MASTER

Condition-based maintenance under uncertain failure mechanisms

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Condition-based maintenance under uncertain failure mechanisms

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Abstract

Objective: The objective of this work is to model and compute the condition-based maintenance (CBM) costs for systems with an unobservable (hidden) failure mechanism, viz., the system is said to fail when the observed condition/ degradation process exceeds this (otherwise unobservable) failure threshold. Thus, the goal of this work is threefold: 1. to develop a mathematical framework for the modelling of such phenomena; 2. to quantify for the purpose of CBM the cost of information, i.e., what is the cost benefit in the decision-making of the observing (fully or partial) the failure mechanism; 3. to investigate how the mathematical model can be used to derive CBM policies, as well as policies based on the remaining useful lifetime (RUL).
Design/Methodology/Approach: The most general mathematical framework for the objective at hand, appropriate to model the degradation of a system over time, are the so-called Lévy processes. Such processes can be approximated by the independent sum of a Compound Poisson Process (CPP) and a Wiener drift process. We further assume that the unobservable dynamic failure mechanism (threshold) is also modelled by a Lévy process partially coupled to the degradation process. We provide an estimation procedure for the parameters involved in case of both discrete and continuously monitored systems. In addition, we demonstrate how to compute (sub-)optimal stationary policies minimizing the average cost criterion using the lifetime of a system. Using simulation, we investigate the performance of piece-wise linear in time threshold policies.
Findings: We formulate a stochastic optimization problem, the solution of which produces the optimal policy. We demonstrate that the unobservable process results in the deterministic stationary policies (control-limit policy) to be sub-optimal. To overcome this hurdle, we propose and devise a computation scheme for the superior deterministic policies (such as linear control-limit policies). Furthermore, we demonstrate, through a series of well-chosen examples of Lévy processes, conditions under which the optimal policy dictates to perform corrective maintenance (CM). Exact results are illustrated by mathematically tractable examples such as the Poisson or Wiener drift degradation processes.
Managerial insights: We compute, using simulation, the cost of CBM for various levels of information (full, partial and no information) for the failure mechanism, a direct comparison of the savings in these three cases puts a value on the worth of information.

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Contents

1 Introduction

1.1 Motivation

We are on the verge of the next global industrial revolution called 'Industry 4.0'. In essence, Industry 4.0 is concerned with the digitalization and automation of industrial processes. As operations have become more complex than ever before, the need and availability for smart sensors, data analytics, algorithms and (real-time) decision-making increases considerably. It is crucial to understand how the degradation of a system leads to failure in order to determine cost-optimal maintenance actions. The degradation of a system is described by its condition, and this condition is used as a measure to quantify 'how far from failure this component is'. In current literature on condition-based maintenance (CBM), it is typically assumed that this is the difference between some known threshold and the observed condition while this may not always be the case (see [33] and [5]). Environmental factors, such as heating, exposure to chemicals or dependent failure modes affect control limits, which are not reflected in the measured condition. Within material sciences, certain metals are known to harden or weaken after exposure to heat which can be seen as a hidden process affecting the condition. Such underlying processes have been taken into account, for example, Kenbeek et al. [19] improve wind turbine maintenance practice using adaptive alarm thresholds whilst correcting for environmental factors and other relevant parameters. Thus, moving from static thresholds to dynamic thresholds can lead to significant saving of maintenance costs.

The procedure of performing CBM can be roughly divided into three parts: data gathering, development of a model to measure the condition of a component (diagnostics) and determination of a cost-effective maintenance strategy (policy). Often this is an expensive procedure, requiring industrial adjustments such as measurement devices, model and policy development and staff training [27]. Hashemian and Bean [16] estimated that as much as 70% of industrial equipment does benefit from CBM despite these investment costs. As such, the value of information can be considered a trade-off between investment costs and potentially improved CBM planning.

Current literature [27] suggests that CBM approaches can be categorized into three groups: data-driven approaches, model-based approaches and hybrid approaches. Data-driven approaches employ machine learning and data mining methods to automatically learn system behaviors. Model-based approaches rely on mathematical methods to represent the state and behaviors of the system. The advantage of the model-based approach is that much more physical understanding of the system dynamics can be incorporated [27]. The hybrid approach combines automatic learning and the model-based approach to construct or learn system dynamics based on physical motivations. In this work, we employ the hybrid approach by using statistical methods on data to improve CBM models based on physical interpretation.

What all approaches to CBM have in common is that they follow the same procedures, although employing different techniques. A few illustrative examples of these techniques are listed in Table 1.

Procedure	Techniques
Data processing	- Fourier analysis
	- Genetic algorithms
	- Statistical pattern recognition
	- Hidden Markov model
Diagnostics	- Reliability theory
	- Time series data analysis
Policy	- Markov Decision Process (MDP)
	- Stochastic optimization
	- Mathematical simulation

Table 1: Several techniques used in condition-based maintenance [27].

1.2 Problem statement

A typical implementation of a physically motivated model is that the condition of the component can be modelled by some stochastic process, the so-called *degradation process* [6]. Failure then occurs if the degradation process exceeds a known, fixed threshold. It is, however, this assumption that turns out to be false in practice. For example, in the case of vibration readings for bearings, the business practice suggests that the use of a static failure threshold (typically set at approximately $2g$) does not account for the environmental circumstances that impact the failure mechanisms [19]. Moreover, in material science, it is well known that small shocks (depicted with small jumps in the degradation process) might result in local strengthening of the material (elasticity), while large shocks can cause a catastrophic impact or rupture [38]. As such, small shocks might cause the failure threshold to increase, while a large shock might all together cross the dynamic threshold leading to a failure. All in all, field experts and other research suggest that in order to determine whether a more accurate depiction of the physical behavior of the system at hand is a worthy investment, it is paramount to investigate the effect on the cost of the maintenance when adapting a stochastic failure threshold. As this is still a CBM model, the procedures will remain the same, however, we will now have to incorporate this new assumption in our techniques. The exact knowledge of the failure threshold at the epochs of corrective maintenance (CM) is generally unknown and for that reason we propose three levels of information, namely full information (always observable), partial information (observations at CM as well as knowledge of the initial value of the failure threshold) and no information (only the initial value) respectively, for each of which we compute the (sub-)optimal policy and compare the cost of the found policies.

For complex systems, we might need multiple models to accurately describe the connection between the observed degradation process and the physical properties which are involved in events leading to failure. In this work and for the sake of simplicity, we assume that the condition of a component can be modelled by a one-dimensional condition index (typically referred to as a ‘health index’ or ‘degradation process’ - we opt here for the latter) starting at a perfect state (referred to in the sequel as state “0”) and with initial failure threshold ξ_0. The dynamics describing the time (stochastic) evolution of the degradation and the failure process, as well as their dependence over time are assumed to have a sufficiently simple form which naturally generalizes to more difficult relations. These are made precise in section 3.1.2.

The construction of a degradation process is system-specific and this is outside the scope of this work. Here, we are more interested in the aspect of policy making: *how does uncertainty in the failure mechanism change the policy of preventively maintaining the system under a cost structure?*

From the perspective of application, our objective is to develop a general approach incorporating uncertainty in the failure dynamics. More concretely, we investigate:

1. the creation of a mathematical model for the failure mechanism that captures environmental conditions, elasticity, etc.,
2. the estimation procedure (fitting a model) for the failure mechanism in 1.,
3. the derivation of cost-optimal policies for preventively maintaining the system for the failure model developed in 1. and 2.,
4. a comparison of various policies, dependening on the level of information available about the failure mechanism.

The levels of information that we have about these systems will be reflected in the outcome of the policies for preventively maintaining the system, in particular the computed maintenance strategy and its performance. In the case where the failure threshold is not always observable, it is not possible to fully determine the nature of the underlying stochastic process nor is it possible to compute (due to issues of identifiability) the model parameters, meaning that the optimal policy cannot be derived. As such, it is of paramount importance to investigate the error of sub-optimal policies, when the Lévy process modelling the failure threshold process is approximated. Different scenario evaluations can thus be used to determine economical valuation of the level of information, i.e., the difference between the maintenance costs of the optimal policy and sub-optimal policies. As such, it can be determined if the cost of gathering and pre-processing information is justified by the potential benefits.

1.3 Overview

This section provides a general overview of what each of the following sections entails. Section 2 contains an overview of the relevant literature in the current field of maintenance research and where our work is positioned. Section 3 provides a framework to describe the dynamics of the degradation process, as well as the failure threshold processes, and it proposes approximate processes for their mathematical modelling (the independent sum of pure jump processes and Wiener drift processes). Furthermore, it discusses the levels of available information with regard to the failure mechanism in detail. In this section, we state several assumptions and propose the estimation procedures for the scenarios accordingly to the levels of information and the data availability. Computations, properties and approximations of sub-optimal (non-)stationary threshold policies, remaining useful lifetime (RUL) and RUL based policies for pure jump processes and Wiener drift processes are studied in section 4 and 5 respectively. Section 6 discusses the independent sum of these stochastic processes and provides an approach to extend the results to the larger class of Lévy processes.

2 Literature review

Maintenance models have been widely studied in the literature. Typical goals in maintenance optimization are to ensure maximal reliability and availability against a minimal cost of ownership. For this purpose, scientific methods to determine optimal preventive maintenance (PM) policies have been developed. In practice, PM policies can roughly be divided into two approaches: time-based maintenance (TBM) and condition-based maintenance. TBM policies are based on optimal replacement intervals based on a study of failure times whereas CBM policies recommend maintenance actions based on regular or irregular monitoring of the system condition. Our model is built on the latter concept, as it incorporates knowledge of the condition of the system in the form of an observable degradation process. We include the knowledge about the unobservable degradation processes by computing the lifetime of the system, which is used for the computation of TBM policies. These TBM policies are then converted into a CBM policy. Figure 1 shows an overview of maintenance strategies in practice, in this work, we consider the highlighted ones in particular.

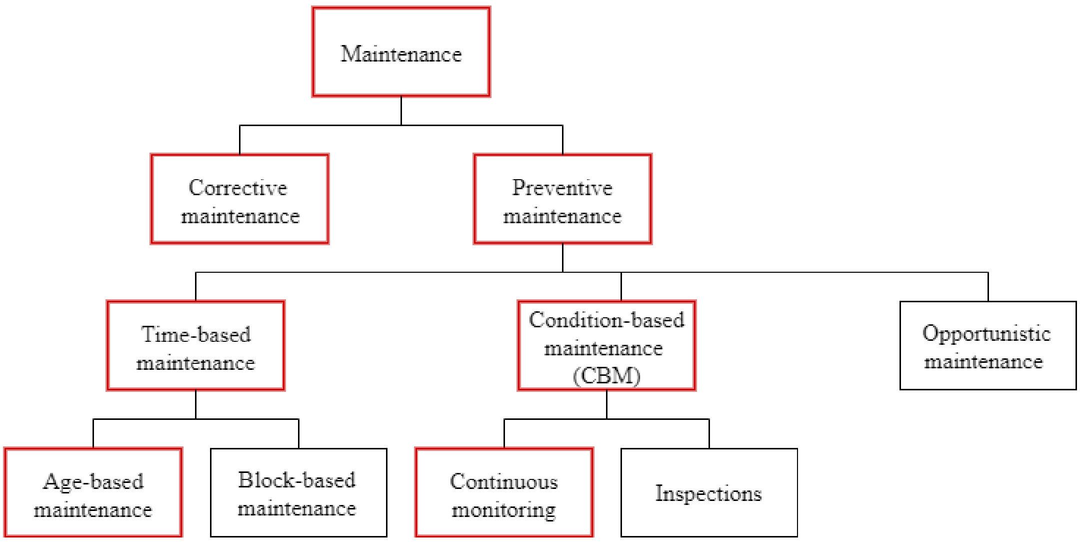

Figure 1: Overview maintenance strategies [12].

Time-based maintenance
TBM assumes that failure behavior, e.g., failure times, are predictable. Thus, maintenance actions are determined by studying the lifetime of a component, which is usually done by a statistical analysis on a set of failure times. Popular lifetime distributions used in practice are Weibull distributions, which are suitable to mimic other distributions, 2-parameter Gamma distributions as they arise naturally and other exponential family distributions such as a Normal or Inverse Gaussian distributions.

Condition-based maintenance
The concept of CBM was introduced in 1940 by the Rio Grande Railway Company [24]. Using simple pressure and temperature readings, successful CBM policies were developed for engine maintenance, leading to an entire industry in CBM strategies. The motivation behind CBM

is that many component failures are preceded by signs of wear or degradation, for example, cracks in rail tracks, thickness of brake pads or oil viscosity.

In maintenance literature, the *P-F interval* is the time between the points where *potential failure* and *functional failure* occur. These terms refer to the points in time where signs of failure (visible degradation) can be detected and when a component fails. The so-called P-F curve describes the condition of an asset over time. The creation of a degradation index is thus related to maximizing the length of the P-F interval, which is mostly done by identifying the physical events causing damage or failure. Typically, the P-F curve is decreasing, indicating that the damage process is increasing. Such increasing processes appear quite naturally in the setting of Lévy processes and they are referred to as subordinator processes. Some classical examples of such processes are the CPP and the Gamma processes, both of which are regularly used in the context of CBM. Degradation such as corrosion, erosion, crack growth and creep can be modelled by Gamma processes whose applications include steel coatings, gates and pressure vessels [2]. Compound Poisson Processes have a finite number of jumps in finite time intervals, whereas Gamma processes have an infinite number of jumps in a finite time interval. More precisely, the Gamma process can be interpreted as the limit distribution of a CPP where the arrival intensity goes to infinity while the Gamma distributed jumps tend to zero [33].

Typically such processes are the result of either the mathematical modelling of a physical mechanism or a data-driven one-dimensional degradation index. In either case, it is reasonable to assume that due to measurement errors there are deviations from the exact measurement, which can be modelled using as a mathematical model a stochastic process based on the Normal distribution, e.g., the Wiener process or the Ornstein–Uhlenbeck process. In order to assure that the underlying mathematical model is general enough, we shall consider here that the degradation process is modelled by a Lévy process and we shall use the result of Asmussen and Rosinsky [7], who show that a Lévy process can be approximated by the sum of an independent Compound Poisson Process (CPP) and a Wiener drift process. In our context, we assume that the CPP models the non-decreasing part of the degradation process and that the Wiener drift process models the measurement errors (in case the drift is in the same direction as the jumps) or the self-healing mechanism (in case the drift is in the opposite direction than that of the jumps). Figure 2 shows the position of our work in recent CBM mathematical optimization literature.

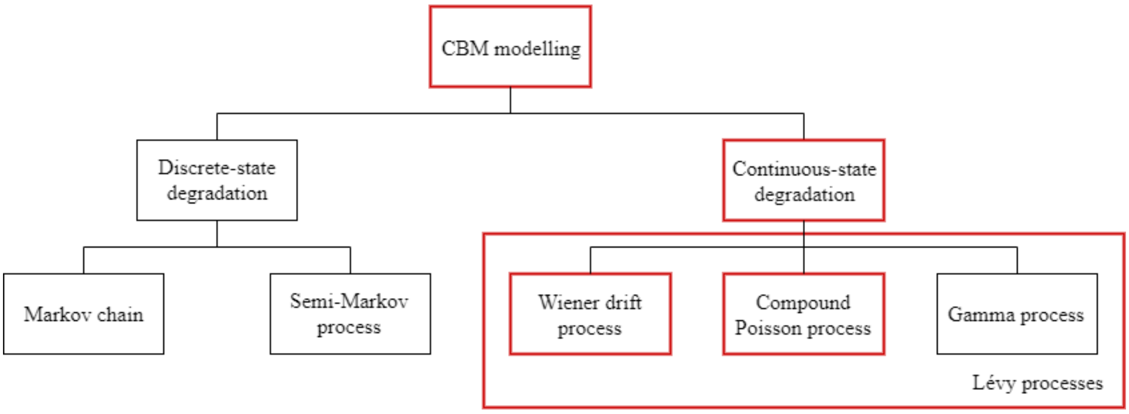

Figure 2: Overview mathematical CBM models [5].

Having established the motivation behind the choice of the degradation model, we now turn our attention to the failure mechanism. More concretely, we discuss in the next section, how the failure mechanism is modelled in the literature and what motivates the choices we make.

Modelling of the failure threshold

For CBM models implementing continuous-state degradation, CM is usually defined as the first hitting time of a threshold. These CM thresholds can either be deterministic and known, random with a known distribution or a stochastic process.

Static CM thresholds have been widely implemented in the maintenance field of research. An example for this is [15], in which Grall et at. study a system subject to periodic, perfect inspections and continuous degradation. In their work, a maintenance policy with a parametric structure is proposed to optimize both the PM threshold and the time to the next inspection whereas in a study of Do Van et al. [35], a CBM and RUL policy are compared for a system subject to periodic inspections.

However, a static CM threshold is said to be too simplistic (see [17]) to capture the uncertainty in the failure mechanism that is predominant in complex systems. For this reason, a random, static threshold is of interest, which is implemented in the following studies: Abdel-Hameed [1] provides a method, using renewal theory, to compute optimal PM policies w.r.t. both the discounted and the average cost criterion for systems subject to an increasing degradation process implementing a random, fixed threshold. In the same manner that addresses the failure thresholds, Van Horenbeek and Pintelon [32] study multi-component systems subject to Gamma degradation and a Weibull failure threshold. The component dependencies included in the model are stochastic, structural and economic dependencies. A dynamic predictive maintenance policy is proposed such that it is updated when new RUL predictions and system state information become available. The policy is compared to TBM and CBM policies and it is shown to assure an optimal policy. Wang et al. [36] propose a degradation model based on an adaptive Wiener drift process, that is, the process parameters are updated using Kalman filtering when monitoring information becomes available. The failure threshold is modelled as a distribution, and at failure instances, a Bayesian approach is used to update the parameters of this distribution. This is shown to improve the accuracy of the RUL estimates and it makes the model more suitable for coping with nonlinear or time-dependent behavior. Xiang et al. [37] model the degradation of system operating in a Markovian environment, that is, the instantaneous degradation rate varies according to a Markov chain. They investigated the accuracy of approximating complex (remaining) lifetime distribution with traditional lifetime distributions. Another remark from this study is that they develop and compare TBM and CBM policies using a failure threshold which follows a Weibull distribution.

Few studies have been devoted to exploring the possibility of a dynamic failure threshold. Tang et al. [30] compute and estimate the RUL for truncated normal thresholds under Wiener degradation processes whereas Jiang [17] combines several condition variables into a composite condition variables using a weighted power model. The weights are determined such that the coefficient of variation of this condition variable at failure instances is minimized, assuming that this maximizes failure predictability. The error is then assumed to be Gaussian, from which they conclude that the threshold evolution follows a Brownian law. In order to determine a PM threshold policy, an equivalent age as the linear combination of the age and the composite variable is defined where the coefficients are chosen such that the failure predictability of this new variable is maximized. For this equivalent age, the PM threshold is determined as the optimal time to replacement.

Other studies include randomness in failure dynamics. For example, Van Noortwijk et al. [34] consider a model where a load process peaks over a stochastic resistance process, modelled by an independent CPP and Gamma process respectively. They derive time-dependent reliability estimates, which are shown to rely on the uncertainty in the failure threshold.

In this work, we assume a (partially coupled) Lévy structure for both (composite) degradation process and failure threshold process and compute sub-optimal PM policies and RUL based on this assumption only. The next section describes the mathematical framework in detail.

3 Model description

In this section, we introduce the necessary mathematical definitions and notations to formally state our problem as a Markov decision process (MDP) with various level of information (fully observable, partly observable, and unobservable). The following introduction to MDPs is adapted from the lecture notes by Kallenberg [21, p.1-2].

An MDP is a model for sequential decision-making under uncertainty, taking into account both the short-term outcomes of current decisions and opportunities for making decisions in the future. While the notion of an MDP may appear quite simple, it encompasses a wide range of applications and has generated a rich mathematical theory. In an MDP model, one can distinguish the following seven characteristics.

1. The state space ($\mathcal{S}$): At any time point at which a decision has to be made, the state of the system is observed by the decision-maker. The set of possible states is called the state space and will be denoted by $\mathcal{S}$. For the systems under investigation in this work, the state is the entire real line, i.e., $\mathcal{S} = \mathbb{R}$. More concretely, there is an underlying Markov process, say $\{\tilde{X}(t),\, t \geq 0\}$, that captures the state of the system, with support $\mathcal{S}$. The process $\{\tilde{X}(t),\, t \geq 0\}$ will be referred to as the degradation process. The degradation processes under consideration are the sum of a Wiener drift process and a Compound Poisson process, which are defined in section 3.1.
2. The action sets ($\mathcal{A}$): When the decision-maker observes that the system is in state $i \in \mathcal{S}$, the decision-maker (from now onward we refer to the decision-maker as 'he') chooses an action from a certain action set that may depend on the observed state: the action set in state $i \in \mathcal{S}$ is denoted by $\mathcal{A}(i)$ and let $\mathcal{A} = \cup_{i \in \mathcal{S}} \mathcal{A}(i)$. Note that, we assume here a finite set of actions, namely $\mathcal{A} = \{$"do nothing", "do PM", "do CM"$\}$. At states of failure, the only available action is the action of "do CM", while at all other states, the objective of this work is to determine when it is optimal to opt for "do nothing" vs "do PM".
3. The decision time points ($t \in \mathcal{T}$): Decisions can be made in discrete epoch times or in continuous time. For the former, the time intervals between the decision points may be constant/deterministic or random. In the first case, the model is said to be a Markov decision process (MDP); when the times between consecutive decision points are random the model is called a semi-Markov decision process (sMDP). In this work, decisions can be made in continuous time, i.e., $\mathcal{T} = \mathbb{R}^+$.
4. The immediate costs (or rewards) ($c_i(a)$): Given the state of the system and the chosen action, an immediate cost is incurred. Moreover, we assume that the action "do nothing" comes at a zero cost, while the action "do PM" comes at a cost c_p and the action "do CM" comes at a cost c_c. All in all, the immediate cost for an action $a \in \mathcal{A}(i)$ in state $i \in \mathcal{S}$ is denoted by

$$c_i(a) = \begin{cases} 0, & a = \text{“do nothing”}, \\ c_p, & a = \text{“do PM”}, \\ c_c, & a = \text{“do CM”}. \end{cases}$$

5. The transition probabilities ($p_{ij}(a)$): Given the state of the system and the chosen action, the state at the next decision time point is determined by a transition law. These transitions only depend on the decision time point $t \in \mathcal{T}$, the observed state $i \in \mathcal{S}$ and the chosen action $a \in \mathcal{A}(i)$ and not on the history, i.e., they possess the Markov property. For the model at hand (CPP+Wiener drift process) the transitions are independent of the time points, the problem is thus stationary, and the transition probabilities are denoted by $p_{ij}(a)$. The probabilities are determined by the degradation processes defined in section 3.1.
6. The planning horizon: The process has a planning horizon, which is the result of the time points at which the system has to be controlled. This horizon is infinite.
7. The optimality criterion: The objective of a (semi-)Markov decision problem is to determine a policy, i.e., a decision rule for each decision time point and each history (including the present state) of the process, that optimizes the performance of the system. The performance is measured by a utility function. This utility function assigns to each policy a value, given the starting state of the process. In this work, we focus on the long-run rate of cost criterion, but we also briefly demonstrate how the results should be adapted in case one considers the discounted cost criterion.

