MASTER

Event-triggered control in presence of measurement noise
a space-regularization approach

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Event-Triggered Control in Presence of Measurement Noise: A Space-Regularization Approach

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Abstract

Controllers are now commonly implemented digitally. In this context, when using periodic sampling strategies, the achievable performance of the closed loop system is related to the amount of data that can be processed. Hence, in applications with limited computational or energy resources, traditional periodic sampling strategies might not meet the desired performance criteria while satisfying these constraints. Event-triggered control aims to solve this problem by only sampling when the system needs attention. Due to this aperiodic sampling strategy, computational burden and communication bandwidth may be reduced while preserving the desired stability and performance properties.

Event-triggered control algorithms often sample based on state information. Due to sensor imperfections and limited precision, perfect state information is in reality often unavailable, and an open problem in event-triggered control is how to deal with these measurement noises. A known issue when dealing with measurement noise in event-triggered control is the occurrence of Zeno behavior, which requires an infinite number of transmissions in finite time. Some solutions for the measurement noise problem exists, however, these often impose constraints such as, e.g., requiring “unnatural” assumptions, such as differentiability, on the character of the measurement noise.

In this thesis, general conditions for set stabilization of (distributed) event-triggered control systems affected by measurement noises are presented. It is shown that, under these conditions, both static and dynamic triggering conditions can be designed such that the closed-loop system ensures an input-to-state practical set stability property. Additionally, by proper choice of the tuning parameters, the system does not exhibit Zeno behavior. Contrary to various results in the event-triggered control literature, the measurement noises do not have to be differentiable in the proposed setup. The general results are applied to point stabilization and consensus problems as particular cases. Simulations illustrate the strengths of the results.

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Chapter 1

Introduction

For systems in which the communication energy consumption, communication bandwidth or computation power is constrained, traditional periodic sampling and/or controller updates might require resources that are not available to obtain the desired system performance. To this end, event-triggered control (ETC) can be applied, see, e.g., [1] and the references therein, to reduce computational burden and/or the communication bandwidth of the control strategies, while preserving important stability and performance properties. In ETC, the sampling instant is decided on the basis of a triggering condition, hence, sampling instants are not necessarily periodic.

In general, most literature on ETC assumes that perfect state or output information is available for control, even though in most physical systems, this is often not the case due to sensor limitations. Since sensors are susceptible to measurement noises, exact state or output information is therefore typically not available. It is known that under these circumstances, the design of triggering conditions that do not require an infinite number of controller updates in finite time, i.e., Zeno behavior, is in general a hard problem, see, e.g., [2]. Several solutions have been proposed in the literature to address this problem, see, e.g., [3, 4]. However, these require differentiability conditions on the noise, $\mathcal{L}_\infty$-bounds on the derivative of the noise and the ensured input-to-state stability (ISS) or $\mathcal{L}_p$-stability of the closed-loop system holds with respect to the noise and its time-derivative. When dealing with real sensors, the differentiability condition and global boundedness of the derivative of the noise may not be natural assumptions. The observer-based approaches, see, e.g., [5], on the other hand, overcome this issue, but these results only apply to linear systems and require multiple additional internal models, thereby requiring extra processing power and energy to run. In [6], a periodic event-triggered controller (PETC) is run simultaneously with a continuous event-triggered controller (CETC), and transmission occurs when the triggering conditions of both controllers hold. The difference between PETC and CETC is that for the former, the triggering condition is checked periodically while the triggering condition for the latter has to be monitored constantly. The downside to this particular method is that if the state is close to the origin, periodic sampling is obtained, hence, the communication benefit of ETC is not preserved. This issue is even harder when designing distributed event-triggered controllers for consensus [7]. We know of only one paper dealing with measurement noise in this context, [8], where the control input is integrated to estimate an upper-bound for the error. Since a conservative estimate is used and due to the absolute triggering condition, the amount of controller updates (network bandwidth) required is relatively large compared to other ETC consensus algorithms, see, e.g., [9].

In this thesis, we present a general framework to address the measurement noise problem, based on space-regularized (fixed threshold) ETC, in line with classical event generators, such as [10, 11, 12]. For this, we present a new hybrid model, in which we use a hybrid system for which a jump models a transmission. The model does not involve the derivative of the noise as opposed to [3, 4]. We then provide general prescriptive conditions, under which both dynamic and static triggering rules are designed to ensure an input-to-state practical stability property, while ruling out Zeno phenomena. In particular, we show that applying space-regularization, i.e., artificially enlarging the “flow set” of the hybrid model, needs to be done with care to ensure the existence of strictly positive minimum inter-event times, which only requires that an upper-bound of the noise is known. The existence of these strictly positive inter-event times are necessary to rule out Zeno phenomena. These results are written for the general scenario where N plants, possibly interconnected, are controlled by N event-triggered controllers, hence covering both classical point stabilization problems as in, e.g., [10, 11, 12, 13] and consensus problems as in, e.g., [14] in a unified way. We then demonstrate the relevance of our technique by showing that it can be applied to design event-triggering strategies robust to measurement noise. In particular, we explain how to modify the triggering rules presented in [10, 11] to be applicable in presence of measurement noise. We also apply it to consensus seeking problems, where we show that we can maintain long inter-event times even in the presence of measurement noise. We show this, for instance, in the methods of [9, 15]. Lastly, we use simulations to show the effectiveness of our technique and to demonstrate the implications of applying space-regularization.

This thesis is structured as follows. In Chapter 2 we present the necessary preliminaries and notational conventions. Chapter 3 contains the problem statement. We present a hybrid model and the framework, including the main results, in Chapter 4. We apply our framework to several interesting case studies in Chapter 5. Finally, we illustrate the obtained results by simulating some case studies in Chapter 6, and provide conclusions in Chapter 7.

Part of this work has been submitted to the 59^{th} IEEE Conference on Decision and Control.

Chapter 2

Preliminaries

2.1 Notation

The sets of all non-negative and positive integers are denoted $\mathbb{N}$ and $\mathbb{N}_{>0}$, respectively. The field of all reals and all non-negative reals are indicated by $\mathbb{R}$ and $\mathbb{R}_{\geqslant 0}$, respectively. The identity matrix of size $N \times N$ is denoted by I_N, and the vectors in $\mathbb{R}^N$ whose elements are all ones or zeros are denoted by $\mathbf{1}_N$ and $\mathbf{0}_N$, respectively. For N vectors $x_i \in \mathbb{R}^{n_i}$, the vector obtained by stacking all vectors into one column vector $x \in \mathbb{R}^n$ with $n = \sum_{i=1}^{N} n_i$ is denoted as $(x_1, x_2, \ldots, x_N)$, i.e., $(x_1, x_2, \ldots, x_N) = \begin{bmatrix} x_1^\top & x_2^\top & \ldots & x_N^\top \end{bmatrix}^\top$. By $\langle \cdot, \cdot \rangle$ and $|\cdot|$ we denote the usual inner product of real vectors and the Euclidean norm, respectively. For a measurable signal $w : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{n_w}$, we denote by $\|w\|_\infty = \operatorname{ess\,sup}_{t \in \mathbb{R}_{\geqslant 0}} |w(t)|$ its $\mathcal{L}_\infty$-norm, provided it exists and is finite, we then write $w \in \mathcal{L}_\infty$. A function $w : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^{n_w}$ is said to be càdlàg, denoted by $w \in \mathcal{PC}$, if there exists a sequence $\{t_i\}_{i \in \mathbb{N}}$ with $t_{i+1} > t_i > t_0 = 0$ for all $i \in \mathbb{N}$ and $t_i \to \infty$ when $i \to \infty$ such that w is a continuous function on (t_i, t_{i+1}) where $\lim_{t \uparrow t_i} w(t)$ exists for all $i \in \mathbb{N}_{>0}$ and $\lim_{t \downarrow t_i} w(t)$ exists for all $i \in \mathbb{N}$ with $\lim_{t \downarrow t_i} w(t) = w(t_i)$, i.e., w is right-continuous and left limits exist for each $i \in \mathbb{N}_{>0}$. For any $x \in \mathbb{R}^N$, the distance to a closed non-empty set $\mathcal{A}$ is denoted by $|x|_\mathcal{A} := \inf_{y \in \mathcal{A}} |x - y|$. The closure of a set $\mathcal{A}$ is denoted by $\overline{\mathcal{A}}$.

A continuous function $\alpha : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a class-$\mathcal{K}$ function if it is strictly increasing and $\alpha(0) = 0$ and it is a class-$\mathcal{K}_\infty$ function if, in addition, $\alpha(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is a class-$\mathcal{KL}$ function if, for each fixed $s \geqslant 0$, the mapping $\beta(\cdot, s)$ is a class-$\mathcal{K}$ function and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

2.2 Graph theory

A weighted graph $\mathcal{G} := (\mathcal{V}, \mathcal{E}, A)$ consists of a vertex set $\mathcal{V} := \{1, 2, ..., N\}$, a set of edges $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ and an adjacency matrix $A \in \mathbb{R}^{N \times N}$. An ordered pair $(i, j) \in \mathcal{E}$, with $i, j \in \mathcal{V}$, is an edge from i to j. For an edge $(i, j) \in \mathcal{E}$, i is called the *in*-neighbor of j, and j is called the *out*-neighbor of i. All $(i, j) \in \mathcal{E}$ have an associated weight, denoted $w_{ij} \in \mathbb{R}_{>0}$. The adjacency matrix $A := (a_{i,j})$, $i, j \in \mathcal{V}$ of a graph is defined as

$$a_{i,j} := \begin{cases} w_{ij} & \text{if } (i, j) \in \mathcal{E}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

The set $\mathcal{V}_i^{\text{in}}$ of the in-neighbors of i is defined as $\mathcal{V}_i^{\text{in}} := \{j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$ and the set of out-neighbors as $\mathcal{V}_i^{\text{out}} := \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$. An undirected graph is a graph where, for any edge $(i, j) \in \mathcal{E}$, (j, i) is also in $\mathcal{E}$. A sequence of edges $(i, j) \in \mathcal{E}$ connecting two vertices is called a directed path. For a connected graph $\mathcal{G}$, there exists a path between any two vertices in $\mathcal{V}$. The in-degree is defined as $d_i^{\text{in}} := \sum_{j\in\mathcal{V}_i^{\text{in}}} w_{ji}$ and the out-degree as $d_i^{\text{out}} := \sum_{j\in\mathcal{V}_i^{\text{out}}} w_{ij}$. The in-degree matrix D^{in} and out-degree matrix D^{out} are diagonal matrices with d_i^{in} respectively d_i^{out} as the ith diagonal element. A weight-balanced digraph (directed graph) is a digraph where $d_i^{\text{out}} = d_i^{\text{in}}$ for all i. The Laplacian L of a graph $\mathcal{G}$ is defined as $L := D^{\text{out}} - A$. For an undirected graph, $D^{\text{in}} := D^{\text{out}}$.

2.3 Hybrid systems

We model hybrid systems using the formalism of [16, 17]. As such, we consider systems $\mathcal{H}(F, \mathcal{C}, G, \mathcal{D})$ of the form

$$\begin{cases} \dot{\xi} \in F(\xi, w) & (\xi, w) \in \mathcal{C}, \\ \xi^+ \in G(\xi, w) & (\xi, w) \in \mathcal{D}, \end{cases} \tag{2.2}$$

where $\xi \in \mathbb{R}^{n_\xi}$ denotes the state, $w \in \mathbb{R}^{n_w}$ a disturbance, $\mathcal{C} \in \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_w}$ the flow set, $\mathcal{D} \in \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_w}$ the jump set, $F : \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_w} \rightrightarrows \mathbb{R}^{n_\xi}$ the flow map and $G : \mathbb{R}^{n_\xi} \times \mathbb{R}^{n_w} \rightrightarrows \mathbb{R}^{n_\xi}$ the jump map, where the maps F and G are possibly set-valued. Loosely speaking, while $(\xi, w) \in \mathcal{C}$, the state can flow continuously according to $\dot{\xi} \in F(\xi, w)$. If $(\xi, w) \in \mathcal{D}$, the state can jump as $\xi^+ \in G(\xi, w)$. If $(\xi, w) \in \mathcal{C} \cap \mathcal{D}$, the system can either flow or jump, flow is only allowed if flowing keeps the solution in $\mathcal{C}$. See [16] for more details on the adopted hybrid terminology. Given a hybrid system $\mathcal{H}$, its solutions are hybrid arcs ϕ (see [16, Def. 2.4] for a formal definition of hybrid arcs) that satisfy the following definition.

Definition 2.1. *A hybrid arc ϕ is a solution to $\mathcal{H}$ for $w \in \mathcal{PC}$, if $(\phi(0,0), w(0)) \in \overline{\mathcal{C}} \cup \mathcal{D}$, and*

(S1) for all $j \in \mathbb{N}$ and almost all t such that $(t, j) \in \operatorname{dom}\phi$, $(\phi(t, j), w(t)) \in \mathcal{C}$ and $\dot{\phi}(t, j) \in F(\phi(t, j), w(t))$;

(S2) for all $(t, j) \in \operatorname{dom}\phi$ such that $(t, j+1) \in \operatorname{dom}\phi$, $(\phi(t, j), w(t)) \in \mathcal{D}$ and $(\phi(t, j+1), w(t)) \in G(\phi(t, j), w(t))$.

A solution ϕ is non-trivial if $\operatorname{dom}\phi$ contains at least two points.

Definition 2.2. *For $a \in \mathbb{R}_{\geqslant 0}$ define $S_a : \mathcal{PC} \to \mathcal{PC}$ for $w \in \mathcal{PC}$ by $S_a(w) = \widetilde{w}$, where $\widetilde{w}(t) = w(t + a)$ for all $t \geqslant 0$. Moreover, for $\mathbb{W} \subset \mathcal{PC}$ and $a \in \mathbb{R}_{\geqslant 0}$, we define $S_a(\mathbb{W}) := \{S_a(w) \mid w \in \mathbb{W}\}$. $\mathbb{W}$ is called* time-invariant *if $S_a(\mathbb{W}) = \mathbb{W}$ for all $a \in \mathbb{R}_{\geqslant 0}$.*

Loosely speaking, the set of time-invariant functions is a collection of functions where a particular function at a specific time $t \geqslant 0$ can always be written as a different function of the set at $t = 0$. For example, the set of the single function $\mathbb{W} = \{\sin(\cdot)\}$ is *not* time invariant, since by applying the shift operator S_a, we obtain the set $\{\widetilde{w} \in \mathcal{PC} \mid \widetilde{w}(t) = \sin(t + a)\}$, $t \in \mathbb{R}_{\geqslant 0}$, which only contains the same function as the original set if a is a multiple of 2π, but not for all $a \geqslant 0$. We can extend $\mathbb{W}$ in a time-invariant set in this case by using, for example, $\mathbb{W} = \cup_{\phi \in [0, 2\pi)} \{w \in \mathcal{PC} \mid w(t) \in \sin(t + \phi)\}$. Indeed, for this set it holds that $\mathbb{W} = S_a(\mathbb{W})$ for all $a \geqslant 0$.

