

MASTER

Crossing Numbers of Beyond-Planar Graphs

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Department of Mathematics and Computer Science
Algorithms, Geometry, and Applications Cluster

Crossing Numbers of Beyond-Planar Graphs

Master Thesis

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Abstract

Graph drawing beyond planarity focuses on drawings of high visual quality for non-planar graphs which are characterized by certain forbidden edge configurations. The (rectilinear) crossing number of a drawing is a well-established criterion for its quality. Forbidding an edge configuration may drastically increase the crossing number for some graphs. Chimani *et al.* [GD'19] gave bounds for the supremum of the ratio between the crossing number of three classes of beyond-planar graphs and the unrestricted crossing number. Lower bounds on this supremum imply that there is a graph for which the crossing number increases by some factor, while upper bounds imply that the crossing number can never increase more than certain factor.

In this thesis we extend the results by Chimani *et al.* to the main currently known classes of beyond-planar graphs characterized by forbidden topological patterns and answer several of their open questions. We achieve linear lower bounds for all the classes of beyond-planar graphs we consider, yielding tight bounds for the 1-gap-planar and the skewness-1 crossing ratio. Additionally, we extend our constructions to the forbidden configurations that are dependent on some variable k .

Chapter 1

Introduction

A central topic in graph drawing are good drawings of graphs which are not necessarily planar. A natural criterion for the quality of a drawing is the number of edge crossings. While empirical studies suggest that the number of crossings is not the only factor that influences human understanding of a drawing, it is nevertheless one of the most relevant aesthetic indicators [20, 25, 26, 33]. However, it is NP-hard to compute the minimum number of crossings of a graph G across all possible drawings of G , also referred to as the *crossing number* of G , $\text{cr}(G)$ [18]. Meaningful upper and lower bounds, as well as heuristic, approximation, and parameterized algorithms, have hence been a major focus [10].

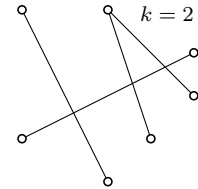
Drawing edges as straight-line segments aligns well with another important aesthetic criterion: minimizing the number of bends. One classic variant is hence the *rectilinear crossing number* $\overline{\text{cr}}(G)$ of a graph G , that is, the minimum number of crossings across all straight-line drawings of G . Clearly $\text{cr}(G) \leq \overline{\text{cr}}(G)$ for any graph G . The straight-line restriction increases the crossing number arbitrarily: for every k there exists a graph G such that $\text{cr}(G) = 4$ and $\overline{\text{cr}}(G) = k$ [9]. On the other hand, if $G = (V, E)$ is a graph with maximum degree Δ then $\overline{\text{cr}}(G) = O(\Delta \cdot \text{cr}^2(G))$ [8, 30] and when $|E| \geq 4|V|$ this bound can be improved to $\overline{\text{cr}}(G) = O(\Delta \text{cr}(G) \log \text{cr}(G))$; see Schaefer's book [29] on crossing numbers. Computing $\overline{\text{cr}}(G)$ is also NP-hard (actually, it is $\exists\mathbb{R}$ -complete [7, 28]) but in polynomial time it can be approximated to $\overline{\text{cr}}(G) + o(n^4)$ [17].

Empirical studies suggest that not only the number of crossings matters for human understanding, but also the topological and geometric properties of the drawing. In recent years there has been particular interest in such drawings of *beyond-planar* graphs, which are characterized by certain forbidden crossing configurations of the edges; see the recent survey by Didimo *et al.* [16]. It is hence natural to consider also the relation between the (unrestricted) crossing number and the crossing number across beyond-planar drawings.

Chimani *et al.* [15] gave bounds on the supremum ρ of the ratio between the 1-planar, quasi-planar, and fan-planar crossing number of a graph G ($\text{cr}_{1\text{-pl}}(G)$, $\text{cr}_{\text{quasi}}(G)$, and $\text{cr}_{\text{fan}}(G)$, respectively) and the unrestricted crossing number $\text{cr}(G)$. These results show that there exist graphs that have significantly larger crossing numbers when drawn with the beyond-planar restrictions. We extend their results giving a more general lower bound for k -planarity. Furthermore, we answer several of their open questions, including the extension to other classes of beyond-planar graphs and realizing their lower bounds as well as our lower bounds in the straight-line setting (i.e. when edges are drawn as straight-line segments). The bounds are summarized in Table 1.1; all can be achieved with straight-line drawings.

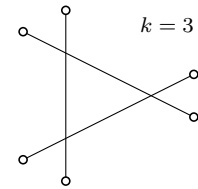
Below we define all beyond-planar graph families that are considered in this thesis and show examples of configurations that are forbidden for that family. A graph belongs to a family \mathcal{F} if it admits a drawing that belongs to \mathcal{F} , that is, the forbidden configuration of \mathcal{F} does not occur in the drawing. These definitions correspond to the definitions as described in Section 3.1 in the survey by Didimo *et al.* [16].

k -planar drawings, do not contain an edge crossed more than k times. The notion of 1-planar drawings was introduced in 1965 [27] and they have since been widely studied. The study of k -planar graphs ($k > 1$) was introduced to find lower bounds on the crossing number of a graph [23].



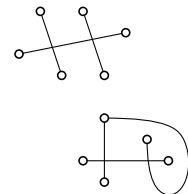
We define the k -planar crossing number of a k -planar graph G , $cr_{k\text{-pl}}(G)$, as the minimum number of crossings over all k -planar drawings of G . We define the k -planar crossing ratio, $\rho_{k\text{-pl}}$, as the supremum of $cr_{k\text{-pl}}(G)/cr(G)$ over all k -planar graphs G .

k -quasi-planar drawings, do not have k mutually crossing edges. We use the term *quasi-planar* to refer to 3-quasi-planar. Early results on this family aimed to determine the maximum number of edges for these graphs [4, 24]. These results answered the question of what the maximum number of edges is of a geometric graph that contains no 3 pairwise disjoint edges posed by Avital and Hanani [5]. The problem of determining or estimating the maximum number of edges in a k -quasi-planar graph for $k \geq 3$ was raised by Kupitz [21] and by Perles in unpublished notes.



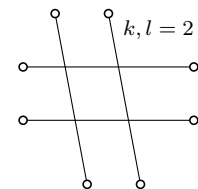
We define the k -quasi-planar crossing number of a k -quasi-planar graph G , $cr_{k\text{-quasi}}(G)$, as the minimum number of crossings over all k -quasi-planar drawings of G . We define the k -quasi-planar crossing ratio, $\rho_{k\text{-quasi}}$, as the supremum of $cr_{k\text{-quasi}}(G)/cr(G)$ over all k -quasi-planar graphs G . Similarly, we denote the quasi-planar crossing number as cr_{quasi} and the quasi-planar crossing ratio as ρ_{quasi} .

Fan-planar drawings, do not contain two independent edges that cross a third one or two adjacent edges that cross a third edge from different sides. The latter means that the common endpoint of the two adjacent edges is on different sides of the third edge [19]. This family of drawings can be viewed as opposite to the class of fan-crossing-free drawings. Kaufmann and Ueckerdt achieved a tight bound on the maximum number of edges for fan-planar graphs [19].



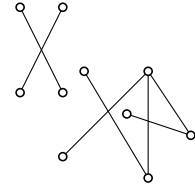
We define the fan-planar crossing number of a fan-planar graph G , $cr_{\text{fan}}(G)$, as the minimum number of crossings over all fan-planar drawings of G . We define the fan-planar crossing ratio, ρ_{fan} , as the supremum of $cr_{\text{fan}}(G)/cr(G)$ over all fan-planar graphs G .

(k, l) -grid-free drawings, do not contain a (k, l) -grid: a set of k edges all crossing each edge in a set of l edges [22]. If the k edges are incident to the same vertex the (k, l) -grid is *radial*; if the l edges are also incident to the same vertex the (k, l) -grid is *biradial*. A (k, l) -grid is natural if all its edges are independent and no two edges in the same group cross. Pach *et al.* gave an upper bound on the maximum number of edges in a (k, l) -grid-free graph [22].



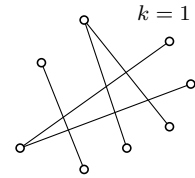
We define the (k, l) -grid-free crossing number of a (k, l) -grid-free graph G , $cr_{k,l\text{-grid-free}}(G)$, as the minimum number of crossings over all (k, l) -grid-free drawings of G . We define the (k, l) -grid-free crossing ratio, $\rho_{k,l\text{-grid-free}}$, as the supremum of $cr_{k,l\text{-grid-free}}(G)/cr(G)$ over all (k, l) -grid-free graphs G .

Planarly connected drawings, are such that each pair of crossing edges is independent and there is a crossing-free edge that connects their endpoints [2]. In terms of forbidden configuration a planarly connected drawings does not contain a pair of crossing edges that do not have their endpoints connected by a crossing-free edge. Ackerman *et al.* introduced the class of graphs because they are related to quasi-planar graphs and to maximal 1-planar and fan-planar graphs [2]. The authors achieved a linear upper bound on the maximum number of edges in planarly connected graphs.



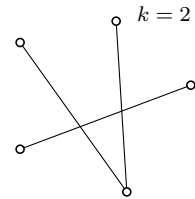
We define the planarly connected crossing number of a planarly connected graph G , $cr_{pl-con}(G)$, as the minimum number of crossings over all planarly connected drawings of G . We define the planarly connected crossing ratio, ρ_{pl-con} , as the supremum of $cr_{pl-con}(G)/cr(G)$ over all planarly connected graphs G .

k -gap planar drawings, allow to map each crossing to one of the two corresponding edges such that each edge has at most k crossings mapped to it [6]. In terms of forbidden configuration a k -gap-planar drawing may not have more than k crossings mapped to an edge when optimally mapping each crossing to a corresponding edge. In a *cased* drawing of a graph, each crossing is resolved by locally interrupting one of the two crossing edges. A k -gap-planar graph can equivalently be defined as graph that admits a cased drawing in which each edge has at most k gaps. Bae *et al.* achieved a tight bound on the maximum number of edges in gap-planar graphs [6].



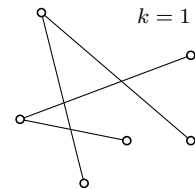
We define the k -gap planar crossing number of a k -gap planar graph G , $cr_{k-gap}(G)$, as the minimum number of crossings over all k -gap planar drawings of G . We define the k -gap planar crossing ratio, ρ_{k-gap} , as the supremum of $cr_{k-gap}(G)/cr(G)$ over all k -gap planar graphs G .

k -fan-crossing-free drawings, do not contain an edge that crosses k adjacent edges. Edges are *adjacent* if they all have an endpoint in common. We use *fan-crossing-free* to refer to 2-fan-crossing-free [13]. The class of k -fan-crossing-free graphs coincides with the class of radial $(k, 1)$ -grid-free graphs, for $k \geq 2$. Cheong *et al.* achieved a tight bound on the maximum number of edges in a fan-crossing-free graph [13].



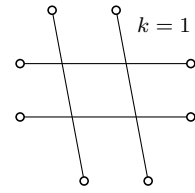
We define the k -fan-crossing-free crossing number of a k -fan-crossing-free graph G , $cr_{k-fan-free}(G)$, as the minimum number of crossings over all k -fan-crossing-free drawings of G . We define the k -fan-crossing-free crossing ratio, $\rho_{k-fan-free}$, as the supremum of $cr_{k-fan-free}(G)/cr(G)$ over all k -fan-crossing-free graphs G . Similarly, we denote the fan-crossing-free crossing number as $cr_{fan-free}$ and the fan-crossing-free crossing ratio as $\rho_{fan-free}$.

