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Clustering Based Model Reduction Method for Interconnected Port Hamiltonian Systems

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# Clustering Based Model Reduction Method for Interconnected Port Hamiltonian Systems

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*Abstract*—This paper proposes different dissimilarity measures that are especially relevant for port Hamiltonian systems. Computational schemes are developed for clustering methods that are based on these dissimilarity measures. Comparisons of the results are given in a number of examples.

*Index Terms*—Order Reduction; port-Hamiltonian Systems; Structure Preserving; Large-Scale Systems

## I. INTRODUCTION

The port Hamiltonian framework has become a powerful tool for modelling and control of complex physical systems over the years [21], [14] and [17]. It makes use of the kinetic and potential energy equations to model the system, which enables a multi-physics model to capture the physical properties of the system such as stability and passivity. The port Hamiltonian modelling approach is also able to model mechanical, electrical and other forms of physical systems together. Interconnection of various port-Hamiltonian systems together is known as physical network system. It is evident from [7], [4], [12], [1] that, the physical network system can be treated as a graphical network consisting of nodes and edges. A graphical network is of special interest because the information about the interconnections between nodes gives the physical interpretation to the states in the state space model.

Figure 1 depicts a graphical network system sharing information among the nodes and interacting with the environment. A connected graph is represented as $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a (finite) set denoting the vertices (nodes) of the network and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ describes the edges (interaction with environment or nodes). Throughout this paper, we consider networks without self loops. This means that there are no elements $\mathcal{V} \in \mathcal{V}$ such that $(\mathcal{V}, \mathcal{V}) \in \mathcal{E}$.

In case of a physical network system, the vertices represent individual port Hamiltonian systems and the edges represent the energy exchange between two port Hamiltonian systems. Increasing the number of port Hamiltonian systems leads to an increase in complexity of such a physical network system, causing difficulty in its analysis and control. Hence, in order to simplify such network systems model reduction techniques are used.

There are mainly two ways to simplify a network system:
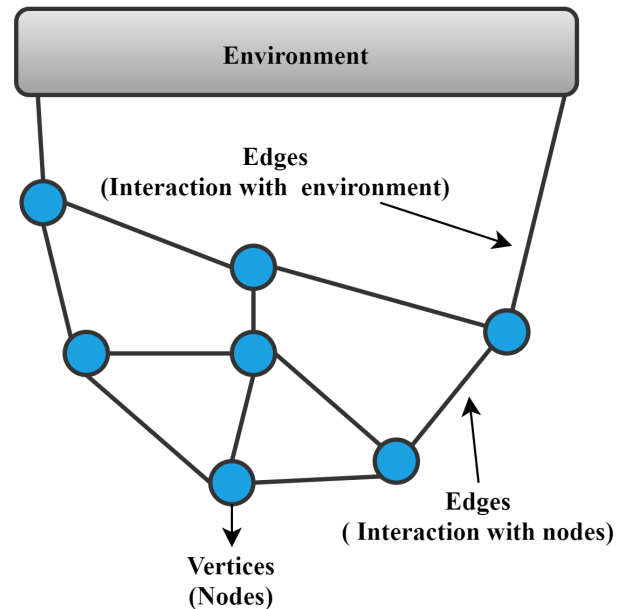- Nodal approximation



Figure 1. Graphical network systems

- Topological simplification

In nodal approximation [18], each node (physical system) is simplified by using Krylov-based and Gramian-based model reduction. In topological simplification [9] we simply remove less important systems and update the network structure. In this research we focus on topological simplification of physical network system.

Clustering techniques are one of the popular ways to do topological simplifications on graphical network systems, as presented in [9]. In a clustering technique a simplified network is obtained, by grouping similar behaving neighboring systems (which form the vertices of the network graph) and merge their states to obtain a single system that represents the original cluster. Each new resulting cluster is then treated as a node in a simplified network. The simplified network generates an updated interconnection topology which has a clear physical interpretation. In order to cluster similar behaving systems together into a single cluster, we must define how different each system is from one another. The

difference in the behaviour between two systems are given by dissimilarity. The systems placed in same cluster will have low dissimilarity when compared to the systems from other clusters. The main focus of this research work is to explore various energy properties of port Hamiltonian systems to define the dissimilarity measurement and finally develop an algorithm to simplify a physical network system.

The paper has been organised in the following way. First the notations used in the paper has been explained in section II. In section III, literature related to clustering technique have been explored and research questions that will be answered through this paper is presented. Preliminaries regarding port-Hamiltonian systems and physical network systems are provided in section IV. The problem setting is stated in section V. In Section VI, various ways to define dissimilarities have been explored, which is followed by brief explanation about hierarchical clustering algorithm in section VII. The topological simplification for a physical network system has been presented in section VIII. Next, the error between a full network and a simplified network for different dissimilarity measurements is discussed in section IX. Finally, various dissimilarity measurements have been compared in section X followed by the conclusions in section XI and the possible extensions of the research have been provided in section XII.

## II. NOTATION

- $\dot{H}$ represents the time derivative of function $H : \mathbb{R} \to \mathbb{R}$.
- $\mathbf{0}_n$ is a zero matrix of size $n \times n$.
- $\mathbf{0}_{n \times m}$ is a zero matrix of size $n \times m$.
- $\langle a, b \rangle$ refers to the Euclidean inner product of vectors $a$ and $b$ of the same dimension.
- $I_n$ is the identity matrix of size $n \times n$.

## III. LITERATURE AND PROBLEM FORMULATION

A considerable amount of work has been done with reference to clustering of graphical network system. Until now various mathematical properties have been used for clustering of graphical network systems. In [9], [4], [8], [6] and [5] dissimilarity between two nodes is defined as the $\mathcal{H}_2$−norm of their transfer function deviation. The difference of states in a semi-stable system has been used to define the dissimilarity in [15]. In [11] the dissimilarity between two subsystems of a bi-directional network is given by $H_\infty$-norm of their transfer function. In [2] the controlablity and observability property has been used to cluster the network system. Lastly [13] focuses on data driven reduced-order models. Here, the snapshot of a trajectory is used to create clusters. From the literature provided it can be observed that, none of these techniques address the utilization of a physical property to simplify a network system. An important physical property such as energy can provide a lot of information regarding energy distribution in the network. This information can be utilized to keep high energy subsystems and remove low energy subsystems from simplified network system. As the main interest lies in developing a clustering algorithm that

makes use of the energy properties associated to the network, we formulate our research questions for physical network systems as follows:

How to simplify a network of neutrally (lossless) interconnected port Hamiltonian systems to get

1) a simplified network that preserves the port Hamiltonian property ?
2) a simplified network in which the interconnections preserve their neutrality?
3) a simplified network, where the energy distribution (Hamiltonian energy function) still represents physical energy?

Since, our main research questions are with respect to interconnected port Hamiltonian system, in next section the preliminaries of the same is provided.

## IV. PRELIMINARIES

### A. Linear port Hamiltonian system

A linear port Hamiltonian system excluding algebraic constraints and without feedthrough term takes the following form [10]

$$\Sigma : \begin{cases} \dot{x} = (J - R)Qx(t) + Bu(t) \\ y(t) = B^\top Qx(t) \end{cases} \tag{1}$$

where, $H(x(t)) = \frac{1}{2}x(t)^\top Qx(t)$ is total energy (Hamiltonian), with $Q = Q^\top > 0$ as the energy matrix and $R = R^\top \geq 0$ being the dissipation matrix. The matrix $J$ and $B$ represent the internal interconnection structure of the port Hamiltonian system, with $J = -J^\top$. Thus, we have $J$ as skew-symmetric and $R$ as positive semi definite. The state variables $x(t) \in \mathbb{R}^n$ of port Hamiltonian system are known as energy variables because the total energy $H(x(t))$ is expressed as a function of these variables. Additionally, the variables $u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^m$ represent the power variables, since their product $u(t)^\top y(t) = \langle u(t), y(t) \rangle$ equals the power supplied to the system. For all trajectories $(u(t), x(t), y(t))$ that satisfy (1), the rate of change of energy with respect to time is given by (2).

$$\dot{H}(x(t)) = \langle Qx(t), \dot{x}(t) \rangle \tag{2}$$

$$\dot{H}(x(t)) = \langle Qx(t), (J - R)Qx(t) + Bu(t) \rangle \tag{3}$$

$$\dot{H}(x(t)) = \langle y(t), u(t) \rangle - \langle Qx(t), RQx(t) \rangle \tag{4}$$

Next observation is that in (1), $H(x(t)) = \frac{1}{2}x(t)^\top Qx(t)$ satisfies the dissipation inequality given by (5).

$$\dot{H}(x(t)) = \langle u(t), y(t) \rangle - x(t)^\top Q^\top RQx(t) \leq \langle u(t), y(t) \rangle \tag{5}$$

Now we define the passivity property of the port Hamiltonian system:

**Definition 1:** A system $\Sigma$ as in (1) is said to be passive if there exists a differentiable storage function $H(x) : \mathbb{R}^n \to \mathbb{R}^n$

Q:why exclude feedthrough term?
A: Effect of feedthrough not yet included in any of the papers.

satisfying $H(x) \geq 0$ and $H(0) = 0$ such that it satisfies the dissipation inequality given in (5).

Now we consider two special cases corresponding to $R = 0$ or $R > 0$. If $R = 0$, we have no internal dissipation and the dissipation inequality (5) reduces to equality

$$\dot{H}(x(t)) = \langle u(t), y(t) \rangle \tag{6}$$

When $R > 0$, we have a energy dissipating system and the inequality (5) reduces to strict inequality

$$\dot{H}(x(t)) = \langle u(t), y(t) \rangle - x(t)^\top Q^\top R Q x(t) < \langle u(t), y(t) \rangle \tag{7}$$

In this research work the main focus will be on energy dissipating systems. In order to give a better idea about port Hamiltonian system, the physical representation of such systems is provided in Figure 2.

In Figure 2, every input-output pair represents a power $(P_i)$ delivering port, with $P_i = y_i u_i$. Here $(u_i, y_i)$ are components in the input, and output vector $(u, y)$ of equal dimension. Then $(u_i, y_i)$ define the port-variables at the $i^{th}$ port of the system and $P_i := u_i^\top y_i$ denotes the power delivered to the system through its $i^{th}$ port. The total power delivered to the system $\Sigma$ is given by:

$$P(t) = u^\top(t)y(t) = \sum_{i=1}^{n} P_i(t) \tag{8}$$

Where, $n$ is number of ports, inputs $u(t) = col(u_1, ...u_n)$ and outputs $y(t) = col(y_1, ...y_n)$. The ports shown in Figure 2 will be used for interconnection with other port Hamiltonian systems. According to [3] and [16], physical systems of the form (1) can be interconnected using gyrator interconnections or (ideal) transformer interconnections.
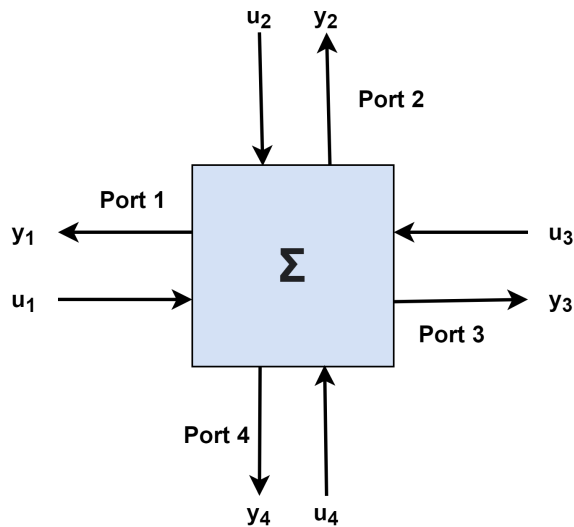
### B. Interconnected port Hamiltonian systems

The gyrator interconnection of two port Hamiltonian systems is done in the following steps.