The following proposition provides conditions for the existence of non-trivial solutions for hybrid system (2.2).

Proposition 2.1. *Consider the hybrid system $\mathcal{H}$ with $\mathbb{W} \subset \mathcal{PC}$ given. Consider $\xi \in \mathbb{R}^{n_\xi}$ and $w \in \mathbb{W}$ with $(\xi, w(0)) \in \overline{\mathcal{C}} \cup \mathcal{D}$. If $(\xi, w(0)) \in \mathcal{D}$ or*

(VC) there exist $\epsilon > 0$ and an absolutely continuous function $z : [0, \epsilon] \to \mathbb{R}^{n_\xi}$ such that $z(0) = \xi$, $\dot{z}(t) \in F(z(t), w(t))$ and $(z(t), w(t)) \in \mathcal{C}$ for almost all $t \in [0, \epsilon]$,

then there exists a non-trivial solution ϕ to $\mathcal{H}$ for w with $\phi(0,0) = \xi$. If (VC) holds for every $\xi \in \mathbb{R}^{n_\xi}$ and $w \in \mathbb{W}$ with $(\xi, w(0)) \in \overline{\mathcal{C}} \setminus \mathcal{D}$, then there exists a non-trivial solution to $\mathcal{H}$ for every $(\xi, w(0)) \in \overline{\mathcal{C}} \cup \mathcal{D}$, and every maximal solution satisfies exactly one of the following properties:

(a) *ϕ is complete;*

(b) $\operatorname{dom}\phi$ *is bounded and, with $J = \sup_j \operatorname{dom}\phi$, the interval $I^J := \{t : (t,j) \in \operatorname{dom}\phi\}$ has non-empty interior and is open to the right, and there does not exist an absolutely continuous function $z : [a,b] \to \mathbb{R}^{n_\xi}$ satisfying $\dot{z}(t) \in F(z(t), w(t))$ for almost all $t \in [a,b]$, $(z(t), w(t)) \in \mathcal{C}$ for almost all $t \in (a,b)$, and such that $I^J \subset [a,b)$ and $z(t) = \phi(t,J)$ for all $t \in I^J$;*

(c) $\operatorname{dom}\phi$ *is bounded and $(\phi(T,J), w(T)) \notin \overline{\mathcal{C}} \cup \mathcal{D}$, where $(T,J) = \sup\operatorname{dom}\phi$.*

Proof. The first statements regarding the existence of a non-trivial solution follows directly from the definition of a solution to $\mathcal{H}$. For regarding the properties of maximal solutions, suppose that ϕ is a maximal solution that is not complete, i.e., $\operatorname{dom}\phi$ is bounded. Let $(T,J) = \sup\operatorname{dom}\phi$. If $(T,J) \in \operatorname{dom}\phi$ and $(\phi(T,J), w(T)) \in \overline{\mathcal{C}} \cup \mathcal{D}$, then either $(\phi(T,J), w(T)) \in \mathcal{D}$ in which case ϕ can be extended via a jump, or $(\phi(T,J), w(T)) \in \overline{\mathcal{C}} \setminus \mathcal{D}$ in which case ϕ can be extended via flow, thanks to (VC). Thus, either (c) holds or $(T,J) \notin \operatorname{dom}\phi$. If the latter holds, then the interior of I^J is non-empty, since we could not get there via a jump as then $(T,J) \in \operatorname{dom}\phi$, and (b) must hold to ensure maximality of ϕ. Indeed, if (b) would fail, ϕ could be extended to a solution to $\mathcal{H}$ for w on $\overline{\operatorname{dom}\phi}$. ■

Remark 2.1. *Due to the use of the time-invariant set $\mathbb{W} \subset \mathcal{PC}$, (VC) does not depend on the initial time. Consequently, (VC) only has to be proven for a single point in time and not for all $t \in \mathbb{R}_{\geqslant 0}$.*

Remark 2.2. *We would like to emphasize that we require some extra care for item (c) above. In hybrid systems without inputs, it would be sufficient to prove that $G(D) \subset \overline{\mathcal{C}} \cup \mathcal{D}$ to exclude (c), however, when dealing with discontinuous inputs as defined above, discontinuities in w could also result in (c) occurring.*

To illustrate the rationale behind our choice for $w \in \mathcal{PC}$, suppose that we define w such that $w(t) = 0$ for all $t \neq 1$, and $w(t) = 1$ for $t = 1$. In this case, verifying whether $(\xi, w(1)) \in \overline{\mathcal{C}} \setminus \mathcal{D}$ is not a clear indication of whether (VC) holds. Indeed, by enforcing that w is càdlàg, i.e., $w \in \mathcal{PC}$, we can avoid such issues, and in this case point-wise evaluation of $(\xi, w) \in \overline{\mathcal{C}} \setminus \mathcal{D}$ is an indication of the behavior for a (small) neighborhood after the evaluated point due to right-continuity. Note that this implies that we only have to check if (VC) holds for continuous inputs w, due to w being right-continuous and the choice of $\{t_i\}_{i \in \mathbb{N}}$, in the definition of $\mathcal{PC}$.

We are mainly interested in systems $\mathcal{H}$ that are persistently flowing as defined below.

Definition 2.3. *A hybrid system $\mathcal{H}$ is persistently flowing if, for any $w \in \mathbb{W}$, all maximal solutions ϕ are unbounded in t-direction, i.e., $\sup_t \operatorname{dom}\phi = \infty$.*

We focus on the following stability definitions in this thesis.

Definition 2.4. *When $\mathcal{H}$ is persistently flowing, we say that a non-empty closed set $\mathcal{A} \subset \mathbb{R}^{n_\xi}$ is* input-to-state practically stable *(ISpS) if there exist $\gamma \in \mathcal{K}$, $\beta \in \mathcal{KL}$ and $d \in \mathbb{R}_{\geqslant 0}$ such that for any solution pair (ξ, w) with $w \in \mathcal{L}_\infty \cap \mathcal{PC}$*

$$|\xi(t,j)|_{\mathcal{A}} \leqslant \beta(|\xi(0,0)|_{\mathcal{A}}, t) + \gamma(\|w\|_\infty) + d, \tag{2.3}$$

for all $(t, j) \in \operatorname{dom} \xi$. *If* (2.3) *holds with* $d = 0$, *then* $\mathcal{A}$ *is said to be* input-to-state stable *(ISS) for* $\mathcal{H}$.

To prove that a given non-empty closed set $\mathcal{A}$ is IS(p)S, we will use the following Lyapunov conditions.

Proposition 2.2. *Consider a persistently flowing system* $\mathcal{H}$ *and let* $\mathcal{A} \subset \mathbb{R}^{n_\xi}$ *be a non-empty closed set. If there exist a continuously differentiable* $V : \mathbb{R}^{n_\xi} \to \mathbb{R}_{\geqslant 0}$, $\alpha, \underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}$ *and* $c \in \mathbb{R}_{\geqslant 0}$ *such that*

1. *for any* $(\xi, w) \in \mathcal{C} \cup \mathcal{D}$,
$$\underline{\alpha}(|\xi|_{\mathcal{A}}) \leqslant V(\xi) \leqslant \overline{\alpha}(|\xi|_{\mathcal{A}}),$$

2. *for all* $(\xi, w) \in \mathcal{C}$ *and* $f \in F(\xi, w)$,
$$\langle \nabla V(\xi), f \rangle \leqslant -\alpha(|\xi|_{\mathcal{A}}) + \gamma(|w|) + c,$$

3. *for all* $(\xi, w) \in \mathcal{D}$ *and any* $g \in G(\xi, w)$,
$$V(g) - V(\xi) \leqslant 0,$$

then $\mathcal{A}$ *is ISpS, and it is ISS if* $c = 0$.

Sketch of proof. Let (ξ, w) be a hybrid solution to $\mathcal{H}$, $(t, j) \in \operatorname{dom} \xi$ and $0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_{j+1} = t$ satisfy

$$\operatorname{dom} \xi \cap ([0, t] \times \{0, \ldots, j\}) = \bigcup_{i \in \{0, \ldots, j\}} [t_i, t_{i+1}] \times \{i\}. \tag{2.4}$$

For each $i \in \{1, 2, \ldots, j\}$ and for almost all $s \in [t_i, t_{i+1}]$, item 2) of Proposition 2 implies that $\left\langle \nabla V(\xi(s, i)), \dot{\xi}(s, i) \right\rangle \leqslant -\alpha(|\xi(s, i)|_{\mathcal{A}}) + \gamma(|w(s)|) + c$. We can then invoke similar arguments as in [18, Lemma 2.14] to obtain the desired result as: (i) V does not increase at jumps according to item 3) of Proposition 2, (ii) item 1) holds, and (iii) $\mathcal{H}$ is persistently flowing. ■

Chapter 3

Problem formulation

We consider a collection of $N \in \mathbb{N}_{>0}$ interconnected plants $P_1, P_2, \ldots, P_N$. Each plant P_i, $i \in \mathcal{N} := \{1, 2, \ldots, N\}$, is equipped with a sensor that communicates its state (with measurement noise) to the controllers $C_1, C_2, \ldots, C_N$ via a digital network. Plant P_i, $i \in \mathcal{N}$, has a state $x_i \in \mathbb{R}^{n_x^i}$ with dynamics

$$\dot{x}_i = f_i(x, u_i), \tag{3.1}$$

where $u_i \in \mathbb{R}^{n_u^i}$ is the control input of P_i, $x := (x_1, x_2, \ldots, x_N)$ is the concatenated state variable, and $f_i : \mathbb{R}^n \times \mathbb{R}^{n_u^i} \to \mathbb{R}^{n_x^i}$ is a continuous function, with $n = \sum_{i \in \mathcal{N}} n_x^i$. Note that f_i may depend on the states of other plants, i.e., physical couplings are allowed. The controllers C_i, $i \in \mathcal{N}$, take the form, in absence of noise,

$$u_i = k_i(x), \tag{3.2}$$

with $k_i : \mathbb{R}^n \to \mathbb{R}^{n_u^i}$ a continuous map. We assume that plants $P_1, P_2, \ldots, P_N$ in closed loop with the controllers $C_1, C_2, \ldots, C_N$ satisfy desired control objectives in the absence of a network, as formalized in the following, see Section 4.2.

We investigate the scenario where the values of each state x_i, $i \in \mathcal{N}$, are broadcasted by the corresponding sensors to the controllers $C_1, C_2, \ldots, C_N$, which depend on it, via a digital network, as illustrated in Fig. 3.1. The corresponding transmissions occur at some time instants t_k^i, $k \in \mathbb{N}$, which are generated by a local triggering condition. Moreover, the measurements are affected by noise. To model the obtained feedback law in this context, we introduce $\widetilde{x}_i$, the noisy measurement of x_i, for $i \in \mathcal{N}$, as

$$\widetilde{x}_i := x_i + w_i, \tag{3.3}$$

where $w_i \in \mathbb{R}^{n_x^i}$ is an (additive) bounded piecewise continuous measurement noise, which is assumed to satisfy the following assumption.

Assumption 3.1. *For each $i \in \mathcal{N}$, $w_i \in \mathcal{PC}$ and $w_i(t) \in \mathcal{W}_i$ for all $t \in \operatorname{dom} w$, where $\mathcal{W}_i := \left\{ w_i \in \mathbb{R}^{n_x^i} \mid |w_i| \leqslant \overline{w}_i \right\}$ for some $\overline{w}_i \in \mathbb{R}_{\geqslant 0}$.*

Note that Assumption 1 constrains the signals $w_i \in \mathcal{PC}$, $i \in \mathcal{N}$, such that the norm of its range is bounded for all t. Since we do not restrict ourselves to a specific class of functions, and since the bound on the norm is constant, this subset of $\mathcal{PC}$-functions is time-invariant. To illustrate this, suppose that $w_i \in \mathbb{R}$. Then, $w_i(t) \in [-\overline{w}_i, \overline{w}_i]$ for all $t \in \mathbb{R}_{\geqslant 0}$. If we apply the shift operator we obtain $w_i(t + a) \in [-\overline{w}_i, \overline{w}_i]$ for all $t, a \in \mathbb{R}_{\geqslant 0}$. Since these sets are equivalent, we indeed conclude that $\{\mathcal{W}\} = \{\mathcal{W}\}_a$ and, consequently, $\mathcal{W}$ is time-invariant.

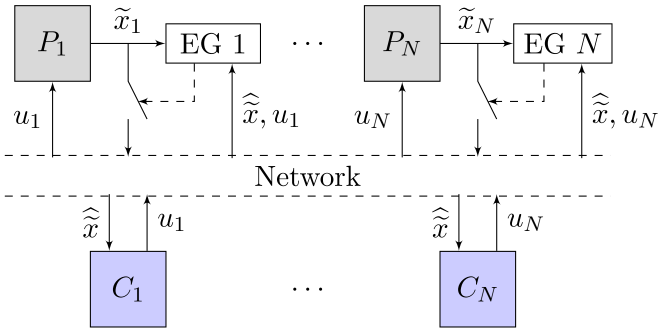

Figure 3.1: Networked control setup with event generators (EG). EG i determines when the current state $\widetilde{x}_i$ is transmitted over the network.

Because of the packet-based communication over the network, the controllers, which depend on the state of P_i, do not have continuous access to $\widetilde{x}_i$ in (3.3), but only to its discrete networked version, $\widehat{\widetilde{x}}_i := \widehat{x}_i + \widehat{w}_i$, where

$$\begin{aligned} \widehat{x}_i(t) &= x_i(t_k^i) \qquad \text{for } t \in [t_k^i, t_{k+1}^i),\ k \in \mathbb{N}, \\ \widehat{w}_i(t) &= w_i(t_k^i) \qquad \text{for } t \in [t_k^i, t_{k+1}^i),\ k \in \mathbb{N}. \end{aligned} \tag{3.4}$$

To keep the definitions consistent with the existing literature, we define the network-induced error $e = (e_1, e_2, \ldots, e_N)$ as the difference between the sampled state $\widehat{x} := (\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_N)$ *without* the measurement noise and the current state *without* measurement noise, i.e.,

$$e := \widehat{x} - x. \tag{3.5}$$

We also introduce the *measured* network-induced error $\widetilde{e} = (\widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_N)$ as the difference between the most recently transmitted state and the currently measured state, which are both affected by noise, i.e.,

$$\widetilde{e} := \widehat{x} + \widehat{w} - x - w = e + \widehat{w} - w. \tag{3.6}$$

Note that e_i is not known by event generator i, and therefore, cannot be used by the corresponding local triggering condition for determining t_k^i, $k \in \mathbb{N}$. However, the event generators do have access to $\widetilde{e}_i$.