Skewness- k drawings, can be made planar by removing at most k edges. In terms of forbidden configuration this means that the drawing does not contain a set of crossings not covered by at most k edges. Here, a crossing is covered by an edge, if that edge is part of the crossing. Skewness- k graphs are mainly studied for $k = 1$ under the name of *almost planar* or *near planar* graphs. Remarkably, computing the unrestricted crossing number of almost planar graphs is NP-hard [11, 12]. On the other hand, it has been shown that computing a minimum-crossing drawing for the insertion of an edge in a planar graph can be done in linear time. This implies that skewness-1 crossing number can be computed in polynomial time.



We define the skewness- k crossing number of a skewness- k graph G , $cr_{skew-k}(G)$, as the minimum number of crossings over all skewness- k drawings of G . We define the skewness- k crossing ratio, ρ_{skew-k} , as the supremum of $cr_{skew-k}(G)/cr(G)$ over all skewness- k graphs G .

k -apex drawings, A k -apex can be made planar by removing at most k vertices. Accordingly, the drawing does not contain a set of crossings not covered by at most k vertices. Here, a crossing is covered by an vertex, if that vertex is the endpoint of an edge that is part of the crossing. It is immediate to see that a skewness- k drawing is also a k -apex drawing. Generally, 1-apex graphs are mainly recognized as *apex graphs*. Similarly to the edge insertion problem, it has been shown that computing a minimum-crossing drawing for the insertion of an vertex and some edge connected to that vertex in a planar graph can be done in polynomial time [14]. This implies that 1-apex crossing number can be computed in polynomial time.



We define the k -apex crossing number of a k -apex graph G , $cr_{k\text{-apex}}(G)$, as the minimum number of crossings over all k -apex drawings of G . We define the k -apex crossing ratio, $\rho_{k\text{-apex}}$, as the supremum of $cr_{k\text{-apex}}(G)/cr(G)$ over all k -apex graphs G .

Table 1.1: Bounds for the supremum ρ of the ratio between crossing numbers.

Family	Forbidden Configurations	Lower	Upper
k -planar ($k \geq 1$)	An edge crossed more than k times	$\Omega(n/k)$ (Section 2.2)	$O(k\sqrt{kn})$ (Section 2.2)
k -quasi-planar ($k \geq 3$)	k pairwise edges crossing	$\Omega(n/k^3)$ [15]	$f(k)n^2 \log^2 n$ [15]
Fan-planar	Two independent edges that cross a third one or two adjacent edges that cross another edge from different "sides"	$\Omega(n)$ [15]	$O(n^2)$ [15]
(k, l) -grid-free	Set of k edges such that each edge crosses each edge from a set of l edges.	$\Omega\left(\frac{n}{kl(k+l)}\right)$ (Section 3.1)	$g(k, l)n^2$ (Section 3.1)
Planarly connected	Two crossing edges that do not have two of their endpoint connected by a crossing-free edge or two adjacent edges crossing	$\Omega(n)$ (Section 3.2)	$O(n^2)$ (Section 3.2)
k -gap-planar	More than k mappings to an edge when optimally mapping each crossing to a corresponding edge	$\Omega(n/k^3)$ (Section 3.3)	$O(k\sqrt{kn})$ (Section 3.3)
k -fan-crossing-free	An edge that crosses k adjacent edges	$\Omega(n/k^2)$ (Section 3.4)	$O(k^2 n^2)$ (Section 3.4)
Skewness- k	Set of crossings not covered by at most k edges	$\Omega(n/k)$ (Section 4.1)	$O(kn + k^2)$ (Section 4.1)
k -apex	Set of crossings not covered by at most k vertices	$\Omega(n/k)$ (Section 4.2)	$O(k^2 n^2 + k^4)$ (Section 4.2)

Chapter 2

Revisiting k -planar, quasi-planar and fan-planar graphs

Chimani *et al.* addressed three different classes of beyond-planar graphs: 1-planar, quasi-planar and fan-planar graphs [15]. They studied the crossing ratios of the classes. The crossing ratio $\rho_{\mathcal{F}}$ is the largest ratio $\text{cr}_{\mathcal{F}}(G)/\text{cr}(G)$ possible over all graphs G that are in \mathcal{F} . In other words it indicates how much larger the crossing number can get by enforcing the forbidden configuration of \mathcal{F} in the worst case.

They showed lower bounds on the crossing ratios, which show that there does exist a graph that has an increase in crossing number by a certain factor. Furthermore, they showed upper bounds on the crossing ratios, which indicate that there are no graphs which have an increase by more than a certain factor. Additionally, they used their constructions to prove a lower bound for the k -planar crossing ratio and the k -quasi planar crossing ratio.

We generalise the lower bound for the k -planar crossing ratio. For completeness, we recall the upper bounds and we recall the remaining lower bounds. Additionally, we provide upper bounds for the k -planar crossing ratio and the k -quasi planar crossing ratio, as they were not mentioned by Chimani *et al.*

We can show a lower bound on a crossing ratio $\rho_{\mathcal{F}}$ by presenting a graph with $\text{cr}(G) \leq x$ and $\text{cr}_{\mathcal{F}}(G) \geq y$. This then implies the lower bound $\Omega(y/x)$.

Our upper bounds on the *crossing ratio* $\rho_{\mathcal{F}}$ for the beyond-planar graph families \mathcal{F} that we studied are all rather straightforward. They are based on (i) the definition of the beyond-planar class \mathcal{F} , (ii) the fact that $\text{cr}_{\mathcal{F}}(G) > \text{cr}(G)$ only holds for non-planar graphs, (iii) the known bounds on the maximum number of edges of graphs in each class, and (iv) the fact that drawings minimizing the crossing number are simple (see for example Lemma 1.3 in [29]). A drawing is *simple*¹ if every pair of edges has at most one point in common (either a common endpoint or a proper crossing). Thus, in crossing-minimizing drawings two edges cross at most once.

Chimani *et al.* [15] asked whether their lower bounds carry over into the straight-line setting. We show that every lower bound also holds in the straight-line setting. We argue our lower bounds on the *rectilinear crossing ratio* $\bar{\rho}_{\mathcal{F}}$ for several beyond-planar graph families \mathcal{F} . In the following, $\bar{\rho}_{\mathcal{F}}$ denotes the supremum of $\bar{\text{cr}}_{\mathcal{F}}(G)/\bar{\text{cr}}(G)$ over all graphs G that admit a straight-line drawing in \mathcal{F} . Additionally, $\bar{\text{cr}}_{\mathcal{F}}(G)$ denotes the minimum number of crossings over all straight-line drawings in \mathcal{F} of a graph G . In other words, we argue that the same lower bounds hold when we can only draw edges as straight-line segments.

¹Simple drawings are also known as *good drawings*.

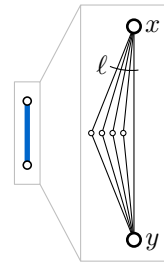
The straight-line setting imposes an additional constraint on the drawings. Therefore, all upper bounds directly apply in the straight-line setting. Lemma 1 proves that it is sufficient to draw each of our lower bound constructions with straight lines to establish the same bounds. We say that two drawings of a graph are *weakly isomorphic* if there is a one-to-one correspondence between their vertices and edges that is incidence-preserving and crossing-preserving.

Lemma 1. *Let G be a graph and D be a drawing of G with x crossings, showing that $\text{cr}(G) \leq x$. Moreover, let \mathcal{F} be a beyond-planar family of graphs such that $\text{cr}_{\mathcal{F}}(G) \geq y$. If there is a straight-line drawing of G weakly isomorphic to D and a straight-line drawing of G in \mathcal{F} , then $\overline{\text{cr}}(G) \leq x$ and $\overline{\text{cr}}_{\mathcal{F}}(G) \geq y$.*

Proof. Since there is a straight-line drawing of D with the same number of crossings it immediately follows that $\overline{\text{cr}}(G) \leq x$. Drawing graphs in a straight-line setting imposes an extra constraint on the drawing. Thus, the number of crossings cannot decrease. Since G has a straight-line drawing in \mathcal{F} it follows that $\overline{\text{cr}}_{\mathcal{F}}(G) \geq y$. \square

2.1 Compound edges

For two of their proofs Chimani *et al.* [15] use the notion of extended edges, which we refer to as ℓ -compound edges: a ℓ -compound edge (x, y) is a combination of the edge (x, y) and a set Π_{xy} of $\ell - 1$ paths of length two all connecting x and y (see figure). They show that if two ℓ -compound edges cross, then D contains at least ℓ crossings (Lemma 5 in [15], quoted below). Furthermore, they use the fact that a ℓ -compound edge is crossed to prove that a drawing contains at least ℓ crossings. We indicate ℓ -compound edges with thick blue lines.



Lemma 2 (Lemma 5 in [15]). *Let G be a graph containing two independent edges (u, v) and (w, z) . Suppose that u and v (w and z resp.) are connected by a set Π_{uv} (Π_{wz} resp.) of $\ell - 1$ paths of length two. Let Γ be a drawing of G . If (u, v) and (w, z) cross in Γ , then Γ contains at least ℓ crossings.*

Proof [15]. Suppose that (u, v) and (w, z) cross. If each of the $\ell - 1$ paths Π_{wz} crosses (u, v) , then the claim follows. Assume otherwise that at least one of these paths does not cross (u, v) . This path forms a 3-cycle with t with (w, z) ; the $\ell - 1$ paths of Π_{uv} all cross at least one edge of t , which proves the claim. \square

2.2 k -planar graphs

2.2.1 1-planar graphs

Chimani *et al.* showed a tight bound on the 1-planar crossing ratio $\rho_{1\text{-pl}} = n/2 - 1$ [15]. This means that this is the highest possible 1-planar crossing ratio.

Theorem 3 (Theorem 1 in [15]). *For every $\ell \geq 7$ there exists a 1-planar graph G_{ℓ} with $n = 11\ell + 2$ vertices such that $\text{cr}_{1\text{-pl}}(G_{\ell}) = n - 2$ and $\text{cr}(G_{\ell}) = 2$ which yields the largest possible 1-planar crossing ratio.*

They showed that G_{ℓ} admits a drawing with 2 crossings (see Figure 2.1a where the red subgraph can be drawn planarly), and only one 1-planar drawing, which admits 11ℓ crossings (see Figure 2.1b).

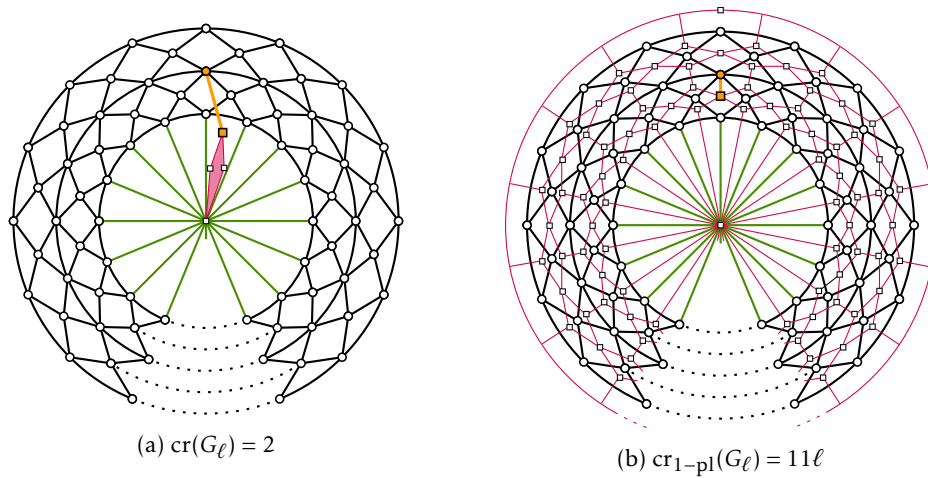


Figure 2.1: Drawings of the graph G_ℓ as reasoned in the proof of Theorem 3 (original figure in [15]).