We split the model description into three parts, namely the description of the degradation processes which determines the transition probabilities, followed by an overview of the model assumptions and its dynamics and lastly, the problem definition as an MDP.

3.1 Degradation process

The degradation processes under consideration are the sum of a Wiener drift process and a Compound Poisson process, which are defined as follows

Definition 3.1 (Wiener drift process)
A Wiener process $W(t)$ is characterized by the following properties:

- *W(0) = 0 almost surely.*
- ***Independent increments:** For any sequence $0 \leq t_1 < \ldots < t_n < \infty$, it holds that $\forall i, j \geq 1, i \neq j$, $W(t_j) - W(t_{j-1})$ and $W(t_i) - W(t_{i-1})$ are independent.*
- ***Gaussian increments:** For any $0 \leq s < t$, we have that $W(t) - W(s) \stackrel{d}{=} N(0, t-s)$*
- *$W(t)$ has continuous sample paths.*

Here, $\mathcal{N}(0,t)$ denotes a Normal distribution with mean 0 and variance t. Let $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$, the stochastic process $X(t)$ defined by $X(t) = \mu t + \sigma W(t)$ is called a Wiener drift process.

From now on, we refer to a Wiener drift process with $W(t)$.

Definition 3.2 (Compound Poisson Process)
The stochastic counting process $\{N(t), t \geq 0\}$ is called a Poisson process with intensity λ if is satisfies the following properties:

- $N(0) = 0$ *almost surely.*
- ***Independent increments:*** *For any sequence $0 \leq t_1 < \ldots < t_n < \infty$, it holds that $\forall i, j \geq 1, i \neq j$ $N(t_j) - N(t_{j-1})$ and $N(t_i) - N(t_{i-1})$ are independent.*
- $\mathbb{P}(N(t) = k) = e^{-\lambda t}\frac{(\lambda t)^k)}{k!}$.

The last property of course implies that $N(t) \sim$ Poisson(λt). Furthermore, assume that $\{X_i, i = 1, 2, \ldots\}$ are identical random variables, which are independent of each other and $N(t)$. We then call the stochastic process defined by $CP(t) = \sum_{i=1}^{N(t)} X_i$ a Compound Poisson process.

Note that both these processes belong to the larger class of Lévy processes, for which the definition can be found in appendix A. Given an observable degradation process $X(t)$ and hidden failure mechanism $\xi(t)$, the system is said to fail when the sample paths of $\{X(t), t \geq 0\}$ and $\{\xi(t), t \geq 0\}$ meet, after which the system is replaced at cost c_c and restored to its initial condition $(X(0), \xi(0)) = (0, \xi_0)$, with ξ_0 a known positive constant.

The observable degradation process of a system is described in the following section.

3.1.1 Observable degradation process

Define the observable degradation process $X(t)$ as

$$X(t) \stackrel{\text{def}}{=} CP_L(t) + W_L(t), \tag{1}$$

where $CP_L(t)$ is a CPP with intensity λ_L and jump distribution X_i. $W_L(t)$ denotes an independent Wiener drift process with drift μ_L and infinitesimal variance σ_L^2.

The CPP models the size of the shocks $X_i \geq 0$ which arrive according to a Poisson process $N_L(t)$ whereas a Wiener drift process is used to model continuous degradation or healing of a system in the presence of noise. An example of such system in which a Wiener drift process can be applied is solar panels, whose performance is known to degrade linearly over time;

or self-healing polymers, which have the ability to repair themselves in the event of damage. Hence, as the condition of a system might change over time, we incorporate the Wiener drift component in the specification of the observable degradation process.

In the case of run-to-failure maintenance with a fixed, known threshold $\xi_0 > 0$, system failure is defined as the first passage time of $X(t)$ and ξ_0, However, we now assume that the failure threshold is a stochastic process, say $\{\xi(t), t \geq 0\}$ with $\xi(0) = \xi_0$. A description of the structure of this stochastic process follows now.

3.1.2 (Unobservable) threshold process

In a similar manner, the failure threshold process $\xi(t)$ is defined as

$$\xi(t) \stackrel{\text{def}}{=} CP^c_T(t) + CP^d_T(t) + W_T(t), \tag{2}$$

here $W_T(t)$ is an independent Wiener drift process with drift μ_T and infinitesimal variance σ_T^2. $CP^d(t)$ denotes an independent CPP with intensity λ_T and i.i.d. jump size distribution $Y_{c,i}$. The failure threshold is coupled through $CP^c(t)$, which is defined as

$$CP^c_T(t) \stackrel{\text{def}}{=} \sum_{i=0}^{N_L(t)} \beta X_i.$$

In this work, we assume $\beta \geq 0$ to ensure positive jumps. Note that this naturally generalizes using multiple parameters which couple jumps in the failure threshold to jumps of a specific size in the observable degradation process. It can be seen that in addition to fitting the degradation process which comes with CBM models, we also have to estimate the parameters which appear in this threshold process. An outline for these statistical procedures for continuous monitoring and discrete inspection times is given in section 3.4, section 4 discusses policies for CPP degradation processes, while section 4 focusses on policies for Wiener degradation processes. Decision-making for the most general model is discussed in section 6. We first state the assumptions, and then describe the model dynamics in more details.

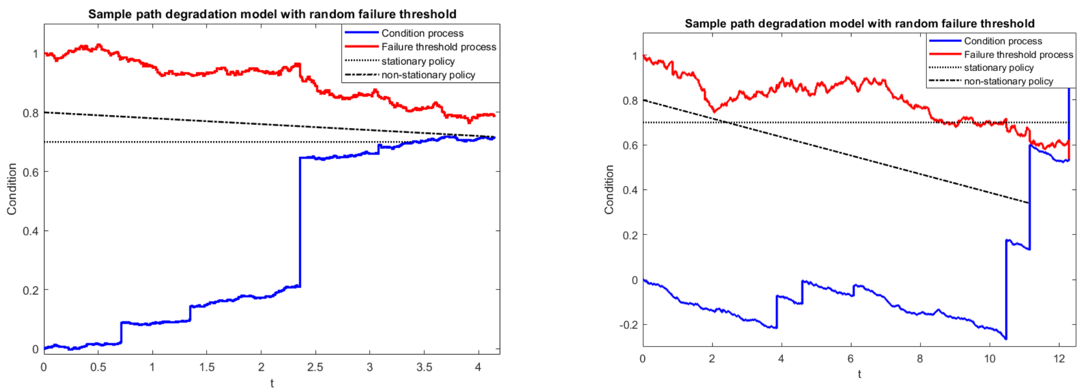

Figure 3: Sample paths $(X(t), \xi(t))$ of a degrading (left) and self-healing (right) system.

Figure 3 shows sample paths of a degrading and a self-healing system which are described by the proposed model together with a stationary and linear threshold policy which will trigger PM. The system fails at the point where the stochastic processes meet for the first time.

3.2 Model assumptions

In reality, we have to solve the decision problems without the full knowledge of the underlying distributions, especially in the threshold process. We distinguish between three levels of information:

(L1) **Full information:** $\xi(t)$ is observed for all $t \geq 0$.

(L2) **Partial information:** at $t = 0$, we observe $\xi(0) = \xi_0$ and at $t = T$ (T denotes the time of failure), we observe $\xi(T)$.

(L3) **No information:** only at $t = 0$, $\xi(0) = \xi_0$ is observed.

Note that information level (L1) is equivalent to the traditional maintenance model with a static, known threshold, this is because we can "add" two Lévy processes and obtain again a Lévy process. Assuming less information requires the choice of a failure threshold model, as well as accounting for the corresponding estimation procedure. In section 3.4, we outline the estimation procedures and model choices for these three levels of information given continuous or discrete measurements. Throughout this work, the model under investigation satisfies the following five assumptions

(AS1) Each condition measurement reveals the state of the system perfectly, i.e, we observe $\{X(t), t \geq 0\}$.

(AS2) Failures are self-announcing and the system is instantaneously replaced directly after failure. Moreover, after PM or CM the system is instantaneously restored to its initial condition $(X(0), \xi(0)) = (0, \xi_0)$.

(AS3) ξ_0 is a known positive constant (homogeneity).

(AS4) The dynamics of the stochastic process $X(t)$ are fully known, that is all the parameters and distributions which appear in the stochastic processes on the r.h.s. of Equation (1).

(AS5) The jumps $X_i, Y_{d,i}$ are positive and have finite mean and variance.

Moreover, we assume a simple, preventive cost structure with corrective cost c_c and preventive cost c_p. The cost ratio c_r is given by $\frac{c_c}{c_p} > 1$. In maintenance literature values of c_r exceeding 10 are rarely observed. In a study case of the Swedish railways, a cost-benefit analysis shows the benefit of PM is positive with a benefit–cost ratio estimated at 3.3, however highly dependent on data input [29]. We limit ourselves in our discussion to $c_r \in [2, 10]$, although calculations are valid for all $c_r > 1$. Without loss of generality, we normalise the values used for the numerical experiments to $c_p = 1$ and $(X(0), \xi(0)) = (0, 1)$.

3.3 Model dynamics

In order to mathematically define the maintenance optimization problem, we first need the definition of a policy and the set of policies under consideration. In the most general scenario, the decision-maker is allowed to choose any action at any time $t \geq 0$ using the history $h_t \in \mathcal{H}_t$, here $\mathcal{H}_t$ denotes the set of all possible histories up to time t.

Definition 3.3 (Policy)
A policy is a function τ which at time t assigns a probability to an action, given the history of the system:

$$\tau(t) : \mathcal{H}_t \times \mathcal{A} \to [0, 1].$$

The set of all such policies is denoted with $\mathcal{C}$. In this work, we consider non-random policies which can only depend on the information at time, e.g., $(t, X(t), \xi(t))$. Such policies are called memoryless policies and the set of all such policies is denoted with $\mathcal{C}(M) \subset \mathcal{C}$. Note that this limitation is justified by the stationary increments of $(X(t), \xi(t))$. In particular, the class of threshold policies is of interest:

Definition 3.4 (Threshold policy)
Let $s \in \mathcal{S}$. Given two actions $a_0, a_1 \in \mathcal{A}0$ with $a_0 \neq a_1$, a threshold policy $\tau(t) \in \mathcal{C}(M)$ is a policy such that $\exists \tilde{s}(t) \in \mathcal{S} : \tau(t, s, a_0) = 1$ if $s < \tilde{s}(t)$, and if $s \geq \tilde{s}(t)$ we have that $\tau(t, s, a_1) = 1$. Furthermore, denote the set of all threshold policies with $\mathcal{C}(M') \subset \mathcal{C}(M)$.

We identify threshold policies with the threshold value and in particular let $\tau(t) = \tau_0 - \alpha t$ where $\tau_0 > 0$ and $\alpha \in \mathbb{R}$ denote a linear threshold policy.

Definition 3.5 (Gap)
The gap $\Delta(t)$ can be interpreted as the real condition of the system and is defined by

$$\Delta(t) \stackrel{def}{=} \xi_0 - \xi(t) - X(t) \text{ and similarly,}$$
$$\Delta_\tau(t) \stackrel{def}{=} \tau(t) - X(t).$$

It can be easily seen that the gap is again described by the sum of a CPP and a Wiener drift process, given by

$$\Delta(t) = \xi_0 - \sum_{i=1}^{N(t)} Z_i - \mu t - \sigma W_t. \tag{3}$$

Here, $\mu = \mu_L + \mu_T$ and $\sigma = \sqrt{\sigma_L^2 + \sigma_T^2}$, where $N(t)$ is a Poisson Process with intensity $\lambda = \lambda_L + \lambda_T$ and Z_i are iid random variables given by

$$Z_i \stackrel{\text{def}}{=} \begin{cases} (1+\beta)X_i & \text{w.p. } \lambda_L/\lambda \\ Y_{d,i} & \text{w.p. } \lambda_T/\lambda \end{cases}. \tag{4}$$

Thus, define the stochastic process $\{\tilde{X}(t), t \geq 0\}$ which models the true degradation of the system as

$$\tilde{X}(t) = X(t) + \xi(t). \tag{5}$$

We can directly observe that

$$\begin{aligned}
\mathbb{E}[X(t)] &= (\lambda_L \mathbb{E}[X_1] + \mu_L)t, \\
\mathbb{E}[\xi(t)] &= (\lambda_L \mathbb{E}[X_1] + \beta\lambda_T \mathbb{E}[Y_{d,1}s] + \mu_T)t, \\
\mathbb{V}[X(t)] &= (\lambda_L \mathbb{E}[X_1^2] + \sigma_L^2)t, \\
\mathbb{V}[\xi(t)] &= (\lambda_L \beta^2 \mathbb{E}[X_1^2] + \lambda_T \mathbb{E}[Y_{d,1}^2] + \sigma_T^2)t \text{ and} \\
\mathbb{V}[\tilde{X}(t)] &= (\lambda_L (1+\beta)^2 \mathbb{E}[X_1^2] + \lambda_T \mathbb{E}[Y_{d,1}^2] + \sigma^2)t.
\end{aligned}$$

We proceed with introducing the necessary notation to describe CM and PM.

Definition 3.6 (Time to CM/PM threshold)
The time to CM and PM thresholds, T, T_τ resp. are defined as first passage times (FPT)

$$\begin{aligned}
T &\overset{def}{=} \inf\{t \geq 0 : \tilde{X}(t) \geq \xi_0\}, \\
T_\tau &\overset{def}{=} \inf\{t \geq 0 : X(t) \geq \tau(t)\}.
\end{aligned}$$

Failure thus happens if $T \leq T_\tau$, thus we can define the time to CM and PM as follows

Definition 3.7 (Time to CM/PM)
The time to CM/PM is defined as the time to the CM/PM threshold, given CM/PM occurs first.

$$\begin{aligned}
T_{CM} &\overset{def}{=} T | T \leq T_\tau, \\
T_{PM} &\overset{def}{=} T_\tau | T > T_\tau.
\end{aligned}$$

Note that these random variables are not independent. In order to proceed further, we first introduce the notion of overshoot and undershoot.

Definition 3.8 (Overshoot at CM/PM threshold)
The overshoot at the first hitting time of the CM/PM threshold is given by $-\Delta(T) = \tilde{X}(T) - \xi_0$ and $-\Delta_\tau(T_\tau) = X(T_\tau) - \tau(T_\tau)$ respectively.

If $\tau \in C(\mathbb{R}, \mathcal{S})$, meaning that τ is continuous a.e., then the Wiener part of a Lévy process cannot cause an overshoot due to creeping to the failure threshold. As such, an overshoot can be caused by the CPP, this complicates the analysis as it can happen that at an observable jump instance, $\{X(t), t \geq 0\}$ crosses both $\{\xi(t), t \geq 0\}$ and $\tau(t)$. In other words, $\mathbb{P}(T \leq T_\tau)$ can be non-zero. The gap is $\tilde{X}(T) - \tilde{X}(T^-)$, while the undershoot is the lower part of the jump.

Definition 3.9 (Undershoot before CM/PM threshold)
The undershoot before CM is given by the left limit at failure instance, written as $\Delta(T^-)$*. For any* $t > 0$*, we define the left limit as*

$$\Delta(t^-) \overset{def}{=} \lim_{s \uparrow t} \Delta(s).$$

Similarly, the undershoot before crossing the PM threshold is defined as $\Delta_\tau(T_\tau^-)$*.*

Now that the necessary definitions have been established, we shall proceed to prove the following theorem.

Theorem 3.1 (Expected time to CM/PM threshold for linear threshold policies)
Let $\tau(t) = \tau_0 - \alpha t$ *a linear threshold policy. If* $\mathbb{E}[\tilde{X}(1)] > 0$ *and* $\mathbb{E}[X(1)] + \alpha > 0$*, then the expectations of* T, T_τ *satisfy the following relations*

$$\mathbb{E}[T] = \frac{\xi_0 - \mathbb{E}[-\Delta(T)]}{\lambda_L(1+\beta)\mathbb{E}[X_1] + \lambda_T \mathbb{E}[Y_{d,1}] + (\mu_L + \mu_T)}, \text{ and} \tag{6}$$

$$\mathbb{E}[T_\tau] = \frac{\tau_0 + \mathbb{E}[-\Delta_\tau(T_\tau)]}{\lambda_L \mathbb{E}[X_1] + (\mu_L + \alpha)}. \tag{7}$$

Proof. Note that using the linearity of the expectation, we have

$$\mathbb{E}[\tilde{X}(T)] = \mathbb{E}\left[\sum_i^{N(T)} Z_i + \mu T + \sigma_T W_T\right] = \mathbb{E}\left[\sum_i^{N(T)} Z_i\right] + \mu \mathbb{E}[T] + \sigma_T \mathbb{E}[W(T)].$$

Now, we first prove that T is a stopping time. Let $t \geq 0$, then we have that

$$T \leq t \iff \sup_{s \in [0,t]} \tilde{X}(s) \geq \xi_0,$$

implying that indeed the decision to stop the stochastic process at time t only depends on the history up to time t. Similarly, T_τ is also a stopping time. Note that $\mathbb{E}[\tilde{X}(1)] > 0 \implies \mathbb{E}[T] < \infty$ almost surely, hence $\mathbb{E}[W(T)] = 0$ by the optional stopping theorem. Assumption (AS5) ensures that Wald's identity is applicable, which gives

$$\mathbb{E}\left[\sum_i^{N(T)} Z_i\right] = \lambda \mathbb{E}[Z_1]\mathbb{E}[T].$$

Using that $\lambda \mathbb{E}[Z_1] = \lambda_L(1+\beta)\mathbb{E}[X_1] + \lambda_T \mathbb{E}[Y_{d,1}]$ and $\mathbb{E}[\tilde{X}(T)] - \xi_0 = \mathbb{E}[-\Delta(T)]$, we get

$$\mathbb{E}[-\Delta(T)] = (\lambda_L(1+\beta)\mathbb{E}[X_1] + \lambda_T \mathbb{E}[Y_{d,1}] + \mu)\mathbb{E}[T] - \xi_0.$$

Isolating $\mathbb{E}[T]$ directly yields the desired result. The proof for Equation (7) is similar and omitted here. □

The theorem states that the mean time to the CM/PM threshold is equal to the mean time to reach the mean overshoot. Computation of the overshoot is often problematic and only in a few cases its distribution is known. Theorem 3.1 provides us with a relation in the case one of the expectations is known or can be estimated. In section 3.4, we discuss a method for approximating the expected overshoot.

3.4 Parameter estimation failure threshold process

This section addresses the distinct scenarios which arise as combinations of different levels of assumed information and available data. Typically, in maintenance engineering, data availability is either continuous, discrete or on demand. Here, we restrict ourselves to the first two cases. Note (or assume) that with continuous data, we can identify the jumps $N_L(t)$, however the jumps $N_T(t)$ remain unobserved if the level of information is not (L1). For discrete data, no jumps can be identified. These observations motivate the choices for the failure threshold models per scenario, which can be found in Table 2.

Level of information \ Data availability	Continuous data (D1)	Discrete data (D2)
Full information (L1)	$CP^c + CP^d + W$	W (approx.)
Partial information (L2)	CP^c $(+\ CP^d) + W$	W (approx.)
No information (L3)	CP^c $(+\ CP^d) + W$	W (approx.)

Table 2: Failure threshold model choice based on data availability and level of information.

When estimating the parameters of the unobserved jump process CP^d, it is not possible to distinguish between drift and jump size. As such, expert knowledge is required. In this work, we propose the estimation procedures for the scenarios in which no information (L3) or partial information (L2) is assumed and the available data is continuous which are outlined in subsections 3.4.1 and 3.4.2 respectively. The estimation procedures in case of discrete data are discussed in section 3.4.3. Note that the estimation procedure for the scenarios corresponding to (L1) is not discussed as in such scenarios, full information is known, implying that there is no underlying mechanism of uncertainty, which is outside the scope of this work.

3.4.1 Continuous data (D1) and no information (L3)

The estimation procedure for (L2) and (L3) are addressed in a similar manner. From the information of the observed degradation process, we start with constructing an approximation for $\xi(T)$. This places us in the (L2) setting and as such the remaining procedures are identical.

Suppose we have continuously monitored M systems until failure. Denote with $X_i(t)$ the value of the condition process at time t and $n_i \overset{def}{=} N_L^i(T_i) \in \mathbb{N}$ the number of jumps until failure time $T_i \in \mathbb{R}^+$ of the i-th component, respectively. The observed jumps are denoted with $X_1^i, \ldots, X_{n_i}^i$ together with their realizations $x_1^i, \ldots, x_{n_i}^i$. Moreover, define $\delta_i \overset{def}{=} N_L^i(T_i) - N_L^i(T_i^-) \in \{0,1\}$ indicating whether the failure is caused by an observable jump. Our proposed estimation procedure requires data for $\{T_i, X_i(T_i), n_i, \delta_i, i = 1, \ldots, M\}$.

In order to approximate $\xi_i(T_i)$, we need to take into account the overshoot caused by a jump, meaning that the observations $X_i(T_i)$ can be considered censored. Generally, the distribution of the i-th overshoot $\Delta_i(T_i)$ is unknown and therefore correcting the data points for which $\delta_i = 1$ is already challenging. In general, the following properties can be deduced:

(1) If $\delta_i = 0$ then $-\Delta_i(T_i) \in [0, y^i_{2,N_T^i(T_i)}]$.

(2) If $\delta_i = 1$ then $-\Delta_i(T_i) \in [0, (1+\beta)x^i_{n_i}]$.

The first observation is due to *creeping* of the Wiener drift process or occurrence of an independent jump. As we cannot identify between them, we will not correct for these points.

