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H∞ and H2 optimal sampled-data controller synthesis

a hybrid systems approach with generalised disturbance and performance channels

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$\mathcal{H}_\infty$ and $\mathcal{H}_2$ Optimal Sampled-data Controller Synthesis: A Hybrid Systems Approach with Generalised Disturbance and Performance Channels

H.J. Dreef

Abstract— **Discrete-time controllers, implemented on digital platforms, are generally used to control continuous-time plants using sampled-measurements. For these sampled-data systems, the frequency on which samples are taken and control inputs are updated is essential for stability and performance of the system. Still, the available design methods either do not explicitly account for or make assumptions impairing practical application. In this paper, a tractable synthesis method is proposed that gives guarantees for stability and performance in terms of the $\mathcal{H}_\infty$- and $\mathcal{H}_2$-norm, while taking the effect of sampling explicitly into account. Furthermore, the method is extended to bridge the gap towards continuous-time design techniques like $\mathcal{H}_\infty$ loop shaping by considering generalised disturbance and performance channels, where both discrete and continuous signals are weighted by using designed filters. The hybrid system approach allows formulating Linear Matrix Inequalities using the explicit solution to the Riccati differential equation. Furthermore, a new approximation method for the $\mathcal{H}_2$-norm is proposed. The resulting controllers are validated on academic examples and on a more practical design example.**

I. Introduction

The vast majority of modern control problems can be considered as sampled-data systems, in which a continuous-time plant is controlled using sampled measurements and a (discrete-time) controller is implemented on a digital platform. The frequency on which samples are taken and control inputs are updated is essential for the performance and even the stability of the system, which are not relevant in controller design when the sampling frequency is chosen sufficiently large, see e.g., [1], [2]. However, taking sampling explicitly into account can give performance and stability guarantees when resources are limited. In a way, traditional methods, described in [1], [3], [4], fall into a two-step category, being sampling/discretisation and controller design/synthesis in arbitrary order, illustrated by dashed lines in Fig. 1. A direct method, indicated with a solid line, would give guarantees on stability and desired behaviour, while helping the selection of the sampling/update frequency.

There are numerous tools available for controller design of discrete- and continuous-time plants in different domains, see e.g., [1], [3], [4]. Nowadays, complex systems with interaction, uncertainty, disturbances and multiple in- and outputs are encountered and require advanced controller design techniques. Computer algorithms can be employed to deal with these concepts and obtain controllers that satisfy *a priori* formulated performance specifications, which is called synthesis. Usually performance is formulated in terms of the $\mathcal{H}_\infty$- and $\mathcal{H}_2$-norm, while most design methods are presented with respect to the $\mathcal{H}_\infty$-norm for continuous-time systems, see e.g., [3], [4]. Using weighting filters, a weighted plant model can be formulated, with which the desired output for the expected magnitudes of the input signals can be described. Synthesis with respect to this model effectively shapes the closed-loop transfers and is referred to as $\mathcal{H}_\infty$ loop or sensitivity shaping, see e.g., [3], [4].

Extensive research has been done on the topic of sampled-data controller synthesis with respect to the $\mathcal{H}_\infty$-norm in the last few decades, see e.g., [5]–[13]. The proposed methods can be categorised in two groups, namely results using the “lifting” technique and another with a hybrid systems approach. The “lifting” technique, which derives a norm-equivalent discrete-time system on which well-known synthesis techniques can be applied, was the first tool to solve the sampled-data problem with the main disadvantage that measurement noise cannot be present, see e.g., [5]–[8] and references therein. The hybrid system approach describes the combined continuous- and discrete-time behaviour, see e.g., [14]. Using this approach, results are presented in terms of Riccati differential and difference equations in, e.g., [9], [10]. It should be noted that solving these Riccati equations is not trivial, which inspired [11] to reformulate them into Differential Linear Matrix Inequalities (DLMIs). The advantage of this method is that DLMIs can be converted into Linear Matrix Inequalities (LMIs) by making (conservative) piecewise linear approximations of the solution to the Riccati

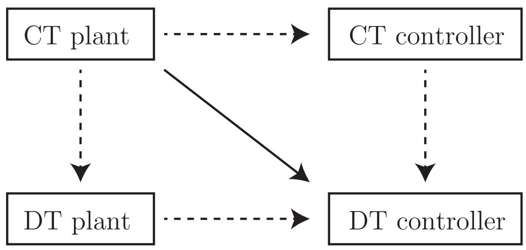


Fig. 1: The contrast between traditional discretisation and design/synthesis steps taken in control design (dashed) and the direct synthesis of a discrete-time controller for a continuous-time plant (solid).

differential equation. Moreover, although LMIs are sparsely present in the literature, there are results in, for example state feedback, see e.g., [12] and output feedback control, see e.g., [13], where again no measurement noise is considered. Despite all the presented results, the problem of how to formulate a practical design problem as an $\mathcal{H}_\infty$ sampled-data synthesis problem has, to the author's knowledge, not been considered. For example in all the reviewed literature, no reference tracking problem is found, for which it should be noted that consideration of a measurement disturbance is essential.

