_k^{cb}. \tag{16}$$

**Remark 1** *To give an indication of the complexity of the design of the compensators in case of two, serially connected, lossy network links, consider the state feedback case $u_k^c = K x_k^c$ (i.e., $C = I_n$) such that (9) becomes*

$$\mathcal{C}_c^* : \begin{cases} x_{k+1}^c = A x_k^c + B u_k^c + \Delta_k L_{j_{k-1}}^c (x_k - x_k^c) \\ y_k^c = \begin{cases} x_k & \text{if} \quad \Delta_k = 1 \\ x_k^c & \text{if} \quad \Delta_k = 0 \,. \end{cases} \end{cases} \tag{17}$$

*Even though the full state is transmitted to the compensator $\mathcal{C}_c^*$, having a perfect estimate at some time $\bar{k}$, $\bar{k} \in \mathbb{N}$, i.e, $x_{\bar{k}}^c = x_{\bar{k}}$, does not necessarily imply that $x_k^c = x_k$, for all $k > \bar{k}$ (as is normally the case for observers). The reason is that the input to the plant $u_k^a$ is not available at $\mathcal{C}_c^*$ if the transmission in the controller-to-actuator channel fails. Indeed, typically $u_{\bar{k}}^c \neq u_{\bar{k}}^a$ if $\delta_{\bar{k}} = 0$ for $\bar{k} \in \mathbb{N}$. Due to $\mathcal{C}_c^*$ not knowing the control value $u_k^a$ at the actuator, it can not perform exact updates of the states according to $x_{k+1} = A x_k + B u_k^a$ in this case. This can cause $x_{\bar{k}+1}^c \neq x_{\bar{k}+1}$ even though $x_{\bar{k}}^c = x_{\bar{k}}$. The same complexity arises for the compensator*

$$\mathcal{C}_c^{**} : \begin{cases} x_{k+1}^c = \begin{cases} A x_k + B u_k^c & \text{if} \quad \Delta_k = 1 \\ A x_k^c + B u_k^c & \text{if} \quad \Delta_k = 0 \end{cases} \\ y_k^c = \begin{cases} x_k & \text{if} \quad \Delta_k = 1 \\ x_k^c & \text{if} \quad \Delta_k = 0 \,, \end{cases} \end{cases} \tag{18}$$

*which instead of innovation, exactly sets the estimate $x_{\bar{k}}^c$ to $x_{\bar{k}}$ in case the full measurement is received at $\bar{k} \in \mathbb{N}$, i.e., $\Delta_{\bar{k}} = 1$. Note that $\mathcal{C}_a$ suffers from a similar problem as $\mathcal{C}_c$ (or $\mathcal{C}_c^*$, $\mathcal{C}_c^{**}$). Also here $x_{\bar{k}}^a = x_{\bar{k}}$ for some $\bar{k} \in \mathbb{N}$ does not imply $x_k^a = x_k$ for all $k > \bar{k}$. Indeed, even though $x_{\bar{k}}^a = x_{\bar{k}}$, the innovation term $\delta_{\bar{k}} L_{i_{\bar{k}-1}}^a (u_{\bar{k}}^c - K C x_{\bar{k}}^a)$ can be non-zero when $\delta_{\bar{k}} = 1$, as $u_{\bar{k}}^c \neq K C x_{\bar{k}}^a$ due to $x_{\bar{k}}^c \neq x_{\bar{k}}^a$. Consequently, the update of $x_{\bar{k}+1}^a$ in (10) is not equal to the exact update according to $x_{\bar{k}+1} = A x_{\bar{k}}^a + B u_{\bar{k}}^a = A x_{\bar{k}} + B u_{\bar{k}}^a$. This shows that the compensator design problem in case of two lossy network links (even in case of full state measurements) is highly complex. Note that in case of only one lossy link, the issues mentioned above do not occur.*

### *2.2 Dropout Models*

To evaluate the three strategies mentioned above, we need to introduce suitable models for the dropout behavior. In fact, packet dropouts in both network links can be described through different model types. In the first dropout model used here, and explained in Section 2.2.1, one assumes that there exists a worst-case bound on the number of successive dropouts, as was also used for instance in [1,7,26,38]. A second class of models employs more detailed stochastic information on the occurrence of dropouts. The simplest stochastic models assume that the dropouts are realizations of a Bernoulli process in case of a memoryless channel [29], or of the well known Gilbert-Elliott models [14,17], which use finite-state Markov chains to include correlation between successive dropouts [32].

In the next two subsections we will discuss these models in more detail, as both these situations will be studied in this paper.

#### *2.2.1 Worst-Case Bound Model*

The worst-case bound model is based on an upper bound on the number of successive dropouts in each of the channels given by $\bar{\delta} \in \mathbb{N}$ and $\bar{\Delta} \in \mathbb{N}$, for the controller-to-actuator and sensor-to-controller channel, respectively.

$$A^{cb}_{\delta,\Delta,i,j} = \begin{bmatrix} A+BKC & -(1-\delta)BKC & -\delta(1-\Delta)BKC \\ O_{n\times n} & A-\delta L^a_i KC & \delta(1-\Delta)L^a_i KC \\ O_{n\times n} & -(1-\delta)BKC & A+(1-\delta)(1-\Delta)BKC-\Delta L^c_j \end{bmatrix} \tag{13}$$

$$E_{\delta,\Delta,i,j} = \begin{bmatrix} A-\delta L^a_i KC & \delta(1-\Delta)L^a_i KC \\ -(1-\delta)BKC & A+(1-\delta)(1-\Delta)BKC-\Delta L^c_j \end{bmatrix} \tag{14}$$

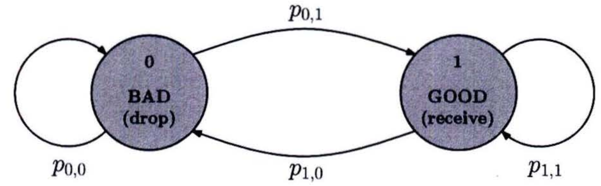



Fig. 5. Gilbert-Elliott model of a lossy network link.

This imposes the following constraint on $i_k$ and $j_k$ as defined in (11): $i_k \in \{0,1,\dots,\bar{\delta}\}$ and $j_k \in \{0,1,\dots,\bar{\Delta}\}$, $k \in \mathbb{N}$. Hence, it holds for $k \in \mathbb{N}$, that

$$i_{k+1} \in g_{\bar{\delta}}(i_k), \qquad j_{k+1} \in g_{\bar{\Delta}}(j_k), \tag{19}$$
$$\delta_{k+1} \in h_{\bar{\delta}}(i_k), \qquad \Delta_{k+1} \in h_{\bar{\Delta}}(j_k), \tag{20}$$

where the parameterized set-valued maps $g_r : \{0,\dots,r\} \rightrightarrows \{0,\dots,r\}$ and $h_r : \{0,\dots,r\} \rightrightarrows \{0,1\}$, with $r \in \mathbb{N}$, are given by

$$g_r(s) := \begin{cases} \{s+1,0\} & , s \in \{0,1,\dots,r-1\} \\ \{0\} & , s = r, \end{cases} \tag{21}$$

and

$$h_r(s) := \begin{cases} \{0,1\} & , s \in \{0,1,\dots,r-1\} \\ \{1\} & , s = r. \end{cases} \tag{22}$$

We combine the maps in (19) and (20) to obtain the updates for $\mu$ as in (15), which leads to

$$\mu_{k+1} \in G_{\bar{\delta},\bar{\Delta}}(\mu_k) \tag{23}$$

for all $k \in \mathbb{N}$, where the set-valued map $G_{\bar{\delta},\bar{\Delta}} : \mathcal{M} \rightrightarrows \mathcal{M}$ is defined as

$$G_{\bar{\delta},\bar{\Delta}}(\mu) := g_{\bar{\delta}}(i) \times g_{\bar{\Delta}}(j) \times h_{\bar{\delta}}(i) \times h_{\bar{\Delta}}(j), \tag{24}$$

with $\mu = (\delta,\Delta,i,j) \in \mathcal{M} := \{0,1\}^2 \times \{0,\dots,\bar{\delta}\} \times \{0,\dots,\bar{\Delta}\}$.

#### *2.2.2 Stochastic Models*

The simplest model of random packet losses over each of the network channels is to describe the packet loss as a Bernoulli process. A packet sent over the network from controller to actuator can be lost with probability $p^a \in [0,1]$ and can arrive with probability $1-p^a$, i.e., $\mathbb{P}(\delta_k = 0) = p^a$ and $\mathbb{P}(\delta_k = 1) = 1-p^a$, $k \in \mathbb{N}$. Similarly for the packets sent from sensor to controller, we have $\mathbb{P}(\Delta_k = 0) = p^c$, $p^c \in [0,1]$ and $\mathbb{P}(\Delta_k = 1) = 1-p^c$, $k \in \mathbb{N}$. Hence, $p^a$ and $p^c$ denote the packet loss probabilities in the channel between the controller and actuator and sensor and controller, respectively. This setup models a memoryless channel, since the probability of dropouts at time $k+1$ is independent of the channel's dropout history.

