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Inventory Model with Depletion

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BACHELOR THESIS

Inventory Model with Depletion

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"Festina Lente - Make haste slowly"

Anonymous

Eindhoven University of Technology

Abstract

Department of Mathematics and Computer Science

Bachelor

Inventory Model with Depletion

by RALPH VAN IERLAND

In this report we study a queueing / inventory model, based on both an M/M/1 queue and an M/G/1 queue. The server works on a constant rate, building up an inventory once it has become idle. According to a Poisson process with a rate depending on the inventory level, the inventory is completely depleted. Based on research in Albrecher et al., 2016 and Boxma, Essifi, and Janssen, 2016, we will determine the stationary densities of the workload and inventory level according to different functions for the depletion rate for the M/M/1 queue. We will mainly discuss the model where the inventory is split at a certain threshold level, with a different, constant depletion rate above this level and below this level. Some numerical results of this model will be discussed and compared to the model with constant depletion rate without threshold level. Furthermore, a simulation is considered to analyze the model with general service distribution. Finally, this simulation is used to analyze an optimization problem for a certain profit model.

Keywords *Inventory, Workload, Threshold Level, Inventory Depletion*

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Contents

Chapter 1

Introduction

In industry, a lot of queueing processes take place and analyzing these can be very useful in order to optimize them. This can be done in terms of money, time or any other quantity that one wants to be improved. In this report, we will discuss such a queueing system, but this time combined with inventory theory. This means we do not look at a model where the server is getting idle once all service requirements are resolved, but at a model where the server starts building up an inventory. These kind of processes can be found a lot in for example the commercial market, where customers do not want to wait for their orders too long. Having an inventory built up then is a great advantage, but how large should this inventory be? A process where the inventory builds up infinitely large if there are no arriving orders would be quite unrealistic and mathematically rather trivial, as this system would not give us a stable process at all.

More interesting are for example processes that have a limited storage available or processes where the inventory may lose its value. Think about the manufacturing of certain electronics. At some moment in time, the software or hardware might become obsolete due to a new model or update, causing the built up inventory to be (almost) worthless.

Therefore we will consider the following queueing / inventory model where the base of the model is an M/G/1 queue where the server starts building up inventory once he has become idle, but with the small addition that if the server is building up inventory, on certain random moments the whole inventory is depleted. This happens with a certain rate $\omega(x)$ which might depend on the size of the inventory x.

Certain variations of this model have already been analyzed in Albrecher et al., 2016 and Boxma, Essifi, and Janssen, 2016. Here several different functions for $\omega(\cdot)$ are given, such as a constant and linear growth. In Albrecher et al., 2016 this is based on an M/M/1 queueing system, while in Boxma, Essifi, and Janssen, 2016 also some results with these functions are obtained based on an M/G/1 queueing system.

In this report, we will go further on this, but with some differentiation in the function for $\omega(\cdot)$. We will consider the process where there is a certain threshold level from which onward the depletion rate ω is changed. In other words, the depletion rate for the inventory is given by ω_1 as long as the inventory is below a fixed level x_1, but once the inventory exceeds this level x_1, the inventory is depleted with a different (often larger) rate ω_2.

Think about a scenario where competing companies are more likely to invest in innovating the product when they notice to be already behind in manufacturing this product. Another example could be as follows. A manufacturer is building products with a large profit margin in a competitive field. The manufacturer has limited own (free) storage, but (almost) infinitely available external (paid) storage. As the products have a large profit margin, the manufacturer wants to process as much as possible. However, at certain moments, competitors decide to sell similar products in one moment all for stunt prices. At these moments the manufacturer has to decide whether to go along with them, to sell his complete stock at once, but for a less attractive profit margin. One could imagine that the manufacturer is more willing to do so, when it has a lot of inventory which is partly stocked externally, than when it has only a bit of inventory which perfectly fits in his own (free) storage.

The organization of this report is as follows. First we will describe the base model - so without specifying the function of $\omega(\cdot)$ - in a more mathematical way. In Chapter 3 we will then analyze the variants of the model described in the articles of Albrecher et al., 2016 and Boxma, Essifi, and Janssen, 2016. In Chapter 4 we will then discuss our own model with the certain threshold, based on an M/M/1 queue. Here, we will try to derive some interesting mathematical results about this variant by using techniques used in article Albrecher et al., 2016. Furthermore we will take a quick look at some other related variations also based on the M/M/1 queue.
In the last chapter (Chapter 5) we will then discuss variations like in Chapter 4, but now based on an M/G/1 queue. This part will be done based on a simulation, where we will also consider an example of an optimization program, based on the discussed model.

Chapter 2

Mathematical Model

In this chapter, we will discuss a model which combines queueing and inventory theory. In the next section, we describe the general model we use, which is obtained from the underlying model of Albrecher et al., 2016. In Chapter 3 we will analyze how this model is applied in their article, and we will discuss the corresponding results. In the chapters following, we will eventually use techniques from this article to analyze slightly different specific applications of the model, corresponding with the problem described in Chapter 1.

2.1 Description of General Model

Consider the following queueing/inventory model. Jobs arrive according to a Poisson process with rate λ. The corresponding services are independently and identically distributed according to a certain distribution $B(\cdot)$. The server is processing continuously with a fixed speed which we normalize to 1. If there are no service requirements anymore, the server keeps processing and therefore builds up an inventory, which can be interpreted as a negative workload. When the inventory is at level $x > 0$, the inventory is depleted according to a Poisson process with rate $\omega(x)$, so depending on the level of the inventory. At these times, the inventory is set back to zero again and the process continues. We denote the load of the system by $\rho := \lambda\mathbb{E}[B]$ and assume $\rho < 1$. This way we ensure the steady state distribution of the workload exists.
We write $V_+(x), x \geq 0$ as the steady state distribution of the workload, with density $v_+(x)$. When the inventory level is positive, there is an increasing inventory drift of $1 - \rho$. However, assuming $\exists\ \hat{x} > 0$ such that $\forall x > \hat{x} : \omega(x) > 0$, we know the inventory level will always return to zero at a certain moment, so the steady state distribution of the inventory exists. We will write $V_-(x), x \geq 0$ for this distribution with density $v_-(x)$.

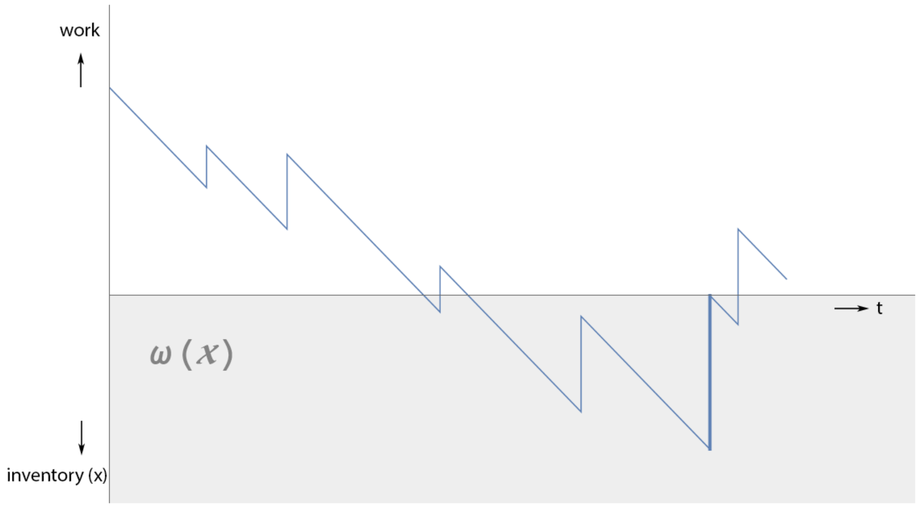

FIGURE 2.1: Work and Inventory Process

In Figure 2.1 an example of the process is given. We see a constant drift downward which are the diagonal lines, and the arriving jobs, visualized by the vertical lines. The thick vertical line is an inventory depletion which causes the inventory level to return to zero.

Remark. *Please note that we are considering the workload and the inventory separately, so for both we consider the levels to be positive* ($x > 0$)*. i.e. when in positive workload, we define the inventory to be zero and vice versa.*

2.1.1 Base Equations of the Model

We want to describe the model in two base equations, namely expressions for densities $v_+(x)$ and $v_-(x)$. To determine these, we will use a level crossing technique which is also used in Albrecher et al., 2016.

Constructing Equation for $v_+(x)$

We know V_+ is a continuous steady state distribution. Therefore, each level $x > 0$ the workload reaches, in the limit, is crossed as often from above, as from below. Hence, for the limiting distribution, it is enough to determine the density of how often each level is crossed from either below or above. This is what is called the level crossing technique. We will construct the equation by considering the crossing of levels from below as this is in the limit equal to the crossings from above. These down crossings can only happen in case of a job arrival. As jobs arrive with intensity λ we get:

$$v_+(x) = \lambda \cdot \mathbb{P}(\text{the job creates a jump from below } x \text{ to above } x) \tag{2.1}$$

which can be split up in:

$$\begin{aligned} v_+(x) = &\lambda \cdot \mathbb{P}(\text{job initiates a jump from work level } \hat{x} \in (0, x) \text{ to above } x) \\ &+ \lambda \cdot \mathbb{P}(\text{job initiates a jump from an Inventory level to above } x) \end{aligned} \tag{2.2}$$

Constructing Equation for $v_-(x)$

Also for V_- we know it is a continuous steady state distribution and therefore we can again use the same level crossing technique. Just as for V_+, we will consider the crossings from each level from below to above with respect to the orientation of Figure 2.1. i.e. a jump upward means a jump from an inventory level $x_1 > 0$ to a level x_2 for which $0 < x_2 < x_1$. For the inventory level, there are two ways a level $x > 0$ can be crossed from below: either a job arrives which creates a jump from $x_1 > x$ to a level $x_2 < x$ or an inventory depletion occurs from a level $x_3 > x$. Jobs still arrive with rate λ and an inventory depletion occurs with rate $\omega(y)$ depending on the inventory level $y > x$ at that moment. Then this gives:

$$\begin{aligned} v_-(x) = &\lambda \cdot \mathbb{P}(\text{job initiates a jump from Inventory level } \hat{x} \in (x, \infty) \text{ to above } x) \\ &+ \int_x^\infty \omega(\hat{x}) \cdot \mathbb{P}(\text{the system is at inventory level } \hat{x} > x) \mathrm{d}x \end{aligned} \tag{2.3}$$

Mathematical Expression for Equations

Now by rewriting respectively equations (2.2) and (2.3), we obtain:

$$v_+(x) = \lambda \int_0^x \mathbb{P}(B > x - y) v_+(y) \mathrm{d}y + \lambda \int_0^\infty \mathbb{P}(B > x + y) v_-(y) \mathrm{d}y, \quad x \geq 0, \tag{2.4}$$

and

$$v_-(x) = \lambda \int_x^\infty \mathbb{P}(B > y - x) v_-(y) \mathrm{d}y + \int_x^\infty \omega(y) v_-(y) \mathrm{d}y, \quad x \geq 0 \tag{2.5}$$

This gives us the base equations for our model, where B is still some general distribution for the jobs arriving in the system. In articles in Chapter 3, some assumptions are made about both this distribution B and the function $\omega(\cdot)$. In Chapter 4 and Chapter 5 we will then discuss the model with the slightly different functions for $\omega(\cdot)$ for specified distribution B and general distributions.

Chapter 3

Analysis of Literature

3.1 A Queueing Model with Randomized Depletion of Inventory

In this section we will discuss the article of Albrecher et al., 2016.
The model is the same as discussed in the previous chapter, but in this model the service times are exponentially distributed as well. Therefore the system looks a lot like an M/M/1 queue, but, just as in the other model, with a negative half space (inventory) which always returns to zero for a sufficiently large enough inventory with a certain rate $\omega(x)$.
The main equations here are like in chapter 2 given by

$$v_+(x) = \lambda \int_0^x \mathbb{P}(B > x - y) v_+(y) \mathrm{d}y + \lambda \int_0^\infty \mathbb{P}(B > x + y) v_-(y) \mathrm{d}y, \quad x \geq 0, \tag{3.1}$$

and

$$v_-(x) = \lambda \int_x^\infty \mathbb{P}(B > y - x) v_-(y) \mathrm{d}y + \int_x^\infty \omega(y) v_-(y) \mathrm{d}y, \quad x \geq 0. \tag{3.2}$$

3.1.1 Exponential Services

In the model description above, we did not use the fact that the services are exponentially distributed at all, this will be done in this subsection. Since $B \sim \mathrm{Exp}(\mu)$, we know $\mathbb{P}(B > x) = e^{-\mu x}$. This gives us for (3.1) and (3.2) the expression

$$v_+(x) = \lambda e^{-\mu x} \int_0^x e^{\mu y} v_+(y) \mathrm{d}y + \lambda e^{-\mu x} \int_0^\infty e^{-\mu y} v_-(y) \mathrm{d}y \tag{3.3}$$

and

$$v_-(x) = \lambda e^{\mu x} \int_x^\infty e^{-\mu y} v_-(y) \mathrm{d}y + \int_x^\infty \omega(y) v_-(y) \mathrm{d}y. \tag{3.4}$$

Now we are going to use the transformations $z_+(x) = e^{\mu x} v_+(x),\ x \geq 0$ and $z_-(x) = e^{-\mu x} v_-(x),\ x \geq 0$ to eventually make life easier. We now get

$$z_+(x) = \lambda \int_0^x z_+(y) \mathrm{d}y + \lambda \int_0^\infty z_-(y) \mathrm{d}y \tag{3.5}$$

and

$$z_-(x) = \lambda \int_x^\infty z_-(y) \mathrm{d}y + e^{-\mu x} \int_x^\infty \omega(y) e^{\mu y} z_-(y) \mathrm{d}y. \tag{3.6}$$

Now we take a closer look at (3.5) to get a nice expression for $z_+(x)$. We differentiate both sides with respect to x, which makes the right part of the expression vanish.

Therefore we get

$$z'_+(x) = \lambda z_+(x). \tag{3.7}$$

From theory of differential equations, we know the solution of this equation is of the form

$$z_+(x) = \tilde{C}e^{\lambda x},$$

where $\tilde{C}$ is a constant. For convenience of the following steps, we use $C = \frac{\tilde{C}}{\lambda}$ to get

$$z_+(x) = C\lambda e^{\lambda x}, \tag{3.8}$$

with C still a constant.
To obtain constant C, we will substitute (3.8) in (3.5) to get

$$C\lambda e^{\lambda x} = \lambda \int_0^x C\lambda e^{\lambda y}\mathrm{d}y + \lambda \int_0^\infty z_-(y)\mathrm{d}y. \tag{3.9}$$

Dividing the left hand side and the right hand side of this equation by λ, and integrating the first integral on the right hand side, gives

$$\begin{aligned} Ce^{\lambda x} &= C\Big[e^{\lambda y}\Big]_0^x + \int_0^\infty z_-(y)\mathrm{d}y \\ &= C(e^{\lambda x} - 1) + \int_0^\infty z_-(y)\mathrm{d}y. \end{aligned} \tag{3.10}$$

Simplifying this gives

$$C = \int_0^\infty z_-(y)\mathrm{d}y. \tag{3.11}$$

Therefore we get for $v_+(x)$,

$$v_+(x) = e^{-\mu x} z_+(x) = C\lambda e^{-(\mu-\lambda)x}, \quad x > 0. \tag{3.12}$$

Integrating this gives

$$\begin{aligned} \int_0^\infty v_+(x)dx &= C\lambda \int_0^\infty e^{-(\mu-\lambda)x}dx \\ &= C\lambda \left[\frac{e^{-(\mu-\lambda)x}}{-(\mu-\lambda)}\right]_0^\infty \\ &= C\frac{\frac{\lambda}{\mu}}{1-\frac{\lambda}{\mu}}. \end{aligned} \tag{3.13}$$

Using that $\rho = \lambda\mathbb{E}[B] = \frac{\lambda}{\mu}$, we get

$$\int_0^\infty v_+(x)dx = C\frac{\rho}{1-\rho}. \tag{3.14}$$

Now we have an expression for $v_+(x)$, from (3.14) we see the expression is, up to a constant equal to the density of an exponential distribution if $\mu > \lambda$. This constraint is also the constraint for V_+ to have a steady state distribution, so this is satisfied.
In (3.14) we get the fraction of time the system has a positive workload. This is still depending on the distribution of V_-. We will now consider this distribution.

Determining $\mathbf{v}_-(x)$

Theorem (Leibniz Integral Rule).