The second observation states that the real value of the failure threshold lies in the interval determined by the size of the last jump in the condition process and the corresponding coupled jump. In order to correct for this bias, common practice is to correct with the mean residual, see [17]. This reduces to the computation of $\mathbb{E}[\Delta_i(T_i) \mid X^i_{n_i} = x^i_{n_i}]$. Under the assumption that $X^i_{n_i}$ are positive random variables and the real value of the failure threshold is uniformly distributed in the interval, expressions for the n-th moment of the excess time are given by (e.g., see [3, p. 71])

$$\mathbb{E}[R^n_{X_1}] = \frac{\mathbb{E}[X_1^{(n+1)}]}{(n+1)\mathbb{E}[X_1]}, \ n \in \mathbb{N}.$$

Assuming that β is small, an approximation $\tilde{\xi}_i(T_i)$ for $\xi_i(T_i)$ is then obtained by subtracting the expected excess time from the observed condition, meaning that

$$\tilde{\xi}_i(T_i) \approx \xi_0 - \big(X_i(T_i) - \min\{\mathbb{E}[R_{X_1}], x^i_{n_i}\}\mathbb{1}\{\delta_i\}\big). \tag{8}$$

We can also use this approximation for Equations (6) and (7) to obtain the approximations for the mean time to the CM and PM thresholds, thus

$$\mathbb{E}[-\Delta(T)] \approx \frac{\frac{\lambda_L}{\lambda}(1+\beta)^2\mathbb{E}[X_1^2] + \frac{\lambda_T}{\lambda}\mathbb{E}[Y^2_{d,1}]}{\frac{\lambda_L}{\lambda}(1+\beta)\mathbb{E}[X_1] + \frac{\lambda_T}{\lambda}\mathbb{E}[Y_{d,1}]} \text{ and}$$

$$\mathbb{E}[-\Delta_\tau(T_\tau)] \approx \frac{\mathbb{E}[X_1^2]}{2\mathbb{E}[X_1]}.$$

The found approximations $\tilde{\xi}_i(T_i)$ can be used instead of the uncensored observations $\xi_i(T_i)$ in the remaining estimation procedure for continuous data and partial information, which is discussed in the following subsection.

3.4.2 Continuous data (D1) and partial information (L2)

As previously established, for scenarios that involve partial information and continuous data, the assumed failure threshold model consists of a coupled CPP and an independent Wiener drift process (see Table 2). The goal of this section is to provide a method to estimate the parameters for such failure threshold processes, for which we use maximum likelihood estimators (MLE), as they determine the parameter values for which the data is most probable. Taking into account that the coupled jumps are caused by jumps in the condition process, a parameter for modelling the relation between coupled jumps is included, which we refer to as β. Denote the cumulative jumps by $\bar{x}_i(T_i) = \sum_{j=1}^{n_i} x_j^i$. As an approximation, we assume that the observations $\{(\xi_i(t_i)|X_i(T_i) = x_i, T_i = t_i), i = 1, \dots, M\}$ are independent and from Gaussian distributions $\mathcal{N}(\beta\bar{x}_i(t_i) + \mu_T t_i, \sigma_T^2 t_i)$. Like [17], we construct the maximum likelihood estimators for σ_T, μ_T and β.
The likelihood function is given by

$$\mathcal{L}(\sigma_T, \mu_T, \beta) = e^{-\frac{1}{2\sigma_T^2}\sum_{i=1}^M \frac{(\xi_i(T_i)-\beta\bar{x}_i(T_i)-\mu_T T_i)^2}{T_i}} (2\pi)^{-M/2}(\sigma_T^2)^{-M/2}\Pi_{i=1}^M(T_i)^{-1/2},$$

and thus the log-likelihood equals

$$\tilde{\mathcal{L}}(\mu_T, \sigma_T, \beta) = -\frac{1}{2\sigma_T^2}\sum_{i=1}^M \frac{(\xi_i(T_i) - \beta\bar{x}_i(T_i) - \mu_T T_i)^2}{T_i} - \frac{M}{2}\ln(2\pi) - \frac{M}{2}\ln(\sigma_T^2) - \sum_{i=1}^M \frac{1}{2}\ln(T_i).$$

Solving for $\frac{d\tilde{\mathcal{L}}}{d\mu_T} = 0$ gives

$$-\frac{1}{2\sigma_T^2}\sum_{i=1}^M \frac{-2T_i}{T_i}(\xi_i(T_i) - \beta\bar{x}_i(T_i) - \mu_T T_i) = 0 \iff \sum_{i=1}^M(\xi_i(T_i) - \beta\bar{x}_i(T_i) - \mu_T T_i) = 0,$$

Which yields a linear relation between μ_T and β given by

$$\mu_T = \frac{\sum_{i=1}^M \xi_i(T_i) - \beta\sum_{i=1}^M \bar{x}_i(T_i)}{\sum_{i=1}^M T_i}. \tag{9}$$

Solving for $\frac{\tilde{\mathcal{L}}}{d\beta} = 0$ gives

$$-\frac{1}{2\sigma_T^2}\sum_{i=1}^M \frac{-2\bar{x}_i(T_i)}{T_i}(\xi_i(T_i) - \beta\bar{x}_i(T_i) - \mu_T T_i) = 0 \iff \sum_{i=1}^M \frac{\bar{x}_i(T_i)}{T_i}(\xi_i(T_i) - \beta\bar{x}_i(T_i) - \mu_T T_i) = 0,$$

Which gives us a second linear relation, namely

$$\mu_T = \frac{\sum_{i=1}^M \bar{x}_i(T_i)T_i^{-1}\xi_i(T_i) - \beta\sum_{i=1}^M \bar{x}_i(T_i)^2 T_i^{-1}}{\sum_{i=1}^M \bar{x}_i(T_i)}. \tag{10}$$

Equating the r.h.s. of Equations (9) and (10) gives the following MLE for β

$$\hat{\beta} = \frac{\left(\sum_{i=1}^M \bar{x}_i(T_i)\right)^{-1}\sum_{i=1}^M \bar{x}_i(T_i)T_i^{-1}\xi_i(T_i) - \left(\sum_{i=1}^M T_i\right)^{-1}\sum_{i=1}^M \xi_i(T_i)}{\left(\sum_{i=1}^M \bar{x}_i(T_i)\right)^{-1}\sum_{i=1}^M \bar{x}_i(T_i)^2 T_i^{-1} - \left(\sum_{i=1}^M T_i\right)^{-1}\sum_{i=1}^M \bar{x}_i(T_i)}, \tag{11}$$

after which the MLE $\hat{\mu}_T$ can be derived using Equations (11) and (9). Note that

$$\frac{d^2\tilde{\mathcal{L}}}{d\mu_T^2} = -\frac{1}{\sigma_T^2}\sum_{i=1}^{M} T_i < 0$$

and

$$\frac{d^2\tilde{\mathcal{L}}}{d\beta^2} = -\frac{1}{\sigma_T^2}\sum_{i=1}^{M} \bar{x}_i(T_i)^2 T_i^{-1} < 0,$$

which imply the found MLE's indeed maximize the likelihood function. Lastly, solving $\frac{d\tilde{L}}{d\sigma_T^2} = 0$ returns the following MLE

$$\hat{\sigma}_T = \sqrt{\frac{1}{n}\sum_{i=1}^{n} \frac{(\tilde{\xi}_i(T_i) - \hat{\mu}_T T_i - \hat{\beta}\bar{x}_i(T_i))^2}{T_i}}. \tag{12}$$

3.4.3 Discrete data (D2)

The estimation procedure for discrete data is in fact a special case of the above procedures. In the situation of no information (L3), Equation (8) still provides a method to correct for possibly censored observations. The only information about observations being censored, is given by the dynamics of $X(t)$. When $X(t)$ is a pure jump process, all of them are censored whilst this might not be the case in the presence of a Wiener process.

The maximum likelihood estimators in this scenario can be constructed by substituting $\beta = 0$ in Equations (9) and (12).

3.4.4 Numerical example

Throughout this work, we use several (rather arbitrarily chosen) examples with Weibull jumps to demonstrate the methods and illustrate the results. The parameters and the expected increments per time unit are given in Tables 3 and 4.

Parameter	λ_L	λ_T	$\mathbb{E}[X_1]$	$\mathbb{V}(X_1)$	β	$\mathbb{E}[Y_{d,1}]$	μ_L	μ_T	σ_L	σ_T
Value	1	1	$\frac{1}{8}$	$\frac{1}{64}$	$\frac{1}{10}$	$\frac{1}{16}$	$\frac{1}{50}$	$\frac{1}{100}$	$\frac{1}{50}$	$\frac{1}{100}$

Table 3: Drift parameters of the degradation process.

For the variance of the threshold jumps, we consider three examples (EX) with low variance (LV), medium variance (MV) and high variance (HV). The variance of the independent part $\xi_d \equiv CP_T^d + W_T$ is also shown in the first row.

Example / θ	(EX1LV)	(EX2MV)	(EX3HV)
$(\mathbb{V}[Y_{d,1}], \mathbb{V}[\xi_d(1)])$	$(\frac{1}{256}, 0.008)$	$(\frac{1}{128}, 0.012)$	$(\frac{1}{64}, 0.020)$
$(\mathbb{E}[X(1)], \mathbb{E}[\xi(1)])$	$(0.145, 0.085)$	$(0.145, 0.085)$	$(0.145, 0.085)$
$(\mathbb{V}[X(1)], \mathbb{V}[\xi(1)])$	$(0.032, 0.008)$	$(0.032, 0.012)$	$(0.032, 0.020)$
$(\mathbb{E}[\tilde{X}(1)], \mathbb{V}[\tilde{X}(1)])$	$(0.230, 0.046)$	$(0.230, 0.050)$	$(0.230, 0.058)$
$(\mathbb{V}[\xi_d(1)], \mathbb{V}[\xi(1)])$	$(0.032, 0.008)$	$(0.032, 0.012)$	$(0.032, 0.020)$
$(\mathbb{E}[\tilde{X}(1)], \mathbb{V}[\tilde{X}(1)])$	$(0.230, 0.046)$	$(0.230, 0.050)$	$(0.230, 0.058)$

Table 4: Variance threshold jumps and expected increments per unit of time.

To investigate the error caused by removing bias using Equation (8), Tables 5 and 6 show the estimated parameters for simulated data using models described by model parameters listed in Tables 3 and 4 for information levels (L2) and (L3). The quality of the approximation is assessed using the mean squared error (MSE).

$$MSE(\xi, \tilde{\xi}) = \frac{1}{M} \sum_{i=1}^{M} \left(\xi_i(t_i) - \tilde{\xi}_i(t_i) \right)^2 .$$

Table 5 shows the computed MLEs using simulated data where the independent part ($CP_T^d + W_T$) of the failure threshold is approximated (using the first two moments) by a single Wiener process, thus $CP_T^c + \tilde{W}_T$ remains. The variables $\mu_d \overset{def}{=} \mathbb{E}[\xi_d(1)])$ and $\sigma_d^2 \overset{def}{=} \mathbb{V}[\xi_d(1)]$ denote the drift and variance explained by the independent part, respectively.

Level	est.	Value	(EX1LV)-S	(EX1LV)-L	(EX2MV)-S	(EX2MV)-L	(EX3HV)-S	(EX3HV)-L
(L2)	μ_d	0.073	0.081	0.071	0.144	0.075	0.133	0.083
	β	0.1	0.065	0.110	-0.525	0.079	-0.387	0.045
	σ_d^2	(Tbl 4)	0.014	0.008	0.023	0.012	0.024	0.022
(L3)	μ_d	0.073	0.090	0.078	0.143	0.080	0.123	0.086
	β	0.1	-0.033	0.040	-0.523	0.019	-0.251	0.003
	σ_d^2	(Tbl 4)	0.014	0.010	0.024	0.013	0.024	0.022
	MSE$(\xi, \tilde{\xi})$	–	0.005	0.008	0.004	0.008	0.003	0.005

Table 5: Estimated parameters $CP^c + W$ failure threshold for small (S) number of data points $n = 10$ and for large (L) number of data points $n = 1000$.

As the MSE is small, it follows that Equation (8) provides a good approximation method for correcting censored observations. However, the number of observations M is inversely proportional to the fluctuations in the MLEs. Given a sufficient number of data points, we can estimate the parameters reasonably well. Table 6 shows the computed estimators in the other case where unobservable jumps are present.

Level \ Example		Value	EX1LV-S	EX1LV-L	EX2MV-S	EX2MV-L	EX3HV-S	EX3HV-L
(L2)	μ_d	0.073	0.075	0.069	0.038	0.073	0.057	0.076
	β	0.1	0.056	0.129	0.227	0.121	0.142	0.060
	σ_d^2	(Tbl 4)	0.003	0.010	0.011	0.020	0.002	0.053
(L3)	μ_d	0.073	0.088	0.074	0.037	0.077	0.063	0.071
	β	0.1	-0.120	0.043	0.244	0.034	0.002	0.021
	σ_d^2	(Tbl 4)	0.003	0.012	0.019	0.018	0.004	0.038
	MSE$(\xi, \tilde{\xi})$	–	0.013	0.012	0.010	0.014	0.033	0.023

Table 6: Estimated parameters $CP^d + CP^c + W$ failure threshold for small (S) number of data points $n = 10$ and for large (L) number of data points $n = 1000$.

As expected, introducing unobservable jumps leads to a higher MSE, which complicates the estimation procedure for β, given information level (L3).

In order to compute (sub-)optimal maintenance policies, an understanding of the parameters of the unobservable threshold process is required. The next section discusses multiple cost criterions which characterize such optimal policies.

3.5 Problem formulation as a (Partially Observable) MDP

Depending on the level of the available information, one can formulate this problem very differently: More concretely, if we are in the case of full information, then the problem reduces to a classical MDP in which decisions are made at continuous time. This mimics the characteristics of stochastic control problems, see, e.g., [39]. In this case, the derivation of the optimal policy reduces to solving the so-called HJB equations and using these equations to prove that there exists an optimal policy in the class of stationary deterministic policies. In the case of partial or no information, the problem can be viewed on two distinctive levels: on the one hand, one can assume that the threshold process is given by Equation (2), but as we do not observe it, we have a belief on the state of the threshold at a given time. This implies that the problem can be viewed as a partially observable MDP (POMDP), see, e.g., [8,28,40]. On the other hand, one may assume that the failure threshold is static/dynamic. As such, the problem again reduces to a classical MDP in which decisions are made at continuous time, thus there is an optimal policy which is (stationary and) deterministic.

In the case of discrete availability (with periodic inspection times), similar results hold. In the special case of a classical MDP, one could discretize the state space (say N states which are uniformly spaced), compute the corresponding transition probabilities and use the result in [20] to prove that the optimal policy τ_N^* are in the class of stationary and deterministic replacement rules. The optimal replacement policy τ^* for the continuous state space is then obtained as the limit of N to infinity. In the other case of partial information, the problem can again be viewed as a POMDP, as described above.

The aim of this work is to derive the optimal, non-stationary threshold policy. To this extent, we investigate the smaller classes of stationary, deterministic policies and linear deterministic policies. For these classes of policies, we consider the long-run rate of cost criterion and using a renewal argument, we derive the long run rate of cost as a function of the policy.

Define the stochastic process $\{K(t), t \geq 0\}$ as the number of maintained components (preventively or correctively under some policy τ (which is a function of time t)) at time t. Thus, $K(t)$ is a renewal process (see [26], Ch. 7) with i.i.d. interarrival times $(T \wedge T_\tau)_i, i \in \mathbb{N}$. Define

$$c(X(t_i), \xi(t_i); \tau) = \begin{cases} c_c, & \text{if CM,} \\ c_p, & \text{if PM.} \end{cases} \text{ given state } (X(t_i), \xi(t_i)), \text{ policy } \tau \text{ and let } t_0 = 0.$$

Define the renewal reward process $R(t)$ as the maintenance cost up to time t by

$$R(t) = \sum_{i=1}^{K(t)} c(X(t_i), \xi(t_i); \tau).$$

Here, t_i denotes the epoch of the i-th renewal until time t. $c(X(t_i), \xi(t_i); \tau) = c_c$ if system i is replaced correctively and $c(X(t_i), \xi(t_i); \tau) = c_p$ otherwise. Then, one can define two optimization criteria: Namely the long-run rate of cost, which is given in Definition 3.10, and the discounted cost criterion, which is presented in Definition 3.11.

Definition 3.10 (Long-run rate of cost)

$$g(\tau) = \lim_{t\to\infty} \frac{\mathbb{E}[R(t)]}{t}.$$

Using results from renewal theory (see [26], Ch. 7.4), g can be shown to be equal to

$$g(\tau) = \frac{\mathbb{E}[C(i)]}{\mathbb{E}[T \wedge T_\tau]} = \frac{c_c \mathbb{P}_\tau(\text{failure}) + c_p(1 - \mathbb{P}_\tau(\text{failure}))}{\mathbb{E}[T \wedge T_\tau]}. \tag{13}$$

Expanding the denominator by conditioning on the event of failure and using that $\mathbb{P}_\tau(\text{failure}) = \mathbb{P}(T \leq T_\tau)$, we have that

$$g(\tau) = \frac{c_c \mathbb{P}(T \leq T_\tau) + c_p(1 - \mathbb{P}(T \leq T_\tau))}{\mathbb{E}[T_{CM}] \cdot \mathbb{P}(T \leq T_\tau) + \mathbb{E}[T_{PM}](1 - \mathbb{P}(T \leq T_\tau))}. \tag{14}$$

Definition 3.11 (Discounted cost criterion)
The optimal discounted cost is given by

$$g_\alpha(x_0) = \inf_{\tau\in\mathcal{C}} \left\{ \mathbb{E}\left[\sum_{i=0}^{\infty} \alpha^{t_i} c(X(t_i), \xi(t_i); \tau) | X(t_0) = x_0 \right] \right\}.$$

Here, τ is a policy, $\mathcal{C}$ is a set of policies and $\alpha \in [0, 1]$ denotes the discount factor.

The following theorem shows the similarity between the two cost criteria.

Theorem 3.2 (Discounted cost of a policy τ)
The discounted cost of a policy $\tau \in \mathcal{C}$, denoted by $g_\alpha^\tau(x_0)$, satisfies

$$g_\alpha^\tau(x_0) = \frac{c_c \mathbb{E}[\alpha^T \mathbb{1}\{T \leq T_\tau\}|X(0) = x_0] + c_p \mathbb{E}[\alpha^{T_\tau} \mathbb{1}\{T > T_\tau\}|X(0) = x_0]}{1 - \mathbb{E}[\alpha^T \mathbb{1}\{T \leq T_\tau\}|X(0) = x_0] - \mathbb{E}[\alpha^{T_\tau} \mathbb{1}\{T > T_\tau\}|X(0) = x_0]}. \tag{15}$$

Proof. Using a first step argument and conditioning

$$\begin{aligned} g_\alpha^\tau(x_0) &= \mathbb{E}\left[\alpha^T (c_c + g_\alpha(x_0))\mathbb{1}\{T \leq T_\tau\} + \alpha^{T_\tau}(1 + g_\alpha(x_0))\mathbb{1}\{T > T_\tau\}|X(0) = x_0\right] \\ &= c_c \mathbb{E}\left[\alpha^T \mathbb{1}\{T \leq T_\tau\}|X(0) = x_0\right] + g_\alpha^\tau(x_0))\mathbb{E}\left[\alpha^T \mathbb{1}\{T \leq T_\tau\}|X(0) = x_0\right] \\ &\quad + c_p \mathbb{E}\left[\alpha^{T_\tau} \mathbb{1}\{T > T_\tau\}|X(0) = x_0\right] + g_\alpha^\tau(x_0))\mathbb{E}\left[\alpha^{T_\tau} \mathbb{1}\{T > T_\tau\}|X(0) = x_0\right]. \end{aligned}$$

Isolating $g_\alpha^\tau(x_0)$ on the l.h.s. yields Equation (15). □

Remark
Note that for $X(0) = x_0$ (w.p. 1), we have that $\lim_{\alpha \to 1}(1 - \alpha) g_\alpha^\tau(x_0) = g(\tau)$.

We proceed with the average cost criterion as in Definition 3.10. However, Theorem 3.2 shows that we can extend the analysis in a straightforward manner by computing the Laplace transforms of the time to CM/PM, instead of the probabilities and expectations. The goal is thus to find or approximate the policy τ^* which is a global minimizer of the average cost metric, meaning that

$$\tau^* = \arg\min_{\tau \in \mathcal{C}(M')} g(\tau).$$

The core complexity of computing the cost of a policy τ lies in deriving the joint distribution of (T, T_τ) and to derive the distribution of T_{CM} and T_{PM}, which we need to compute the quantities in Equation (14). The next section is devoted to illustrating the complexity of this matter in the context of pure jump processes.

4 Compound Poisson degradation processes

In this section, we assume a pure jump process for both degradation and threshold process. This relates to the situation where μ_L, σ_L, μ_T and σ_T are equal to 0. After estimating β and assuming continuous data (D1), we use the information about the coupled unobservable jump by considering $(1+\beta)X_i$ instead of X_i. This allows us to treat $X(t)$ and $\xi(t)$ as independent processes, which will prove useful in the computation of sub-optimal policies and RUL.

First, we investigate the basic properties of the model at hand and provide a glimpse at the complexity of determining the probabilities and the expectation appearing in Equation (14).

Theorem 4.1 (Time to the CM threshold)
Let $F_{Z_1}^{*k}(\cdot)$ denote the k-fold convolution of Z_1 as defined in Equation (4). Then, the distribution of T is given by

$$\mathbb{P}(T < t) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \bar{F}_{Z_1}^{*k}(\xi_0).$$

Proof. The result follows directly from conditioning on the value of the underlying Poisson process. For the full proof, see appendix A. □

A threshold policy τ might be seen as an upper boundary for the condition process, for which the distribution of T_τ can be derived using similar techniques as above. Otherwise, under the additional assumptions which seem reasonable in practice, Gallot [13] provides an expression for the FPT of CPP in a general upper boundary. The special case of a non-increasing (linear) threshold policy is easier.

Theorem 4.2 (Time to the PM threshold)
Assume $\tau(t) = \tau_0 - \alpha t, \alpha \geq 0$. Then, the distribution of T_τ is given by

$$\mathbb{P}(T_\tau < t) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda_L t)^k}{k!} \bar{F}_{X_1}^{*k}(\tau_0 - \alpha t).$$

Proof. As $X(t)$ is non-decreasing, exceeding $\inf_{s\in[0,t]} \tau(s)$ will inevitably lead to crossing the PM threshold. Since τ is decreasing and $[0,t]$ is compact, the minimum is attained at the end point of the interval. Therefore, following a similar reasoning as in Theorem 4.1 yields the result. □

Remark
The reasoning in Theorem 4.2 extends to any non-increasing policy $\tau(t)$.

4.1 Stationary threshold policies

The class of stationary policies is interesting because such policies are simple and achieve high usability. We first focus on computing the optimal stationary policy for a simplistic example in section 4.1.1 and then, as this is a difficult problem, illustrate how well we can approximate it. Section 4.1.4 provides an insight into the validity of the approximations proposed in sections 4.1.2 and 4.1.3 for this example.

Consider an example with Poisson degradation processes, that is, with deterministic jumps of size 1 and initial failure threshold $\xi_0 \in \mathbb{N}$. This refers to a degradation model where the number of events required for failure is fixed, but not all of them are observed. Figure 4 shows a sample path of this model.

