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Random walks in random environments
applications on a ladder graph

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Random Walks in Random Environments,

Applications on a Ladder Graph

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Abstract

The main conclusion in this report is that there are possibilities of random walks on a ladder graph which are transient to the left, but for which both layers of the random walk are transient to the right. In this report we will start giving a short introduction and some theorems. After that we will go into random walks in random environments (RWRE) in one dimension. We will explain the random environment, give some applications of RWRE's and give and prove the criteria for recurrence and transience of an RWRE in one dimension. In the end we will explain random walks on a ladder graph and show an example of a random walk on a ladder graph which is transient to the left, but for which both layers are transient to the right. Although the behaviour of recurrence and transience of random walks on a ladder graph is more complicated than the behaviour of recurrence and transience of RWRE's in one dimension, we will see that the criteria for recurrence and transience of RWRE's in one dimension are useful in understanding the behaviour of recurrence and transience in our given example of a random walk on a ladder graph.

Contents

1 Introduction to Random Walks

1.1 Stochastic Processes

Most sciences are not completely black and white at all. In these sciences there has to be space for some randomness instead. Examples of these sciences are economics, psychology, physics, chemistry and biology. In order to bring this randomness stochastic processes have been introduced. Stochastic processes can be defined as followed [4].

Definition 1.1.1 (Stochastic Processes). *A stochastic process X is a family of random variables $\{X_t, t \in T\}$, indexed by a set T, for which in a discrete-time process $T = \{0, 1, 2, ...\}$.*

The random variables of which a stochastic process contains evolve in a random, but prescribed, manner and are, therefore, useful for these studies. In this report one category of stochastic processes will be discussed in detail: random walks. Random walks describe a certain path in space, consisting of random steps. The idea of a random walk is that for a starting point x, there is a chance ω of going to the right (from x to $x+1$) and a chance $1-\omega$ of going to the left (from x to $x-1$). A random walk $(X_n)_n$ is called recurrent if the following holds:

$$\limsup_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s. and} \liminf_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s.}$$

In these equations X_n gives the position of the random walk after n steps.
A recurrent random walk reaches its starting position infinitely many times with probability 1.
A random walk is called transient if, given enough time, every finite set of the state space will be left forever. In one dimension the only possibility for this is that the random walk goes to the left or to the right. A random walk is transient to the right if

$$\lim_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s.}$$

and a random walk is transient to the left if

$$\lim_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s.}$$

Note that this is about the convergence of random variables, so the limits are almost sure (a.s.) limits. As random walks are also part of Markov processes, which form a subcategory of stochastic processes, we will look at these Markov processes in the next paragraph.

1.2 Markov Processes

A Markov process is a stochastic process with the property that it only depends on the state attained in the previous time step. This property of a Markov process is called the Markov property. It refers to the memorylessness of a Markov process. A Markov process can be defined as the following [4].

Definition 1.2.1 (Markov Process). *A stochastic process X is a Markov process if it satisfies the Markov property, defined as:*

$$\mathbb{P}(X_n = s | X_0 = x_0, X_1 = x_1, ..., X_{n-1} = x_{n-1}) = \mathbb{P}(X_n = s | X_{n-1} = x_{n-1})$$

for all $s, x_0, ..., x_{n-1} \in \Omega$ for Ω a countable set such that $\mathbb{P}(X_0 = x_0, ..., X_{n-1} = x_{n-1}) > 0$. The set Ω is called the state space of the random walk.

In the next paragraph we will give some theorems which will be used in this report.

1.3 Important Theorems

In this paragraph some general theorems will be mentioned. These theorems will return later on in this report [4].

Theorem 1.3.1 (Law of Total Probability). *Let $B_1, B_2, ..., B_n$ be a partition of the sample space Ω such that $\mathbb{P}(B_i) > 0$ for all i. Then*

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A|B_i) \cdot \mathbb{P}(B_i).$$

Theorem 1.3.2 (Continuity of Probability Measure). *Let $A_1, A_2, ...$ be an increasing sequence of events, such that $A_1 \subseteq A_2 \subseteq A_3 \subseteq ...$ Then*

$$\text{For } A = \bigcup_{i=1}^{\infty} A_i : \mathbb{P}(A) = \lim_{i\to\infty} \mathbb{P}(A_i).$$

Similarly, let $B_1, B_2, ...$ be a decreasing sequence of events, so that $B_1 \supseteq B_2 \supseteq B_3 \supseteq ...$ Then

$$\text{For } B = \bigcap_{i=1}^{\infty} B_i : \mathbb{P}(B) = \lim_{i\to\infty} \mathbb{P}(B_i).$$

Theorem 1.3.3 (Law of Large Numbers). *Let $X_1, X_2, ...$ be a sequence of independent identically distributed (i.i.d.) random variables with finite means $\mathbb{E}[X_i] = \mu$. Then*

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[n\to\infty]{} \mu, P_\omega\text{-a.s.}$$

Theorem 1.3.4 (Central Limit Theorem). *Let $X_1, X_2, ...$ be a sequence of i.i.d. random variables with finite means μ and finite non-zero variance σ^2 and define S_n such that $S_n = \sum_{i=1}^{n} X_i$. Then*

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow[n\to\infty]{D} \mathcal{N}(0,1).$$

Theorem 1.3.5 (Law of the Iterated Logarithm [5]). *Let $X_1, X_2, ...$ be a sequence of i.i.d. random variables with $\mathbb{E}[X_i] = 0$ and $\mathbb{V}(X_i) = 1$. Let $S_n = \sum_{i=1}^{n} X_n$. Then*

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log(\log(n))}} = 1, P_\omega\text{-a.s. and } \liminf_{n\to\infty} \frac{S_n}{\sqrt{2n\log(\log(n))}} = -1, P_\omega\text{-a.s.}$$

In the next paragraph we will explain simple random walks.

1.4 Simple Random Walks

A simple random walk $(X_n)_n$ has a chance of going to the right of ω and a chance of going to the left of $1-\omega$. In a simple random walk the value of this ω does not differ per point, so ω is the same for each node on the random walk. For $\omega = \frac{1}{2}$ a simple random walk is recurrent and

$$\limsup_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s. and} \liminf_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s.}$$

A simple random walk with $\omega = \frac{1}{2}$ is called symmetric.

For $\omega > \frac{1}{2}$ a simple random walk is transient to the right and

$$\lim_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s.}$$

and for $\omega < \frac{1}{2}$ is transient to the left and

$$\lim_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s.}$$

Simple random walks are too restrictive for the subjects of this report, so we will not go deeper into this topic. In the next chapter we will explain random walks in random environments (RWRE's) in one dimension. This seems to be the solution for problems for which simple random walks are too restrictive. We will explain a random environment, give some applications of RWRE's and give the criteria for recurrence and transience of an RWRE in one dimension.

Later in this report we will use these criteria in order to fulfill the main goal of this report, namely understanding the behaviour of recurrence and transience of a random walk on a ladder graph. This will be explained in chapter 4.

2 Random Walks in Random Environments in One Dimension

2.1 A Random Environment

From now on in this report when we talk about random walks, we will mean random walks in random environments (RWRE's), unless stated otherwise. For a function $f : \mathbb{X} \to \mathbb{R}$ we know for $x \in \mathbb{X}$, x random, that $f(x)$ is a random variable as well. At first we will define an environment [8].

Definition 2.1.1 (Environment)**.** *An environment is an element $\omega = \{\omega_x\}_{x\in\mathbb{Z}}$, for which it is possible to construct a Markov chain $(X_n)_{n\geq 0}$ for n a natural number. The state space Ω of ω will be equal to*

$$\Omega = [0,1]^{\mathbb{Z}}.$$

For $\omega \in \Omega$ (fixed) we define the probability distribution P_ω^x of a random walk $(X_n)_n$ in the environment ω by:

$$P_\omega^x(X_0 = x) = 1 \text{ and } P_\omega^x(X_{n+1} = z|X_n = y) = \begin{cases} \omega_y & \text{for } z = y+1, \\ 1-\omega_y & \text{for } z = y-1, \\ 0 & \text{otherwise.} \end{cases}$$

Now define also a probability distribution P on Ω. Then the environment $\omega \in \Omega$ becomes random as well. From now on there is a random environment and the random walk $(X_n)_n$ has become a random walk in a random environment. Next we will show a transition model of a random walk in a random environment and afterwards an example [8].

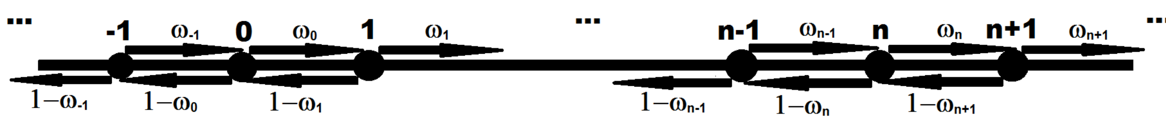


Figure 1: Transition model of a random walk in a random environment.

Example 2.1.1. *For ω_x i.i.d., a random walk $(X_n)_n$ with probability distribution P with*

$$P\left(\omega_x = \frac{3}{4}\right) = p \text{ and } P\left(\omega_x = \frac{1}{3}\right) = 1-p \text{ for some } p \in (0,1).$$

has a random environment. Here the value of ω_x is not the same in every node on the random walk.

So in a random walk in a random environment the value of ω depends on the position x in the random walk. Because random walks in this report often start at $x = 0$, from now on we will use the notation of P_ω for P_ω^0.

For any fixed event G for the random walk $P_\omega^x(G)$ is a $[0,1]-$valued random variable, since ω is random. The distribution P_ω for a fixed environment is called the quenched law of the RWRE [8]. Thus, we can define another probability measure $\mathbb{P}^x$ on $(X_n)_n$ by $\mathbb{P}^x(\cdot) = E_P[P_\omega^x(\cdot)]$. The distribution $\mathbb{P}$ is called the averaged/annealed law for the RWRE.