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f(x,b(x))\cdot\frac{d}{dx}b(x) - f(x,a(x))\cdot\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt$$

We are going to use Leibniz Integral Rule to determine the derivative (with respect to x) of

$$z_-(x) = \lambda\int_x^\infty z_-(y)\mathrm{d}y + e^{-\mu x}\int_x^\infty \omega(y)e^{\mu y}z_-(y)\mathrm{d}y. \tag{3.15}$$

Using Leibniz Integral Rule gives us for the first term on the right hand side, we get

$$\frac{d}{dx}\left[\lambda\int_x^\infty z_-(y)\mathrm{d}y\right] = -\lambda z_-(x), \tag{3.16}$$

and for the second term,

$$\begin{aligned}\frac{d}{dx}\left[e^{-\mu x}\int_x^\infty \omega(y)e^{\mu y}z_-(y)\mathrm{d}y\right] &= \frac{d}{dx}\left[\int_x^\infty e^{-\mu x}\omega(y)e^{\mu y}z_-(y)\mathrm{d}y\right]\\ &= -e^{-\mu x}\omega(x)e^{\mu x}z_-(x) + \int_x^\infty \frac{\partial}{\partial x}e^{-\mu x}\omega(y)e^{\mu y}z_-(y)\mathrm{d}y\\ &= -\omega(x)z_-(x) + \int_x^\infty \frac{\partial}{\partial x}e^{-\mu x}\omega(y)e^{\mu y}z_-(y)\mathrm{d}y\\ &= -\omega(x)z_-(x) + \int_x^\infty -\mu e^{-\mu x}\omega(y)e^{\mu y}z_-(y)\mathrm{d}y\\ &= -\omega(x)z_-(x) - \mu\int_x^\infty e^{-\mu x}\omega(y)e^{\mu y}z_-(y)\mathrm{d}y.\end{aligned} \tag{3.17}$$

Now we substitute (3.15) in (3.17) and combine this with (3.16) to get

$$z_-'(x) = -\lambda z_-(x) - \omega(x)z_-(x) - \mu\left[z_-(x) - \lambda\int_x^\infty z_-(y)dy\right]. \tag{3.18}$$

To make the integral on the right side vanish, we will differentiate once more, which gives us

$$\begin{aligned}z_-''(x) &= \Big[-\lambda z_-(x)\Big]' + \Big[-\omega(x)z_-(x)\Big]' - \mu\left[z_-(x) - \lambda\int_x^\infty z_-(y)dy\right]'\\ &= \Big[-\lambda z_-'(x)\Big] + \Big[-\omega(x)z_-'(x) - \omega'(x)z_-(x)\Big] - \mu\Big[z_-'(x) - \lambda\cdot - z_-(x)\Big].\end{aligned} \tag{3.19}$$

Now in the specific case where $\omega(x) = \omega$ is a constant we can simplify this to

$$z_-''(x) + (\lambda+\omega+\mu)z_-'(x) + \mu\lambda z_-(x) = 0. \tag{3.20}$$

We are now left with a homogeneous second order differential equation. For these type of differential equations we know the solutions for this equation are of the form

$$z_{-}(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}, \tag{3.21}$$

where $r_{1,2}$ are given by the roots of the characteristic polynomial

$$r^2 + (\lambda + \omega + \mu)r + \mu\lambda = 0,$$

which are

$$\begin{aligned} r_1 &= \frac{-(\lambda+\omega+\mu) - \sqrt{(\lambda+\omega+\mu)^2 - 4\lambda\mu}}{2}, \\ r_2 &= \frac{-(\lambda+\omega+\mu) + \sqrt{(\lambda+\omega+\mu)^2 - 4\lambda\mu}}{2}. \end{aligned} \tag{3.22}$$

Since $v_{-}(x) = e^{\mu x} z_{-}(x)$ we define

$$s_i := r_i + \mu = \frac{-(\lambda+\omega-\mu) \mp \sqrt{(\lambda+\omega+\mu)^2 - 4\lambda\mu}}{2}. \tag{3.23}$$

So

$$v_{-}(x) = C_1 e^{s_1 x} + C_2 e^{s_2 x}. \tag{3.24}$$

Now we wonder whether the s_is are negative for all possible parameters $0 < \lambda < \mu$ and $\omega > 0$. We notice that

$$\begin{aligned} (\lambda+\mu+\omega)^2 - 4\lambda\mu &= \lambda^2 + \mu^2 + \omega^2 + 2(\lambda\mu + \lambda\omega + \mu\omega) - 4\lambda\mu \\ &= \lambda^2 + \mu^2 + \omega^2 + 2(-\lambda\mu + \lambda\omega + \mu\omega) \\ &> \lambda^2 + \mu^2 + \omega^2 + 2(-\lambda\mu + \lambda\omega - \mu\omega) \\ &= (\lambda - \mu + \omega)^2. \end{aligned} \tag{3.25}$$

So if $(\lambda - \mu + \omega) \leq 0$, we get

$$s_1 < \frac{-(\lambda-\mu+\omega) - |\lambda-\mu+\omega|}{2} = 0, \tag{3.26}$$

and for $(\lambda - \mu + \omega) > 0$,

$$s_1 < \frac{-2(\lambda-\mu+\omega)}{2} < 0. \tag{3.27}$$

However, for s_2 we find for $(\lambda - \mu + \omega) \leq 0$ that

$$s_2 > \frac{-(\lambda-\mu+\omega) + |\lambda-\mu+\omega|}{2} = \frac{|\lambda-\mu+\omega| + |\lambda-\mu+\omega|}{2} \geq 0. \tag{3.28}$$

Hence, s_1 is negative everywhere, but s_2 is not necessarily. As C_2 is just a constant, and $\lim_{x\to\infty} v_{-}(x) = 0$, we conclude that $C_2 = 0$. This gives us that

$$z_{-}(x) = C_1 e^{r_1 x}. \tag{3.29}$$

Combining (3.29) with (3.11) we get

$$C \overset{(3.11)}{=} \int_0^\infty z_{-}(x) \overset{(3.29)}{=} \int_0^\infty C_1 e^{r_1 x} dx = \frac{C_1}{-r_1}. \tag{3.30}$$

Moreover we will use that

$$\int_0^\infty v_+(x) + \int_0^\infty v_-(x) = 1. \tag{3.31}$$

So

$$\begin{aligned}
\int_0^\infty v_+(x) + \int_0^\infty v_-(x) &= C\frac{\rho}{1-\rho} + \int_0^\infty C_1 e^{s_1 x} dx \\
&= \frac{C_1\rho}{-r_1(1-\rho)} + \frac{C_1}{-s_1} \\
&= \frac{C_1\lambda}{(\mu-s_1)(\mu-\lambda)} + \frac{C_1}{-s_1} \\
&= \frac{-s_1 C_1\lambda}{-s_1(\mu-s_1)(\mu-\lambda)} + \frac{C_1(\mu-s_1)(\mu-\lambda)}{-s_1(\mu-s_1)(\mu-\lambda)} \\
&= \frac{-s_1 C_1\lambda + C_1(\mu-s_1)(\mu-\lambda)}{-s_1(\mu-s_1)(\mu-\lambda)} = 1,
\end{aligned} \tag{3.32}$$

and therefore

$$\begin{aligned}
C_1 &= \frac{-s_1(\mu-s_1)(\mu-\lambda)}{-s_1\lambda + (\mu-s_1)(\mu-\lambda)} \\
&= \frac{-s_1(\mu-s_1)(\mu-\lambda)}{-s_1\lambda + \mu^2 - \lambda\mu - s_1\mu + s_1\lambda} \\
&= \frac{s_1(\mu-s_1)(\mu-\lambda)}{\mu(s_1-\mu+\lambda)}.
\end{aligned} \tag{3.33}$$

Numerical Results
In the article, also several numerical results are given. We will discuss these in Section 4.3, in comparison to our own variation of the model.

3.2 A Queueing/Inventory and an Insurance Risk Model

In this article of Albrecher et al., 2016, the same model as described in Chapter 2 is discussed, but this time based on an M/G/1 queueing system. As this implies that the theoretical parts are getting quite complex, we will discuss this article quite briefly. Moreover, in this article it is tried to combine the queueing/inventory model with a bankruptcy problem as these models look quite similar. However, we will not discuss the latter case any further.

The goal of this section is to give an idea of what is needed to obtain theoretical results if the service distribution is general. In Chapter 5 we will again take a look at the M/G/1 queueing system, but there it will only be discussed in terms of a simulation.

Analysis $\omega(\cdot)$ Constant

At first, the model is discussed for $\omega(\cdot)$ constant. Just as in the previous paper, we want to gain more insight in what V_+ and V_- look like. This is done by considering the Laplace transforms of V_+ and V_-, which are respectively $\phi_+(s)$ and $\phi_-(s)$. As also in the structure of the previous paper, we notice that V_+ depends on V_-, but not vice versa. To ensure a good analysis of the model, we therefore first take a look at the distribution of V_-.

$\mathbf{V}_-$ Analyzing the transforms further gives an expression for $\phi_-(s)$ in which the Laplace Stieltjes Transform of an exponential distribution is recognized. This means that for constant $\omega(x) = \omega$ the steady-state distribution for the inventory $V_-(x)$ is exponentially distributed.
As it is quite remarkable that $V_-(x)$ turns out to be exponentially distributed - it is quite a nice result for a M/G/1 queue - Also a more heuristic argument is given:

As also illustrated in Figure 3.1, the M/G/1 queue is first rewritten as a G/M/1 queue, by considering service distribution as the arrival distribution and vice versa. The original M/G/1 (upper graph) is now rewritten as the middle G/M/1 queue. For the upper graph, we want to show the distribution of the x_is are exponential, because this would imply $V_-(x)$ is exponentially distributed, as follows from PASTA (Poisson Arrival See Time Average). As the work rate is assumed to be 1, it results the diagonal lines in the figure to be at $45°$.
In the lower graph, we can find the number of orders in the system for the G/M/1 queue. Since the diagonal lines in the middle figure are at $45°$, we can conclude the red painted lines in the lower graph represent the x_is of the middle and upper graph. Moreover we know these red lines to be the waiting times of the G/M/1 queue. Now the key is that we know for G/M/1 queues that - given the waiting time is greater than zero - these waiting times are exponentially distributed. As these times correspond with the x_is, we know those to be exponentially distributed as well. As we have noted before, this eventually means $V_-(x)$ is exponentially distributed.

$\mathbf{V}_+$ So a nice result for V_- is obtained, however, V_+ turns out to be not that simple. First we define the busy period of this system by the period the workload is positive. This means the system is called "empty" if the workload $x \leq 0$.
Since we know the jobs arrive according to a Poisson process with rate λ and have i.i.d. service times $B_1, B_2, \ldots$, we can consider it as an adapted M/G/1 queue with a different distribution for the first service of a busy period.
As the busy period starts with an arrival that cannot be fully served from the inventory, it starts with a "residual" service - with a different distribution of course. This immediately makes it quite complex to determine V_+. In the article, an expression is found for the residual service, after which also an expression for V_+ is found.

Other Functions for $\omega(\cdot)$

More details about this and other functions of $\omega(\cdot)$ can be found in the article Boxma, Essifi, and Janssen, 2016 and will not be discussed here, as they are less relevant for the results obtained in this report.

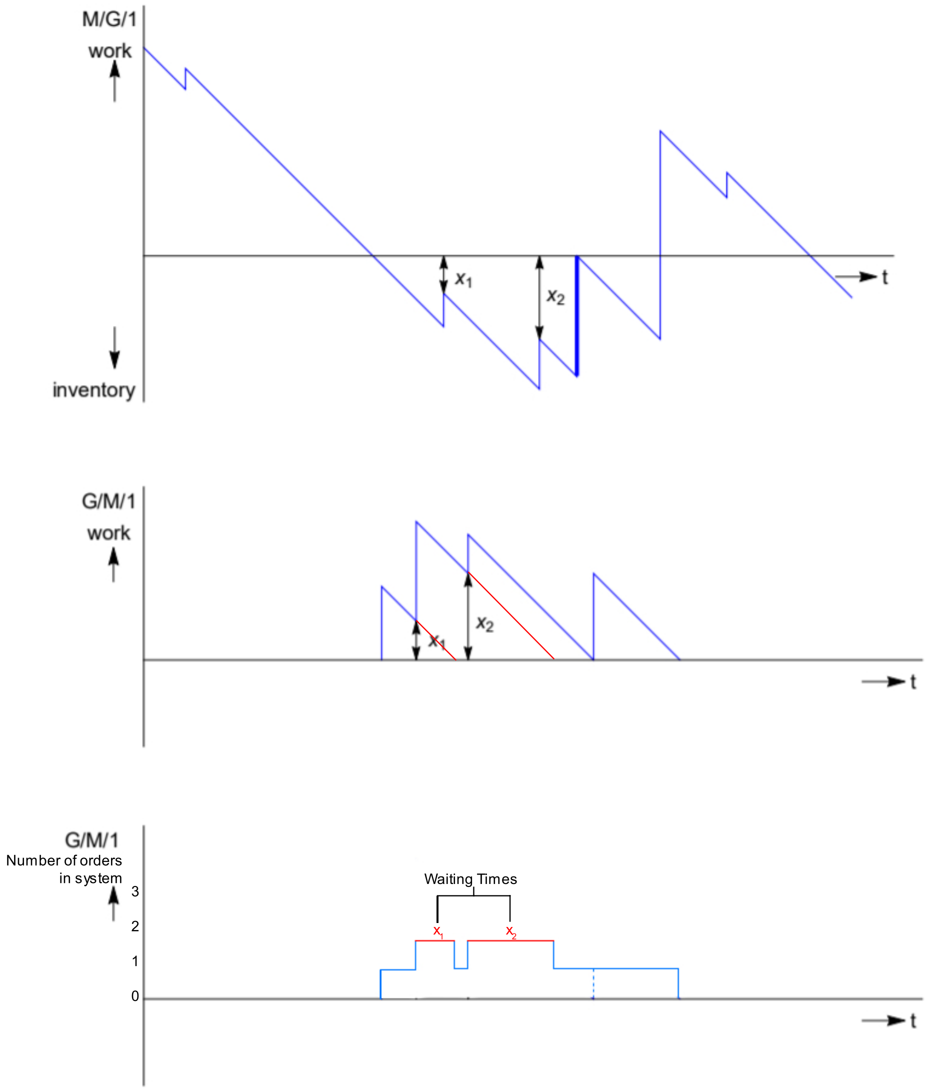

FIGURE 3.1: The workload and inventory process as described in Boxma, Essifi, and Janssen, 2016. Above the standard process. In the middle the process rewritten to a G/M/1 queue. Below the number of jobs in the system with the corresponding waiting time.

Chapter 4

Model with Threshold

4.1 Model Description

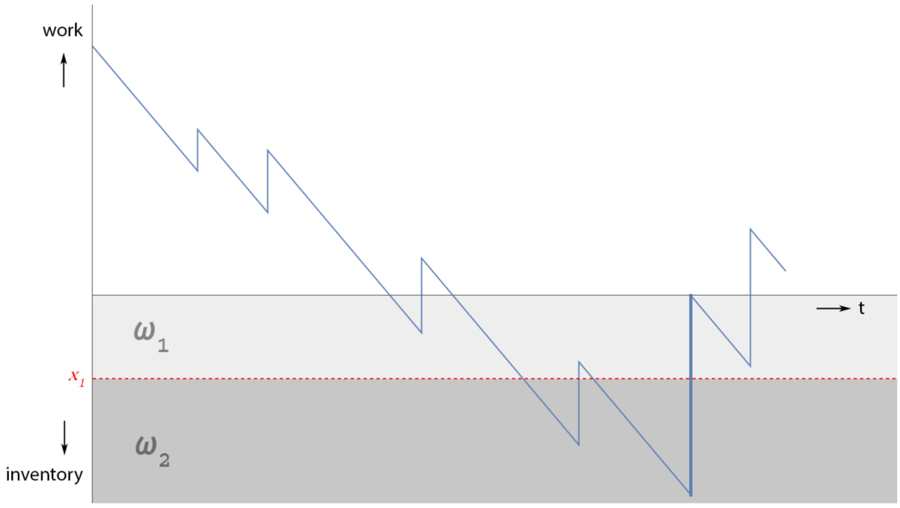

FIGURE 4.1: Work and Inventory Process with Threshold

Now assume there is a certain threshold, where the rate of clearing the inventory changes. This point we call x_1, and we can write $\omega(x)$ as:

$$\omega(x) = \mathbb{1}_{\{0<x<x_1\}}\omega_1 + \mathbb{1}_{\{x\geq x_1\}}\omega_2 \tag{4.1}$$

For this model, we will get the same kind of main equations as in Section 3.1, but now we will have, next to the v_+ a v_{-1} and a v_{-2}. To make the notation in some places a bit easier, we define

$$v_-(x) = \mathbb{1}_{\{0\leq x<x_1\}}v_{-1}(x) + \mathbb{1}_{\{x\geq x_1\}}v_{-2}(x), \tag{4.2}$$

where the main equations will then be:

$$v_+(x) = \lambda \int_0^x \mathbb{P}(B > x - y)v_+(y)\mathrm{d}y + \lambda \int_0^\infty \mathbb{P}(B > x + y)v_-(y)\mathrm{d}y, \quad x > 0, \tag{4.3}$$

$$v_{-1}(x) = \lambda \int_x^\infty \mathbb{P}(B > y - x)v_-(y)\mathrm{d}y + \int_x^\infty \omega(y)v_-(y)\mathrm{d}y, \quad 0 \leq x \leq x_1, \tag{4.4}$$

$$v_{-2}(x) = \lambda \int_x^\infty \mathbb{P}(B > y - x) v_{-2}(y) \mathrm{d}y + \int_x^\infty \omega_2 v_{-2}(y) \mathrm{d}y, \quad x \geq x_1. \tag{4.5}$$

We see eq. (4.3) is exactly the same as eq. (3.1), only the $v_-(\cdot)$ has become a bit harder. For eq. (4.5) we see this is of exactly the same form as eq. (3.2). The strategy to determine the three distributions, will be to first determine v_{-2}, which is only depending on itself. Then we will determine v_{-1}, which depends on only v_{-2} and at last we can find a closed expression for v_+, by using v_{-1} and v_{-2}.