The situation in which packet losses occur in bursts can not be captured with this memoryless model [17]. Therefore, in this paper we also consider the packet losses in each of the two channels being governed by different two-state Markov chains. An often adopted Markov chain in this case is depicted in Fig. 5. This model is known as the Gilbert-Elliott model for fading channels and consists of a good and a bad network state. The probability of packet loss at time $k+1$ now depends on the success or failure of the transmission at time $k$, i.e.,

$$\begin{aligned} \mathbb{P}(\delta_{k+1} = \delta | \delta_k = \delta^-) &= p^a_{\delta^-,\delta}, \\ \mathbb{P}(\Delta_{k+1} = \Delta | \Delta_k = \Delta^-) &= p^c_{\Delta^-,\Delta}, \end{aligned} \tag{25}$$

where $p^a_{\delta^-,\delta}$ and $p^c_{\Delta^-,\Delta}$ denote the transition probabilities in the controller-to-actuator and sensor-to-controller channel, respectively, for $\delta,\delta^-,\Delta,\Delta^- \in \{0,1\}$. Obviously, $p^a_{\delta^-,0} + p^a_{\delta^-,1} = 1$ and $p^c_{\Delta^-,0} + p^c_{\Delta^-,1} = 1$ for all $\delta^-,\Delta^- \in \{0,1\}$. As for each channel the packet loss is modeled by a separate Gilbert-Elliott model, the transition probabilities in one channel are independent of the other channel, so we can use that

$$\begin{aligned} &\mathbb{P}(\delta_{k+1} = \delta \text{ and } \Delta_{k+1} = \Delta | \delta_k = \delta^- \text{ and } \Delta_k = \Delta^-) = \\ &\mathbb{P}(\delta_{k+1} = \delta | \delta_k = \delta^-)\mathbb{P}(\Delta_{k+1} = \Delta | \Delta_k = \Delta^-) = \\ &p^a_{\delta^-,\delta} p^c_{\Delta^-,\Delta}, \end{aligned} \tag{26}$$

where $\delta,\delta^-,\Delta,\Delta^- \in \{0,1\}$.

**Remark 2** *Note that the Bernoulli model is a special case of the Gilbert-Elliott model. Indeed by taking* $\mathbb{P}(\delta_{k+1} = 0 | \delta_k = \delta^-) = p^a_{\delta^-,0} = p^a$, *for* $\delta^- \in \{0,1\}$, *and* $\mathbb{P}(\delta_{k+1} = 1 | \delta_k = \delta^-) = p^a_{\delta^-,1} = 1 - p^a$, *for* $\Delta^- \in \{0,1\}$, *the Gilbert-Elliott model reduces to the Bernoulli model.*

### 2.3 Problem Formulation

The main objectives of this paper are to study the stability properties of the NCS with the compensation-based strategy, as presented in Section 2.1.3, for both worst-case bound and stochastic dropout models, as presented in Section 2.2.1 and Section 2.2.2, respectively. In addition, we aim at deriving efficient design conditions for the compensator gains $L_i^a$ and $L_j^c$ leading to the largest regions of stability in terms of the largest maximum number of subsequent dropouts $\bar{\delta}, \bar{\Delta}$, or the largest dropout probabilities that can be allowed while still guaranteeing stability. In particular, our aim is to obtain stability for the designed compensation-based strategies with a significantly larger robustness with respect to packet dropouts compared to the zero strategy and the hold strategy, as in presented in Section 2.1.1 and Section 2.1.2, respectively. Note that stability of the zero strategy and the hold strategy is well studied in the literature and various stability conditions are available, see, e.g., [28–30,35].

## 3 Stability Analysis for Worst-Case Bound Model

In this section, we consider the stability analysis and the design of the NCS with the compensation-based strategy, with packet loss modeled using worst-case bounds on the number of successive dropouts as described in Section 2.2.1. In particular, we are interested in proving global asymptotic stability (GAS) of (16), where $\boldsymbol{\mu} = \{\mu_k\}_{k\in\mathbb{N}}$ satisfies (23) for $i_{-1} = j_{-1} = 0$. Let us first formalize the adopted stability notion.

To do so, we denote the solution of (16) at time $k \in \mathbb{N}$ with initial state $\xi_0^{cb}$ and sequence $\boldsymbol{\mu} = \{\mu_k\}_{k\in\mathbb{N}}$ satisfying (23) for $i_{-1} = j_{-1} = 0$ by $\xi^{cb}(k, \xi_0^{cb}, \boldsymbol{\mu})$.

**Definition 3** *System* (16) *with* (23)*, is globally asymptotically stable (GAS) for given bounds* $\bar{\Delta}, \bar{\delta}$*, if there exists a* $\mathcal{KL}$*-function* $\beta$*, such that for all* $\xi_0^{cb} \in \mathbb{R}^{3n}$ *and all sequences* $\boldsymbol{\mu} = \{\mu_k\}_{k\in\mathbb{N}}$ *satisfying* (23) *with* $i_{-1} = j_{-1} = 0$*, the corresponding solution* $\xi^{cb}(\,\cdot\,, \xi_0^{cb}, \boldsymbol{\mu})$ *satisfies*

$$\|\xi^{cb}(k, \xi_0^{cb}, \boldsymbol{\mu})\|_2 \leq \beta\left(\|\xi_0^{cb}\|_2, k\right) \tag{27}$$

*for all* $k \in \mathbb{N}$.

In order to guarantee GAS of (16) with (23), we observe that in the closed-loop description of the compensation-based NCS, as given in (16) with $A_\mu^{cb}$, for $\mu = (\delta, \Delta, i, j) \in \mathcal{M}$, as in (13), $e_{k+1}^a$ and $e_{k+1}^c$ are independent of $x_k$. Therefore, we can split (16) in two subsystems, one related to the dynamics of the plant state $x_k$, the other to the dynamics of the estimation errors $e_k := \left[(e_k^a)^T \;\; (e_k^c)^T\right]^T$, for $k \in \mathbb{N}$. This yields the two subsystems

$$x_{k+1} = \bar{A}x_k + \bar{B}_{\delta_k,\Delta_k} e_k, \tag{28a}$$
$$e_{k+1} = E_{\delta_k,\Delta_k,i_{k-1},j_{k-1}} e_k, \tag{28b}$$

where

$$\bar{A} := A + BKC, \tag{29}$$
$$\bar{B}_{\delta,\Delta} := \begin{bmatrix} -(1-\delta)\,BKC & -\delta\,(1-\Delta)\,BKC \end{bmatrix}, \quad \Delta \in \{0,1\}, \delta \in \{0,1\}, \tag{30}$$

and $E_\mu$ as in (14), for $\mu = (\delta, \Delta, i, j) \in \mathcal{M}$. To prove GAS of system (28) with (23), it will be shown that if the $e$-system (28b) with (23) is GAS and if the $x$-system (28a) with (23) is input-to-state stable (ISS), the cascaded system (28) with (23) is GAS. To do so, let us define the concept of input-to-state stability (ISS) [23,33] of (28a) with (23), and note that GAS of (28b) with (23) can be defined similarly as GAS of (16) with (23) as in Definition 3.

For introducing ISS we denote the solution of (28a) at time $k \in \mathbb{N}$ with initial state $x_0 \in \mathbb{R}^n$, input $\mathbf{e} = \{e_k\}_{k\in\mathbb{N}}$ and sequences $\boldsymbol{\mu} = \{\mu_k\}_{k\in\mathbb{N}}$ satisfying (23) with $i_{-1} = j_{-1} = 0$ by $x(k, x_0, \boldsymbol{\mu}, \mathbf{e})$.

**Definition 4** *System* (28a) *with* (23) *is input-to-state stable (ISS) if there exist a* $\mathcal{KL}$*-function* $\beta$ *and a* $\mathcal{K}$*-function* $\gamma$ *such that, for each input* $\mathbf{e} = \{e_k\}_{k\in\mathbb{N}} \in \ell_\infty^{2n}$*, each* $x_0 \in \mathbb{R}^n$*, and each sequence* $\boldsymbol{\mu} = \{\mu_k\}_{k\in\mathbb{N}}$ *satisfying* (23) *with* $i_{-1} = j_{-1} = 0$*, the corresponding solution* $x(\,\cdot\,, x_0, \boldsymbol{\mu}, \mathbf{e})$ *satisfies*

$$\|x(k, x_0, \boldsymbol{\mu}, \mathbf{e})\|_2 \leq \beta(\|x_0\|_2, k) + \gamma(\|\mathbf{e}\|) \tag{31}$$

*for all* $k \in \mathbb{N}$.

To prove now GAS of (16) with (23), three theorems will be presented. Theorem 5 will state conditions under which system (28a) with (23) is ISS, Theorem 6 will state conditions under which system (28b) with (23) is GAS and Theorem 7 will indicate how the results of Theorem 5 and Theorem 6 can be combined to obtain GAS of (16) with (23).