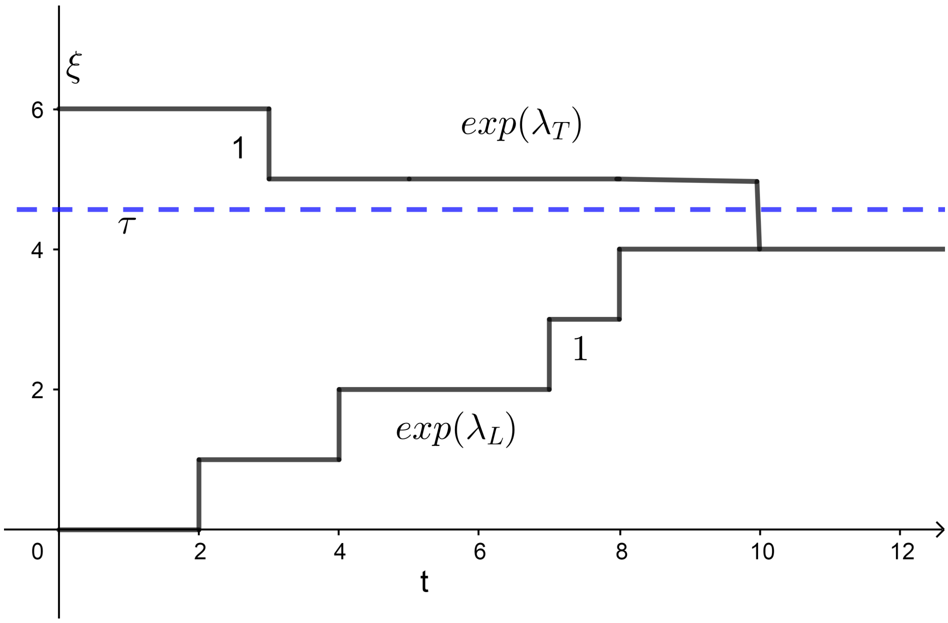

Figure 4: Sample path independent Poisson degradation processes.

The next corollaries describe the dynamics of this example.

Corollary 4.2.1 (Time to the CM threshold)
As a result of Theorem 4.1, we have that $T \sim Gamma(\xi_0, \lambda)$.

It is obvious that this lifetime distribution has an increasing failure rate (IFR). Using similar arguments, we obtain an expression for the time to the PM threshold.

Corollary 4.2.2 (Time to the PM threshold)
Let $\tau \in [0, \xi_0]$ *integer, then for the time to the PM threshold we have that* $T_\tau \sim Gamma(\tau, \lambda_L)$.

Having established the basic dynamics of the example, the next section is devoted to computing the optimal stationary policy for this example.

4.1.1 Optimal stationary threshold policies for Poisson degradation processes

For non-decreasing degradation processes, note that as soon as $\xi_0 - \xi(t) \leq \tau$, failure will happen with probability 1. Therefore, it is useful to investigate this particular event.

Definition 4.1 (Hidden failure)
The time to a 'hidden failure' is defined as

$$T_\xi = inf\{t \geq 0 : \xi_0 - \xi(t) \leq \tau\}.$$

By the law of total probability, we have that

$$\begin{aligned}\mathbb{P}(T \leq T_\tau) &= \mathbb{P}(T \leq T_\tau, T_\xi < T_\tau) + \mathbb{P}(T \leq T_\tau, T_\tau < T_\xi) \\ &= \mathbb{P}(T_\xi < T_\tau) + \mathbb{P}(T \leq T_\tau, T_\tau < T_\xi).\end{aligned}$$

The last term covers the situation where $X(t)$ jumps over both $\xi_0 - \xi(t)$ and τ_0. In the example, this happens with probability 0 and since T_τ and T_ξ are independent

$$\begin{aligned}\mathbb{P}_\tau(\text{failure}) &= \mathbb{P}(T_\tau < T_\xi) \\ &= \mathbb{P}(\text{Erlang}(\lambda_L, \tau) < \text{Erlang}(\lambda_T, \xi_0 - \tau)) \\ &= \sum_{k=\tau}^{\xi_0 - 1} \binom{\xi_0 - 1}{k} (\frac{\lambda_L}{\lambda})^k (\frac{\lambda_T}{\lambda})^{\xi_0 - k - 1}.\end{aligned}$$

For a proof, see appendix A. In particular, this implies that $T \leq T_\tau \iff T_\xi < T_\tau$. This can be used to compute the distribution of T_{PM}. Using the definition of conditional probability, we have

$$\begin{aligned}\mathbb{P}(T_\tau < x | T_\tau < T_\xi) &= \frac{\mathbb{P}(T_\tau < x, T_\tau < T_\xi)}{1 - \mathbb{P}_\tau(\text{failure})} \\ &= \frac{1}{1 - \mathbb{P}_\tau(\text{failure})} \int_0^x f_{T_\tau}(y) \bar{F}_{T_\xi}(y) dy\end{aligned}$$

Taking the derivative w.r.t. x gives

$$f_{T_\tau | T_\tau < T_\xi}(x) = \frac{f_{T_\tau}(x) \bar{F}_{T_\xi}(x)}{1 - \mathbb{P}_\tau(\text{failure})}.$$

Using this, we can compute the conditional expectation

$$\mathbb{E}[T_{PM}] = (1 - \mathbb{P}(T \leq T_\tau))^{-1} \frac{\tau (\lambda_L/\lambda)^\tau}{\tau! \lambda} \sum_{k=0}^{\xi_0 - \tau - 1} \frac{(\lambda_T/\lambda)^k (\tau + k)!}{k!}.$$

For the complete derivation, see appendix A. The following figures shows the dependence on the policy and the approximation resulting by using the unconditional expectation.

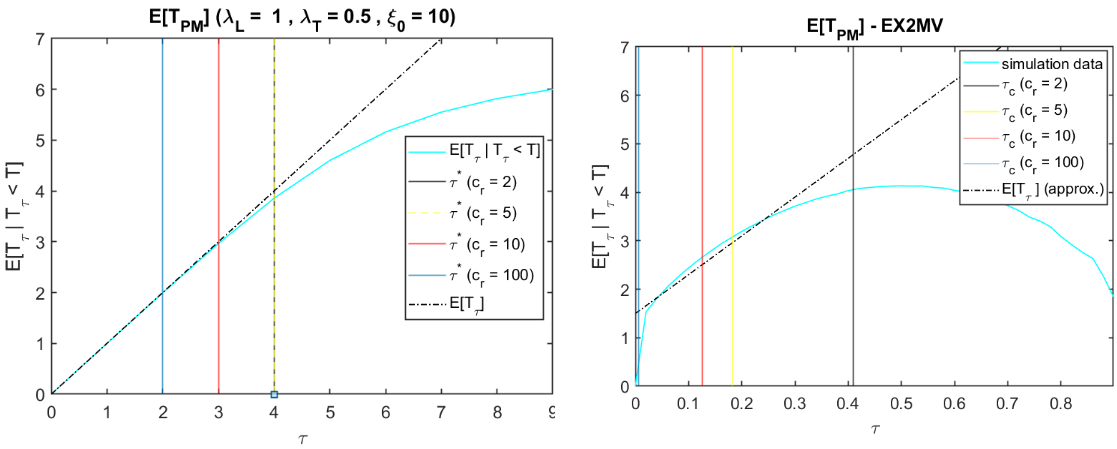

Figure 5: Expected time to preventive maintenance.

For the computation of $\mathbb{E}[T_{CM}]$, we assume that $\mathbb{E}[T_{CM}] \approx \mathbb{E}[T]$. Note that for $\tau \to \xi_0$, this becomes accurate. Figure 6 demonstrates a comparison of this approximation to the simulated values.

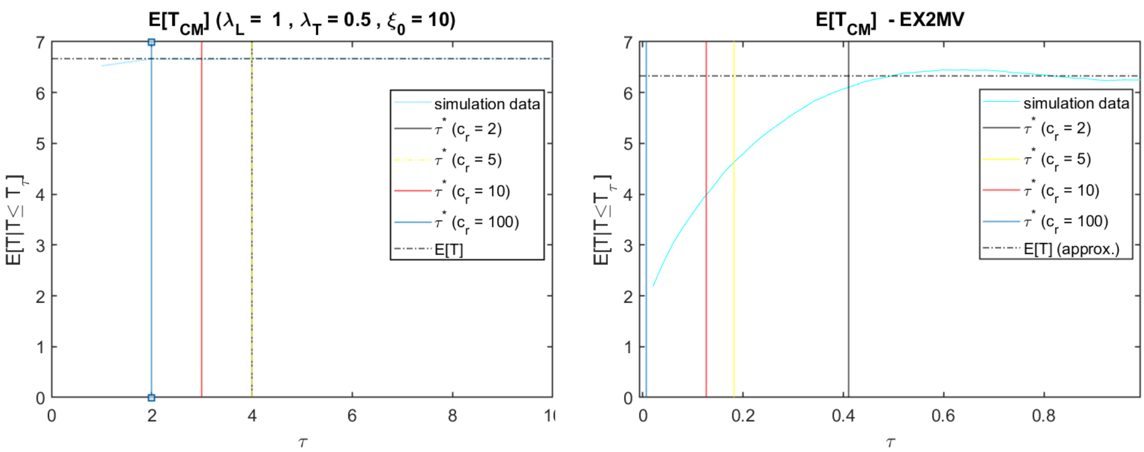

Figure 6: Expected time to corrective maintenance

The approximation is reasonable around τ^* for the example, whilst for the general CPP models this is not the case. The different behavior is explained by the presence of (large) jumps leading to (instant) failure. Note from the simulation data that for small τ, both $\mathbb{E}[T_{CM}]$ and $\mathbb{P}(T \leq T_\tau)$ become small, hence their product too. The approximation given by $\mathbb{E}[T_{CM}]\mathbb{P}(T \leq T_\tau) \approx \mathbb{E}[T]\mathbb{P}(T \leq T_\tau)$ preserves this property as well. Figure 7 shows the structure of the computed optimal stationary policies for a grid of arrival intensities using this approximation only in the denumerator of Equation (14).

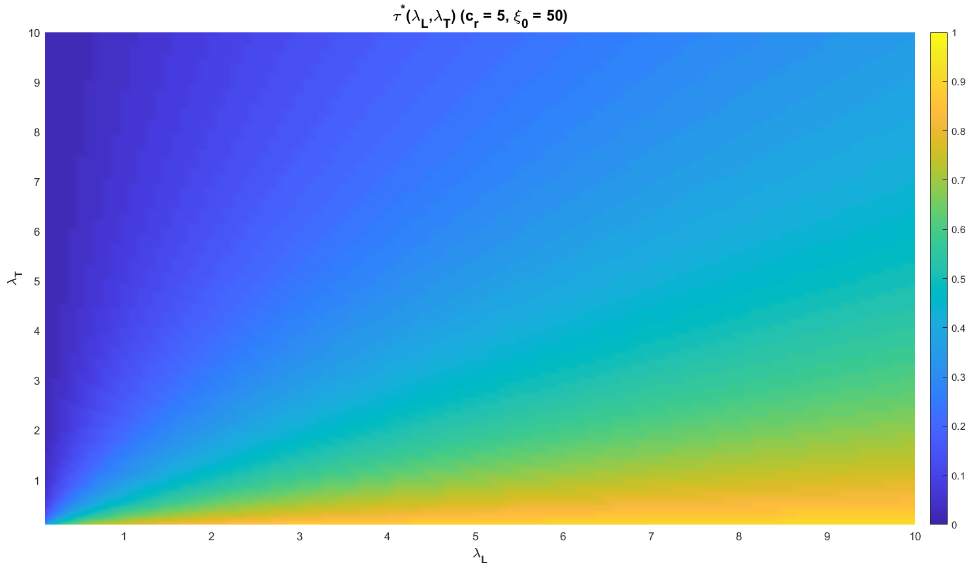

Figure 7: Optimal stationary CBM policies.

As can be observed by the rays, the optimal stationary policy is relatively robust when the fraction $\frac{\lambda_L}{\lambda_T}$ is preserved. Similar behavior, as expected, occurs for other cost ratios. These plots can be found in appendix B. We proceed with computing approximations for the optimal, stationary policy.

4.1.2 TBM approximations of optimal stationary threshold policies

An interesting subset of threshold policies is given by the TBM policies in which PM is performed after a deterministic time $\tau_{TBM} \in \mathbb{R}^+$, regardless of the state of the system. Any TBM policy can be mapped to a condition threshold policy τ_{CBM} by setting

$$\tau_{CBM}(t) = \mathbb{E}[X(\tau_{TBM})] = \lambda_L \tau_{TBM} \mathbb{E}[X_1]. \tag{16}$$

Considering such approximations is interesting because the computation of (sub-)optimal TBM policies has a few advantages, as we can express all terms in Equation (14) in terms of the lifetime T, meaning that

$$g(\tau) = \frac{c_r F_T(\tau) + \bar{F}_T(\tau)}{\int_0^\tau x dF_T(x) + \tau \bar{F}_T(\tau)}. \tag{17}$$

Finding the roots of the derivative of $g(\tau)$ with respect to τ is equivalent (after cumbersome, but straightforward computations) to obtaining the roots of the following equation

$$g(\tau) = (c_r - 1)\frac{f_T(\tau)}{\bar{F}_T(\tau)} \equiv (c_r - 1)h_T(\tau), \tag{18}$$

which is often done by using numerical methods. We will look briefly at the error made by Equation (16) for (sub-)optimal τ_{TBM} policies.

TBM approximations using the Law of Large Numbers (LLN)

Using LLN, the TBM approximation approximates T by its expectation $\mathbb{E}[T]$:

$$g(\tau) \approx \frac{c_r \mathbb{1}\{\mathbb{E}[T] \leq \tau\} + \mathbb{1}\{\mathbb{E}[T] > \tau\}}{\mathbb{E}[T]\mathbb{1}\{\mathbb{E}[T] \leq \tau\} + \tau \mathbb{1}\{\mathbb{E}[T] > \tau\}} = \begin{cases} \frac{c_r}{\mathbb{E}[T]} & \text{if } \mathbb{E}[T] \leq \tau, \\ \frac{1}{\tau} & \text{if } \mathbb{E}[T] > \tau. \end{cases}$$

This is discontinuous with no local minima. However, the minimizer can be approximated by $\tau_{TBM1} = \mathbb{E}[T] - \epsilon$ for ϵ small but positive. As this approach yields no interesting structure, we need to incorporate more information, for example, by a non-deterministic approximation.

TBM approximations using the Central Limit Theorem (CLT)

The FPT can be seen as a (random) sum of interarrival times, and thus motivated by the Central Limit Theorem, it can be approximated by a normal distribution. On average, the number of jumps until failure is given by

$$\mathbb{E}[N(T)] = \lambda \mathbb{E}[T] = \frac{\lambda(\xi_0 - \mathbb{E}[\Delta(T)])}{\lambda_L(1+\beta)\mathbb{E}[X_1] + \lambda_T \mathbb{E}[Y_{d,1}]} := n_T.$$

Then, T is approximated by a Gamma distribution with shape parameter n_T and with scale parameter $\frac{1}{\lambda}$. Assuming that n_T is sufficiently large we have that $\frac{T - \frac{n_T}{\lambda}}{\frac{n_T}{\lambda^2}} \sim \mathcal{N}(0,1)$ by CLT and thus $T \approx \mathcal{N}(\frac{n_T}{\lambda}, \frac{n_T}{\lambda^2})$. We deduce

$$\mathbb{P}(T < \tau) = \mathbb{P}(\frac{T - \frac{n_T}{\lambda}}{\sqrt{\frac{n_T}{\lambda^2}}} < \frac{\tau - \frac{n_T}{\lambda}}{\sqrt{\frac{n_T}{\lambda^2}}}) \approx \Phi(\frac{\tau - \frac{n_T}{\lambda}}{\sqrt{\frac{n_T}{\lambda^2}}}) = \Phi(\frac{\lambda\tau - n_T}{\sqrt{n_T}}). \tag{19}$$

As T is differentiable, one could approximate its density by

$$f_T(t) = \frac{d}{dt}\mathbb{P}(T < t) \approx \frac{\lambda}{\sqrt{n_T}} \phi(\frac{\lambda\tau - n_T}{\sqrt{n_T}}). \tag{20}$$

After substituting the expressions for F_T and f_T given by Equation (19) and (20) into Equation (17), one can use numerical methods to find a sub-optimal policy using Equation (18).

Of course, one can compute the exact distribution of T using Theorem 4.1 and afterwards, compute the optimal TBM policy τ^*_{TBM} using Equation (18). Figure 8 show the error of the approximation $\tau(t)$ approximation using the optimal TBM policy τ^*_{TBM}.

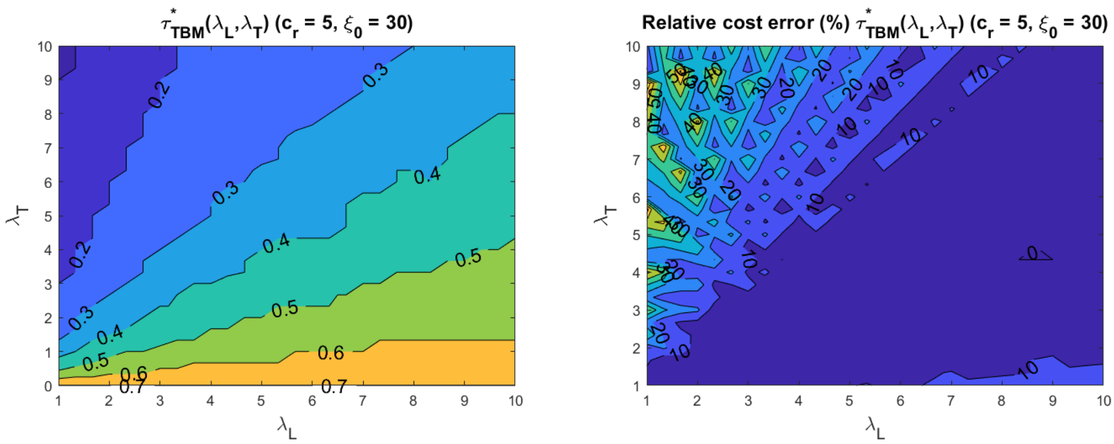

Figure 8: CBM approximations using the optimal TBM policy.

Largely, the error is below 5%, while for edge cases where $\frac{\lambda_L}{\lambda_T}$ is small, this can increase up to 30%.

4.1.3 CBM approximations of threshold policies

Instead of approximating the optimal CBM policy using a (sub-)optimal TBM policy, we can also approximate the CBM policy in a more direct way. At $t > 0$, on average $\lfloor \mathbb{E}[N_L(t)] \rfloor$ observable jumps will have occured. Using that $\mathbb{E}[X(t)] = \lambda_L t \mathbb{E}[X_1]$, $\mathbb{V}[X(t)] = \lambda_L t \mathbb{E}[X_1^2]$ and $\lambda_L t$ is sufficiently large, an approximation based on the Central Limit Theorem is given by

$$X(t) \overset{d}{\approx} \mathcal{N}(\lambda_L \mathbb{E}[X_1]t, \lambda_L \mathbb{E}[X_i^2]t).$$

And using similar arguments we find an approximation (in distribution) for the failure threshold given by

$$\xi(t) \overset{d}{\approx} \mathcal{N}(\lambda_T \mathbb{E}[Y_{d,1}]t, \lambda_T \mathbb{E}[Y_{d,1}^2]t).$$

Normal approximations of a (random) sum of random variables are common in many fields such as risk theory, stock prices and service times, see, e.g., [22] for detailed studies about the accuracy of such approximations. The probability of failure is then approximated by the probability that PM has not occurred before the expected FPT $\mathbb{E}[T]$, e.g.,

$$\begin{aligned} \mathbb{P}(\text{failure}) &\approx \mathbb{P}(X(\mathbb{E}[T]) < \tau) \\ &\approx \Phi\Big(\frac{\tau - \mathbb{E}[X(\mathbb{E}[T])]}{\sqrt{\mathbb{V}[X(\mathbb{E}[T])]}}\Big). \end{aligned} \tag{21}$$

Then, we optimize

$$g(\tau) \approx \frac{c_r \Phi\Big(\frac{\tau - \mathbb{E}[X(\mathbb{E}[T])]}{\sqrt{\mathbb{V}[X(\mathbb{E}[T])]}}\Big) + \Big(1 - \Phi\Big(\frac{\tau - \mathbb{E}[X(\mathbb{E}[T])]}{\sqrt{\mathbb{V}[X(\mathbb{E}[T])]}}\Big)\Big)}{\mathbb{E}[T]\Phi\Big(\frac{\tau - \mathbb{E}[X(\mathbb{E}[T])]}{\sqrt{\mathbb{V}[X(\mathbb{E}[T])]}}\Big) + \mathbb{E}[T_\tau]\Big(1 - \Phi\Big(\frac{\tau - \mathbb{E}[X(\mathbb{E}[T])]}{\sqrt{\mathbb{V}[X(\mathbb{E}[T])]}}\Big)\Big)}. \tag{22}$$

Remark
This reasoning also extends to all non-increasing threshold policies $\tau(t)$ *by replacing* $\tau = \tau(\mathbb{E}[T])$ *in Equation 21.*

For the example with Poisson processes, Figure 9 shows the resulting policies and relative errors.

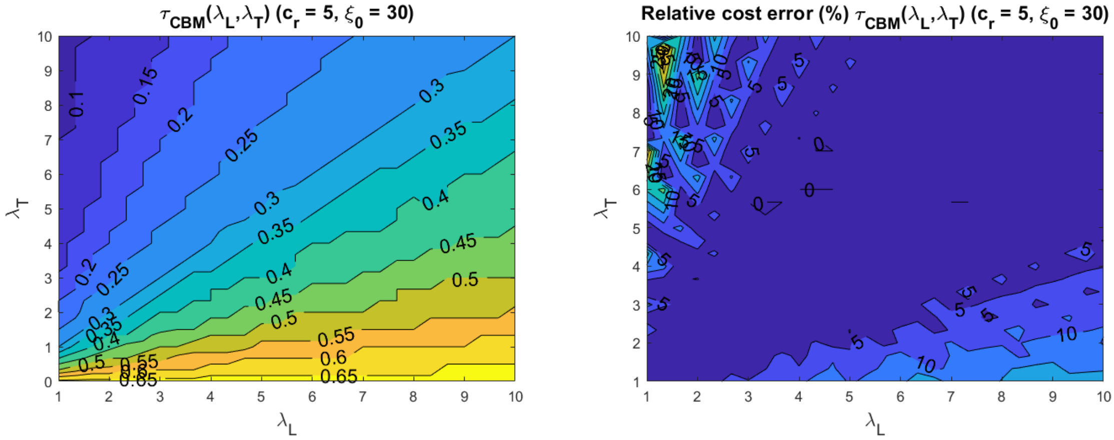

Figure 9: CBM approximations using normal approximations of degradation processes.