Due to the network and the noisy measured states, the feedback law in C_i applied to plant P_i is, for $i \in \mathcal{N}$,

$$u_i = k_i(x + e + \widehat{w}). \tag{3.7}$$

Our objective is to determine the transmission times t_k^i, $k \in \mathbb{N}$, for any $i \in \mathcal{N}$, to ensure that:

(i) the combined closed-loop system (3.1), (3.7) satisfies an input-to-state practical stability property in the presence of measurement noise;

(ii) there exists a strictly positive time between any two transmissions generated by the triggering condition of plant P_i, i.e., for any initial condition there exists a $T_i > 0$ such that $t_{k+1}^i - t_k^i \geqslant T_i$ for all $k \in \mathbb{N}$, $i \in \mathcal{N}$.

Chapter 4

General results

4.1 Hybrid model

We model the overall system as a hybrid system $\mathcal{H}$ for which a jump corresponds to the broadcasting of one of the noisy states $\widetilde{x}_i$, $i \in \mathcal{N}$, over the network. We allow the local triggering conditions to depend on a local variable denoted $\eta_i \in \mathbb{R}_{\geqslant 0}$, $i \in \mathcal{N}$, as in the dynamic triggering of [11, 13]. We also consider static triggering conditions in the sequel. We define $\eta := (\eta_1, \eta_2, \ldots, \eta_N) \in \mathbb{R}^N_{\geqslant 0}$, and stack the "physical" variables in $\chi := (x, e, \widehat{w})$. The full state for $\mathcal{H}$ becomes $\xi := (\chi, \eta) = (x, e, \widehat{w}, \eta)$ and is defined as

$$\begin{aligned} \dot{\xi} &= F(\xi, w), \qquad (\xi, w) \in \mathcal{C}, \\ \xi^+ &\in G(\xi, w), \qquad (\xi, w) \in \mathcal{D}, \end{aligned} \tag{4.1}$$

where the flow map is given, for all $(\xi, w) \in \mathbb{X} \times \mathcal{W}$, where $\mathbb{X} := \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{W} \times \mathbb{R}^N_{\geqslant 0}$, $\mathcal{W} := \mathcal{W}_1 \times \mathcal{W}_2 \times \ldots \times \mathcal{W}_N$ and $\mathcal{W}_i$ comes from Assumption 1, by

$$F(\xi, w) := (F_\chi(\chi), \Psi(o)), \tag{4.2}$$

with $\Psi(o) := (\Psi_1, \Psi_2, \ldots, \Psi_N)$ the to-be-designed dynamics of the dynamic variables η and $o := (o_1, o_2, \ldots, o_N)$ collects $o_i \in \mathbb{R}^{n_o^i}$, $i \in \mathcal{N}$, being the information locally available to plant i. In (4.2),

$$F_\chi(\chi) := (f(x, k(x + e + \widehat{w})), -f(x, k(x + e + \widehat{w})), \mathbf{0}_n), \tag{4.3}$$

for $\chi \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{W}$. Let, for $i \in \mathcal{N}$,

$$\mathcal{C}_i := \{(\xi, w) \in \mathbb{X} \times \mathcal{W} \mid \eta_i + \theta_i \Psi_i(o_i) \geqslant 0\} \tag{4.4}$$

with $\theta_i \in \mathbb{R}_{\geqslant 0}$ a design parameter. The flow set for the overall system is given by

$$\mathcal{C} := \bigcap_{i \in \mathcal{N}} \mathcal{C}_i. \tag{4.5}$$

The jump set corresponding to a transmission of $\widetilde{x}_i$ generated by triggering condition $i \in \mathcal{N}$ is defined as

$$\mathcal{D}_i := \big\{(\xi, w) \in \mathbb{X} \times \mathcal{W} \mid \eta_i + \theta_i \Psi_i(o_i) \leqslant 0 \text{ and } \Psi_i(o_i) \leqslant 0\big\}. \tag{4.6}$$

Note that, with respect to [11], we require the additional condition $\Psi_i(o_i) \leqslant 0$ to ensure that Zeno behavior does not occur when $\theta_i = 0$. By selecting a $\theta_i > 0$, we trigger earlier than the "pure" dynamic case (i.e., when $\theta_i = 0$). Generally, this results in faster convergence but shorter inter-event times, which allows us to tune bandwidth usage versus performance, see [11] for more details. The jump set for the overall system is defined as

$$\mathcal{D} := \bigcup_{i\in\mathcal{N}} \mathcal{D}_i. \tag{4.7}$$

Note that both $\mathcal{C}$ and $\mathcal{D}$ are closed sets. The jump map for triggering condition i is now defined as

$$G_i(\xi, w) := \begin{cases} \{(G_{\chi,i}(\chi, w), \eta)\}, & \text{if } (\xi, w) \in \mathcal{D}_i \\ \emptyset, & \text{if } (\xi, w) \notin \mathcal{D}_i, \end{cases} \tag{4.8}$$

where

$$G_{\chi,i}(\chi, w) := (x, \overline{\Gamma}_i e, \overline{\Gamma}_i \widehat{w} + \Gamma_i w), \tag{4.9}$$

with Γ_i the block diagonal matrix where the i-th block is $I_{n_x^i}$ and all other blocks are $\mathbf{0}_{n_x^j \times n_x^j}$, $j \in \mathcal{N} \setminus \{i\}$, and $\overline{\Gamma}_i := I_n - \Gamma_i$. Map (4.8) simply means that a jump due to triggering condition i resets e_i to 0 and $\widehat{w}_i$ to w_i (essentially, $\widehat{w}_i^+ \in \mathcal{W}_i$), leaving the other variables unchanged. The complete jump map is given by

$$G(\xi, w) := \bigcup_{i\in\mathcal{N}} G_i(\xi, w). \tag{4.10}$$

For future use we also define the jump map for $(\chi, w) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathcal{W} \times \mathcal{W}$ as

$$G_\chi(\chi, w) := \bigcup_{i\in\mathcal{N}} G_{\chi,i}(\chi, w). \tag{4.11}$$

Because of the selected state variables, system (4.1) does not depend on the time-derivative of w as in [3, 4], which allows us to work under more general and more natural assumptions on the measurement noise.

The goal is to design the dynamics of η_i, $\mathcal{C}_i$ and $\mathcal{D}_i$, i.e., the functions Ψ_i, for all $i \in \mathcal{N}$, such that a given set $\mathcal{A}$ is ISpS, see Definition 2.4. To formalize objective (ii) stated at the end of Chapter 3, we introduce, for any solution ξ to $\mathcal{H}$ for $w \in \mathcal{W}$ and $i \in \mathcal{N}$, the set

$$\begin{aligned} \mathcal{T}_i(\xi, w) := \Big\{(t, j) \in \operatorname{dom}\xi \mid (\xi(t, j), w(t)) \in \mathcal{D}_i \text{ and} \\ (\xi(t, j+1), w(t)) \in G_i(\xi(t, j), w(t))\Big\}. \end{aligned} \tag{4.12}$$

Hence, $\mathcal{T}^i(\xi, w)$ contains all hybrid times belonging to the hybrid time domain of a solution ξ to $\mathcal{H}$ for $w \in \mathcal{W}$ at which a jump occurs due to triggering condition i ($\mathcal{D}_i$ and G_i). We introduce the following definition.

Definition 4.1. *Given a closed set $\mathcal{A} \subset \mathbb{R}^{2n} \times \mathcal{W}$, system* (4.1) *has a* semi-global individual minimum inter-event time (SGiMIET) *with respect to $\mathcal{A}$, if, for all $\Delta \geqslant 0$ and all $i \in \mathcal{N}$, there exists a $\tau^i_{\text{MIET}} > 0$ such that ξ is a solution of $\mathcal{H}$ for any $w \in \mathcal{W}$ with $|\xi(0,0)|_{\mathcal{A}} \leqslant \Delta$, for all $(t, j), (t', j') \in \mathcal{T}_i(\xi, w)$,*

$$t + j < t' + j' \Rightarrow t - t' \geqslant \tau^i_{\text{MIET}}. \tag{4.13}$$

If τ^i_{MIET} can be chosen independently of Δ for all $i \in \mathcal{N}$, then we say that $\mathcal{H}$ has a global individual minimum inter-event time (GiMIET)*.*

Definition 3 means that the (continuous) time between two successive transmission instants due to a trigger of condition i are spaced by at least τ^i_{MIET} units of time, and that τ^i_{MIET} depends on the size of the initial conditions. Hence, the problem formulation at the end of Section II can be formally stated as for a given set $\mathcal{A}$, synthesize the sets $\mathcal{C}_i$ and $\mathcal{D}_i$, $i \in \mathcal{N}$ such that $\mathcal{A}$ is ISpS and $\mathcal{H}$ has a SGiMIET w.r.t. $\mathcal{A}$.

4.2 Design and analysis

We assume that controllers C_i, $i \in \mathcal{N}$, are designed such that Assumption 4.1 below holds. We show in Chapter 5 how this assumption is naturally obtained for specific scenarios.

Assumption 4.1. *There exist $\alpha, \underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}$, $\beta_i \in \mathcal{K}$ and $\delta_i : \mathbb{R}^{n_o^i} \to \mathbb{R}_{\geqslant 0}$ continuous for all $i \in \mathcal{N}$, a closed non-empty set $\mathcal{A}$ and a continuously differentiable function $V : \mathbb{R}^{2n} \times \mathcal{W} \to \mathbb{R}_{\geqslant 0}$ such that*

i) for any $\chi \in \mathbb{R}^{2n} \times \mathcal{W}$

$$\underline{\alpha}(|\chi|_{\mathcal{A}}) \leqslant V(\chi) \leqslant \overline{\alpha}(|\chi|_{\mathcal{A}}), \tag{4.14}$$

ii) for all $\chi \in \mathbb{R}^{2n} \times \mathcal{W}$ and $w \in \mathcal{W}$,

$$\langle \nabla V(\chi), F_\chi(\chi) \rangle \leqslant -\alpha(|\chi|_{\mathcal{A}}) + \gamma(|w|) + \sum_{i \in \mathcal{N}} \beta_i(|\widetilde{e}_i|) - \delta_i(o_i), \tag{4.15}$$

iii) for any $\chi \in \mathbb{R}^{2n} \times \mathcal{W}$, $w \in \mathcal{W}$ and $g \in G_\chi(\chi, w)$,

$$V(g) - V(\chi) \leqslant 0, \tag{4.16}$$

iv) for any $\Delta > 0$, there exists $M_\Delta \geqslant 0$ such that for any $\chi \in \mathbb{R}^{2n} \times \mathcal{W}$ satisfying $|\chi|_{\mathcal{A}} \leqslant \Delta$,

$$|F_\chi(\chi)| \leqslant M_\Delta. \tag{4.17}$$

Assumption 4.1 imposes Lyapunov conditions on the χ-system. Item i) means that V is positive definite and radially unbounded with respect to $\mathcal{A}$. Item (ii) is an input-to-state stability property of set $\mathcal{A}$ for the flow dynamics, but not the desired one as it involves the error $\widetilde{e}_i$. Item iii) implies that the Lyapunov function does not increase at jumps and item iv) imposes boundedness conditions on f_i and k_i. Assumption 4.1 implies that, in the absence of a digital network (and thus, $\widetilde{e}_i = 0$ and $\widehat{w} = w$), the set $\mathcal{A}$ is input-to-state stable with respect to input w. Again, examples of systems verifying Assumption 4.1 are provided in Chapter 5.

The next theorem explains how to design Ψ_i, $i \in \mathcal{N}$, arising in the flow map, and the flow and jump set definitions to ensure the desired objectives are met.

Theorem 4.1. *Consider system* (4.1) *and suppose Assumptions 3.1 and 4.1 hold. We define for all $i \in \mathcal{N}$, $\xi \in \mathbb{X}$ and $w \in \mathcal{W}$*

$$\Psi_i(o_i) := \delta_i(o_i) - \beta_i(|\widetilde{e}_i|) - \epsilon_i \eta_i + c_i, \tag{4.18}$$

with $c_i > \beta_i(2\overline{w}_i)$ and $\epsilon_i \in \mathbb{R}_{>0}$ tuning parameters. The set $\mathcal{A}^d := \{\xi : \chi \in \mathcal{A} \text{ and } \eta = 0\}$ is ISpS and system (4.1) *has a SGiMIET.*

Theorem 4.1 provides the expressions of Ψ_i, $i \in \mathcal{N}$, which ensure that ISS of set $\mathcal{A}$ guaranteed by Assumption 4.1 in the absence of network is approximately preserved in the presence of the digital network. Moreover, the existence of a strictly positive lower-bound on the inter-event time of each triggering mechanism is guaranteed. The interest of Theorem 4.1 lies in its simplicity, generality and in revealing the main concepts as a “prescriptive framework.”

The expression of Ψ_i in (4.18) is based on so-called space-regularization, as by introducing c_i, we enlarge the flow set to ensure the existence of a SGiMIET. While space-regularization is well known in the hybrid systems literature and has been used under different forms in event-triggered control [2, 12, 19, 20], we have to be careful when designing c_i, because a priori the non-Zenoness only holds if c_i satisfies the condition mentioned in Theorem 4.1.

Remark 4.1. *We would like to note that in [2, Remark V.3], the same lower-bound for the space-regularization constant has been derived based on their trigger style, however, the results that we obtain here are more general. Indeed if β_i is the identity function as in [2], we recover the lower-bound $c_i > 2\overline{w}_i$.*

The consequence of $c_i > \beta_i(2\overline{w}_i)$ is that we obtain practical stability, i.e., the constant d in (2.3) will be non-zero, see Remark 4.3 below for more details. On the other hand, Theorem 4.1 does not require to make assumptions on the differentiability of w_i, and a fortiori on boundedness properties of $\dot{w}_i$, as in various works in ETC considering measurement noise, see, e.g., [3, 4]. Additionally, we may exploit the structure present in specific scenarios or ETC mechanisms to obtain less conservative bounds for the parameters c_i and, in some cases, a GiMIET, as opposed to semi-global one in Theorem 4.1, as will be illustrated in Chapter 5.

Proof of Theorem 4.1. The first part of the proof consists of showing that the conditions of Proposition 2.2 hold. To this end, we introduce a Lyapunov candidate U, defined for all $\xi \in \mathbb{X}$ as

$$U(\xi) := V(\chi) + \sum_{i\in\mathcal{N}} \eta_i. \tag{4.19}$$

Lyapunov conditions. Due to item i) of Assumption 4.1, there exist class-$\mathcal{K}_\infty$ functions α_1, α_2 such that $\alpha_1(|\xi|_{\mathcal{A}^d}) \leqslant U(\xi) \leqslant \alpha_2(|\xi|_{\mathcal{A}^d})$ for all $\xi \in \mathcal{C} \cup \mathcal{D}$, and thus item 1) of Proposition 2.2 holds. Next, let $(\xi, w) \in \mathcal{C}$, in view of (4.15) and (4.18),

$$\begin{aligned}
\langle \nabla U(\xi), F(\xi, w))\rangle &= \langle \nabla V(\chi), F_\chi(\chi))\rangle + \sum_{i\in\mathcal{N}} \Psi_i(o_i) \\
&\leqslant -\alpha(|\chi|_{\mathcal{A}}) + \gamma(|w|) + \sum_{i\in\mathcal{N}} \beta_i(|\widetilde{e}_i|) - \delta_i(o_i) + \Psi_i(o_i) \\
&= -\alpha(|\chi|_{\mathcal{A}}) + \gamma(|w|) + \sum_{i\in\mathcal{N}} c_i - \epsilon_i \eta_i \\
&\leqslant -\alpha^d(|\xi|_{\mathcal{A}^d}) + \gamma(|w|) + c
\end{aligned} \tag{4.20}$$

with $c := \sum_{i\in\mathcal{N}} c_i$ and for some $\alpha^d \in \mathcal{K}_\infty$. Hence, item 2) holds. Since $\eta^+ = \eta$ and due to item iii) of Assumption 4.1, we note that for any $(\xi, w) \in \mathcal{D}$ and all $g \in G(\xi, w)$,

$$U(g) - U(\xi) \leqslant 0, \tag{4.21}$$

thus, item 3) also holds. Hence, we are left with proving that $\mathcal{H}$ is persistently flowing.