**Step 1:**

Let us consider two generic port Hamiltonian systems of the form (1). First input and output ports are divided into its internal and external components. The internal ports are used to couple two port Hamiltonian systems and the external ports are used to interact with the environment. The equation of the form (1) will be rewritten after splitting its internal and external ports.

$$\Sigma_i : \begin{cases} \dot{x}_i = (J_i - R_i)Q_i x_i + B_i^{int} u_i^{int} + B_i^{ext} u_i^{ext} \\ y_i^{int} = B_i^{int\top} Q_i x_i \\ y_i^{ext} = B_i^{ext\top} Q_i x_i \end{cases} \quad \forall i = 1, 2. \tag{9}$$

where :

- $J_i = -J_i^\top$ represents internal interconnections of the system $\Sigma_i$.
- $x_i \in \mathbb{R}^{n_i}$ are the energy variables.
- $R_i = R_i^\top \succeq 0$ and $Q_i = Q_i^\top > 0$ are the dissipation and energy matrix of $\Sigma_i$ respectively.
- $u_i^{int}, y_i^{int} \in \mathbb{R}^{m_i^{int}}$, where $m_i^{int}$ is the dimension of internal inputs and outputs.
- $u_i^{ext}, y_i^{ext} \in \mathbb{R}^{m_i^{ext}}$, where $m_i^{ext}$ is the dimension of external inputs and outputs.

**Step 2:**

In a gyrator interconnection the internal output $(y_1^{int})$ of system $\Sigma_1$ goes to the internal input $(u_2^{int})$ of the system $\Sigma_2$ and internal output $(y_2^{int})$ of system $\Sigma_2$ is fed back to internal input $(u_1^{int})$ of the system $\Sigma_1$ as shown in Figure 3.

The gyrator interconnection gives us the relation as shown in (10).

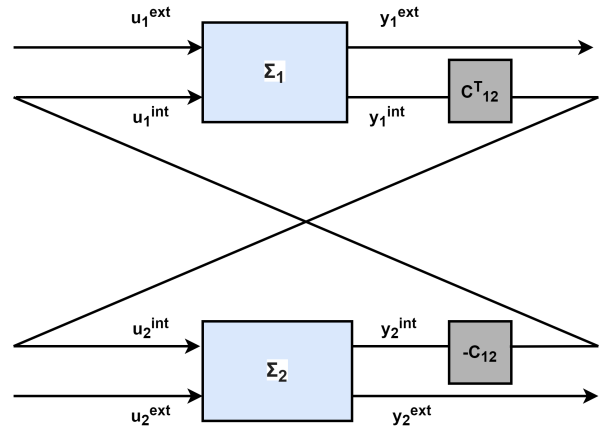$$\begin{aligned} u_1^{int} &= -C_{12} y_2^{int} \\ u_2^{int} &= C_{12}^\top y_1^{int} \end{aligned} \tag{10}$$



Figure 2. Port Hamiltonian system



Figure 3. Gyrator Interconnection of port Hamiltonian system

Here $C_{12}$ is the gain factor, with which the output of $\Sigma_1$ is fed to the input of $\Sigma_2$. Putting (10) in matrix form leads to (11).

$$\begin{bmatrix} u_1^{int} \\ u_2^{int} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -C_{12} \\ C_{12}^T & 0 \end{bmatrix}}_{S} \begin{bmatrix} y_1^{int} \\ y_2^{int} \end{bmatrix} \qquad (11)$$

where :

- $C_{12}$ is a $m_1^{int} \times m_2^{int}$ real-valued matrix.
- The interconnection between two port Hamiltonian systems are given by matrix $S$ in (11), which is of the size $(m_2^{int} + m_1^{int}) \times (m_2^{int} + m_1^{int})$.
- The $S$ matrix is a skew symmetric matrix .

**Note 1:** It must be noted that, while $S$ from (11) gives the relation between the internal input and internal output variables, we use this relation to convert internal input variables in terms of energy variables i.e, $x_i$ and it is represented in $M$ matrix as shown in (12).

$$\begin{bmatrix} u_1^{int} \\ u_2^{int} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{0}_{n_1} & -C_{12}B_2^{int\top} \\ C_{12}^\top B_1^{int\top} & \mathbf{0}_{n_2} \end{bmatrix}}_{M} \begin{bmatrix} Q_1 & \mathbf{0}_{n_1} \\ \mathbf{0}_{n_2} & Q_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (12)$$

**Note 2:** It should be noted that S in (11) represents the ~~internal~~ interconnection between two port Hamiltonian systems.

The gyrator interconnection is called lossless interconnection because it holds the following property [3], [16]

$$\langle u_1^{int}, y_1^{int} \rangle + \langle u_2^{int}, y_2^{int} \rangle = 0 \qquad (13)$$

**Step 3:**

We will replace the internal inputs of $\Sigma_i$ in (9) with the internal output of port Hamiltonian system using relation from (11).

$$\mathbf{\Sigma_1} : \begin{cases} \dot{x}_1 = (J_1 - R_1)Q_1 x_1 + B_1^{int}(-C_{12})y_2^{int} + B_1^{ext}u_1^{ext} \\ y_1^{int} = B_1^{int\top}Q_i x_1 \\ y_1^{ext} = B_1^{ext\top}Q_i x_1 \end{cases} \qquad (14)$$

**Step 4:**

Replace the internal output of the second subsystem with its state relation from (9) in (14)

$$\mathbf{\Sigma_1} : \begin{cases} \dot{x}_1 = (J_1 - R_1)Q_1 x_1 - B_1^{int}C_{12}B_2^{int\top}Q_2 x_2 + B_1^{ext}u_1^{ext} \\ y_1^{int} = B_1^{int\top}Q_i x_1 \\ y_1^{ext} = B_1^{ext\top}Q_i x_1 \end{cases} \qquad (15)$$

**Step 5:**

Repeat **Step 3** and **Step 4** for subsystem $\Sigma_2$ and then we can write the overall state equation for two interconnected port Hamiltonian system as shown in (16)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} J_1 - R_1 & -B_1^{int}C_{12}B_2^{int\top} \\ B_2^{int}C_{12}^\top B_1^{int\top} & J_2 - R_2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} Q_1 & \mathbf{0}_{n_2} \\ \mathbf{0}_{n_1} & Q_2 \end{bmatrix}}_{Q} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$+ \underbrace{\begin{bmatrix} B_1^{ext} & \mathbf{0}_{n_2 \times m_2^{int}} \\ \mathbf{0}_{n_1 \times m_1^{int}} & B_2^{ext} \end{bmatrix}}_{B} \begin{bmatrix} u_1^{ext} \\ u_2^{ext} \end{bmatrix}$$

$$\begin{bmatrix} y_1^{ext} \\ y_2^{ext} \end{bmatrix} = \underbrace{\begin{bmatrix} B_1^{ext\top} & \mathbf{0}_{n_1 \times m_1^{int}}^\top \\ \mathbf{0}_{n_2 \times m_2^{int}}^\top & B_2^{ext\top} \end{bmatrix}}_{B^\top} \begin{bmatrix} Q_1 & \mathbf{0}_{n_2} \\ \mathbf{0}_{n_1} & Q_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(16)$$

Where

- $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$ are the vectors of energy variables for $\Sigma_1$ and $\Sigma_2$ respectively. Here $n_1$ and $n_2$ are dimension of $\Sigma_1$ and $\Sigma_2$ respectively.
- $Q_1$ and $Q_2$ are the energy matrices of $\Sigma_1$ and $\Sigma_2$ respectively.

Due to lossless interconnection as shown in (13), the Hamiltonian energy function of the overall system is given by (17).

$$H(x) = H(x_1, x_2) = \frac{1}{2}[x_1^\top Q_1 x_1 + x_2^\top Q_2 x_2] \qquad (17)$$

**Note 3:** The internal ports from (9) used for interconnection totally disappears from input and output matrix in (16). As a result, the rate of change of energy in the system depends only on the external ports.

*C. Graphical representation of interconnected port Hamiltonian system*

As mentioned previously, we focus on topological simplification of the physical network system. Now, we will find the graphical representation from the interconnected port Hamiltonian systems. First, let us consider $A$ matrix from (16). We rewrite the $A$ matrix as shown in (18)

$$A = A_1 - A_2 \qquad (18)$$

Where, $A_1$ is skew symmetric part and $A_2$ is symmetric part of matrix $A$. We have $A_1 = J + K$, $A_2 = diag(R_1, R_2)$ as shown in (19).

$$A = \underbrace{\begin{bmatrix} J_1 & \mathbf{0}_{n_2} \\ \mathbf{0}_{n_1} & J_2 \end{bmatrix}}_{J} + \underbrace{\begin{bmatrix} \mathbf{0}_{n_1} & -B_1^{int}C_{12}B_2^{int\top} \\ B_2^{int}C_{12}^\top B_1^{int\top} & \mathbf{0}_{n_2} \end{bmatrix}}_{K} - \underbrace{\begin{bmatrix} R_1 & \mathbf{0}_{n_2} \\ \mathbf{0}_{n_1} & R_2 \end{bmatrix}}_{R} \qquad (19)$$

Where,

- $J$ is skew-symmetric matrix containing the stacked internal interconnections of individual port Hamiltonian systems.
- $R$ contains the stacked dissipation matrices $R_1$ and $R_2$.

- $K$ is a skew symmetric matrix containing the interconnections between each port Hamiltonian system.
- $K$ is defined by property 1.

**Property 1.**

$$K_{12} = -K_{21}^\top \tag{20}$$

*Proof.* Consider $K$ from (19), we can see that $B_1^{int} C_{12} B_2^{int\top} = (B_2^{int} C_{12}^\top B_1^{int\top})^\top$. Let us consider

$$\begin{aligned} K_{21} &= B_2^{int} C_{12}^\top B_1^{int\top} \\ K_{12} &= -B_1^{int} C_{12} B_2^{int\top} \end{aligned} \tag{21}$$

Where, $K_{21} \in \mathbb{R}^{n_2} \times \mathbb{R}^{n_1}$ and $K_{12} \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, with $n_1$ and $n_2$ being the dimensions of systems $\Sigma_1$ and $\Sigma_2$ respectively.