In the next paragraph we will have a look at applications of RWRE's and after that we will give a theorem that describes the criteria for recurrence and transience of RWRE's in one dimension.
Before giving this theorem we have to make the following two assumptions in order to avoid degeneracy complications and to make the proof easier [8]:

(a) The distribution P is such that $\{\omega_x\}_x$ is an i.i.d. sequence and

(b) There has to exist a $c > 0$ such that $P(\omega_x \in [c, 1-c]) = 1$.

2.2 Applications of Random Walks in Random Environments

In this paragraph we will elaborate on applications for which random walks in random environments can be useful or even necessary.

An important application for random walks in random environments nowadays is in modeling the stock indices of different nations and/or companies [2]. But, because stock indices are never below 0, the exponential of an RWRE is used in order to model stock indices. In the picture below the stock market of the USA (NASDAQ) is given from 1970 until 2012.

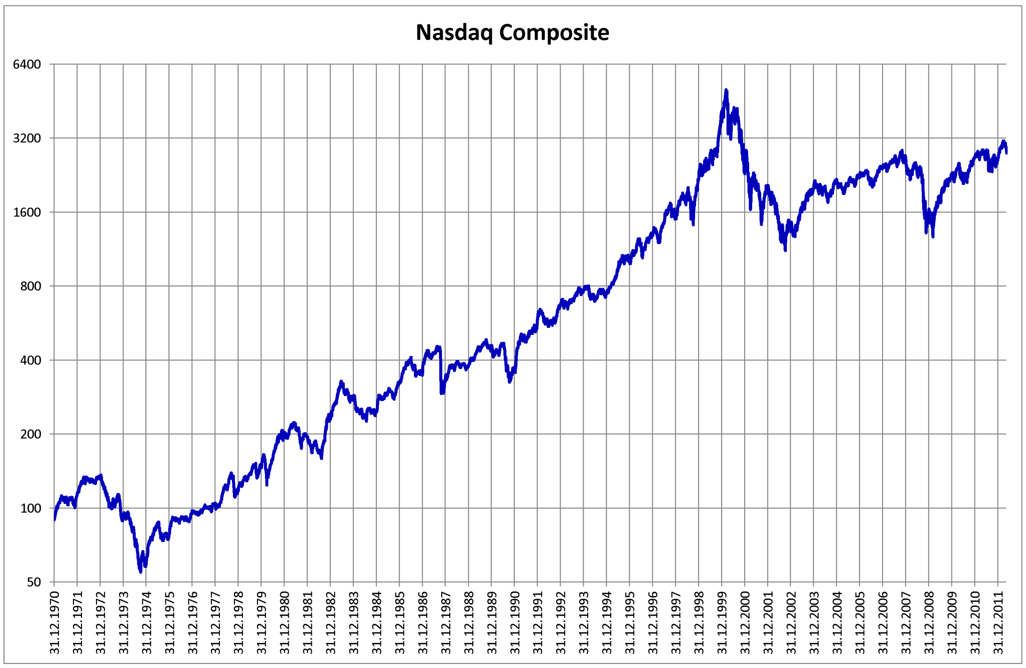


Figure 2: Stock market of the USA (NASDAQ) from 1970 until 2012[1].

Random walks in random environments have been proposed as a model for DNA replication around the seventies [10].
Random walks in random environments have an application in understanding and modeling how the diffusion behaviour of a biomolecule is affected by other moving macromolecules, organelles, and so on, inside a living cell [3].
Random walks in random environments are used as a method to model the movement of electrons in physics [7].

[1]"Nasdaq Composite.png" by Jashuah is licensed under the Creative Commons Attribution-Share Alike 3.0 Unported license.

2.3 Recurrence and Transience

Now that we have explained random walks in random environments, in this paragraph we will give criteria for recurrence and transience of an RWRE. As said in the introduction (paragraph 1.1), there are three possibilities with respect to recurrence and transience of an RWRE $(X_n)_n$:

(a) Transience to the right. In this case:

$$\lim_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega.$$

(b) Transience to the left. In this case:

$$\lim_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega.$$

(c) Recurrence. In this case:

$$\limsup_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s. and } \liminf_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega.$$

In Solomon's seminal paper on RWRE [9], Solomon gave an explicit criterion for recurrence or transience. Examples which are not part of (a), (b) or (c) do not occur in this theorem, due to the two assumptions discussed in the end of paragraph 2.1. An example of a random walk which is neither recurrent nor transient is the following.

Example 2.3.1 (Random walk in random environment neither recurrent nor transient)**.** *Define a random walk $(X_n)_n$ consisting of the following three types of nodes:*

(i) Type A with $P(\omega_x = 1) = \frac{1}{10}$,

(ii) Type B with $P(\omega_x = 0) = \frac{1}{10}$,

(iii) Type C with $P\left(\omega_x = \frac{1}{2}\right) = \frac{8}{10}$.

This random walk $(X_n)_n$ is neither recurrent nor transient. It is not transient to the left, because nodes of Type A will act as boundaries for going to the left, and it is not transient to the right, because nodes of Type B will act as boundaries for going to the right.
Nevertheless, $\limsup_{n\to\infty} X_n$ and $\liminf_{n\to\infty} X_n$ will be finite, because of the same fact that nodes of Type A will act as boundaries for going to the left and nodes of Type B will act as boundaries for going to the right. The random walk will neither reach nodes left of the first node of Type A it finds in the left of its starting position nor nodes right of the first node of Type B it finds in the right of its starting position, so the random walk is also not recurrent. The random walk might look like the following:

Figure 3: Example which is neither recurrent nor transient.

The random walk in example 2.3.1 does not meet assumption (b) in paragraph 2.1 that there has to exist a $c > 0$ such that $P(\omega_x \in [c, 1-c]) = 1$. We made these assumptions so that we do not have to take examples such as this, which are neither recurrent nor transient, into account in the criteria for recurrence and transience we will give in this paragraph.

In his paper Solomon states that the recurrence or transience of random walks in random environments is determined by the quantity of $E_P[\log(\rho_x)]$, with $\rho_x = \frac{1-\omega_x}{\omega_x}$, for $x \in \mathbb{Z}$.
The theorem is the following [9].

Theorem 2.3.1. *Make sure that the assumptions from the paragraph 2.1 hold true. Then for a random walk* $(X_n)_n$*, for* $\rho_x = \frac{1-\omega_x}{\omega_x}$, $x \in \mathbb{Z}$ *the following holds:*

(a) $E_P[\log(\rho_x)] < 0 \Longrightarrow \lim_{n\to\infty} X_n = +\infty, P_\omega$*-a.s., for P almost all* ω*, so the random walk is transient to the right,*

(b) $E_P[\log(\rho_x)] > 0 \Longrightarrow \lim_{n\to\infty} X_n = -\infty, P_\omega$*-a.s., for P almost all* ω*, so the random walk is transient to the left and*

(c) $E_P[\log(\rho_x)] = 0 \Longrightarrow \limsup_{n\to\infty} X_n = +\infty, P_\omega$*-a.s. and* $\liminf_{n\to\infty} X_n = -\infty, P_\omega$*-a.s., for P almost all* ω*, so the random walk is recurrent.*

At first we will show that this theorem also holds for simple random walks. For simple random walks $\omega_x = \omega$ and $\rho_x = \frac{1-\omega}{\omega}$.

(a) A simple random walk with $\omega > \frac{1}{2}$ is transient to the right. This is in line with the theorem, because for $\omega > \frac{1}{2}$:
$$\rho_x = \frac{1-\omega}{\omega} < 1 \text{ and therefore } E_P[\log(\rho_x)] < 0.$$

(b) A simple random walk with $\omega < \frac{1}{2}$ is transient to the left. This is in line with the theorem, because for $\omega < \frac{1}{2}$:
$$\rho_x = \frac{1-\omega}{\omega} > 1 \text{ and therefore } E_P[\log(\rho_x)] > 0.$$

(c) A simple random walk with $\omega = \frac{1}{2}$ is recurrent. This is in line with the theorem, because for $\omega = \frac{1}{2}$:
$$\rho_x = \frac{1-\omega}{\omega} = 1 \text{ and therefore } E_P[\log(\rho_x)] = 0.$$

So, despite the fact that the recurrence and transience of simple random walks is easier than the recurrence and transience of RWRE's, it is also possible to find the recurrence and transience of simple random walks with theorem 2.3.1.
Next the proof of the theorem will be given. In this proof we will use the lecture notes on random walks in random environments of Jonathan Peterson [8].

Proof. We will start this proof by defining some new concepts.

For a fixed environment ω, we define the potential V of the environment as followed:

$$V(k) = \begin{cases} \sum_{i=0}^{k-1} \log(\rho_i) & \text{for } k \geq 1, \\ 0 & \text{for } k = 0, \\ -\sum_{i=k}^{-1} \log(\rho_i) & \text{for } k \leq -1. \end{cases}$$

Since the environment is i.i.d. due to the assumptions made in paragraph 2.1 the law of large numbers (Theorem 1.3.3) tells us that
$$V(i) \sim E_P[\log(\rho_x)] \cdot i \text{ as } i \to \pm\infty.$$

The hitting time T_x of the point x is the first moment in time at which the random walk hits the point x. For $x \in \mathbb{Z}$ define the hitting time T_x as followed:

$$T_x = \inf\{n \geq 0 : X_n = x\}.$$

Choose a and b such that $0 \leq a < b$ and choose the function g such that

$$g(x) = P^x_\omega(T_a < T_b).$$

The goal is to find an explicit function h for which

$$h(x) = g(x) \text{ for all } x \in \mathbb{Z}.$$

Next we are going to show that for

$$h(x) = \frac{\sum\limits_{i=x+1}^{b} e^{V(i)}}{\sum\limits_{i=a+1}^{x} e^{V(i)} + \sum\limits_{i=x+1}^{b} e^{V(i)}}$$

we have found this explicit function h. We will show that this is the only possibility of h using the maximum principle.