**Theorem 5** *[23] System* (28a) *with* (23) *is ISS if* $K$ *is chosen such that* $\bar{A} = A + BKC$ *is a Schur matrix, i.e., all eigenvalues of* $\bar{A}$ *are contained in the open unit disk.*

**Theorem 6** *Consider system* (28b) *with* (23)*, if there exists a set of symmetric matrices* $\{P_\mu | \mu \in \mathcal{M}\}$ *satisfying*

$$\begin{bmatrix} P_{\mu^-} & \star \\ P_\mu E_\mu & P_\mu \end{bmatrix} \succ 0, \;\; \mu \in G_{\bar{\delta},\bar{\Delta}}(\mu^-), \;\; \mu^- \in \mathcal{M}, \tag{32}$$

*then* (28b) *with* (23)*, is GAS.*

**PROOF.** Theorem 6 results from showing that

$$V(e_k, \mu_{k-1}) := e_k^T P_{\mu_{k-1}} e_k, \tag{33}$$

$k \in \mathbb{N} \setminus \{0\}$, is a parameter-dependent Lyapunov function for the system (28b), see [11]. To show this, we will prove that for $e_k \neq 0$, $V(e_{k+1}, \mu_k) < V(e_k, \mu_{k-1})$ holds, which due to (33) is equivalent to

$$e_k^T E_{\mu_k}^T P_{\mu_k} E_{\mu_k} e_k < e_k^T P_{\mu_{k-1}} e_k, \tag{34}$$

for all $e_k \neq 0$ and all $\mu_k \in G_{\bar{\delta},\bar{\Delta}}(\mu_{k-1})$. Obviously, (34) is satisfied as it results from (32) by pre- and post-multiplication by $\mathrm{diag}(I_{2n}, P_{\mu}^{-1})$, followed by taking a Schur complement, thereby showing the strict decrease of the Lyapunov function at each step as in (34). In addition, we observe that due to (32) it holds that $P_\mu \succ 0$, $\mu \in \mathcal{M}$, and thus there exist $0 < c_1 \leq c_2$ such that $c_1 \|e\|_2^2 \leq V(e, \mu) \leq c_2 \|e\|_2^2$. By standard Lyapunov arguments these facts show GAS of (28b) with (23). □

**Theorem 7** *[23] If the e-system* (28b) *with* (23) *is GAS and the x-system* (28a) *with* (23) *is ISS, then the cascaded system* (28) *with* (23) *is GAS.*

We now combine Theorem 5, Theorem 6 and Theorem 7 to obtain one of our main results, which formulates conditions under which (28) with (23) is GAS.

**Theorem 8** *Consider system* (28) *with* (23), *if $\bar{A} = A + BKC$ is a Schur matrix and there exist a set of symmetric matrices $\{P_\mu | \mu \in \mathcal{M}\}$ satisfying* (32), *then the cascaded system* (28) *with* (23) *is GAS.*

Using Theorem 8, one can *analyse* stability of (28) with (23) for *given* compensator gains $L_i^a$ and $L_j^c$, $i \in \{0, \dots, \bar{\Delta}\}$, $j \in \{0, \dots, \bar{\Delta}\}$. Since we are interested in designing $L_i^a$ and $L_j^c$ to guarantee stability with large values of $\bar{\delta}$ and $\bar{\Delta}$, Theorem 9 states LMI-based conditions for the *synthesis* of $L_i^a$ and $L_j^c$, based on Theorem 8.

**Theorem 9** *Consider system* (28) *with* (23). *Suppose $\bar{A} = A + BKC$ is a Schur matrix, and there exist a set of symmetric matrices $\{P_\mu | \mu \in \mathcal{M}\}$ with $P_\mu$ of the form*

$$P_\mu = \begin{bmatrix} P_{\bar{\delta},i}^a & O_{n\times n} \\ O_{n\times n} & P_{\bar{\Delta},j}^c \end{bmatrix}, \quad \mu \in \mathcal{M},$$

*and a set of matrices $\{R_\mu | \mu \in \mathcal{M}\}$ with $R_\mu$ of the form*

$$R_\mu = \begin{bmatrix} R_i^a & O_{n\times n} \\ O_{n\times m} & R_j^c \end{bmatrix}, \quad \mu \in \mathcal{M},$$

*satisfying*

$$\begin{bmatrix} P_{\mu^-} & \star \\ \Omega_\mu & P_\mu \end{bmatrix} \succ 0, \ \mu \in G_{\bar{\delta},\bar{\Delta}}(\mu^-), \ \mu^- \in \mathcal{M}, \tag{35}$$

*with $\Omega_\mu$ as in* (37). *Then the system* (28) *with* (23) *is GAS for the compensator gains $L_i^a$ and $L_j^c$ given by*

$$\begin{aligned} L_i^a &= (P_{1,i}^a)^{-1} R_i^a, \quad & i = 0, \dots, \bar{\delta}, \\ L_j^c &= (P_{1,j}^c)^{-1} R_j^c, \quad & j = 0, \dots, \bar{\Delta}. \end{aligned} \tag{36}$$

**PROOF.** From (36) we obtain

$$\begin{aligned} R_i^a &= P_{1,i}^a L_i^a, \quad & i = 0, \dots, \bar{\delta}, \\ R_j^c &= P_{1,j}^c L_j^c, \quad & j = 0, \dots, \bar{\Delta}. \end{aligned} \tag{38}$$

Since substitution of (38) in (37) yields $\Omega_\mu = P_\mu E_\mu$, it is clear that (35) implies (32). Hence, we recovered the hypotheses of Theorem 8, and thus the system (28) with (23) is GAS for the compensator gains in (36). □

## 4 Stability analysis for Stochastic Dropout Models

In this section, we again consider the stability analysis and design of the NCS with the compensation-based strategy, but now for the case where the lossy channels are described by the Gilbert-Elliott models as discussed in Section 2.2.2. The Bernoulli case can be handled in a similar manner, as was also indicated in Remark 2.

In the case of the worst-case bound models as treated in the previous section, the compensator gains $L_i^a$ and $L_j^c$ depend on $i_{k-1}$ and $j_{k-1}$, $k \in \mathbb{N}$, the number of subsequent dropouts experienced until time $k-1$ in the sensor-to-controller and controller-to-actuator channel, respectively. In the case of worst-case bounds the values $i_{k-1}$ and $j_{k-1}$ remain bounded, and thus only a a finite number of these gains have to be used. This is no longer the case when the Gilbert-Elliott models are adopted, as these allow the occurrence of an infinite number of successive dropouts in the controller-to-actuator channel if $p_{0,0}^a \neq 0$, and in the sensor-to-controller channel if $p_{0,0}^c \neq 0$. This would lead to designing infinitely many compensator gains $L_i^a$ and $L_j^c$, $i \in \mathbb{N}$, $j \in \mathbb{N}$, as we do not have the restrictions $i_k \leq \bar{\delta}$ and $j_k \leq \bar{\Delta}$ as in the case of the worst-case bound models of Section 2.2.1. Clearly, for practical reasons it is desirable to have a finite number of compensator gains, and therefore we reduce the flexibility of the compensators by introducing saturated

$$\Omega_{\delta,\Delta,i,j} = \begin{bmatrix} P^a_{\delta,i}A - \delta R^a_i KC & \delta(1-\Delta)R^a_i KC \\ -(1-\delta)P^c_{\Delta,j}BKC & P^c_{\Delta,j}A + (1-\delta)(1-\Delta)P^c_{\Delta,j}BKC - \Delta R^c_j \end{bmatrix} \tag{37}$$

dropout counters $\tilde{i}_k$ and $\tilde{j}_k$ subject to the saturation levels $\tilde{\delta}$ and $\tilde{\Delta}$, respectively, i.e.,

$$\begin{aligned} \tilde{i}_k &= \min(i_k, \tilde{\delta}), \\ \tilde{j}_k &= \min(j_k, \tilde{\Delta}), \end{aligned} \tag{39}$$

for $k \in \mathbb{N}$, where $i_k$ and $j_k$ are defined as in (11). Instead of letting the compensator gains in (9)-(10) depend on $i_{k-1}$ and $j_{k-1}$, we let them depend on $\tilde{i}_{k-1}$ and $\tilde{j}_{k-1}$. Therefore, we replace the compensator gains $L^a_i$, $i \in \mathbb{N}$, in (10) by $L^a_{\tilde{i}}$, $\tilde{i} \in \{0, \ldots, \tilde{\delta}\}$ and the compensator gains $L^c_j$, $j \in \mathbb{N}$, in (9) by $L^c_{\tilde{j}}$, $\tilde{j} \in \{0, \ldots, \tilde{\Delta}\}$, leading to

$$\mathcal{C}_c : \begin{cases} x^c_{k+1} = Ax^c_k + Bu^c_k + \Delta_k L^c_{\tilde{j}_{k-1}}(y^s_k - Cx^c_k) \\ y^c_k = \begin{cases} Cx_k\ (= y^s_k) & \text{if} \quad \Delta_k = 1 \\ Cx^c_k & \text{if} \quad \Delta_k = 0\ , \end{cases} \end{cases} \tag{40}$$

$$\mathcal{C}_a : \begin{cases} x^a_{k+1} = Ax^a_k + Bu^a_k + \delta_k L^a_{\tilde{i}_{k-1}}(u^c_k - KCx^a_k) \\ u^a_k = \begin{cases} Ky^c_k\ (= u^c_k) & \text{if} \quad \delta_k = 1 \\ KCx^a_k & \text{if} \quad \delta_k = 0\ . \end{cases} \end{cases} \tag{41}$$

The number of compensators gains, $L^a_{\tilde{i}}$, $\tilde{i} \in \{0, \ldots, \tilde{\delta}\}$, and $L^c_{\tilde{j}}$, $\tilde{j} \in \{0, \ldots, \tilde{\Delta}\}$, to be designed for each channel is now finite. The exact number can be chosen freely by selecting $\tilde{\delta}$ and $\tilde{\Delta}$ in a desirable manner. A direct consequence of these choices is that for all $i_k \geq \tilde{\delta}$, $k \in \mathbb{N}$ we apply the same gain $L^a_{\tilde{\delta}}$ in (41). Similarly, for all $j_k \geq \tilde{\Delta}$, $k \in \mathbb{N}$ we apply the same gain $L^c_{\tilde{\Delta}}$ in (40).