From (21) we get:

$$K_{12} = -B_1^{int} C_{12} B_2^{int\top} = -(B_2^{int} C_{12}^\top B_1^{int\top})^\top = -K_{21}^\top \tag{22}$$

$\square$

The terms $K_{12}$ and $K_{21}$ in $K$ can be treated as weight matrix for bi-directional edge connection between two nodes as shown in Figure 4.

Substituting (21) in (19) leads to (23)

$$A = \underbrace{\begin{bmatrix} J_1 & \mathbf{0}_{n_2} \\ \mathbf{0}_{n_1} & J_2 \end{bmatrix}}_{J} + \underbrace{\begin{bmatrix} \mathbf{0}_{n_1} & K_{12} \\ K_{21} & \mathbf{0}_{n_2} \end{bmatrix}}_{K} - \underbrace{\begin{bmatrix} R_1 & \mathbf{0}_{n_2} \\ \mathbf{0}_{n_1} & R_2 \end{bmatrix}}_{R} \tag{23}$$

Since both $J$ and $K$ in (23) are skew symmetric, we can say that the interconnected port Hamiltonian system is also a port Hamiltonian system.

Now that the interconnection between two port Hamiltonian systems has been established, next we extend the two interconnected port Hamiltonian systems to a physical network system consisting of $N$ interconnected port Hamiltonian system. Each of these port Hamiltonian systems are of the form as given in (9). Now, $i^{th}$ and $j^{th}$ port Hamiltonian systems are interconnected using gyrator interconnection relation given in (11), with $C_{ij}$ possibly port dependent (not the same for every port). Then we eliminate the internal input - output ports using relation given in (12). This gives a physical network system
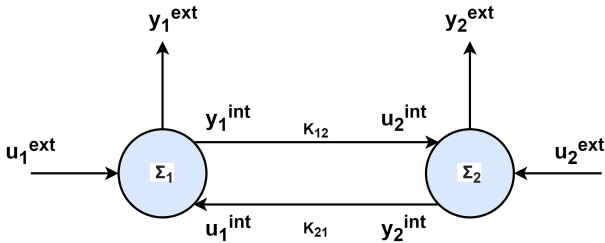


Figure 4. Two node representation of two interconnected port Hamiltonian system

consisting of $N$ interconnected port Hamiltonian system as given by (24).

$$\begin{aligned} \dot{x} &= [J + K - R] Q x + B u^{ext} \\ y^{ext} &= B^\top Q x \end{aligned} \tag{24}$$

Where, $x = [x_1, \ldots, x_N]$, $u^{ext} = [u_1^{ext}, \ldots, u_N^{ext}]$ and $y^{ext} = [y_1^{ext}, \ldots, y_N^{ext}]$. Expanding $J$, $R$, $K$, $Q$ and $B$ accordingly leads to (25). Now that the network structure of the interconnected port Hamiltonian system has been shown, in the next section the problem setting for the network simplification of physical network system (24) will be elaborated.

$$J = \begin{bmatrix} J_1 & \mathbf{0}_{n_2} & \mathbf{0}_{n_3} & \ldots & \mathbf{0}_N \\ \mathbf{0}_{n_1} & J_2 & \mathbf{0}_{n_3} & \ldots & \mathbf{0}_N \\ \mathbf{0}_{n_1} & \mathbf{0}_{n_2} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \mathbf{0}_N \\ \mathbf{0}_{n_1} & \mathbf{0}_{n_2} & \ldots & \mathbf{0}_{N-1} & J_N \end{bmatrix}$$

$$R = \begin{bmatrix} R_1 & \mathbf{0}_{n_2} & \mathbf{0}_{n_3} & \ldots & \mathbf{0}_N \\ \mathbf{0}_{n_1} & R_2 & \mathbf{0}_{n_3} & \ldots & \mathbf{0}_N \\ \mathbf{0}_{n_1} & \mathbf{0}_{n_2} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \mathbf{0}_N \\ \mathbf{0}_{n_1} & \mathbf{0}_{n_2} & \ldots & \mathbf{0}_{N-1} & R_n \end{bmatrix}$$

$$K = \begin{bmatrix} \mathbf{0}_{n_1} & K_{12} & K_{13} & \ldots & \color{red}{K_{1n}} \\ K_{21} & \mathbf{0}_{n_2} & \ldots & K_{2(n-1)} & \color{red}{K_{2n}} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ K_{(n-1)1} & K_{(n-1)2} & \ldots & \mathbf{0}_{N-1} & \color{red}{K_{(n-1)n}} \\ K_{n1} & \ldots & \ldots & K_{n(n-1)} & \mathbf{0}_N \end{bmatrix}$$

K_{1N}

$$Q = \begin{bmatrix} Q_1 & \mathbf{0}_{n_2} & \mathbf{0}_{n_3} & \ldots & \mathbf{0}_N \\ \mathbf{0}_{n_1} & Q_2 & \mathbf{0}_{n_3} & \ldots & \mathbf{0}_N \\ \mathbf{0}_{n_1} & \mathbf{0}_{n_2} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \mathbf{0}_N \\ \mathbf{0}_{n_1} & \mathbf{0}_{n_2} & \ldots & \mathbf{0}_{N-1} & Q_N \end{bmatrix}$$

$$B = \begin{bmatrix} B_1^{ext} & \mathbf{0}_{m_2^{ext} \times n_1} & \ldots & \mathbf{0}_{m_N^{ext} \times n_N} \\ \mathbf{0}_{m_1^{ext} \times n_1} & B_2^{ext} & \ldots & \mathbf{0}_{m_N^{ext} \times n_N} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{0}_{m_1^{ext} \times n_1} & \mathbf{0}_{m_2^{ext} \times n_2} & \ldots & B_N^{ext} \end{bmatrix} \tag{25}$$

## V. Problem setting

In this research we consider a special case of network systems known as multi agent systems, which assumes that all the systems in the network are identical in nature. Hence $\Sigma_i$

in (9) are considered as identical port Hamiltonian systems. Rewriting (9) for identical systems we get (26)

$$
\Sigma_i : \begin{cases}
\dot{x}_i = (J_i - R_i)Q x_i + \Sigma_{i=0}^{j} B_i^{int} u_i^{int} + \Sigma_{i=1}^{k} B_i^{ext} u_i^{ext} \\
y_1^{int} = B_1^{int\top} Q_i x_i \\
\vdots \\
y_j^{int} = B_j^{int\top} Q_i x_i \\
y_i^{ext} = B_1^{ext\top} Q_i x_i \\
\vdots \\
y_k^{ext} = B_k^{ext\top} Q_i x_i
\end{cases}
$$

(26)

where,

- $j$ is the number subsystems with which $\Sigma_i$ is interconnected and $k$ is the number of external inputs.
- $x_i(t) \in \mathbb{R}^n$ is the vector of energy variables.
- $n$ is the dimension of identical port Hamiltonian systems.
- $u_i^{int}, y_i^{int} \in \mathbb{R}^{m^{int}}$, where $m^{int}$ is the dimension of internal inputs. and outputs
- $u_i^{ext}, y_i^{ext} \in \mathbb{R}^{m^{ext}}$, where $m^{ext}$ is the dimension of the external inputs. and outputs
- $J_i$ is $n \times n$ skew-symmetric matrix and $R_i > 0$ is a positive definite matrix of same size.

Since we are considering strictly dissipative systems, that is $R > 0$, it follows from (5) that

$$
\dot{H}(x(t)) = \langle u^{ext}, y^{ext} \rangle - x^\top Q^\top R Q x < \langle u^{ext}, y^{ext} \rangle \quad (27)
$$

Next, in order to define the stability of the system we consider $u^{ext} = 0$ in (27), which leads to (28).

$$
\dot{H}(x(t)) = -x^\top Q^\top R Q x < 0 \quad (28)
$$

From definition 1 we can say that $H(x)$ serves as a lyapunov function. With $H(x) > 0$ for $x \neq 0$ and (28) we can say that $\dot{H}(x) < 0$ along the solutions of the differential equation. Thus, the origin $x = 0$ is asymptotically stable.

The network of $N$ port Hamiltonian ~~sub~~systems will be of the form (24). In order to do network simplification, first the definition of dissimilarity must be established. In the following section energy based dissimilarity will be explored.

## VI. Energy Based dissimilarity measurement

Let us consider two identical port Hamiltonian systems $\Sigma_i$ and $\Sigma_j$ of the form (9) with

- Hamiltonian energy functions $H_i$ and $H_j$.
- Internal interaction given by skew symmetric matrices $J_i$ and $J_j$.
- Dissipation matrices $R_i$ and $R_j$
- Energy variables $x_i \in \mathbb{R}^n$ and $x_j \in \mathbb{R}^n$

In this paper the dissimilarity between $\Sigma_i$ and $\Sigma_j$ is defined in the three following ways.

a) Dissipation matrix.
b) Dissipation gramian.
c) Asymptotic hamiltonian energy.

### A. Dissipation Matrix

As mentioned in Section IV, the dissipation matrix $R$ refers to the internal dissipation. It contains all the coefficients of resisting or damping components. For example, if an electrical network contains resistances, then the resistances of individual sub-systems will be stored in dissipation matrix $R$. Hence we can define the dissimilarity between two port Hamiltonian systems as the difference of their resistances, which will be given by sum matrix norm of the difference between dissipation matrices $R_i$ and $R_j$.

According to [19] sum matrix norm is given by the sum of element magnitudes, as shown in (29)

$$
\|A\|_{\text{sum}} = \sum_{i,j} |a_{ij}| \quad (29)
$$

**Proposition 1.** *The dissimilarity between $\Sigma_i$ and $\Sigma_j$ is given by $D_{ij}$, as shown in (30).*

$$
D_{ij} = \|R_i - R_j\|_{sum} \quad (30)
$$

$D$ is the $N \times N$ dissimilarity matrix consisting of dissimilarities between all the port Hamiltonian systems and $D_{ij}$ is the element in $i^{th}$ row and $j^{th}$ column giving dissimilarity between $\Sigma_i$ and $\Sigma_j$. In (30), $D_{ij}$ represents the sum of differences between all the resistances. $R_i$ and $R_j$ are the dissipation matrices of $\Sigma_i$ and $\Sigma_j$ respectively.

### B. Dissipation Gramian

Here, the dissipated energy is used to define the dissimilarity between $\Sigma_i$ and $\Sigma_j$. Since the port Hamiltonian system under consideration dissipates energy over time, the dissipation provided in (27) is considered first

$$
\dot{H}(x(t)) = u^{ext\top} y^{ext} - x(t)^\top Q^\top R Q x(t) \quad (31)
$$

Where,

- $Q = diag(Q_1, \ldots, Q_N)$
- $x(t) = col(x_1(t), \ldots, x_N(t))$ and $x(t)$ is the state variable generated in the model (24) of the network system.