For the function $g(x)$ the following two points should hold:

(i) (1) $g(a) = 1$. This is because in $g(a)$ the random walk starts in the point a, so the hitting time of the point a is 0 and is always less than the hitting time of the point b, because $a \neq b$.

(2) $g(b) = 0$. This is because in $g(b)$ the random walk starts in the point b, so the hitting time of the point b is 0 and is always less than the hitting time of the point a, because $a \neq b$.

(ii) Because of the law of total probability (Theorem 1.3.1):

$$\omega_x \cdot g(x+1) + (1-\omega_x) \cdot g(x-1) = g(x).$$

Now we will check if these points also hold for the function h:

(i) (1) $h(a) = 1$:

$$h(a) = \frac{\sum\limits_{i=a+1}^{b} e^{V(i)}}{\sum\limits_{i=a+1}^{a} e^{V(i)} + \sum\limits_{i=a+1}^{b} e^{V(i)}} = 1, \text{ because } \sum_{i=a+1}^{a} e^{V(i)} = 0 \text{ and } \sum_{i=a+1}^{b} e^{V(i)} \neq 0 \text{ and}$$

(2) $h(b) = 0$:

$$h(b) = \frac{\sum\limits_{i=b+1}^{b} e^{V(i)}}{\sum\limits_{i=a+1}^{b} e^{V(i)} + \sum\limits_{i=b+1}^{b} e^{V(i)}} = 0, \text{ because } \sum_{i=b+1}^{b} e^{V(i)} = 0 \text{ and } \sum_{i=a+1}^{b} e^{V(i)} \neq 0.$$

(ii) $\omega_x \cdot h(x+1) + (1-\omega_x)\cdot h(x-1) = h(x)$:

$$\omega_x \cdot h(x+1) + (1-\omega_x)\cdot h(x-1)$$

$$=\omega_x \cdot \frac{\sum\limits_{i=x+2}^{b} e^{V(i)}}{\sum\limits_{i=a+1}^{x+1} e^{V(i)} + \sum\limits_{i=x+2}^{b} e^{V(i)}} + (1-\omega_x)\cdot \frac{\sum\limits_{i=x}^{b} e^{V(i)}}{\sum\limits_{i=a+1}^{x-1} e^{V(i)} + \sum\limits_{i=x}^{b} e^{V(i)}} = \omega_x \cdot \frac{\sum\limits_{i=x+2}^{b} e^{V(i)}}{\sum\limits_{i=a+1}^{b} e^{V(i)}} + (1-\omega_x)\cdot \frac{\sum\limits_{i=x}^{b} e^{V(i)}}{\sum\limits_{i=a+1}^{b} e^{V(i)}}$$

$$=\frac{\omega_x \cdot \sum\limits_{i=x+2}^{b} e^{V(i)} + (1-\omega_x)\cdot \sum\limits_{i=x}^{b} e^{V(i)}}{\sum\limits_{i=a+1}^{b} e^{V(i)}} = \frac{\omega_x \cdot \sum\limits_{i=x+2}^{b} e^{V(i)} + (1-\omega_x)\cdot \left(\sum\limits_{i=x}^{x+1} e^{V(i)} + \sum\limits_{i=x+2}^{b} e^{V(i)}\right)}{\sum\limits_{i=a+1}^{x} e^{V(i)} + \sum\limits_{i=x+1}^{b} e^{V(i)}}$$

$$=\frac{\sum\limits_{i=x+2}^{b} e^{V(i)} + (1-\omega_x)\cdot \sum\limits_{i=x}^{x+1} e^{V(i)}}{\sum\limits_{i=a+1}^{x} e^{V(i)} + \sum\limits_{i=x+1}^{b} e^{V(i)}}.$$

From now on the denominator will be the same, so we will just discuss the calculations in the numerator. Because

$$V(i) = \sum_{j=0}^{i-1} \log(\rho_j) \text{ for } i \geq 1$$

we know that

$$\sum_{i=x+2}^{b} e^{V(i)} + (1-\omega_x)\cdot \sum_{i=x}^{x+1} e^{V(i)}$$

$$= \sum_{i=x+2}^{b} e^{V(i)} + (1-\omega_x)\cdot \sum_{i=x}^{x+1} e^{\sum\limits_{j=0}^{i-1}\log(\rho_j)} = \sum_{i=x+2}^{b} e^{V(i)} + (1-\omega_x)\cdot \sum_{i=x}^{x+1}\prod_{j=0}^{i-1}\frac{1-\omega_j}{\omega_j}$$

$$= \sum_{i=x+2}^{b} e^{V(i)} + (1-\omega_x)\cdot \left(\left(\frac{1-\omega_0}{\omega_0}\right)\cdot\left(\frac{1-\omega_1}{\omega_1}\right)\cdot ... \left(\frac{1-\omega_{x-1}}{\omega_{x-1}}\right) + \left(\frac{1-\omega_0}{\omega_0}\right)\cdot\left(\frac{1-\omega_1}{\omega_1}\right)\cdot ... \left(\frac{1-\omega_x}{\omega_x}\right)\right)$$

$$= \sum_{i=x+2}^{b} e^{V(i)} + \left(\frac{1-\omega_0}{\omega_0}\right)\cdot\left(\frac{1-\omega_1}{\omega_1}\right)\cdot ... \left(\frac{1-\omega_x}{\omega_x}\right)\cdot\omega_x + \left(\frac{1-\omega_0}{\omega_0}\right)\cdot\left(\frac{1-\omega_1}{\omega_1}\right)\cdot ... \left(\frac{1-\omega_x}{\omega_x}\right)\cdot(1-\omega_x)$$

$$= \sum_{i=x+2}^{b} e^{V(i)} + \left(\frac{1-\omega_0}{\omega_0}\right)\cdot\left(\frac{1-\omega_1}{\omega_1}\right)\cdot ... \left(\frac{1-\omega_x}{\omega_x}\right) = \sum_{i=x+2}^{b} e^{V(i)} + \prod_{j=0}^{x}\frac{1-\omega_j}{\omega_j}.$$

$\frac{1-\omega_j}{\omega_j}$ is the definition of ρ_j, so

$$\sum_{i=x+2}^{b} e^{V(i)} + \prod_{j=0}^{x}\frac{1-\omega_j}{\omega_j}$$

$$= \sum_{i=x+2}^{b} e^{V(i)} + \prod_{j=0}^{x}\rho_j = \sum_{i=x+2}^{b} e^{V(i)} + e^{\sum\limits_{j=0}^{x}\log(\rho_j)} = \sum_{i=x+2}^{b} e^{V(i)} + e^{V(x+1)} = \sum_{i=x+1}^{b} e^{V(i)}$$

When adding the denominator again the following happens:

$$\frac{\sum_{i=x+2}^{b} e^{V(i)} + (1-\omega_x)\cdot\sum_{i=x}^{x+1} e^{V(i)}}{\sum_{i=a+1}^{x} e^{V(i)} + \sum_{i=x+1}^{b} e^{V(i)}}$$

$$= \frac{\sum_{i=x+1}^{b} e^{V(i)}}{\sum_{i=a+1}^{x} e^{V(i)} + \sum_{i=x+1}^{b} e^{V(i)}} = h(x)$$

So h and g satisfy:

(i) $g(a) = h(a) = 1$ and $g(b) = h(b) = 0$ and

(ii) $\omega_x \cdot h(x+1) + (1-\omega_x)\cdot h(x-1) = h(x)$ and $\omega_x \cdot g(x+1) + (1-\omega_x)\cdot g(x-1) = g(x)$.

We claim that this means that $g = h$.

In order to prove this we will look at the function d, defined as the difference between h and g.

$$d(a) = h(a) - g(a) = 1 - 1 = 0 \text{ and } d(b) = h(b) - g(b) = 0 - 0 = 0$$

and also

$$\begin{aligned}
&\omega_x \cdot d(x+1) + (1-\omega_x)\cdot d(x-1)\\
=&\omega_x \cdot (h(x+1) - g(x+1)) + (1-\omega_x)\cdot(h(x-1) - g(x-1))\\
=&(\omega_x \cdot g(x+1) + (1-\omega_x)\cdot g(x-1)) + (\omega_x \cdot h(x+1) + (1-\omega_x)\cdot h(x-1))\\
=&h(x) - g(x) = d(x)
\end{aligned}$$

From the last we conclude that

$$\min\{d(x-1), d(x+1)\} \leq d(x) \leq \max\{d(x-1), d(x+1)\}$$

and therefore also that $d(x)$ is between $d(x-1)$ and $d(x+1)$. But because $d(a) = d(b) = 0$, this means that $d(x) = 0$ for all $x \in \mathbb{Z}$.

So

$$h(x) = g(x) \text{ for all } x \in \mathbb{Z}$$

and therefore

$$P_\omega^x(T_a < T_b) = \frac{\sum_{i=x+1}^{b} e^{V(i)}}{\sum_{i=a+1}^{x} e^{V(i)} + \sum_{i=x+1}^{b} e^{V(i)}} \text{ for all } x \in \mathbb{Z}.$$

Now we will discuss the three possibilities of recurrence and transience.

2.3.1 Transience to the right

In this part we will prove that for $E_P[\log(\rho_x)] < 0$ the random walk $(X_n)_n$ is transient to the right and

$$\lim_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega.$$

In order to prove this we have to show that for $E_p[\log(\rho_x)] < 0$

$$\limsup_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s. and } \liminf_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega.$$

When these statements are proven, we know that the random walk is transient to the right.