Straight-line setting. The graph G_ℓ constructed by Chimani *et al.* [15] without the special edge is planar. Thus, by Fary's theorem it admits a plane straight-line drawing. Moreover, as in Figure 2.1a, G_ℓ can be drawn with straight-line edges and such that $\overline{cr}(G_\ell) = 2$. Thomassen showed that for every 1-planar drawing D there exists a straight-line drawing weakly isomorphic to D if and only if D does not contain a B - or W -configuration [32]. The 1-planar drawing in Figure 2.1b (which is redrawn from Chimani *et al.* [15]) contains neither a B - or a W -configuration. Hence, there is straight-line drawing weakly isomorphic to it. Thus, by Lemma 1, the statement of Theorem 3 also holds in the straight-line setting and $\overline{\rho}_{1\text{-pl}} = n/2 - 1$

Corollary 3.1. *For every $\ell \geq 7$ there exists a 1-planar graph G_ℓ with $n = 11\ell + 2$ vertices such that $\overline{cr}_{1\text{-pl}}(G_\ell) = n - 2$ and $\overline{cr}(G_\ell) = 2$ which yields the largest possible rectilinear 1-planar crossing ratio.*

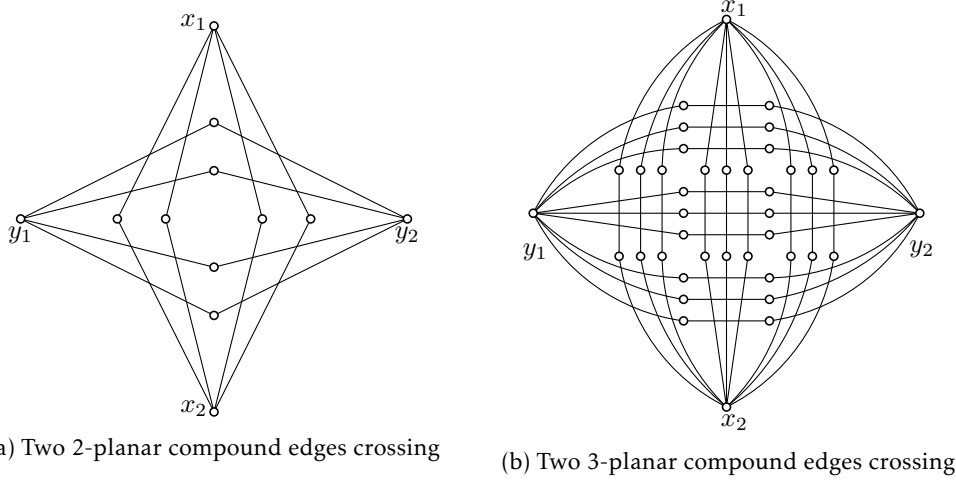
2.2.2 k -planar graphs

Lower bound. In a k -planar drawing no edge can be involved in more than k crossings. Chimani *et al.* [15] proved a tight bound on the crossing ratio for 1-planarity, $\rho_{1\text{-pl}} \geq n/2 - 1$. They also noted that the same arguments hold for k -planarity, if one allows parallel edges (multiple edges between two vertices). They achieve a $k(n - 2)/2$ lower bound on $\rho_{k\text{-pl}}$ for graphs with n vertices by replacing all edges except for one in their construction for 1-planarity by a bundle of k parallel edges. We show that parallel edges are not needed to extend their proof to k -planarity. However, the dependence on k in the lower bound that we achieve is worse than the lower bound show by Chimani *et al.* using parallel edges.

To extend their construction to k -planarity we introduce k -planar compound edges (see Figure 2.2), which exhibit essentially the same behavior as k parallel edges. One k -planar compound edge consists of k^2 parallel edges, each subdivided $k - 1$ times (so it consists of a total of k^3 edges). To prove the bounds for k -planarity, we replace each edge except for one in the 1-planar construction in [15] with a k -planar compound edge. In a k -planar setting, each k -planar compound edge can cross exactly one other k -planar compound edge (see Figure 2.2).

Theorem 4. *For every $\ell \geq 7$ there exists a k -planar graph G_ℓ^k with $n = 23\ell(k - 1)k^2 + 11\ell + 2$ vertices such that $\overline{cr}(G_\ell^k) \leq 2k^2$ and $\overline{cr}_{k\text{-pl}}(G_\ell^k) \geq k^4(11\ell)$, thus $\rho_{k\text{-pl}} \in \Omega(n/k)$.*

Proof. We construct the graph G_ℓ as described in Section 2 by Chimani *et al.* [15] (see Figure 2.1).


 Figure 2.2: Examples of k -planar compound edges.

The construction of G_ℓ consists of three parts: a rigid graph P (black edges, round vertices); its dual P^* (red edges, square vertices); a set of *binding edges* (green edges); and one *special edge* (orange edge). The graph G_ℓ has $11\ell + 2$ vertices and $23\ell + 1$ edges.

Subsequently, we replace each edge in G_ℓ except the special edge with a k -planar compound edge. This means we remove every edge and replace it by k^2 edges parallel edges, all split by $k - 1$ vertices, yielding G_ℓ^k . Thus, in total we have $n = 23\ell(k - 1)k^2 + 11\ell + 2$ vertices in G_ℓ^k . Two k -planar compound edges may cross while maintaining k -planarity (see Figure 2.2). G_ℓ^k admits a drawing with $2k^2$ crossings, when the graph is drawn as in Figure 2.1 a, thus $\text{cr}(G_\ell^k) \leq 2k^2$.

We show that $\text{cr}_{k\text{-pl}}(G_\ell^k) \geq k^4(11\ell)$. Suppose we have a k -planar, crossing minimal drawing Γ of G_ℓ^k . We now transform each k -planar compound edge in Γ such that each path of length k now becomes one edge. Thus, each k -planar compound edge becomes k^2 parallel edges, yielding a drawing Γ' of a multigraph G_{ℓ,k^2} . Since all edges remain in the same position and only subdivisions are removed, it follows that the number of crossings in Γ' is equal to the number of crossings in Γ . Additionally, it follows that G_{ℓ,k^2} is a modification of G_ℓ in which each edge except the special edge is replaced with k^2 parallel edges. This multigraph G_{ℓ,k^2} with $11\ell + 2$ vertices is actually the same graph used in Corollary 3 in [15] for their construction showing a tight bound on $\rho_{k^2\text{-pl}}$ when k^2 parallel edges are allowed. In their proof they show that $\text{cr}_{k^2\text{-pl}}(G_{\ell,k^2}) = k^4(11\ell)$.

In Γ each edge in each path of length k had at most k crossings. Thus, in Γ' each edge has at most k^2 crossings (the special edge has at most k crossings). Therefore, Γ' is k^2 -plane drawing of G_{ℓ,k^2} . Thus, there are at least $k^4(11\ell)$ crossing in Γ' and at least $k^4(11\ell)$ crossing in Γ . Since by definition Γ is a crossing minimal drawing of G_ℓ^k , we have shown that $\text{cr}_{k\text{-pl}}(G_\ell^k) \geq k^4(11\ell)$. Recall that $\text{cr}(G_\ell^k) \leq 2k^2$ and $n = 23\ell(k - 1)k^2 + 11\ell + 2$. It follows that $\rho_{k\text{-pl}} \in \Omega(n/k)$. \square

Upper bound. A k -planar graph with n vertices can have at most $3.81\sqrt{kn}$ edges [1]. Each edge can have at most k crossings. Thus, a crossing-minimal k -plane drawing cannot have more than $k \cdot 3.81\sqrt{kn}$ crossings. This yields $\rho_{k\text{-pl}} \in O(k\sqrt{kn})$.

Straight-line setting. We argue that the graph G_ℓ constructed by Chimani *et al.* [15] still admits the correct straight-line drawings when edges are replaced with k -planar compound edges.

The graph G_ℓ without the special edge is planar. Thus, by Fary's theorem it admits a plane straight-line drawing. Moreover, as in Figure 2.1a, G_ℓ can be drawn with straight-line edges and

such that $\overline{\text{cr}}(G_\ell) = 2$. Since our k -planar compound edges can be drawn with straight-line edges and arbitrarily close to the original edges they replace, we have that G_ℓ^k (a modification of the graph G_ℓ in which every edge except for the special edge was replaced by a k -planar compound edge) admits a straight-line drawing with $2k^2$ crossings. Thus $\overline{\text{cr}}(G_\ell^k) \leq 2k^2$.

Thomassen showed that for every 1-planar drawing D there exists a straight-line drawing weakly isomorphic to D if and only if D does not contain a B - or W -configuration [32]. The 1-planar drawing in Figure 2.1b (which is redrawn from Chimani *et al.* [15]) contains neither a B - or a W -configuration. Hence there is straight-line drawing weakly isomorphic to it. Since again k -planar compound edges can be drawn with straight-line edges and arbitrarily close to the original edges they replace, there is a straight-line drawing of G_ℓ^k that is also k -plane. Thus, by Lemma 1, the statement of Theorem 4 also holds in the straight-line setting.

Corollary 4.1. *For every $\ell \geq 7$ there exists a k -planar graph G_ℓ^k with $n = 23\ell(k-1)k^2 + 11\ell + 2$ vertices such that $\overline{\text{cr}}_{k\text{-pl}}(G_\ell^k) = 11\ell k^4$ and $\overline{\text{cr}}(G_\ell^k) \leq 2k^2$. Thus, $\overline{\rho}_{k\text{-pl}} \in \Omega(n/k)$.*

2.3 k -quasi-planar graphs

Lower bound. Chimani *et al.* showed a linear lower bound on the quasi-planar crossing ratio ρ_{quasi} [15]. They asked whether there exist quasi-planar graphs whose crossing ratio is $\Omega(n^2)$. This question remains open.

Theorem 5 (Theorem 4 in [15]). *For every $\ell > 2$, there exists a quasi-planar graph G_ℓ with $n = 12\ell - 5$ vertices such that $\text{cr}_{\text{quasi}}(G_\ell) \geq \ell$ and $\text{cr}(G_\ell) \leq 3$, thus $\rho_{\text{quasi}} \in \Omega(n)$.*

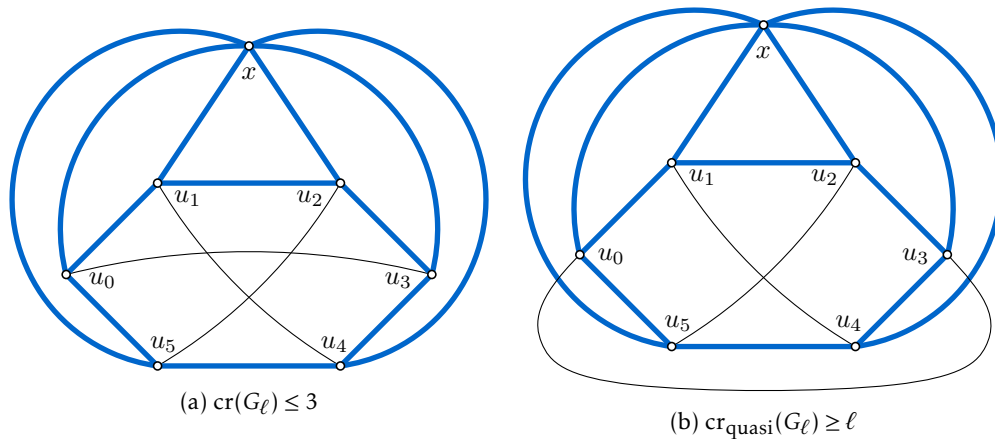


Figure 2.3: Drawings of graph G_ℓ to illustrate the proof provided by Chimani *et al.* [15] for Theorem 5. Thick blue edges are ℓ -compound edges (original figure in [15]).

They argued that their proof could be extended to k -quasi-planar graphs by using exactly the same construction in which the cycle C has length $2k$.