As it can be seen, mostly the relative error is within 10% from the optimal solution as well. The next subsection discusses the validity of these approximations in more detail.

4.1.4 Validity of approximations

In this part, we assess the validity of the proposed policies, i.e., the extent to which such policies can approximate the optimal solution depending on the arrival rates and the cost ratio for the example with the Poisson processes. An approximation is considered valid if it is within 10% of the optimal solution. Under this remark, it can be observed that the proposed policies are not valid for all the model configurations. Appendix B contains figures illustrating the performance of the aforementioned approximations. Table 7 provides an overview of the quality of the proposed approximations of the optimal decision rules. Note that the TBM approximation using the Gamma distribution is not included, as it coincides with the approximation using the optimal TBM policy for Poisson degradation processes.

Approximation	Model	Validity (c_r)	Validity $\lambda_L\lambda_T^{-1}$
$\tau = \mathbb{E}[X(\tau^*_{TBM})]$	TBM	[2, 10]	[1/2, 5]
$\tau = \mathbb{E}[X(\tau_{CLT})]$	TBM	[2, 10]	[1, 4]
$\tau = \mathbb{E}[X(\tau_{CBM})]$	CBM	[2, 5]	[2, 5]

Table 7: Overview approximations to optimal decision rules.

We now focus our attention on the larger class of non-stationary, deterministic threshold policies.

4.2 Non-stationary threshold policies

Since condition-monitoring only monitors $X(t)$ and $\xi(t)$ is unobservable for $t > 0$, it follows that the optimal policy is no longer a stationary policy as the only available information about $\xi(t)$ are $X(t), t$ and ξ_0. Time-dependent policies with a simple structure, for example, step functions or (piece-wise) linear functions are discussed in this section.

4.2.1 Linear threshold policies

An intuitive initial guess is to consider a linear policy $\tau(t)$ with slope equal to $\mathbb{E}[\xi(t)]$. The remaining problem is to determine the intercept of $\tau(t)$, say $\tau_0 \in [\tau^*, \xi_0]$, which can be done by means of simulation.

If we do not fix the slope, the complexity of the optimization problem increases since we have two free parameters. Figures 10 and 11 suggest that there is no reason to consider other slopes and it contains enough evidence to conclude that there are time-dependent policies which have a significantly better cost than the stationary policies.

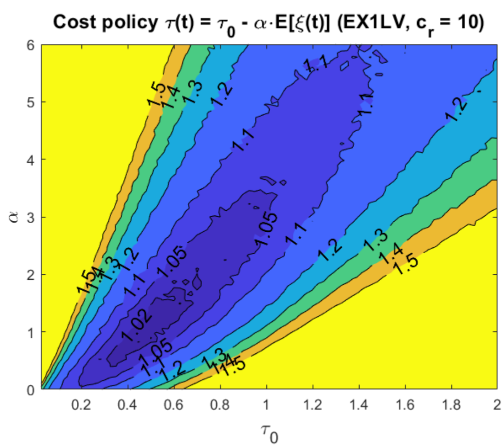

Figure 10: Cost linear policies for (EX1LV).

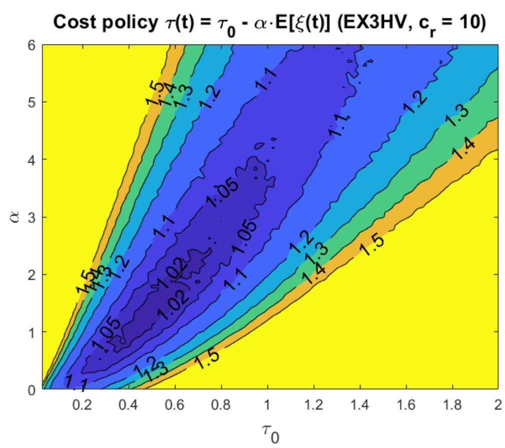

Figure 11: Cost linear policies for (EX3HV).

The level curves show the cost relative to the cost of the optimal linear policy found. For $\alpha = 0$, note that the linear policy reduces to a stationary policy. As α increases, there seems to be more room for error in choosing the value for τ_0. This raises the question of whether there exists policies within the class of piece-wise linear policies which achieve a lower cost.

4.2.2 A piece-wise linear threshold policy

A more general class of policies is those of the step functions. In analysis, step functions are often used to approximate elements in function spaces. From an engineering perspective, step functions allow for a trivial check whether PM or CM has occured. For general functions, this often involves numerical root-finding methods, which are too slow for our purpose. Building on the previous class of policies, we aim to capture the evolution of the stochastic processes in the evolution of the policy. At time t, we observe $X(t) = x_t$. As an approximation, we assume that $\{\xi(t)|\xi(t) < \xi_0 - x_t\} \approx \mathbb{E}[\xi(t)]$ and therefore $\{\Delta(t)|X(t) = x_t, \xi(t) < \xi_0 - x_t\} \approx \xi_0 - x_t - \mathbb{E}[\xi(t)]$. Let $X'(s) \equiv X(s) = X(t+s) - X(t)$, $\xi'(s) \equiv \xi(s) = \xi(t+s) - \xi(t)$ and $\tilde{X}'(s) = X'(s) + \xi'(s)$. Define $T'_{x_t} = inf\{s \geq 0 : \tilde{X}'(s) \geq \xi_0 - x_t - \mathbb{E}[\xi(t)]\}$, let $\tau \in \mathbb{R}^+$ and similarly, define $T'_\tau = inf\{s \geq 0 : X'(s) \geq \tau\}$

$$\tilde{g}(\tau, t, x_t) = \frac{c_c \mathbb{P}(T'_{x_t} \leq T'_\tau) + c_p \tilde{\mathbb{P}}(T'_{x_t} > T'_\tau)}{\mathbb{E}[T'_{x_t} \wedge T'_\tau] + t}.$$

Thus, $g(\tau, t)$ can be interpreted as the long-term rate of costs with respect to the (expected) gap at time t, given that the system has been alive for t time units. The proposed policies are then:

1. $\tau_1(t, x_t) = x_t + \underset{\tau \in [0, \xi_0 - x_t - \mathbb{E}[\xi(t)]]}{\arg\min} \tilde{g}(\tau, t, x_t)$,
2. $\tau_2(t) = \mathbb{E}[X(t)] + \underset{\tau \in [0, \mathbb{E}[\Delta(t)]]}{\arg\min} \tilde{g}(\tau, t, \mathbb{E}[X(t)])$.

Note that $\tau_1(t)$ requires real-time data, whilst $\tau_2(t)$ does not and can thus be computed prior to installation. In this work, we shall only consider τ_2. It can be easily seen that $\tau_1(0) = \tau_2(0) = \tau^*$. In general, the policy first decreases and then increases. The moment the policy starts increasing is interesting because it indicates that the optimization problem has returned the most conservative policy (i.e., the arg max is $\tau = 0$). However, since the gap is non-increasing, this will result in an increasing policy, which is undesirable. Two ways to tackle this behavior are to forbid such behavior, e.g., work with the minimum value (denote this policy with τ_{min}) or perform PM at the turning point (denote this policy with $\tau_0(t)$), which is the true interpretation of the policy at that time. In order to investigate the performance of these policies, we use the coefficient of variation c_v, which is defined as the ratio of the standard deviation to the mean. Note that the cost in a cycle C_τ is given by $C_\tau = c_c \mathbb{1}_\tau\{\text{CM}\} + c_p \mathbb{1}_\tau\{\text{PM}\}$. Subsequently, the coefficient of variation can be expressed as

$$\begin{aligned} c_v &= \sqrt{\mathbb{V}[C_\tau]}\mathbb{E}[C_\tau]^{-1} \\ &= (c_c - c_p)\sqrt{\mathbb{P}(T \leq T_\tau)\mathbb{P}(T > T_\tau)}\big(c_c \mathbb{P}(T \leq T_\tau) + c_p \mathbb{P}(T > T_\tau)\big)^{-1}. \end{aligned}$$

Table 8 provides a comparison of the performance of these policies and the optimal stationary and linear policy.

Policy \ Example		(EX1LV)		(EX2MV)		(EX3HV)	
		(μ, c_v)	95%-CI	(μ, c_v)	95%-CI	(μ, c_v)	95%-CI
$c_r = 2$	τ_{CM}	(0.359, 0)	[0.349, 0.369]	(0.356, 0)	[0.346, 0.365]	(0.351, 0)	[0.341, 0.361]
	τ^*	(0.303, 0.353)	[0.293, 0.313]	(0.306, 0.350)	[0.296, 0.316]	(0.302, 0.353)	[0.291, 0.314]
	τ^*_{lin}	(0.291, 0.349)	[0.281, 0.302]	(0.297, 0.352)	[0.286, 0.307]	(0.298, 0.354)	[0.287, 0.308]
	$\tau_2(t)$	(0.324, 0.339)	[0.315, 0.334]	(0.321, 0.341)	[0.311, 0.333]	(0.316, 0.342)	[0.305, 0.323]
	$\tau_{min}(t)$	(0.304, 0.347)	[0.293, 0.314]	(0.306, 0.347)	[0.292, 0.317]	(0.305, 0.350)	[0.293, 0.317]
	$\tau_0(t)$	(0.369, 0.201)	[0.360, 0.377]	(0.390, 0.214))	[0.381, 0.400]	(0.378, 0.255)	[0.367, 0.388]
$c_r = 10$	τ_{CM}	(1.794, 0)	[1.747, 1.841]	(1.780, 0)	[1.723, 1.836]	(1.755, 0)	[1.701, 1.808]
	τ^*	(0.544, 1.310)	[0.493, 0.593]	(0.583, 1.350)	[0.529, 0.634]	(0.629, 1.418)	[0.574, 0.683]
	τ^*_{lin}	(0.460, 1.222)	[0.421, 0.498]	(0.509, 1.336)	[0.463, 0.556]	(0.561, 1.393)	[0.512, 0.611]
	$\tau_2(t)$	(1.141,1.050)	[1.094, 1.187]	(1.125, 1.064)	[1.076, 1.174]	(1.085, 1.097)	[1.023, 1.142]
	$\tau_{min}(t)$	(0.543, 1.286)	[0.500, 0.586]	(0.584, 1.348)	[0.534, 0.633]	(0.628, 1.375)	[0.570, 0.686]
	$\tau_0(t)$	(0.839, 0.423)	[0.809, 0.870]	(0.852, 0.557)	[0.816, 0.889]	(0.887, 0.801)	[0.836, 0.936]

Table 8: Performance of (non-)stationary policies for (EX1LV), (EX2MV), and (EX3HV) with each entry consisting of mean μ, coefficient of variation c_v and 95% CI obtained by simulation.

At any time, the update algorithm bases the new policy on an infinite horizon, which is proved to be too conservative. We see that none of these policies outperform τ^*, although note that the hybrid policy $\tau_0(t)$ achieves a lower coefficient of variation, which is not uncommon for TBM policies.

In order to tackle this, the update algorithm should include the next point in time where the policy is updated. This directs us to the policies based on the *remaining useful lifetime.*

4.3 Computation of the remaining useful lifetime

In this section, our focus is on the computation of the remaining useful lifetime (RUL) and decision-making using the RUL estimates. RUL is defined as the time to failure, given that the system has not failed yet at time $t > 0$.

Definition 4.2 (Remaining useful lifetime)
Define RUL as $T_t = (T|T > t) - t$.

It can be seen that $\mathbb{P}(T_t > s) = \bar{F}_T(t+s)\bar{F}_T(t)^{-1}$, which is given in Equation 4.1. Now, at an inspection time t, we observe the observable degradation process $X(t)$. This information is particularly of help for improving the RUL estimates.

Definition 4.3 (Remaining useful lifetime with observation)
Define RUL with observation as $T_{t,x} = (T|X(t) = x, T > t) - t$.

$$T_{t,x} := inf\{s > 0 : X(t+s) \geq \xi_0 - \xi(t+s)|X(t) = x, X(u) < \xi_0 - \xi(u), u \in [0,t]\}.$$

We proceed with computing the distribution of $T_{t,x}$ in the next theorem.

Theorem 4.3
The distribution of $T_{t,x}$ is given by

$$F_{T_{t,x}}(s) = 1 - F_{\xi(t)}(\xi_0 - x)^{-1} \int_0^{\xi_0 - x} f_{\xi(t)}(u) F_{\tilde{X}(s)}(\xi_0 - x - u) du.$$

Proof. Note that

$$\bar{F}_{T_{t,x}}(s) = \mathbb{P}(T_{t,x} > s) = \mathbb{P}\big(\sup_{u \in [0,s]} \tilde{X}(t+u) < \xi_0 | X(t) = x, X(u) < \xi_0 - \xi(u), u \in [0,t]\big).$$

In the case of a.s. non-decreasing processes, we have

$$X(u) < \xi_0 - \xi(u), u \in [0,t] \iff \sup_{u \in [0,t]} \tilde{X}(u) < \xi_0 \iff X(t) < \xi_0 - \xi(t),$$

as $[0,t]$ is compact and thus $\tilde{X}(u)$ attains its supremum at the edge of the interval. Using the same argument, we get

$$\begin{aligned}
&\bar{F}_{T_{t,x}}(s) = \mathbb{P}\big(\sup_{u \in [0,s]} \tilde{X}(t+u) < \xi_0 | X(t) = x, \xi(t) < \xi_0 - x\big) \\
&= \mathbb{P}\big(\tilde{X}(t+s) < \xi_0 | X(t) = x, \xi(t) < \xi_0 - x\big) \\
&= \mathbb{P}\big(X(t+s) - x + \xi(t+s) < \xi_0 - x | X(t) = x, \xi(t) < \xi_0 - x\big).
\end{aligned}$$

Now, $\xi(t)$ and $X(t)$ are independent, the support of $\xi(t)|\xi(t) < \xi_0 - x$ is precisely $[0, \xi_0 - x)$ and its density is given by

$$f_{\xi(t)|\xi(t)<\xi_0 - x}(u) = f_{\xi(t)}(u) F_{\xi(t)}(\xi_0 - x)^{-1}.$$

Let $\tilde{X}'(s) \equiv \tilde{X}(s) = \tilde{X}(t+s) - \tilde{X}(t)$, then

$$\begin{aligned}
&\mathbb{P}\Big(X(t+s) - x + \xi(t+s) < \xi_0 - x | X(t) = x, \xi(t) < \xi_0 - x\Big) \\
&= F_{\xi(t)}(\xi_0 - x)^{-1} \int_0^{\xi_0 - x} f_{\xi(t)}(u) \mathbb{P}\Big(X(t+s) - x + \xi(t+s) - u < \xi_0 - x - u | X(t) = x, \xi(t) = u\Big) du \\
&= F_{\xi(t)}(\xi_0 - x)^{-1} \int_0^{\xi_0 - x} f_{\xi(t)}(u) \mathbb{P}\Big(\tilde{X}'(s) < \xi_0 - x - u\Big) du.
\end{aligned}$$

Using above expression and the fact that $F_{T_{t,x}}(s) = 1 - \bar{F}_{T_{t,x}}(s)$ yields the result. □

Remark
As the pdf and cdfs in Equation 4.3 might be difficult to compute, one could use the approximations proposed in sections 4.1.2 and 4.1.3.

In application, one would like to use decision-making based on the RUL to maintain a certain reliability level.

Definition 4.4 (RUL with reliability $1 - q$)
Let $q \in (0,1)$. The $RUL(t,q)$ is defined as the maximum time such that the system will survive with probability at least $1 - q$, thus

$$RUL(t,q) = sup\{s > 0 : \mathbb{P}(T_{t,x} < s) \leq q\}.$$

Before we can completely clarify the matter, we first need the definition of a quantile function.

Definition 4.5 (Quantile function)
Let F_X be the distribution function of a random variable X and $p \in [0,1]$. The quantile function F_X^{-1} returns

$$F_X^{-1}(p) = inf\{x \in \mathbb{R} : F_X(x) \geq p\}.$$

The following corollary can be used to compute RUL(t, q).

Corollary 4.3.1
Using Theorem 4.3, we have that

$$RUL(t, q) = F_{T_{t,x}}^{-1}(q). \tag{23}$$

As distribution functions are non-decreasing, this can be solved using standard numerical methods. As mentioned, the RUL can be used to construct maintenance policies.

Definition 4.6 (RUL decision-making with reliability $1 - q$)
Let $I = \{t_i | i = 1, 2, \ldots\}$ be a set of periodic inspection times and denote the inspection interval with $\Delta t = t_2 - t_1$. Furthermore, let $q \in (0, 1)$ and $c_s < c_p$ model the maintenance setup cost. Define the RUL maintenance policy as

$$\tau_{RUL}(t; q, \Delta t) = \begin{cases} \text{“do CM”}, & \text{if } t = T \text{ at cost } c_c, \\ \text{“do PM”}, & \text{else if } t \in I \text{ and } RUL(t, q) < \Delta t \text{ at cost } c_p, \\ \text{“do nothing”} & \text{else if } t \in I \text{ and } RUL(t, q) \geq \Delta t \text{ at cost } c_s. \end{cases}$$

Note that this is an *online* policy, meaning that at every inspection time an RUL estimate has to be computed. The time to the PM threshold is thus defined as

Definition 4.7 (Time to PM threshold: RUL)
The time the PM threshold under an RUL policy is defined as the first time the policy returns “do PM”.

$$T_\tau^{RUL} \stackrel{def}{=} inf\{i \in \mathbb{N} : \tau(t_i; q, \Delta t) = \text{“do PM”}\}$$

The optimal RUL policy τ_{RUL}^* is then defined as the solution of the following optimization problem:

$$\tau_{RUL}^*(t_i; q) = \arg\min_{\Delta t > 0} g_{RUL}\big(\tau_{RUL}(t_i; q, \Delta t)\big),$$

where

$$g_{RUL}\big(\tau_{RUL}(t_i; q, \Delta t)\big) = \frac{c_r \mathbb{P}(T \leq T_\tau^{RUL}) + (1 - \mathbb{P}(T \leq T_\tau^{RUL}))}{\mathbb{E}[T_{CM}^{RUL}] \cdot \mathbb{P}(T \leq T_\tau^{RUL}) + \mathbb{E}[T_{PM}^{RUL}](1 - \mathbb{P}(T \leq T_\tau^{RUL}))}.$$

Here $T_{CM}^{RUL} = T | T \leq T_\tau^{RUL}$ and $T_{CM}^{RUL} = T_\tau^{RUL} | T > T_\tau^{RUL}$ denote the time to CM/PM in the case of an RUL policy respectively.

Remark

If the state of the system is not observed, one can alter the definition of this policy in a straightforward manner by using Definition 4.2 instead, i.e., replacing $T_{t,x}$ *by* T_t *where required.*

5 Wiener degradation processes

This section discusses Wiener degradation processes. The Lévy–Khintchine formula (see [25, p. 31]) suggests that any Lévy process exhibiting continuous sample paths must be a Wiener drift process. These processes are of special interest in this work as we aim to approximate the sum of a CPP and Wiener drift process with a single Wiener drift process. In this section, the degradation processes are independent and described by

$$X(t) = \mu_L t + \sigma_T W_t,$$

and

$$\xi(t) = \mu_T t + \sigma_T \tilde{W}_t.$$

Thus by Equation (5)

$$\Delta(t) \overset{\mathrm{d}}{=} \xi_0 - \mu t - \sigma W_t,$$

where $\mu = \mu_L + \mu_T$ and $\sigma = \sqrt{\sigma_L^2 + \sigma_T^2}$. Throughout this section, we assume that μ_L, $\mu > 0$, ensuring that the time to failure is finite almost surely. Note that

$$T = \inf\{t \geq 0 : \Delta(t) \geq 0\} = \inf\left\{t \geq 0 : W_t \geq \frac{\xi_0 - \mu t}{\sigma}\right\},$$

and for any linear policy $\tau(t) = \tau_0 - \alpha t$, we have that

$$T_\tau = \inf\{t \geq 0 : X(t) \geq \tau(t)\} = \inf\left\{t \geq 0 : W_t \geq \frac{\tau_0 - (\alpha + \mu_L)t}{\sigma_L}\right\}.$$

Recall that the property of continuous sample paths ensures that $\Delta(T) = 0 = \Delta_\tau(T_\tau)$ w.p. 1, i.e., both the overshoot and undershoot are zero. Thus, by Theorem 3.1, we have

$$\mathbb{E}[T] = \frac{\xi_0}{\mu} \text{ and } \mathbb{E}[T_\tau] = \frac{\xi_0}{\mu_L + a}.$$

A simple-form expression for the FPT density of a standard Wiener process $\{W_s(t), t \geq 0\}$ with linear boundaries is available and is known to follow an Inverse Gaussian (IG) distribution, see, e.g., [18, p. 238]. In the case of a single linear upper boundary given by $\tau(t) = \tau_0 - \alpha t$, $\tau_0 > 0$, the probability of no up-crossing up to time t is given by

$$\mathbb{P}(\sup_{u \in [0,t]} W_s(u) - y(u) < 0) = \Phi\Big(\frac{-\alpha t + \tau_0}{\sqrt{t}}\Big) - e^{2\alpha\tau_0}\Phi\Big(\frac{-\alpha t - \tau_0}{\sqrt{t}}\Big),$$

which is continuously differentiable and hence we obtain its density

$$f_T(t) = \frac{\tau_0}{\sqrt{2\pi t^3}} e^{-\frac{(\tau_0 - \alpha t)^2}{2t}}.$$

This implies that T and T_τ follow IG distributions with parameters

$$T_\tau \sim \mathrm{IG}\Big(\frac{\tau_0}{\mu_L + \alpha}, \big(\frac{\tau_0}{\sigma_L}\big)^2\Big),$$

and

$$T \sim \text{IG}\Big(\frac{\xi_0}{\mu}, \Big(\frac{\xi_0}{\sigma}\Big)^2\Big). \tag{24}$$

IG lifetime models are often used to model systems which often show early failures and prove to be more reliable later on, for instance, due to a manufacturing error. A detailed study about IG lifetimes has been done by Chhikara and Folks [10], including a full derivation of the maximum likelihood estimators and a description of the behavior of the hazard rate. It increases from zero to a modal value, then decreases, approaching its asymptotic value of, in our context, $\frac{1}{2}\mu^2\sigma^{-2}$. A study about the application of IG lifetimes in maintenance is done by Lemeshko et al. [23]. The pdf, cdf and reliability function of T (and T_τ) are easily obtained and given by

$$f_T(t) = \frac{\xi_0}{\sigma}\sqrt{\frac{1}{2\pi t^3}}e^{-\frac{\left(t-\frac{\xi_0}{\mu}\right)^2}{2\left(\frac{\xi_0}{\mu}\right)^2 t}},$$

and

$$F_T(t) = \Phi\Big(\frac{\xi_0}{\sigma}\frac{1}{\sqrt{t}}\Big(\frac{\mu t}{\xi_0} - 1\Big)\Big) + e^{2\frac{\xi_0\mu}{\sigma^2}}\Phi\Big(-\frac{\xi_0}{\sigma}\frac{1}{\sqrt{t}}\Big(\frac{\mu t}{\xi_0} + 1\Big)\Big). \tag{25}$$

We shall use these results in order to investigate the structural properties of optimal TBM and (stationary) CBM policies, which are discussed in the next section.