4.2 Main Equations

For $\mathbf{v_{-2}}$

Since our services are again exponentially distributed, we get:

$$v_{-2}(x) = \lambda e^{\mu x} \int_x^\infty e^{-\mu y} v_{-2}(y) \mathrm{d}y + \int_x^\infty \omega(y) v_{-2}(y) \mathrm{d}y, \quad x > x_1 \tag{4.6}$$

We again use the translation $z_{-2}(x) = e^{-\mu x} v_{-2}(x)$ for $x > x_1$. This gives us:

$$z_{-2}(x) = \lambda \int_x^\infty z_{-2}(y) \mathrm{d}y + e^{-\mu x} \int_x^\infty \omega(y) e^{\mu y} z_{-2}(y) \mathrm{d}y, \quad x > x_1 \tag{4.7}$$

We will now use the same technique as in Chapter 3, as this equation looks almost the same as eq. (3.15). Moreover we use that $\omega(\cdot)$ is a constant and equal to ω_1. This means we will differentiate two times which gives us, completely analogue to eq. (3.20),

$$z''_{-2}(x) + (\lambda + \omega_2 + \mu) z'_{-2}(x) + \mu\lambda z_{-2}(x) = 0. \tag{4.8}$$

We again conclude that

$$z_{-2}(x) = C_{2,1} e^{r_{2,1} \cdot x} + C_{2,2} e^{r_{2,2} \cdot x}, \tag{4.9}$$

where this time

$$\begin{aligned} r_{2,1} &= \frac{-(\lambda + \omega_2 + \mu) - \sqrt{(\lambda + \omega_2 + \mu)^2 - 4\lambda\mu}}{2}, \\ r_{2,2} &= \frac{-(\lambda + \omega_2 + \mu) + \sqrt{(\lambda + \omega_2 + \mu)^2 - 4\lambda\mu}}{2}. \end{aligned} \tag{4.10}$$

We again define

$$s_{i,j} := r_{i,j} + \mu, \qquad j \in \{1, 2\}, \tag{4.11}$$

to write

$$v_{-2}(x) = C_{2,1} e^{s_{2,1} \cdot x} + C_{2,2} e^{s_{2,2} \cdot x}. \tag{4.12}$$

With the same arguments as in Chapter 3 , we conclude $C_{2,2} = 0$ what gives us

$$v_{-2}(x) = C_{2,1} e^{s_{2,1} \cdot x}, \quad x > x_1, \tag{4.13}$$

and

$$z_{-2}(x) = C_{2,1} e^{r_{2,1} \cdot x}, \quad x > x_1, \tag{4.14}$$

and therefore,

$$\begin{aligned}\int_{x_1}^{\infty} v_{-2}(x) &= \int_{x_1}^{\infty} C_{2,1}e^{s_{2,1}\cdot x}\\ &= \left[\frac{C_{2,1}}{s_{2,1}}e^{s_{2,1}\cdot x}\right]_{x_1}^{\infty}\\ &= -\frac{C_{2,1}}{s_{2,1}}e^{s_{2,1}\cdot x_1}.\end{aligned} \tag{4.15}$$

For $\mathbf{v_{-1}}$

We first rewrite eq. (4.4) to

$$\begin{aligned}v_{-1}(x) = \lambda e^{\mu x}&\left(\int_x^{x_1} e^{-\mu y}v_{-1}(y)\mathrm{d}y + \int_{x_1}^{\infty} e^{-\mu y}v_{-2}(y)\mathrm{d}y\right)\\ &+ \int_x^{x_1}\omega(y)v_{-1}(y)\mathrm{d}y + \int_{x_1}^{\infty}\omega(y)v_{-2}(y)\mathrm{d}y, \quad 0 \le x \le x_1.\end{aligned} \tag{4.16}$$

Now using the translation $z_{-2}(x) = e^{-\mu x}v_{-2}(x)$ for $x \ge x_1$ and $z_{-1}(x) = e^{-\mu x}v_{-1}(x)$ for $0 < x < x_1$ gives

$$\begin{aligned}z_{-1}(x) = \lambda&\left(\int_x^{x_1} z_{-1}(y)\mathrm{d}y + \int_{x_1}^{\infty} z_{-2}(y)\mathrm{d}y\right)\\ &+ e^{-\mu x}\left(\int_x^{x_1}\omega_1 e^{\mu y}z_{-1}(y)\mathrm{d}y + \int_{x_1}^{\infty}\omega_2 e^{\mu y}z_{-2}(y)\mathrm{d}y\right), \quad 0 \le x \le x_1.\end{aligned} \tag{4.17}$$

Again, we will differentiate and use Leibniz Integral Rule to get

$$\begin{aligned}z'_{-1}(x) &= -\lambda z_{-1}(x) - e^{-\mu x}\omega_1 e^{\mu x}z_{-1}(x)\\ &\quad - \mu e^{-\mu x}\int_x^{x_1}\omega_1 e^{\mu y}z_{-1}(y)\mathrm{d}y - \mu e^{-\mu x}\int_{x_1}^{\infty}\omega_2 e^{\mu y}z_{-2}(y)\mathrm{d}y.\end{aligned} \tag{4.18}$$

So

$$\begin{aligned}z'_{-1}(x) &= -\lambda z_{-1}(x) - \omega_1 z_{-1}(x)\\ &\quad - \mu\left[e^{-\mu x}\left(\int_x^{x_1}\omega_1 e^{\mu y}z_{-1}(y)\mathrm{d}y + \int_{x_1}^{\infty}\omega_2 e^{\mu y}z_{-2}(y)\mathrm{d}y\right)\right].\end{aligned} \tag{4.19}$$

Now substituting eq. (4.17) gives

$$\begin{aligned}z'_{-1}(x) &= -\lambda z_{-1}(x) - \omega_1 z_{-1}(x)\\ &\quad - \mu\left[z_{-1}(x) - \lambda\left(\int_x^{x_1} z_{-1}(y)\mathrm{d}y + \int_{x_1}^{\infty} z_{-2}(y)\mathrm{d}y\right)\right].\end{aligned} \tag{4.20}$$

Differentiating once more gives us

$$z''_{-1}(x) = -\lambda z'_{-1}(x) - \omega_1 z'_{-1}(x) - \mu\left[z'_{-1}(x) + \lambda z_{-1}(x)\right]. \tag{4.21}$$

So

$$z''_{-1}(x) + (\lambda + \omega_1 + \mu)z'_{-1}(x) + \mu\lambda z_{-1}(x) = 0. \tag{4.22}$$

So we have a homogeneous differential equation which looks identical to eq. (4.8), except for a different ω. therefore the solution for the differential equation is

$$z_{-1}(x) = C_{1,1}e^{r_{1,1}x} + C_{1,2}e^{r_{1,2}x}, \tag{4.23}$$

where this time

$$\begin{aligned} r_{1,1} &= \frac{-(\lambda+\omega_0+\mu) - \sqrt{(\lambda+\omega_0+\mu)^2 - 4\lambda\mu}}{2}, \\ r_{1,2} &= \frac{-(\lambda+\omega_0+\mu) + \sqrt{(\lambda+\omega_0+\mu)^2 - 4\lambda\mu}}{2}. \end{aligned} \tag{4.24}$$

Furthermore, to get a relation between $C_{1,1}$, $C_{1,2}$ and $C_{2,1}$, we will substitute eq. (4.14) and eq. (4.23) in eq. (4.17).
This gives us

$$\begin{aligned} C_{1,1}e^{r_{1,1}x} + C_{1,2}e^{r_{1,2}x} = \lambda \int_x^{x_1} & C_{1,1}e^{r_{1,1}y} + C_{1,2}e^{r_{1,2}y}\mathrm{d}y \\ & + e^{-\mu x}\int_x^{x_1} \omega_1 e^{\mu y}(C_{1,1}e^{r_{1,1}y} + C_{1,2}e^{r_{1,2}y})\mathrm{d}y \\ & - \lambda\frac{C_{2,1}}{r_{2,1}}e^{r_{2,1}x_1} - \omega_2\frac{C_{2,1}}{s_{2,1}}e^{s_{2,1}\cdot x_1}\cdot e^{-\mu x}. \end{aligned} \tag{4.25}$$

To determine two relations between the constants, we fill in two different values for x.
First we take the easy $x \to x_1$, so then

$$C_{1,1}e^{r_{1,1}x_1} + C_{1,2}e^{r_{1,2}x_1} = -\lambda\frac{C_{2,1}}{r_{2,1}}e^{r_{2,1}x_1} - \omega_2\frac{C_{2,1}}{s_{2,1}}e^{r_{2,1}x_1}. \tag{4.26}$$

To get another equation, we substitute another value for x, this time $x \to 0$. So then

$$\begin{aligned} C_{1,1} + C_{1,2} = \lambda\int_0^{x_1} & C_{1,1}e^{r_{1,1}y} + C_{1,2}e^{r_{1,2}y}\mathrm{d}y \\ & + \int_0^{x_1} \omega_1 e^{\mu y}(C_{1,1}e^{r_{1,1}y} + C_{1,2}e^{r_{1,2}y})\mathrm{d}y \\ & - \lambda\frac{C_{2,1}}{r_{2,1}}e^{r_{2,1}x_1} - \omega_2\frac{C_{2,1}}{s_{2,1}}e^{s_{2,1}x_1}. \end{aligned} \tag{4.27}$$

So

$$\begin{aligned} C_{1,1} + C_{1,2} = \lambda & \left(\frac{C_{1,1}}{r_{1,1}}e^{r_{1,1}x_1} + \frac{C_{1,2}}{r_{1,2}}e^{r_{1,2}x_1} - \frac{C_{1,1}}{r_{1,1}} - \frac{C_{1,2}}{r_{1,2}}\right) \\ & + \omega_1\left(\frac{C_{1,1}}{s_{1,1}}e^{s_{1,1}x_1} + \frac{C_{1,2}}{s_{1,2}}e^{s_{1,2}x_1} - \frac{C_{1,1}}{s_{1,1}} - \frac{C_{1,2}}{s_{1,2}}\right) \\ & - \lambda\frac{C_{2,1}}{r_{2,1}}e^{r_{2,1}x_1} - \omega_2\frac{C_{2,1}}{s_{2,1}}e^{s_{2,1}x_1}. \end{aligned} \tag{4.28}$$

For $\mathbf{v}_+$

Since the service times are exponentially distributed, we get the exact same expression for $v_+(\cdot)$ as in previous chapter, namely eq. (3.12), where $z_-(\cdot)$ is defined as

$$z_-(x) = e^{-\mu x}\Big(\mathbb{1}_{\{0<x<x_1\}}v_{-1} + \mathbb{1}_{\{x\geq x_1\}}v_{-2}\Big). \tag{4.29}$$

This gives us, like in the article of Albrecher et al., 2016

$$\begin{aligned} C &= \int_0^\infty z_-(x)dx = \int_0^{x_1} z_{-1}(x)dx + \int_{x_1}^\infty z_{-2}(x)dx \\ &= \int_0^{x_1} C_{1,1}e^{r_{1,1}\cdot x} + C_{1,2}e^{r_{1,2}\cdot x}dx + \int_{x_1}^\infty C_{2,1}e^{r_{2,1}\cdot x}dx \\ &= \frac{C_{2,1}}{r_{2,1}}e^{r_{1,1}\cdot x_1} + \frac{C_{1,2}}{r_{1,2}}e^{r_{1,2}\cdot x_1} - \frac{C_{1,1}}{r_{1,1}} - \frac{C_{1,2}}{r_{1,2}} - \frac{C_{2,1}}{r_{2,1}}e^{r_{2,1}\cdot x_1}. \end{aligned} \tag{4.30}$$

At last, we use the fact that

$$\int_0^\infty v_+(x)dx + \int_0^{x_1} v_{-1}(x)dx + \int_{x_1}^\infty v_{-2}(x)dx = 1, \tag{4.31}$$

so

$$\left[C\frac{\rho}{1-\rho}\right] + \left[\frac{C_{1,1}}{s_{1,1}}e^{s_{1,1}x_1} + \frac{C_{1,2}}{s_{1,2}}e^{s_{1,2}x_1} - \frac{C_{1,1}}{s_{1,1}} - \frac{C_{1,2}}{s_{1,2}}\right] + \left[-\frac{C_{2,1}}{s_{2,1}}e^{s_{2,1}x_1}\right] = 1. \tag{4.32}$$

Now with the system of equations (4.26), (4.28) and (4.32) (where we also substitute (4.30) in (4.32)), we can find the values for $C_{1,1}$, $C_{1,2}$ and $C_{2,1}$. Therefore we now also know the value of C, which makes it possible to find the value of the mean of work as we have seen in (3.14) where

$$\int_0^\infty v_+(x)dx = C\frac{\rho}{1-\rho}$$

gives the fraction of time the system has a positive workload.
As we can solve the system and find values for the different constants, we can also determine other quantities like:

- The mean work level:

$$\mathbb{E}[X] = \int_0^\infty xv_+(x)dx = \int_0^\infty x\cdot(C\lambda e^{-(\mu-\lambda)x})dx. \tag{4.33}$$

- The mean inventory level:

$$\begin{aligned} \mathbb{E}[I] &= \int_0^\infty xv_-(x)dx = \int_0^{x_1} xv_{-1}(x)dx + \int_{x_1}^\infty xv_{-2}(x)dx \\ &= \int_0^{x_1} xC_{1,1}e^{s_{1,1}x} + xC_{1,2}e^{s_{1,2}x}dx + \int_{x_1}^\infty xC_{2,1}e^{s_{2,1}x}dx. \end{aligned} \tag{4.34}$$

- The probability a job arriving in the system can be fully served, directly from the inventory:

$$\begin{aligned}\int_0^\infty v_-(x)\mathbb{P}(B<x)dx &= \int_0^{x_1} v_{-1}(x)(1-e^{-\mu x})dx + \int_{x_1}^\infty v_{-2}(x)(1-e^{-\mu x})dx \\ &= \int_0^{x_1}(1-e^{-\mu x})(C_{1,1}e^{s_{1,1}x}+C_{1,2}e^{s_{1,2}x})dx \\ &+ \int_{x_1}^\infty (1-e^{-\mu x})C_{2,1}e^{s_{2,1}x}dx.\end{aligned} \tag{4.35}$$

4.3 Numerical Results

So we have derived a system of equations from which we can determine exact results for, for example, the mean work level, mean inventory level and the probability a job can be served completely from the inventory. In Chapter 3 we saw how this was done for the model without a threshold and for $\omega(x) = ax$ and in the previous section we saw the results of our new model with a threshold.
The goal of this section is to learn something from this new model and to gain insight in the behaviour of it. Moreover, we will try to find out whether this revised model is an improvement, relative to the model without threshold, in certain scenarios.
We will do this by considering some numerical results by solving the system of equations (4.26), (4.28) and (4.32) using `Mathematica`'s solving algorithms. In order to do so, we will first explain the experiment we set up.

4.3.1 Experiment

As we want to compare the two models that were just mentioned, we first have to conclude which of all parameters are variable and which of those are fixed by the system. As the difference in the models only lies in the chosen function for $\omega(\cdot)$, it is reasonable to assume the λ and μ to be fixed by the system and the function for $\omega(\cdot)$ to be changeable. In our case, this means we are going to look at the impact of changing the values of ω_1, ω_2 and x_1 for specific λ and μ.
Now the remaining question is which quantities we should consider to be able to conclude whether the revised model is actually better. This brings us back to the scenarios described in Chapter 1. In most supply chain models, waiting time for a product to be delivered is a very important quantity. In familiar queueing terminology this would be called the 'sojourn time': the time from placing the order until actual delivery. This means it could be interesting to reduce the mean waiting time (and therefore the sojourn time), but probably even more interesting is to increase the probability of delivering straight from inventory. i.e. the probability that the sojourn time is zero. From now on we will denote this probability with $\mathbb{P}_I$.
So in the experiment, we will try to find out whether changing the threshold and ω_1 and ω_2 increases the probability of leaving the system immediately.
But just increasing this $\mathbb{P}_I$ is not challenging at all. Simply let ω go to zero and this probability will get to one if $\lambda < \mu$. However, doing so also implies the mean inventory level to go towards infinity, which is actually something we do not want. In the described scenarios namely, it was discussed that a very large inventory probably causes problems with either storage room or storage costs.
This brings us to the formulation of the experiment we would like to test:

"Can we increase $\mathbb{P}_I$ by varying the values of the parameters ω_1, ω_2 and x_1 without increasing the mean inventory level, compared to a standard case where ω is a fixed constant?"