The above considerations modify (12) into the closed-loop system representation

$$\xi^{cb}_{k+1} = A^{cb}_{\delta_k, \Delta_k, \tilde{i}_{k-1}, \tilde{j}_{k-1}} \xi^{cb}_k, \tag{42}$$

with $A^{cb}_{\delta,\Delta,i,j}$ as in (13), for $\delta \in \{0,1\}$, $\Delta \in \{0,1\}$, $i \in \{0, \ldots, \tilde{\delta}\}$ and $j \in \{0, \ldots, \tilde{\Delta}\}$. For ease of notation, we define

$$\tilde{\mu}_k := (\delta_k, \Delta_k, \tilde{i}_{k-1}, \tilde{j}_{k-1}) \in \tilde{\mathcal{M}}, \tag{43}$$

where $\tilde{\mathcal{M}} = \{0,1\}^2 \times \{0, \ldots, \tilde{\delta}\} \times \{0, \ldots, \tilde{\Delta}\}$. This allows a compact representation of (42), i.e.,

$$\xi^{cb}_{k+1} = A^{cb}_{\tilde{\mu}_k} \xi^{cb}_k. \tag{44}$$

As in the case of worst-case bounds, also here not all transitions from $\tilde{\mu}_k \in \tilde{\mathcal{M}}$ to $\tilde{\mu}_{k+1} \in \tilde{\mathcal{M}}$ are possible. In fact, for $k \in \mathbb{N}$, it holds that

$$\tilde{i}_{k+1} = \tilde{g}_{\tilde{\delta}}(\tilde{i}_k, \delta_k), \qquad \tilde{j}_{k+1} = \tilde{g}_{\tilde{\Delta}}(\tilde{j}_k, \Delta_k) \tag{45}$$

$$\delta_{k+1} \in \{0,1\}, \qquad \Delta_{k+1} \in \{0,1\} \tag{46}$$

where the parameterized set-valued map $\tilde{g}_r : \{0, \ldots, r\} \times \{0,1\} \rightrightarrows \{0, \ldots, r\}$ is given by

$$\tilde{g}_r(s_1, s_2) := \begin{cases} 0 & ,\ s_2 = 1 \\ s_1 + 1 & ,\ s_2 = 0\ ,\ s_1 \in \{0, \ldots, r-1\}\ , \\ s_1 & ,\ s_2 = 0\ ,\ s_1 = r. \end{cases} \tag{47}$$

We combine the maps in (45) and (46) to obtain

$$\tilde{\mu}_{k+1} \in G_{\tilde{\delta},\tilde{\Delta}}(\tilde{\mu}_k) \tag{48}$$

for all $k \in \mathbb{N}$, where the set-valued map $G_{\tilde{\delta},\tilde{\Delta}} : \tilde{\mathcal{M}} \rightrightarrows \tilde{\mathcal{M}}$ is defined as

$$G_{\tilde{\delta},\tilde{\Delta}}(\tilde{\mu}) := \{\tilde{g}_{\tilde{\delta}}(\tilde{i}, \delta)\} \times \{\tilde{g}_{\tilde{\Delta}}(\tilde{j}, \Delta)\} \times \{0,1\}^2, \tag{49}$$

with $\tilde{\mu} = (\delta, \Delta, \tilde{i}, \tilde{j}) \in \tilde{\mathcal{M}}$.

A final step to model the complete NCS with the compensation-based strategy using the compensators $\mathcal{C}_c$ and $\mathcal{C}_a$, as defined in (40) and (41), respectively, is to include the transition probabilities from $\tilde{\mu}_k$ to $\tilde{\mu}_{k+1} \in G_{\tilde{\delta},\tilde{\Delta}}(\tilde{\mu}_k)$ based on the Gilbert-Elliott models for the dropout behavior in each channel. These probabilities combined with (44) and (48) will lead to a so-called Markov Jump Linear System (MJLS), which are well known in the literature, see, e.g., [10]. To obtain these probabilities, observe that the probability of going from $\tilde{\mu}_k = (\delta_k, \Delta_k, \tilde{i}_{k-1}, \tilde{j}_{k-1})$ to $\tilde{\mu}_{k+1} = (\delta_{k+1}, \Delta_{k+1}, \tilde{i}_k, \tilde{j}_k)$ is completely determined by the probability of going from $\delta_k$ to $\delta_{k+1}$ and $\Delta_k$ to $\Delta_{k+1}$ as already expressed in (26). As a consequence, the probability $p_{\tilde{\mu}^-,\tilde{\mu}}$ of going from $\tilde{\mu}^- = (\delta^-, \Delta^-, i^-, j^-) \in \tilde{\mathcal{M}}$ to $\tilde{\mu} = (\delta, \Delta, i, j) \in G_{\tilde{\delta},\tilde{\Delta}}(\tilde{\mu}^-)$ is given by $p^a_{\delta^-,\delta} p^c_{\Delta^-,\Delta}$, and thus we obtain the transition probabilities

$$p_{\tilde{\mu}^-,\tilde{\mu}} = \begin{cases} p^a_{\delta^-,\delta} p^c_{\Delta^-,\Delta}, & \text{when } \tilde{\mu}^- \in \tilde{\mathcal{M}},\ \tilde{\mu} \in G_{\tilde{\delta},\tilde{\Delta}}(\tilde{\mu}^-), \\ 0 & ,\text{when } \tilde{\mu}^- \in \tilde{\mathcal{M}},\ \tilde{\mu} \in \tilde{\mathcal{M}} \backslash G_{\tilde{\delta},\tilde{\Delta}}(\tilde{\mu}^-). \end{cases} \tag{50}$$

Note that with these probabilities a new Markov chain with state $\tilde{\mu} \in \tilde{\mathcal{M}}$ is obtained. The discrete-time system (44) with $A^{cb}_{\mu}$ as in (13) combined with the Markov chain (50) forms the overall model of the NCS in the form of a MJLS, with initial conditions $\xi^{cb}_0 \in \mathbb{R}^{3n}$ and $\mu_0 \in \tilde{\mathcal{M}}$. We denote this MJLS for shortness by $\Sigma_{MJLS}$.

Let us first define several forms of stability for discrete-time jump linear systems of the form $\Sigma_{MJLS}$, see [9,30].

**Definition 10** *The Markov jump linear system given by $\Sigma_{MJLS}$ is:*

*(1) mean-square stable (MSS) if for every initial state $(\xi_0^{cb}, \tilde{\mu}_0)$, $\lim_{k\to\infty} \mathbb{E}\big[\|\xi_k^{cb}\|_2^2 \,|\, \xi_0^{cb}, \tilde{\mu}_0\big] = 0$;*
*(2) stochastically stable (SS) if for every initial state $(\xi_0^{cb}, \tilde{\mu}_0)$, $\mathbb{E}\big[\sum_{k=0}^{\infty} \|\xi_k^{cb}\|_2^2 \,|\, \xi_0^{cb}, \tilde{\mu}_0\big] < \infty$;*
*(3) exponentially mean square stable (EMSS) if for every initial state $(\xi_0^{cb}, \tilde{\mu}_0)$, there exist constants $0 \leq \alpha < 1$ and $\beta \geq 0$ such that for all $k \geq 0$, $\mathbb{E}\big[\|\xi_k^{cb}\|_2^2 \,|\, \xi_0^{cb}, \tilde{\mu}_0\big] < \beta\alpha^k \|\xi_0^{cb}\|_2^2$;*
*(4) uniformly exponentially mean square stable (UEMSS), EMSS but $\alpha$ and $\beta$ independent of $\xi_0^{cb}$ and $\tilde{\mu}_0$;*
*(5) almost surely stable (ASS) if for every initial state $(\xi_0^{cb}, \tilde{\mu}_0)$, we have that $\mathbb{P}\left[\lim_{k\to\infty} \|\xi_k^{cb}\| = 0\right] = 1$.*

It is shown in [9] that the first four stability properties in Definition 10 are equivalent and any one implies almost-sure stability, i.e.,

$$MSS \Leftrightarrow SS \Leftrightarrow EMSS \Leftrightarrow UEMSS \Rightarrow ASS. \quad (51)$$

In the remainder of this section, we present conditions under which the Markov jump linear system $\Sigma_{MJLS}$ is EMSS.