Completeness of maximal solutions. To prove that all maximal solutions are complete we will use Proposition 2.1. To this end, we first have to prove that there exists a non-trivial solution by showing that (VC) holds for all $(\xi, w) \in \mathcal{C} \setminus \mathcal{D}$. Note that due to $\mathcal{C}$ being closed, $\overline{\mathcal{C}} = \mathcal{C}$. Let $t_1 > t_0 = 0$ denote the time at which the next discontinuity in w occurs. By definition, w is continuous on $[0, t_1)$. Suppose that $(\xi, w(0))$ is in the interior of $\mathcal{C} \setminus \mathcal{D}$. Then, due to the continuity of both w and ξ, we have suitable continuity properties on z for local existence of solutions such that (VC) is satisfied.

In view of (4.4) and (4.6), for any $(\xi, w(0))$ not in the interior of $\mathcal{C} \setminus \mathcal{D}$, we can distinguish four cases, which may or may not hold simultaneously:

1) $|\widehat{w}_i| = \overline{w}_i$ for some $i \in \mathcal{N}$,
2) $\eta_i + \theta_i \Psi_i(o_i) = 0$ and $\Psi_i(o_i) > 0$ for some $i \in \mathcal{N}$,
3) $\eta_i = 0$ for some $i \in \mathcal{N}$,
4) $|w(0)| = \overline{w}_i$ for some $i \in \mathcal{N}$.

For item 1), we note that, in view of (4.3), $\frac{d}{dt}\widehat{w}_i = 0$, and (VC) holds for this case. For item 2), we note that $(\xi, w) \in \mathcal{C} \setminus \mathcal{D}$ implies that $\xi \in \mathbb{X}$, which implies $\eta_i \geqslant 0$. Hence, item 2) can only hold if $\eta_i = 0$ and $\theta_i = 0$. In view of (4.4), we note that $\dot{\eta}_i = \Psi_i(o_i) > 0$. Due to continuity of ξ and w, (VC) holds. For item 3), we can follow a similar argument as before since, if $(\xi, w) \in \mathcal{C} \setminus \mathcal{D}$, $\Psi_i(o_i)$ has to be strictly positive if $\eta_i = 0$. Hence, due to $\dot{\eta}_i = \Psi_i(o_i) > 0$ and due to the continuity of both w and ξ, (VC) holds for this case. For item 4), (VC) holds trivially due to Assumption 3.1. Thus, there exists a non-trivial solution ξ to $\mathcal{H}$ for all $w \in \mathcal{W}$ such that $(\xi, w) \in \mathcal{C} \cup \mathcal{D}$, and all maximal solutions satisfy exactly one of three cases (a), (b) or (c) in Proposition 2.1.

Item (b) cannot occur for the following reason. Suppose that for a maximal solution, $\operatorname{dom}\xi$ is bounded. As items 1)-3) of Proposition 2.2 hold, there exist $\theta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that $|\xi(t,j)|_\mathcal{A} \leqslant \theta(|\xi(0,0)|_\mathcal{A}, t) + \gamma(c)$ for all $(t,j) \in \operatorname{dom}\xi$. Let $\mu > 0$, and suppose $\xi(0,0)$ satisfies $|\xi(0,0)|_\mathcal{A} \leqslant \mu$. Then, for all $(t,j) \in \operatorname{dom}\xi$, $|\xi(t,j)|_\mathcal{A} \leqslant \beta(\mu, 0) + \gamma(c) =: \Delta$ and $|F_\chi(\chi(t,j))| \leqslant M_\Delta$ for some $M_\Delta \geqslant 0$ in view of item iv) of Assumption 4.1. Since the dynamics of x, e and $\widehat{w}$ are bounded (i.e. F_χ is bounded), the dynamics of η are also bounded. Consequently, since ξ is bounded and does not have a finite escape time, we can close the right-open interval. Item (c) only occurs if either $G(\mathcal{D}) \not\subset \mathcal{C} \cup \mathcal{D}$ or if $(\xi, w) \notin \mathcal{C} \cup \mathcal{D}$ due to a discontinuity in w. For the former, we note that $\mathcal{C} \cup \mathcal{D} = \mathbb{X} \times \mathcal{W}$. In view of (4.9), we note that $\widehat{w}_i^+ = w_i$ if i broadcasts its state and $\widehat{w}_i^+ = \widehat{w}_i$ otherwise. Additionally, $\eta_i^+ = \eta_i$. Consequently, $G(\mathcal{D}) \subset \mathcal{C} \cup \mathcal{D}$. Furthermore, since $\mathcal{C} \cup \mathcal{D} = \mathbb{X} \times \mathcal{W}$, item (c) cannot occur due to a discontinuity in signal w, since by Assumption 3.1, $w \in \mathcal{W}$. Thus we deduce that all maximal solutions ξ to the hybrid system $\mathcal{H}$ are complete for any $w \in \mathcal{W}$. Now that we have proved that any maximal solution is complete, we show that maximal solutions are also t-complete.

Semi-global individual minimum inter-event time. We prove t-completeness by showing that system (4.1) has the SGiMIET property. To this end, we examine the time between two successive jumps generated by triggering condition $i \in \mathcal{N}$. Recall that we trigger when $\eta_i + \theta_i \Psi_i(o_i) \leqslant 0$ and $\Psi_i(o_i) \leqslant 0$. By [11, Prop. 2.3], we know that, after a first triggering instant has occurred,

$$\delta_i(o_i) + c_i - \beta_i(|\widetilde{e}_i|) \leqslant 0 \text{ and } \eta_i \geqslant 0 \tag{4.22}$$

is always satisfied before $\eta_i + \theta_i \Psi_i(o_i) \leqslant 0$, $\Psi_i(o_i) \leqslant 0$ and $\eta_i \geqslant 0$ is. Hence, we can analyze when (4.22) holds to obtain a lower-bound for the inter-event times. Since δ_i takes non-negative values, we can under-estimate the inter-event times for triggering condition i by analyzing when

$$c_i = \beta_i(|\widetilde{e}_i|). \tag{4.23}$$

Rewriting this, we obtain the condition

$$\beta_i^{-1}(c_i) = |\widetilde{e}_i|. \tag{4.24}$$

Note that we can upper-bound the right-hand side of (4.24) as, in view of Assumption 1,

$$\beta_i^{-1}(c_i) = |\widetilde{e}_i| \leqslant |e_i| + |\widehat{w}_i| + |w_i| \leqslant |e_i| + 2\overline{w}_i. \tag{4.25}$$

Hence, we can under-estimate the triggering times by analyzing when

$$\beta_i^{-1}(c_i) - 2\overline{w}_i = |e_i|. \tag{4.26}$$

Recall that, by the condition on c_i in Theorem 4.1, we have $c_i > \beta_i(2\overline{w}_i)$, thus, the left-hand side of (4.26) is always positive. In view of (4.26), we define

$$\overline{c}_i := \beta_i^{-1}(c_i) - 2\overline{w}_i > 0. \tag{4.27}$$

Since $|e_i|$ is 0 after a transmission due to triggering rule i, the inter-event time for triggering rule i is lower bounded by the time it takes for $|e_i|$ to grow from 0 to $\overline{c}_i$ in view of (4.26). Note that the bound in (4.27) is *not* dependent on actual values of w_i, only on the upper-bounds presented in $w_i \in \mathcal{W}_i$, $i \in \mathcal{N}$. In the following, we provide a lower-bound on this inter-event time. Let $\mu > 0$ and consider (ξ, w) such that $|\xi(0,0)|_{\mathcal{A}^d} \leqslant \mu$. Note that by (4.20), (4.21) and the satisfaction of item (i) of Assumption 4.1, $|\xi(t,j)|_{\mathcal{A}^d} \leqslant \Delta$ for some $\Delta > 0$ (dependent on μ but not on $|\xi(0,0)|_{\mathcal{A}^d}$) and any $(t,j) \in \operatorname{dom}\xi$. Hence, in view of item iv) of Assumption 4.1, $|F_\chi(\chi(t,j))| \leqslant M_\Delta$. Thus, for almost all $j \in \mathbb{N}_{\geqslant 0}$ and almost all $t \in I^j$ where $I^j = \{t : (t,j) \in \operatorname{dom}\xi\}$, $\frac{d|e_i(t)|}{dt} \leqslant M_\Delta$. Consequently, the time between any two transmissions generated by triggering rule i is larger than or equal to $\overline{c}_i / M_\Delta$. Hence, $\mathcal{H}$ has the SGiMIET property and thus solutions are persistently flowing.

Since the system is persistently flowing, we also have that $\mathcal{H}$ is ISpS w.r.t. the set $\mathcal{A}^d$. ■

We can derive similar results when the triggering conditions are static, i.e., when no variable η_i is introduced to define the transmission instants. In this case, we obtain the hybrid system $\mathcal{H}^s$ defined as

$$\begin{aligned} \dot{\chi} &= F_\chi(\chi), & (\chi, w) &\in \mathcal{C}^s, \\ \chi^+ &\in G_\chi(\chi, w), & (\chi, w) &\in \mathcal{D}^s, \end{aligned} \tag{4.28}$$

where

$$\mathcal{C}^s := \bigcap_{i \in \mathcal{N}} \mathcal{C}_i^s, \quad \mathcal{D}^s := \bigcup_{i \in \mathcal{N}} \mathcal{D}_i^s \tag{4.29}$$

with the sets $\mathcal{C}_i^s, \mathcal{D}_i^s$ as

$$\begin{aligned} \mathcal{D}_i^s &:= \{(\chi, w) \in \mathbb{R}^{2n} \times \mathcal{W} \times \mathcal{W} \mid \Psi_i^s(o_i) \leqslant 0\}, \\ \mathcal{C}_i^s &:= \{(\chi, w) \in \mathbb{R}^{2n} \times \mathcal{W} \times \mathcal{W} \mid \Psi_i^s(o_i) \geqslant 0\}, \end{aligned} \tag{4.30}$$

where $\Psi_i^s(o_i)$ is a static triggering condition, which is designed according to the next result.

Corollary 4.1. *Consider system* (4.28) *and suppose Assumptions 1 and 2 hold. We define for all $i \in \mathcal{N}$, $\chi \in \mathbb{R}^{2n} \times \mathcal{W}$ and $w \in \mathcal{W}$*

$$\Psi_i^s(o_i) := \delta_i(o_i) + c_i - \beta_i(|\widetilde{e}_i|) \tag{4.31}$$

with $c_i > \beta_i(2\overline{w}_i)$ tuning parameters. The set $\mathcal{A}$ is ISpS and system (4.28) *has a SGiMIET.*

The proof of Corollary 4.1 follows similar steps as the proof of Theorem 4.1, and is therefore omitted.

Remark 4.2. *Assumption 4.1 and Corollary 4.1 allow us to consider the case where any δ_i is equal to zero. In this case, Ψ_i^s is given by $\Psi_i^s(o_i) := c_i - \beta_i(|\widetilde{e}_i|)$, with $c_i > \beta_i(2\overline{w}_i)$ tuning parameters. Note that triggering conditions of this form are often called absolute triggering conditions in the event-triggered control literature, see e.g. [2, 19, 20].*

Remark 4.3. *The parameters c_i in Theorem 4.1 and Corollary 4.1 are directly related to the constant d in the ISpS definition (2.3). Note that (2.3) holds with $d = \theta(c)$ for some $\theta \in \mathcal{K}_\infty$, where $c = \sum_{i\in\mathcal{N}} c_i$. Hence, for a tighter ultimate bound on $|\xi(t,j)|_{\mathcal{A}^d}$, we require that c_i is as small as possible. Note, however, that due to Theorem 4.1, c_i is lower-bounded by $\beta_i(2\overline{w}_i)$, and thus the minimum value of d is $d_{\min} = \theta(\sum_{i\in\mathcal{N}} \beta_i(2\bar{w}_i))$ to ensure proper SGiMIET and ISpS properties. On the other hand, selecting a small c_i implies a small lower-bound on the SGiMIET in view of (4.27). Hence, there exists a trade-off between large lower-bounds on the inter-event times and "asymptotic closeness" to $\mathcal{A}^d$ in terms of d, which is tunable through selection of c_i, $i \in \mathcal{N}$.*

Chapter 5

Case studies

In this section, we investigate several existing event-triggering techniques in the literature and show how to modify these to handle measurement noise. We want to stress that we just took a non-exhaustive sample of a few well-known techniques, but many more can be handled by our general framework. We prove for this purpose that Assumption 4.1 is verified, which allows to directly apply Theorem 4.1 or Corollary 4.1.

5.1 Stabilization of a single system

In this section we aim to stabilize a single system, similarly to [10, 11]. First we treat the general non-linear case, and hereafter we will provide some additional insights for the linear case, which are not necessarily true in the general setting.

5.1.1 Non-linear stabilization of the origin

A single plant P and a single controller C are considered here. In particular, the plant is given by

$$\dot{x} = f(x, u) \tag{5.1}$$

and the feedback controller by

$$u = k(x). \tag{5.2}$$

As in [10, 11], we assume that the following properties hold.

Assumption 5.1. *Maps f and k are Lipschitz continuous on compacts. Additionally, there exist $\underline{\alpha}, \overline{\alpha}, \alpha, \gamma \in \mathcal{K}_\infty$ and a continuously differentiable Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying, for any $x, v \in \mathbb{R}^n$,*

$$\begin{aligned} \underline{\alpha}(|x|) \leqslant V(x) \leqslant \overline{\alpha}(|x|), \\ \langle \nabla V(x), f(x, k(x+v)) \rangle \leqslant -\alpha(|x|) + \gamma(|v|), \end{aligned} \tag{5.3}$$

implying that the origin of $\dot{x} = f(x, k(x+v))$ is ISS with respect to v.

We derive the following result from Assumption 5.1.