4.3.2 Behaviour of System

To get a better feeling of the behaviour of both models, we will first discuss Figure 4.2 and Figure 4.3 . We fix $\mu = 4$ and take a look at the behaviour of the model as λ increases - and therefore $\rho \to 1$. For $\omega(\cdot)$ we consider:

- The standard model without threshold with $\omega = 3$ constant
- The model with threshold with $\omega_1 = 0.0001$, $\omega_2 = 8$ and $x_1 = 1$. These values are a bit arbitrarily chosen, but the main remark is that in this case the value for ω_1 is significantly lower than the standard ω, while the value of ω_2 is significantly higher.
- The model with threshold with $\omega_1 = 8$ $\omega_2 = 0.01$ and $x_1 = 1$. Also these values are a bit arbitrarily chosen, but the main remark is that in this case the value for ω_1 is significantly higher than the standard ω, while the value of ω_2 is significantly lower.

Then we get the following figures:

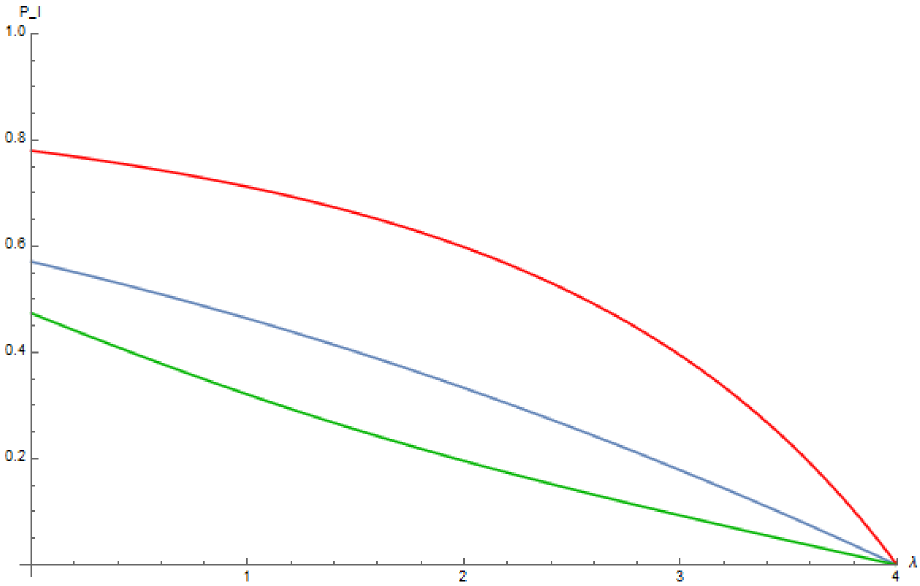

FIGURE 4.2: $\mathbb{P}_I$ for model, $\mu = 4$ fixed, without threshold ($\omega = 3$) and models with threshold ($\omega_1 = 0.0001$, $\omega_2 = 8$, $x_1 = 1$ and $\omega_1 = 8$, $\omega_2 = 0.01$, $x_1 = 1$)

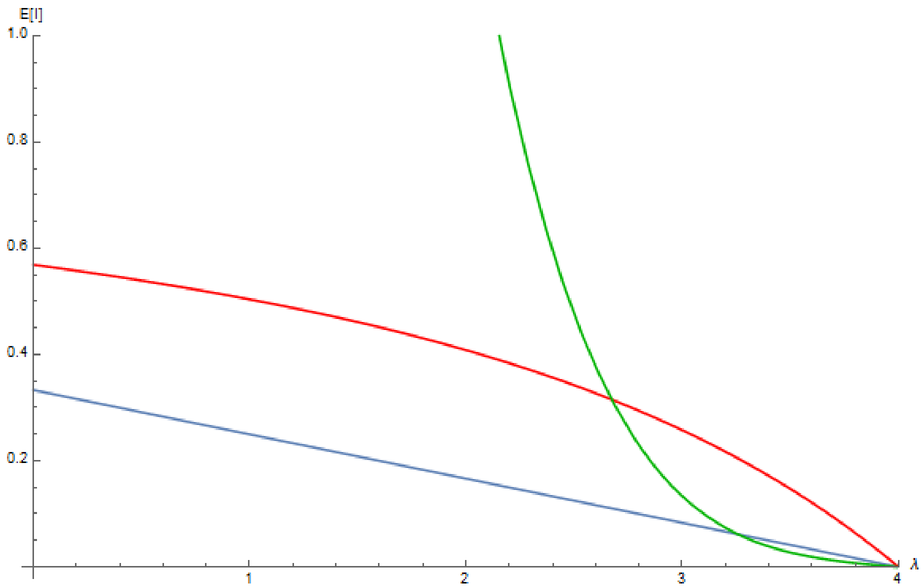

FIGURE 4.3: Mean Inventory for model, $\mu = 4$ fixed, without threshold ($\omega = 3$) and models with threshold ($\omega_1 = 0.0001, \omega_2 = 8, x_1 = 1$ and $\omega_1 = 8, \omega_2 = 0.01, x_1 = 1$)

Remark. *Although Figure 4.3 might suggest the green model is unstable for small values of λ, this is not the case. the value of $\mathbb{E}[I]$ increases very much as λ gets smaller, but it has a finite value for $\mathbb{E}[I]$ for all $\lambda \in [0, 4]$.*

At first, we see the green model does not improve the system at all. If we consider this system in a heuristic way, we also quickly conclude why this model is not an improvement.
In this system, we namely chose ω_1 to be quite large and we chose ω_2 to be smaller than the regular ω. In practise, this means that when we only have a small inventory, we are very willing to deplete it all, while when having a very large inventory, we are a lot less willing to do so. As this does not make a lot of sense, we will more consider the system with the values for ω_1 and ω_2 the other way around. This means a less likely depletion of the inventory when there is little in stock, and a more likely depletion when there is a lot in stock.
To answer the question of our experiment which we stated in the previous section, we want to find out whether there are values for $\omega_1 < \omega < \omega_2$ and x_1 such that the red line in Figure 4.2 stays above the blue line, while the red line in Figure 4.3 gets below or on the blue line.

4.3.3 Find Improving Model

In order to find such values, we first try to simplify the problem. This means we will just take values $\omega_1 = 0.0001$ and $\omega_2 = 8$ and let x_1 vary. This is not necessarily the optimal solution, but if we can show that

$$\forall_{0<\lambda<\mu} \ \exists_{x_1(\lambda)>0} \text{ such that } \mathbb{P}_I^{(2)} > \mathbb{P}_I^{(1)} \text{ and } \mathbb{E}[I]^{(2)} < \mathbb{E}[I]^{(1)},$$

- where $^{(2)}$ denotes the quantity of the model with threshold (red) and $^{(1)}$ denotes the one of the model without threshold (blue) - then there for sure is an improvement of the model.
To conclude whether this is true, we made a 3D plot (Figure 4.4) using `Mathematica`,

where we plot the functions

$$f(\lambda, x_1) = \mathbb{P}_I^{(2)} - \mathbb{P}_I^{(1)} \\ \text{and} \\ g(\lambda, x_1) = \mathbb{E}[I]^{(1)} - \mathbb{E}[I]^{(2)}. \tag{4.36}$$

Then the set of (λ, x_1) for which $f(\lambda, x_1) > 0$ and $g(\lambda, x_1) > 0$ shows us which x_1 is suitable at a certain arrival rate λ.

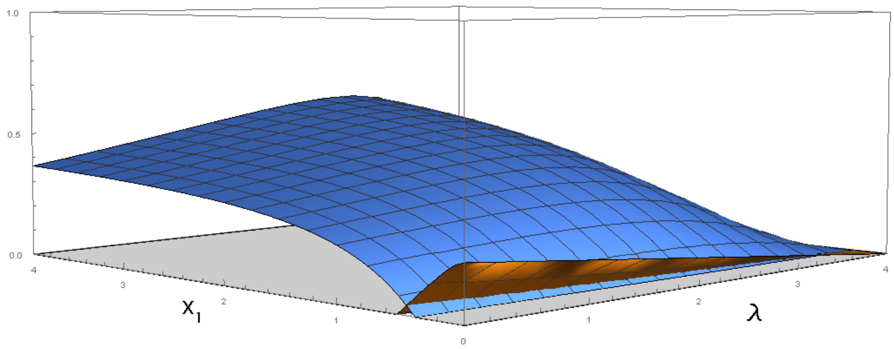

FIGURE 4.4: Functions $f(\lambda, x_1)$ and $g(\lambda, x_1)$

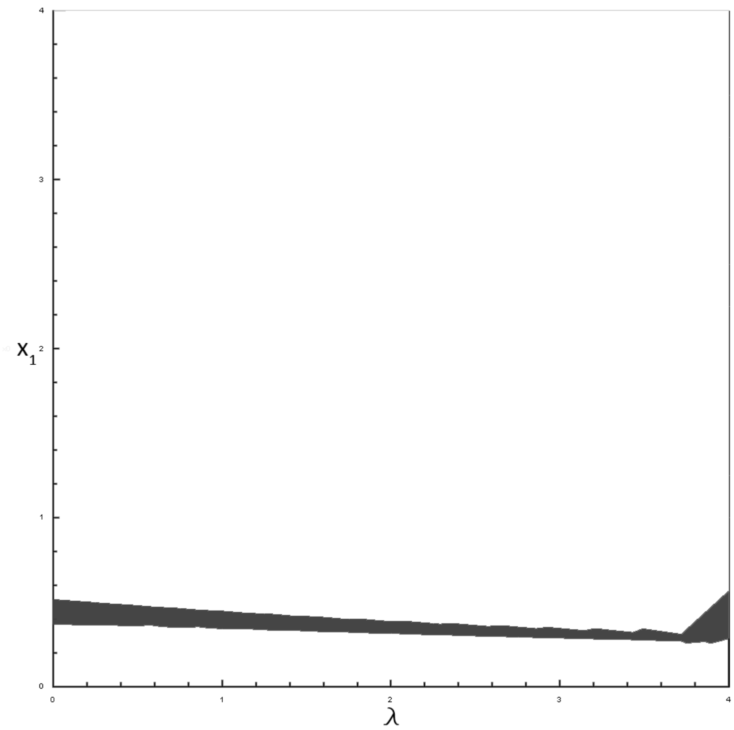

FIGURE 4.5: In gray the plane of (λ, x_1) for which $f(\lambda, x_1),\ g(\lambda, x_1) > 0$

In Figure 4.4 we see a plot of both functions $f(\lambda, x_1)$ and $g(\lambda, x_1)$. In this figure we see for which values of (λ, x_1) both functions are positive. in the most ideal case, this stroke of values would be perpendicular on the x_1-axis, as this would mean choosing the value of x_1 is independent from λ. However, this is not the case. This means, for different values of λ, an other value of x_1 should be chosen. Nevertheless, considering the value $x_1 = 0.41$ is almost a suitable value. Using this for plotting figures like in Figure 4.2 and Figure 4.3, we get a good sense of the improvement of the model. The figures with this value for x_1 then become:

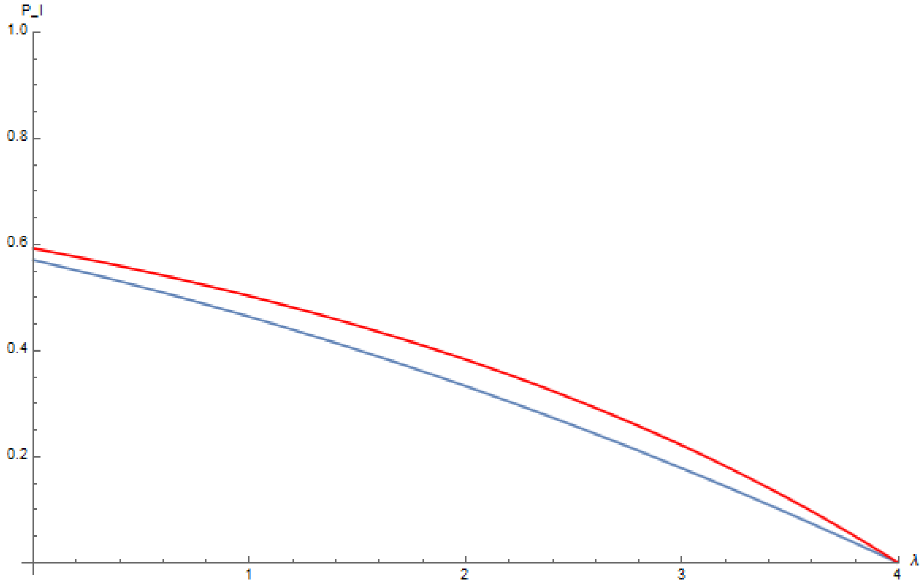

FIGURE 4.6: $\mathbb{P}_I$ for model, $\mu = 4$ fixed, without threshold ($\omega = 3$) and models with threshold ($\omega_1 = 0.0001$, $\omega_2 = 8$, $x_1 = 0.41$)

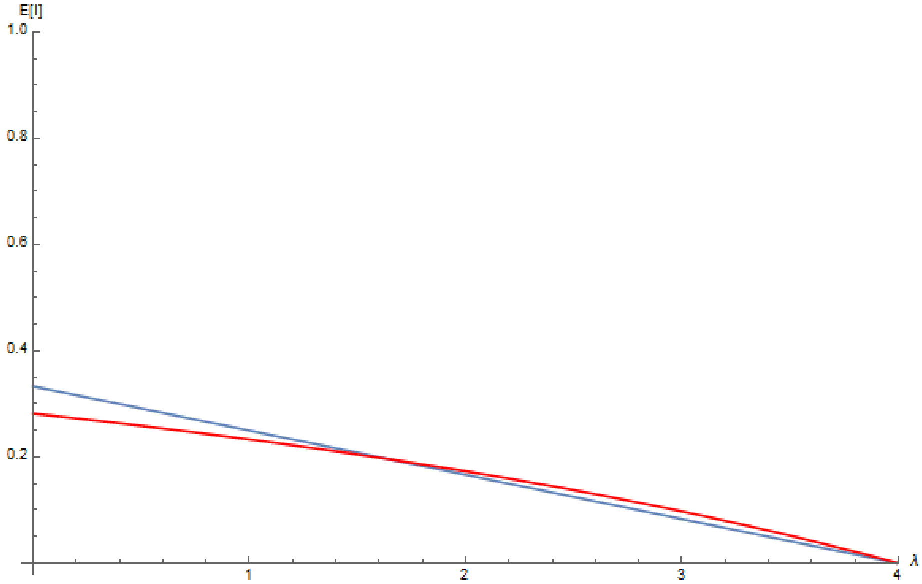

FIGURE 4.7: Mean Inventory for model, $\mu = 4$ fixed, without threshold ($\omega = 3$) and models with threshold ($\omega_1 = 0.0001$, $\omega_2 = 8$, $x_1 = 0.41$)

So we see the new model gives a small, but noticeable and still significant improvement of the system. However, one might wonder whether the system can also be

improved with different values of the constant ω, as we only showed there is an improvement compared to the standard case where $\omega = 3$.
To show this is the case, we will now consider the 3D plot where we fix $\lambda = 2$ and let the parameter ω vary (the values for ω_1, ω_2 and μ stay the same). We then get the following plot (Figure 4.8):

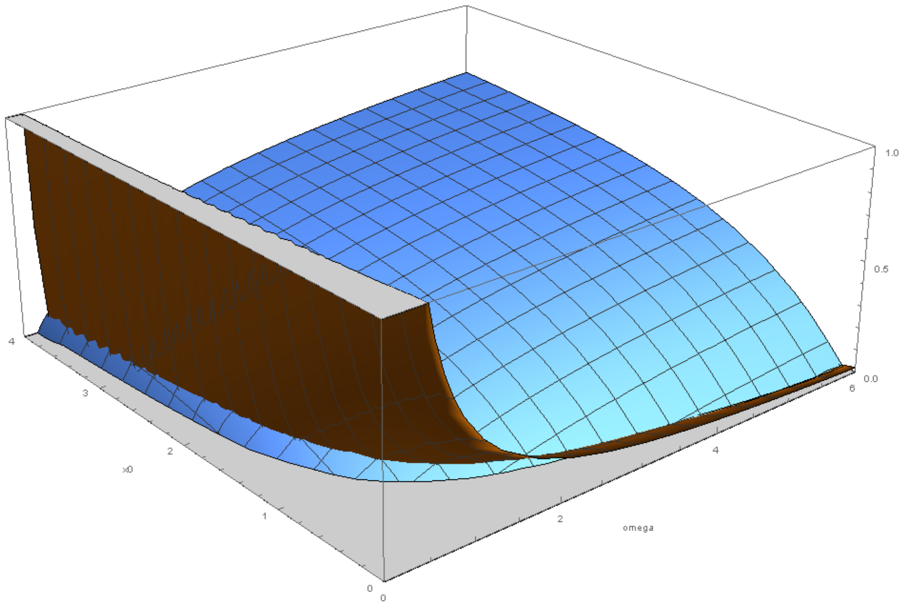

FIGURE 4.8: Functions $f(\omega, x_1)$ and $g(\omega, x_1)$ for $\lambda = 2$, $\mu = 4$, $\omega_1 = 0.0001$ and $\omega_2 = 9$.