Corollary 5.1 (Corollary 6 in [15]). *For every $\ell \geq 2$ and $k \geq 3$, there exists a k -quasi-planar graph $G_{\ell,k}^{\ell,k}$ with $n = 2k(\ell+1)$ vertices such that $\text{cr}_{\text{quasi}}(G_{\ell,k}^{\ell,k}) \geq \ell$ and $\text{cr}(G_{\ell,k}^{\ell,k}) \leq k(k-1)/2$, thus $\rho_{k\text{-quasi}} \in \Omega(n/k^3)$.*

Upper bound. The maximum number of edges in quasi-planar graphs with n vertices is $6.5n - 20$ [3]. Thus, as stated in [15], $\rho_{\text{quasi}} \in O(n^2)$.

The maximum number of edges in k -quasi-planar graphs with n vertices is $c_k n \log n$, where c_k depends only on k [31]. Thus, as stated in [15], $\rho_{k\text{-quasi}} \leq f(k) \cdot n^2 \log^2 n$.

Straight-line setting. Chimani *et al.* [15] showed that $\rho_{\text{quasi}} \in \Omega(n)$. Figure 2.4 is a straight-line version of Figure 2 in [15]. Thus, by Lemma 1, the statement of Theorem 4 in [15] also holds in the straight-line setting.

Corollary 5.2. *For every $\ell \geq 2$ there exists a quasi-planar graph G_ℓ with $n = 12\ell - 5$ vertices such that $\overline{\text{cr}}_{\text{quasi}}(G_\ell) \geq \ell$ and $\overline{\text{cr}}(G_\ell) \leq 3$, thus $\rho_{\text{quasi}} \in \Omega(n)$.*

Chimani *et al.* [15] showed that $\rho_{k\text{-quasi}} \in \Omega(n/k^3)$. For a different k we can easily expand the drawings by adding more vertices between u_2 and u_3 and between u_4 and u_5 and rearranging the edges such that they all mutually cross.

Corollary 5.3. *For every $\ell \geq 2$ there exists a k -quasi-planar graph G_ℓ with $n = 12\ell - 5$ vertices such that $\overline{\text{cr}}_{k\text{-quasi}}(G_\ell) \geq \ell$ and $\overline{\text{cr}}(G_\ell) \leq 3$, thus $\rho_{k\text{-quasi}} \in \Omega(n/k^3)$.*

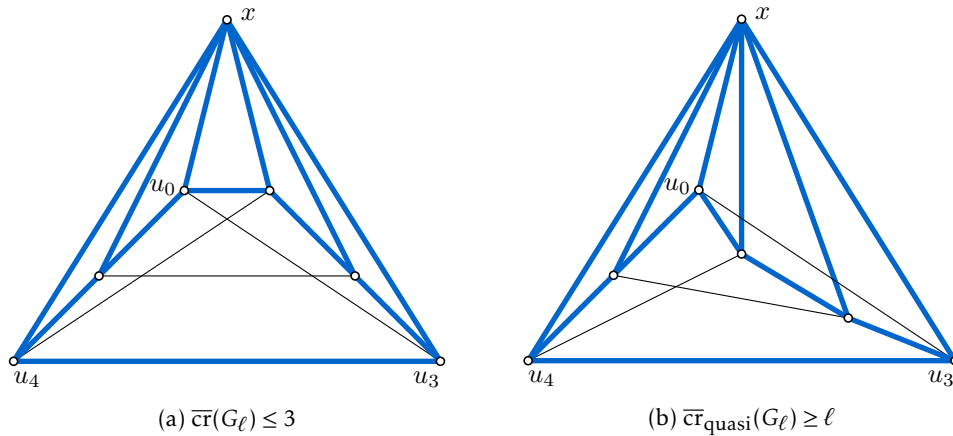


Figure 2.4: Straight-line drawings of graph G_ℓ that are weakly isomorphic to the drawings in Figure 2.3 to support the reasoning of Corollary 5.2. Thick blue edges are ℓ -compound edges.

2.4 Fan-planar graphs

Lower bound. Chimani *et al.* [15] gave a linear lower bound on the supremum ρ_{fan} of the ratio between the fan-planar and the unrestricted crossing number. They asked whether there exist fan-planar graphs whose crossing ratio is $\Omega(n^2)$. This question remains open.

Theorem 6 (Theorem 7 in [15]). *For every $\ell > 2$, there exists a fan-planar graph G_ℓ with $n = 9\ell + 1$ vertices such that $\text{cr}_{\text{fan}}(G_\ell) = \ell$ and $\text{cr}(G_\ell) = 3$, thus $\rho_{\text{fan}} \in \Omega(n)$.*

Upper bound. (For completeness we repeat the upper bound argument in Theorem 6.) A fan-planar graph with n vertices can have at most $5n - 10$ edges [19]. Thus, a crossing-minimal fan-planar drawing cannot have more than $(5n - 10)^2$ crossings. Thus, as stated in [15], $\rho_{\text{fan}} \in O(n^2)$.

Straight-line setting. Chimani *et al.* [15] showed that $\rho_{\text{fan}} \in \Omega(n)$. Figure 2.6 is a straight-line version of Figure 3 in [15]. Thus, by Lemma 1, the statement of Theorem 7 in [15] also holds in the straight-line setting.

Corollary 6.1. *For every $\ell > 2$, there exists a fan-planar graph G_ℓ with $n = 9\ell + 1$ vertices such that $\overline{\text{cr}}_{\text{fan}}(G_\ell) = \ell$ and $\overline{\text{cr}}(G_\ell) = 3$, thus $\overline{\rho}_{\text{fan}} \in \Omega(n)$.*

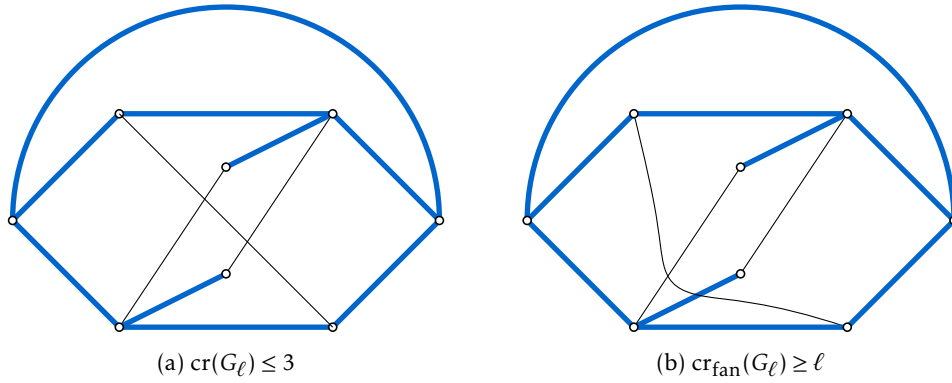


Figure 2.5: Drawings of graph G_ℓ to illustrate the proof provided by Chimani *et al.* [15] for Theorem 6. Thick blue edges are ℓ -compound edges (original figure in [15]).

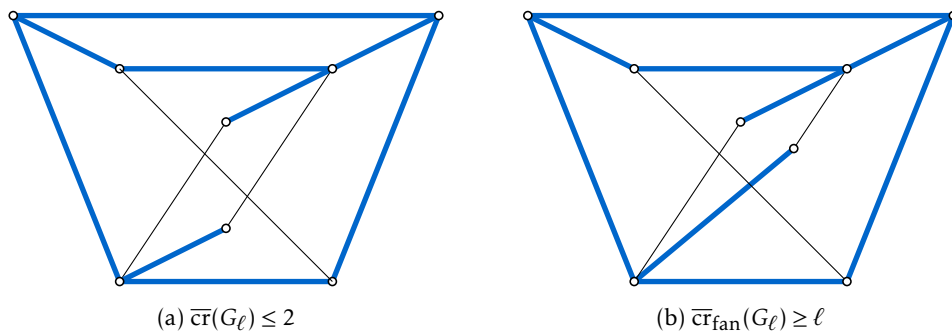


Figure 2.6: Straight-line drawings of graph G_ℓ that are weakly isomorphic to the drawings in Figure 2.5 to support the reasoning of Corollary 6.1. Thick blue edges are ℓ -compound edges.

Chapter 3

Additional beyond-planar graphs

Chimani *et al.* conjectured that their results could be extended to further families of beyond-planar graphs. In this section we extend their results, using similar proof structures, to four more classes of beyond-planar graphs: k, l -grid-free graphs, planarly connected graphs, k -gap-planar graphs, and k -fan-crossing-free graphs. For all four of the classes we achieve lower bounds on the crossing ratio which are linear in the number of vertices n . For k -gap-planar and k -fan-crossing-free crossing ratios, we first provide a lower bound for the base cases (1-gap-planar and 2-fan-crossing-free crossing ratios) and then show how to extend the proof to the general class. For the 1-gap-planar crossing ratio this results in a tight bound. For k, l -grid-free crossing ratio we provide a general proof, and additionally a figure of a small example ($k = 2, l = 3$).

3.1 k, l -grid free graphs

Lower bound. In a (k, l) -grid-free drawing it is forbidden to have a set of k edges all crossing each edge in a set of l edges. We show that $\rho_{k, l\text{-grid-free}} \in \Omega(n)$ by creating a graph where a (k, l) -grid occurs when we draw all regular edges (black) in the inner face of a cycle of ℓ -compound edges (see Figure 3.1). Thus, a regular edge must be drawn (at least partially) in the outer face and has to cross at least one ℓ -compound edge (blue), which results in at least ℓ crossings.

Theorem 7. *For every $\ell \geq 2, k > 1, l > 1$ there exists a (k, l) -grid free graph G_ℓ with $n = 2(k + l)\ell + 2(k + l) - 19$ vertices such that $\text{cr}_{k, l\text{-grid-free}}(G_\ell) \geq \ell$ and $\text{cr}(G_\ell) \leq k \cdot l$.*

Thus, $\rho_{k, l\text{-grid-free}} \in \Omega\left(\frac{n}{kl(k+l)}\right)$.

Proof. Let graph G_ℓ be constructed as follows: We start with a cycle C of length $2k + 2l$, $C = \langle u_1, \dots, u_{2k+2l} \rangle$ and a vertex x connected to each vertex of C , yielding G' . Make each edge in G' a ℓ -compound edge by adding $\ell - 1$ disjoint paths of length two between its endpoints. Finally we add the regular edges $(u_i, u_{2k+l-i+1})$ for $i = 1, \dots, k$ and we add the regular edges $(u_{k+i}, u_{2k+2l-i+1})$ for $i = 1, \dots, l$ (see Figure 3.1 for an example with $k = 2$ and $l = 3$). In general, the resulting graph G_ℓ has $n = 2(k + l)\ell + 2(k + l) - 19$ vertices.

If no regular edge crosses G' , they must all be drawn within the unique face of size $2k + 2l$ in G' . Then, the drawing would not be (k, l) -grid free, as there would be a group of k edges that all split the face such that for each edge in a group of l edges, its endpoints lie on a different plane. Summarily, there would be a group of k edges such that each edge crosses all edges of a group of l edges. The drawing then has $k \cdot l$ crossings but is not (k, l) -grid free.

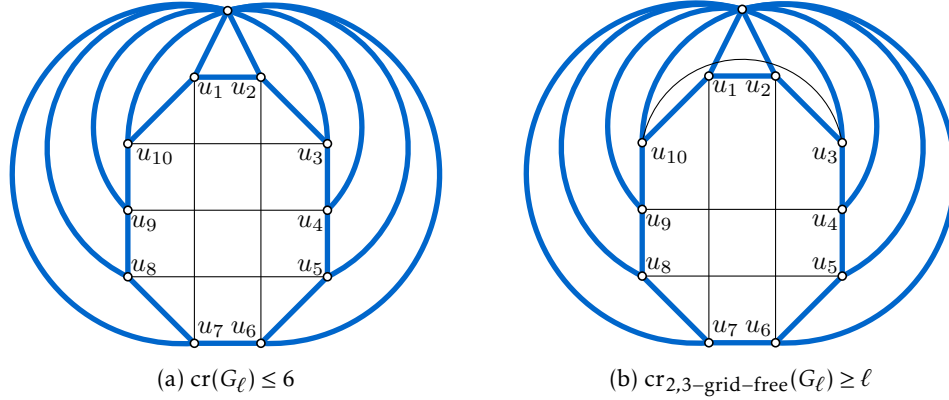


Figure 3.1: Drawings of graph G_ℓ to illustrate the proof for Theorem 7. Thick blue edges are ℓ -compound edges.