5.1 Stationary threshold policies

For the same reasons given in section 4.1, in this part we aim to construct (sub-)optimal stationary threshold policies for Wiener degradation processes. Similarly structured as the preceding sections, we first investigate the optimal TBM policies and in order to cope with the structural and computational complexities, we provide approximations for the optimal stationary threshold policy and formulate a rule of thumb under which this policy is expected to outperform τ_{CM}.

5.1.1 Optimal TBM policies

Derivation of the optimal TBM policy τ^*_{TBM} is rather straightforward and can be obtained by solving Equation (18) using the lifetime distribution given in Equation (25). As the tail of the hazard rate is decreasing, it might be that the optimal TBM policy is to correctively replace the system. Figure 12 shows the structural dependence of τ^*_{TBM} on the stochastic volatility σ.

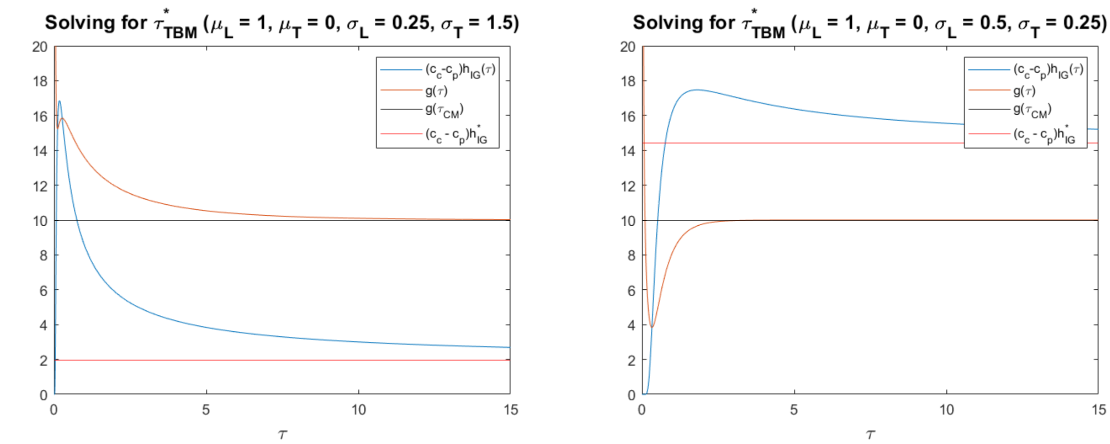

Figure 12: Intersection hazard function and cost function for IG lifetimes with asymptotics.

As can be seen, the intersection points do not necessarily indicate a global minimum of the cost function and thus $\tau^*_{TBM} = \infty$ if $g(\tau_i) > \frac{c_c \mu}{\xi_0}$ for all τ_i such that $g(\tau_i) = (c_c - c_p)h_{IG}(\tau_i)$. Note that for any non-deterministic failure threshold process, from Equation (24) it can be deduced that this behavior only depends on the total drift and stochastic volatility, i.e., it depends only on the parameters of $\tilde{X}(t)$ and not on the individual parameters of $X(t)$ and $\xi(t)$.

Figure 13 shows two examples of the computation of optimal stationary policies, one where μ is fixed, and one where $\sigma = \sqrt{|\mu|}$, which can be seen as the Wiener approximation of a Poisson process based on the first two moments. Observe that, opposed to τ^*_{TBM}, the turning point when τ_{CM} outperforms $\tau^* \in [0, \xi_0]$, actually depends on both σ_L and σ_T. The white region in the plots correspond to $\tau^* = \tau_{CM}$.

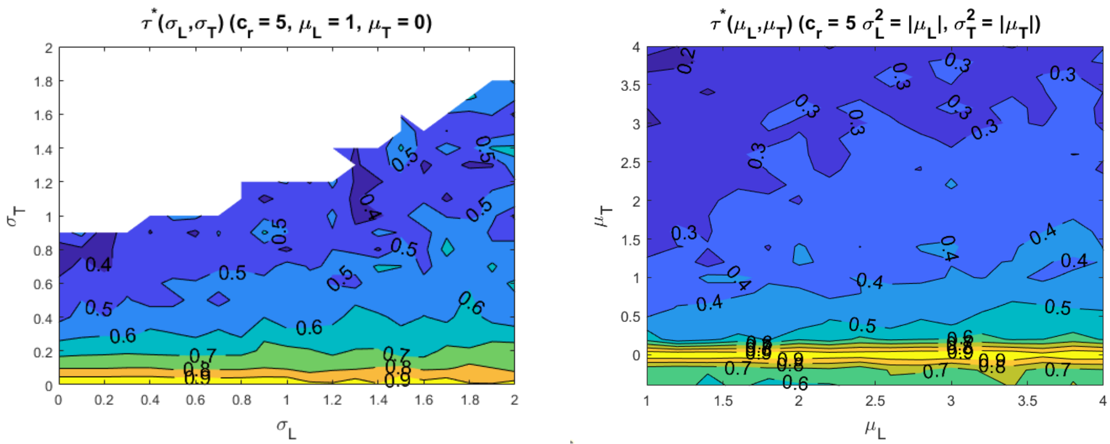

Figure 13: Optimal stationary CBM policies for Wiener degradation processes, obtained by simulation.

For standard Wiener processes, as a rule of thumb, if $c_v = \sigma\mu^{-1} \gg 1$, the optimal policy is given by τ_{CM}. This makes the approximation of $CP^d + W \sim \tilde{W}$ a dangerous matter, as the level curves indicate that structural properties of the optimal stationary threshold policies for Wiener degradation processes and CPPs are very different.

To conclude this section, the sub-optimal stationary policy can then be defined as in Equation (16), e.g., $\tau_{CBM} = \mathbb{E}[X(t^*_{TBM})] = \mu_L t^*_{TBM}$. We proceed with extending the reasoning in section 4.1.3 to Wiener degradation processes.

5.1.2 CBM approximations of threshold policies

Recall the proposed approximation for the probability of failure in Equation (21). In the case of Wiener degradation processes, this becomes

$$\begin{aligned}\mathbb{P}_\tau\big(\text{failure}\big) &\approx \mathbb{P}\big(\sup_{u\in[0,\mathbb{E}[T]]} X(u) < \tau\big)\\ &= \bar{F}_{T_\tau}(\mathbb{E}[T]\big)\end{aligned} \tag{26}$$

Replacing the found probability in Equation (22) yields

$$g(\tau) \approx \frac{c_c \bar{F}_{T_\tau}(\mathbb{E}[T]) + c_p F_{T_\tau}(\mathbb{E}[T])}{\mathbb{E}[T]\bar{F}_{T_\tau}(\mathbb{E}[T]) + \mathbb{E}[T_\tau]F_{T_\tau}(\mathbb{E}[T])},$$

which again can be optimized using numerical methods.

Remark
This reasoning also extends to (piece-wise) linear policies $\tau(t)$, which can be used to approximate nonlinear policies. In particular, one could use this to find the intersect of a linear policy with fixed drift.

The next section focusses on extending the results on the RUL to Wiener degradation processes.

5.2 Approximation of the remaining useful lifetime for Wiener processes

The computation of the RUL for Wiener degradation processes is similar to the procedures in section 4.3. The classical definition of RUL, as in Definition 4.2, is straightforward to compute using Equation (25). The computations in case of using Definition 4.3 are slightly different. Note that in the case of Wiener drift processes, we only have that

$$\sup_{u\in[0,t]} \tilde{X}(u) < \xi_0 \implies X(t) < \xi_0 - \xi(t).$$

Assume that $\mu_L >> \sigma_L$ and $\mu_T >> \sigma_T$. In this case, the supremum is most likely to be found at the edge of the interval and, as an approximation, we replace the stronger condition by the weaker one. This implies that

$$\begin{aligned}\mathbb{P}(T_{t,x} < s) &= \mathbb{P}(\sup_{u\in[0,s]} \tilde{X}(t+u) \geq \xi_0 | X(t) = x, X(u) \geq \xi_0 - \xi(u), u \in [0,t])\\ &\approx \mathbb{P}(\sup_{u\in[0,s]} \tilde{X}(t+u) \geq \xi_0 | X(t) = x, \xi(t) < \xi_0 - x).\end{aligned}$$

Clearly, as $\sigma_L, \sigma_T \to 0$ this approximation is accurate. Conditioning on $\{\xi(t)|\xi(t) < \xi_0 - x\} = u$, the property of independent, stationary increments yields that the remaining lifetime follows an IG distribution

$$(T_{t,x}|\xi(t) = u) \sim \mathrm{IG}\Big(\frac{\xi_0 - x - u}{\mu}, (\frac{\xi_0 - x - u}{\sigma})^2\Big).$$

Using this in the proof of Theorem 4.3, we obtain the following approximation

$$\begin{aligned}F_{T_{t,x}}(s) &\approx [F_{\xi(t)}(\xi_0 - x)]^{-1} \int_{-\infty}^{\xi_0 - x} f_{\xi(t)}(u)\mathbb{P}\big(\sup_{r\in[0,s]} \tilde{X}(r) \geq \xi_0 - x - u\big)du\\ &= F_{\xi(t)}(\xi_0 - x)^{-1} \int_{-\infty}^{\xi_0 - x} f_{\xi(t)}(u)\mathbb{P}\Big(\mathrm{IG}\Big(\frac{\xi_0 - x - u}{\mu}, (\frac{\xi_0 - x - u}{\sigma})^2\Big) < s\Big)du.\end{aligned} \tag{27}$$

An approximation for the RUL(t, q) can be obtained by evaluating the quantile function, as in Equation (23) and used in the proposed RUL policy in Definition 4.6.

Having investigated both CPP and Wiener degradation processes, we now turn our attention to the larger class of Lévy degradation processes.

6 Lévy degradation processes

Recall that any Lévy process[1] can be approximated by the independent sum of a CPP and a Wiener drift process. The goal of this section is to use the previous computations in order to compute (sub-)optimal maintenance policies for Lévy degradation processes. We start by briefly discussing the optimal TBM policy, for which we need the lifetime distribution.

6.1 Computation of (sub-)optimal TBM policies

We first investigate the exact expression for the FPT to see how a meaningful approximation can be constructed, which we can then use in decision-making. Rewrite Equation (3.3) as

$$\tilde{X}_\sigma(t) := \frac{1}{\sigma}\tilde{X} = \sum_{i=1}^{N(t)} \frac{Z_i}{\sigma} + \frac{\mu}{\sigma}t + W_t.$$

and note that

$$\begin{aligned} T &= \inf\left\{ t \geq 0 : \xi_0 - \sum_{i=1}^{N(t)} Z_i - \mu t - \sigma W_t \leq 0 \right\} \\ &\stackrel{d}{=} \inf\left\{ t \geq 0 : \sum_{i=1}^{N(t)} \frac{Z_i}{\sigma} + \frac{\mu}{\sigma}t + W_t \geq \frac{\xi_0}{\sigma} \right\}. \end{aligned}$$

The lifetime is thus the FPT of $\tilde{X}_\sigma(t)$ to $\xi_0\sigma^{-1}$. Denote with T_n the sequence of jump times, thus $T_n \sim \text{Erlang}(n, \lambda)$, where $n \in \mathbb{N}$. Write $\frac{1}{\sigma}Z_1 = Z_\sigma$. The distribution of the FPT T is differentiable and we can use the expression Coutin and Dorobantu [11, p. 1129] have derived in their study. Their main result is repeated in the following theorem.

Theorem 6.1 (Time to the CM/PM threshold for Lévy processes)
The distribution function of T has a right derivative at $t = 0$ and is differentiable for every $t > 0$. At $t = 0$, the density has value

$$f_T(0) = \lambda \bar{F}_{Z_\sigma}(\xi_0\sigma^{-1}) = \lambda \bar{F}_{Z_1}(\xi_0),$$

and for $t > 0$, its density is given by

$$f_T(t) = \lambda \mathbb{E}\Big[\mathbb{1}\{T > t\}\bar{F}_{Z_\sigma}\Big(\xi_0\sigma^{-1} - \tilde{X}_\sigma(t)\Big)\Big] + \mathbb{E}\Big[\mathbb{1}\{T > T_{N(t)}\}\tilde{f}\Big(t - T_{N(t)}, \xi_0\sigma^{-1} - \tilde{X}_\sigma(T_{N(t)})\Big)\Big].$$

Proof. See [11, p. 1129-1133]. □

Remark
The validity of this theorem extends to Lévy processes with negative jumps as well. Moreover, $T < \infty$ a.s. $\iff$ $\mathbb{E}[\tilde{X}(1)] > 0$. For linear policies, this theorem can be used to derive the FPT density of T_τ as well.

[1] See Definition A.1 in appendix A.

Note that $\mathbb{1}\{T > t\} = \mathbb{1}\{\sup_{0<s<t} \tilde{X}_\sigma(t) < \xi_0\sigma^{-1}\}$ and $\tilde{f}$ denotes the density of an $\text{IG}\Big(\frac{\xi_0\sigma^{-1}-\tilde{x}_\sigma}{\mu\sigma^{-1}}, (\xi_0\sigma^{-1} - \tilde{x}_\sigma)^2\Big)$ distribution, which has a natural interpretation as failure due to creeping in the remaining time $t - T_{N(t)}$ with initial gap $\xi_0\sigma^{-1} - \tilde{x}_\sigma$ e.g., as described by Equation (24). Computing the last term is done by first conditioning on the Poisson process $N(t)$, then on the timestamp $T_{N(t)}$ of the last jump and lastly on the joint value of $(\tilde{X}_\sigma(T_{N(t)}), \sup_{u\in[0,T_{N(t)}]} \tilde{X}_\sigma(u))$.

To summarize, evaluating this density, in general, is very complicated. As such, we are interested in an approximation. The expression given above suggests that the density comes from failure due to a jump, or due to creeping. Based on a theorem of Tijms [31, p. 163], we note that any lifetime distribution can be approximated by a finite mixture distribution[2]. Thus, motivated by the preceding sections, we propose an approximation (in distribution of T), say $S \overset{d}{\approx} T$, that is $f_T \approx f_S$, where f_S is given by

$$f_S = wf_{\text{G}} + (1 - w)f_{IG}. \tag{28}$$

Here, $w \in [0, 1]$ denotes the weight, $G(a, b)$ is a Gamma distribution with shape parameter a and scale parameter b. $IG(c, d)$ denotes an Inverse Gaussian distribution with parameters c and d, thus S can be interpreted as a mixture distribution with underlying (independent) lifetime distributions $G(a, b)$ and $IG(c, d)$. Determining the parameters $\theta = (a, b, c, d, w)$ can be done by the method of moments. From the theory of mixture distribution, we know that

$$\mathbb{E}[S(\theta)^j] = w\mathbb{E}[G(a, b)^j] + (1 - w)\mathbb{E}[\text{IG}(c, d)^j].$$

Using Laplace transforms, it can be shown that

$$\mathbb{E}[G(a, b)^j] = \frac{b^j\Gamma(a + j - 1)}{\Gamma(a - 1)},$$

and the j-th moment of an Inverse Gaussian r.v. satisfies a recursive relation given by

$$\begin{aligned}
\mathbb{E}[\text{IG}(c, d)^j] &= \frac{(2j - 1)c^2}{d}\mathbb{E}[\text{IG}(c, d)^{j-1}] + c^2\mathbb{E}[\text{IG}(c, d)^{j-2}], \text{ where} \\
\mathbb{E}[\text{IG}(c, d)] &= c \\
\mathbb{E}[\text{IG}(c, d)^2] &= \frac{c^2(c + d)}{d}.
\end{aligned}$$

Assuming that the first five moments of T are known, for example, using a short simulation, the problem of estimating θ reduces to a nonlinear optimization problem given by

$$\begin{aligned}
\min_\theta \quad & \sum_{j=1}^{5} \mathbb{E}[S^j - T^j]^2 && \text{(29a)} \\
\text{subject to} \quad & a \geq 1, && \text{(29b)} \\
& b, c, d \geq 0, \text{ and} && \text{(29c)} \\
& 0 \leq w \leq 1. && \text{(29d)}
\end{aligned}$$

[2] See Theorem A.1 in appendix A.

Equation (29b) ensures that the Gamma hazard rate is constant ($a = 1$) or IFR ($a > 1$). The hazard rate of S is given by

$$h_S(t) = \frac{f_S(t)}{\bar{F}_S(t)} = \frac{wf_G(t) + (1-w)f_{\text{IG}}(t)}{w\bar{F}_G(t) + (1-w)\bar{F}_{\text{IG}}(t)}.$$

The analysis of such hazard rates can be done by analyzing the individual hazard rates of the subpopulations, see the paper by Block et al. [9] for example, in particular Corollary 2.1. Define the individual hazard rates as h_G and h_{IG}. For finite mixtures, the asymptotic hazard rate converges to the asymptotic hazard rate of the strongest subpopulation. The individual asymptotic hazard rates are given by

$$\lim_{t\to\infty} h_{\text{G}}(t) = b^{-1},$$

and

$$\lim_{t\to\infty} h_{\text{IG}}(t) = d(2c^2)^{-1}.$$

Thus, we conclude that

$$\lim_{t\to\infty} h_S(t) = \min\{b^{-1}, d(2c^2)^{-1}\}.$$

Clearly, $f_{\text{IG}}(0) = 0$, implying that $f_S(0) = wf_{\text{G}}(0) = wb^{-1}$ if $a = 1$ and 0 otherwise. Thus, initially, the hazard rate is IFR and it will converge to $\min\{b^{-1}, d(2c^2)^{-1}\}$. In between, however, the hazard rate might have a bathtub shape or even a so-called *rollercoaster* shape. Figure 14 shows some examples of fitting the parameters to the simulated data and computing τ^*_{TBM} w.r.t. the lifetime S.

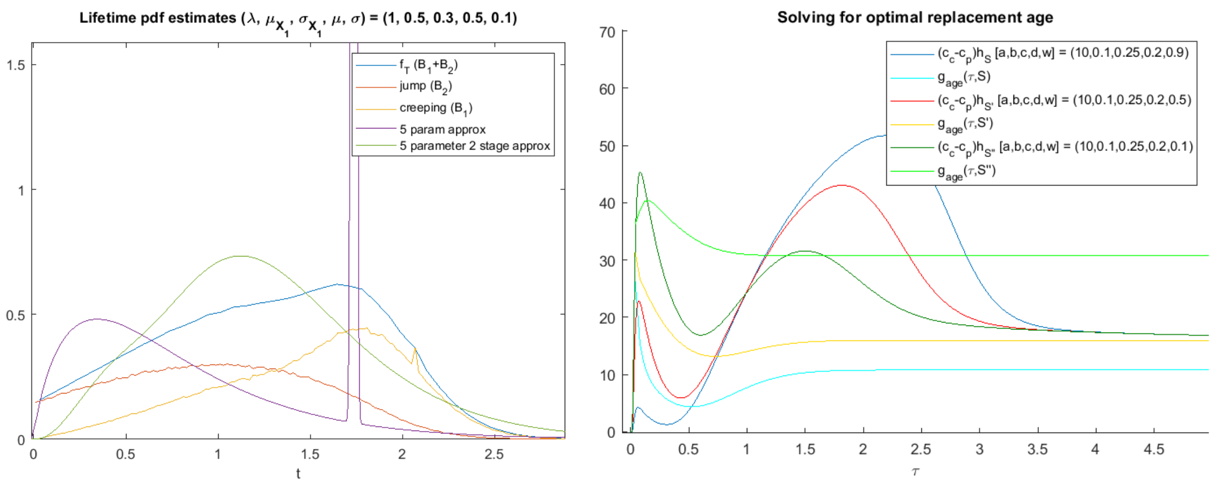

Figure 14: Approximation with mixture lifetime distributions.

It can be seen that solving the nonlinear problem (29a) can result in a bimodal distribution which still results in a low residual error, meaning that the optimization algorithm finds a local minimum. From the simulated data, we can perform two stages, first fit the individual distributions to the decomposition and then determine the weight afterwards. Again, the optimal TBM policy is only a global minimizer if $g(t_i^*, S) < c_c\mathbb{E}[S]^{-1}$ for all solutions t_i^* of Equation (18).

Given that this is satisfied, the computation of (sub-)optimal CBM policies using TBM policies can be done in a similar way as in section 4.1.2. Having established a computationally tractable approximation for the lifetime, we proceed with constructing CBM approximations of threshold policies.

6.2 CBM approximation of threshold policies

Using the approximations for the CPPs given in section 4.1.3 and using that the sum of Wiener processes is again a Wiener process, we obtain the following approximations (in distribution) for the degradation processes:

$$X(t) \overset{d}{\approx} \mathcal{N}((\lambda_L \mathbb{E}[X_1] + \mu_L)t, (\lambda_L \mathbb{E}[X_i^2] + \sigma_L^2)t),$$

and

$$\xi(t) \overset{d}{\approx} \mathcal{N}((\lambda_T \mathbb{E}[Y_{d,1}] + \mu_T)t+, (\lambda_T \mathbb{E}[Y_{d,1}^2] + \sigma_T^2)t).$$

Then, $\mathbb{P}_\tau(\text{failure})$ can either be approximated using Equation (21) or (26), depending on whether the CPP or the Wiener process is dominant, i.e., which part determines the characteristics of the optimal CBM/TBM policies. The policy is then obtained by minimizing the corresponding cost function.

The proposed approximation in distribution of the unobservable degradation process will prove useful in the next section, which discusses an approximation for the computation of the RUL.