In Figure 4.8 we see that for each value of ω, there is a value for x_1 such that both planes are greater than zero in that coordinate. This means it is very reasonable the improvement will work for all different kind of values for ω and λ.

Conclusion
So the conclusion and therefore answer to the question in the experiment is as follows. Yes, it is possible to vary the values of the parameters of ω_1, ω_2 and x_1, without increasing the mean inventory level, in order to increase $\mathbb{P}_I$ compared to a standard case where ω is a fixed constant.
Furthermore we gained insight in the behaviour of the new model, and learned how a good guesstimation can be made for the different values of the parameters in order to improve the system.
In the next paragraph, we will consider two again slightly different models related to our threshold model.

4.4 Different Models

4.4.1 Depletion Back to Threshold Level

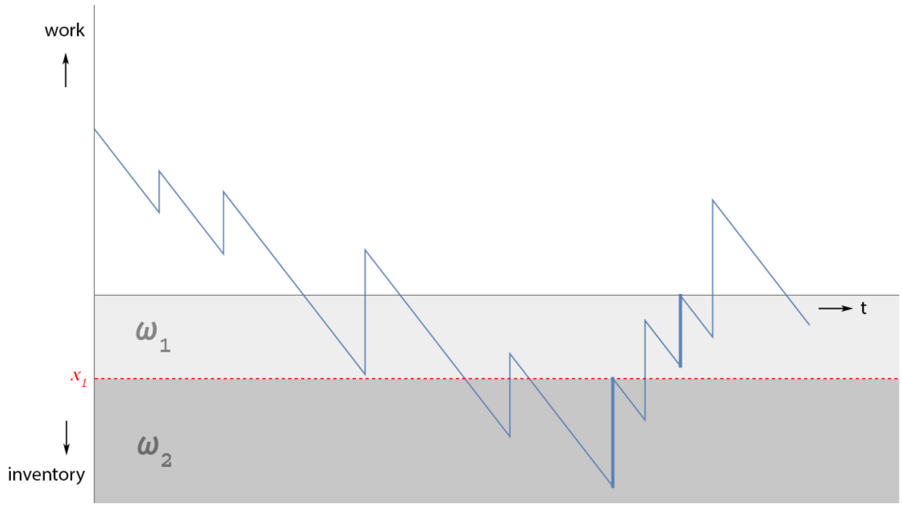

FIGURE 4.9: Work and Inventory Process with return to threshold level x_1

We have now considered a model with a threshold at which the depletion rate increased (or decreased). However, it might also be interesting to consider a model where above a certain inventory level x_1, the inventory is set back to level x_1 again (instead of zero) with rate ω_2. This scenario could be interesting for models with only a limited amount of stock capacity S. Then we could consider this new model with for example $x_1 = S$ and $\omega_2 = \infty$.
The main equations of this model are the same as of the model in section 4.1, namely equations (4.3), (4.4) and (4.5). Only for eq. (4.4), we get a different integration domain in the last term on the right hand side, so the equation becomes

$$v_{-1}(x) = \lambda \int_x^\infty \mathbb{P}(B > y - x) v_-(y)\mathrm{d}y + \int_x^{x_1} \omega(y) v_-(y)\mathrm{d}y, \quad 0 < x < x_1. \tag{4.37}$$

Now in a same way as in the previous section we get

$$\begin{aligned} v_{-1}(x) = \lambda e^{\mu x}\bigg(\int_x^{x_1} e^{-\mu y} v_{-1}(y)\mathrm{d}y + \int_{x_1}^\infty e^{-\mu y} v_{-2}(y)\mathrm{d}y\bigg) \\ + \int_x^{x_1} \omega(y) v_{-1}(y)\mathrm{d}y, \quad 0 < x < x_1, \end{aligned} \tag{4.38}$$

so

$$\begin{aligned} z_{-1}(x) = \lambda\bigg(\int_x^{x_1} z_{-1}(y)\mathrm{d}y + \int_{x_1}^\infty z_{-2}(y)\mathrm{d}y\bigg) \\ + e^{-\mu x}\int_x^{x_1} \omega_1 e^{\mu y} z_{-1}(y)\mathrm{d}y, \quad 0 < x < x_1. \end{aligned} \tag{4.39}$$

Now differentiating with respect to x gives us:

$$\begin{aligned} z'_{-1}(x) = -\lambda z_{-1}(x) - e^{-\mu x}\omega_1 e^{\mu x} z_{-1}(x) \\ - \mu e^{-\mu x}\int_x^{x_1} \omega_1 e^{\mu y} z_{-1}(y)\mathrm{d}y \end{aligned} \tag{4.40}$$

where we can substitute eq. (4.39) which gives us exactly the same equation as in eq. (4.20), which gives the same solution as in eq (4.23).
Following the same steps as in Section 4.2 we then get a system of equations consisting of

$$C_{1,1}e^{r_{1,1}x_1} + C_{1,2}e^{r_{1,2}x_1} = -\lambda\frac{C_{2,1}}{r_{2,1}}e^{r_{2,1}x_1}, \tag{4.41}$$

$$\begin{aligned} C_{1,1} + C_{1,2} = \lambda\left(\frac{C_{1,1}}{r_{1,1}}e^{r_{1,1}x_1} + \frac{C_{1,2}}{r_{1,2}}e^{r_{1,2}x_1} - \frac{C_{1,1}}{r_{1,1}} - \frac{C_{1,2}}{r_{1,2}}\right) \\ + \omega_1\left(\frac{C_{1,1}}{s_{1,1}}e^{s_{1,1}x_1} + \frac{C_{1,2}}{s_{1,2}}e^{s_{1,2}x_1} - \frac{C_{1,1}}{s_{1,1}} - \frac{C_{1,2}}{s_{1,2}}\right) \\ - \lambda\frac{C_{2,1}}{r_{2,1}}e^{r_{2,1}x_1}, \end{aligned} \tag{4.42}$$

and equation (4.32) with (4.30) substituted in it.

4.4.2 Numerical Results

Now we want to compare this model to the model we described in section 4.2. For our numerical results, we will again fix $\omega_1 = 0.0001$ and $\omega_2 = 8$. However, taking the same value for x_1 would not make sense. As in this model the dumps above the threshold level get back to the threshold level itself, instead of going all the way back to zero, we would expect the proper x_1 for this value to be lower than in the model of section 4.2. To determine a good value, we will again take a look at the functions defined in (4.36), where $^{(2)}$ again denotes the model with threshold, however this time with the jumps back to the threshold level instead of zero. We then get the following figures:

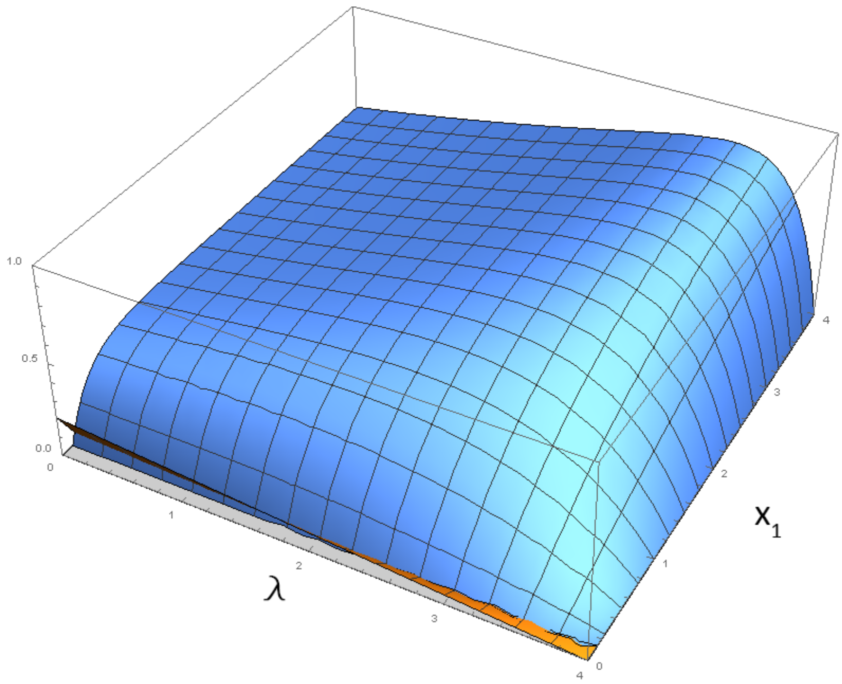

FIGURE 4.10: Functions $f(\lambda, x_1)$ and $g(\lambda, x_1)$

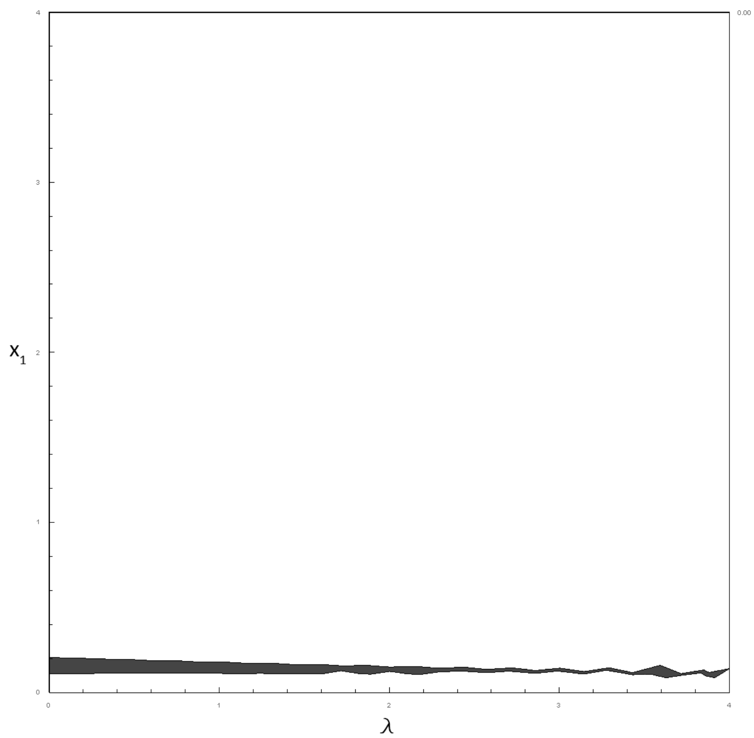

FIGURE 4.11: In gray the plane of (λ, x_1) for which $f(\lambda, x_1),\ g(\lambda, x_1) > 0$

From figure 4.11 we see a proper value for x_1 is around 0.18. Again, there is not one x_1 which suits for all λ, but fixing $x_1 = 0.18$, we get:

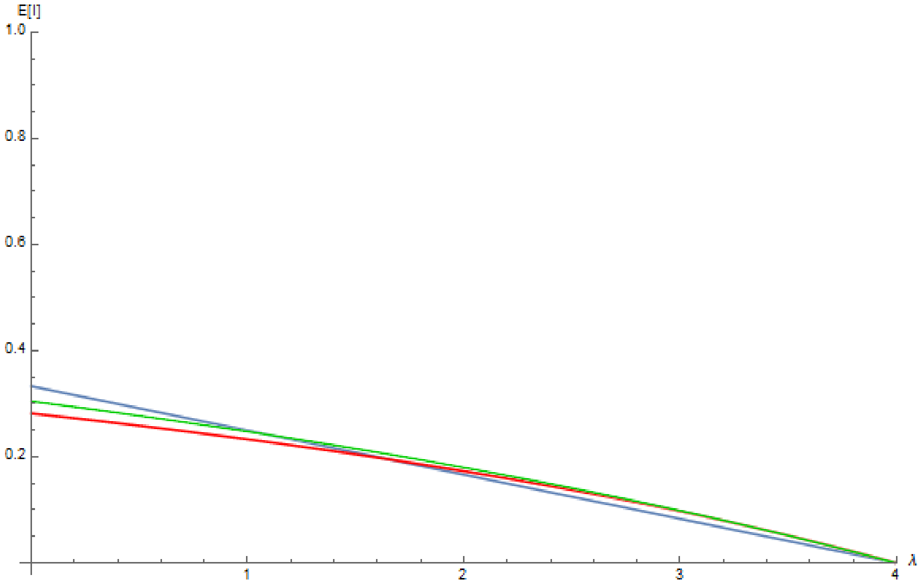

FIGURE 4.12: Mean Inventory of fig. 4.2 compared to with return to threshold level x_1 with ($\mu = 4, \omega_1 = 0.0001, \omega_2 = 8, x_1 = 0.18$)

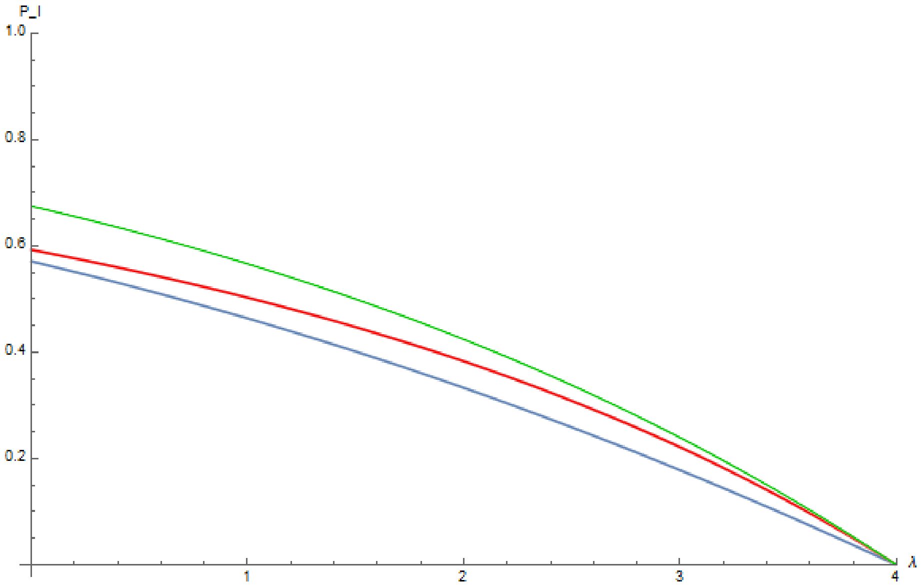

FIGURE 4.13: Probability of serving completely from inventory for model of fig. 4.3 compared to with return to threshold level x_1 with ($\mu = 4, \omega_1 = 0.0001, \omega_2 = 8, x_1 = 0.18$)

We see the mean inventory is almost the same as the other models, while the probability a job could be served directly from the inventory is a lot higher, mainly for small λ.
It is not surprising that for the determined objectives, this model works best of all. As the inventory depletion below x_1 does not go back all the way to zero, but just a bit higher, an arriving job is almost never larger than the present inventory. At the same time, the intensity of depleting the inventory is quite high above $x_1 = 0.25$. Therefore the inventory also almost never gets too large.
Of course one is not always free to choose their own model and one should take a look at the possibilities within the boundaries of their system. However, if someone

is free to choose their model, the model described in this section will probably be the best of the models we considered.