The state response is given by (32)

$$
x(t) = e^{At} x_0 + \int_{t_0}^{t} e^{A(t-\tau)} B u^{ext}(\tau) d\tau, t \geq t_0 \quad (32)
$$

In order to compare the dissipation through every port Hamiltonian system, the external input is set to zero in (31) and (32). Then, all the port Hamiltonian systems are set to be starting from some initial condition $x_0$ at time $t = t_0$ and let the energy dissipate over time. Since the external input is zero, (32) becomes

$$
x(t) = e^{At} x_0 \quad (33)
$$

**Note 4:** It should be noted that here we are considering dissipation equality of the whole network. As mentioned previously in (12), the internal input appear in $M$ matrix after applying gyrator interconnection. Thus, we set only external inputs to be zero and not the internal inputs. The internal

input represents the energy flow between the port Hamiltonian systems.

Substituting (33) in (31) we get

$$\dot{H}(x(t)) = -\left(e^{At}x_0\right)^\top Q^\top R Q \left(e^{At}x_0\right) \tag{34}$$

$$\dot{H}(x(t)) = -x_0^\top e^{A^\top t} Q^\top R Q e^{At} x_0 \tag{35}$$

Here, (35) represents energy decay due to dissipation only. Integrating both sides with respect to time from $t_0$ to $t_1$

$$\int_{t_0}^{t_1} \dot{H}(x) = -x_0^\top \underbrace{\left(\int_{t_0}^{t_1} e^{A^\top t} Q^\top R Q e^{At} dt\right)}_{D_g(t_0,t_1)} x_0 \tag{36}$$

$$H(x(t_1)) - H(x_0) = -x_0^\top \underbrace{\left(\int_{t_0}^{t_1} e^{A^\top t} Q^\top R Q e^{At} dt\right)}_{D_g(t_0,t_1)} x_0 \tag{37}$$

It is observed from (37) that, the amount of dissipation over a time depends on the gramian $D_g(t_0, t_1)$ and initial condition $x_0$ at time $t = t_0$. Here we will treat the matrix $D_g(t_0, t_1)$ as a dissipation gramian matrix, which indicates the dissipation capacity of the port Hamiltonian systems.

$$D_g(t_0, t_1) = \int_{t_0}^{t_1} e^{A^\top t} Q^\top R Q e^{At} dt \tag{38}$$

The dissipation gramian in (38) depends on time $t_0$ and $t_1$. Thus in order to remove dependence of time from (38), we take $t_0 = 0$ and $t_1 \longrightarrow \infty$. $D_g$ is the infinite dissipation gramian having block structure, where each block show how much energy can be dissipated through each port Hamiltonian system over infinite time horizon.

$$D_g = \int_0^\infty e^{A^\top t} Q^\top R Q e^{At} dt \tag{39}$$

**Lemma 1.** *The dissipation gramian for infinite time horizon is the solution of the lyapunov equation given by (40).*

$$A^\top D_g + D_g A = -Q^\top R Q \tag{40}$$

*Proof.*

$$A^\top D_g + D_g A = \int_0^\infty A^\top e^{A^\top t} Q^\top R Q e^{At} dt \\ + \int_0^\infty e^{A^\top t} Q^\top R Q e^{At} A dt \tag{41}$$

$$A^\top D_g + D_g A = \int_0^\infty \frac{d}{dt}\left(e^{A^\top t} Q^\top R Q e^{At}\right) dt \tag{42}$$

$$A^\top D_g + D_g A = \left[e^{A^\top t} Q^\top R Q e^{At}\right]_0^\infty \tag{43}$$

$$A^\top D_g + D_g A = \left[e^{A^\top t} Q^\top R Q e^{At}\right]_0^\infty \tag{44}$$

From (27), it follows that our system is asymptotically stable. As the dissipative system is asymptotically stable we get $e^{At} \longrightarrow 0$ as $t \longrightarrow \infty$

Thus (44) for asymptotic stability becomes:

$$A^\top D_g + D_g A = -Q^\top R Q \tag{45}$$

$\square$

The dissipated energy from (37) is used to define the dissimilarity between $\Sigma_i$ and $\Sigma_j$. Since the dissipated energy in (37) is dependent on $x_0$, we will make the following assumption,

**Assumption 1.** *The initial condition for $\Sigma_i$ and $\Sigma_j$ are same i.e. $x_0^i = x_0^j = \alpha$. Where $x_0^i$ and $x_0^j$ are the initial conditions of $\Sigma_i$ and $\Sigma_j$ and $\alpha$ is a fixed non-zero initial condition for all subsystems.*

The dissipated energy from (37) gives us the total energy dissipated through the entire network. In order to extract the energy dissipated through only $\Sigma_i$ we set all the initial conditions to zero, except that of $\Sigma_i$ in $x_0$ of (37). The energy dissipated over infinite time horizon is given by (46).

$$H_i(x(\infty)) - H_i(x(0)) = -e_i^\top D_g e_i \tag{46}$$

where, $e_i$ is a vector of size $1 \times (n \times N)$, with initial conditions of all except that of $\Sigma_i$ set to zero. For example, if we consider dissipation energy of $4^{th}$ port Hamiltonian system with $n = 2$ we set $e_4 = col(0,0,0,0,0,0,\alpha,\alpha,0,0...)$. Here, $D_g$ is the solution of dissipation gramian from (45).

**Note 5:** Here it is important to note that dissipated energy depends quadratically on $\alpha$.

**Proposition 2.** *The dissimilarity measurement can be given as the difference in asymptotic dissipation energy between each port Hamiltonian system and is given by (47).*

$$D_{ij} = |e_i^\top D_g e_i - e_j^\top D_g e_j| \tag{47}$$

Since the dissipated energy depends quadratically on $\alpha$, the dissipated energy difference, that is, dissimilarity between $\Sigma_i$ and $\Sigma_j$ also depends quadratically on $\alpha$. However this does not effect the clustering because of the assumption that initial condition for every port Hamiltonian system is considered same. Hence the dissipated energy through every port Hamiltonian system is quadratically scaled by $\alpha$.

### C. Asymptotic Hamiltonian Energy

In this section, the Hamiltonian energy is used to define the dissimilarity measurement. The Hamiltonian energy function will be used to find out the energy distribution throughout the network. Our main interest lies in the energy distribution as $t \longrightarrow \infty$. In order to do so, step input through the external ports are applied to (32) and we compute $\lim_{t\to\infty} x(t)$. Here $x(t)$ includes the states of whole network ,that is, $x(t) \in \mathbb{R}^{N \times n}$. Applying $\lim_{t\longrightarrow\infty}$ to the hamiltonian energy function for the whole network leads to (48).

$$\lim_{t\to\infty} H(x) = \lim_{t\to\infty} \frac{1}{2} x(t)^\top Q x(t) \tag{48}$$

**Theorem 1.** *The asymptotic hamiltonian energy function of the whole network is given by* (49)

$$\lim_{t \to \infty} H(x(t)) = 1_m^\top B^\top A^{-\top} Q A^{-1} B 1_m \qquad (49)$$

*Proof.* The external step input $u^{ext}(\tau)$ that is applied to (32) is a vector of dimension $m$ for all time $t$, where $m$ is the total number of external input applied to the whole network. The step input to every $k^{th}$ external input channel is given as

$$u_k^{ext}(t) = \begin{cases} 1 \text{ if } k = i, t \geq 0 \\ 0 \text{ if } k \neq i, t \geq 0 \end{cases} \qquad (50)$$

Hence we get the external input for all time $t$ as a vector of size $m$.

$$u^{ext}(\tau) = 1_m \qquad (51)$$

Taking zero initial condition and substituting (51) in (32) we get

$$x(t) = \int_0^t e^{A(t-\tau)} B 1_m d\tau \qquad (52)$$

Let $t - \tau = \sigma$, we get $-d\tau = d\sigma$.
Taking upper limit $\tau = t$, we get $\sigma = t - t = 0$
Taking lower limit $\tau = 0$, we get $\sigma = t - 0 = t$
Using change of variables to integrate (52)

$$x(t) = \int_t^0 -e^{A(\sigma)} B 1_m d\sigma \qquad (53)$$

Switching limits we get:

$$x(t) = \int_0^t e^{A(\sigma)} B 1_m d\sigma \qquad (54)$$

$$x(t) = A^{-1}[e^{A\sigma}]_0^t B 1_m \qquad (55)$$

$$x(t) = A^{-1}[e^{At} - I] B 1_m \qquad (56)$$

Since we are interested in $\lim_{t \to \infty} x(t)$, we take $t \longrightarrow \infty$. According to (27) the given network of system is asymptotically stable, thus $e^{At} \longrightarrow 0$, as $t \longrightarrow \infty$. Applying this limits to (56) we get

$$\lim_{t \to \infty} x(t) = -A^{-1} B 1_m \qquad (57)$$

Here, $\lim_{t \to \infty} x(t)$ is a column vector of size $N \times n$
Substituting (57) to (48) we get

$$\lim_{t \to \infty} H(x(t)) = (A^{-1} B 1_m)^\top Q A^{-1} B 1_m \qquad (58)$$

$$\lim_{t \to \infty} H(x(t)) = 1_m^\top B^\top A^{-\top} Q A^{-1} B 1_m \qquad (59)$$

$\square$

Let us denote $\lim_{t \to \infty} x(t)$ as $x(\infty)$ for ease of notation. The asymptotic Hamiltonian energy of $i^{th}$ port Hamiltonian system is given by

$$\lim_{t \to \infty} H_i = x_i(\infty)^\top Q_i x_i(\infty) \qquad (60)$$

Here, $x_i(\infty)$ is the column vector of states belonging to $\Sigma_i$ from (57).

**Proposition 3.** *The dissimilarity is given by the asymptotic energy difference between $\Sigma_i$ and $\Sigma_j$*

$$D_{ij} = |\lim_{t \to \infty} H_i - \lim_{t \to \infty} H_j| \qquad (61)$$

As the measurement of dissimilarity has been defined in three different ways, the next step is to cluster similar behaving port Hamiltonian systems together into single cluster. In this research work, the hierarchical clustering algorithm is employed to cluster two similar behaving port Hamiltonian systems. The hierarchical clustering algorithm will be elaborated in the next section.

## VII. HIERARCHICAL CLUSTERING ALGORITHM

Hierarchical clustering is an algorithm that groups similar objects into groups called clusters. The definition of cluster with respect to a connected graph is given as follows:

**Definition 2:** Consider a connected graph $\mathcal{G}_r = (\mathcal{V}_r; \mathcal{E}_r)$. Here $\mathcal{G}_r$ represents the simplified network. A non-empty index subset of $\mathcal{V}_r$ is called a cluster of graph $\mathcal{G}_r$. $\mathcal{V}_r$ is a collection of clusters $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_r\}$ and $\mathcal{C}_i$ is the $i^{th}$ cluster which is created by merging similar behaving port Hamiltonian systems.