Proposition 5.1. *Consider system* (5.1) *with controller* (5.2) *and suppose Assumption 5.1 holds. Then all conditions of Assumption 4.1 are met for $\mathcal{A} = \{\chi : x = \mathbf{0}\}$ with $\beta(s) = \gamma(2s)$ for any $s \geqslant 0$, $\delta(o) = \sigma\alpha(\frac{1}{2}|\widetilde{x}|)$ for any $\widetilde{x} \in \mathbb{R}^n$ and V as in Assumption 5.1, with $\sigma \in (0, 1)$ a tuning parameter.*

Proposition 5.1 implies that, for any bounded measurement noise as defined by Assumption 3.1, the triggers defined in Theorem 4.1 and Corollary 4.1 render the origin of the closed-loop system ISpS with the SGiMIET property.

As a result, we have redesigned the triggering conditions of [10, 11] to be applicable in presence of measurement noises.

Proof. By Assumption 5.1, item i) of Assumption 4.1 holds trivially. Let $x, e, \widehat{w} \in \mathbb{R}^{2n} \times \mathcal{W}$. In view of Assumption 3, by substituting v with $e + \widehat{w}$, we obtain

$$\langle \nabla V(x), f(x, k(x + e + \widehat{w})) \rangle \leqslant -\alpha(|x|) + \gamma(|e + \widehat{w}|). \tag{5.4}$$

By using (3.6), i.e., $e + \widehat{w} = \widetilde{e} + w$, we obtain

$$\langle \nabla V(x), f(x, k(x + e + \widehat{w})) \rangle \leqslant -\alpha(|x|) + \gamma(|\widetilde{e} + w|). \tag{5.5}$$

Next, we use the weak triangle inequality, $\gamma(a + b) \leqslant \gamma(2a) + \gamma(2b)$, see [21], to obtain

$$\langle \nabla V(x), f(x, k(x + e + \widehat{w})) \rangle \leqslant -\alpha(|x|) + \gamma(2|\widetilde{e}|) + \gamma(2|w|).$$

Then, for any $\sigma \in (0, 1)$,

$$\begin{aligned}\langle \nabla V(x), f(x, k(x + e + \widehat{w})) \rangle &\leqslant -(1-\sigma)\alpha(|x|) - \sigma\alpha(|x|) - \sigma\alpha(|w|) + \sigma\alpha(|w|) \\ &\quad + \gamma(2|\widetilde{e}|) + \gamma(2|w|) \\ &\leqslant -(1-\sigma)\alpha(|x|) - \sigma\alpha\left(\tfrac{1}{2}(|x| + |w|)\right) + \gamma(2|\widetilde{e}|) \\ &\quad + \gamma(2|w|) + \sigma\alpha(|w|) \\ &\leqslant -(1-\sigma)\alpha(|x|) - \sigma\alpha\left(\tfrac{1}{2}|\widetilde{x}|\right) + \gamma(2|\widetilde{e}|) \\ &\quad + \gamma(2|w|) + \sigma\alpha(|w|) \\ &\leqslant -(1-\sigma)\alpha(|x|) + \zeta(|w|) + \gamma(2|\widetilde{e}|) - \sigma\alpha(\tfrac{1}{2}|\widetilde{x}|),\end{aligned} \tag{5.6}$$

for some $\zeta \in \mathcal{K}$, hence item ii) of Assumption 4.1 holds. Since the Lyapunov function V does not depend on e or $\widehat{w}$, for all $(\xi, w) \in \mathcal{D}$ and $g \in G(\xi, w)$, $V(g) - V(\xi) = 0$, and item iii) holds. Since f and k are Lipschitz continuous on compacts, for any $|\xi| \leqslant \Delta$ there exists a constant $L > 0$ such that $|f(x, k(x+e+\widehat{w}))| \leqslant L|\xi| = L\Delta$ and item iv) holds. ■

5.1.2 The linear single-system case

If the results are tailored to linear dynamics, we can assume stronger properties with respect to the MIET. In particular, we obtain a GiMIET for any linear system.

Consider any linear system with dynamics

$$\dot{x} = Ax + Bu, \tag{5.7}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and matrices A, B of appropriate dimensions, and with a linear feedback

$$u = Kx, \tag{5.8}$$

with K of appropriate dimensions. We assume the linear feedback renders the closed-loop system globally asymptotically stable in the absence of a network. In this case, there exists a quadratic ISS-Lyapunov function with respect to additive measurement errors $v \in \mathbb{R}^n$. For such a quadratic Lyapunov function, there exist constant $\underline{a}, \overline{a}, a, b \in \mathbb{R}_{>0}$ such that, for any $x, v \in \mathbb{R}^n$,

$$\begin{gathered}\underline{a}|x|^2 \leqslant V(x) \leqslant \overline{a}|x|^2, \\ \langle \nabla V(x), (A + BK)x + BKv \rangle \leqslant -a|x|^2 + b|v||x|.\end{gathered} \tag{5.9}$$

The following result will illustrate the stronger properties with respect to the general, non-linear case.

Proposition 5.2. *Consider system* (5.7) *with a stabilizing controller* (5.8). *For this system, all conditions of Assumption 4.1 are met with* $\beta(s) = \frac{1}{2d}bs^2$ *for any* $s \geqslant 0$ *and* $\delta(o) = \frac{1}{2}\sigma(a-bd)|\widetilde{x}|^2$ *where* $d \in (0, \frac{a}{b})$ *and* $\sigma \in (0,1)$ *are tuning parameters. Moreover, for* $c \geqslant \frac{1}{2}\sigma(a-bd)|\overline{w}|^2 + \frac{1}{d}|2\overline{w}|^2$, *the hybrid system* $\mathcal{H}$ *has a global minimum inter-event time* τ_{MIET}.

Proof. Due to (5.9), item i) of Assumption 4.1 directly holds. By substituting $v = e + \widehat{w} = \widetilde{e} + w$ and by applying Young's inequality, we obtain

$$\begin{aligned}\langle \nabla V(x), K(x+e+\widehat{w})\rangle &\leqslant -a|x|^2 + b|e+\widehat{w}||x| = -a|x|^2 + b|\widetilde{e}+w||x| \\ &\leqslant -a|x|^2 + b|\widetilde{e}||x| + b|w||x| \\ &\leqslant -(a-bd)|x|^2 + \frac{1}{2d}b|\widetilde{e}|^2 + \frac{1}{2d}b|w|^2,\end{aligned} \tag{5.10}$$

for any $d \in (0, \frac{a}{b})$. Note that, due to Young's inequality, for any $p, q \in \mathbb{R}$ it holds that

$$\tfrac{1}{2}(p+q)^2 \leqslant \tfrac{1}{2}(p^2+q^2) + |p||q| \leqslant p^2 + q^2, \tag{5.11}$$

hence, for any $\sigma \in (0,1)$, we have

$$\begin{aligned}\langle \nabla V(x), K(x+e+\widehat{w})\rangle &\leqslant -(1-\sigma)(a-bd)|x|^2 - \sigma(a-bd)|x|^2 \\ &\quad - \sigma(a-bd)|w|^2 + \sigma(a-bd)|w|^2 + \frac{1}{2d}b|\widetilde{e}|^2 + \frac{1}{2d}b|w|^2 \\ &\leqslant -(1-\sigma)(a-bd)|x|^2 - \tfrac{1}{2}\sigma(a-bd)(|x|+|w|)^2 \\ &\quad + \sigma(a-bd)|w|^2 + \frac{1}{2d}b|\widetilde{e}|^2 + \frac{1}{2d}b|w|^2 \\ &\leqslant -(1-\sigma)(a-bd)|x|^2 + \sigma(a-bd)|w|^2 + \frac{1}{2d}b|w|^2 \\ &\quad - \tfrac{1}{2}\sigma(a-bd)|\widetilde{x}|^2 + \frac{1}{2d}b|\widetilde{e}|^2 \\ &\leqslant -\alpha(|x|) + \gamma(|w|) - \tfrac{1}{2}\sigma(a-bd)|\widetilde{x}|^2 + \frac{1}{2d}b|\widetilde{e}|^2\end{aligned} \tag{5.12}$$

for some $\alpha \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}$, hence, item ii) of Assumption 4.1 holds. Since $V(x)$ does not depend on e or $\widehat{w}$, item iii) holds trivially. Additionally, due to the linear dynamics, item iv) also holds trivially.

Next we prove the existence of a global minimum inter-event time. Note that the system is ISpS for any disturbance $w \in \mathcal{W}$, and we know that for every solution ξ to $\mathcal{H}$ for any $w \in \mathcal{W}$ there exists a forward invariant set $\mathcal{I} := \{\xi \in \mathbb{R}^{2n} \times \mathcal{W} \mid V(x) \leqslant \mu\}$. Hence, if at some time $(t,j) \in \operatorname{dom}\xi$, $\xi(t,j) \in \mathcal{I}$, then for all $(t',j') \in \operatorname{dom}\xi$ with $t+j \leqslant t'+j'$, $\xi(t',j') \in \mathcal{I}$. We pick the smallest μ for which the forward invariance condition holds. Recall from (4.22) that we can under-estimate the inter-event times by analyzing when

$$\frac{1}{2}\sigma(a-bd)|\widetilde{x}|^2 + c - \frac{1}{2d}|\widetilde{e}|^2 = 0. \tag{5.13}$$

Due to Young's inequality, for any $p, q \in \mathbb{R}$ it holds that

$$\begin{aligned}(p+q)^2 &= p^2 + q^2 + 2pq \geqslant p^2 + q^2 - 2|p||q| \\ &\geqslant p^2 + q^2 - \epsilon p^2 - \frac{1}{\epsilon}q^2\end{aligned} \tag{5.14}$$

for any $\epsilon > 0$. By picking $\epsilon = \frac{1}{2}$, we obtain $(p+q)^2 \geqslant \frac{1}{2}p^2 - q^2$. By using (5.11) and (5.14), for two vectors $r, s \in \mathbb{R}^n$ it holds that

$$\begin{aligned} |r+s|^2 &= \sum_{i=1}^{n}(r_i+s_i)^2 \geqslant \sum_{i=1}^{n} \frac{1}{2}r_i^2 - s_i^2 = \frac{1}{2}|r|^2 - |s|^2, \\ |r+s|^2 &= \sum_{i=1}^{n}(r_i+s_i)^2 \leqslant \sum_{i=1}^{n} 2r_i^2 + 2s_i^2 = 2|r|^2 + 2|s|^2. \end{aligned} \tag{5.15}$$

With these inequalities, we can bound $|\widetilde{x}|^2$ and $|\widetilde{e}|^2$ as

$$\begin{aligned} \frac{1}{2}|x|^2 - |\overline{w}|^2 \leqslant \frac{1}{2}|x|^2 - |w|^2 \leqslant |x+w|^2 = |\widetilde{x}|^2, \\ -2|e|^2 - 2|2\overline{w}|^2 \leqslant -2|e|^2 - 2|\widehat{w} - w|^2 \leqslant -|e + \widehat{w} - w|^2 = |\widetilde{e}|^2. \end{aligned} \tag{5.16}$$

We then obtain by substitution of (5.16) in (5.13) that

$$\frac{1}{4}\sigma(a-bd)|x|^2 - \frac{1}{d}|e|^2 + c - \frac{1}{2}\sigma(a-bd)|\overline{w}|^2 - \frac{1}{d}|2\overline{w}|^2 \leqslant \frac{1}{2}\sigma(a-bd)|\widetilde{x}|^2 + c - \frac{1}{2d}|\widetilde{e}|^2. \tag{5.17}$$

By selecting c such that

$$c \geqslant \frac{1}{2}\sigma(a-bd)|\overline{w}|^2 + \frac{1}{d}|2\overline{w}|^2, \tag{5.18}$$

we can under-estimate the inter-event times by analyzing when

$$\frac{1}{4}\sigma(a-bd)|x|^2 - \frac{1}{d}|e|^2 = 0. \tag{5.19}$$

We split the global minimum inter-event time analysis in two parts. Let ξ be a hybrid solution to $\mathcal{H}$ for some w, $(t,j) \in \operatorname{dom}\xi$ and $0 = t_0 \leqslant t_1 < \ldots < t_{j+1} = t$ satisfy

$$\operatorname{dom}\xi \cap ([0,t] \times \{0,\ldots,j\}) = \bigcup_{i\in\{0,\ldots,j\}} [t_i, t_{i+1}] \times \{i\}. \tag{5.20}$$

First, we analyze the inter-event times when $\xi(t,j)$ is not in $\mathcal{I}$, i.e., the restriction of ξ to the domain ϕ_1 defined as

$$\phi_1 := \bigcup_{i\in\{0,\ldots,j\}} \{[t_i, t_{i+1}] \times \{i\}\} \qquad \text{if } \xi(t_{i+1}, i) \notin \mathcal{I}. \tag{5.21}$$

For any solution ξ to the hybrid system $\mathcal{H}$ for some w, we consider t_i, t_{i+1} such that $(t_i, i), (t_{i+1}, i+1) \in \phi_1$ with $i \geqslant 1$. We can under-estimate the inter-event times $t_{i+1} - t_i$ by checking when (5.19) holds. Note that this condition can be rewritten to

$$\frac{d}{4}\sigma(a-bd) = \frac{|e|^2}{|x|^2}, \tag{5.22}$$

which, since the left-hand side is positive, can be reformulated as

$$\sqrt{\frac{d}{4}\sigma(a-bd)} = \frac{|e|}{|x|} \tag{5.23}$$

Following the reasoning of [10], we have that for all $(t, j) \in \phi_1$ it holds that $V(x) > \mu$, and, in view of the first item of (5.9), $|x|^2 \geqslant \overline{a}^{-1}V(x) > \overline{a}^{-1}\mu$.

$$\begin{aligned}\frac{d}{dt}\frac{|e|}{|x|} &\leqslant \left(1+\frac{|e|}{|x|}\right)\frac{|\dot{x}|}{|x|} \\ &\leqslant \left(1+\frac{|e|}{|x|}\right)\frac{|A+BK||x|+|BK||e|+|BK||\widehat{w}|}{|x|} \\ &= \left(1+\frac{|e|}{|x|}\right)\left(|A+BK|+|BK|\frac{|e|}{|x|}+|BK|\frac{|\widehat{w}|}{|x|}\right) \\ &\leqslant \left(1+\frac{|e|}{|x|}\right)\left(|A+BK|+|BK|\frac{|e|}{|x|}+|BK|\overline{w}\sqrt{\frac{\overline{a}}{\mu}}\right) \\ &\leqslant \alpha+\beta\gamma+(\alpha+\beta\,(1+\gamma))\,\frac{|e|}{|x|}+\beta\left(\frac{|e|}{|x|}\right)^2\end{aligned} \tag{5.24}$$

with $\alpha := |A + BK|$, $\beta := |BK|$ and $\gamma := \overline{w}\sqrt{\frac{\overline{a}}{\mu}}$. Then, for all $(t_i, i), (t_{i+1}, i + 1) \in \phi_1$, the inter-event times $t_{i+1} - t_i$ are lower-bounded by the time $\tau_{\text{MIET},1}$ satisfying

$$\psi(\tau_{\text{MIET},1}, 0) = \sqrt{\frac{d}{4}\sigma(a - bd)} \tag{5.25}$$

where $\psi(t, \psi_0)$ is the solution to

$$\dot{\psi} = \alpha + \beta\gamma + (\alpha + \beta\,(1 + \gamma))\,\psi + \beta\psi^2. \tag{5.26}$$

Next, we analyze all $(t_i, i), (t_{i+1}, i + 1) \in \phi_2$ with $i \geqslant 1$, where ϕ_2 is defined as

$$\phi_2 := \bigcup_{i\in\{0,\ldots,j\}} \{[t_i, t_{i+1}] \times \{i\}\} \qquad \text{if } \xi(t_{i+1}, i) \in \mathcal{I}. \tag{5.27}$$

Note that $\mathcal{I}$ is a compact set, and hence, by extension and due to the triggering condition, there exists a compact set $\mathcal{J}$ such that for all $(t, j) \in \phi_2$, $\chi(t, j) \in \mathcal{J}$. Due to the compactness of the set $\mathcal{J}$, we can use the results of Theorem 4.1 to obtain a MIET for all $(t, j) \in \phi_2$, i.e., $\tau_{\text{MIET},2} := \overline{c}/M_\Delta$. We take the minimal inter-event time for $\phi_1 \cup \phi_2$ as

$$\tau_{\text{MIET}} := \min(\tau_{\text{MIET},1}, \tau_{\text{MIET},2}). \tag{5.28}$$

Then, for any $(t, j) \in \operatorname{dom}\phi_1 \cup \phi_2$, it holds that $t_{i+1} - t_i \geqslant \tau_{\text{MIET}}$ for all $i \geqslant 1$. By noting that $\operatorname{dom}\phi_1 \cup \phi_2 = \operatorname{dom}\xi$, we prove that τ_{MIET} is a global minimum-inter event time for any solution ξ to the hybrid system $\mathcal{H}$ for some w. ■

5.2 Consensus for multi-agent systems

A specific field of interest for ETC is consensus of multi-agent systems. We study several event-triggering control schemes in this context next. We focus here on single integrator systems, where each plant P_i, which we call agent in this section, has single integrator dynamics, i.e., $\dot{x}_i = u_i$, with $x_i, u_i \in \mathbb{R}$. However, the ideas of this work apply in more general settings.