4.4.3 More Threshold levels

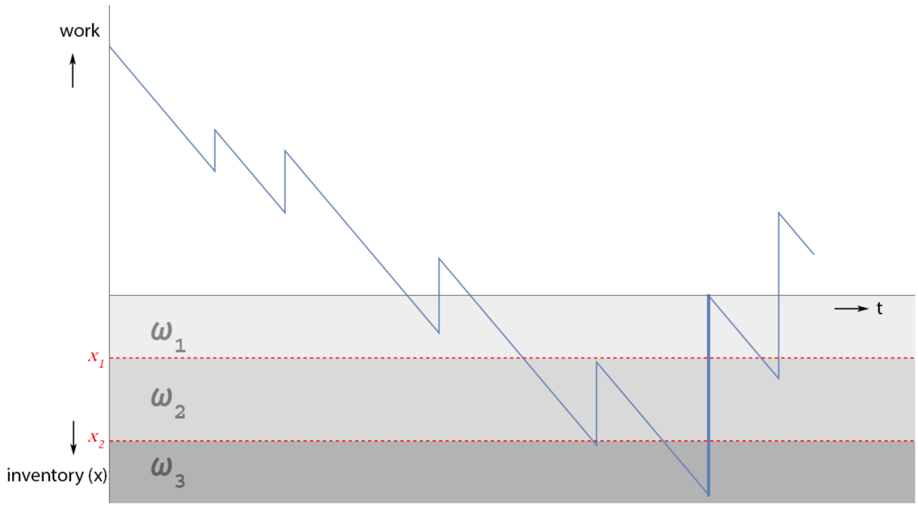

FIGURE 4.14: Work and Inventory Process with several Threshold levels

In Section 4.1 and 4.2 we considered our model with a certain threshold level. This level divides the inventory space in two areas with a different depletion rate. Of course it could also be interesting to consider a model with multiple threshold levels, which divides the inventory space in multiple areas with different depletion rate. In figure 4.14 a graphic example is given for two different threshold levels. In this subsection, we will show how to derive the system of equations from which one can determine quantities like the mean inventory, mean work, et cetera, like in the previous sections.

Model Description

First, we assume there are n threshold levels x_j (where $x_1 < x_2 < \cdots < x_n$) and therefore depletion rates ω_i for $i \in \{1, \ldots, n+1\}$. The same way as in previous sections we can write down the main equations, using the definitions

$$\omega(x) := \mathbb{1}_{\{0 \le x < x_1\}} \omega_1 + \mathbb{1}_{\{x_1 \le x < x_2\}} \omega_2 + \cdots + \mathbb{1}_{\{x \ge x_n\}} \omega_{n+1} \tag{4.43}$$

and

$$v_-(x) := \mathbb{1}_{\{0 \le x < x_1\}} v_{-1}(x) + \mathbb{1}_{\{x_1 \le x < x_2\}} v_{-2}(x) + \cdots + \mathbb{1}_{\{x \ge x_n\}} v_{-n-1}(x). \tag{4.44}$$

Moreover we define $x_0 := 0$ and $x_{n+1} := \infty$, then we get our main equations

$$v_+(x) = \lambda \int_0^x \mathbb{P}(B > x - y) v_+(y) \mathrm{d}y + \lambda \int_0^\infty \mathbb{P}(B > x + y) v_-(y) \mathrm{d}y, \quad x > 0, \tag{4.45}$$

and

$$v_{-i}(x) = \lambda \int_x^\infty \mathbb{P}(B > y - x) v_-(y) \mathrm{d}y + \int_x^\infty \omega(y) v_-(y) \mathrm{d}y, \quad x_{i-1} < x < x_i, \\ \text{for } i \in \{1, \dots, n+1\}. \tag{4.46}$$

As we assumed the services to be exponentially distributed and using the same definition as in previous sections for $z_+(x) = e^{\mu x} v_+(x)$ and $z_{-i}(x) = e^{-\mu x} v_{-i}(x)$ where $i \in \{1, \dots, n+1\}$, we can write

$$z_+(x) = \lambda \int_0^x z_+(y) \mathrm{d}y + \lambda \int_0^\infty z_-(y) \mathrm{d}y, \quad x > 0, \tag{4.47}$$

and

$$\begin{aligned} z_{-i}(x) = \lambda \Bigg(&\int_x^{x_i} z_{-i}(y) \mathrm{d}y + \sum_{j=i}^{n} \int_{x_j}^{x_{j+1}} z_{-j-1}(y) \mathrm{d}y \Bigg) \\ &+ e^{-\mu x} \Bigg(\int_x^{x_i} \omega_i e^{\mu y} z_{-i}(y) \mathrm{d}y + \sum_{j=i}^{n} \int_{x_j}^{x_{j+1}} \omega_{j+1} e^{\mu y} z_{-j-1}(y) \mathrm{d}y \Bigg) \\ &\text{for} \quad x_{i-1} \leq x \leq x_i \quad i \in \{1, \dots, n+1\}. \end{aligned} \tag{4.48}$$

$\mathbf{v_+(x)}$
Now as this $z_+(x)$ is exactly the same as in equation (3.5), we also get for $v_+(x)$ the same expression as in equation (3.12).

$\mathbf{v_{-i}(x)}$
Now by differentiating equation (4.48) two times and solving the differential equation, we get - in an analogue way as from equations (4.17) till (4.23) -

$$z_{-i}(x) = C_{i,1} e^{r_{i,1} x} + C_{i,2} e^{r_{i,2} x}, \quad i \in \{1, \dots, n+1\}, \tag{4.49}$$

where, following the same arguments as in equation (3.29), we know $C_{n+1,2} = 0$ and where

$$\begin{aligned} r_{i,1} &:= \frac{-(\lambda + \omega_i + \mu) - \sqrt{(\lambda + \omega_i + \mu)^2 - 4\lambda\mu}}{2}, \\ r_{i,2} &:= \frac{-(\lambda + \omega_i + \mu) + \sqrt{(\lambda + \omega_i + \mu)^2 - 4\lambda\mu}}{2}, \\ &\text{for } i \in \{1, \dots, n+1\}. \end{aligned} \tag{4.50}$$

Now to create the system of equations like before, we will substitute (4.49) in (4.48) and, like we did in equations (4.26) and (4.27), we will let $x \to x_i$ and $x \to x_{i-1}$ for all $i \in \{1, \dots, n\}$ (so not for $z_{-(i+1)}$). We then get equations

$$\begin{aligned} C_{i,1} e^{r_{i,1} \cdot x_i} + C_{i,2} e^{r_{i,2} \cdot x_i} = \lambda &\sum_{j=i}^{n} \int_{x_j}^{x_{j+1}} C_{j+1,1} e^{r_{j+1,1} \cdot y} + C_{j+1,2} e^{r_{j+1,2} \cdot y} \mathrm{d}y \\ &+ e^{-\mu x_i} \sum_{j=i}^{n} \omega_{j+1} \int_{x_j}^{x_{j+1}} C_{j+1,1} e^{s_{j+1,1} \cdot y} + C_{j+1,2} e^{s_{j+1,2} \cdot y} \mathrm{d}y, \end{aligned} \tag{4.51}$$

and

$$
\begin{aligned}
C_{i,1}e^{r_{i,1}\cdot x_{i-1}} + C_{i,2}e^{r_{i,2}\cdot x_{i-1}} &= \lambda \int_{x_{i-1}}^{x_i} C_{i,1}e^{r_{i,1}\cdot y} + C_{i,2}e^{r_{i,2}\cdot y}\mathrm{d}y \\
&+ e^{-\mu x_{i-1}} \int_{x_{i-1}}^{x_i} \omega_i C_{i,1}e^{s_{i,1}\cdot y} + C_{i,2}e^{s_{i,2}\cdot y}\mathrm{d}y \\
&+ \lambda \sum_{j=i}^{n} \int_{x_j}^{x_{j+1}} C_{j+1,1}e^{r_{j+1,1}\cdot y} + C_{j+1,2}e^{r_{j+1,2}\cdot y}\mathrm{d}y \\
&+ e^{-\mu x_{i-1}} \sum_{j=i}^{n} \omega_{j+1} \int_{x_j}^{x_{j+1}} C_{j+1,1}e^{s_{j+1,1}\cdot y} + C_{j+1,2}e^{s_{j+1,2}\cdot y}\mathrm{d}y,
\end{aligned}
\tag{4.52}
$$

where we define $s_{i,j}$ as we did in (4.11).

This now gives us $2n$ independent equations. Furthermore we use the fact that

$$
\begin{aligned}
C = \int_0^\infty z_-(x)dx &= \sum_{j=0}^{n} \int_{x_j}^{x_{j+1}} z_{-j-1}(y)\mathrm{d}y \\
&= \sum_{j=0}^{n} \int_{x_j}^{x_{j+1}} C_{j+1,1}e^{r_{j+1,1}\cdot y} + C_{j+1,2}e^{r_{j+1,2}\cdot y}\mathrm{d}y,
\end{aligned}
\tag{4.53}
$$

and the normalization

$$
\begin{aligned}
&\int_0^\infty v_+(x)dx + \sum_{j=0}^{n} \int_{x_j}^{x_{j+1}} v_{-j-1}(y)\mathrm{d}y = 1 \quad \text{, so} \\
&C\frac{\rho}{1-\rho} + \sum_{j=0}^{n} \int_{x_j}^{x_{j+1}} C_{j+1,1}e^{s_{j+1,1}\cdot y} + C_{j+1,2}e^{s_{j+1,2}\cdot y}(y)\mathrm{d}y = 1.
\end{aligned}
\tag{4.54}
$$

This gives in total a system of $2n+2$ equations and as we also have $2n+2$ unknown parameters, $\{C,\ C_{n+1,1},\ C_{i,j} \mid i \in \{1,\dots,n\},\ j \in \{1,2\}\}$, this means we can solve the system for all parameters. In a same way as in previous sections, one can then use the values of these parameters to determine the values of certain quantities like the mean inventory, mean work, probability of leaving the system without having to wait et cetera.

In this report, we will not discuss these any further, as we are not quite interested in the results for specific cases, but more in the behaviour of the model in general. For specific cases, one can simply follow the steps we used in Section 4.2 and Section 4.3.

Chapter 5

Simulation

In the previous chapter we obtained results from our model with one threshold value based on an M/M/1 system (with exponentially distributed service times). In article Boxma, Essifi, and Janssen, 2016 we see the model with a general distribution for the service times becomes mathematically very complex, even for $\omega(\cdot)$ constant. To make it easier analyzing our model with threshold for different service distributions, we therefore create a simulation which will give us a good impression of the behaviour of the model. This simulation can operate as a tool to create a good insight in your current process and can be used to optimize a certain profit. In this chapter, we will first explain the set up of the simulation and the way it calculates the values of the desired quantities. Then, in section 5.3 we will discuss some results of our simulation for different service distributions. In section 5.4 we will explain how a model with a linear growth of the depletion rate can be analyzed using the simulation. Finally, in Section 5.5, we will give an example of how the simulation can be used to tweak free parameters to improve a certain profit.

5.1 Simulation Setup

At first, we will categorize the events that can happen during the simulation. In Figure 5.1, a schematic version of the process is given. Here also all different events are indicated, which are important in our simulation.

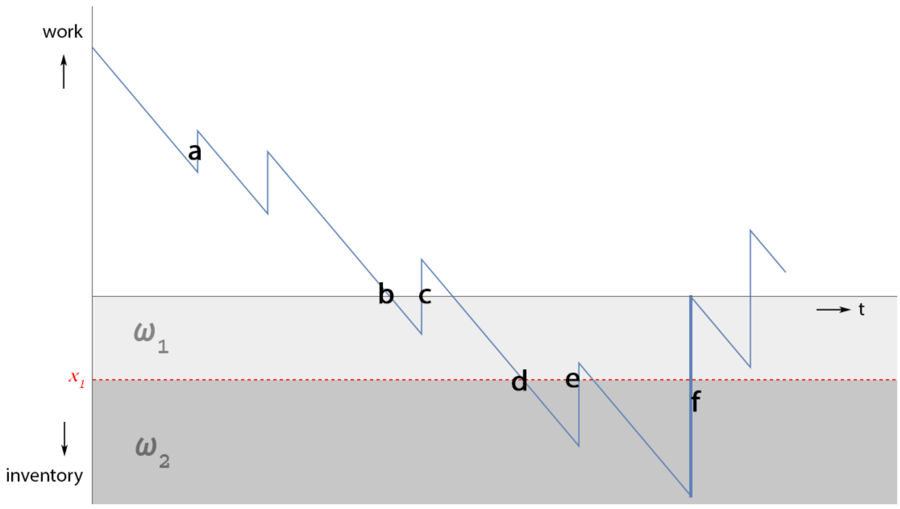

FIGURE 5.1

Corresponding Letter in Figure

a An arrival in the work phase.

b The system proceeds from the work phase to the inventory phase.

c An arrival in inventory phase which cannot be completely delivered from the inventory and therefore the process enters the work phase.

d The system proceeds from above the threshold level (where $\omega = \omega_1$) to below the threshold level (where $\omega = \omega_2$).

e The system proceeds from below the threshold level (where $\omega = \omega_2$) to above the threshold level (where $\omega = \omega_1$).

f An inventory dump takes place.

5.1.1 Algorithm of Simulation

Now, we will describe how we simulate using a pseudo-code. The algorithm of this simulation is based on the algorithms used in Van Ierland and Pastoor, 2017 and in Chu and Van Ierland, 2017.

Algorithm 1 Simulation

procedure SIMULATIONPROCEDURE
    work $\leftarrow 0$
    $x1 \leftarrow -$(threshold level) ▷ Threshold level is positive, so $x1$ is negative
    $T \leftarrow$ length of Simulation
    $ServDist \leftarrow$ Distribution of the Service heights
    $ArrDist \leftarrow$ Distribution of the inter-arrival times of customers
    $DumpDist1 \leftarrow$ Distribution of inter-arrival times of Inventory Dumps above Threshold ▷ Where $\omega(x) = \omega_1$
    $DumpDist2 \leftarrow$ Distribution of inter-arrival times of Inventory Dumps below Threshold ▷ Where $\omega(x) = \omega_2$
    Construct $Event\ set$ ▷ consisting of: **Arrival**: new customer in system, **Inventory**: crossing the zero line from work to inventory, **Thershold**: crossing the threshold level from above (where $\omega(x) = \omega_1$) to below (where $\omega(x) = \omega_2$) **Empty**: inventory dump

top:
    $d = ArrDist.next.Random$
    $new.Event.Arrival(d)$
    $d = DumpDist1.next.Random$
    $new.Event.Empty(d)$
    $new.Event.Threshold(x1)$
    $t = 0$
    $t_{old} = 0$

```

loop: WHILE(t < T):
    Get next.Event
    t = time of Event

    if next.Event.type == ARRIVAL then
        Event.remove(Type == Inventory||Type == Threshold)    ▷ Remove all entries of type 'Inventory' and 'Threshold'
        add = ServDist.next.Random
        work = work − (t − t_old) + add
        d = ArrDist.next.Random
        new.Event.Arrival(d + t)
        if 0 < work < d then
            new.Event.Inventory(work + t)
        if x1 < work < d + x1 then
            new.Event.Threshold(work − x1 + t)
        if (work + (t − t_old) − add < x1) AND (x1 < work < 0) then    ▷ Jump from below threshold level to above in inventory
            Event.remove(Type == Empty)    ▷ Remove all entries of type 'Empty'
            d = DumpDist1.next.Random
            new.Event.Empty(d + t)
        else if (work + (t − t_old) − add < 0) AND (work > 0) then    ▷ Jump from Inventory to Work
            Event.remove(Type == Empty)    ▷ Remove all entries of type 'Empty'
        t_old = t

    else if next.Event.type == INVENTORY then
        Event.remove(Type == Empty)    ▷ Remove all entries of type 'Empty'
        d = DumpDist1.next.Random
        new.Event.Empty(d + t)

    else if next.Event.type == THRESHOLD then
        Event.remove(Type == Empty)    ▷ Remove all entries of type 'Empty'
        d = DumpDist2.next.Random
        new.Event.Empty(d + t)

    else if next.Event.type = EMPTY then
        Event.remove(Type == Inventory||Type == Threshold)    ▷ Remove all entries of type 'Inventory' and 'Threshold'
        work = 0
        t_old = t
        d = DumpDist1.next.Random
        new.Event.Empty(d + t)
        new.Event.Threshold(t − x1)

    goto loop.
```

5.2 Calculating Results from Simulation

We are interested in the same kind of results as obtained before. We will discuss the way of calculating these result by result.