As in Section 3, we note that in the closed-loop description of the resulting NCS, as given in (44) with $A_{\tilde{\mu}}^{cb}$ given in (13) for $\tilde{\mu} = (\delta, \Delta, \tilde{i}, \tilde{j}) \in \tilde{\mathcal{M}}$, the states $e_{k+1}^a$ and $e_{k+1}^c$ are independent of $x_k$. Therefore, we can also adopt a cascaded system decomposition similar to (28) to obtain

$$x_{k+1} = \bar{A}x_k + w_k, \quad (52a)$$
$$e_{k+1} = E_{\delta_k,\Delta_k,\tilde{i}_{k-1},\tilde{j}_{k-1}} e_k, \quad (52b)$$

where $w_k := \bar{B}_{\delta_k,\Delta_k} e_k$. $k \in \mathbb{N}$, $\bar{A}$ is given in (29), $\bar{B}_{\delta,\Delta}$ in (30) and $E_{\tilde{\mu}}$ in (14) for $\tilde{\mu} \in \tilde{\mathcal{M}}$. To prove that $\Sigma_{MJLS}$ is EMSS, we again use a 'cascade' reasoning. In Theorem 11, we will provide a result that can be used to conclude that if $\bar{A} = A + BKC$ is a Schur matrix and if the $e$-system (52b) with (50) is EMSS, then the system $\Sigma_{MJLS}$ given by (52) with (50) is EMSS. Note that all stability properties in Definition 10 can be defined similarly for (52b) with (50), and moreover, note that Theorem 11 is the stochastic equivalent of Theorem 7, which, to the best of the authors' knowledge, is not available in the literature. In Theorem 12 we will present necessary and sufficient matrix inequality conditions for EMSS of the $e$-system (52b) with (50), which are proven in [9]. Combining Theorems 11 and 12 will result in EMSS of $\Sigma_{MJLS}$ as will be formulated in Theorem 13.

**Theorem 11** *Consider the system (52a) where $\{w_k\}_{k\in\mathbb{N}}$ is a sequence of random variables with the property that for some $c_1 \geq 0$ and $0 \leq \rho < 1$ it holds that, for any $w_0 \in \mathbb{R}^{2n}$, $\mathbb{E}\big[\|w_k\|_2^2\big] \leq c_1\rho^k\|w_0\|_2^2$, $k \in \mathbb{N}$. If $\bar{A} = A + BKC$ is a Schur matrix, then there exist $c_2 \geq 0$, $c_3 \geq 0$ and $0 \leq r < 1$ such that*

$$\mathbb{E}\big[\|x_k\|_2^2 \,|\, x_0\big] \leq c_2 r^k \|x_0\|_2^2 + c_3 r^k \|w_0\|_2^2 \quad (53)$$

*for all $x_0, w_0$, $k \in \mathbb{N}$.*

**PROOF.** The proof is given in Appendix A.

**Theorem 12** *[9] The MJLS given by (52b) with (50), is EMSS if and only if there exists a set $\{P_{\tilde{\mu}} | \tilde{\mu} \in \tilde{\mathcal{M}}\}$ of positive definite matrices satisfying*

$$P_{\tilde{\mu}^-} - \sum_{\tilde{\mu} \in G_{\tilde{\delta},\tilde{\Delta}}(\tilde{\mu}^-)} p_{\tilde{\mu}^-,\tilde{\mu}} E_{\tilde{\mu}}^T P_{\tilde{\mu}} E_{\tilde{\mu}} \succ 0, \; \tilde{\mu}^- \in \tilde{\mathcal{M}}. \quad (54)$$

We now combine Theorem 11 and Theorem 12 to obtain one of our main results, which formulates conditions under which $\Sigma_{MJLS}$ is EMSS.

**Theorem 13** *Consider system $\Sigma_{MJLS}$ given by (52) with (50). System $\Sigma_{MJLS}$ is EMSS if and only if there exists a set $\{P_{\tilde{\mu}} | \tilde{\mu} \in \tilde{\mathcal{M}}\}$ of positive definite matrices satisfying (54) and $\bar{A} = A + BKC$ is a Schur matrix.*

**PROOF.** We first show the sufficiency. From Theorem 12 we have that if (54) is satisfied, then the MJLS (52b) with (50) is EMSS, i.e., for some $c_4 \geq 0$ and $0 \leq \rho < 1$, $\mathbb{E}\big[\|e_k\|_2^2\big] \leq c_4\rho^k\|e_0\|_2^2$, for all $e_0 \in \mathbb{R}^{2n}$ and all $k \in \mathbb{N}$. Since $w_k = \bar{B}_{\delta_k,\Delta_k} e_k$, $k \in \mathbb{N}$, this implies that for some $c_1 \geq 0$ and $0 \leq \rho < 1$, $\mathbb{E}\big[\|w_k\|_2^2\big] \leq c_1\rho^k\|w_0\|_2^2$, for all $w_0 \in \mathbb{R}^{2n}$ and all $k \in \mathbb{N}$. Now we can invoke Theorem 11 to obtain a bound as in (53), since $\bar{A}$ is Schur. The above facts imply that $\Sigma_{MJLS}$ is MSS and thus also EMSS.

To show necessity, take $e_0 = 0$ which implies that $e_k = 0$ for all $k \in \mathbb{N}$, and thus also $w_k = 0$ for all $k \in \mathbb{N}$. Hence, for $e_0 = 0$ (52a) reduces to the linear system $x_{k+1} = \bar{A}x_k$. Since $\Sigma_{MJLS}$ is EMSS, it must hold that $\lim_{k\to\infty} x_k = 0$ for any $x_0$ and consequently, $\bar{A}$ must be Schur. Finally, note that if $\Sigma_{MJLS}$ is EMSS, then the MJLS given by (52b) with (50) is EMSS as well. Since Theorem 12 presents necessary and sufficient conditions for (52b) with (50) to be EMSS, it follows that there must exist a set $\{P_{\tilde{\mu}} | \tilde{\mu} \in \tilde{\mathcal{M}}\}$ of matrices that satisfy $P_{\tilde{\mu}} \succ 0$, $\tilde{\mu} \in \tilde{\mathcal{M}}$, and (54). This completes the proof. □

Using Theorem 13, one can *analyse* stability of $\Sigma_{MJLS}$ for *given* compensator gains $L_{\tilde{i}}^a$ and $L_{\tilde{j}}^c$, $\tilde{i} \in \{0, \ldots, \tilde{\Delta}\}$,

$\tilde{j} \in \{0, \ldots, \tilde{\Delta}\}$. Since we are interested in designing $L^a_{\tilde{i}}$ and $L^c_{\tilde{j}}$ to obtain stability with a large robustness with respect to dropouts, Theorem 14 will state LMI-based conditions for the *synthesis* of $L^a_{\tilde{i}}$ and $L^c_{\tilde{j}}$, based on Theorem 13.

**Theorem 14** *Consider the system $\Sigma_{MJLS}$ given by (52) with (50). Suppose $\bar{A} = A + BKC$ is a Schur matrix, and there exist a set $\{P_{\tilde{\mu}} | \tilde{\mu} \in \tilde{\mathcal{M}}\}$ of symmetric matrices, with $P_{\tilde{\mu}}$ of the form*

$$P_{\tilde{\mu}} = \begin{bmatrix} P^a_{\delta,\tilde{i}} & O_{n\times n} \\ O_{n\times n} & P^c_{\Delta,\tilde{j}} \end{bmatrix}, \quad \tilde{\mu} \in \tilde{\mathcal{M}},$$

*and a set $\{R_{\tilde{\mu}} | \tilde{\mu} \in \tilde{\mathcal{M}}\}$ of matrices, with $R_{\tilde{\mu}}$ of the form*

$$R_{\tilde{\mu}} = \begin{bmatrix} R^a_{\tilde{i}} & O_{n\times n} \\ O_{n\times m} & R^c_{\tilde{j}} \end{bmatrix}, \quad \tilde{\mu} \in \tilde{\mathcal{M}},$$

*satisfying*

$$\begin{bmatrix} P_{\tilde{\mu}^-} & \star \\ \Xi_1(\tilde{\mu}^-) & \Xi_2(\tilde{\mu}^-) \end{bmatrix} \succ 0, \quad \tilde{\mu}^- \in \tilde{\mathcal{M}} \tag{55}$$

*with for $\tilde{\mu}^- = (\delta^-, \Delta^-, \tilde{i}^-, \tilde{j}^-)$*

$$\Xi_1(\tilde{\mu}^-) := \begin{bmatrix} \sqrt{p^a_{\delta^-,0} p^c_{\Delta^-,0}}\, \Omega_{0,0,\tilde{i},\tilde{j}} \\ \sqrt{p^a_{\delta^-,0} p^c_{\Delta^-,1}}\, \Omega_{0,1,\tilde{i},\tilde{j}} \\ \sqrt{p^a_{\delta^-,1} p^c_{\Delta^-,0}}\, \Omega_{1,0,\tilde{i},\tilde{j}} \\ \sqrt{p^a_{\delta^-,1} p^c_{\Delta^-,1}}\, \Omega_{1,1,\tilde{i},\tilde{j}} \end{bmatrix},$$

$$\Xi_2(\tilde{\mu}^-) := \mathrm{diag}(P_{0,0,\tilde{i},\tilde{j}},\ P_{0,1,\tilde{i},\tilde{j}},\ P_{1,0,\tilde{i},\tilde{j}},\ P_{1,1,\tilde{i},\tilde{j}}),$$