For a network topology described by a connected weight-balanced digraph $\mathcal{G}$ with Laplacian L, it is known that agents achieve consensus when the control law

$$u_i = \sum_{j \in \mathcal{V}_i^{\text{out}}} (x_i - x_j), \tag{5.29}$$

with $\mathcal{V}_i^{\text{out}}$ the out-neighbors of agent i, is applied, see [22]. In vector notation, this is written as $u = -Lx$, where $u = (u_1, u_2, \ldots, u_N)$. We use the noisy sampled states for each agent instead of the actual states, resulting in the control law

$$u = -L(x + e + \widehat{w}). \tag{5.30}$$

Hence, the closed-loop system dynamics are

$$\dot{x} = -Lx - Le - L\widehat{w}, \tag{5.31}$$

which results in the dynamics for the hybrid system as

$$F_\chi(\chi) = (-Lx - Le - L\widehat{w}, Lx + Le + L\widehat{w}, \mathbf{0}_N). \tag{5.32}$$

We are interested in stability properties of the consensus set

$$\mathcal{A} := \left\{ \chi \in \mathbb{R}^{2n} \times \mathcal{W} \mid x_1 = x_2 = \ldots = x_N \right\}. \tag{5.33}$$

We show that our results extend the works of [15] and [9], to render the ETC schemes robust to measurement noise.

5.2.1 Decentralized strategy for undirected graphs

We consider a similar triggering style as in [15] first. For this case we consider an undirected, connected graph. This event generator is of particular interest, since the original paper does *not* have a non-Zeno proof, as also noted in [7]. By applying our results, we can design two robust triggers, one static and one dynamic, that have the SGiMIET property and thus no Zeno behavior.

The proposition below contains the functions required to design a trigger for (5.31) such that Assumption 4.1 holds.

Proposition 5.3. *Assumption 4.1 holds for F_χ defined as in* (5.32) *and G_χ as in* (4.11) *with $\beta_i(s) = \frac{1}{a} N_i s^2$ and $\delta_i(o_i) = \sigma_i(1 - 2aN_i)u_i^2$, where N_i denotes the number of neighbors of agent i and $a \in (0, \frac{1}{2N_i})$, $\sigma_i \in (0, 1)$ are tuning parameters.*

Proposition 5.3 implies that, for any bounded measurement noise as defined by Assumption 1, the triggers defined in Theorem 4.1 and Corollary 4.1 render the hybrid system (4.1) ISpS w.r.t. $\mathcal{A}^d$ with the SGiMIET property.

Proof. First we note that due to the undirected graph, $L^\top = L$. We introduce the Lyapunov function $V(\chi) = x^\top Lx$ for any $\chi \in \mathbb{R}^{2n} \times \mathcal{W}$. For this Lyapunov function, item i) of Assumption 4.1 holds (see [9, Lemma 1]). Additionally, items iii) and iv) hold trivially (the latter because the dynamics are linear). We are left with proving that we can obtain the form of item ii). To this end, note that we can write the derivative of V along flow as, for any $\chi \in \mathbb{R}^{2n} \times \mathcal{W}$,

$$\begin{aligned}\langle \nabla V(\chi), F_\chi(\chi)\rangle &= -(x+e+\widehat{w})^\top L^\top L x \\ &= -(x+e+\widehat{w})^\top L^\top L(x+e+\widehat{w}-e-\widehat{w}) \\ &= -u^\top u - u^\top L e - u^\top L\widehat{w} = -u^\top u - u^\top L\widetilde{e} - u^\top L w,\end{aligned} \tag{5.34}$$

where we use (3.6) to substitute $e+\widehat{w}$ by $\widetilde{e}+w$. Following [15], using Young's inequality, we obtain for any $a \in (0, \frac{1}{2N_i})$

$$\langle \nabla V(\chi), F_\chi(\chi)\rangle \leqslant \textstyle\sum_{i\in\mathcal{N}} -(1-2aN_i)u_i^2 + \frac{1}{a}N_i\left(\widetilde{e}_i^2 + w_i^2\right).$$

Note that u_i is a quantity we have access to locally, since it includes the sampled measurement noise. Then, for any $\sigma_i \in (0,1)$, it holds that

$$\begin{aligned}\langle \nabla V(\chi), F_\chi(\chi)\rangle &\leqslant \textstyle\sum_{i\in\mathcal{N}} -(1-\sigma_i)(1-2aN_i)u_i^2 + \frac{1}{a}N_i w_i^2 - \sigma_i(1-2aN_i)u_i^2 + \frac{1}{a}N_i\widetilde{e}_i^2 \\ &\leqslant -\alpha(|u|) + \gamma(|w|) + \textstyle\sum_{i\in\mathcal{N}} \sigma_i(1-2aN_i)u_i^2 + \frac{1}{a}N_i\widetilde{e}_i^2\end{aligned}$$

for some $\alpha \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}$. Hence, item ii) holds. ■

5.2.2 Decentralized strategy including time-regularization for undirected graphs

Next we analyze the trigger designed in [9] *without* transmission delays to avoid blurring the exposition with too many technicalities. For this case we consider an undirected, connected graph. For the scheme of [9] we require that each agent has an internal clock, $\tau_i \in \mathbb{R}_{\geqslant 0}$, such that $\dot{\tau}_i = 1$ on flows and $\tau_i^+ = 0$ at any triggering instant of agent i, i.e., we reset the clock if agent i transmits its state. We denote the hybrid system in which these clocks are integrated in $\mathcal{H}$ (4.1)-(4.11) with $\mathcal{H}_{\text{clock}}$. Hence, the state for the hybrid system can be written as $\xi = (\chi, \tau, \eta)$ where $\chi = (x, e, \widehat{w})$ is unchanged and $\tau := (\tau_1, \tau_2, \ldots, \tau_N)$.

The proposition below contains the functions required to design a trigger for system (5.31) such that Assumption 4.1 holds.

Proposition 5.4. *Assumption 4.1 holds for* (4.11) *and* (5.32) *with* $\mathcal{A} = \{\chi \in \mathbb{R}^{2n} \times \mathcal{W} \mid x_i = x_j \text{ for all } i,j \in \mathcal{N}, e = \mathbf{0}\}$, $\beta_i(\widetilde{e}_i, \tau_i) = (1-\omega_i(\tau_i))\gamma_i^2\left(\frac{1}{\alpha_i\sigma_i}\lambda_i^2 + 1\right)\widetilde{e}_i^2$ *and* $\delta_i(o_i) = (1-\alpha_i)\sigma_i u_i^2$, *where* $\sigma_i := (1-\varrho)(1-2aN_i)$, $\gamma_i := \sqrt{\frac{1}{a}N_i + \mu_i}$, $d_i := \varrho(1-2aN_i)$,

$$\omega_i(\tau_i) := \begin{cases}\{1\}, & \text{when } \tau_i \in [0, \tau_{\text{MIET}}^i),\\ [0,1], & \text{when } \tau_i = \tau_{\text{MIET}}^i,\\ \{0\}, & \text{when } \tau_i > \tau_{\text{MIET}}^i,\end{cases}$$

$$\tau_{\text{MIET}}^i := -\frac{\sqrt{\alpha_i\sigma_i}}{\gamma_i}\arctan\left(\frac{(\lambda_i^2-1)\sqrt{\alpha_i\sigma_i}}{\lambda_i(\alpha_i\sigma_i+1)}\right),$$

with $\alpha_i \in (0,1)$, $\varrho \in (0,1)$, $\mu_i \in \mathbb{R}_{>0}$ *and* $\lambda_i \in (0,1)$ *tuning parameters.*

Proposition 5.4 implies that, for any bounded measurement noise as defined by Assumption 1, the triggers defined in Theorem 4.1 and Corollary 4.1 render the hybrid system (4.1) ISpS w.r.t. $\mathcal{A}_{\text{clock}}^d := \{(\xi, \tau) : \xi \in \mathcal{A}^d \text{ and } \tau \in \mathbb{R}_{\geqslant 0}^N\}$. Let us note that, due to the inclusion of the timer-dependent function ω_i in the triggers, the system has a GiMIET (instead of a SGiMIET) in this particular case. Additionally, there is no requirement (i.e., no lower bound) on the space-regularization constants c_i, and, in fact, if $c_i = 0$ for all $i \in \mathcal{N}$, we obtain ISS w.r.t. $\mathcal{A}^d$ (instead of ISpS).

Proof. We analyze the Lyapunov function candidate $V(x) = \frac{1}{2}x^\top Lx$. Using (5.31), the derivative of V along flow can be written as

$$\langle \nabla V(x), F(\xi, w)\rangle = -x^\top LL(x + e + \widehat{w}) = -x^\top LLx - x^\top LLe - x^\top LL\widehat{w}, \tag{5.35}$$

or as (5.34). Using Young's inequality, we obtain for some $a \in (0, \frac{1}{2N_i})$

$$\begin{aligned}\langle \nabla V(x), F(\xi, w)\rangle &\leqslant \sum_{i\in\mathcal{N}} -(1-2aN_i)z_i^2 + \frac{1}{a}N_i\left(e_i^2 + \widehat{w}_i^2\right),\\ \langle \nabla V(x), F(\xi, w)\rangle &\leqslant \sum_{i\in\mathcal{N}} -(1-2aN_i)u_i^2 + \frac{1}{a}N_i\left(e_i^2 + \widehat{w}_i^2\right),\end{aligned} \tag{5.36}$$

where $z_i = (Lx)_i$. Combining these two inequalities results in

$$\langle \nabla V(x), F(\xi, w)\rangle \leqslant \sum_{i\in\mathcal{N}} -d_i z_i^2 - \sigma_i u_i^2 + (\gamma_i^2 - \mu_i)e_i^2 + \frac{1}{a}N_i\widehat{w}_i^2, \tag{5.37}$$

with $\sigma_i := (1-\varrho)(1-2aN_i)$, $\gamma_i := \sqrt{\frac{1}{a}N_i + \mu_i}$ and $d_i := \varrho(1-2aN_i)$ and where α_i, $\varrho \in (0,1)$ and $\mu_i \in \mathbb{R}_{>0}$ are tuning parameters. Additionally, we define

$$\omega_i(\tau_i) := \begin{cases}\{1\}, & \text{when } \tau_i \in [0, \tau^i_{\text{MIET}}),\\ [0,1], & \text{when } \tau_i = \tau^i_{\text{MIET}},\\ \{0\}, & \text{when } \tau_i > \tau^i_{\text{MIET}},\end{cases} \tag{5.38}$$

with constant τ^i_{MIET} as

$$\tau^i_{\text{MIET}} = -\frac{\sqrt{\alpha_i\sigma_i}}{\gamma_i}\arctan\left(\frac{(\lambda_i^2 - 1)\sqrt{\alpha_i\sigma_i}}{\lambda_i(\alpha_i\sigma_i + 1)}\right) \tag{5.39}$$

where $\lambda_i \in (0,1)$ is a tuning parameter. We are interested in the stability of the set

$$\mathcal{A} = \{\xi \in \mathbb{X} \mid x_i = x_j \text{ for all } i, j \in \mathcal{N}, e = \mathbf{0}\}. \tag{5.40}$$

To this end, we analyze the Lyapunov function

$$U(\xi) = V(x) + \sum_{i\in\mathcal{N}} \gamma_i\phi_i(\tau_i)e_i^2 \tag{5.41}$$

with

$$\frac{d\phi_i}{d\tau_i} = -\omega_i(\tau_i)\gamma_i\left(\frac{1}{\alpha_i\sigma_i}\phi_i^2(\tau_i) + 1\right). \tag{5.42}$$

As stated in [9], for this Lyapunov function, there exist $\mathcal{K}_\infty$-functions α_1, α_2 such that $\alpha_1(|\xi|_\mathcal{A}) \leqslant U(\xi) \leqslant \alpha_2(|\xi|_\mathcal{A})$, hence, item i) of Assumption 4.1 holds. We can upper-bound the derivative of U along flow as, for any $(\xi, w) \in \mathcal{C}$,

$$\begin{aligned}
\langle \nabla U(\xi), F(\xi, w)\rangle &\leqslant \langle \nabla V(x), F(\xi, w)\rangle + \sum_{i\in\mathcal{N}} \gamma_i \frac{d\phi_i}{d\tau_i} e_i^2 + 2\gamma_i\phi_i e_i u_i \\
&\leqslant \sum_{i\in\mathcal{N}} -d_i z_i^2 - \sigma_i u_i^2 + (\gamma_i^2 - \mu_i) e_i^2 + \frac{1}{a} N_i \widehat{w}_i^2 \\
&\quad + \gamma_i \frac{d\phi_i}{d\tau_i} e_i^2 + \gamma_i^2 \frac{1}{\alpha_i\sigma_i} \phi_i^2 e_i^2 + \alpha_i \sigma_i u_i^2 \\
&\leqslant \sum_{i\in\mathcal{N}} -d_i z_i^2 - \mu_i e_i^2 + \frac{1}{a} N_i \widehat{w}_i^2 \\
&\quad - (1-\alpha_i)\sigma_i u_i^2 + (1-\omega_i(\tau_i))\gamma_i^2 \left(\frac{1}{\alpha_i\sigma_i}\lambda_i^2 + 1\right) e_i^2.
\end{aligned} \tag{5.43}$$