In this research, the number of clusters refers to the number of port Hamiltonian systems in the simplified network. If we have $N$ interconnected port Hamiltonian systems in the full network, then we have $r$ port Hamiltonian systems in simplified network such that $N > r$. Hierarchical clustering typically works by sequentially merging similar clusters. In the previous section, the dissimilarity between two port Hamiltonian systems which serves as the distance between two port Hamiltonian systems was discussed. After obtaining the dissimilarity matrix, it is necessary to determine from where can the distance be computed. For example, it can be computed between the two most similar parts of a cluster (single-linkage), the two least similar bits of a cluster (complete-linkage), the center of the clusters (mean or average-linkage), or some other criterion [20].

A small example of hierarchical clustering is provided in Figure 5. Here it is seen that, first the top two red clusters are merged together into a single cluster as they are the closest. Then the two red clusters on the right are merged together. In the end there is a single cluster, which contains all the clusters.

After creating clusters for a given $r$ using hierarchical clustering technique, we compile all the information regarding the port Hamiltonian systems present in various clusters in the aggregation matrix $P$. Aggregation matrix is defined by (62).

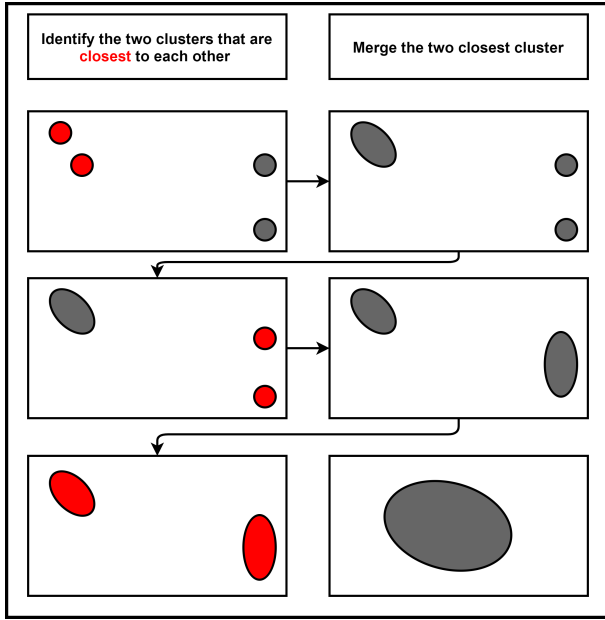$$P := col(p(\mathcal{C}_1), p(\mathcal{C}_2), \ldots, p(\mathcal{C}_r)) \qquad (62)$$

Figure 5. Hierarchical Clustering

Where,

- $p(\mathcal{C}_i)$ is a tall matrix of size $(n \times N) \times n$. Recalling from section V, $n$ is the dimension of identical port Hamiltonian system and $N$ is the number of interconnected port Hamiltonian systems.
- The $k^{th}$ block of $p(\mathcal{C}_i)$ is of size $n \times n$. The $k^{th}$ block is equal to an identity matrix $I_n$ when $k^{th}$ port Hamiltonian system $\in \mathcal{C}_i$ or $0_n$ otherwise.

The aggregation matrix is used for topological simplification of the network system with $N$ port Hamiltonian systems to $r$ port Hamiltonian systems, which is discussed in the following section.

## VIII. TOPOLOGICAL SIMPLIFICATION OF PHYSICAL NETWORK SYSTEM

Now, we consider the model reduction problem of the network system of the form (24). We use the graph clustering method proposed in [9]. We recall some basic definitions in graph clustering from [9].

For a given aggregation matrix $P$, the reduced order system is given by

$$\hat{\Sigma} \begin{cases} \dot{x}_r = \underbrace{\left(\hat{J} + \hat{K} - \hat{R}\right)\hat{Q}}_{\hat{A}} x_r + \hat{B}u^{ext} \\ \\ y^{ext} = \hat{B}^\top \hat{Q} x_r \end{cases} \tag{63}$$

Where,

- $\hat{J} = P^\top J P$ is a skew matrix matrix.
- $\hat{K} = P^\top K P$ is a skew matrix matrix.
- $\hat{R} = P^\top R P$ is a positive definite matrix.
- $\hat{B} = P^\top B$.
- $\hat{Q} = P^\top Q P$ is a positive definite matrix.

The matrix property preservation of reduced order matrix is provided in appendix C.

### Example 1

Consider four port Hamiltonian systems interconnected as shown in Figure 6. The $J$, $K$, $R$ and $B$ matrices of the system under consideration are given by

$$J = \begin{bmatrix} J_1 & \mathbf{0}_n & \mathbf{0}_n & \mathbf{0}_n \\ \mathbf{0}_n & J_2 & \mathbf{0}_n & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n & J_3 & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n & \mathbf{0}_n & J_4 \end{bmatrix}$$

$$R = \begin{bmatrix} R_1 & \mathbf{0}_n & \mathbf{0}_n & \mathbf{0}_n \\ \mathbf{0}_n & R_2 & \mathbf{0}_n & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n & R_3 & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n & \mathbf{0}_n & R_4 \end{bmatrix}$$

$$\tag{64}$$

$$K = \begin{bmatrix} \mathbf{0}_n & K_{12} & K_{13} & K_{14} \\ K_{21} & \mathbf{0}_n & \mathbf{0}_n & K_{24} \\ K_{31} & \mathbf{0}_n & \mathbf{0}_n & K_{34} \\ K_{41} & K_{42} & K_{43} & \mathbf{0}_n \end{bmatrix}$$

$$B = \begin{bmatrix} B_1^{ext} & \mathbf{0}_{n \times m^{int}} \\ \mathbf{0}_{n \times m^{int}} & B_2^{ext} \\ \mathbf{0}_{n \times m^{int}} & \mathbf{0}_{n \times m^{int}} \\ \mathbf{0}_{n \times m^{int}} & \mathbf{0}_{n \times m^{int}} \end{bmatrix}$$

The graphical representation of the system in Figure 6 is given by Figure 7.

Here the network is simplified to three interconnected port Hamiltonian systems. Let the three clusters be defined as $C_1 = \{\Sigma_1, \Sigma_2\}$, $C_2 = \{\Sigma_3\}$ and $C_3 = \{\Sigma_4\}$. The projection
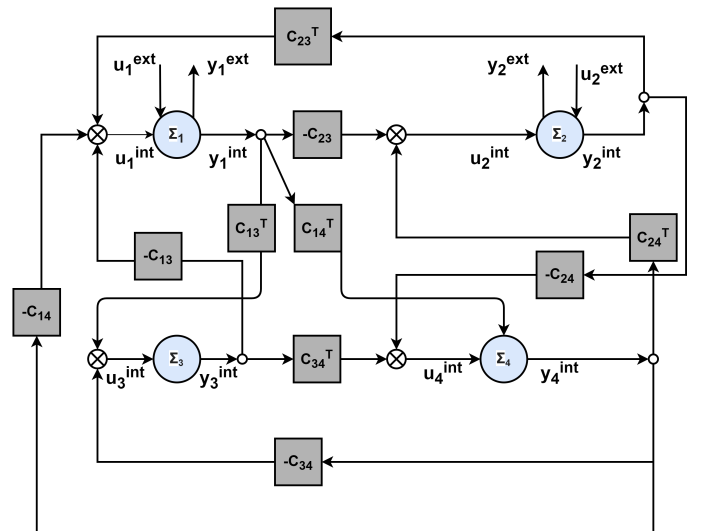


Figure 6. Four port Hamiltonian systems interconnection
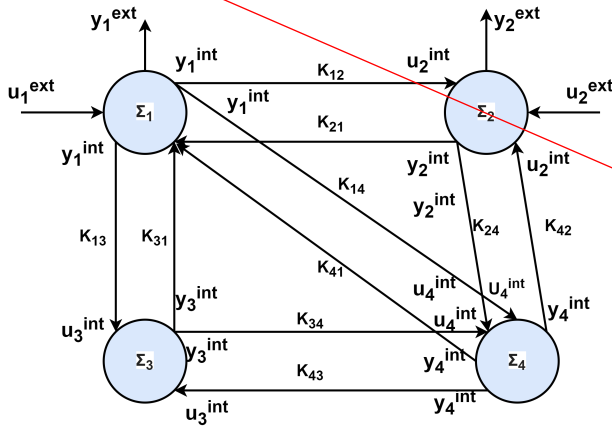
Figure 7.  Graphical representation of example 1

Figure 8.  $\Sigma_{12}$ cluster

matrix for the clsuters is given by:

$$P = \begin{bmatrix} I_n & \mathbf{0}_n & \mathbf{0}_n \\ I_n & \mathbf{0}_n & \mathbf{0}_n \\ \mathbf{0}_n & I_n & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n & I_n \end{bmatrix} \tag{65}$$

Using the aggregation matrix in (65) to obtain the simplified network of the form (63), we get (66).

$$
\begin{aligned}
\hat{K} &= \begin{bmatrix} K_{12}+K_{21} & K_{13} & K_{14}+K_{24} \\ K_{31} & \mathbf{0}_n & K_{34} \\ K_{41}+K_{42} & K_{43} & \mathbf{0}_n \end{bmatrix} \\[1em]
\hat{J} &= \begin{bmatrix} J_1+J_2 & \mathbf{0}_n & \mathbf{0}_n \\ \mathbf{0}_n & J_3 & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n & J_4 \end{bmatrix} \\[1em]
\hat{R} &= \begin{bmatrix} R_1+R_2 & \mathbf{0}_n & \mathbf{0}_n \\ \mathbf{0}_n & R_3 & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n & R_4 \end{bmatrix} \\[1em]
\hat{B} &= \begin{bmatrix} B_{1,ext} & B_{2,ext} \\ \mathbf{0}_{n \times m^{int}} & \mathbf{0}_{n \times m^{int}} \\ \mathbf{0}_{n \times m^{int}} & \mathbf{0}_{n \times m^{int}} \end{bmatrix}
\end{aligned}
\tag{66}
$$

The $\hat{J}$ and $\hat{K}$ is rewritten as

$$
\begin{aligned}
\hat{K} &= \begin{bmatrix} \mathbf{0}_n & K_{13} & K_{14}+K_{24} \\ K_{31} & \mathbf{0}_n & K_{34} \\ K_{41}+K_{42} & K_{43} & \mathbf{0}_n \end{bmatrix} \\[1em]
\hat{J} &= \begin{bmatrix} J_1+J_2+K_{12}+K_{21} & \mathbf{0}_n & \mathbf{0}_n \\ \mathbf{0}_n & J_3 & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n & J_4 \end{bmatrix}
\end{aligned}
\tag{67}
$$