$\boldsymbol{\limsup_{n\to\infty} X_n = +\infty, P_\omega}$**-a.s.:**

In this part we want to prove that the probability of all points $b > 0$ of being hit eventually is equal to 1, P_ω-a.s., for P almost all ω. The probability of a point $b > 0$ of being hit eventually is equal to $P_\omega(T_b < \infty)$, so the probability of all points $b > 0$ of being hit eventually is, due to the continuity of probability measure (Theorem 1.3.2), equal to

$$P_\omega\left(\bigcap_{b>0}\{T_b < \infty\}\right) = \lim_{b\to\infty} P_\omega(T_b < \infty).$$

This probability is equal to the probability that there exists a point $a < 0$ such that this point b will be hit before a, so

$$\lim_{b\to\infty} P_\omega(T_b < \infty) = \lim_{b\to\infty} P_\omega\left(\bigcup_{a<0}\{T_b < T_a\}\right).$$

Due to the continuity of probability measure (Theorem 1.3.2):

$$\begin{aligned}
&\lim_{b\to\infty} P_\omega\left(\bigcup_{a<0}\{T_b < T_a\}\right)\\
&= \lim_{b\to\infty} 1 - P_\omega\left(\bigcap_{a<0}\{T_a < T_b\}\right) = \lim_{b\to\infty}\left(\lim_{a\to-\infty} 1 - P_\omega(T_a < T_b)\right)\\
&= \lim_{b\to\infty}\left(\lim_{a\to-\infty}\left(1 - \frac{\sum_{i=1}^{b} e^{V(i)}}{\sum_{i=a+1}^{0} e^{V(i)} + \sum_{i=1}^{b} e^{V(i)}}\right)\right) = \lim_{b\to\infty}\left(\lim_{a\to-\infty}\frac{\sum_{i=a+1}^{0} e^{V(i)}}{\sum_{i=a+1}^{0} e^{V(i)} + \sum_{i=1}^{b} e^{V(i)}}\right) = 1, P_\omega\text{-a.s., because}
\end{aligned}$$

$\sum_{i=a+1}^{0} e^{V(i)} \xrightarrow[a\to-\infty]{} +\infty$, P_ω-a.s. and $\sum_{i=1}^{b} e^{V(i)}$ is finite for all b, b a fixed integer.

$\sum_{i=a+1}^{0} e^{V(i)} \xrightarrow[a\to-\infty]{} +\infty$, P_ω-a.s., because of the following:

For $i \le -1$: $V(i) = -\sum_{j=i}^{-1}\log(\rho_j)$, and for $i = 0$: $V(i) = 0$.

The law of large numbers (Theorem 1.3.3) tells us that:

$$\frac{1}{-i}V(-i) = -\frac{1}{-i}\sum_{j=i}^{-1}\log(\rho_j) \xrightarrow[i\to-\infty]{} -E_P[\log(\rho_x)] > \delta, P_\omega\text{-a.s. for some } \delta > 0.$$

So there is an $M < 0$ such that for $i < M$: $V(i) > -i\delta > 0$. So

$$\sum_{i=a+1}^{0} e^{V(i)} = \sum_{i=a+1}^{M} e^{V(i)} + \sum_{i=M+1}^{0} e^{V(i)} > \sum_{i=a+1}^{M} e^{-i\delta} + \sum_{i=M+1}^{0} e^{V(i)} \xrightarrow[a\to-\infty]{} +\infty, P_\omega\text{-a.s.},$$

because $\sum_{i=M+1}^{0} e^{V(i)}$ is finite for all $M < 0$, M fixed, and $\lim_{a\to-\infty} \sum_{i=a+1}^{M} e^{-i\delta} = +\infty$, P_ω-a.s., for all $M < 0$, M fixed, because $\sum_{i=a+1}^{M} e^{-i\delta} = \sum_{i=a+1}^{M} (e^{-\delta})^i$ is a geometric sum with $e^{-\delta} > 1$ for $\delta < 0$.

So

$$\limsup_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega.$$

$\liminf_{n\to\infty} X_n = +\infty$, P_ω-a.s.:

This part of the proof works by contradiction. Assume

$$\limsup_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s., but } \liminf_{n\to\infty} X_n \neq +\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega.$$

Then with positive probability there is an x which is hit infinitely often, P_ω-a.s. When hitting this x there is a positive probability of going one step to the left, so to the point $x-1$. In other words $1 - \omega_x > 0$. So the point $x-1$ is hit infinitely often as well, P_ω-a.s. In the same way the points $x-2, ..., x-n$ are a.s. hit infinitely often.

So if

$$\limsup_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s., but } \liminf_{n\to\infty} X_n \neq +\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega,$$

then all points are hit infinitely often and $P_\omega(T_a < \infty) > 0$, no matter how far a is chosen under 0.

But the probability of a point $a < 0$ of being hit eventually is equal to $P_\omega(T_a < \infty)$, so the probability of all points $a < 0$ of being hit eventually is, due to the continuity of probability measure (Theorem 1.3.2), equal to

$$P_\omega\left(\bigcap_{a<0} \{T_a < \infty\}\right) = \lim_{a\to-\infty} P_\omega(T_a < \infty).$$

This probability is equal to the probability that there exists a point $b > 0$ such that this point a will be hit before b, so

$$\lim_{a\to-\infty} P_\omega(T_a < \infty) = \lim_{a\to-\infty} P_\omega\left(\bigcup_{b>0} \{T_a < T_b\}\right).$$

Due to the continuity of probability measure (Theorem 1.3.2):

$$
\begin{aligned}
&\lim_{a\to-\infty} P_\omega\left(\bigcup_{b>0}\{T_a < T_b\}\right)\\
&= \lim_{a\to-\infty}\left(\lim_{b\to\infty} P_\omega(T_a < T_b)\right) = \lim_{a\to-\infty}\left(\lim_{b\to\infty}\frac{\sum\limits_{i=1}^{b} e^{V(i)}}{\sum\limits_{i=a+1}^{0} e^{V(i)} + \sum\limits_{i=1}^{b} e^{V(i)}}\right)\\
&= \lim_{a\to-\infty}\frac{\sum\limits_{i=1}^{\infty} e^{V(i)}}{\sum\limits_{i=a+1}^{0} e^{V(i)} + \sum\limits_{i=1}^{\infty} e^{V(i)}} = 0, P_\omega\text{-a.s., because}
\end{aligned}
$$

$\sum\limits_{i=a+1}^{0} e^{V(i)} \xrightarrow[a\to-\infty]{} +\infty$ and $\sum\limits_{i=1}^{\infty} e^{V(i)}$ is finite for $E_P[\log(\rho_x)] < 0$.

The fact that $\sum\limits_{i=a+1}^{0} e^{V(i)} \xrightarrow[a\to-\infty]{} +\infty$ has been proven in the part $\limsup\limits_{n\to\infty} X_n = +\infty, P_\omega$-a.s. in this paragraph.

We can show that $\sum\limits_{i=1}^{\infty} e^{V(i)}$ is finite for $E_P[\log(\rho_x)] < 0$ with the law of large numbers (Theorem 1.3.3). For $i \geq 1 : V(i) = \sum\limits_{j=0}^{i-1} \log(\rho_j)$. The law of large numbers tells us that

$$\frac{1}{i}V(i) = \frac{1}{i}\sum_{j=0}^{i-1}\log(\rho_j) \xrightarrow[i\to\infty]{} E_P[\log(\rho_x)] < \delta, P_\omega\text{-a.s. for some } \delta < 0.$$

So there is an $M > 0$, such that for $i > M$: $V(i) < i\delta < 0$. So

$$\sum_{i=1}^{\infty} e^{V(i)} = \sum_{i=1}^{M} e^{V(i)} + \sum_{i=M+1}^{\infty} e^{V(i)} < \sum_{i=1}^{M} e^{V(i)} + \sum_{i=M+1}^{\infty} e^{i\delta} \text{is finite,}$$

because $\sum\limits_{i=1}^{M} e^{V(i)}$ is finite for all $M > 0$, M fixed, and $\sum\limits_{i=M+1}^{\infty} e^{i\delta} = \sum\limits_{i=M+1}^{\infty} (e^{\delta})^i$ is finite as well for all $M > 0$, M fixed, because it is a geometric sum with $e^{\delta} < 1$ for $\delta < 0$.

The fact that

$$\lim_{a\to-\infty} P_\omega(T_a < \infty) = \lim_{a\to-\infty}\frac{\sum\limits_{i=1}^{\infty} e^{V(i)}}{\sum\limits_{i=a+1}^{0} e^{V(i)} + \sum\limits_{i=1}^{\infty} e^{V(i)}} = 0, P_\omega\text{-a.s.}$$

is a contradiction to the fact that for all a we showed that $P_\omega(T_a < \infty) > 0$, no matter how far a is chosen under 0.

So

$$\liminf_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega.$$

So, because for $E_P[\log(\rho_x)] < 0$

$$\limsup_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s. and } \liminf_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega,$$

also

$$\lim_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega$$

and therefore the random walk $(X_n)_n$ is transient to the right.

2.3.2 Transience to the left

In this part we will prove that for $E_P[\log(\rho_x)] > 0$ the random walk $(X_n)_n$ is transient to the left and

$$\lim_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega.$$

In order to prove this we have to show that for $E_P[\log(\rho_x)] > 0$

$$\liminf_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s. and } \limsup_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s., for P almost all } \omega.$$

When these statements are proven, we know that the random walk is transient to the left. Because some steps of this proof are similar to the steps of the proof of transience to the right in paragraph 2.3.1, these steps will be left out.