Thus, in order to make a (k, l) -grid-free drawing, at least one regular edge $r = (u_i, u_j)$ must cross and edge (a, b) of G' . Consider the closed (possible self-intersecting) curve \mathcal{L} composed of r plus the subpath of C connecting u_i to u_j and containing none of the vertices a and b . This curve partitions the the plane into two or more regions, and a and b lie in different regions. Thus (a, b) and the $\ell - 1$ paths connecting a and b cross \mathcal{L} , yielding ℓ crossings in Γ , as desired. \square

Upper bound. The maximum number of edges in (k, l) -grid-free graphs with n vertices is $c_{k,l}n$, where $c_{k,l}$ depends only on k and l [22]. Thus, $\rho_{k,l\text{-grid-free}} \leq g(k, l) \cdot n^2$.

Straight-line setting. Figure 3.2 is a straight-line version of Figure 3.1. Thus, by Lemma 1, the statement of Theorem 7 also holds in the straight-line setting for $k = 2, l = 3$. For a different k we can easily expand the drawings by adding more vertices between u_1 and u_2 and between u_6 and u_7 . For a different l we can do the same between u_3 and u_4 and between u_9 and u_{10} .

Corollary 7.1. For every $\ell \geq 2, k > 1, l > 1$ there exists a k, l -grid free graph G_ℓ with $n = 2(k+l)\ell - 19 + 2(k+l)$ vertices such that $\overline{\text{cr}}_{k,l\text{-grid-free}}(G_\ell) \geq \ell$ and $\overline{\text{cr}}(G_\ell) \leq k \cdot l$, thus $\rho_{k,l\text{-grid-free}} \in \Omega\left(\frac{n}{kl(k+l)}\right)$.

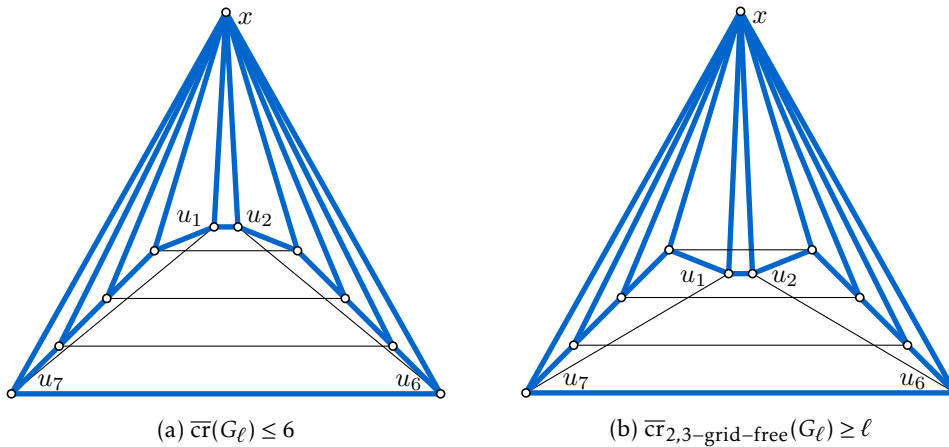


Figure 3.2: Straight-line drawing of graph G_ℓ that are weakly isomorphic to the drawings in Figure 3.1 to support the reasoning of Corollary 7.1. Thick blue edges are ℓ -compound edges.

3.2 Planarly-connected graphs

Lower bound. In a planarly connected drawing it is forbidden for two edges to be crossing when their endpoints are not connected by a crossing-free edge (i.e. an edge that is not crossed). We show that $\rho_{\text{pl-con}} \in \Omega(n^2)$ by creating a graph G_ℓ in which a crossing occurs between two regular edges (black) whose endpoints are not connected by a crossing-free edge when we draw all the regular edges the inner face of a cycle of ℓ -compound edges (see Figure 3.3). Thus, at least one regular edge is drawn in the outer face and has to cross at least one ℓ -compound edge (blue), which results in at least ℓ crossings.

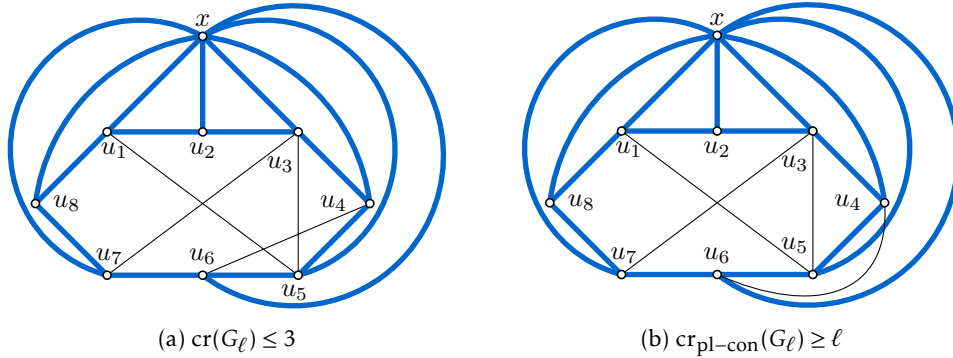


Figure 3.3: Drawings of graph G_ℓ to illustrate the proof for Theorem 8. Thick blue edges are ℓ -compound edges.

Theorem 8. For every $\ell \geq 2$ there exists a planarly connected graph G_ℓ with $n = 16\ell - 7$ vertices such that $\text{cr}_{\text{pl-con}}(G_\ell) \geq \ell$ and $\text{cr}(G_\ell) \leq 3$. Thus, $\rho_{\text{pl-con}} \in \Omega(n)$.

Proof. Let graph G_ℓ be constructed as follows (see Figure 3.3). Start with an 8-cycle $C = \langle u_1, u_2, \dots, u_8 \rangle$, and a vertex x connected to each vertex of C , yielding G' . Make each edge in G' a ℓ -compound edge by adding $\ell - 1$ disjoint paths of length two between its endpoints. Finally add regular edges $(u_1, u_5), (u_3, u_7), (u_3, u_5)$ and (u_4, u_6) .

The resulting graph G_ℓ has $n = 16(\ell - 1) + 9 = 16\ell - 7$ vertices and admits a drawing with 3 crossings, so $\text{cr}(G_\ell) \leq 3$ (see Figure 3.3a). As we can see in Figure 3.3b, G_ℓ admits a planarly connected drawing with $\ell + 1$ crossings. We prove that $\text{cr}_{\text{pl-con}}(G_\ell) \geq \ell$. If two ℓ -compound edges of G' cross each other, then the claim follows from Lemma 5 in [15].

If no regular edge crosses G' , they must all be drawn within the unique face of size 8 in G' . Then, (u_1, u_5) would cross (u_3, u_7) and (u_4, u_6) would cross (u_3, u_5) . This would contradict planar connectivity, as there would be no crossing-free edge that connects endpoints of (u_1, u_5) and (u_3, u_7) .

Thus, in order to make a planarly connected drawing, at least one regular edge $r = (u_i, u_j)$ must cross an edge (a, b) of G' . Consider the closed (possibly self-intersecting) curve \mathcal{L} composed of r plus the subpath of C connecting u_i to u_j and containing none of the vertices a and b . This curve partitions the plane into two or more regions, and a and b lie in different regions. Thus, (a, b) and the $\ell - 1$ paths connecting a and b cross \mathcal{L} , yielding ℓ crossings in Γ , as claimed. \square

Upper bound. (For completeness we repeat the upper bound argument in Theorem 8.) A planarly connected graph with n vertices can have at most cn edges, where c is an absolute constant [2]. Thus, a crossing-minimal planarly connected drawing can have at most $c^2 n^2$ crossings. This yields $\rho_{\text{pl-con}} \in O(n^2)$.

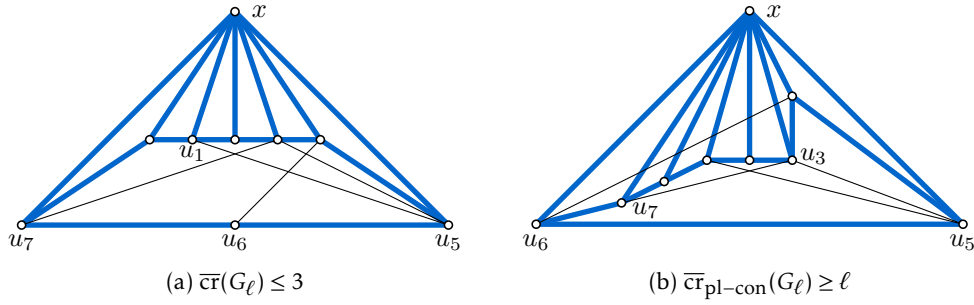


Figure 3.4: Straight-line drawings of graph G_ℓ that are weakly isomorphic to the drawings in Figure 3.3 to support the reasoning of Corollary 8.1. Thick blue edges are ℓ -compound edges.

Straight-line setting. Figure 3.4 is a straight-line version of Figure 3.3. Thus, by Lemma 1, the statement of Theorem 8 also holds in the straight-line setting.

Corollary 8.1. *For every $\ell \geq 2$ there exists a planarly connected graph G_ℓ with $n = 16\ell - 7$ vertices such that $\overline{cr}_{\text{pl-con}}(G_\ell) \geq \ell$ and $\overline{cr}(G_\ell) \leq 3$. Thus, $\overline{\rho}_{\text{pl-con}} \in \Omega(n)$.*

3.3 k -gap-planar graphs

Lower bound. In a k -gap-planar drawing it is forbidden to have more than k mappings to an edge when optimally mapping each crossing to a corresponding edge. We show that $\rho_{k\text{-gap}} \in \Omega(n/k^3)$ by creating a graph where there are too many crossings to map to the corresponding edges when we draw all regular edges (black) in the inner face (see Figure 3.5). Thus, at least one regular edge is drawn in the outer face and has to cross a ℓ -compound edge (blue), which results in at least ℓ crossings. We first show that the bound holds for 1-gap-planarity and then how the construction can be extended to k -gap-planarity.

Theorem 9. *For every $\ell \geq 2$ there exists a 1-gap-planar graph G_ℓ with $n = 16\ell - 7$ vertices such that $cr_{1\text{-gap}}(G_\ell) \geq \ell$ and $cr(G_\ell) \leq 6$. Thus, $\rho_{1\text{-gap}} \in \Theta(n)$.*

Proof. The upper bound $\rho_{1\text{-gap}} \in O(n)$ follows directly from the fact that a 1-gap-planar graph with n vertices can have at most $5n - 10$ edges [6]. Since each edge can have at most 1 crossings assigned to it, the number of crossings cannot exceed the number of edges. Thus, a 1-gap-planar graph can have no more than $5n - 10$ crossings. Clearly, if $cr_{1\text{-gap}}(G) > cr(G)$ it must hold that $cr(G) \geq 2$. This yields $\rho_{1\text{-gap}} \in O(n)$.

For the lower bound $\rho_{1\text{-gap}} \in \Omega(n)$ we first construct the graph G_ℓ , drawn in Figure 3.5. Start with an 8-cycle $C = \langle u_1, u_2, \dots, u_8 \rangle$, and a vertex x connected to each vertex of C , yielding G' . Make each edge in G' a ℓ -compound edge by adding $\ell - 1$ disjoint paths of length two between its endpoints. Finally add regular edges $(u_i, u_{i+4}), i = 1, 2, 3, 4$.