In (67) it should be noted that, $\hat{J}$ contains the term $K_{12}$ and $K_{21}$ with the internal interconnections of $\Sigma_1$ and $\Sigma_2$. This shows that the dynamics of the new cluster $\Sigma_{12}$ contains the dynamics of the interconnection as well. This is represented in
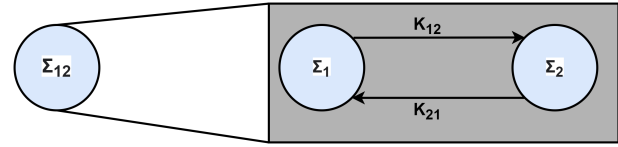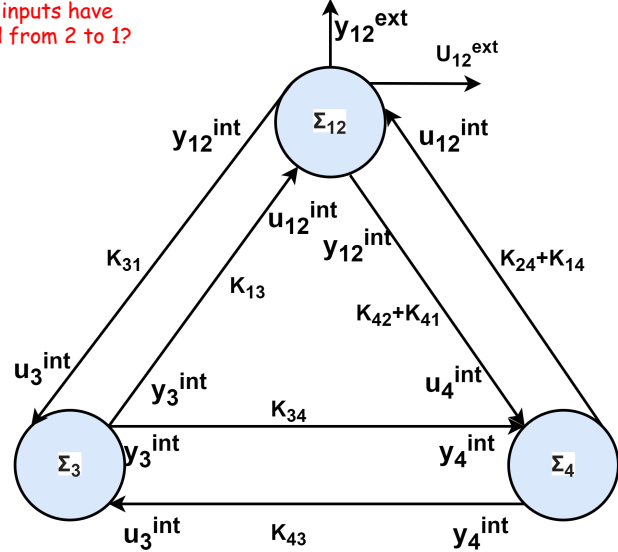


Figure 9.  Reduced graphical representation of example 1

Figure 8. The Figure 9 shows the new graphical representation for the simplified network.

Finally the clustering algorithm for the interconnected port Hamiltonian system is presented as follows:

**Input:** $J$, $K$, $R$, $Q$ and $B$ for $n$ interconnected port Hamiltonian systems and desired number of port Hamiltonian systems $r$ in simplified network.

**Output:** $\hat{J}$, $\hat{K}$, $\hat{R}$, $\hat{Q}$ and $\hat{B}$ of a simplified network.

1) Select the measurement used for dissimilarity.
2) Based on the three choices discussed in Section VI:
   **Based on dissipation matrix**
   a) Compute the dissimilarity matrix based on the dissipation matrix (30).
   **Based on dissipation gramian**
   a) Compute the dissipation gramian $D_g$.
   b) Compute the dissimilarity matrix (47).
   **Based on Asymptotic Hamiltonian energy**
   a) Compute the $\lim_{t \to \infty} x(t)$ (57).
   b) Compute the dissimilarity matrix as shown in (61).
3) Implement hierarchical clustering algorithm using the dissimilarity matrix.
4) Construct aggregation matrix $P$ based on the clusters generated by hierarchical clustering algorithm.
5) Generate $\hat{J}$, $\hat{K}$, $\hat{R}$, $\hat{B}$ and $\hat{Q}$ using the aggregation matrix as shown in (63).

It must be noted that different choices of $r^{th}$ order clustering will lead to different simplified networks. Hence, it is a challenge to compute the most suitable aggregation matrix $P$ such that the error between full network and simplified network $|\Sigma - \hat{\Sigma}|$ is small. Hence in next section we look into error computation between full network and simplified network.

## IX. ERROR APPROXIMATION

When we reduce a model using model reduction techniques, the reduced order model is the approximated model of full order model. Thus we will get discrepancies between the full network and the simplified network. Based on different dissimilarity measurements, we get different reduced order model and hence different modelling errors, with the full network system $\Sigma$ and simplified network $\hat{\Sigma}$ given by (24) and (63) respectively. Let $u^{ext}$ and $\hat{u}^{ext}$, $y^{ext}$ and $\hat{y}^{ext}$ denote the input and output of full network and simplified network respectively. We define inputs and outputs of the full network and simplified network as shown in (68).

$$
\begin{aligned}
u^{ext} &= \hat{u}^{ext} \\
e &= y^{ext} - \hat{y}^{ext}
\end{aligned}
\tag{68}
$$

Where, $e$ is the error. Considering the error system $\Sigma_e$ mapping $u^{ext} \longrightarrow e$ given by (69).

$$
\Sigma_{\text{e}} : \begin{cases} \dot{x}_e(t) = \mathcal{A}\mathcal{Q}x_e + \mathcal{B}u^{ext} \\ y_e^{ext} = \mathcal{B}^\top \mathcal{Q}x_e \end{cases}
\tag{69}
$$

where, state vector $x_e = [x, x_r]^\top$ , $y_e^{ext} = y^{ext} - \hat{y}^{ext}$ and

$$
\begin{aligned}
\mathcal{A} &= \begin{bmatrix} J + K - R & \mathbf{0}_{N_s \times r} \\ \mathbf{0}_{N_s \times n} & \hat{J} + \hat{K} - \hat{R} \end{bmatrix} \\
\mathcal{B} &= \begin{bmatrix} B \\ \hat{B} \end{bmatrix} \\
\mathcal{Q} &= \begin{bmatrix} Q & \mathbf{0}_{N_s \times r} \\ \mathbf{0}_{N_s \times n} & \hat{Q} \end{bmatrix}
\end{aligned}
\tag{70}
$$

Now, the error approximation for three types of dissimilarity measurements are shown.

### A. *Dissipation matrix*

**Theorem 2.** *The approximation error between the full network and the simplified network is given by* (71)

$$
Error = \|R - P\hat{R}P^\top\|_{sum}
\tag{71}
$$

*Proof.*

$$
\hat{R} = P^\top R P
\tag{72}
$$

Pre and post multiplying by $P$ and $P^\top$ we get

$$
P\hat{R}P^\top = PP^\top RPP^\top
\tag{73}
$$

Since $PP^\top \neq I$ we therefore have $P\hat{R}P^\top \neq R$. Thus projection of $\hat{R}$ is not equal to $R$.

$\square$

Since $R$ and $\hat{R}$ does not have the same dimension. In order to find the error between dissipation matrix of full

network and simplified network, we project back the reduce order dissipation matrix to full order and then take their difference to find the error.

### B. *Dissipation gramian*

**Theorem 3.** *The approximation error between the full network and simplified network will be the difference between the asymptotic energy dissipated, as given by* (74).

$$
Error = |x_n^\top D_g x_n - x_r^\top D_{gr} x_r|
\tag{74}
$$

*where, $D_{gr}$ is the dissipation gramian of the reduced order model. $x_n$ is the initial condition and $x_r$ is the projected initial condition given by* (75)

$$
x_r = P^\top x_n
\tag{75}
$$

According to proof provided in appendix A, dissipation gramian of projected system is not equal to projection of dissipation gramian , that is, $D_{gr} \neq P^\top D_g P$. Hence, $D_{gr}$ needs to be calculated from the lyapunov equation of the form (45), which is given by

$$
\hat{A}^\top D_{gr} + D_{gr}\hat{A} = -\hat{Q}^\top \hat{R}\hat{Q}
\tag{76}
$$

**Note 6:** The initial condition $x_n$ can be scaled up or down by the factor $\alpha$. Thus, we can make an important conclusion that is, even though the clustering is not affected by the factor $\alpha$ as shown in (47), the approximation error depends quadratically on the value $\alpha$. However, the trend of decay in error with increasing number of cluster will remain the same for different values of $\alpha$, as shown in Figure 14, Figure 15 and Figure 16 in section X.

### C. *Asymptotic Hamiltonian Energy*

**Theorem 4.** *The approximation error between the full network and the simplified network will be the difference in their asymptotic energy, given by* (77).

$$
Error = |x(\infty)^\top Q x(\infty) - x_r(\infty)^\top \hat{Q} x_r(\infty)|
\tag{77}
$$

According to the proof provided in appendix B $x_r(\infty) \neq P^\top x(\infty)$. Thus, $x_r(\infty)$ must be computed from an equation of the form (57).

$$
x_r(\infty) = -\hat{A}^{-1}\hat{B}1_m
\tag{78}
$$

## X. RESULTS

In this section the clustering algorithm developed is applied to RLC circuits interconnected to each other as a ladder as shown in Figure 10. For ease of visualization we have selected number of interconnected port Hamiltonian systems to be 20. First we do the modelling of the interconnected port Hamiltonian system. For simplicity purpose, the modelling is done for three RLC circuits interconnected to each other.
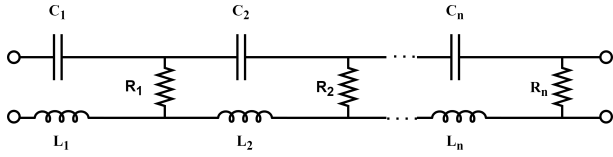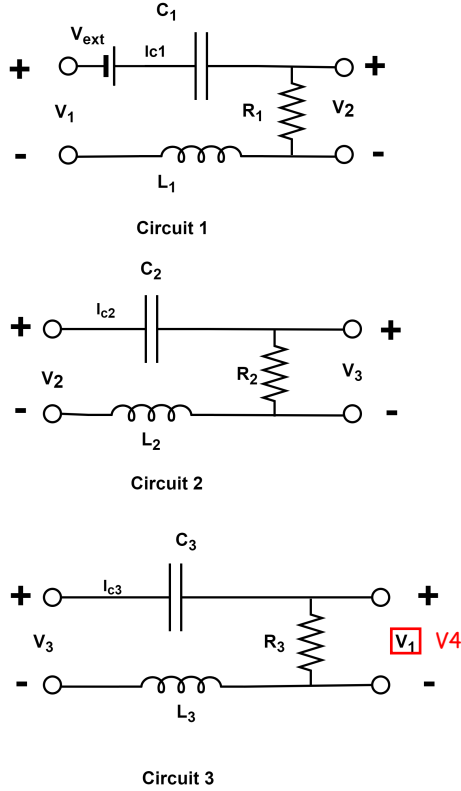
Figure 10. Ladder RLC circuit



Circuit 1



Circuit 2



Circuit 3

Figure 11. Separated RLC circuit

## A. MODELLING

First we will break down the ladder circuit presented in Figure 10 and treat each of the circuits as a separate port Hamiltonian system as shown in Figure 11. Next we make the gyrator interconnection structure between them as shown in Figure 12.
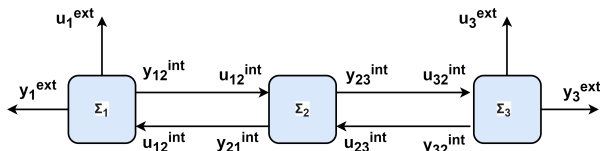


Figure 12. Node diagram of Figure 10

**Step 1:**

Let us consider circuit 1 in Figure 11 and write down its equations using port Hamiltonian formulation and Kirchhoff's laws, as shown in (79).

$$\begin{bmatrix} I_{C_1} \\ V_{L_1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -R_1 \end{bmatrix} \begin{bmatrix} V_{C_1} \\ I_{L_1} \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V_{left} \\ I_1 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} V_{C_1} \\ I_{L_1} \end{bmatrix}$$

(79)

Where,

$$\begin{bmatrix} V_{C_1} \\ I_{L_1} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{C} & 0 \\ 0 & \frac{1}{L} \end{bmatrix}}_{Q_1} \begin{bmatrix} q_1 \\ \phi_1 \end{bmatrix}$$

(80)

Here $C$ is capacitance of capacitor $L$ is inductance of the inductor and $Q_1$ is the energy matrix of the RLC circuit 1.