$\boldsymbol{\liminf_{n\to\infty} X_n = -\infty, P_\omega}$**-a.s.:**

In this part we want to prove that the probability of all points $a < 0$ of being hit eventually is equal to 1, P_ω-a.s., for P almost all ω. The probability of a point $a < 0$ of being hit eventually is equal to $P_\omega(T_a < \infty)$. In a similar way as in the part $\liminf_{n\to\infty} X_n = +\infty, P_\omega$-a.s. in paragraph 2.3.1, the probability of all points $a < 0$ of being hit eventually is, due to the continuity of probability measure (Theorem 1.3.2), equal to

$$\begin{aligned}
&\lim_{a\to-\infty} P_\omega\left(T_a < \infty\right) \\
&= \lim_{a\to-\infty}\left(\lim_{b\to\infty} P_\omega\left(T_a < T_b\right)\right) = \lim_{a\to-\infty}\left(\lim_{b\to\infty}\frac{\sum\limits_{i=1}^{b} e^{V(i)}}{\sum\limits_{i=a+1}^{0} e^{V(i)} + \sum\limits_{i=1}^{b} e^{V(i)}}\right) \\
&=1, P_\omega\text{-a.s., because}
\end{aligned}$$

$\sum\limits_{i=1}^{b} e^{V(i)} \xrightarrow[b\to\infty]{} +\infty, P_\omega$-a.s. and $\sum\limits_{i=a+1}^{0} e^{V(i)}$ is finite for all a, a a fixed integer.

The proof of $\sum\limits_{i=1}^{b} e^{V(i)} \xrightarrow[b\to\infty]{} +\infty, P_\omega$-a.s. is the similar to the proof that $\sum\limits_{i=a+1}^{0} e^{V(i)} \xrightarrow[a\to-\infty]{} +\infty, P_\omega$-a.s. in paragraph 2.3.1, but in here $V(i) = \sum\limits_{j=0}^{i-1} e^{V(i)}$, because $i \geq 1$.

So

$$\liminf_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega.$$

$\limsup\limits_{n\to\infty} X_n = -\infty$, P_ω**-a.s.:**

This part of the proof works by contradiction. Assume

$$\liminf_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s., but } \limsup_{n\to\infty} X_n \neq -\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega.$$

Then with positive probability there is an x which is hit infinitely often, P_ω-a.s. When hitting this x there is a positive probability of going one step to the right, so to the point $x+1$. In other words $\omega_x > 0$. So the point $x+1$ is hit infinitely often as well, P_ω-a.s. In the same way the points $x+2, ..., x+n$ are a.s. hit infinitely often.

So if

$$\liminf_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s., but } \limsup_{n\to\infty} X_n \neq -\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega,$$

then all points are hit infinitely often and $P_\omega(T_b < \infty) > 0$, no matter how far b is chosen above 0.

But the probability of a point $b > 0$ of being hit eventually is equal to $P_\omega(T_b < \infty)$, so, in a similar way as in the part $\limsup\limits_{n\to\infty} X_n = +\infty$, P_ω-a.s. in paragraph 2.3.1, the probability of all points $b > 0$ of being hit eventually is, due to the continuity of probability measure (Theorem 1.3.2), equal to

$$\begin{aligned}
&\lim_{b\to\infty} P_\omega(T_b < \infty) = \\
&= \lim_{b\to\infty}\left(1 - \lim_{a\to-\infty} P_\omega(T_a < T_b)\right) = \lim_{b\to\infty}\left(\lim_{a\to-\infty} \frac{\sum\limits_{i=a+1}^{0} e^{V(i)}}{\sum\limits_{i=a+1}^{0} e^{V(i)} + \sum\limits_{i=1}^{b} e^{V(i)}}\right) \\
&= \lim_{b\to\infty} \frac{\sum\limits_{i=-\infty}^{0} e^{V(i)}}{\sum\limits_{i=-\infty}^{0} e^{V(i)} + \sum\limits_{i=1}^{b} e^{V(i)}} = 0, P_\omega\text{-a.s., because}
\end{aligned}$$

$\sum\limits_{i=1}^{b} e^{V(i)} \xrightarrow[b\to\infty]{} +\infty$ and $\sum\limits_{i=-\infty}^{0} e^{V(i)}$ is finite for $E_P[\log(\rho_x)] > 0$.

The proof that $\sum\limits_{i=-\infty}^{0} e^{V(i)}$ is finite for $E_P[\log(\rho_x)] > 0$ is similar to the proof that $\sum\limits_{i=1}^{\infty} e^{V(i)}$ is finite for $E_P[\log(\rho_x)] < 0$ in paragraph 2.3.1 and the proof of $\sum\limits_{i=1}^{b} e^{V(i)} \xrightarrow[b\to\infty]{} +\infty$ is similar to the proof that $\sum\limits_{i=a+1}^{0} e^{V(i)} \xrightarrow[a\to-\infty]{} +\infty$, P_ω-a.s. in paragraph 2.3.1, but in here $V(i) = \sum\limits_{j=0}^{i-1}$, because $i \geq 1$.

The fact that

$$\lim_{b\to\infty} \frac{\sum\limits_{i=-\infty}^{0} e^{V(i)}}{\sum\limits_{i=-\infty}^{0} e^{V(i)} + \sum\limits_{i=1}^{b} e^{V(i)}} = 0, P_\omega\text{-a.s.}$$

is a contradiction to the fact that for all b we showed that $P_\omega(T_b < \infty) > 0$, no matter how far b is chosen above 0.

So

$$\limsup_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega.$$

So, because for $E_P[\log(\rho_x)] > 0$

$$\liminf_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s. and } \limsup_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega,$$

also

$$\lim_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega$$

and therefore the random walk $(X_n)_n$ is transient to the left.

2.3.3 Recurrence

In this part we will prove that for $E_P[\log(\rho_x)] = 0$ the random walk $(X_n)_n$ is recurrent and

$$\liminf_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s. and } \limsup_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega.$$

It is important to keep in mind that

$$V(i) = -\sum_{j=i}^{-1} \log(\rho_j) = -\sum_{j=1}^{|i|} \log(\rho_{-j}) \text{ for } i < 0$$

is of the form

$$V(n) = \sum_{j=1}^{n} Y_j \text{ with } n = |i| \text{ and } Y_j = -\log\left(\rho_{-j}\right).$$

Y_j is i.i.d. for every j, because ρ_j is i.i.d. for every j.

Now we are going to use the law of the iterated logarithm (Theorem 1.3.5), which is actually in between the law of large numbers (Theorem 1.3.3) and the central limit theorem (Theorem 1.3.4).

For using the law of the iterated logarithm on a sequence of i.i.d. random variables $\{X_i\}$ the following two things should hold:

(a) $\mathbb{E}[X_i] = 0$

(b) $\mathbb{V}(X_i) = 1$

For $Y_j = -\log\left(\rho_{-j}\right)$ the point (a) has been met:

$$\mathbb{E}[Y_j] = E_P[\log\left(\rho_j\right)] = 0.$$

In order to make the point (b) hold, we should rescale Y_j by dividing it by the square root of the variance of Y_j. After doing that both point (a) and point (b) have been met:

$$\mathbb{E}\left[\frac{Y_j}{\sqrt{\mathbb{V}\left(Y_j\right)}}\right] = \mathbb{E}[Y_j] = 0 \text{ and } \mathbb{V}\left[\frac{Y_j}{\sqrt{\mathbb{V}\left(Y_j\right)}}\right] = 1.$$

Now the law of the iterated logarithm tells us that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2n\log(\log(n))}} \cdot \frac{\sum_{j=1}^{n} Y_j}{\sqrt{\mathbb{V}(Y_j)}} = 1, P_\omega\text{-a.s.}$$

Since $V(n) = \sum_{j=0}^{n-1} Y_j$ for $n \geq 1$ also

$$\limsup_{n\to\infty} \frac{1}{\sqrt{2n\log(\log(n))}} \frac{V(n)}{\sqrt{\mathbb{V}(Y_j)}} = 1, P_\omega\text{-a.s.}$$

From this we can see that infinitely many of the values $V(1), V(2), \ldots$ are positive, so in particular infinitely many of the values $e^{V(1)}, e^{V(2)}, \ldots$ are larger than 1. So $\sum_{i=1}^{\infty} e^{V(i)}$ is infinite.

The law of the iterated logarithm also tells us that

$$\liminf_{n\to\infty} \frac{1}{\sqrt{2n\log(\log(n))}} \cdot \frac{\sum_{j=1}^{n} Y_j}{\sqrt{\mathbb{V}(Y_j)}} = -1, P_\omega\text{-a.s.}$$

From this we can conclude that $\sum_{j=1}^{n} Y_j$ is negative infinitely often and, since $V(-n) = -\sum_{j=1}^{n} Y_j$ for $n \geq 1$, we can see that infinitely many of the values $V(-1), V(-2), \ldots$ are positive, so in particular infinitely many of the values $e^{V(-1)}, e^{V(-2)}, \ldots$ are larger than 1. So $\sum_{i=1}^{\infty} e^{V(-i)}$ is infinite as well.

Now we will prove that

$$\liminf_{n\to\infty} X_n = -\infty, P_\omega\text{-a.s. and } \limsup_{n\to\infty} X_n = +\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega.$$

After having proven these statements, we have proven that a random walk is recurrent for $E_p[\log(\rho_x)] = 0$.