The resulting graph G_ℓ has $n = 16(\ell - 1) + 9 = 16\ell - 7$ vertices and admits a drawing with six crossings, so $cr(G_\ell) \leq 6$ (see Figure 3.5a). As we can see in Figure 3.5b, G_ℓ admits a 1-gap-planar drawing with $3\ell + 3$ crossings. We prove that $cr_{1\text{-gap}}(G_\ell) \geq \ell$. If two edges of G' cross each other, then the claim follows from Lemma 5 in [15].

If no regular edge crosses G' , they must all be drawn within the unique face of size eight in G' . Then, they would mutually cross, because every regular edge exactly splits the face such that for every other regular edge its endpoints are on different planes. This would contradict 1-gap-planarity, as there would be 6 crossings having to be mapped to 4 edges. Thus, in order to make a 1-gap-planar drawing, at least one regular edge $r = (u_i, u_j)$ must cross and edge (a, b) of G' .

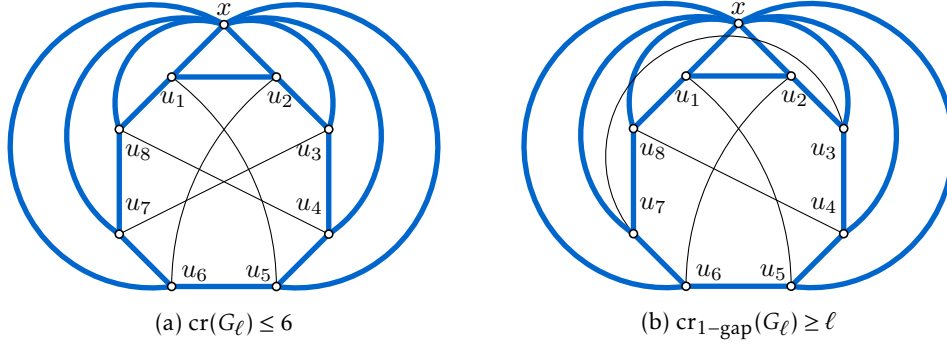


Figure 3.5: Drawings of graph G_ℓ to illustrate the proof for Theorem 9. Thick blue edges are ℓ -compound edges.

Consider the closed (possible self-intersecting) curve \mathcal{L} composed of r plus the subpath of C connecting u_i to u_j and containing none of the vertices a and b . This curve partitions the the plane into two or more regions, and a and b lie in different regions. Thus, (a, b) and the $\ell - 1$ paths connecting a and b cross \mathcal{L} , yielding ℓ crossings in Γ , as desired. \square

The proof on the lower bound can be extended to k -gap-planarity by making the cycle of length $8k$, but in this case the dependency on k of the lower bound that we obtain does not match the upper bound.

Theorem 10. *For every $\ell \geq 2$ there exists a k -gap-planar graph G_ℓ^k with $n = 16k\ell - 8k + 1$ vertices such that $\text{cr}_{k\text{-gap}}(G_\ell^k) \geq \ell$ and $\text{cr}(G_\ell^k) \leq 8k^2 - 2k$. Thus, $\rho_{k\text{-gap}} \in \Omega(n/k^3)$.*

Proof. Let graph G_ℓ^k be constructed as follows: We start with a cycle $C = \langle u_1, u_2, \dots, u_{8k} \rangle$ of length $8k$. We connect each vertex of C to a new vertex x . Let this graph be G^k . Next, we replace all edges with ℓ -compound edges. Finally, we add the *regular* edges $(u_i, u_{i+4k}), 1 \leq i \leq 4k$. If no regular edge crosses G^k , they must all be drawn within the unique face of size $8k$ in G^k . Then, they would mutually cross, because every regular edge exactly splits the face such that for every other regular edge its endpoints are on different planes.

A cycle of length $8k$ has $4k$ edges mutually crossing, which results in $\binom{4k}{2} = \frac{1}{2}(4k)(4k-1) = 8k^2 - 2k$ crossings to map onto $4k$ edges. For to k -gap-planarity to hold, their may not be more than k crossings mapped to any edge. Thus, if the number of crossings exceeds the number of edges times k , k -gap-planarity does not hold. Clearly, $8k^2 - 2k > 4k^2$ for $k > \frac{1}{2}$. Thus, for every $k \geq 1$ a cycle of $8k$ will, in the inner face, have more crossings than edges to map them to.

Thus, in order to make a k -gap-planar drawing, at least one regular edge $r = (u_i, u_j)$ must cross and edge (a, b) of G' . Consider the closed (possible self-intersecting) curve \mathcal{L} composed of r plus the subpath of C connecting u_i to u_j and containing none of the vertices a and b . This curve partitions the the plane into two or more regions, and a and b lie in different regions. Thus (a, b) and the $\ell - 1$ paths connecting a and b cross \mathcal{L} , yielding ℓ crossings in Γ , as desired. \square

Upper bound. (For completeness we repeat the upper bound argument in Theorem 9.) A 1-gap-planar graph with n vertices can have at most $5n - 10$ edges [6]. Since each edge can have at most 1 crossing assigned to it, the number of crossings cannot exceed the number of edges. Thus, a 1-gap-planar graph can have no more than $5n - 10$ crossings. This yields $\rho_{1\text{-gap}} \in O(n)$.

A k -gap-planar graph with n vertices can have at most $O(\sqrt{kn})$ edges [6]. Since each edge can have at most k crossings assigned to it, the number of crossings cannot exceed the number of

edges times k . Thus, a k -gap-planar graph can have no more than $O(k\sqrt{kn})$ crossings. This yields $\rho_{k\text{-gap}} \in O(k\sqrt{kn})$.

Straight-line setting. Figure 3.6 is a straight-line version of Figure 3.5. Thus, by Lemma 1, the statement of Theorem 9 also holds in the straight-line setting.

Corollary 10.1. *For every $\ell \geq 2$ there exists a 1-gap-planar graph G_ℓ with $n = 16\ell - 7$ vertices such that $\overline{\text{cr}}_{1\text{-gap}}(\ell) \geq \ell$ and $\overline{\text{cr}}(G_\ell) \leq 6$. Thus, $\overline{\rho}_{1\text{-gap}} \in \Omega(n)$.*

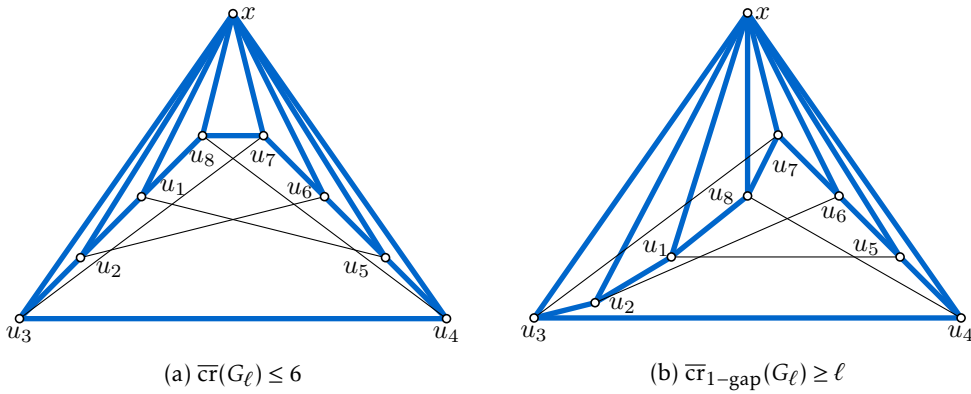


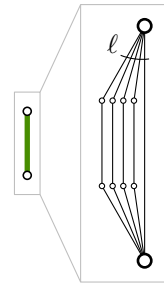
Figure 3.6: Straight-line drawings of graph G_ℓ that are weakly isomorphic to the drawings in Figure 3.5 to support the reasoning of Corollary 10.1. Thick blue edges are ℓ -compound edges.

For a different k we can easily expand the drawings by adding more vertices between u_2 and u_3 and between u_4 and u_5 and rearranging the edges such that they all mutually cross. Thus, by Lemma 1, we argue that the statement of Theorem 10 also holds in the straight-line setting.

Corollary 10.2. *For every $\ell \geq 2$ there exists a k -gap-planar graph G_ℓ^k with $n = 16k\ell - 8k + 1$ vertices such that $\overline{\text{cr}}_{k\text{-gap}}(G_\ell) \geq \ell$ and $\overline{\text{cr}}(G_\ell) \leq 8k^2 - 2k$. Thus, $\overline{\rho}_{k\text{-gap}} \in \Omega(n/k^3)$.*

3.4 k -fan-crossing-free graphs

Lower bound. In a k -fan-crossing-free drawing it is forbidden for an edge to cross k adjacent edges. Clearly, in a k -fan-crossing-free drawing it is forbidden for an edge to cross an ℓ -compound edge with $\ell \geq k$. Thus, we introduce the notion k -fan-crossing-free ℓ -compound edges. A k -fan-crossing-free ℓ -compound edge (x, y) is the combination of the edge (x, y) and $\ell - 1$ disjoint paths of length 3 connecting x and y (see figure). In a k -fan-crossing-free drawing Γ a k -fan-crossing-free ℓ -compound edge may be crossed. By the same proof that was provided for Lemma 5 in [15], it follows that if two k -fan-crossing-free ℓ -compound edges cross in Γ , then Γ contains at least ℓ crossings.



We show that $\rho_{k\text{-fan-free}} \in \Omega(n/k^2)$ by creating a graph where a regular edge (black) crosses k adjacent regular edges when we draw them in the inner face (see Figure 3.7). Thus, at least one regular edge is drawn in the outer face and has to cross at least one k -fan-crossing-free ℓ -compound edge (green), which results in at least ℓ crossings. We first show that this bound holds for 2-fan-crossing-freedom and then how the construction can be extended to k -fan-crossing-freedom.

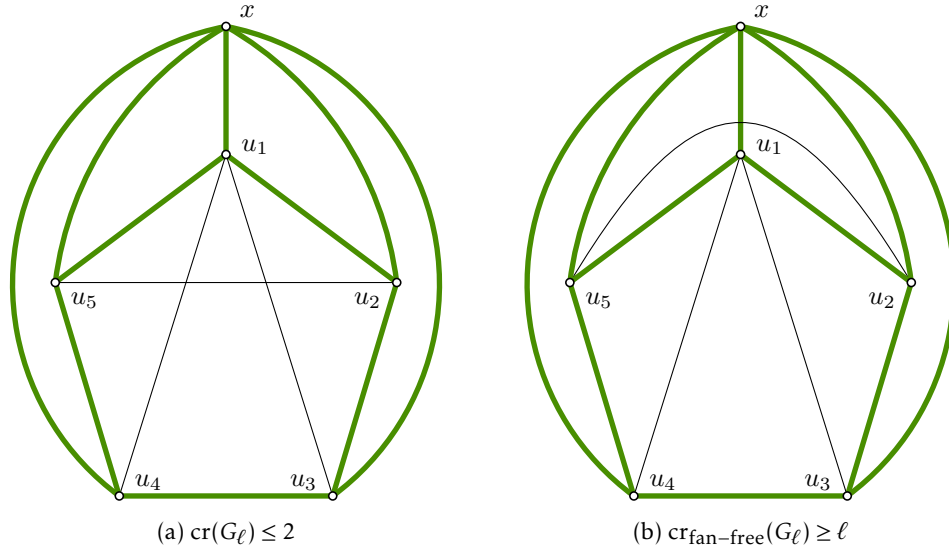


Figure 3.7: Drawings of graph G_ℓ to illustrate the proof for Theorem 11. Thick green edges are k -fan-crossing-free ℓ -compound edges.

Theorem 11. For every $\ell \geq 2$ there exists a fan-crossing-free graph G_ℓ with $n = 20\ell - 14$ vertices such that $\text{cr}_{\text{fan-free}}(G_\ell) \geq \ell$ and $\text{cr}(G_\ell) \leq 2$. Thus, $\rho_{\text{fan-free}} \in \Omega(n)$

Proof. Let graph G_ℓ be constructed as follows; see Figure 3.7. Start with an 5-cycle $C = \langle u_1, u_2, \dots, u_5 \rangle$, and a vertex x connected to each vertex of C , yielding G' . Make each edge in G' a fan-crossing-free compound edge by adding $\ell - 1$ disjoint paths of length three between its endpoints. This makes them into fan-crossing-free compound edges. Finally add regular edges $(u_1, u_3), (u_1, u_4)$ and (u_2, u_5) .