6.3 Approximation of the remaining useful lifetime for Lévy processes

The computation of the RUL for Lévy degradation processes are along the same line of reasoning as in section 5.2. The classical definition of the RUL given Definition 4.2 can be approximated using the mixture lifetime approximation defined in Equation (28). For the computation of the RUL as in Definition 4.3, we build on the reasoning established in section 5.2. Similar to the case of Wiener degradation processes, we only have that

$$\sup_{u\in[0,t]} \tilde{X}(u) < \xi_0 \implies X(t) < \xi_0 - \xi(t).$$

Again, as an approximation, the stronger condition is replaced by the weaker one. Recall the approximation for the RUL for Wiener processes given in Equation (27):

$$F_{T_{t,x}}(s) \approx F_{\xi(t)}(\xi_0 - x)^{-1} \int_{-\infty}^{\xi_0 - x} f_{\xi(t)}(u)\mathbb{P}\Big(\text{IG}\Big(\frac{\xi_0 - x - u}{\mu}, (\frac{\xi_0 - x - u}{\sigma})^2\Big) < s\Big)du.$$

Note that the lifetime in the integrand depends on u, which makes it rather complicated to use the approximated lifetime distribution S in a direct manner. The parameters $\theta = \theta(\xi_0)$ are defined as the solution of a nonlinear optimization problem w.r.t. the initial gap ξ_0, thus $F_{S(\theta(\xi_0-u-x))}(u, x)$ is not *a priori* guaranteed to be an integrable function w.r.t. u. As such, we circumvent integrating and instead approximate

$$\begin{aligned}\{\xi(t)|\xi(t) < \xi_0 - x\} &\approx \mathbb{E}[\xi(t)|\xi(t) < \xi_0 - x] \\ &= F_{\xi(t)}(\xi_0 - x)^{-1} \int_{-\infty}^{\xi_0 - x} u f_{\xi(t)}(u) du =: \xi_x.\end{aligned}$$

The pdf and cdf in the above expression can be approximated using the approximation (in distribution) of $\xi(t)$ proposed in the previous section. Finally, we approximate the RUL as

$$T_{t,x} \overset{d}{\approx} S(\theta(\xi_0 - x - \xi_x)).$$

An approximation for RUL(t, q) can be obtained by evaluating the quantile function of the proposed approximation and used in the RUL based policy given in Definition 4.6.

7 Conclusions and discussion

Conclusions: Due to the increase in demand for (real-time) decision-making in the field of industrial maintenance, determining a cost-effective maintenance policy is of importance and highly prioritized. In this work, we investigate phenomena where a system is caused to fail by a random (unobservable) failure mechanism.

In section 3, we proposed the mathematical framework for the modeling of a random (unobservable) failure mechanism for the planning of cost-optimal CBM. To this purpose, we consider various levels of information about the failure mechanism and that both unobservable and observable degradation dynamics are governed by, not necessarily independent, Lévy processes. Lévy processes are a convenient mathematical model due their generality and also from a practical perspective bare all necessary features to capture the behavior of the degradation path. What makes them very appealing for the purpose at hand is the fact that Lévy processes can be accurately approximated by the independent sum of a compound Poisson process and a (drift) Wiener process. This approximation is particularly useful in the context of computing the (sub-) optimal CBM policy, as well as when computing approximations. To compute such approximations, we investigate time-based maintenance policies. To implement the model in practice, we proposed an MLE approach, per level of information, to estimate the parameters of the unobservable degradation process.

Afterwards, per level of information (full, partial or no information), we formulate the maintenance problem as a (PO)MDP in section 3.5. The cost-optimal policies, per scenario, are then defined as the optimal policy to the proposed (PO)MDP. As such, the cost-benefit of information is quantified as the cost difference of the (sub-)optimal threshold policy of the MDP given full information and the (proposed) sub-optimal threshold policies of the POMDP, given partial or no information. The base cost can be considered to be the (sub-)optimal policy of the MDP where the threshold process is assumed to be static. Thus, the value of information can be investigated by conducting a scenario analysis. A numerical study in section 3.4 conjectures that the cost-benefit between partial or no information is negligible as a consequence of bias and uncertainty about the underlying distribution, however more research is needed.

In sections 4 - 6, we show how to compute the (sub-)optimal time-based maintenance and CBM policies for continuously monitored systems where the underlying degradation processes are compound Poisson, Wiener and Lévy processes, respectively.

A numerical study in section 4.2.1 shows that, in the case of partial information, non-stationary linear threshold policies are superior to the traditional stationary, deterministic policies (aka control limit policies). Moreover, from the numerical examples and the analysis performed in this work, we conjecture that deriving the optimal linear threshold policy is, computation-wise, equally expensive as deriving the optimal threshold policy, although the linear policies are defined by two degrees of freedom. However, the slope achieving minimal cost can be computed in constant time, since this is the expected drift of the unobservable degradation process.

In section 5.1, we show that structural properties of both age replacement and CBM policies for Wiener degradation processes are very different, as high stochastic volatility may lead to the corrective maintenance policy to be cost-optimal. This is a reason for concern since, in the case of partial or no information, a Wiener processes is assumed to approximate the underlying Lévy process and as such there is no guarantee that the optimal CBM policy is invariant under such an approximation. Section 6 combines the knowledge of the preceding sections and proposes an approximation of the lifetime distribution as the solution of a nonlinear optimization problem. We then analyze this lifetime and use it to construct age replacement policies. Lastly, methods and approximations for decision-making based on the remaining useful lifetime have been established in sections 4.3, 5.2 and 6.3 for compound Poisson, Wiener and Lévy degradation processes respectively.

Discussion: Throughout this work, we assume various levels of information about the failure threshold for which we provide estimation procedures. However, in practise, biased information is often the problem that we need to face. Therefore, a method for dealing with bias, especially bias induced by a policy, would be of practical value to improve the estimation procedures proposed in section 3.4. Moreover, in case the unobservable failure threshold process contains jumps, an estimation procedure which takes this into account is desirable.

As real-time decision-making requires fast and efficient means of computation, another matter lies within the class of linear policies, which is to determine the coefficients independently of the currently used simulation methods. It remains a conjecture that for such policies, the optimal drift (w.r.t. the average cost criterion) is equal to $\mathbb{E}[\xi(1)]$. Another concern of a more theoretical nature is among the processes with jumps, where the reliability function and density have shown to lead to the computation of a k-fold convolution which is computationally expensive. Computing the law of Lévy processes involving such structures is difficult and requires more research. In the current work, only spectrally positive processes have an increasing failure rate. More research should be also done when negative jumps are allowed. As an extension, for cases in which there are pure jump processes only, we have described the distribution of the time to preventive maintenance in detail for non-stationary non-increasing policies. Although it is considered more limited, i.e., with finite state space and the jump distribution with finite amount of values, this allows for the approximation of the distribution of the time to preventive maintenance for increasing policies as well.

Besides the previously discussed theoretical and practical reflections, there are a few points to consider that might give directions for improvements to further research in the field of maintenance. The model can be expanded by, for example, adding a Markovian process which causes the system to switch between multiple degradation processes (see, e.g., [14]). Switching between states models the response of a system to an event and can either be coupled to the degradation processes, it can be completely independent or even deterministic. Of course, this opens up to new challenges in statistical methods and decision-making, for example by considering state-dependent policies. Lastly, the problem formulation as a partially observable Markov decision process can be investigated further by considering policies which interact with the environment and by using data monitoring, the policy is updated by updating the probability distribution of the underlying Lévy processes, e.g., adaptive policies incorporating Bayesian inference.

References

[1] M. Abdel-Hameed. Optimal predictive maintenance policies for a deteriorating system: the total discounted cost and the long-Run average cost cases, *Communication in Statistics-Theory and Methods*, vol. 33, no. 3, pp. 735-745, 2004.

[2] M. Abdel-Hameed. *Lévy Processes And Their Applications In Reliability And Storage.* Springer, 2014.

[3] I. Adan & J. Resing. *Queueing Systems*, 2015. Lecture notes, retrieved from https://www.win.tue.nl/ iadan/queueing.pdf

[4] R. Ahmad & S. Kamaruddin. An overview of time-based and condition-based maintenance in industrial application, *Computers & Industrial Engineering*, vol. 63, no. 1, pp. 135-149, 2012.

[5] S. Alaswad & Y. Xiang. A review on condition-based maintenance optimization models for stochastically deteriorating system, *Reliability Engineering & System Safety*, vol. 157, pp. 54-63, 2017.

[6] J. J. Arts. *Maintenance Modeling And Optimization*, 2017. Lecture notes, retrieved from https://pure.tue.nl/ws/portalfiles/portal/60891240/wp_526.compressed_1.pdf

[7] S. Asmussen & J. Rosiński. Approximations of small jumps of Lévy processes with a view towards simulation, *Journal of Applied Probability*, vol. 38, no. 2, pp. 482-493, 2001.

[8] K. J. Åström. Optimal control of Markov processes with incomplete state information. *Journal of Mathematical Analysis and Applications*, vol. 10, pp. 174–205, 1965.

[9] H. Block, Y. Li & T. Savits. Initial and final behaviour of failure rate functions for mixtures and systems, *Journal of Applied Probability*, vol. 40, no.3, pp. 721-740, 2003.

[10] R. S. Chhikara & J. L. Folks. The Inverse Gaussian distribution as a lifetime model, *Technometrics*, vol. 19, no.4, pp. 461-468, 1977.

[11] L. Coutin & D. Dorobantu. First passage time law for some Lévy processes with compound Poisson : Existence of a density, *Bernoulli*, vol. 17, no. 4, pp. 1127-1135, 2011.

[12] B. de Jonge. *Maintenance Optimization Based On Mathematical Modeling*, 2017. PhD thesis, retrieved from https://rug.nl/research/portal/files/39447651/Complete_thesis.pdf

[13] S. F. L. Gallot. Absorption and first-Passage times for a Compound Poisson process in a general upper boundary, *Journal of Applied Probability*, vol. 30, no. 4, pp. 835-850, 1993.

[14] M. Gösgens. *Dynamically Adaptive Age-based Maintenance Policies*, 2018. Bachelor thesis, retrieved from pure.tue.nl/ws/portalfiles/portal/113781963/Bep_Martijn_G_sgens.pdf

[15] A. Grall, C. Bérenguer & L. Dieulle. A condition-based maintenance policy for stochastically deteriorating systems, *Reliability Engineering and System Safety*, vol. 76, pp. 167-180, 2002.

[16] H. M. Hashemian & W. C. Bean. State-of-the-art predictive maintenance techniques, *IEEE Transactions on Instrumentation and Measurement*, vol. 60, no. 10, pp. 3480-3492, 2011.

[17] R. Jiang. A multivariate CBM model with a random and time-dependent failure threshold, *Reliability Engineering and System Safety*, vol. 119, pp. 178-185, 2013.

[18] Z. Jin & L. Wang. First passage time for Brownian motion and piecewise linear boundaries, *Methodology and Computing in Applied Probability*, vol. 19, no. 1, pp. 237-253, 2015.

[19] T. Kenbeek, S. Kapodistria & A. Di. Bucchianico. Data-driven online monitoring of wind turbines. *Proceedings of the 12th EAI International Conference on Performance Evaluation Methodologies and Tools.* ACM, 2019.

[20] P. Kolesar. Minimum Cost Replacement under Markovian Deterioration. *Management Science*, vol. 12, no. 9, series A, pp. 694-706, 1966.

[21] L. Kallenberg. *Markov Decision Processes*, 2009. Lectures Notes, retrieved from https://www.math.leidenuniv.nl/ kallenberg/Lecture-notes-MDP.pdf

[22] V. Korolev. On the accuracy of Normal approximation for the distributions of sums of a random number of independent random variables, *Theory of Probability and Its Applications*, vol. 33, no.3, pp. 577-581, 1988.

[23] B. Y. Lemeshko, S. B. Lemeshko, K. A. Akushkina, M. S. Nikulin & N. Saaidia. Inverse Gaussian model and its applications in reliability and survival analysis, *Mathematical and Statistical Models and Methods in Reliability. Statistics for Industry and Technology*, pp. 433-453. Birkhäuser, Boston, MA, 2010.

[24] A. Prajapati, J. Bechtel & S. Ganesan. Condition based maintenance: a survey, *Journal of Quality in Maintenance Engineering*, vol. 18, no. 4, pp. 384-400, 2012.

[25] P. E. Protter. *Stochastic Integration and Differential Equations*, 2nd ed. Springer, 2005.

[26] S. M. Ross. *Introduction to Probability Models*, 10th ed. Academic Press, 2009.

[27] J. Shin & H. Jun. On condition based maintenance policy, *Journal of Computational Design and Engineering*, vol. 2, pp. 119-127, 2015.

[28] E. J. Sondik. The optimal control of partially observable Markov processes over the infinite horizon: discounted cost. *Operations Research*, vol. 26, no. 2, pp. 282–304, 1978.

[29] C. Stenström, P. Norrbin, A. Parida & U. Kumar. Preventive and corrective maintenance-cost comparison and cost-benefit analysis, *Structure and Infrastructure Engineering*, vol. 12, no. 5, pp. 603-617, 2016.

[30] S. Tang, C. Yu & Y. Feng. Remaining useful life estimation based on Wiener degradation processes with random failure threshold, *Journal of Central South University*, vol. 23, no. 9, pp. 2230-2241, 2016.

[31] H. Tijms. *Stochastic Models, an Algorithmic Approach.* Wiley, 1994.

[32] A. Van Horenbeek & L. Pintelon. A dynamic predictive maintenance policy for complex multi-component systems, *Reliability Engineering and System Safety*, vol. 120, pp. 39-50, 2013.

[33] J. M van Noortwijk. A survey of the application of gamma process maintenance, *Reliability Engineering and System Safety*, vol. 94, pp. 2–21, 2009.

[34] J. M. van Noortwijk, J. A. M. van der Weide, M. J. Kallen & M. D.Pandey. Gamma processes and peaks-over-threshold distributions for time-dependent reliability, *Reliability Engineering & System Safety*, vol. 92, no. 12, pp. 1651-1658, 2006.

[35] P. Do Van, E. Levrat, A. Voisin, & B. Iung. Remaining useful life (RUL) based maintenance decision making for deteriorating systems, *IFAC Proceedings*, vol. 45, no. 31, pp. 66-72, 2012.

[36] W. Wang, M. Carr, W. Xu & K. Kobbacy. A model for residual life prediction based on Brownian motion with an adaptive drift, *Microelectronics Reliability*, vol. 51, pp. 285-293, 2011.

[37] Y. Xiang, R. Cassady & E. Pohl. Optimal maintenance policies for systems subject to a Markovian operating environment, *Computers & Industrial Engineering*, vol. 62, pp. 190-197, 2012.

[38] M. J. Yadav, A. N. Jinoop, C. Danduk & S. K. Subbu. Laser shock processing: process physics, parameters, and applications, *Materials Today: Proceedings*, vol. 4, no. 8, pp. 7921-7930, 2017.

[39] J. Yong & X. Y. Zhou. *Stochastic Controls: Hamiltonian Systems and HJB Equations*. Springer, 1999.

[40] Y. Zhou, L. Ma, J. Mathew, Y. Sun & R. Wolff. Maintenance strategy optimization using a continuous-state partially observable semi-Markov decision process. *Microelectronics Reliability*, vol. 51, no. 2, pp. 300-309, 2011.

Appendices

A Definitions, theorems and proofs

A.1 Definitions

Definition A.1 (Lévy process)
Let $\{X(t), t \geq 0\} : \mathbb{R}^+ \rightarrow \mathbb{R}$ denote a stochastic process. $X(t)$ is called a Lévy process if it satisfies:

- *$X(0) = 0$ almost surely.*
- *For any sequence $0 \leq t_1 < .. < t_n < \infty$, it holds that $\forall i, j \geq 1, i \neq j$ $X(t_j) - X(t_{j-1})$ and $X_(t_i) - X(t_{i-1})$ are independent. (Independent increments)*
- *For any $s < t$ we have that $X(t) - X(s) \stackrel{d}{=} X(t-s)$ (Stationary increments)*
- *$X(t)$ has right-continuous sample paths.*

A.2 Theorems

Theorem A.1 (Tijms, [31, p. 163])
The class of mixtures of Erlang distributions with a common scale parameter is dense in the space of distributions on $\mathbb{R}^+$. More specifically, let $F(x)$ be the cumulative distribution function of a positive random variable. Define the following cumulative distribution function of a mixture of Erlang distributions with a common scale parameter $\theta > 0$,

$$F(x; \theta) = \sum_{j=1}^{\infty} \alpha_j(\theta) F(x; j, \theta),$$

where $F(x; j, \theta)$ denotes the cumulative distribution function of an Erlang distribution with shape j and scale θ,

$$F(x; j, \theta) = 1 - \sum_{n=0}^{j-1} e^{-x/\theta} \frac{(x/\theta)^n}{n!},$$

and the mixing weights are given by

$$\alpha_j(\theta) = F(j\theta) - F((j-1)\theta) \qquad \text{for } j=1,2,\ldots.$$

Then

$$\lim_{\theta \rightarrow 0} F(x; \theta) = F(x),$$

for each point x at which $F(\cdot)$ is continuous.

A.3 Proofs

Proof of Lemma 4.1

Proof. Note that $\mathbb{P}(T > t) = \mathbb{P}(X(t) + \xi(t) < \xi_0)$.
Define $\tilde{X}(t) = X(t) + \xi(t)$ and $N(t) = N_L(t) + N_T(t)$. Then, $N(t)$ is a again Poisson process and $\tilde{X}(t)$ is again a compound Poisson process. There are several ways to prove this result, however, the most insightful way is to directly prove the definition. The sum of Poisson processes is again Poisson and the distribution of arrival types given the total number of arrivals for the combined Poisson process is binomial. This last fact will help us to construct a sequence of random variables Z_i which is independent of the combined Poisson process. We define Z_i as in Equation (4).

$$Z_i = \begin{cases} X_i & \text{w.p.}\lambda_L/\lambda \\ Y_i & \text{w.p.}\lambda_T/\lambda \end{cases}$$

Let $n \in \mathbb{N}$ and let $X_i, i = 1, .., n$ independent continuous random variables with density $f_{X_i}(x)$ and moment generating function $\phi_{X_i}(x)$. Define $Y = X_1 + X_2$. Then, Y has the following properties:

1. Y has a density $f_Y(x) = \int_{\mathbb{R}} f_{X_1}(y) f_{X_2}(x-y)dy = (f_{X_1} * f_{X_2})$.
2. Y has moment generating function $\phi_Y(t) = \phi_{X_1}(t) \cdot \phi_{X_2}(t)$.

This generalizes rather easily. Denote with $*$ the convolution operator, define $Y = \sum_{i=1}^{n} X_i$. Then, Y has the following properties:

1. Y has density function $f_Y(x) = (f_{X_1} * .. * f_{X_n})$.
2. Y has moment generating function $\phi_Y(t) = \phi_{X_1}(t) \cdot .. \cdot \phi_{X_n}(t)$.

The moment generating function of $X(t)$ is given by

$$\phi_{X(t)} = e^{\lambda_L t(\phi_{X_1} - 1)}.$$

Therefore, the mgf of $\tilde{X}(t)$ is given by

$$\phi_{\tilde{X}(t)} = e^{\lambda t \left(\lambda_L/\lambda \phi_{X_1} + \lambda_T/\lambda \phi_{Y_1} - 1\right)},$$

which we recognize as the moment generating function of a compound Poisson process with arrival intensity λ and the jump size distribution has moment generating function $\lambda_L/\lambda\phi_{X_1} + \lambda_T/\lambda\phi_{Y_1}$, which is also the moment generating function of Z_i. As $N(t)$ and Z_i are independent, it follows that $\tilde{X}(t)$ is a compound Poisson process with distribution

$$\mathbb{P}(\tilde{X}(t) < \xi_0) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} F_{Z_1}^{*k}(\xi_0),$$

Where $F^{*k}_{Z_1}(x)$ denotes the k-fold convolution of F_{Z_1}. This completes the proof and in a straightforward manner, this generalizes to the case of any finite sum of independent CPPs.

□

Proof of Erlang inequality

Proof.

$$\begin{aligned}
\mathbb{P}(X < Y) &= \mathbb{P}(\text{Erlang}(\lambda_L, \tau) < \text{Erlang}(\lambda_T, \xi_0 - \tau)) \\
&= \int_{\mathbb{R}^+} f_X(x)\mathbb{P}(Y > x)dx \\
&= \int_{\mathbb{R}^+} \frac{\lambda_L^\tau x^{\tau-1}}{(\tau-1)!} e^{-\lambda_L x} \sum_{k=0}^{\xi_0-\tau-1} e^{-\lambda_T x}\frac{(\lambda_T x)^k}{k!}dx \\
&\overset{\text{Tonelli}}{=} \frac{\lambda_L^\tau}{(\tau-1)!} \sum_{k=0}^{\xi_0-\tau-1} \frac{\lambda_T^k}{k!} \int_{\mathbb{R}^+} x^{\tau+k-1}e^{-\lambda x}dx \\
&= \lambda_L^\tau \sum_{k=0}^{\xi_0-\tau-1} \binom{k+\tau-1}{\tau-1} \lambda_T^k \int_{\mathbb{R}^+} \frac{x^{\tau+k-1}}{(\tau+k-1)!}e^{-\lambda x}dx \\
&= \lambda_L^\tau \sum_{k=0}^{\xi_0-\tau-1} \binom{k+\tau-1}{\tau-1} \lambda_T^k \lambda^{-k-\tau} \\
&= \sum_{k=0}^{\xi_0-\tau-1} \binom{k+\tau-1}{\tau-1} (\frac{\lambda_T}{\lambda})^k (\frac{\lambda_L}{\lambda})^\tau \\
&= \sum_{k=\tau}^{\xi_0-1} \binom{\xi_0-1}{k} (\frac{\lambda_L}{\lambda})^k (\frac{\lambda_T}{\lambda})^{\xi-k-1}.
\end{aligned}$$

To show that these last lines are equivalent, define

$$T \sim \text{NB}(\tau, \lambda_L/\lambda) \text{ and } S \sim \text{Binom}(\xi_0 - 1, \lambda_L/\lambda).$$

Then, the following statements are equivalent

$$\tau \leq T \leq \xi_0 - 1 \iff \tau \leq S \leq \xi_0 - 1.$$

The last two expressions define the probabilities of these events, which shows their equality and concludes the proof [3].