Note that, due to (3.6), we can upper-bound e_i^2 as

$$\begin{aligned}
e_i^2 &= (\widetilde{e}_i - \widehat{w}_i + w_i)^2 = \widetilde{e}_i^2 + \widehat{w}_i^2 + w_i^2 - 2\widetilde{e}_i\widehat{w}_i + 2\widetilde{e}_i w_i - 2\widehat{w}_i w_i \\
&= \widetilde{e}_i^2 + \widehat{w}_i^2 + w_i^2 - 2(e_i + \widehat{w}_i - w_i)\widehat{w}_i + 2(e_i + \widehat{w}_i - w_i) w_i - 2\widehat{w}_i w_i \\
&= \widetilde{e}_i^2 - \widehat{w}_i^2 - w_i^2 + 2\widehat{w}_i w_i - 2 e_i \widehat{w}_i + 2 e_i w_i \\
&\leqslant \widetilde{e}_i^2 - \widehat{w}_i^2 - w_i^2 + \widehat{w}_i^2 + w_i^2 + 2\kappa_i e_i^2 + \frac{1}{\kappa_i}\left(\widehat{w}_i^2 + w_i^2\right) \\
&= \widetilde{e}_i^2 + 2\kappa_i e_i^2 + \frac{1}{\kappa_i}\left(\widehat{w}_i^2 + w_i^2\right)
\end{aligned} \tag{5.44}$$

for any $\kappa_i \in \mathbb{R}_{\geqslant 0}$. Note that we do not use Young's inequality directly on $2\widetilde{e}_i\widehat{w}_i$ and $2\widetilde{e}_i w_i$ because that would result in a more conservative trigger. Then, we chose κ_i such that

$$\kappa_i := \frac{\theta_i \mu_i}{2}\left(\gamma_i^2\left(\frac{1}{\alpha_i\sigma_i}\lambda_i^2 + 1\right)\right)^{-1} \tag{5.45}$$

for any $\theta_i \in (0, 1)$. With this, we can upper-bound U during flow as

$$\begin{aligned}
\langle \nabla U, F(\xi, w)\rangle &\leqslant \sum_{i\in\mathcal{N}} -d_i z_i^2 - \mu_i e_i^2 + \frac{1}{a} N_i \widehat{w}_i^2 - (1-\alpha_i)\sigma_i u_i^2 \\
&\quad + (1-\omega_i(\tau_i))\gamma_i^2\left(\frac{1}{\alpha_i\sigma_i}\lambda_i^2 + 1\right) e_i^2 \\
&\leqslant \sum_{i\in\mathcal{N}} -d_i z_i^2 - (1-\theta_i)\mu_i e_i^2 + \frac{1}{a} N_i \widehat{w}_i^2 + \frac{1}{\kappa_i}\left(\widehat{w}_i^2 + w_i^2\right) \\
&\quad - (1-\alpha_i)\sigma_i u_i^2 + (1-\omega_i(\tau_i))\gamma_i^2\left(\frac{1}{\alpha_i\sigma_i}\lambda_i^2 + 1\right)\widetilde{e}_i^2 \\
&\leqslant \alpha(|\xi|_{\mathcal{A}}) + \gamma(|w|) + \sum_{i\in\mathcal{N}} -(1-\alpha_i)\sigma_i u_i^2 + (1-\omega_i(\tau_i))\gamma_i^2\left(\frac{1}{\alpha_i\sigma_i}\lambda_i^2 + 1\right)\widetilde{e}_i^2.
\end{aligned} \tag{5.46}$$

and indeed, item ii) holds. Additionally, for any $(\xi, w) \in \mathcal{D}$ and $g \in G(\xi, w)$,

$$U(g) - U(\xi) = -\gamma_i\lambda_i e_i^2 \leqslant 0, \tag{5.47}$$

and item iii) also holds. Note that, for dynamic triggers, this implies that we can modify the reset of η_i. We design the trigger reset function such that

$$\eta_i^+(o_i) := \eta_i + \gamma_i\lambda_i\left(\max\left[|\widetilde{e}_i| - 2\overline{w}, 0\right]\right)^2. \tag{5.48}$$

Note that the trigger reset η_i^+ is designed such that we use an estimated lower bound for e_i, i.e.,

$$\eta_i(o_i)^+ \leqslant \eta_i + \gamma_i \lambda_i e_i^2, \tag{5.49}$$

so that item iii) still holds, and that it only uses locally available information (i.e. $\widetilde{e}_i$ and not e_i). Additionally, due to the linear dynamics, item iv) holds trivially. Thus, we satisfy all criteria of Assumption 4.1, and for dynamic triggers we then obtain the trigger dynamics

$$\Psi_i(o_i) = (1-\alpha_i)\sigma_i u_i^2 - (1-\omega_i(\tau_i))\gamma_i^2 \left(1 + \frac{1}{\alpha_i \sigma_i}\lambda_i^2\right) \widetilde{e}_i^2 - \epsilon_i \eta_i + c_i, \tag{5.50}$$

with $c_i \in \mathbb{R}_{\geqslant 0}$ the space-regularization constant. ■

5.2.3 Decentralized strategy for weight-balanced digraphs

Lastly we analyze the triggers designed in [23]. In this case, we consider a network topology described by weight-balanced digraphs. Hence, this scheme requires a less restrictive network topology.

Proposition 5.5. *Assumption 4.1 holds for F_χ defined as in (5.32) and G_χ as in (4.11) with $\beta_i(s) = \left(\frac{d_i^2}{2\theta_i} + \gamma_i\right) s^2$ and $\delta_i(o_i) = \sigma_i(1-2\theta_i)u_i^2$, where d_i denotes the degree of agent i, $\theta_i := \sum_{j \in \mathcal{V}_i^{out}} w_{ij}\alpha_{ij}$, $\gamma_i := \sum_{j\in\mathcal{V}_i^{in}} \frac{w_{ji}}{\alpha_{ji}}$, and with $\alpha_{ij} > 0$ (chosen such that $\theta_i \in (0, \frac{1}{2})$) and $\sigma_i \in (0,1)$ tuning parameters.*

Proof. We start by analyzing the Lyapunov function $V(x) = \frac{1}{2}x^\top L^\top x$. Due to the properties of L respectively $L^\top$, item i) of Assumption 4.1 holds. Additionally, items iii) and iv) hold trivially. Note that from [23], we know that for some additive error v, for any $x, v \in \mathbb{R}^N$, it holds that

$$\langle \nabla V(x), -Lx - Lv\rangle \leqslant \sum_{i\in\mathcal{N}} -(1-\frac{1}{2}\theta_i)u_i^2 - d_i v_i u_i + \frac{1}{2}\gamma_i v_i^2 \tag{5.51}$$

with $\theta_i := \sum_{j\in\mathcal{V}_i^{\text{out}}} w_{ij}\alpha_{ij}$, d_i the degree of agent i, $\gamma_i := \sum_{j\in\mathcal{V}_i^{\text{in}}} \frac{w_{ji}}{\alpha_{ji}}$ and where $\alpha_{ij} > 0$ are tuning parameters. To avoid confusion, we want to note that w_{ij} denotes the weights corresponding to the graph. By substitution of $v = e + \widehat{w} = \widetilde{e} + w$ we obtain

$$\langle \nabla V(x), -Lx - Lv\rangle \leqslant \sum_{i\in\mathcal{N}} -(1-\theta_i)u_i^2 - d_i(\widetilde{e}_i + w_i)u_i + \frac{1}{2}\gamma_i(\widetilde{e}_i + w_i)^2. \tag{5.52}$$

By using Young's inequality and (5.11) we obtain

$$\langle \nabla V(x), -Lx - Lv\rangle \leqslant \sum_{i\in\mathcal{N}} -(1-2\theta_i)u_i^2 + \left(\frac{d_i^2}{2\theta_i} + \gamma_i\right)\widetilde{e}_i^2 + \left(\frac{d_i^2}{2\theta_i} + \gamma_i\right) w_i^2. \tag{5.53}$$

Note that the constants α_{ij} should be chosen such that $\theta_i \in (0, \frac{1}{2})$. Then, for any $\sigma_i \in (0,1)$, it holds that

$$\begin{aligned}\langle \nabla V(x), -Lx - Le - L\widehat{w}\rangle \leqslant \sum_{i\in\mathcal{N}} & -(1-\sigma_i)(1-2\theta_i)u_i^2 + \left(\frac{d_i^2}{2\theta_i} + \gamma_i\right) w_i^2 \\ & - \sigma_i(1-2\theta_i)u_i^2 + \left(\frac{d_i^2}{2\theta_i} + \gamma_i\right)\widetilde{e}_i^2,\end{aligned} \tag{5.54}$$

and item ii) also holds. ■

Chapter 6

Numerical Examples

In this section, we illustrate the results of Section 5.2 with $N = 8$ agents that are connected as described by a graph $\mathcal{G}$ with undirected edges $(1, 2)$, $(1, 8)$, $(2, 3)$, $(2, 7)$, $(3, 4)$, $(3, 6)$, $(4, 5)$, $(5, 6)$, $(5, 8)$ and $(7, 8)$. See Figure 6.1 for a graphical representation of the network topology. We simulate the results of Sections 5.2.1 and 5.2.2. In both cases we include uniformly distributed noise in the interval $[-1\cdot 10^{-4}, 1\cdot 10^{-4}]$ as measurement noise, hence, $\overline{w}_i = 1\cdot 10^{-4}$ for all $i \in \mathcal{N}$. The noise is sampled at a rate of $1\cdot 10^4 Hz$ and a zero-order-hold is applied between samples. Note that this signal indeed satisfies the piecewise continuous requirement.

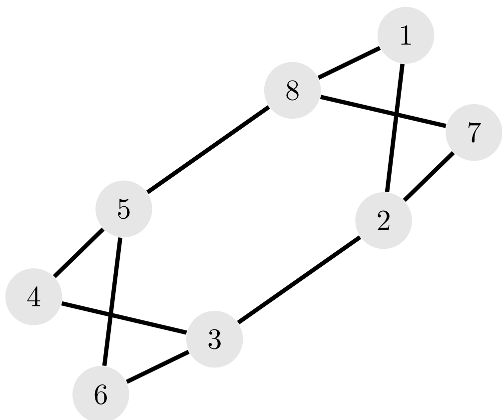

Figure 6.1: The undirected communication topology used in the numerical examples.

6.1 Decentralized static trigger of Proposition 5.3

For the simulation of Proposition 5.3, we use the tuning parameters $\sigma_i = 0.5$ for all $i \in \mathcal{N}$ and $a = 0.1$. Note that, for these parameters, $\max_i(\beta_i(2\overline{w}_i)) = 1.2 \cdot 10^{-6}$, hence we should pick $c_i > 1.2 \cdot 10^{-6}$ to guarantee non-Zenoness. We demonstrate the results of Corollary 4.1, i.e., we apply static triggering. Two cases are simulated, first with no space-regularization for all $i \in \mathcal{N}$ (i.e. $c_i = 0$), to demonstrate that we indeed obtain Zeno-like behavior if c_i is not sufficiently large, and second with $c_i = 2 \cdot 10^{-6} > \max_i(\beta_i(2\overline{w}_i))$. In Figure 6.2, the evolution of the states x_i, $i \in \mathcal{N}$ and the corresponding inter-event times for $c_i = 0$ are shown for the initial condition $x(0, 0) = (8, 6, 4, 2, -2, -4, -6, -8)$, $e(0, 0) = \mathbf{0}_N$, $\widehat{w}(0, 0) = w(0)$ and $\eta(0, 0) = \mathbf{0}_N$. Figure 6.3 depicts the same simulations for $c_i = 2 \cdot 10^{-6}$.

We note that if $c_i = 0$ for all $i \in \mathcal{N}$, we indeed obtain "Zeno-like" behavior, i.e., the inter-event times converge to the sample rate of the noise. Due to the fact that we employ a ZOH between samples (i.e. w_i is constant between samples), each agent can flow for at least

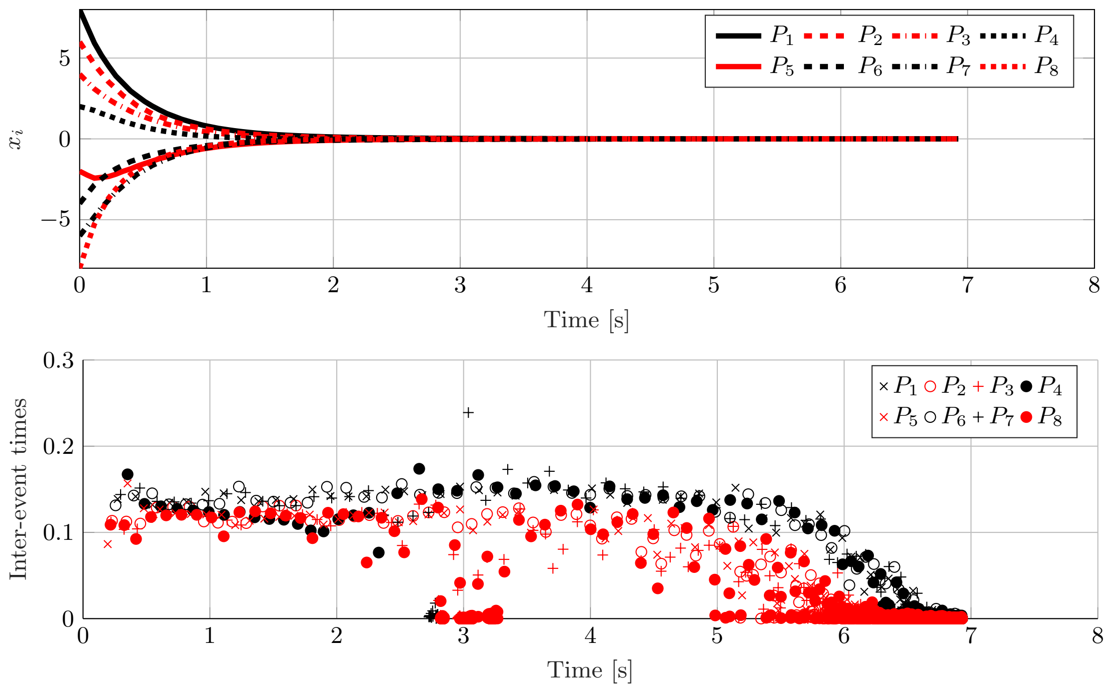

Figure 6.2: Evolution of the states (top) and inter-event times (bottom) of the MAS using the dynamic trigger obtained by applying Corollary 4.1 to Proposition 5.3 with $c_i = 0$ and initial condition $x(0,0) = (8, 6, 4, 2, -2, -4, -8)$.