The resulting graph G_ℓ has $n = 10 \cdot (2\ell - 2) + 6 = 20\ell - 14$ vertices and admits a drawing with 2 crossings, so $\text{cr}(G_\ell) \leq 2$; see Figure 3.7a. As we can see in Figure 3.7b, G_ℓ admits a fan-crossing-free drawing with ℓ crossings. We prove that $\text{cr}_{\text{fan-free}}(G_\ell) \geq \ell$. If two edges of G' cross each other, then the claim follows from Lemma 2.

If no regular edge crosses G' , they must all be drawn within the unique face of size 5 in G' . Then, (u_2, u_5) would cross (u_1, u_3) and (u_1, u_4) , since (u_2, u_5) splits the face such that v_1 is on a different plane than v_3 and v_4 . This would contradict fan-crossing-freedom, as (u_1, u_3) and (u_1, u_4) are adjacent edges. Thus, in order to make a fan-crossing-free drawing, at least one regular edge $r = (u_i, u_j)$ must cross and edge (a, b) of G' . Consider the closed (possible self-intersecting) curve \mathcal{L} composed of r plus the subpath of C connecting u_i to u_j and containing none of the vertices a and b . This curve partitions the the plane into two or more regions, and a and b lie in different regions. Thus (a, b) and the $\ell - 1$ paths connecting a and b cross \mathcal{L} , yielding ℓ crossings in Γ , as desired. \square

The proof can be extended to k -fan-crossing-freedom by making the cycle of length $3 + k$, adding the extra vertices between u_3 and u_4 , and adding more edges like (u_1, u_3) and (u_1, u_4) .

Corollary 11.1. For every $\ell \geq 2$ there exists a fan-crossing-free graph G_ℓ with $n = (6 + 2k)\ell - k - 2$ vertices such that $\text{cr}_{k\text{-fan-free}}(G_\ell) \geq \ell$ and $\text{cr}(G_\ell) \leq k$. Thus, $\rho_{k\text{-fan-free}} \in \Omega(n/k^2)$.

Upper bound. A fan-crossing-free graph with n vertices can have at most $4n - 8$ edges [13]. Thus, a crossing-minimal fan-crossing-free drawing cannot have more than $(4n - 8)^2$ crossings. This yields $\rho_{\text{fan-free}} \in O(n^2)$.

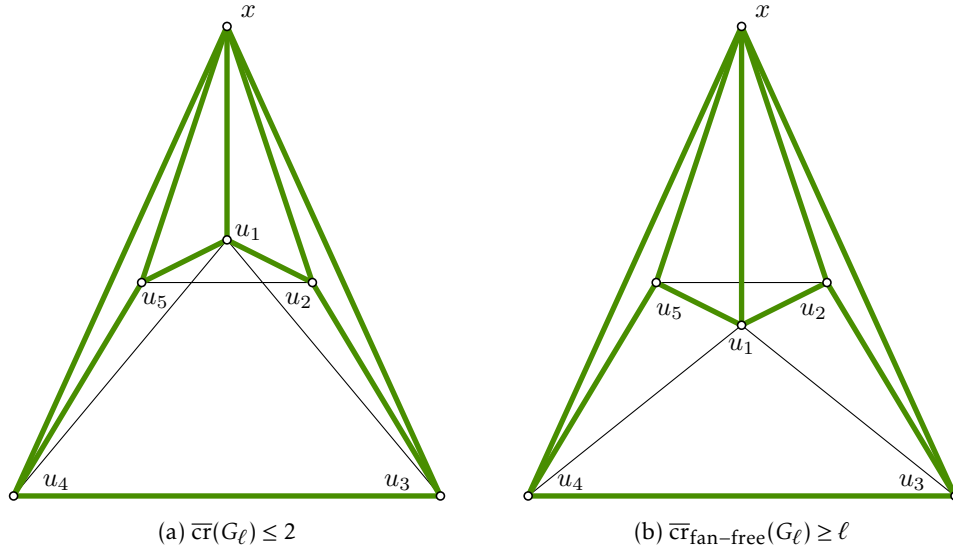


Figure 3.8: Straight-line drawing of graph G_ℓ that are weakly isomorphic to the drawings in Figure 3.7 to support the reasoning of Corollary 11.2. Thick blue edges are ℓ -compound edges. Thick green edges are fan-crossing-free compound edges.

A k -fan-crossing-free graph with n vertices can have at most $3(k-1)(n-2)$ edges [13]. Thus, a crossing-minimal k -fan-crossing-free drawing cannot have more than $(3(k-1)(n-2))^2$ crossings. This yields $\rho_{k\text{-fan-free}} \in O(k^2 n^2)$.

Straight-line setting. Figure 3.8 is a straight-line version of Figure 3.7. Thus, by Lemma 1, the statement of Theorem 11 also holds in the straight-line setting.

Corollary 11.2. *For every $\ell \geq 2$ there exists a fan-crossing-free graph G_ℓ with $n = 10 \cdot (2\ell - 2) + 6 = 20\ell - 14$ vertices such that $\overline{cr}_{\text{fan-free}}(G_\ell) \geq \ell$ and $\overline{cr}(G_\ell) \leq 2$. Thus, $\overline{\rho}_{\text{fan-free}} \in \Omega(n)$.*

For a different k we can easily expand the drawings in Figure 3.8 by making the cycle of length $3+k$, adding the extra vertices between u_3 and u_4 , and adding more edges like (u_1, u_3) and (u_1, u_4) .

Corollary 11.3. *For every $\ell \geq 2$ there exists a fan-crossing-free graph G_ℓ with $n = (6 + 2k)\ell - k - 2$ vertices such that $\overline{cr}_{k\text{-fan-free}}(G_\ell) \geq \ell$ and $\overline{cr}(G_\ell) \leq k$. Thus, $\overline{\rho}_{k\text{-fan-free}} \in \Omega(n/k^2)$.*

Chapter 4

Crossing ratios of skewness- k and k -apex graphs

We prove bounds on the crossing ratio of two more families of beyond-planar graphs: skewness- k graphs and k -apex graphs. Due to the nature of these families we present graph constructions which are different from before. We create a special graph and prove a linear lower bound on the skewness-1 crossing ratio using a case distinction of possible drawings of the graph. We then extend this proof to the 1-apex crossing ratio.

4.1 Skewness- k graphs

In a skewness- k drawing all crossings must be covered by at most k edges. A crossing is covered by an edge when the edge is part of the crossing. We show that $\rho_{\text{skew-1}} \in \Omega(n)$. We do this by creating a graph G_ℓ and showing by a case distinction that in a skewness-1 drawings of G_ℓ there are at least ℓ crossings. We then show how this construction can be extended to skewness- k by combining multiple copies of G_ℓ .

Theorem 12. *For every $\ell \geq 3$ there exists a skewness-1 graph G_ℓ with $n = 11\ell - 4$ vertices such that $\text{cr}_{\text{skew-1}}(G_\ell) \geq \ell$ and $\text{cr}(G_\ell) \leq 3$. Thus, $\rho_{\text{skew-1}} \in \Theta(n)$.*

Proof. The upper bound $\rho_{\text{skew-1}} \in O(n)$ directly follows from the fact that a skewness-1 graph with n vertices can have at most $3n - 5$ edges [16]. Since at most 1 edge is involved in every crossing, there can not be more crossings than the number of edges. Thus, a crossing-minimal skewness-1 drawing cannot have more than $3n - 5$ crossings. This yields $\rho_{\text{skew-1}} \in O(n)$.

For the lower bound $\rho_{\text{skew-1}} \in \Omega(n)$ we first construct the graph G_ℓ , drawn in Figure 4.1. Add 6 vertices $u_1 \dots u_6$ and add the ℓ -compound edges: $(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_5)$ and (u_5, u_6) . Add the vertex x and add the ℓ -compound edges $(u_1, x), (u_2, x), (u_3, x), (u_4, x), (u_5, x)$ and (u_6, x) . Add the regular edges $(u_1, u_4), (u_2, u_5)$ and (u_3, u_6) . This graph is illustrated in Figure 4.1.

The resulting graph G_ℓ has $n = 11\ell - 11 + 7 = 11\ell - 4$ vertices and admits a drawing with 3 crossings, so $\text{cr}(G_\ell) \leq 3$ (see Figure 4.1a). As we can in Figure 4.1b, G_ℓ admits a skewness-1 drawing with ℓ crossings. We prove that $\text{cr}_{\text{skew-1}}(G_\ell) \geq \ell$.

We show there is no skewness-1 drawing with less than ℓ crossings. We denote by P the path $(u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_5), (u_5, u_6)$ of ℓ -compound edges. When in a drawing D an edge

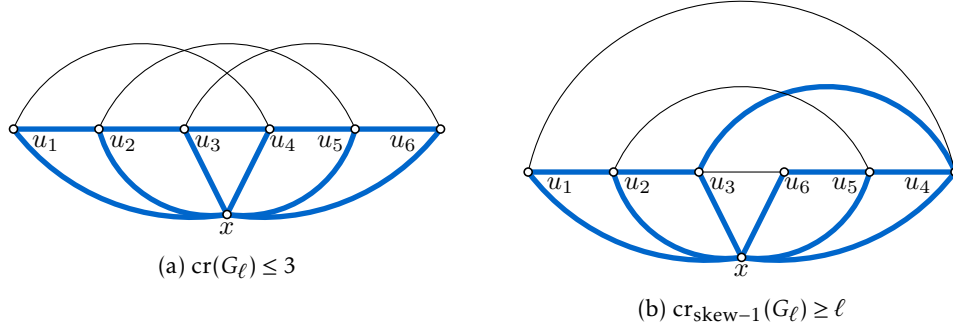


Figure 4.1: Drawings of graph G_ℓ to illustrate the proof for Theorem 12. Thick blue edges are ℓ -compound edges.

crosses P , it holds by Lemma 5 in [15] that D then has at least ℓ crossings. Let *past* vertices at edge (u_x, u_{x+1}) of P be the vertices u_y where $y < x$. Let *future* vertices at edge (u_x, u_{x+1}) of P be the vertices u_y where $y > x + 1$.

If all edges of P make a 3-cycle with x such that all other vertices lie in the same face bounded by the 3-cycle, then all three regular edges (u_1, u_4) , (u_2, u_5) and (u_3, u_6) have to mutually cross, making the drawing not skewness-1. This is illustrated in Figure 4.1a.

If any edge of P makes a 3-cycle with x such that there are future vertices on both the inner face and the outer face of the 3-cycle, or such that there are past vertices on both the inner face and the outer face of the 3-cycle, then P crosses itself and the claim follows by Lemma 2.

The only case remaining is when any edge of P makes a 3-cycle with x such that the 3-cycle divides past vertices and future vertices (i.e. all past vertices lie on the inner face and all future vertices lie on the outer face or vice versa). We then make a case distinction:

1. For the first edge of P (u_1, u_2) there are no past vertices and for the last edge of P (u_5, u_6) there are no future vertices. Thus, these edges cannot divide the past and future vertices.
2. If the second edge of P (u_2, u_3) divides the past and the future vertices, then u_1 and u_4 will be in a different face. This implies that the regular edge (u_1, u_4) has to cross P .
3. If the third edge of P (u_3, u_4) divides the past and the future vertices, then u_2 and u_4 will be in a different face. This implies that the regular edge (u_2, u_4) has to cross P . This is illustrated in Figure 4.1b.
4. If the fourth edge of P (u_4, u_5) divides the past and the future vertices, then u_3 and u_6 will be in a different face. This implies that the regular edge (u_3, u_6) has to cross P .