$\boldsymbol{\liminf_{n\to\infty} X_n = -\infty, P_\omega}$**-a.s.:**

As already shown in paragraph 2.3.2:

$$\begin{aligned}
&P_\omega\left(\liminf_{n\to\infty} X_n = -\infty\right) \\
&= \lim_{a\to-\infty} P_\omega\left(T_a < \infty\right) = \lim_{a\to-\infty}\left(\lim_{b\to\infty} P_\omega(T_a < T_b)\right) \\
&= \lim_{a\to-\infty}\left(\lim_{b\to\infty} \frac{\sum_{i=1}^{b} e^{V(i)}}{\sum_{i=a+1}^{0} e^{V(i)} + \sum_{i=1}^{b} e^{V(i)}}\right) = \lim_{a\to-\infty} \frac{\sum_{i=1}^{\infty} e^{V(i)}}{\sum_{i=a+1}^{0} e^{V(i)} + \sum_{i=1}^{\infty} e^{V(i)}} = 1, P_\omega\text{-a.s., because}
\end{aligned}$$

$\sum_{i=1}^{\infty} e^{V(i)}$ is infinite, so for each a

$$\frac{\sum_{i=1}^{\infty} e^{V(i)}}{\sum_{i=a+1}^{0} e^{V(i)} + \sum_{i=1}^{\infty} e^{V(i)}} = 1$$

and therefore also

$$\lim_{a\to-\infty} \frac{\sum_{i=1}^{\infty} e^{V(i)}}{\sum_{i=a+1}^{0} e^{V(i)} + \sum_{i=1}^{\infty} e^{V(i)}} = 1.$$

So

$$\liminf_{n\to\infty} = -\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega.$$

$\limsup\limits_{n\to\infty} X_n = +\infty, P_\omega$ **-a.s.:**

As already shown in paragraph 2.3.1:

$$\begin{aligned}
&P_\omega\left(\limsup_{n\to\infty} X_n = +\infty\right)\\
&= \lim_{b\to\infty} P_\omega\left(T_b < \infty\right) = \lim_{b\to\infty}\left(\lim_{a\to-\infty} 1 - P_\omega\left(T_a < T_b\right)\right)\\
&= \lim_{b\to\infty}\left(\lim_{a\to-\infty} \frac{\sum\limits_{i=a+1}^{0} e^{V(i)}}{\sum\limits_{i=a+1}^{0} e^{V(i)} + \sum\limits_{i=1}^{b} e^{V(i)}}\right) = \lim_{b\to\infty} \frac{\sum\limits_{i=-\infty}^{0} e^{V(i)}}{\sum\limits_{i=-\infty}^{0} e^{V(i)} + \sum\limits_{i=1}^{b} e^{V(i)}} = 1, P_\omega\text{-a.s., because}
\end{aligned}$$

$\sum\limits_{i=-\infty}^{0} e^{V(i)}$ is infinite and $\sum\limits_{i=1}^{b} e^{V(i)}$ is finite for all b, b a fixed integer.
$\sum\limits_{i=-\infty}^{0} e^{V(i)}$ is infinite, because $\sum\limits_{i=-\infty}^{0} e^{V(i)} = \sum\limits_{i=0}^{\infty} e^{V(-i)}$, which is infinite.

So

$$\limsup_{n\to\infty} = +\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega.$$

So for $\mathbb{E}[Y_i] = E_P[\log(\rho_i)] = 0$

$$\limsup_{n\to\infty} = +\infty, P_\omega\text{-a.s. and } \liminf_{n\to\infty} = -\infty, P_\omega\text{-a.s., for } P \text{ almost all } \omega$$

and therefore the random walk $(X_n)_n$ is recurrent. □

2.4 Application Theorem 2.3.1

To finish this chapter we are going to apply Theorem 2.3.1 to the random walk $(X_n)_n$ in Example 2.1.1. $(X_n)_n$ has been defined such that

$$P\left(\omega_x = \frac{3}{4}\right) = p \text{ and } P\left(\omega_x = \frac{1}{3}\right) = 1 - p \text{ for some } p \in (0,1).$$

In order to describe the behaviour of recurrence and transience of $(X_n)_n$ we should calculate $E_P[\log(\rho_x)]$:

$$\begin{aligned}
&E_P[\log(\rho_x)] \\
=&\log\left(\frac{1-\frac{3}{4}}{\frac{3}{4}}\right)\cdot p + \left(\frac{1-\frac{1}{3}}{\frac{1}{3}}\right)\cdot(1-p) \\
=&\log\left(\frac{1}{3}\right)\cdot p + \log(2)\cdot(1-p) = -\log(3)\cdot p + \log(2)\cdot(1-p)
\end{aligned}$$

So the random walk $(X_n)_n$:

- is transient to the right, and therefore $\lim_{n\to\infty} X_n = +\infty$, P_ω-a.s., for P almost all ω, if

$$E_P[\log(\rho_x)] = -\log(3)\cdot p + \log(2)\cdot(1-p) < 0, \text{ which is the case for } p > \frac{\log(2)}{\log(6)}.$$

- is transient to the left, and therefore $\lim_{n\to\infty} X_n = -\infty$, P_ω-a.s., for P almost all ω, if

$$E_P[\log(\rho_x)] = -\log(3)\cdot p + \log(2)\cdot(1-p) > 0, \text{ which is the case for } p < \frac{\log(2)}{\log(6)}.$$

- is recurrent, and therefore $\limsup_{n\to\infty} X_n = +\infty$ and $\liminf_{n\to\infty} X_n = -\infty$, P_ω-a.s., for P almost all ω, if

$$E_P[\log(\rho_x)] = -\log(3)\cdot p + \log(2)\cdot(1-p) = 0, \text{ which is the case for } p = \frac{\log(2)}{\log(6)}.$$

3 Random Walk on a Ladder graph

Theorem 2.3.1 seems to be useful for the main purpose of this report, which is understanding the behaviour recurrence and transience of a random walk on a ladder graph. In this chapter I have used articles of Erwin Bolthausen and Ilya Goldscheid in the magazine Communications in Mathematical Physics [1], in which the necessary and sufficient conditions for the behaviour of recurrence and transience on a strip are mentioned, and of Michael S. Keane and Silke W.W. Rolles in the magazine Acta Mathematica Hungarica [6], in which is proven that the process of recurrence and transience of a directed-edge-reinforced random walk is equivalent to a random walk in a random environment. Now at first we will explain the idea of a random walk on a ladder graph.

3.1 Introduction to a Random Walk on a Ladder Graph

In a random walk on a ladder graph two random walks in one dimension are linked to each other, which makes it possible to switch between the layers of these random walks.
We define a random walk on a ladder graph $(Z_n)_n$ such that

$$Z_n = (X_n, Y_n) \in \mathbb{Z} \times \{0, 1\},$$

with $X_n \in \mathbb{Z}$ as the horizontal position of the random walk and $Y_n \in \{0, 1\}$ as the vertical position of the random walk. In a random walk on a ladder graph the horizontal position X_n is comparable with the X_n in a random walk in one dimension and for Y_n we define that $Y_n = 0$ means the top layer and $Y_n = 1$ the bottom layer. The state space Ω of a random walk on a ladder graph will be defined as:

$$\Omega = [0, 1]^{\mathbb{Z} \times \{0,1\}}.$$

The next image might give a better idea of what a ladder graph looks like.

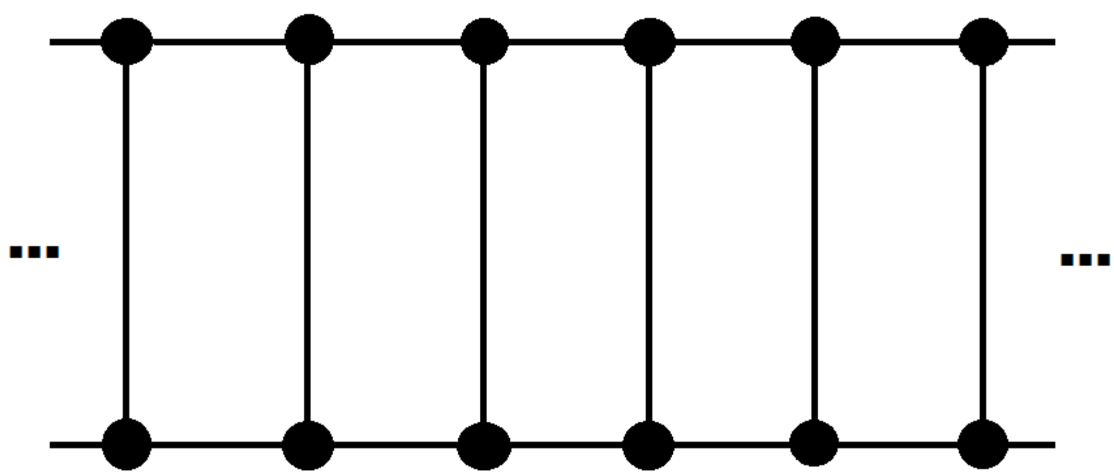


Figure 4: Ladder graph.

In this report the probability distribution of a random walk on a ladder graph will be given by $P_{\omega}^{(x,y)}$ with $x \in \mathbb{Z}$ as the horizontal coordinate in the ladder graph and $y \in \{0,1\}$ the vertical coordinate, for which the case $y = 0$ means the top layer of the ladder graph and $y = 1$ means the bottom layer. In the same way from now on we will define ω as $\omega = \omega_{(x,y)}$. In a certain point $(x, 0)$ on the ladder graph, there are three possibilities:

(i) Switching to the other layer. For both the top layer ($y = 0$) and the bottom layer ($y = 1$) we define the probability for this by γ.

(ii) Going to the right, which is the point $(x + 1, 0)$. The probability for this is

$$\omega_{(x,0)} \cdot (1 - \gamma)\,.$$

(iii) Going to the left, which is the point $(x - 1, 0)$. The probability for this is

$$\left(1 - \omega_{(x,0)}\right) \cdot (1 - \gamma)\,.$$

In the next paragraph we will discuss an example of a random walk on a ladder graph, which might give us insight in the behaviour of recurrence and transience of random walks on a ladder graph.