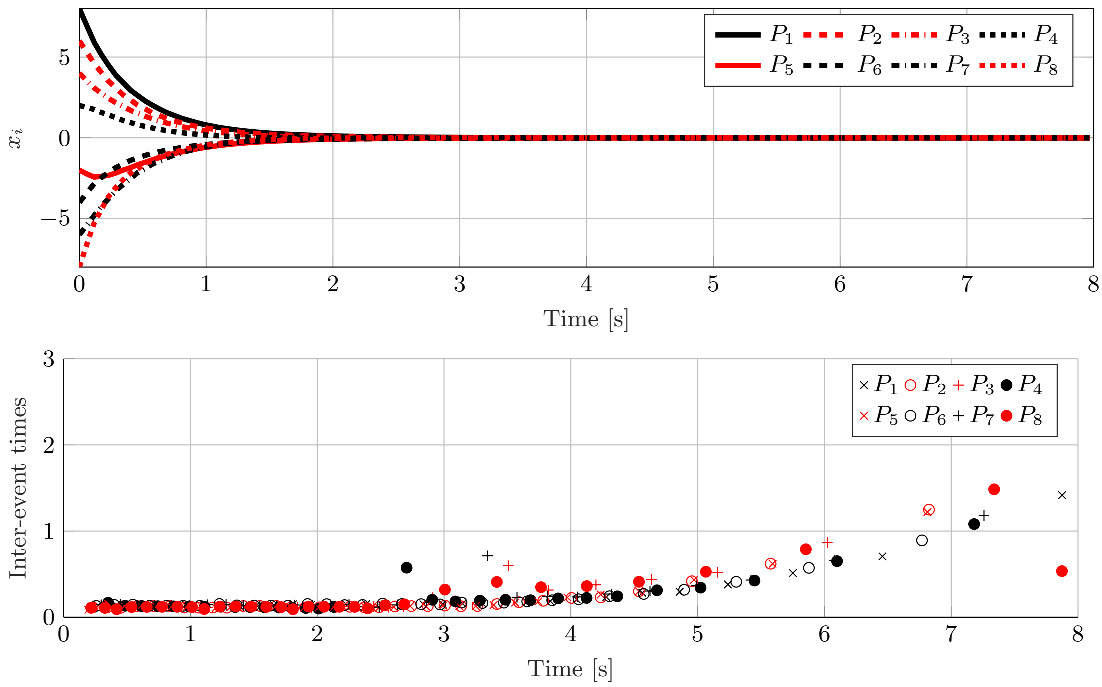

Figure 6.3: Evolution of the states (top) and inter-event times (bottom) of the MAS using the dynamic trigger obtained by applying Corollary 4.1 to Proposition 5.3 with $c_i = 2 \cdot 10^{-6}$ and initial condition $x(0,0) = (8, 6, 4, 2, -2, -4, -8)$.

$1 \cdot 10^{-4}$ seconds before triggering an event. Even though the system is not truly Zeno, any continuous non-constant function between samples would result in Zeno behavior. However, even though the system does not have Zeno behavior, it has undesirable inter-event time characteristics. If the space-regularization constant c_i is designed properly (e.g. as in Figure 6.3), we can see that indeed the inter-event times close to the consensus set remain relatively large, and desirable behavior for the overall system is obtained.

6.2 Decentralized dynamic trigger of Proposition 5.4

For the simulation of Proposition 5.4, the tuning parameters of [9] are used, i.e., $\delta = \mu_i = \epsilon_{\eta,i} = 0.05$, $a = 0.1$ and $\alpha_i = 0.5$ for all $i \in \mathcal{N}$. Given these tuning parameters, we obtain $\gamma_i = 4.478$ and $\sigma_i = 0.76$ for agents $i \in \mathcal{N}$ with two neighbors (i.e., $N_i = 2$, thus agents P_1, P_4, P_6 and P_7) and $\gamma_i = 5.482$ and $\sigma_i = 0.665$ for agents $i \in \mathcal{N}$ with three neighbors (i.e., $N_i = 3$, thus agents P_2, P_3, P_5 and P_8). We choose $\lambda_i = 0.2$ for all agents. For these values, we obtain $\tau^i_{\text{MIET}} = 0.1562$ for agents $i \in \mathcal{N}$ for which $N_i = 2$ and $\tau^i_{\text{MIET}} = 0.1180$ for agents $i \in \mathcal{N}$ for which $N_i = 3$.

We demonstrate the results of Theorem 4.1, i.e., we apply dynamic triggering. Two cases are simulated, first with no space-regularization for all $i \in \mathcal{N}$, for which we obtain ISS w.r.t. the consensus set, second with space-regularization constant $c_i = 1 \cdot 10^{-5}$ for all $i \in \mathcal{N}$, for which we have ISpS w.r.t. the consensus set. To compare the results to [9] (not considering measurement noise), in all cases we select $\theta_i = 0$. In Figure 6.4, the evolution of the states x_i, $i \in \mathcal{N}$, with $c_i = 0$ and the corresponding inter-event times are shown for the initial condition $x(0,0) = (8,6,4,2,-2,-4,-6,-8)$, $e(0,0) = \mathbf{0}_N$, $\widehat{w}(0,0) = w(0)$, $\tau(0,0) = \mathbf{0}_N$ and $\eta(0,0) = \mathbf{0}_N$. Figure 6.5 depicts the same simulations for $c_i = 1 \cdot 10^{-7}$.

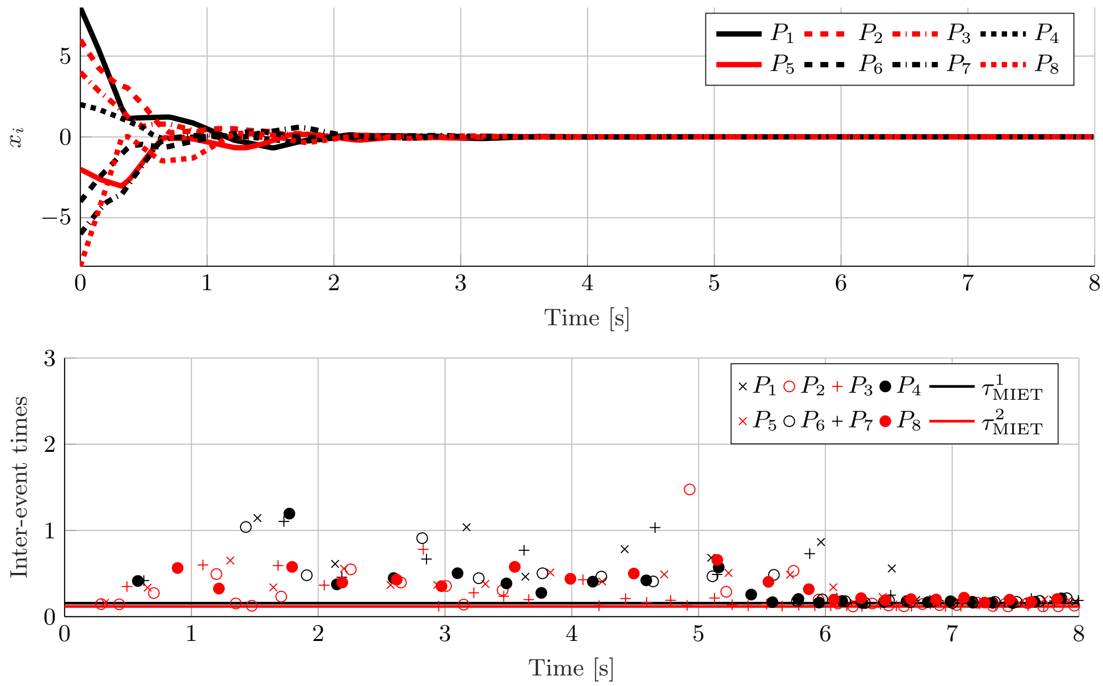

Figure 6.4: Evolution of the states (top) and inter-event times (bottom) of the MAS using the dynamic trigger obtained by applying Theorem 4.1 to Proposition 5.4 with $c_i = 0$ and initial condition $x(0,0) = (8,6,4,2,-2,-4,-8)$.

From the simulations we can make a few observations. Note that, for $c_i = 0$, close to the consensus set the inter-event times are generally close to τ^i_{MIET}. This can be explained from the observation that, in these cases, $\eta_i^+ = 0$ and u_i is generally small, and consequently, the increase in η_i for $\tau \in [0, \tau^i_{\text{MIET}})$ is limited. Additionally, we observe that by selecting a $c_i > 0$, the inter-event times are generally significantly larger than the enforced minimum inter-event time. Moreover, because there is no lower-bound on c_i, a relatively small c_i is often sufficient to obtain desirable average inter-event times. We want to stress that this

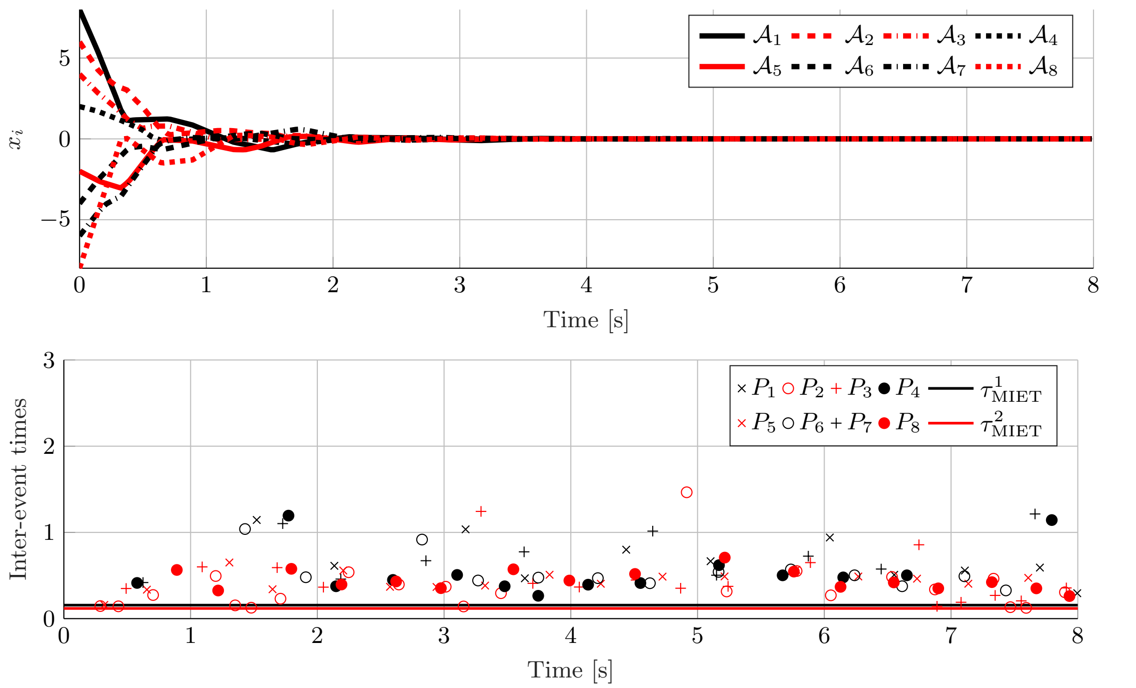

Figure 6.5: Evolution of the states (top) and inter-event times (bottom) of the MAS using the dynamic trigger obtained by applying Theorem 4.1 to Proposition 5.4 with $c_i = 1 \cdot 10^{-7}$ and initial condition $x(0,0) = (8, 6, 4, 2, -2, -4, -8)$.

is a beneficial aspect of this particular scheme, since in general there are constraints on the minimum size of the space-regularization constants c_i to ensure non-Zenoness.

Even though the inclusion of c_i leads to ISpS instead of ISS properties, applying space-regularization leads to triggering conditions that are not only robust to measurement noise, but also have, on average, larger inter-event times. Since ISS only leads to asymptotic behavior of the consensus set for vanishing noise, and since most measurement noise is non-vanishing, practical stability or ISpS with larger inter-event times may be more desirable when having communication limitations in mind.

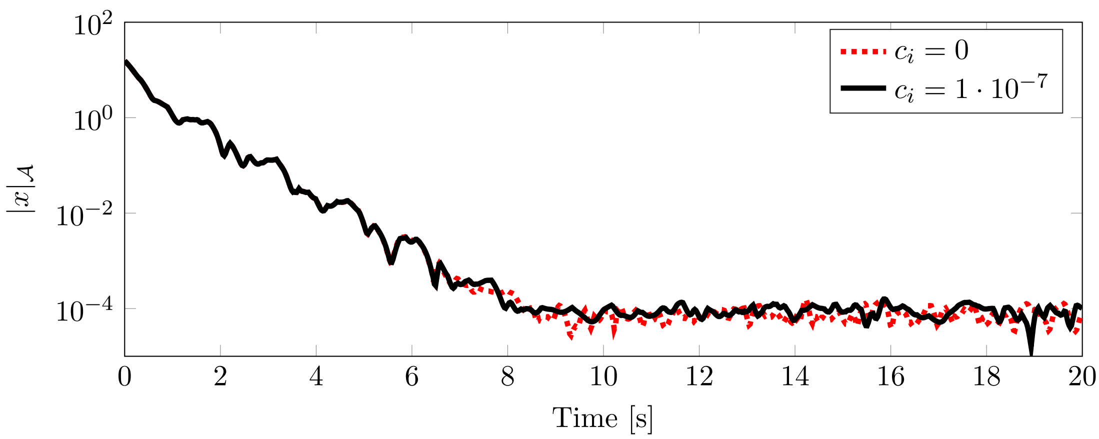

Figure 6.6: Distance of agents to the consensus set $\mathcal{A}$.

In Figure 6.6, the distance to the consensus set, i.e., $|x|_{\mathcal{A}}$, is depicted. We note that even though the inter-event times are more favorable if we apply space-regularization, the remaining distance to the consensus set has the same order of magnitude, which underlines the effectiveness of applying both space- and time-regularization at the same time.

Chapter 7

Conclusions

In this thesis, we presented a general "prescriptive" framework for set stabilization of event-triggered control systems affected by measurement noise. It is shown that, by careful design, we obtain both *dynamic* and *static* triggering conditions that render the closed loop or a set input-to-state (practically) stable with a guaranteed positive (semi-)global individual minimum inter-event time. Key to obtaining this framework is a novel hybrid model that describes the behavior of event-triggered control systems and the application of space-regularization. Due to this model and the space-regularization, differentiability conditions are not required on the measurement noises, as opposed to many works in the literature. The strengths and generality of the framework are demonstrated on several interesting event-triggered control problems, such as the stabilization of the origin for single-plant systems and consensus problems for multi-agent systems, robustifying them for measurement noise.

The framework laid down in this thesis will be extended in future research. We aim to include output-feedback control, dynamic controllers, more general dissipative functions and other holding-functions for the transmitted states in the future. The listed extensions will be the subject of a journal submission.

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