**Step 2:**

Separating internal and external ports from (79) leads to (81).

$$\Sigma_1 \begin{cases} \begin{bmatrix} I_{C_1} \\ V_{L_1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -R_1 \end{bmatrix} \begin{bmatrix} V_{C_1} \\ I_{L_1} \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} I_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} V_{left} \\[2ex] y_{ext}^1 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} V_{C_1} \\ I_{L_1} \end{bmatrix} = I_{L_1} \\[2ex] y_{int}^{12} = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} V_{C_1} \\ I_{L_1} \end{bmatrix} = -V_{C_1} \end{cases}$$

(81)

From Figure 12, the left port of $\Sigma_1$ and right port $\Sigma_3$ are the external ports and rest of the ports are internal ports. The $S$ matrix from (11) for both system 1 to 2 and system 2 to 3 is given by (82)

$$\begin{bmatrix} u_{12}^{int} \\ u_{21}^{int} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{S_1} \begin{bmatrix} y_{12}^{int} \\ y_{21}^{int} \end{bmatrix}$$

$$\begin{bmatrix} u_{23}^{int} \\ u_{32}^{int} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{S_2} \begin{bmatrix} y_{23}^{int} \\ y_{32}^{int} \end{bmatrix}$$

(82)

**Step 3:**

Now we can write the complete state space of the form (24), which is given by (83).

$$\begin{bmatrix} I_{C1} \\ V_{L1} \\ I_{C2} \\ V_{L2} \\ I_{C3} \\ V_{L3} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & -R_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ -1 & 0 & -1 & -R_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -R_3 \end{bmatrix} \begin{bmatrix} V_{C1} \\ I_{L1} \\ V_{C2} \\ I_{L2} \\ V_{C3} \\ I_{L3} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_{left} \\ I_3 \end{bmatrix}$$

$$y_{ext} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} V_{C1} \\ I_{L1} \\ V_{C2} \\ I_{L2} \\ V_{C3} \\ I_{L3} \end{bmatrix}$$

(83)

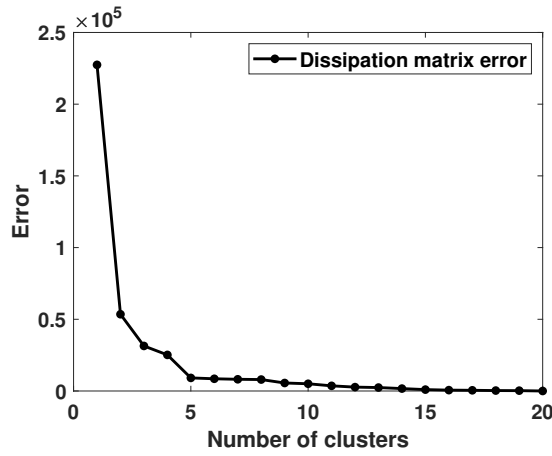The state space for 20 RLC circuits follows the same structure as (83).

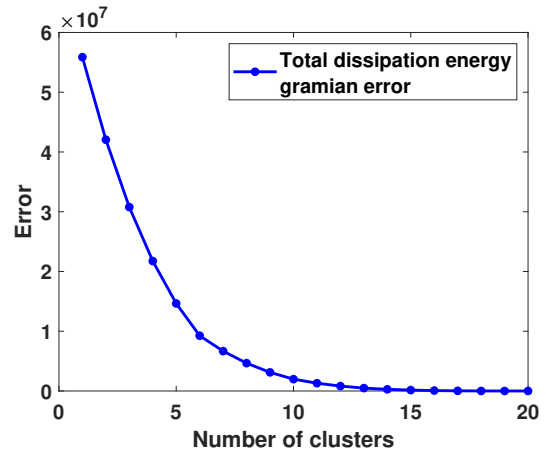Figure 13. Number of clusters vs dissipation matrix error



Figure 15. Number of clusters vs dissipation gramian error for $\alpha=1$
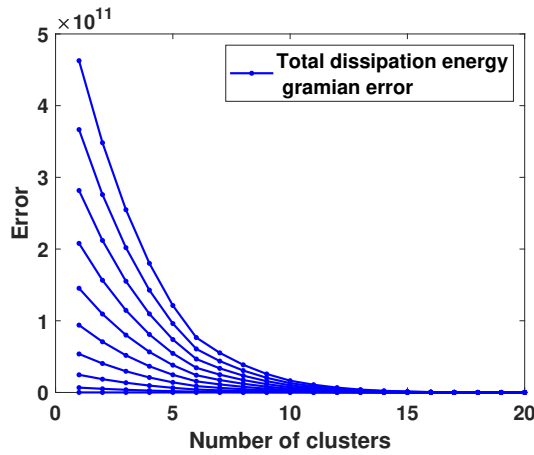


Figure 14. Number of clusters vs dissipation gramian error for 10 different values of $\alpha$ with $0 \leq \alpha \leq 100$
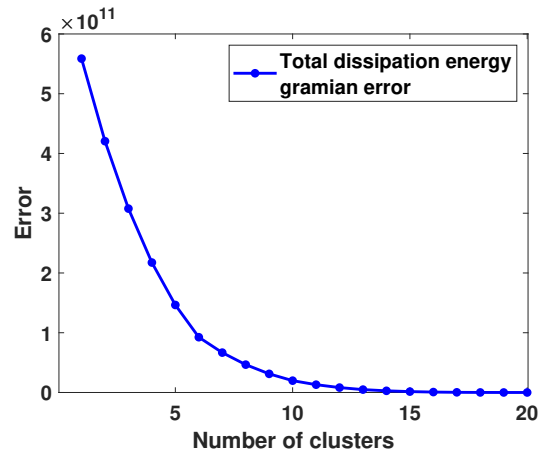


Figure 16. Number of clusters vs dissipation gramian error for $\alpha=100$

### B. SIMULATIONS

Here we compare the clusters created by implementing all three dissimilarity matrices. The clusters are created by taking maximum distance between clusters for hierarchical clustering algorithm. Finally we plot the error with respect to the number of clusters, in order to compare the decay in error.

*1) Dissipation matrix based dissimilarity:* The dissimilarity measure is given by (30). The approximation error is given by (71) and the error with increasing clusters decreases, as shown in Figure 13.

*2) Dissipation gramian based dissimilarity:* The dissimilarity measure is given by (47). The approximation error is given by (74). As we have mentioned previously, the dissimilarity matrix and the error depends on the initial condition and it will be scaled up by the value $\alpha$. We will vary $\alpha$ from one to hundred and plot the approximation error with increasing number of clusters as shown in Figure 14.

The top most curve and lower most curve in Figure 14 represents the $\alpha = 100$ and $\alpha = 1$ respectively. Due to the

quadratic dependence of the error on $\alpha$ and plot scaling it may look like the curves are different. Now we look at the cluster vs error plots for both initial conditions separately. From Figure 15 and Figure 16, the trend of decrease in error is same. Hence it is verified that clustering is independent of initial condition $\alpha$.

*3) Asymptotic energy based dissimilarity:* The dissimilarity measure is given by (61). The approximation error is given by (77) and the error with increasing clusters is shown to decay slowly as shown in Figure 17.

### C. COMPARISON OF NETWORK CLUSTERING

In this section the comparison of clustering using different dissimilarity is shown via network diagram. For the comparison of simplified networks, ten clusters has been selected to be present in the simplified network.

The Figure 18, Figure 19 and Figure 20 shows the ladder structure of interconnection between 20 RLC port Hamiltonian systems. In the mentioned figures the nodes having same colour represent their cluster, that is, the nodes with same color will be placed in same cluster for simplified
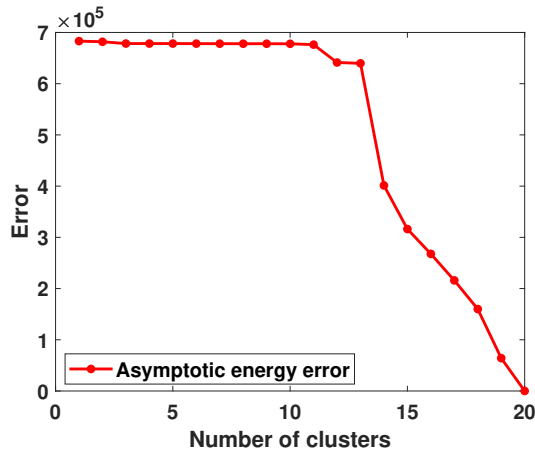
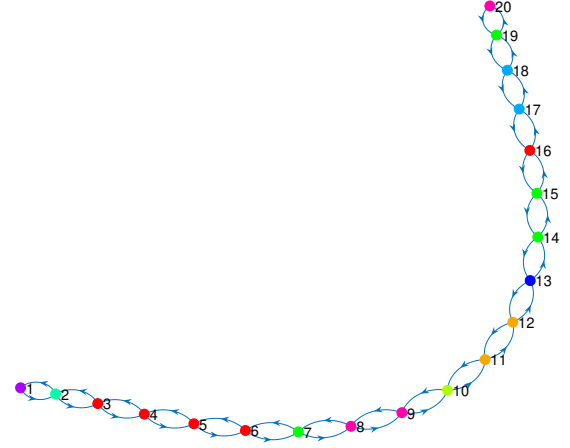Figure 17. Number of clusters vs asymptotic energy error



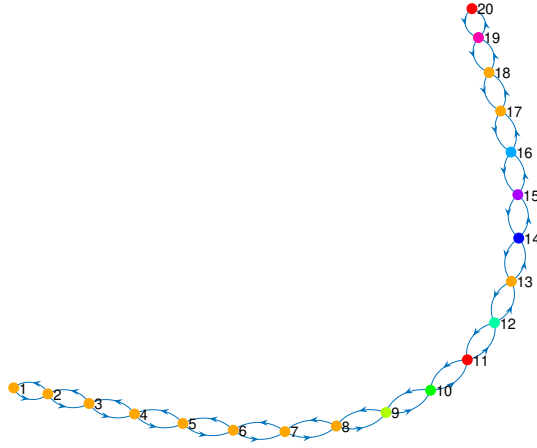Figure 19. Full network for dissipation matrix based dissimilarity



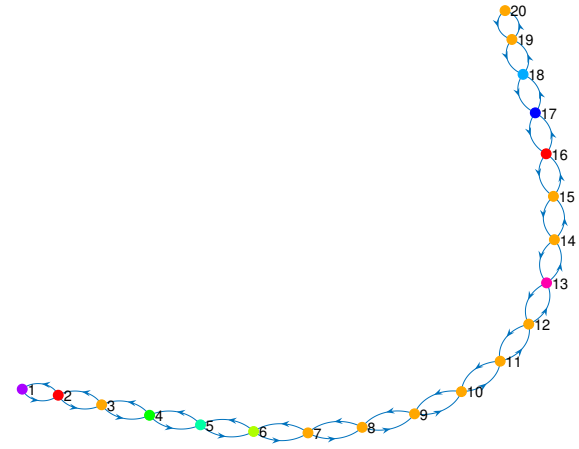Figure 18. Full network for asymptotic energy based dissimilarity



Figure 20. Full network for dissipation gramian based dissimilarity

network. It is evident from Figure 21, Figure 22 and Figure 23 that the simplified network based on different dissimilarity measurements differ from each other. Hence we can see the difference in the error approximation in Figure 13, Figure 17 and Figure 15.