3.2 Example, Random Walk on a Ladder Graph

Assume a random walk on a ladder graph $(Z_n)_n$ with just the following two types of nodes on either the top layer ($y = 0$) or the bottom layer ($y = 1$):

(i) Type A with

$$\omega_{(x,y)} = 1 - \epsilon \text{ with probability } p.$$

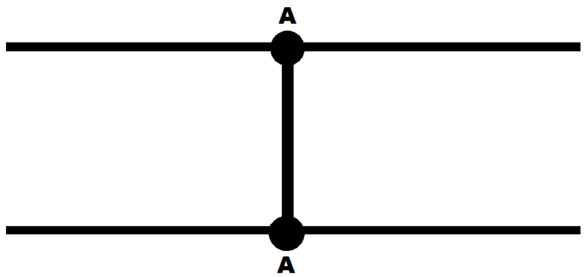
(ii) Type B with

$$\omega_{(x,y)} = \frac{1}{4} \text{ with probability } 1 - p.$$

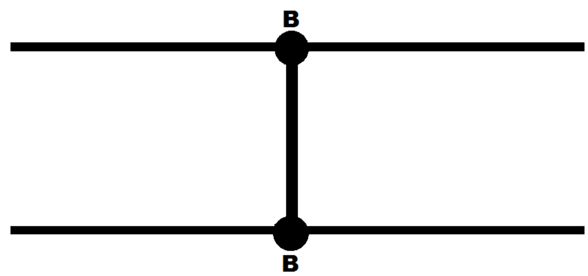
In this example we will only look at values of ϵ and p of:

$$0 < \epsilon < \frac{1}{4} \text{ and } 0 < p < 1.$$

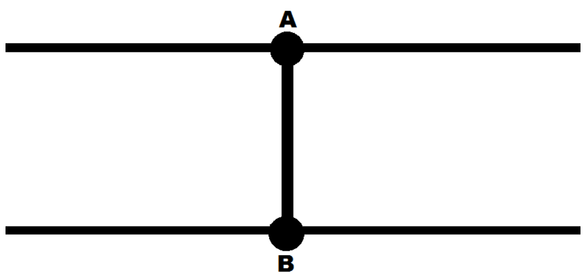
We are going to show that there are values of ϵ, p and γ, such that both layers of the random walk $(Z_n)_n$ themselves are transient to the right, but that the random walk $(Z_n)_n$ is transient to the left. The idea is that for ϵ close to 0 nodes of Type A will act as boundaries for going to the left for both layers of the random walk, but, because a random walk on a ladder graph has a third possibility in switching between the layers, in the random walk on the ladder graph it will be able to "jump over" these boundaries.
In order to show this we will look at the four scenarios on the next page which can occur for two nodes above each other with the same horizontal coordinate x.



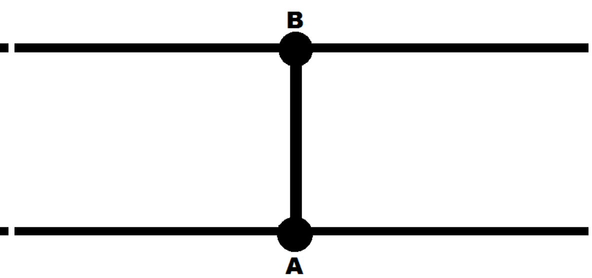
(a) A in top and bottom layer.



(b) B in top and bottom layer.



(c) A in top layer and B in bottom layer.



(d) B in top layer and A in bottom Layer.

Figure 5: Four scenarios for two nodes above each other with the same horizontal coordinate x.

For these scenarios we will look for a worst-case estimate, $\tilde{\omega}_x$, for which

(i) $P_\omega^{(x,0)}$(The first horizontal step is to the right) $\leq \tilde{\omega}_x$ and

(ii) $P_\omega^{(x,1)}$(The first horizontal step is to the right) $\leq \tilde{\omega}_x$.

Define H such that H is the value of n for which the horizontal coordinate changes for the first time.

$$H = \inf\{n : X_n \neq X_0\}.$$

When going to the right at first $X_H = x + 1$ and when going to the left at first $X_H = x - 1$. This happens in the top layer for $Y_H = 0$ and in the bottom layer for $Y_H = 1$.

In scenario 5a with nodes of Type A in both the top layer and the bottom layer, when starting in the top layer:

$$\begin{aligned}
&P_\omega^{(x,0)}(\text{going to the right}) = P_\omega^{(x,0)}(X_H = x+1) \\
=&P_\omega^{(x,0)}(X_H = x+1 | Y_H = 1) \cdot P_\omega^{(x,0)}(Y_H = 1) + P_\omega^{(x,0)}(X_H = x+1 | Y_H = 0) \cdot P_\omega^{(x,0)}(Y_H = 0) \\
=&(1-\epsilon) \cdot P_\omega^{(x,0)}(Y_H = 1) + (1-\epsilon) \cdot P_\omega^{(x,0)}(Y_H = 0) = 1 - \epsilon \\
=&P_\omega^{(x,1)}(X_H = x+1) = P_\omega^{(x,1)}(\text{going to the right}),
\end{aligned}$$

because both layers contain a node of Type A. So for a node of Type A in the top layer and a node of Type A in the bottom layer the chance of going to the right when starting in the top layer is equal to the chance of going to the right when starting in the bottom layer.
So for the worst-case estimate of scenario 5a we choose

$$\tilde{\omega}_x = 1 - \epsilon.$$

In scenario 5b with nodes of Type B in both the top layer and the bottom layer, when starting in the top layer:

$$\begin{aligned}
&P_\omega^{(x,0)}(\text{going to the right}) = P_\omega^{(x,0)}(X_H = x+1)\\
=&P_\omega^{(x,0)}(X_H = x+1|Y_H = 1)\cdot P_\omega^{(x,0)}(Y_H = 1) + P_\omega^{(x,0)}(X_H = x+1|Y_H = 0)\cdot P_\omega^{(x,0)}(Y_H = 0)\\
=&\frac{1}{4}\cdot P_\omega^{(x,0)}(Y_H = 1) + \frac{1}{4}\cdot P_\omega^{(x,0)}(Y_H = 0) = \frac{1}{4}\\
=&P_\omega^{(x,1)}(X_H = x+1) = P_\omega^{(x,1)}(\text{going to the right}),
\end{aligned}$$

because both layers contain a node of Type B. So for a node of Type B in the top layer and a node of Type B in the bottom layer the chance of going to the right when starting in the top layer is equal to the chance of going to the right when starting in the bottom layer, such as in scenario 5a
So for the worst-case estimate of scenario 5b we choose

$$\tilde{\omega}_x = \frac{1}{4}.$$

Scenario 5c with a node of Type A in the top layer and a node of Type B in the bottom layer and scenario 5d with a node of Type B in the top layer and a node of Type A in the bottom layer are more complicated, because in these scenarios the chance of going to the right when starting in the top layer is not equal to the chance of going to the right when starting in the bottom layer. Nevertheless, scenario 5c is similar to scenario 5d.

In scenario 5c with a node of Type A in the top layer and a node of Type B in the bottom layer, when starting in the top layer the chance that the random walk switches coordinates horizontally for the first time in the top layer is:

$$P_\omega^{(x,0)}(Y_H = 0) = (1-\gamma)\cdot\sum_{i=0}^{\infty}\gamma^{2i} = \frac{1}{1+\gamma}$$

and the chance that the random walk switches coordinates horizontally for the first time in the bottom layer is:

$$P_\omega^{(x,0)}(Y_H = 1) = (1-\gamma)\cdot\sum_{i=0}^{\infty}\gamma^{2i+1} = \frac{\gamma}{1+\gamma}.$$

So when starting in the top layer:

$$\begin{aligned}
&P_\omega^{(x,0)}(\text{going to the right}) = P_\omega^{(x,0)}(X_H = x+1)\\
=&P_\omega^{(x,0)}(X_H = x+1|Y_H = 0)\cdot P_\omega^{(x,0)}(Y_H = 0) + P_\omega^{(x,0)}(X_H = x+1|Y_H = 1)\cdot P_\omega^{(x,0)}(Y_H = 1)\\
=&(1-\epsilon)\cdot\frac{1}{1+\gamma} + \frac{1}{4}\cdot\frac{\gamma}{1+\gamma} = \frac{1-\epsilon}{1+\gamma} + \frac{\gamma}{4\,(1+\gamma)}.
\end{aligned}$$

When starting in the bottom layer in scenario 5c the chance that the random walk switches coordinates horizontally for the first time in the top layer is:

$$P_\omega^{(x,1)}(Y_H = 0) = (1-\gamma)\cdot\sum_{i=0}^{\infty}\gamma^{2i+1} = \frac{\gamma}{1+\gamma}$$

and the chance that the random walk switches coordinates horizontally for the first time in the bottom layer is:

$$P_\omega^{(x,1)}(Y_H = 1) = (1-\gamma)\cdot\sum_{i=0}^{\infty}\gamma^{2i} = \frac{1}{1+\gamma}.$$

So when starting in the bottom layer:

$$
\begin{aligned}
&P_\omega^{(x,1)}(\text{going to the right}) = P_\omega^{(x,1)}(X_H = x+1) \\
=&P_\omega^{(x,1)}(X_H = x+1 | Y_H = 1) \cdot P_\omega^{(x,1)}(Y_H = 1) + P_\omega^{(x,1)}(X_H = x+1 | Y_H = 0) \cdot P_\omega^{(x,1)}(Y_H = 0) \\
=&\frac{1}{4} \cdot \frac{1}{1+\gamma} + (1-\epsilon) \cdot \frac{\gamma}{1+\gamma} = \frac{1}{4(1+\gamma)} + \frac{(1-\epsilon)\cdot\gamma}{(1+\gamma)}.
\end{aligned}
$$

Interesting in scenario 5c is that

$$P_\omega^{(x,0)}(\text{going to the right}) \neq P_\omega^{(x,1)}(\text{going to the right}).$$

But we can explain that

$$P_\omega^{(x,0)}(\text{going to the right}) \geq P_\omega^{(x,1)}(\text{going to the right}), \text{ because}$$

in $P_\omega^{(x,1)}$(going to the right) the first point that the random walk finds is a node of Type B, while in $P_\omega^{(x,0)}$(going to the right) the first point that the random walk finds is a node of Type A. The chance of going to the right for a node of Type A is bigger than for a node of Type B.
So for the worst-case estimate $\tilde{\omega}_x$ of scenario 5c we choose

$$\tilde{\omega}_x = P_\omega^{(x,0)}(\text{going to the right}) = \frac{1-\epsilon}{1+\gamma} + \frac{\gamma}{4(1+\gamma)}.$$

Because scenario 5d is similar to scenario 5c, except for the fact that the layers have been turned around, for scenario 5d we choose the worst-case estimate equal to the worst-case estimate of scenario 5c. So for the worst-case estimate $\tilde{\omega}_x$ of scenario 5d we choose

$$\tilde{\omega}_x = \frac{1-\epsilon}{1+\gamma} + \frac{\gamma}{4(1+\gamma)}.$$

The worst-case estimates are summarized in the following table:

	Scenario 5a	Scenario 5b	Scenario 5c and 5d
Worst-case estimate $\tilde{\omega}_x$	$1-\epsilon$	$\frac{1}{4}$	$\frac{1-\epsilon}{1+\gamma} + \frac{\gamma}{4(1+\gamma)}$
Probability of occuring	p^2	$(1-p)^2$	$2p(1-p)$

It is important to realize that we can see the random walk on a ladder graph $(Z_n)_n$ with only nodes of Type A and Type B as a random walk in one dimension with nodes corresponding to the worst-case scenarios from the last table. The explanation for this will follow.