These cases cover all possible drawings, and each possible drawing Γ is either not skewness-1, or contains a crossing between a regular edge $r = (u_i, u_j)$ and an edge (a, b) of P . When such a crossing occurs, r forms a 3-cycle \mathcal{L} with x , which divides the plane into two regions such that a and b are in different regions. Thus (a, b) and the $\ell - 1$ paths connecting a to b has to cross \mathcal{L} , yielding ℓ crossings in Γ , as desired. \square

The proof for skewness-1 can easily be extended to skewness- k .

Theorem 13. *For every $\ell \geq 3$ there exists a skewness- k graph G_ℓ^k with $n = 11\ell k - 5k + 1$ vertices such that $\text{cr}_{\text{skew-}k}(G_\ell^k) \geq \ell k$ and $\text{cr}(G_\ell^k) \leq 3k$. Thus, $\rho_{\text{skew-}k} \in \Omega(n/k)$.*

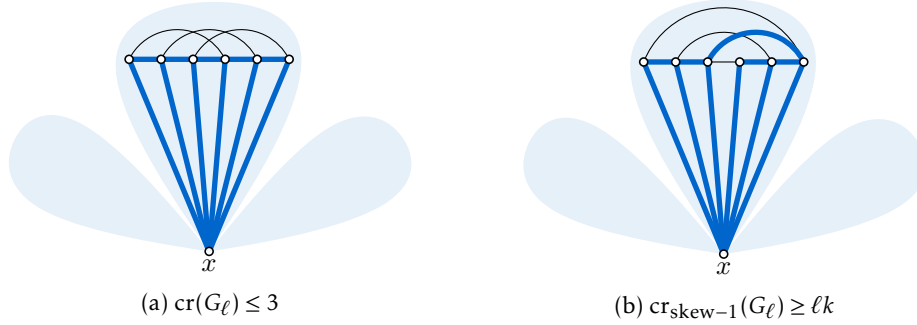


Figure 4.2: Each of the k petals contains a copy of G_ℓ , resulting in a graph G_ℓ^k .

Proof. We simply add k copies of G_ℓ and identify their vertex x . We denote the resulting graph by G_ℓ^k . An example of this is shown for $k = 3$ in Figure 4.2. A drawing of G_ℓ^k is only skewness- k if all k instances of G_ℓ are drawn skewness-1. Therefore, applying the above proof on all instances gives $\text{cr}_{\text{skew-}k}(G_\ell^k) = k\ell$. However, if skewness- k is not enforced, there exists a drawing with $\text{cr}(G_\ell^k) \leq 3k$. That is, when every instance is drawn with three crossings. Therefore, for every k we have shown that $\rho_{\text{skew-}k} \in \Omega(n/k)$. \square

Upper bound. (For completeness we repeat the upper bound argument in Theorem 12.) A skewness-1 graph with n vertices can have at most $3n - 5$ edges [16]. Since at most 1 edge is involved in every crossings, there can not be more crossings than the number of edges. Thus, a crossing-minimal skewness-1 drawing cannot have more than $3n - 5$ crossings. This yields $\rho_{\text{skew-1}} \in O(n)$.

A skewness- k graph with n vertices can have at most $3n - 6 + k$ edges [16]. Since at most k edges are involved in all crossings, there can not be more crossings than the number of edges times k . Thus, a crossing-minimal skewness- k drawing cannot have more than $3kn - 6k + k^2$ crossings. This yields $\rho_{\text{skew-}k} \in O(kn + k^2)$.

Straight-line setting. Figure 4.3 is a straight-line version of Figure 4.1. Thus, by Lemma 1, the statement of Theorem 12 also holds in the straight-line setting.

Corollary 13.1. *For every $\ell \geq 3$ there exists a skewness-1 graph G_ℓ with $n = 11\ell - 4$ vertices such that $\overline{\text{cr}}_{\text{skew-1}}(G_\ell) \geq \ell$ and $\overline{\text{cr}}(G_\ell) \leq 3$. Thus, $\overline{\rho}_{\text{skew-1}} \in \Omega(n)$.*

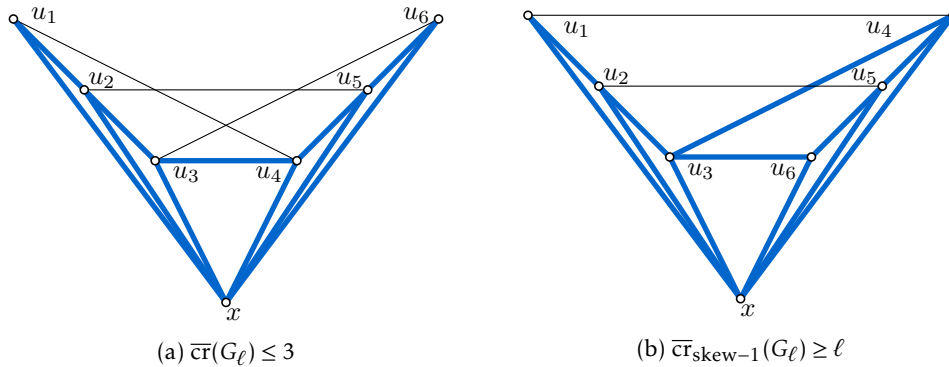


Figure 4.3: Straight-line drawing of graph G_ℓ that are weakly isomorphic to the drawings in Figure 4.1 to support the reasoning of Corollary 13.1. Thick blue edges are ℓ -compound edges.

For a different k we can easily expand the drawings more instances of G_ℓ to x . Thus, by Lemma 1, we argue that the statement of Theorem 13 also holds in the straight-line setting.

Corollary 13.2. *For every $\ell \geq 3$ there exists a skewness- k graph G_ℓ^k with $n = 11\ell k - 5k + 1$ vertices such that $\overline{\text{cr}}_{\text{skew-}k}(G_\ell^k) \geq \ell k$ and $\overline{\text{cr}}(G_\ell^k) \leq 3k$. Thus, $\overline{\rho}_{\text{skew-}k} \in \Omega(n/k)$.*

4.2 k -apex graphs

Lower bound. In a k -apex drawing all crossings must be covered by at most k vertices. A crossing is covered by a vertex when an edge incident to that vertex is part of the crossing. It is clear that every skewness- k drawing is also k -apex. By this observation we know that there is a graph G_ℓ with $n = 11\ell - 4$ vertices which admits a drawing with three crossings, so $\text{cr}(G_\ell) \leq 3$ (see Figure 4.1a), and admits a 1-apex drawing with ℓ crossings (see Figure 4.1b).

Moreover, the proof provided for the lower bound on the skewness-1 crossing ratio also holds for 1-apex drawings: Every time the proof states a drawing is not skewness-1, it is because three independent edges are crossing. Thus, the drawing is then also not 1-apex. It follows that $\text{cr}_{1\text{-apex}}(G_\ell) \geq \ell$.

Corollary 13.3. *For every $\ell \geq 3$ there exists a 1-apex graph G_ℓ with $n = 11\ell - 4$ vertices such that $\text{cr}_{1\text{-apex}}(G_\ell) \geq \ell$ and $\text{cr}(G_\ell) \leq 3$. Thus, $\rho_{1\text{-apex}} \in \Omega(n)$.*

Similarly, by the observation we know that for each k there is a graph G_ℓ^k with $n = 11\ell k - 3k + 1$ vertices which admits a drawing with $3k$ crossings, so $\text{cr}(G_\ell^k) \leq 3k$, and admits a k -apex drawing with $k\ell$ crossings. As before, the proof for skewness- k works for k -apex, because the drawing is only k -apex when every copy of G_ℓ is drawn 1-apex.

Corollary 13.4. *For every $\ell \geq 3$ there exists a k -apex graph G_ℓ^k with $n = 11\ell k - 3k + 1$ vertices such that $\text{cr}_{k\text{-apex}}(G_\ell^k) \geq \ell k$ and $\text{cr}(G_\ell^k) \leq 3k$. Thus, $\rho_{k\text{-apex}} \in \Omega(n/k)$.*

Upper bound. A 1-apex graph with n vertices can have at most $4n - 10$ edges [16]. Thus, a 1-apex crossing-minimal drawing cannot have more than $(4n - 10)^2$ crossings. This yields $\rho_{1\text{-apex}} \in O(n^2)$.

A k -apex graph with n vertices can have at most $3(n - k) - 6 + \sum_{i=1}^k (n - i) = 3(n - k) - 6 + kn - \frac{k(k+1)}{2}$ edges [16]. Thus, a k -apex crossing-minimal drawing has at most $\left(3(n - k) - 6 + kn - \frac{k(k+1)}{2}\right)^2$ crossings. This yields $\rho_{k\text{-apex}} \in O(k^2 n^2 - k^4)$. By the definition of a k -apex drawing it follows that, if $\text{cr}_{k\text{-apex}}(G_\ell) > \text{cr}(G)$ it must hold that $n > k$.

Straight-line setting. Figure 4.3 is a straight-line version of Figure 4.1. Thus, by Lemma 1, the statement of Corollary 13.3 also holds in the straight-line setting.

Corollary 13.5. *For every $\ell \geq 3$ there exists a 1-apex graph G_ℓ with $n = 11\ell - 4$ vertices such that $\overline{\text{cr}}_{1\text{-apex}}(G_\ell) \geq \ell$ and $\overline{\text{cr}}(G_\ell) \leq 3$. Thus, $\overline{\rho}_{1\text{-apex}} \in \Omega(n)$.*

For a different k we can easily expand the drawings more instances of G_ℓ to x . Thus, by Lemma 1, we argue that the statement of Corollary 13.4 also holds in the straight-line setting.

Corollary 13.6. *For every $\ell \geq 3$ there exists a k -apex graph G_ℓ^k with $n = 11\ell k - 3k + 1$ vertices such that $\overline{\text{cr}}_{k\text{-apex}}(G_\ell^k) \geq \ell k$ and $\overline{\text{cr}}(G_\ell^k) \leq 3k$. Thus, $\overline{\rho}_{k\text{-apex}} \in \Omega(n/k)$.*

Chapter 5

Conclusions

We have shown that for each beyond-planar graph family with a topological configuration from the survey by Didimo *et al.* [16], there exist graphs for which the crossing number increases arbitrarily when imposing the constraints. This is an extension on the work by Chimani *et al.* [15], answering most of their open questions. We gave a lower bound on the k -planar crossing ratio in a setting where we do not make use of parallel edges. This lower bound is less tight than the lower bound on the k -planar crossing ratio given by Chimani *et al.*, where they used parallel edges.

Furthermore, we extended their results to all beyond-planar graph families with topological forbidden configurations (as opposed to geometric forbidden configurations). These families are: (k, l) -grid-free graphs, planarly connected graphs, k -gap-planar graphs, k -fan-crossing-free graphs, skewness- k graphs, and k -apex graphs. For the families k -gap-planar graphs, k -fan-crossing-free graphs, skewness- k graphs, and k -apex graphs the forbidden configuration is dependent on one variable k and they have a natural base case, where k has a set value. For planarly connected graphs the forbidden configuration is constant and for (k, l) -grid-free graphs the forbidden configuration is dependent on two variables k, l and there is no useful natural base case, as per the definition, a $(1, 1)$ -grid-free drawing is a planar drawing and there are two variables involved. For the 1-gap-planar and the skewness-1 crossing ratios we presented tight bounds. Furthermore, we achieved lower and upper bounds on the crossing ratio when k (or k, l) is unknown. For these crossing ratios we were not able to achieve tight bounds. These results are presented in Table 1.1 in Chapter 1.

Chimani *et al.* asked whether there exist quasi-planar and fan-planar graphs whose crossing ratio is $\Omega(n^2)$ and conjectured that this bound can be reached [15]. This question remains open. Furthermore, for most of the classes we consider, the bound is not tight. Thus, the question can be asked whether there exist graph families whose crossing ratios can reach the upper bounds, or whether the upper bounds can be lowered.

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