□

[3] Inspiration for the proof from https://math.stackexchange.com/questions/1952342/inequality-between-independent-erlang-random-variables

Proof of expected time to PM (Poisson degradation model)

Proof.

$$\begin{aligned}
(1-\mathbb{P}(T \leq T_\tau))\mathbb{E}[T_\tau | T_\tau < T_\xi] &= \int_{\mathbb{R}^+} t f_{T_\tau}(t) \bar{F}_{T_\xi}(t) dt \\
&= \int_0^\infty t f_{T_\tau}(t) e^{-\lambda_T t} \sum_{k=0}^{\xi-\tau-1} (\lambda_T t)^k / k! dt \\
&\overset{\text{Tonelli}}{=} \frac{\tau \lambda_L^\tau}{\tau!} \sum_{k=0}^{\xi-\tau-1} \frac{\lambda_T^k}{k!} \int_0^\infty t^{\tau+k} e^{-(\lambda_T+\lambda_L)t} dt \\
&= \frac{\tau \lambda_L^\tau}{\tau!} \sum_{k=0}^{\xi-\tau-1} \frac{\lambda_T^k \Gamma(\tau+k+1)}{(\lambda_L+\lambda_T)^{\tau+k+1} k!} \\
&= \frac{\tau (\lambda_L/\lambda)^\tau}{\tau! \lambda} \sum_{k=0}^{\xi-\tau-1} \frac{(\lambda_T/\lambda)^k (\tau+k)!}{k!}.
\end{aligned}$$

□

B Figures

B.1 Plots for τ^*

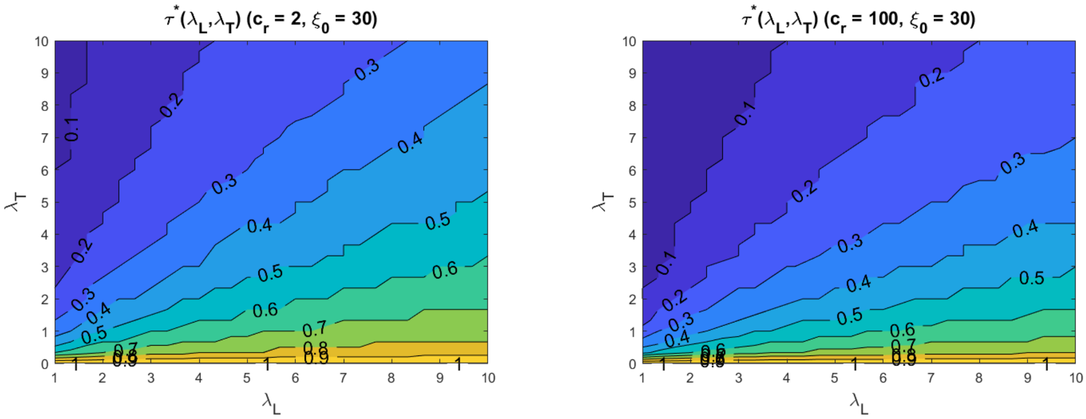

Figure 15: Optimal stationary replacement policies

B.2 Plots for τ_{CLT} approximation

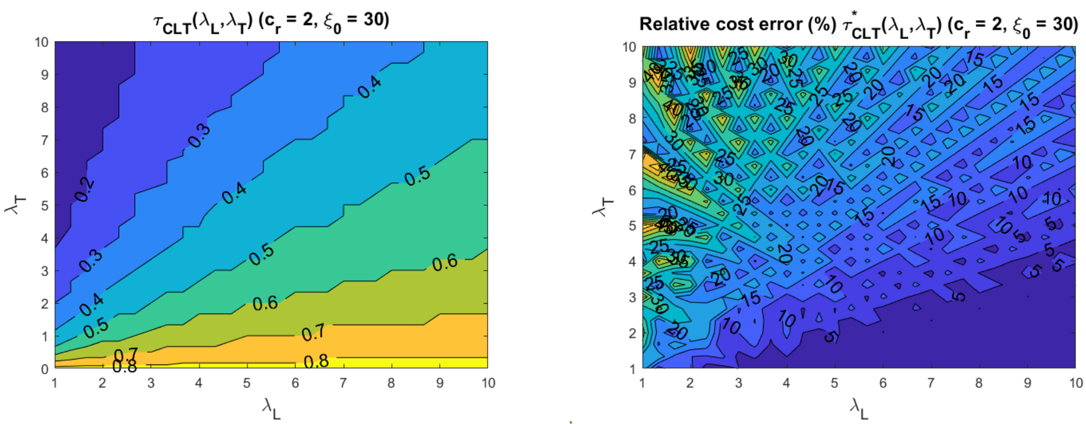

Figure 16: Policy approximation using the Central Limit Theorem for $c_r = 2$.

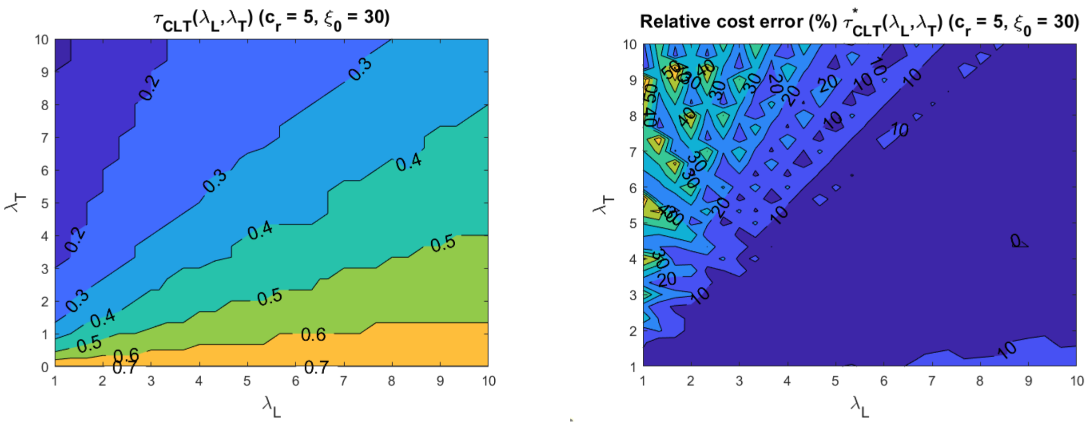

Figure 17: Policy approximation using the Central Limit Theorem for $c_r = 5$.

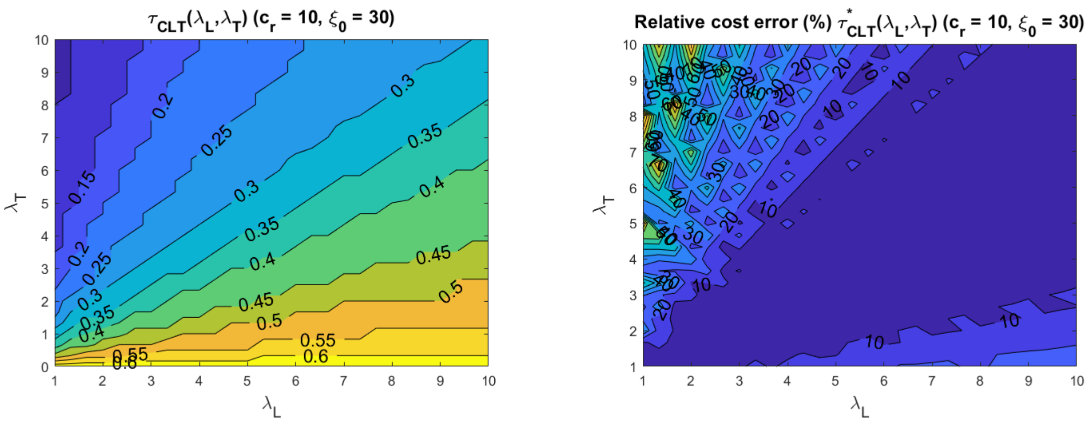

Figure 18: Policy approximation using the Central Limit Theorem for $c_r = 10$.

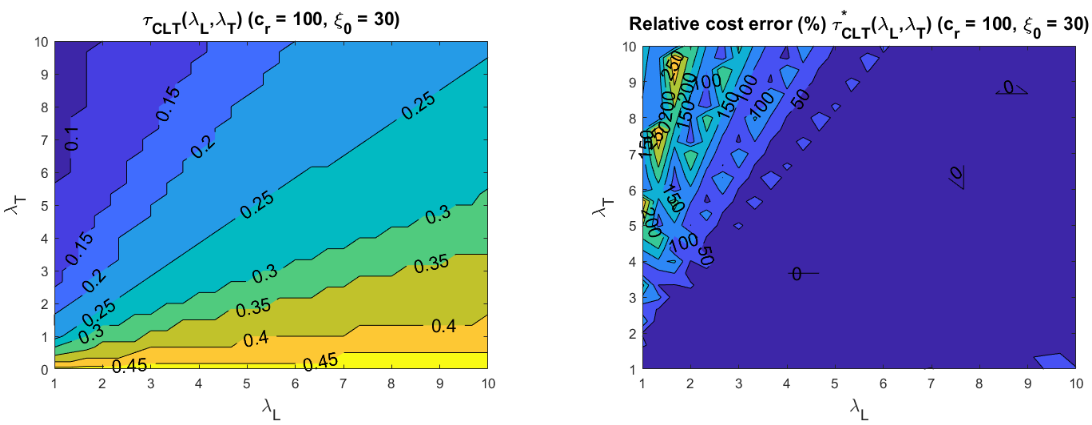

Figure 19: Policy approximation using the Central Limit Theorem for $c_r = 100$.

B.3 Plots for τ^*_{TBM} approximation

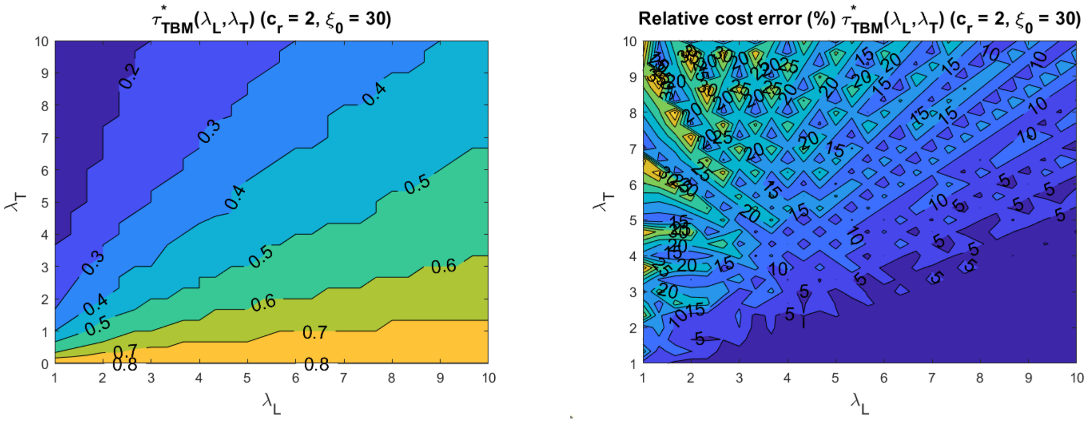

Figure 20: Policy approximation using optimal time to replacement for $c_r = 2$.

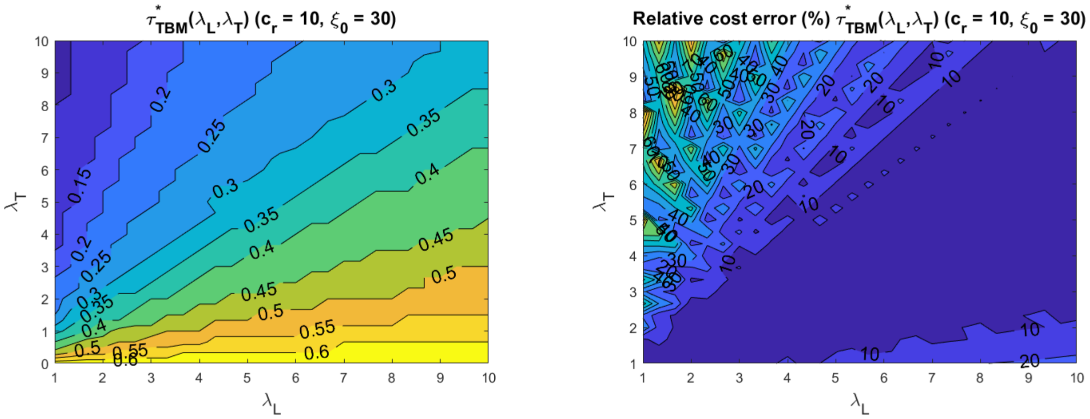

Figure 21: Policy approximation using optimal time to replacement for $c_r = 10$.

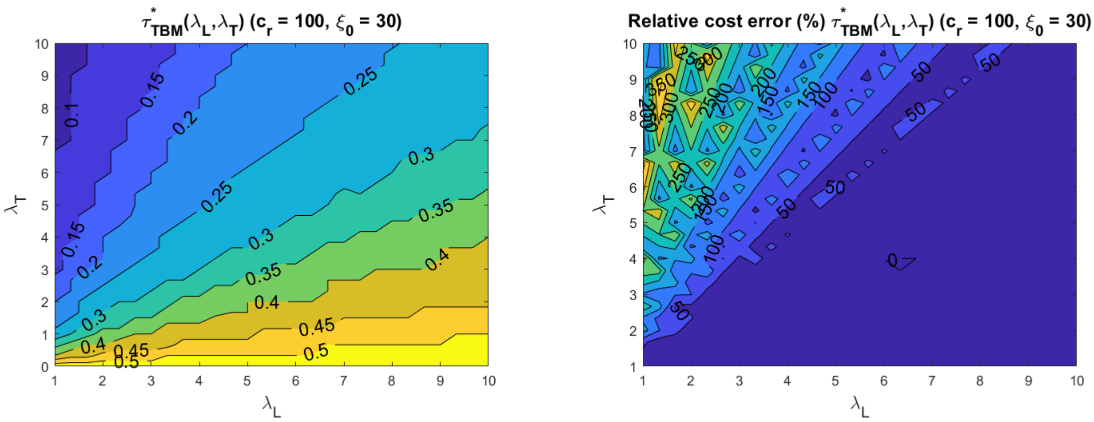

Figure 22: Policy approximation using optimal time to replacement for $c_r = 100$.

B.4 Plots for τ_{CBM} approximation

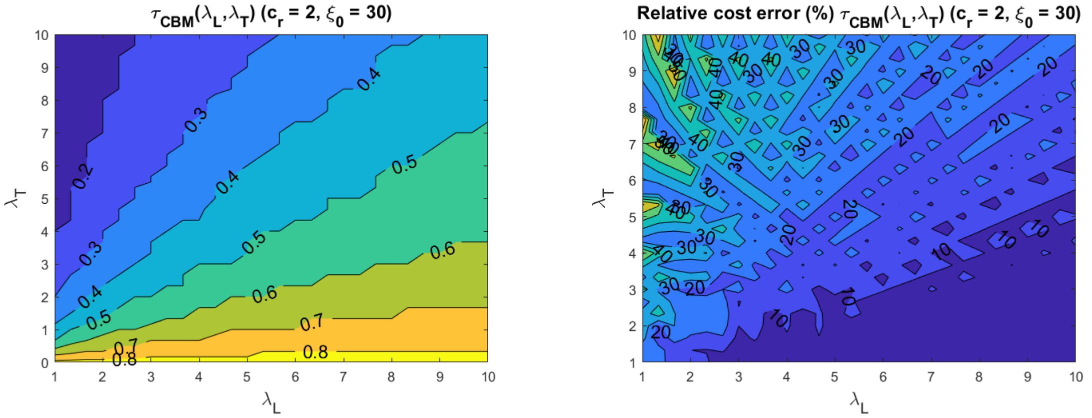

Figure 23: CBM policy approximations for $c_r = 2$.

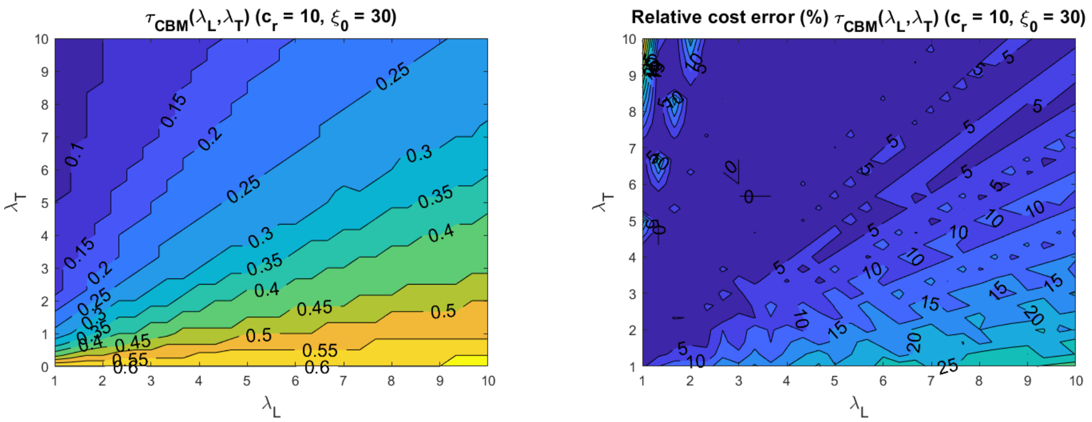

Figure 24: CBM policy approximations for $c_r = 10$.

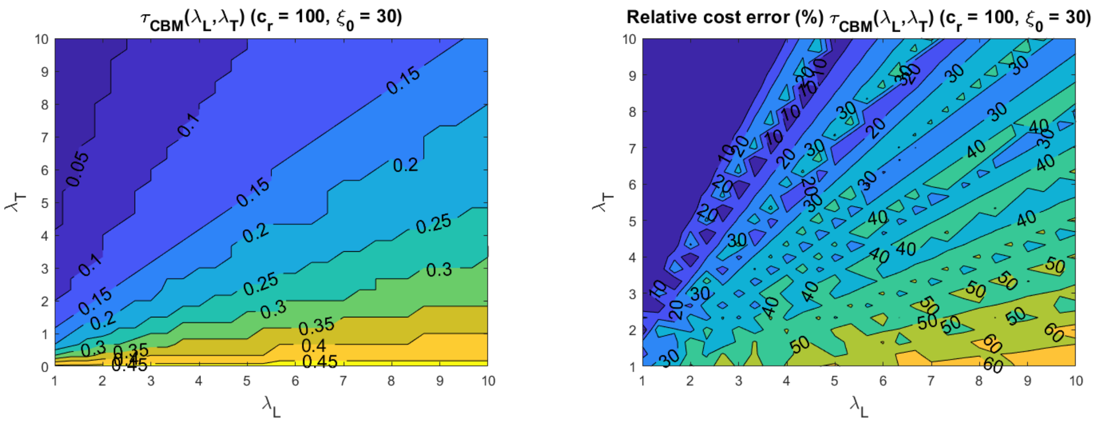

Figure 25: CBM policy approximations for $c_r = 100$.

C Nomenclature

This section contains a list of used definitions.

Nomenclature

$\bar{F}$	Reliability function
$\Delta(t)$	Gap
λ_L	Observable jump arrival intensity
λ_T	Unobservable arrival intensity
$\mathcal{A}$	Action set
$\mathcal{S}$	State space
τ	PM policy
τ^*	Optimal policy
$\tilde{X}(t)$	Total degradation process
$\xi(t)$	Failure threshold process
ξ_0	Initial failure threshold
a	Action
c_c	Corrective cost
$c_i(a)$	Immediate cost
c_p	Preventive cost
c_r	Cost ratio
c_v	Coefficient of variation
$CP(t)$	Compound Poisson process
F	Cumulative distribution function
f	Probability density function
$g(\tau)$	Long-run rate of cost
$g_\alpha(x_0)$	Discounted cost criterion
$i \in \mathcal{S}$	State
$N(t)$	Total arrival process
$N_L(t)$	Observable jump arrival process
$N_T(t)$	Failure threshold arrival process
$p_{ij}(a)$	Transition probabilities
$R(t)$	Renewal reward process
T	Time to CM threshold
t	time point
T_τ	Time to PM threshold
T_ξ	Time to 'hidden' failure
T_t	Remaining useful life
T_{CM}	Time to CM
T_{PM}	Time to PM
$W(t)$	Wiener process
$X(t)$	Observable degradation process

D Simulation description

Algorithm 1 Simulating Compound Poisson process

procedure COMPOUND POISSON
  Inputs:
  ξ_0, λ_L, λ_T, β, F_{X_1}, $F_{Y_{d,1}}$, $\tau(t)$
  Initialize:
  load $\leftarrow 0$
  threshold $\leftarrow \xi_0$
  totalTime $\leftarrow 0$
  while load $<$ threshold and load $< \tau(totalTime)$ **do**
    arrTime $\sim$ Exp$(\lambda_T + \lambda_L)$
    checkPM(load, threshold, τ, totalTime, arrTime) ▷ Check if PM has to be performed
    totalTime $\leftarrow$ totalTime + arrTime
    if Ber$(1, \frac{\lambda_L}{\lambda})$ **then**
      jump $\leftarrow$ sample(F_{X_1})
      load $\leftarrow$ load + jump
      threshold $\leftarrow$ threshold - βjump
    else
      jump $\leftarrow$ sample$(F_{Y_{d,1}})$
      threshold $\leftarrow$ threshold - jump
    end if
  end while
end procedure

Algorithm 2 Simulating Wiener drift process

procedure WIENERDRIFT
    Inputs:
    ξ_0, μ_L, μ_T, σ_L, σ_T , $\tau(t)$, arrTime, step
    Initialize:
    load ← 0
    threshold ← ξ_0
    totalTime ← 0
    nSteps = $\lfloor arrTime/step \rfloor$ ▷ Creates uniform grid with time increments step
    lastStep = arrTime - nSteps*step ▷ The last increment might be a bit smaller
    for i=1,...,nSteps **do**
        checkPM(load, threshold, τ, totalTime, step)
        load ← load + $W_{\mu_L,\sigma_L}(step)$
        threshold ← threshold - $W_{\mu_T,\sigma_T}(step)$
        checkCM(load, threshold, τ, totalTime, step)
        checkPM(load, threshold, τ, totalTime, step)
        totalTime ← totalTime + step
    end for
    if $lastStep > 0$ **then**
        checkPM(load, threshold, τ, totalTime, step)
        load ← load + $W_{\mu_L,\sigma_L}(lastStep)$
        threshold ← threshold - $W_{\mu_T,\sigma_T}(lastStep)$
        checkCM(load, threshold, τ, totalTime, lastStep)
        checkPM(load, threshold, τ, totalTime, lastStep)
        totalTime ← totalTime + step
    end if
end procedure

Remark
Due to discretization of the time interval, if both thresholds are crossed in the same increment, then one needs to check which one was crossed first.

Algorithm 3 Golden-section search

```
procedure FindOptimalPolicy
    Inputs:
    precision. drift
    Initialize:
    initialLoad ← 0
    initialThreshold ← ξ0
    a ← 0                                   ▷ Compute the first interval and boundary values
    b ← ξ0
    gr ← (1 + √(5))/2
    x1 ← b-(b-a)/gr
    x2 ← a+(b-a)/gr
    delta ← b-a;
    status ← 0;
    while delta > precision do
        if status=0 then
            upperValue ← simulateTimePolicy([x2 direction])    ▷ Simulates linear policy
with drift and intercept x2 and returns average cost
            lowerValue ← simulateTimePolicy([x1 direction])
        else if status=1 then
            lowerValue ← upperValue
            upperValue ← simulateTimePolicy([x2 direction])
        else if status=-1 then
            upperValue ← lowerValue
            lowerValue ← simulateTimePolicy([x1 direction])
        end if
        if lowerValue > upperValue then           ▷ meaning x2 is a better policy than x1
            a ← x1;
            x1 ← x2;
            x2 ← a+(b-a)/gr;
            status ← 1;
        else
            b←x2;
            x2 ← x1;
            x1 ← b-(b-a)/gr;
            status ← -1;
        end if
        delta ← b-a;
    end while
    policy ← [(a+b)/2 a b];
end procedure
```