*where $\tilde{i} = \tilde{g}_{\tilde{\delta}}(\tilde{i}^-, \delta^-)$, $\tilde{j} = \tilde{g}_{\tilde{\Delta}}(\tilde{j}^-, \Delta^-)$ and $\Omega_{\tilde{\mu}}$ as in (37). Then $\Sigma_{MJLS}$ is EMSS for the compensator gains $L^a_{\tilde{i}}$ and $L^c_{\tilde{j}}$ given by*

$$\begin{aligned} L^a_{\tilde{i}} &= (P^a_{1,\tilde{i}})^{-1} R^a_{\tilde{i}}, & \tilde{i} &= 0, \ldots, \tilde{\delta}, \\ L^c_{\tilde{j}} &= (P^c_{1,\tilde{j}})^{-1} R^c_{\tilde{j}}, & \tilde{j} &= 0, \ldots, \tilde{\Delta}. \end{aligned} \tag{56}$$

**PROOF.** From (56) we obtain

$$\begin{aligned} R^a_{\tilde{i}} &= P^a_{1,\tilde{i}} L^a_{\tilde{i}}, & \tilde{i} &= 0, \ldots, \tilde{\delta}, \\ R^c_{\tilde{j}} &= P^c_{1,\tilde{j}} L^c_{\tilde{j}}, & \tilde{j} &= 0, \ldots, \tilde{\Delta}. \end{aligned} \tag{57}$$

Since substitution of (57) in (37) yields $\Omega_{\tilde{\mu}} = P_{\tilde{\mu}} E_{\tilde{\mu}}$, it is clear that (55) implies (54), as (54) with $p_{\tilde{\mu}^-,\tilde{\mu}}$ as in (50) results from (55) by pre- and postmultiplication by $\mathrm{diag}(I_{2n}, \Xi_2^{-1}(\tilde{\mu}^-))$, followed by taking a Schur complement. Hence, we recovered the hypothesis of Theorem 13, and thus $\Sigma_{MJLS}$ is EMSS for the compensator gains in (56). □

## 5 Numerical Examples

In this section, we illustrate the presented theory using a well-known benchmark example in the NCS literature [36]. The example, which has been used in many other papers, see, e.g., [5,12,13,19,27], consists of a linearized model of an unstable batch reactor. Here, we will assume that the full state can be measured. We sample the unstable batch reactor as presented in [36] at 100 Hz to obtain a plant of the form (1) with

$$A = \begin{bmatrix} 1.0142 & -0.0018 & 0.0651 & -0.0546 \\ -0.0057 & 0.9582 & -0.0001 & 0.0067 \\ 0.0103 & 0.0417 & 0.9363 & 0.0563 \\ 0.0004 & 0.0417 & 0.0129 & 0.9797 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.0000 & -0.0010 \\ 0.0458 & 0.0000 \\ 0.0123 & -0.0304 \\ 0.0123 & -0.0002 \end{bmatrix}, \qquad C = I_4.$$

In our analysis we will assume that the state feedback gain $K$ in (2) is designed a priori. However, to analyse the influence of the choice of $K$ we consider two different values of $K$, i.e, $K_1$ is designed such that all the eigenvalues of $A + BK_1C$ are 0.4 and $K_2$ is designed such that all the eigenvalues of $A + BK_2C$ are 0.9. In Section 5.1 we will analyse stability of the batch reactor using the results for the worst-case bound modeling as in Section 3, and in Section 5.2 we will analyse stability of the batch reactor using the results for the stochastic modeling as in Section 4. In Section 5.3 we compare the results of the compensation-based strategy obtained for the worst-case bound and stochastic dropout models to the zero strategy and the hold strategy and discuss the influence of the state feedback gain $K$ on the results.

### *5.1 Worst-Case Bound Model*

To obtain maximal robustness of the compensation-based strategy for the worst-case bound dropout model, we design the compensator gains based on Theorem 9 for various values of the maximum number of successive dropouts $\bar{\delta}$ and $\bar{\Delta}$ in each of the channels. If the LMIs provided in Theorem 9 are feasible, then the NCS can be rendered stable by the compensator gains as provided in (36) and all possible sequences of dropouts where the number of subsequent dropouts do not exceed $\bar{\delta}$ and $\bar{\Delta}$, in the controller-to-actuator and sensor-to-controller channels, respectively. To compare the results of the compensation-based strategy with the zero strategy and the hold strategy, we could use sufficient Lyapunov-based tests similar to the ones described in Theorem 8. However, to even better demonstrate the true improvement of the compensation-based strategy with respect to the zero strategy and the hold strategy, we will use necessary conditions for stability of the NCS with the zero strategy and the hold strategy, as they provide an upper bound on the maximum number of successive dropouts that can be guaranteed by any sufficient condition. The necessary conditions consist of performing an eigenvalue test for some admissible periodic dropout sequences, satisfying the upper bounds $\bar{\delta}$ and $\bar{\Delta}$. The selected dropout sequences and the eigenvalue tests performed are explained next.

To select the dropout sequences used to determine upper bounds on the stability regions that can be admitted by the zero strategy and the hold strategy we consider the closed-loop system for the zero strategy as given in (4) with $A^z_{\delta,\Delta}$ as in (5), and the closed-loop system for the hold strategy as given in (7) with $A^h_{\delta,\Delta}$ as in (8). We distinguish three different cases:

(i). Only the sensor-to-controller channel exhibits dropouts, i.e, $\bar{\delta} = 0$ and $\bar{\Delta} > 0$.
(ii). Only the controller-to-actuator channel exhibits dropouts, i.e, $\bar{\delta} > 0$ and $\bar{\Delta} = 0$.
(iii). Both channels exhibit dropouts, i.e., $\bar{\delta} > 0$ and $\bar{\Delta} > 0$.

For case (i) we check (in)stability for a sequence of $\bar{\Delta}$ drops followed by a successful transmission in the sensor-to-controller channel and then repeat this pattern, i.e, we check if $(A^z_{1,0})^{\bar{\Delta}}A^z_{1,1}$ or $(A^h_{1,0})^{\bar{\Delta}}A^h_{1,1}$ is a Schur matrix for the zero strategy and the hold strategy, respectively.

For case (ii) we check (in)stability for a sequence of $\bar{\delta}$ drops followed by a successful transmission in the controller-to-sensor channel and then repeat this pattern, i.e, we check if $(A^z_{0,1})^{\bar{\delta}}A^z_{1,1}$ or $(A^h_{0,1})^{\bar{\delta}}A^h_{1,1}$ is a Schur matrix for the zero strategy and the hold strategy, respectively.

For case (iii) we analyse different dropout sequences for the zero strategy and the hold strategy. For the zero strategy, stability can never be proven for an open-loop unstable system when $\bar{\delta} > 0$ and $\bar{\Delta} > 0$. To demonstrate this, consider the admissible dropout sequence

$$\delta_k = \begin{cases} 0, & \text{if } k \text{ is odd} \\ 1, & \text{if } k \text{ is even} \end{cases} \qquad \Delta_k = \begin{cases} 1, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even,} \end{cases}$$

which, using $A^z_{\delta,\Delta}$ as in (5), results in $x_{k+1} = A^k x_0$. Since $A$ is not a Schur matrix, this implies that if both channels exhibit dropouts, i.e., if $\bar{\delta} > 0$ and $\bar{\Delta} > 0$, the NCS with the zero strategy is never stable. For the hold strategy we consider a dropout sequence where the controller-to-actuator channel drops $\bar{\delta}$ subsequent packets while the sensor-to-controller channel is transmitting successfully, followed by the sensor-to-controller channel dropping $\bar{\Delta}$ subsequent packets while the sensor-to-actuator channel is transmitting successfully, the sequence ends with a successful transmission in both channels and then repeat this pattern, we check if $(A^h_{0,1})^{\bar{\delta}}(A^h_{1,0})^{\bar{\Delta}}A^h_{1,1}$ is a Schur matrix. We also check if $(A^h_{1,0})^{\bar{\Delta}}(A^h_{0,1})^{\bar{\delta}}A^h_{1,1}$ is a Schur matrix, which results from a similar sequence, but where the drops occur in an opposite order.