## XI. CONCLUSION

In this paper a state of the art clustering technique was developed for physical network system. Furthermore three different dissimilarity measurements and approximation error based on asymptotic energy, dissipation matrix and dissipation gramian was proposed. A new aggregation matrix for the clustering of port Hamiltonian system was also formulated. After applying the clustering based model reduction on physical network system we were able to obtain a simplified network which was a port Hamiltonian system. This simplified network had a lossless interconnection and its energy was indeed represented by a Hamiltonian energy function.

From the results it can be concluded that, the trend of approximation error varies based on the choice of dissimilarity and the proposed choices of dissimilarities leads to clustering that are time invariant. That is, the cluster formation only depend on the choice of $r$ and no other variables.

## XII. FUTURE SCOPE

Firstly, instead of gyrator interconnection the transformer interconnection [ [3], [16]] could be used for interconnection of transformer interconnection. Since, this state of the art concept of clustering has a wide range of applications. Hence, this framework can be extended for systems which are described by PDE's (partial differential equations). In this research work we have considered only dissipative network system, further
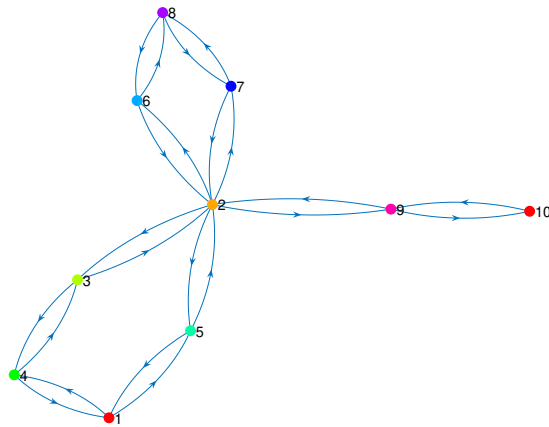
Figure 21. Simplified network after clustering for asymptotic energy based dissimilarity
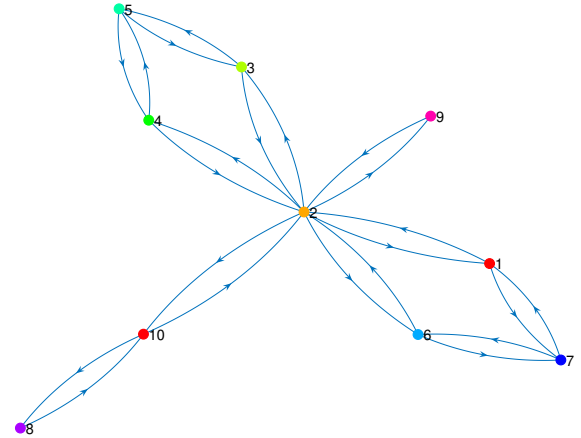


Figure 23. Simplified network after clustering for dissipation gramian based dissimilarity
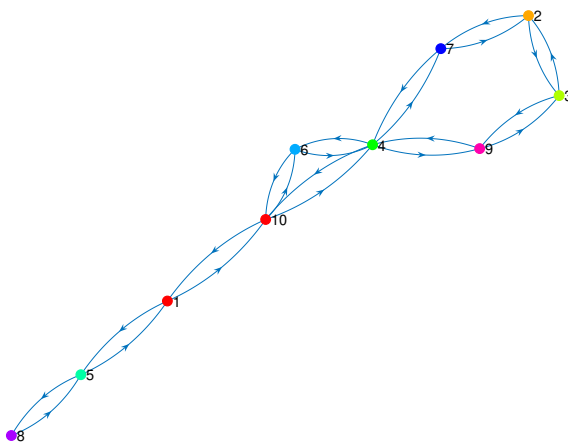


Figure 22. Simplified network after clustering for dissipation matrix based dissimilarity

work needs to be done on conservative or energy generating network systems. Furthermore we can also study how energy gets distributed in the physical network system in frequency domain.

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## APPENDIX

### A. Proof of reduced order dissipation gramian

Starting from the following equation

$$D_{gr} = \int_0^\infty e^{\hat{A}^\top t} \hat{Q}^\top \hat{R} \hat{Q} e^{\hat{A}t} dt \tag{84}$$

Replacing $\hat{A}$, $\hat{Q}$ and $\hat{R}$ in (84)

$$D_{gr} = \int_0^\infty e^{P^\top A^\top P t} P^\top Q^\top P P^\top R P P^\top Q P e^{P^\top A P t} dt \tag{85}$$

$$D_{gr} = \int_0^\infty P^\top e^{A^\top t} P P^\top Q^T P P^\top R P P^\top Q P P^\top e^{At} P dt \tag{86}$$

$$D_{gr} = P^\top \left( \int_0^\infty e^{A^\top t} P P^\top Q^\top P P^\top R P P^\top Q P P^\top e^{At} dt \right) P \tag{87}$$

Since $PP^\top \neq I_{N \times n}$, we can see that infinite dissipation gramian of projected system is not equal to projection of dissipation gramian . Therefore $D_{gr} \neq P^\top D_g P$.

### B. Proof for asymptotic energy error

Applying the external input from (50) and (51) with zero initial condition to state response for simplified network which is of the form (52)

$$x_r(t) = \int_0^t e^{\hat{A}(t-\tau)} \hat{B} 1_m d\tau \tag{88}$$

Let $t - \tau = \sigma$, we get $-d\tau = d\sigma$.
Taking upper limit $\tau = t$, we get $\sigma = t - t = 0$
Taking lower limit $\tau = 0$, we get $\sigma = t - 0 = t$
Using change of variables to integrate (88)

$$x_r(t) = \int_t^0 -e^{\hat{A}\sigma} \hat{B} 1_m d\sigma \tag{89}$$

Switching limits we get:

$$x(t) = \int_0^t e^{\hat{A}\sigma} \hat{B} 1_m d\sigma \tag{90}$$

$$x(t) = \hat{A}^{-1} [e^{\hat{A}\sigma}]_0^t \hat{B} 1_m \tag{91}$$

$$x(t) = \hat{A}^{-1} [e^{\hat{A}t} - I] \hat{B} 1_m \tag{92}$$

As $t \longrightarrow \infty$ we have $e^{\hat{A}t} \longrightarrow 0$, as the system is asymptotically stable. Thus we get

$$\lim_{t \to \infty} x_r(t) = -\hat{A}^{-1} \hat{B} 1_m \tag{93}$$

Substituting $\hat{A}$ and $\hat{B}$ in (93)

$$\lim_{t \to \infty} x_r(t) = -(P^\top A P)^{-1} P^\top B u \tag{94}$$

$$\lim_{t \to \infty} x_r(t) = -P^{-1} A^{-1} P^{-\top} P^\top B u \tag{95}$$

Since $P$ is not invertible, we have $x_r(\infty) \neq P^\top x(\infty)$

### C. Preservation of port Hamiltonian property in simplified network

We will prove the property of preservation separately for $\hat{J}$, $\hat{K}$ and $\hat{R}$. First we show that $\hat{J}$ is skew symmetric matrix. Consider (43) from example 1. We take $\hat{J}$ matrix

$$\hat{J} = \begin{bmatrix} J_1 + J_2 & 0 & 0 \\ 0 & J_3 & 0 \\ 0 & 0 & J_4 \end{bmatrix} \tag{96}$$

In order for the $\hat{J}$ to be skew symmetric, sum of $J_1$ and $J_2$ must be skew symmetric. In the above equation (96) $J_1$ and $J_2$ are skew symmetric matrices.

$$\therefore J_1^\top = -J_1, J_2^\top = -J_2 \tag{97}$$

$$(J_1 + J_2)^\top = J_1^\top + J_2^\top = -J_1 - J_2 = -(J_1 + J_2) \tag{98}$$

So, Sum of two skew symmetric matrices is always skew symmetric matrix. Hence $\hat{J}$ must be skew symmetric matrix.

Next we take $\hat{K}$ matrix from (43)

$$\hat{K} = \begin{bmatrix} K_{12} + K_{21} & K_{13} & K_{14} + K_{24} \\ K_{31} & \mathbf{0}_n & K_{34} \\ K_{41} + K_{42} & K_{43} & \mathbf{0}_n \end{bmatrix} \tag{99}$$

In order for the $\hat{K}$ to be skew symmetric, sum of $K_{12}$ and $K_{21}$ in block diagonal must be skew symmetric as well as the sums in off-diagonal block must be skew symmetric. We have the following relation from (22)

$$K_{12} = -K_{21}^\top \tag{100}$$

Taking sum of $K_{12}$ and $K_{21}$

$$K_{12} + K_{21} = -K_{21}^\top + K_{21} \tag{101}$$

By the skew symmetric matrix property we get

$$K_{21} - K_{21}^\top \in skew_n \tag{102}$$

Considering off-diagonal block sum for $K_{14} + K_{24}$ and $K_{41} + K_{42}$

$$(K_{14} + A_{24})^\top = A_{14}^\top + A_{24}^\top \tag{103}$$

$$K_{14}^\top + K_{24}^\top = -K_{41} - K42 \tag{104}$$

$$K_{14}^\top + A_{24}^\top = -(K_{41} + K_{42}) \tag{105}$$

From (102) and (105) we can say that $\hat{K}$ is also skew symmetric matrix.

Now we will prove that $\hat{R}$ is still positive definite. Take $\hat{R}$ from (43)

$$\hat{R} = \begin{bmatrix} R_1 + R_2 & \mathbf{0}_n & \mathbf{0}_n \\ \mathbf{0}_n & R_3 & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{0}_n & R_4 \end{bmatrix} \qquad (106)$$

In the above equation (106) $R_i > 0, \forall i = 1, 2, 3, 4$. Hence the sum $R_1 + R_2$ will be positive definite.

Thus from the above arguments provided, we can say that the simplified network system with $r$ clusters will have port Hamiltonian property.