5.2.1 Mean of Work Level

The way of calculating the mean of the work level ($\mathbb{E}(X)$) is done by summing the mean of the work at all diagonal edges above zero. So given the diagonal edge (above zero) is from (x_i, y_i) to (x_i+p_i, y_i-p_i), then we add $p_i \cdot \frac{2y_i-p_i}{2}$ to the summation. At the end, we divide this sum over the total time. So, let there be n diagonal edges, then

$$\mathbb{E}[X] = \frac{\sum_{i=1}^{n}(p_i \cdot \frac{2y_i-p_i}{2}|y>0)}{T} \qquad \text{, where } T \text{ is the total running time.} \tag{5.1}$$

5.2.2 Mean of Inventory

This is calculated with the same technique as we calculated the Mean of work level. We again sum over the means of the inventory at all diagonal edges, but now the parts that are completely negative. Then we take the absolute value of it, and divide the sum over the total time. So

$$\mathbb{E}[I] = \frac{\sum_{i=1}^{n}(p_i \cdot \frac{p_i-2y_i}{2}|y<0)}{T} \qquad \text{, where } T \text{ is the total running time.} \tag{5.2}$$

5.2.3 Part of the Time the Inventory Is Empty (p_{empty})

For this we keep track of occasions c and e. We count the moments between getting above the zero axis and getting below. We sum over this time and eventually divide it over the total running time. So

$$p_{\text{empty}} = \frac{\int_0^T \mathbb{1}_{y>0}dy}{T} \qquad \text{, where } T \text{ is the total running time.} \tag{5.3}$$

5.2.4 Part of the Customers That Can Leave the System Without Having to Wait(p_{leave})

Here we need occasion a. We keep track of the amount of times an order can be delivered from the inventory completely and divide it over the total amount of times of arrivals. So let there be a total of n orders of size S_i, $1 < i < n$. And let a job service start at times $x_i + p_i$ with their corresponding work / inventory level $y_i - p_i$ (as described in 5.2.1), then

$$p_{\text{leave}} = \frac{\sum_{i=0}^{n} \mathbb{1}_{S_i+y_i-p_i<0}}{n}. \tag{5.4}$$

Remark. *Note that to obtain correct results of the quantities, the system should be stable. Therefore, $\rho = \lambda\mathbb{E}[B] < 1$ - where $\frac{1}{\lambda}$ is the mean of the arrival distribution and B is the random variable for the job sizes - and $\omega(x) > 0$ for all $x > \hat{x}$, where $0 < \hat{x} < \infty$ is a certain inventory level.*

Remark. *Note that in both the simulation and the calculations of the quantities, nowhere an assumption is made about the distribution of the interarrival times. Therefore - as long as the* *system is stable - one can also use the simulation to analyze the G/G/1 queueing inventory system.*

Remark. *To ensure the results from the simulation are a good representation of the true value, it is important to have a proper length of the simulation, or to do multiple shorter simulations and take the average of these values. In Boon et al., 2017 this is eleborated in detail. For our simulations, we used a simulation length of* 10^6*, which is good as long as the parameters for the dump distribution and arrival distribution are not chosen too small. i.e. not smaller than 0.1.*

5.3 Results with Different Service Distributions

So now we have a simulation which we can use to analyze our model with different service distributions. In this section we will shortly present results obtained with two of these different service distributions and compare these to the results of Figure 4.6.
To ensure the models are comparable, we consider two distributions of the services (B) for which $\mathbb{E}[B] = \mathbb{E}[\hat{B}] = \frac{1}{4}$ where $\hat{B} \sim Exp(4)$ as used in the previous Chapter. From these distributions, one will have a relatively small variation coefficient - compared to $\hat{B}$ - and one a relatively large variation coefficient. Therefore we will analyze the model with an Erlang distribution and a hyperexponential distribution. Suitable parameters for the latter one we determined using theory from Adan and Resing, 2015.The two service distributions will therefore:

$$
\begin{aligned}
&B_1 \sim E_4(16), && \textit{(Erlang distribution with shape 4 and rate 16)}\\
&B_2 \sim H_2[p_1, (1-p_1); 8p_1, 8(1-p_1)], && \textit{(Hyperexponential distribution } H_k(p_1, \ldots, p_k; \mu_1, \ldots, \mu_k)\textit{)}\\
& && \text{where we use } p_1 = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{5}{13}}.
\end{aligned}
$$

We then calculate the values of the mean inventory $\mathbb{E}[I]$ and the probability that a job can be fully served from the inventory $\mathbb{P}_I$ with the same settings as in Figure 4.6, varying λ from 0 to 4 with step size 0.1. This results in the following figure.

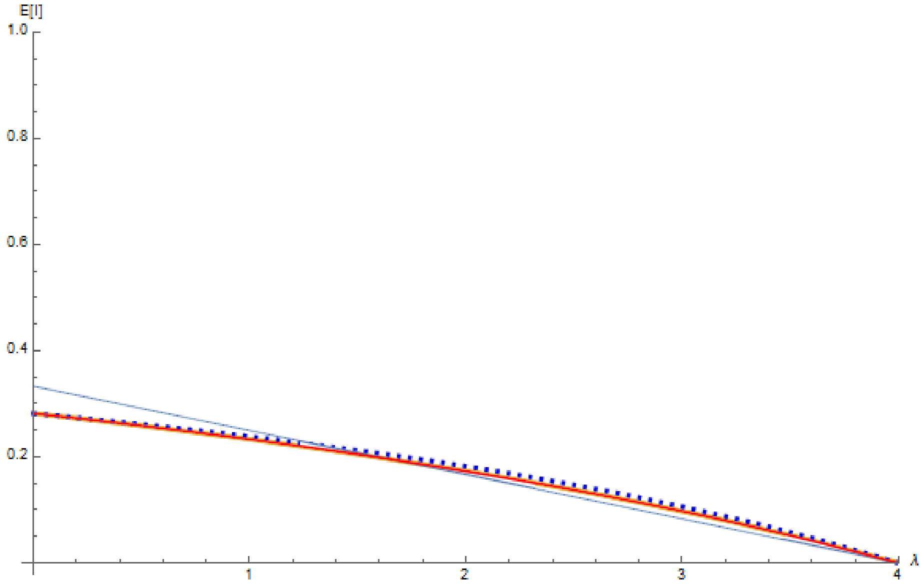

FIGURE 5.2: $\mathbb{E}[I]$ for model, $\mu = 4$ fixed, without threshold ($\omega = 3$) and models with threshold ($\omega_1 = 0.0001$, $\omega_2 = 8$, $x_1 = 0.41$) for services B: $Exp(4)$, $H_2[p_1, (1-p_1); 8p_1, 8(1-p_1)]$, with $p_1 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{5}{13}}$, and $E_4(16)$ (dashed).

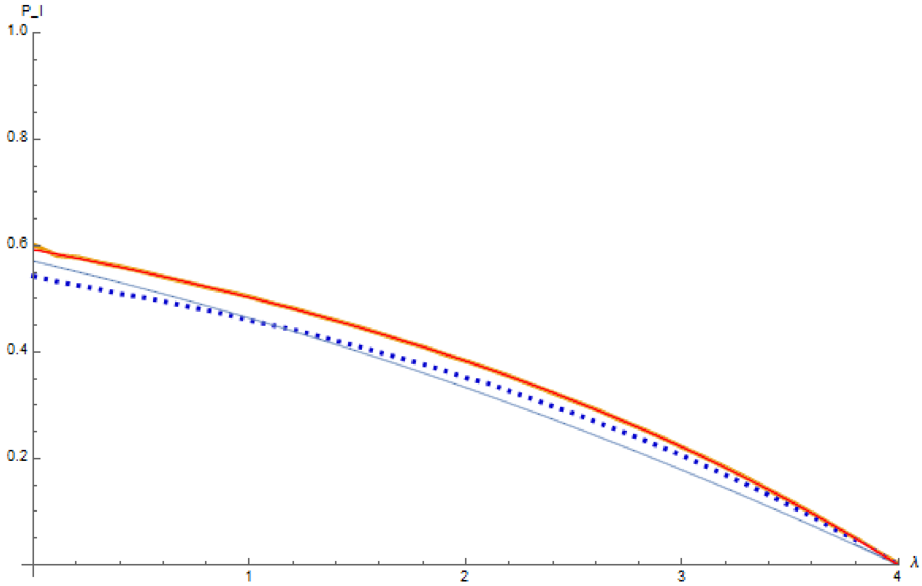

FIGURE 5.3: $\mathbb{P}_I$ for model, $\mu = 4$ fixed, without threshold ($\omega = 3$) and models with threshold ($\omega_1 = 0.0001$, $\omega_2 = 8$, $x_1 = 0.41$) for services B: $Exp(4)$, $H_2[p_1, (1-p_1); 8p_1, 8(1-p_1)]$, with $p_1 = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{5}{13}}$, and $E_4(16)$ (dashed).

First, we take a look at Figure 5.2 which shows the mean inventory for varying λ. It seems remarkable that the lines of all three threshold models (Exponential, hyperexponential and Erlang) are (almost) coincident. A logical explanation for this could be the fact that the means of the services are all the same results in an equal mean inventory for all distributions.

Now we consider Figure 5.3. Here, it is remarkable that the $\mathbb{P}_I$ for the Erlang distributed service times is significantly lower than the $\mathbb{P}_I$ of the Exponentially and hyperexponentially distributed service times, of which the latter two almost coincide. An explanation for this could be the fact that the cumulative distribution function of an exponentially distributed random variable climbs way faster in the beginning. i.e. the probability an hyperexponetial or exponentially distributed random variable is 'very small' is way higher than for the Erlang distribution.
Consider the CDFs of the exponential, Erlang and hyperexponential distribution

$$F_{\hat{B}}(x) = \mathbb{P}(\hat{B} \leq x) = 1 - e^{-4x},$$
$$F_{B_1}(x) = \mathbb{P}(B_1 \leq x) = \frac{\gamma(4, 16x)}{6}, \qquad \text{where } \gamma(\cdot,\cdot) \text{ is the incomplete gamma function,}$$
$$F_{B_2}(x) = \mathbb{P}(B_2 \leq x) = p_1 \cdot (1 - e^{-8p_1 \cdot x}) + (1 - p_1) \cdot (1 - e^{-8x(1-p_1)}).$$

Now let the inventory level be 0.2, then the probability the size of a job is below this level for all three distributions:

$$F_{\hat{B}}(0.2) \approx 0.551$$
$$F_{B_1}(0.2) \approx 0.397$$
$$F_{B_2}(0.2) \approx 0.638$$

Now, from Figure 5.2 with the mean inventory we can conclude the inventory level is usually quite small, which might explain why the $\mathbb{P}_I$ of the Erlang services are lower than the one of the Exponential services. Considering the value of $F_{B_2}(0.2)$, we would then also expect the $\mathbb{P}_I$ of the hyperexponentially distributed service times to be larger than the one of the exponentially distributed services. However, this is not the case. We could not find an explanation for this, so this might be an interesting topic for further research.

5.4 $\omega(x) = ax$

In this section, we consider the case where $\omega(x) = ax$. Our objective is to find a distribution for the time until the inventory dump takes place, given a certain value of the amount of inventory.
We want to determine this distribution by constructing a differential equation to eventually solve it.
First we write T_y as a random variable for the time until the next inventory dump will come, starting from inventory level y. Now we write

$$\mathbb{P}(T_y > x + \delta) = \mathbb{P}(T_y > x) \cdot \mathbb{P}(T_y \notin (x, x + \delta)|T_y > x), \tag{5.5}$$

where we know for δ arbitrarily small,

$$\mathbb{P}(T_y \notin (x, x + \delta)) \approx 1 - a(y + x)\delta \tag{5.6}$$

since for a random variable $X \sim Exp(\lambda)$, the probability that X is in a small interval of length λ is equal to $\lambda \cdot \delta$. Rewriting (5.5) then gives

$$\mathbb{P}(T_y > x + \delta) - \mathbb{P}(T_y > x) \approx \mathbb{P}(T_y > x) \cdot a(y + x)\delta. \tag{5.7}$$

Dividing by δ then gives

$$\frac{\mathbb{P}(T_y > x + \delta) - \mathbb{P}(T_y > x)}{\delta} \approx \mathbb{P}(T_y > x) \cdot a(y + x). \tag{5.8}$$

For $\delta \to 0$, this shows on the left side the derivative with respect to x of $\mathbb{P}(T_y > x)$. So we will write

$$f(x, y) := \mathbb{P}(T_y > x), \tag{5.9}$$

and get

$$\frac{\partial}{\partial x} f(x, y) = -a(y + x) f(x, y). \tag{5.10}$$

Now we know from analysis of differential equations that our solution is of the form

$$f(x, y) = e^{\int -a(y+x)dx} = c \cdot e^{-ayx - \frac{a}{2}x^2}, \tag{5.11}$$

where c is a constant. So we find

$$\mathbb{P}(T_y > x) = c \cdot e^{-ayx - \frac{a}{2}x^2}. \tag{5.12}$$

As we know $\mathbb{P}(T_y > 0) = 1$, we find

$$c \cdot e^{0-0} = 1,$$

so we see $c = 1$ and therefore

$$\mathbb{P}(T_y > x) = e^{-ayx - \frac{a}{2}x^2}. \tag{5.13}$$

So we found a cumulative distribution function for T_y, but to generate a realization of this random variable, we first have to find its inverse.
First we show this CDF has an inverse with respect to x for all $x > 0$ and all $y > 0$ by showing the function is bijective.
For convenience we take the logarithmic function, which is allowed since this is a monotonic increasing function. So

$$\log(\mathbb{P}(T_y > x)) = -ayx - \frac{a}{2}x^2, \tag{5.14}$$

where we see a second order polynomial on the right hand side. We know the top of this function is at

$$x_{top} = \frac{-(-ay)}{2 \cdot \frac{-a}{2}} = -y. \tag{5.15}$$

So the top of the parabola is on the negative half plane for $y > 0$, so the function is a smooth an bijective one and therefore invertible.
Now we want to find the inverse. We write

$$p = e^{-ayx - \frac{a}{2}x^2}, \tag{5.16}$$

so

$$\log(p) = -ayx - \frac{a}{2}x^2, \tag{5.17}$$

then

$$\frac{a}{2}x^2 + ayx + \log(p) = 0, \tag{5.18}$$

so

$$x_{1,2} = \frac{-ay \pm \sqrt{(ay)^2 - 4 \cdot \frac{a}{2} \cdot \log(p)}}{a}, \tag{5.19}$$

where $x_2 < 0$. Furthermore we know, as $p = \mathbb{P}(T_y > x)$, that $0 \leq p \leq 1$. So our inverse is given by

$$x_1 = -y + \sqrt{y^2 - \frac{2}{a} \cdot \log(p)} \qquad \text{, where } 0 \leq p \leq 1. \tag{5.20}$$

Now we use our known random number generator - for the uniform distribution - to generate a p between zero and one, substitute this p in (5.20) and then find x which is a realization of our random variable.

Remark. *Note that we have to change the simulation algorithm a bit to use this distribution for the interarrival times of inventory dumps. In this case, we have to reschedule the event of depleting at every job arrival.*

5.4.1 Results

Now using the theory from the previous section, we can use this in our simulation to analyze the behaviour of the model where $\omega(x) = ax$. For several values of a, we simulated the value of $\mathbb{E}[I]$ and $\mathbb{P}_I$ for variable λ. In Figure 5.4 and Figure 5.5 we find a plot for $a = 6$ and $a = 7$, as these turn out to be comparable with the outcomes in Figure 4.3 and Figure 4.4. i.e. an $\mathbb{E}[I]^{(2)}$ that is not significantly larger than the $\mathbb{E}[I]^{(1)}$ of the model[(1)] with constant ω and without threshold, and a $\mathbb{P}_I^{(2)}$ which is larger than the $\mathbb{P}_I^{(1)}$ of that model[(1)]. To see the change in behaviour of the model as a changes, we also included the lines for $a = 3$ and $a = 12$ in Figure 5.4 and Figure 5.5.

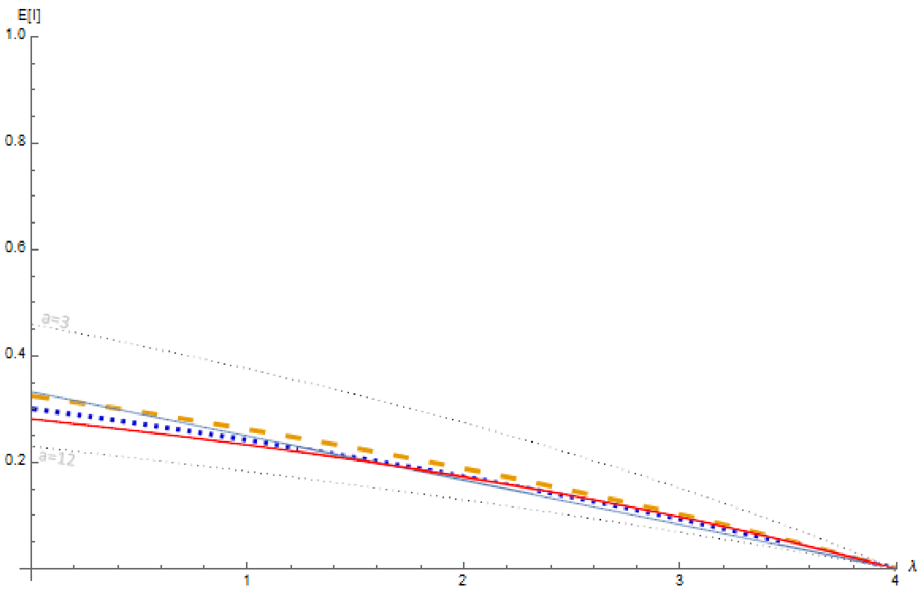

FIGURE 5.4: $\mathbb{E}[I]$ for model, $\mu = 4$ fixed, without threshold ($\omega = 3$) and models with threshold ($\omega_1 = 0.0001$, $\omega_2 = 8$, $x_1 = 0.41$), compared to models with $\omega = ax$ where: $a = 6$ (dashed) and $a = 7$ (dashed). $a = 3$ and $a = 12$ included for reference.