The results obtained by checking the dropout sequences as indicated above for the zero strategy and the hold strategy, and the results for the compensation-based strategy that follow from Theorem 9 are shown in Fig. 6 for all eigenvalues of $A+BKC$ placed at 0.4, and in Fig. 7 for all eigenvalues of $A + BKC$ placed at 0.9. In Fig. 6 we observe that, compared to the zero strategy and the hold strategy, the compensation-based strategy can allow for more successive dropouts for cases (i) and (ii), however no strategy can prove stability for case (iii), in which both channels exhibit dropouts. If the eigenvalues of $A + BKC$ are placed at 0.9, we observe from Fig. 7 that for both the hold strategy and the compensation-based strategy it is possible to prove stability for certain situations complying with case (iii), in which both channels exhibit dropouts. Note that for the results of the zero strategy and the hold strategy, a '$*$' means that a point *might* be stable, as it follows from checking a necessary condition (which might not be related to the 'worst-case' sequence), whereas if there is no '$*$' this means that the NCS can not be stable. For the compensation-based strategy, a '$*$' means that a point is *guaranteed* to be stable, as it follows from a sufficient condition. Hence, although Fig. 6 and Fig. 7 already demonstrate that the compensation-based strategy is in general much more effective in dropout compensation than the zero strategy and the hold strategy, the compensation-based strategy might perform even (much) better compared to the latter strategies than suggested by Fig. 6 and Fig. 7.

### *5.2 Stochastic Model*

Now we assume that the dropouts in the sensor-to-controller and controller-to-actuator channel are governed by Gilbert-Elliott models. For illustrative purposes, we assume that $p^a_{s^-,s} = p^c_{s^-,s} = p_{s^-,s}$, for $s, s^- \in \{0,1\}$. To obtain maximal robustness of the stability property for the compensation-based strategy in case of the stochastic dropout models, we design the compensator gains based on Theorem 14 for various values of $p_{s^-,s}$, $s, s^- \in \{0,1\}$. If we satisfy Theorem 14 for certain $p_{s^-,s}$, then the NCS can be rendered stable by the compensator gains as provided in (56). Note that

Theorem 14 provides sufficient conditions for the existence of stabilizing compensator gains due the imposed structure on the Lyapunov function. We compare the obtained results with the zero strategy and the hold strategy for the counter saturation levels $\tilde{\delta} = \tilde{\Delta} = 1$. To compute the stability regions of the zero strategy and the hold strategy we apply a theorem similar to Theorem 12, which provides necessary and sufficient LMI-based conditions for stability, see, e.g., [30]. This leads to Fig. 8 and Fig. 9, in which we compare the region for which stability can be proven for the different strategies, in case all eigenvalues of $A + BKC$ are placed at 0.4 and 0.9, respectively. The results are based on analysing an equidistant grid of $p_{s^-,s}$, $s, s^- \in \{0, 1\}$, i.e, $p_{0,0} \in \{0, 0.01, \ldots, 0.99, 1\}$, $p_{1,1} \in \{0, 0.01, \ldots, 0.99, 1\}$, $p_{0,1} = 1 - p_{0,0}$ and $p_{1,0} = 1 - p_{1,1}$. Closed-loop stability is guaranteed for all the grid points to the left of each line. Even though the results for the compensation-based strategy are based on sufficient conditions, we observe that the region for which stability can be guaranteed is (much) larger than the regions for the zero strategy and the hold strategy. Only when all eigenvalues of $A + BK_1C$ are placed at 0.4 and the probability of remaining in the good network mode is larger than 0.85 ($p_{1,1} > 0.85$), using the zero strategy yields more robustness with respect to dropouts than using the compensation-based strategy obtained with Theorem 14. Alternative sufficient conditions for the design of stabilizing compensator gains might give even better results than Theorem 14, possibly matching or even improving the results for the zero strategy.

### *5.3 Comparison*

To compare the results of the two different gains $K$ resulting in all eigenvalues of $A + BK_1C$ at 0.4 and all eigenvalues of $A + BK_2C$ at 0.9, we study Figures 6 and 8 and Figures 7 and 9, respectively. Note that we have 'slow' convergence of the closed-loop system without dropouts if the eigenvalues of $A + BKC$ are close to the open unit disc (as for $A+BK_2C$), and, since $K_2 \neq 0$, therefore also the control commands are 'slowly' varying. If the eigenvalues of $A + BKC$ are close to the origin (as for $A+BK_1C$), we have 'fast' convergence of the closed-loop system without dropouts, and since $K_1 \neq 0$, therefore also 'rapidly' varying control commands. For the hold strategy, the actuator always acts based on either new information or information stored in a buffer, whereas for the zero strategy the actuator only acts if new information is received. From the results in Figures 6-9 we observe that, on the one hand, if the control commands are slowly varying, i.e. eigenvalues of $A + BK_2C$ at 0.9, the hold strategy performs better than the zero strategy. This is an intuitive result, as when the control commands are slowly varying, the last successfully received command stored in the buffer is likely to still be adequate. On the other hand, if the control commands vary 'rapidly', i.e. all eigenvalues of $A + BK_1C$ at 0.4, the zero strategy performs better than the hold strategy. This also is an intuitive result, as due to the 'rapidly' varying control commands the last successfully received control command stored in the buffer is likely to be inadequate. Hence, the results of the zero strategy and the hold strategy in Figures 6-9 are conform the expectation.

Most importantly, we observe that in both cases, the NCS with the compensation-based strategy is in general the most robust with respect to dropouts. This shows the importance of the newly proposed class of dropout compensators in this paper.

## 6 Conclusions

In this paper we presented a new compensation-based strategy for the stabilization of a NCS with packet dropouts. The main rationale behind the novel dropout compensators is that they act as model-based, closed-loop observers if information is received and as open-loop predictors if a dropout occurs. These compensators were considered for two dropout models, using either worst-case bounds on the number of subsequent dropouts or stochastic information on the dropout probabilities. For the worst-case bound dropout model we derived sufficient conditions for global asymptotic stability of the closed-loop NCS with the compensation-based strategy. For the stochastic dropout models we derived necessary and sufficient conditions for (exponential) mean square stability of the closed-loop NCS. In addition, for both dropout models we developed LMI-based conditions for the synthesis of the compensator gains that result in a robustly stable closed-loop system. By means of a numerical example, the significant improvements in robustness of stability with respect to packet dropouts for the compensation-based strategy compared to the zero strategy and the hold strategy were demonstrated.

## A Proof of Theorem 11

Consider the system (52a). If $\bar{A}$ is a Schur matrix, then there exists a matrix $P$ satisfying

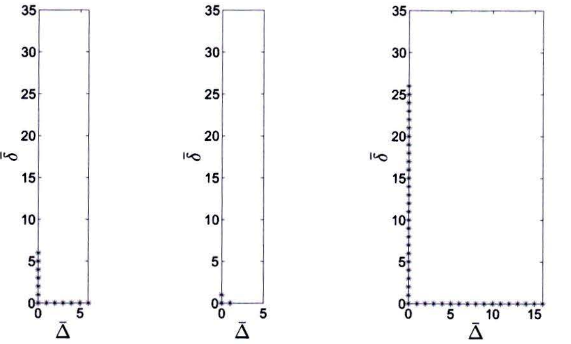
$$\sigma_1 I_n \leq P \leq \sigma_2 I_n \quad \text{with} \quad 0 < \sigma_1 \leq \sigma_2, \qquad \text{(A.1)}$$

such that

$$\bar{A}^T P \bar{A} - P = -I_n. \qquad \text{(A.2)}$$

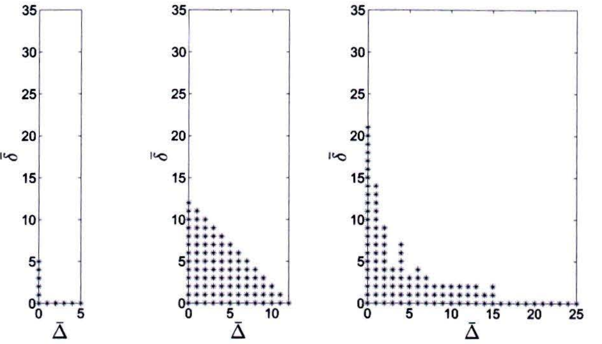
Now we will employ the function $V(x) = x^T P x$ to obtain a bound of the form (53). Hereto, consider the difference in conditional expected value of $V(x_{k+1})$ and $V(x_k)$, for $k \in \mathbb{N}$, which is given by

$$\begin{aligned}\Delta V(x_k) &:= \mathbb{E}[V(x_{k+1})\,|\,x_0] - \mathbb{E}[V(x_k)\,|\,x_0]\,,\\ &= \mathbb{E}\Big[\big(\bar{A}x_k + w_k\big)^T P\big(\bar{A}x_k + w_k\big)\,|\,x_0\Big] - \mathbb{E}\big[x_k^T P x_k\,|\,x_0\big]\,.\end{aligned}$$



(a) zero (b) hold (c) compensation-based

Fig. 6. Results for various compensation strategies and all eigenvalues of $A + BK_1C$ placed at 0.4.



(a) zero (b) hold (c) compensation-based

Fig. 7. Results for various compensation strategies and all eigenvalues of $A + BK_2C$ placed at 0.9.