Consider a random walk in one dimension $(\tilde{Z}_n)_n$ with:

$$\tilde{\omega}_x = \begin{cases} 1-\epsilon & \text{in a node of Type C occuring with probability } p^2, \\ \frac{1}{4} & \text{in a node of Type D occuring with probability } (1-p)^2, \\ \frac{1-\epsilon}{1+\gamma} + \frac{\gamma}{4(1+\gamma)} & \text{in a node of Type E occuring with probability } 2p(1-p). \end{cases}$$

The chance of going to the right for a point on $(Z_n)_n$ is always less than or equal to the chance of going to the right for the corresponding point on $(\tilde{Z}_n)_n$, and therefore the chance of going to the left for a point on $(Z_n)_n$ is always larger than or equal to the chance of going to the left for the corresponding point on $(\tilde{Z}_n)_n$.
So if $(\tilde{Z}_n)_n$ is transient to the left, P_ω-a.s., then for sure $(Z_n)_n$ is transient to the left, P_ω-a.s.

Now we can use this in order to find values of ϵ, p or γ for which $(Z_n)_n$ is transient to the left, but for which both layers of $(Z_n)_n$ are transient to the right. We will do this with theorem 2.3.1.
For $\rho_x = \frac{1-\omega_x}{\omega_x}$ in order to find the behaviour of recurrence and transience of a random walk in one dimension we should calculate $E_p[\log(\rho_x)]$. For both layers of $(Z_n)_n$:

$$E_P[\log(\rho_x)] = \log\left(\frac{1-\frac{1}{4}}{\frac{1}{4}}\right)\cdot(1-p) + \log\left(\frac{1-(1-\epsilon)}{1-\epsilon}\right)\cdot p = \log(3)\cdot(1-p) + \log\left(\frac{\epsilon}{1-\epsilon}\right)\cdot p.$$

The layers of $(Z_n)_n$ are transient to the right for $E_p[\log(\rho_x)] < 0$.

For $\tilde{\rho}_x = \frac{1-\tilde{\omega}_x}{\tilde{\omega}_x}$ for $(\tilde{Z}_n)_n$:

$$\begin{aligned}
&E_p[\log(\tilde{\rho}_x)]\\
&= \log\left(\frac{1-\frac{1}{4}}{\frac{1}{4}}\right)\cdot(1-p)^2 + \log\left(\frac{1-(1-\epsilon)}{1-\epsilon}\right)\cdot p^2 + \log\left(\frac{1-\left(\frac{1-\epsilon}{1+\gamma}+\frac{\gamma}{4(1+\gamma)}\right)}{\frac{1-\epsilon}{1+\gamma}+\frac{\gamma}{4(1+\gamma)}}\right)\cdot 2p(1-p)\\
&= \log(3)\cdot(1-p)^2 + \log\left(\frac{\epsilon}{1-\epsilon}\right)\cdot p^2 + \log\left(\frac{1-\left(\frac{1-\epsilon}{1+\gamma}+\frac{\gamma}{4(1+\gamma)}\right)}{\frac{1-\epsilon}{1+\gamma}+\frac{\gamma}{4(1+\gamma)}}\right)\cdot 2p(1-p).
\end{aligned}$$

$(Z_n)_n$ is transient to the left if $(\tilde{Z}_n)_n$ is transient to the left. This is the case for $E_p[\log(\tilde{\rho}_x)] > 0$.

So in order to find a solution for which $(Z_n)_n$ is transient to the left, when both layers of $(Z_n)_n$ are transient to the right, we should solve the following inequalities:

(i) For both the top layer and the bottom layer of $(Z_n)_n$ itself:

$$E_P[\log(\rho_x)] = \log\left(\frac{1-\frac{1}{4}}{\frac{1}{4}}\right)\cdot(1-p) + \log\left(\frac{1-(1-\epsilon)}{1-\epsilon}\right)\cdot p = \log(3)\cdot(1-p) + \log\left(\frac{\epsilon}{1-\epsilon}\right)\cdot p < 0.$$

(ii) For $(\tilde{Z}_n)_n$:

$$E_p[\log(\tilde{\rho}_x)] = \log(3)\cdot(1-p)^2 + \log\left(\frac{\epsilon}{1-\epsilon}\right)\cdot p^2 + \log\left(\frac{1-\left(\frac{1-\epsilon}{1+\gamma}+\frac{\gamma}{4(1+\gamma)}\right)}{\frac{1-\epsilon}{1+\gamma}+\frac{\gamma}{4(1+\gamma)}}\right)\cdot 2p(1-p) > 0.$$

In solving these inequalities we have found that there are combinations of ϵ, p and γ for which they hold. There are more solutions, but one solution for which the inequalities hold is when choosing for ϵ, p and γ:

$$\epsilon = 0.01,\ p = 0.2 \text{ and } \gamma = 0.99.$$

For these choices of ϵ, p and γ:

(i)

$$
\begin{aligned}
&E_P[\log(\rho_x)]\\
=&\log\left(\frac{1-\frac{1}{4}}{\frac{1}{4}}\right)\cdot(1-p)+\log\left(\frac{1-(1-\epsilon)}{1-\epsilon}\right)\cdot p=\log(3)\cdot(1-p)+\log\left(\frac{\epsilon}{1-\epsilon}\right)\cdot p\\
=&-0.0401341<0 \text{ and}
\end{aligned}
$$

(ii)

$$
\begin{aligned}
&E_p[\log(\tilde{\rho}_x)]\\
=&\log(3)\cdot(1-p)^2+\log\left(\frac{\epsilon}{1-\epsilon}\right)\cdot p^2+\log\left(\frac{1-\left(\frac{1-\epsilon}{1+\gamma}+\frac{\gamma}{4(1+\gamma)}\right)}{\frac{1-\epsilon}{1+\gamma}+\frac{\gamma}{4(1+\gamma)}}\right)\cdot 2p(1-p)\\
=&0.360124>0.
\end{aligned}
$$

So now we have showed that there are possibilities of random walks on a ladder graph for which both layers of the random walk are transient to the right, but the random walk on the ladder graph is transient to the left. As we mentioned before there are more possible solutions for this problem. For these solutions the following should hold:

- ϵ should be small, such that nodes of Type A act as boundaries for going to the left in one dimension, but ϵ can not be too small, because in that case $(Z_n)_n$ itself also becomes transient to the right.
- p should be small, such that nodes of Type A do not occur too many times. In that case $(Z_n)_n$ is not transient to the left. But p should also not be too small, because in that case there are not enough nodes of Type A in order to make that the layers of $(Z_n)_n$ themselves are transient to the right.
- γ should be large enough, such that in $(Z_n)_n$ it is possible to jump over the nodes of type A.

4 Conclusion

Random walks on a ladder graph are more complicated than random walks in random environments in one dimension, which are more complicated than simple random walks. In paragraph 1.4 we have seen that simple random walks are transient to the right for $\omega > \frac{1}{2}$, transient to the left for $\omega < \frac{1}{2}$ and recurrent for $\omega = \frac{1}{2}$. The behaviour of recurrence and transience of random walks in random environments in one dimension is more difficult than in simple random walks. Criteria for recurrence and transience of a random walk in a random environment in one dimension are given in theorem 2.3.1. And the behaviour of recurrence and transience of random walks on a ladder graph is more complicated than both in simple random walks and random walks in random environments in one dimension. A possibility to find recurrence and transience, which works for some examples of random walks on a ladder graph, is to bring it back to one dimension as we did in paragraph 3.2.

The main conclusion of this report is that there are examples for which a random walk on a ladder graph is transient to the left, when both layers of this random walk on the ladder graph themselves are transient to the right. The idea of the example mentioned in paragraph 3.2 is that it is possible to "jump over" points that act as boundaries for going to the left, because in a random walk on a ladder graph, in addition to going to the left or to the right, a third possibility occurs in switching to the other layer.
In this example we found that, for some choices of ϵ, p and γ, one possibility of a random walk on a ladder graph for which both layers themselves are transient to the right, but the random walk on the ladder graph is transient to the left, is a random walk of which both layers contain of nodes of Type A with $\omega_{(x,y)} = 1 - \epsilon$ with probability p and Type B with $\omega_{(x,y)} = \frac{1}{4}$ with probability $1 - p$. Both layers of this random walk are transient to the right for sure, because nodes of Type A will act as boundaries for going to the left when choosing ϵ small enough, but for the random walk on the ladder graph transience to the right is not a certainty for all ϵ, p and γ. It is possible to bring the random walk on the ladder graph back to a random walk in one dimension, using worst-case estimates, and to use Theorem 2.3.1 to check the transience (or recurrence).
One combination of possible values of ϵ, p and γ for which the random walk on the ladder graph is transient to the left is for

$$\epsilon = 0.01,\ p = 0.2 \text{ and } \gamma = 0.99.$$

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