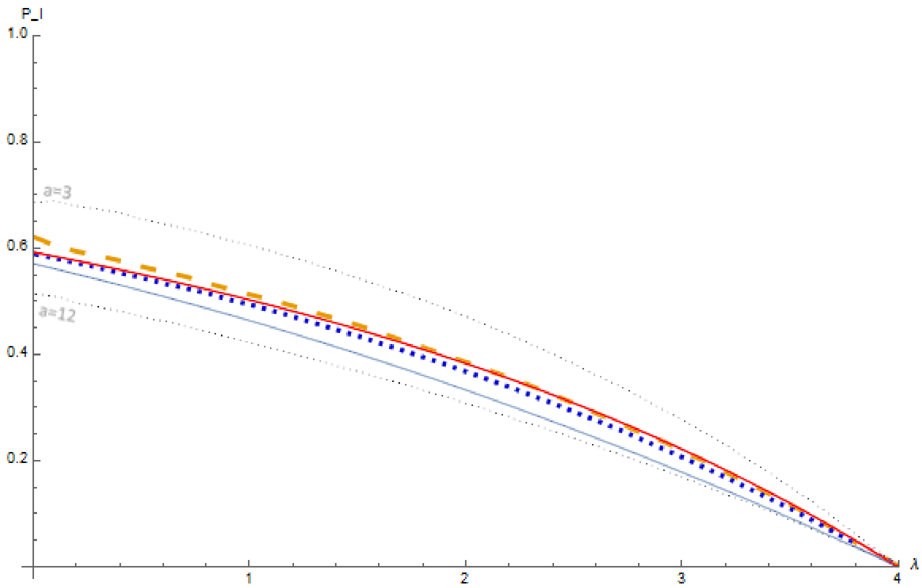

FIGURE 5.5: $\mathbb{P}_I$ for model, $\mu = 4$ fixed, without threshold ($\omega = 3$) and models with threshold ($\omega_1 = 0.0001$, $\omega_2 = 8$, $x_1 = 0.41$), compared to models with $\omega = ax$ where: $a = 6$ (dashed) and $a = 7$ (dashed). $a = 3$ and $a = 12$ included for reference

Now we consider Figure 5.4 and Figure 5.5. First we notice that the model with $a = 6$ gets almost the same result for $\mathbb{P}_I$ as the model with threshold, but the mean inventory is larger than both the model with threshold and the model$^{(1)}$ with constant ω (and no threshold). For the model with $a = 7$, we see the mean inventory is for small λ a little bit larger than the model with threshold and for larger λ just a little bit smaller than the model with threshold. However, the $\mathbb{P}_I$ remains for all λ just a bit smaller than the threshold model.
We can conclude that the model where $\omega(x) = ax$ works better - in terms of our experiment in 4.3.1 - than the model$^{(1)}$, but not just as good as the model with threshold.

It is not quite surprising that this model works better than model$^{(1)}$. In section 4.3 we learned that depleting the inventory with a relatively small rate for a small inventory level and a relatively large rate for a large inventory level works well. Just as the threshold model, also the model with $\omega(x) = ax$ increases its rate when the inventory level increases.

Using the simulation we made, one can also investigate the behaviour of a model with threshold and $\omega(x) = ax$, where the value of a changes at a certain threshold level. Unfortunately there was not enough time in this project to analyze these models in depth, but we have taken a look at a combination of the threshold parameters and the $\omega(x) = ax$ model. We then guesstimated the parameters for this model at: $a_1 = 2,\ a_2 = 15$ and we took $x_1 = 0.41$ like we did in Figure 4.3 and Figure 4.4. We then get:

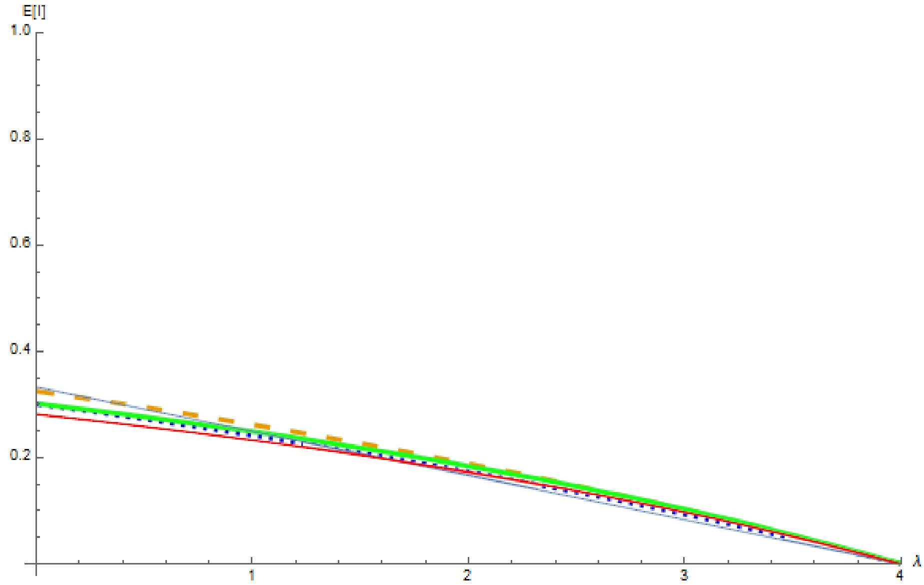

FIGURE 5.6: $\mathbb{E}[I]$ of figure 5.4, compared to model with $\omega(x) = ax$ and threshold for which a changes with $a_1 = 2$, $a_2 = 15$, $x_1 = 0.41$.

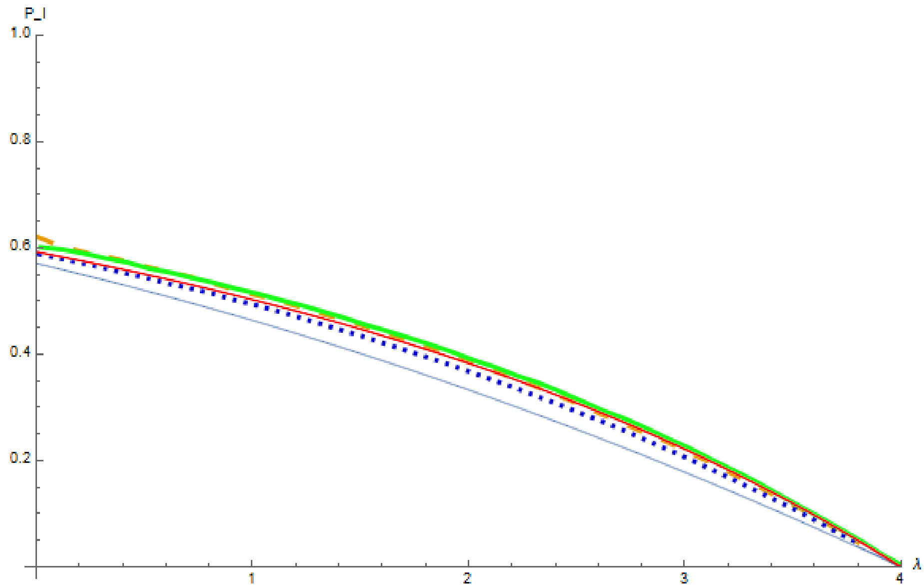

FIGURE 5.7: $\mathbb{P}_I$ of figure 5.5, compared to model with $\omega(x) = ax$ and threshold for which a changes with $a_1 = 2$, $a_2 = 15$, $x_1 = 0.41$.

We see in these figures that it improves relative to the model with a constant a. The $\mathbb{P}_I$ in this model is almost the same as the $\mathbb{P}_I$ of $a = 7$, while the $\mathbb{E}[I]$ is almost the same as the $\mathbb{E}[I]$ of $a = 6$.

Hypothesis for Further Research

Now it might be interesting to have some more research to this model, however our hypothesis is that the optimal model for this threshold for a will not be better than the optimal model for the threshold for ω. It would be reasonable that in our experiment, there is an optimum where one should deplete its inventory. Heuristically, it would not make sense to choose the level at which one depletes random, instead of deterministic on such an optimum. Therefore we think that in this experiment, the optimum is obtained at a certain $\hat{x_1}$ and $\omega_1 = 0$, $\omega_2 = \infty$. Hence, for the model with threshold for a, we would get this same $\hat{x_1}$, and $a_1 = 0$, $a_2 = \infty$.

5.5 Optimization Problem

In this section, we will discuss an optimization problem which can be solved using the simulation described in Section 5.1.
Let us assume the model of Paragraph 4.1, so the model with one threshold level. We are allowed to choose values for $x_1,\ \omega_1$ and ω_2, and we know $\lambda = 2$ and $\mu = 4$ are fixed. Now our goal is to gain the highest profit. We have three types of income:

- I_1 The income per product unit obtained by selling a job completely from the inventory,
- I_2 The income per product unit obtained when a job can not (fully) be served from the inventory (so there is a waiting time for the customer),
- I_3 The income per product unit obtained by selling via an inventory dump.

Furthermore we have the costs for the storage which is K per unit per time unit.

5.5.1 Setup of Optimization

So we have our simulation as described before. To solve the optimization problem, we will calculate the following quantities:

- D_1 The mean amount of product for jobs that are delivered completely from inventory, per time unit,
- D_2 The mean amount of product for jobs which cannot completely be delivered form the inventory, per time unit,
- D_3 The mean amount of product that is dumped, per time unit.
- $\mathbb{E}(I)$ The mean inventory level

Now given certain values for $x_1,\ \omega_1$ and ω_2, we can then easily determine the profit

$$\text{Profit } = -K \cdot \mathbb{E}(I) + \sum_{i=1}^{3} I_i \cdot D_i. \tag{5.21}$$

Now iterating over several values for these parameters $x_1,\ \omega_1$ and ω_2, calculating the profit, gives us a 4D matrix from which one can easily find the maximum Profit and their corresponding parameters.
These parameters then give you the best set up for you inventory depletion function $\omega(y)$, within the single threshold model.
In Table 5.1 we find the optimal values for our parameters $x_1,\ \omega_1$ and ω_2 for the model with exponentially distributed job sizes, $\lambda = 2$ and $\mu = 4$ and initial values for the incomes $(I_1, I_2, I_3) = \frac{1}{6}(3, 2, 1)$, where K is varying.
For obvious reasons, the obtained profit decreases as the costs increase. Now we take a look at the behaviour of our parameters. We first see that x_1 is, generally, decreasing, and ω_1 and ω_2 are increasing all the way from their minimum till their maximum value in the simulation. This behaviour is quite reasonable. As the costs

Exponentially distributed job sizes							
K	$\mathbf{x_1}$	ω_1	ω_2	$\widetilde{\mathbf{x_1}}_{\text{top 10}}$	$\widetilde{\omega_1}_{\text{top 10}}$	$\widetilde{\omega_2}_{\text{top 10}}$	**Profit**
0	5.6	0	0.5	5.75	0	0.5	0.324
0.02	2.4	0	12	2.4	0	12.75	0.288
0.04	1.6	0	16	1.6	0	14.5	0.272
0.06	1.1	0	16.5	1.1	0	11.5	0.261
0.08	0.8	0	15.5	0.8	0	14.75	0.253
0.1	5.2	6	2	0.8	3.5	11	0.248
0.12	0.1	6.5	15	0.1	6	16.25	0.247
0.14	0.1	6.5	17.5	0.1	6	17	0.247
0.16	0.1	6.5	17.5	0.1	6	17	0.246
0.18	0.1	6.5	17.5	0.1	6.25	17	0.245
0.2	0.1	6.5	17.5	0.1	6.25	17	0.244

TABLE 5.1: Profit for the best values of parameters (where $0 < x_1 < 6$; $0 < \omega_1 < 6.5$ and $0.5 < \omega_2 < 17.5$) with certain value of k and fixed: $I_1 = \frac{1}{2}$, $I_2 = \frac{1}{3}$ and $I_3 = \frac{1}{6}$ for model with one threshold level x_1 and exponential services: $\lambda = 2$, $\mu = 4$.

increase, one wants to deplete more. However, as most profit is obtained from types D_1, first ω_2 is increased and x_1 decreased and only at last ω_1 is changed completely to its maximum.

Now we notice a strange row at $K = 0.1$. Here both x_1 and ω_2 do not follow the trend. To investigate what happens here, we have printed the columns $\widetilde{\cdot}_{\text{top 10}}$. These denote the medians of x_1, ω_1 and ω_2 from the top 10 highest profits (so the profits that are almost the same as the best profit). If we consider these columns, we see it is quite reasonable and no real outliers appear. The optimal set of parameters which is given for this $K = 0.1$ therefore very likely has an outlieing value of the profit. This is due to the fact that some parameters correspond with values for the expected profit that are very close to each other. In fact, the set of parameters $(x_1, \omega_1, \omega_2) = (5.2, 6, 2)$ is not that odd to have a good profit, as in practice this is almost a system with a constant depletion rate $\omega = 6$ (as the system will almost never cross the threshold level x_1). However, to obtain a set of parameters for a good profit value, it is recommended to take the median over the top parameter settings and to use these for your system.

So in Table 5.1 we considered the model based on the M/M/1 queueing model, but using the simulation one can obtain similar results for other job size distributions. In Table 5.2 we find the results when assuming $B \sim U[0, 0.5]$. In the table we see the profit for this uniform distribution is lower than for the exponentially distributed services. It is reasonable that we again notice the same trend for x_1, ω_1 and ω_2. However, this model seems to be less susceptible for outliers.

As the simulation and optimization setup will not change significantly with other distributions of the job sizes, we will not discuss the outcomes of other distributions in this report.

Uniformly distributed job sizes							
K	$\mathbf{x_1}$	$\omega_\mathbf{1}$	$\omega_\mathbf{2}$	$\widetilde{\mathbf{x_1}}_{\text{top 10}}$	$\widetilde{\omega_\mathbf{1}}_{\text{top 10}}$	$\widetilde{\omega_\mathbf{2}}_{\text{top 10}}$	**Profit**
0	6	0	0.5	5.65	0	0.5	0.249
0.02	0.8	0	10.5	0.85	0	13.75	0.233
0.04	0.6	0	15	0.6	0	14.75	0.225
0.06	0.5	0	17.5	0.5	0	15.25	0.22
0.08	0.4	0	16	0.4	0	15.25	0.216
0.1	0.3	0	16.5	0.3	0	15.75	0.212
0.12	0.3	0	17.5	0.3	0	16.25	0.209
0.14	0.3	0	17.5	0.3	0.5	17	0.205
0.16	0.1	6.5	17	0.1	6	16.75	0.203
0.18	0.1	6.5	17	0.1	6	17	0.202
0.2	0.1	6.5	17.5	0.1	6	17	0.201

TABLE 5.2: Profit for the best values of parameters (where $0 < x_1 < 6$; $0 < \omega_1 < 6.5$ and $0.5 < \omega_2 < 17.5$) with certain value of k and fixed: $I_1 = \frac{1}{2}$, $I_2 = \frac{1}{3}$ and $I_3 = \frac{1}{6}$ for model with one threshold level x_1, $\lambda = 2$ and uniformly distributed services $[0, 0.5]$.

Chapter 6

Conclusion

In this paper, we further analyzed the models given in the articles Albrecher et al., 2016 and Boxma, Essifi, and Janssen, 2016. We mathematically analyzed the M/M/1 queueing inventory model with a depletion rate which is changed at a certain threshold level. We compared this model to a model without threshold and showed how one can use the derived equations to find parameters that will increase the probability that a job can be fully served straight from the inventory and, at the same time, will decrease the mean inventory. In an analogue way, one can of course use the equations to improve the model considering any of the discussed quantities, like the ones just mentioned or e.g. the mean workload and the probability for an arriving job to find no remaining work (but not necessarily enough inventory such that the job can be delivered straight out of it). Furthermore, we considered some slightly different models such as a model with a depletion from above the threshold level back to the threshold level itself instead of zero, and a model with more than one threshold level.
As it is a rather strong assumption that the job sizes are exponentially distributed, we then also looked at the model for general job size distributions, using a simulation. This simulation forms a tool to analyze behaviour of the model with different values for parameters and different distributions.

6.1 Further Research

Based on the research we discussed in this paper, some topics for future research arose. In the last section of Chapter 5 we discussed an optimization problem with constant values of the income and costs. For example, it might be interesting to investigate the behaviour of the model when the income of customers that have to wait depends on their waiting time. Another example could be a model where the costs per unit of the inventory are depending on the level of the inventory, like when the manufacturer has to buy external storage.
At second, the model with a threshold and $\omega(x) = ax$ could be analyzed, for different values of a below and above the threshold. More research on this could then eventually test the hypothesis we stated in Section 5.4.1.
Finally, it might be interesting to investigate where in industry similar models occur and how one can analyze these using the methods we used in this paper.

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