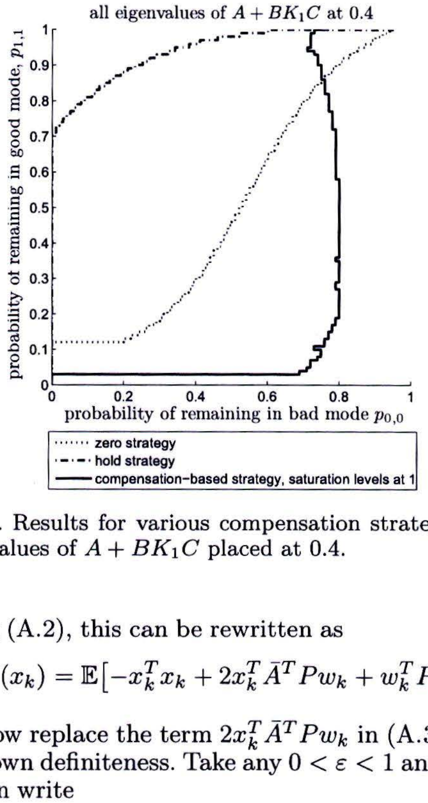


Fig. 8. Results for various compensation strategies and all eigenvalues of $A + BK_1C$ placed at 0.4.

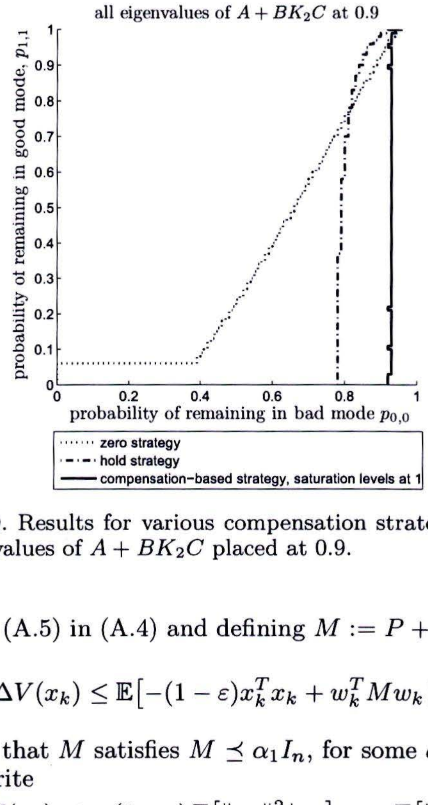


Fig. 9. Results for various compensation strategies and all eigenvalues of $A + BK_2C$ placed at 0.9.

Using (A.2), this can be rewritten as

$$\Delta V(x_k) = \mathbb{E}\big[-x_k^T x_k + 2x_k^T \bar{A}^T P w_k + w_k^T P w_k \,|\, x_0\big]. \qquad \text{(A.3)}$$

We now replace the term $2x_k^T \bar{A}^T P w_k$ in (A.3) by terms of known definiteness. Take any $0 < \varepsilon < 1$ and note that we can write

$$\Delta V(x_k) = \mathbb{E}\Big[-(1-\varepsilon)x_k^T x_k - \varepsilon x_k^T x_k + 2x_k^T \bar{A}^T P w_k - \frac{1}{\varepsilon} w_k^T P \bar{A}\bar{A}^T P w_k + w_k^T \left(P + \frac{1}{\varepsilon} P \bar{A}\bar{A}^T P\right) w_k \mid x_0\Big], \qquad \text{(A.4)}$$

where

$$-\varepsilon x_k^T x_k + 2x_k^T \bar{A}^T P w_k - \frac{1}{\varepsilon} w_k^T P \bar{A}\bar{A}^T P w_k = -\Big\|\sqrt{\varepsilon} x_k - \frac{1}{\sqrt{\varepsilon}} \bar{A}^T P w_k\Big\|_2^2 \leq 0. \qquad \text{(A.5)}$$

Using (A.5) in (A.4) and defining $M := P + \frac{1}{\varepsilon} P \bar{A}\bar{A}^T P$ yield

$$\Delta V(x_k) \leq \mathbb{E}\big[-(1-\varepsilon)x_k^T x_k + w_k^T M w_k \,|\, x_0\big].$$

Using that $M$ satisfies $M \preceq \alpha_1 I_n$, for some $\alpha_1 > 0$, we can write

$$\Delta V(x_k) \leq -(1-\varepsilon)\,\mathbb{E}\big[\|x_k\|_2^2 \,|\, x_0\big] + \alpha_1 \mathbb{E}\big[\|w_k\|_2^2\big]. \qquad \text{(A.6)}$$

From (A.1) we obtain a lower bound on $\mathbb{E}\big[\|x_k\|_2^2 \,|\, x_0\big]$, i.e.,

$$\frac{1}{\sigma_2}\mathbb{E}[V(x_k) \,|\, x_0] \leq \mathbb{E}\big[\|x_k\|_2^2 \,|\, x_0\big].$$

Substituting this lower bound in (A.6) and reordering terms yield

$$\mathbb{E}[V(x_{k+1}) \,|\, x_0] \leq \left(1 - \frac{(1-\varepsilon)}{\sigma_2}\right) \mathbb{E}[V(x_k) \,|\, x_0] + \alpha_1 \mathbb{E}\big[\|w_k\|_2^2\big]. \qquad \text{(A.7)}$$

Let us now introduce the definitions

$$v_k := \mathbb{E}[V(x_k)\,|\,x_0]\,, \qquad q := \left(1 - \frac{(1-\varepsilon)}{\sigma_2}\right). \quad \text{(A.8)}$$

Note that by the hypothesis of the theorem we have that $\mathbb{E}\big[\|w_k\|_2^2\big] \leq c_1\rho^k\|w_0\|_2^2$, for all $w_0$ and all $k \in \mathbb{N}$, and that there exists $0 < \varepsilon < 1$ such that $q \in (0,1)$. Substitution of (A.8) in (A.7) and using the bound on $\mathbb{E}\big[\|w_k\|_2^2\big]$ yields the following difference inequality

$$\begin{aligned} v_{k+1} &\leq q v_k + \alpha_1 \mathbb{E}\big[\|w_k\|_2^2\big]\,, \\ &\leq q v_k + \alpha_1 c_1 \rho^k \|w_0\|_2^2. \end{aligned}$$

By induction arguments, one can see that $v_k$ is upper bounded by $\tilde{v}_k$ (which is a kind of comparison principle, see [18]), i.e., $v_k \leq \tilde{v}_k, k \in \mathbb{N}$, where $\tilde{v}_k$ is the solution of the following difference equality

$$\begin{aligned} \tilde{v}_{k+1} &= q\tilde{v}_k + z_k, & \tilde{v}_0 &= v_0, \\ z_{k+1} &= \rho z_k, & z_0 &= \alpha_1 c_1 \|w_0\|_2^2. \end{aligned} \quad \text{(A.9)}$$

Now note that (A.9) can be written as

$$\begin{bmatrix} \tilde{v}_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} q & 1 \\ 0 & \rho \end{bmatrix} \begin{bmatrix} \tilde{v}_k \\ z_k \end{bmatrix}, \quad \text{(A.10)}$$

and since $q \in (0,1)$ and $\rho \in [0,1)$, we have that $\begin{bmatrix} q & 1 \\ 0 & \rho \end{bmatrix}$ is a Schur matrix. Hence, there exist $\tilde{c}_2 \geq 0$ and $0 \leq r < 1$ such that

$$\left\| \begin{pmatrix} \tilde{v}_k \\ z_k \end{pmatrix} \right\|_2 \leq \tilde{c}_2 r^k \left\| \begin{pmatrix} \tilde{v}_0 \\ z_0 \end{pmatrix} \right\|_2.$$

Since $v_k \leq \tilde{v}_k, k \in \mathbb{N}$, we have that

$$v_k \leq \tilde{c}_2 r^k v_0 + \tilde{c}_2 r^k z_0, \quad k \in \mathbb{N}. \quad \text{(A.11)}$$

From (A.1) we see that $\mathbb{E}[V(x_k)\,|\,x_0]$ is lower bounded by $\sigma_1\mathbb{E}\big[\|x_k\|_2^2\,|\,x_0\big]$, i.e., $\sigma_1\mathbb{E}\big[\|x_k\|_2^2\,|\,x_0\big] \leq \mathbb{E}[V(x_k)\,|\,x_0]$ and that $\mathbb{E}[V(x_0)\,|\,x_0] = V(x_0)$ (note that we evaluate the expected value at $x_0$, given $x_0$) is upper bounded by $\sigma_2\|x_0\|_2^2$, i.e., $V(x_0) \leq \sigma_2\|x_0\|_2^2$. By substitution of these bounds and the definition of $v_k$ from (A.8) and $z_0$ from (A.9) we obtain

$$\mathbb{E}\big[\|x_k\|_2^2\,|\,x_0\big] \leq \tilde{c}_2 r^k \left(\frac{\sigma_2}{\sigma_1}\|x_0\|_2^2 + \frac{1}{\sigma_1}\alpha_1 c_1 \|w_0\|_2^2\right). \quad \text{(A.12)}$$

Now define $c_2 := \tilde{c}_2\frac{\sigma_2}{\sigma_1}$ and $c_3 := \tilde{c}_2\frac{1}{\sigma_1}\alpha_1 c_1$, so that we obtain (53), for all $x_0, w_0$, $k \in \mathbb{N}$. This completes the proof.

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