The $\mathcal{H}_2$-norm problem has also been addressed in the literature, albeit that several definitions are treated. Disturbance impulse interpretations similar to the Linear Time Invariant (LTI) case are derived in e.g., [9], [15]–[17]. However, a tractable solution for this definition has not been found in the reviewed literature. Therefore, approximations are proposed by either presuming that the disturbance impulse occurs during the jump part of the system, see e.g., [11], [12], or by assuming a linear time-dependency in the Lyapunov function, see e.g., [18].

This paper presents a tractable hybrid system based method for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ sampled-data controller synthesis, which is extended to connect to the well-known $\mathcal{H}_\infty$ loop shaping design approach to consider design problems, see e.g., [3]. To allow for $\mathcal{H}_\infty$ loop shaping, a discrete measurement disturbance, which is required for this extension, is also included. The solution is formulated in terms of LMIs, which exploits the explicit solution to the Riccati differential equation that emerges to find a time-dependent Lyapunov function during the flow of the hybrid system. Furthermore, we present an approximation, treating the general case of the $\mathcal{H}_2$-norm. The proposed controllers are validated on academic examples found in [11], but also on a more practical design problem.

The outline of this paper is as follows. Section II presents the considered sampled-data problem, where the hybrid system is introduced and its stability and performance specifications in terms of the $\mathcal{H}_\infty$- and $\mathcal{H}_2$-norm are defined. In Section III, a method is proposed to solve the synthesis problem with respect to the $\mathcal{H}_\infty$-norm using LMIs, which are formulated using the explicit solution to the Riccati differential equation during the flow. The same methodology, along with an approximation method, is applied to the $\mathcal{H}_2$-norm problem in Section IV. With the proposed $\mathcal{H}_\infty$ synthesis method, an extension towards $\mathcal{H}_\infty$ loop shaping is made in Section V, for which an academic example and a more practical example is given in Section VI to show that indeed the specified conditions are satisfied. At last, conclusions will be drawn in Section VII.

Nomenclature: A function $f(t)$ evaluated at $t = t_k \in \mathbb{R}_{\geq 0}$ with $k \in \mathbb{N}$ is denoted as $f[k] = f(t_k)$. For a vector $x \in \mathbb{R}^n$, we denote by $\|x\| := \sqrt{x^\top x}$ its 2-norm. For a signal $x : \mathbb{N} \to \mathbb{R}^n$, we denote by $\|x\|_{\ell_2} = \sqrt{\sum_{k=0}^{\infty} \|x[k]\|^2}$ its ℓ_2-norm and for a signal $x : \mathbb{R} \to \mathbb{R}^n$, we denote by $\|x\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty \|x(t)\|^2 dt}$ its $\mathcal{L}_2$-norm, provided that these quantities are finite. For brevity, we write symmetric matrices of the form $\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix}$ as $\begin{bmatrix} A & \star \\ B^\top & C \end{bmatrix}$. Furthermore, the trace of a square matrix is denoted by $\mathrm{tr}(\cdot)$ and $\mathrm{diag}(A_1, \ldots, A_n)$ denotes a block-diagonal matrix with the entries $A_1, \ldots, A_n$ on the diagonal. At last, $A^\top \in \mathbb{R}^{m \times n}$ denotes the transpose of matrix $A \in \mathbb{R}^{n \times m}$.

II. Problem Statement

Sampled-data systems can generally be represented by the following continuous-time (generalised) plant with sampled in- and outputs:

$$\mathcal{P} : \begin{cases} \dot{x}^p(t) = A^p x^p(t) + B_c^p w_c(t) + B_u^p u(t), \\ z_c(t) = C_c^p x^p(t) + D_{cc}^p w_c(t), \\ z_d[k] = C_d^p x^p[k] + D_{dd}^p w_d[k] + D_{du}^p \hat{u}[k], \\ y[k] = C_y^p x^p[k] + D_{yd}^p w_d[k], \end{cases} \tag{1}$$

in which $t_k = kh$, with sample time $h > 0$, time $t \in \mathbb{R}_+$, sample instants $k \in \mathbb{N}$ and initial condition $x^p(0) = x_0$. In (1), $x^p \in \mathbb{R}^{n_p}$ denotes the states of the plant, $u \in \mathbb{R}^{n_u}$ the continuous input and $\hat{u} \in \mathbb{R}^{n_u}$ the discrete-time input, which is the sampled signal of $u(t)$ with zero-order hold (ZOH), i.e.,

$$u(t) = \hat{u}[k], \text{ for } \; t \in [t_k, t_{k+1}), \tag{2}$$

with $k \in \mathbb{N}$. Furthermore, $y \in \mathbb{R}^{n_y}$ denotes the sampled output, $z_c \in \mathbb{R}^{n_{z_c}}$ the continuous and $z_d \in \mathbb{R}^{n_{z_d}}$ the discrete performance channel, just as $w_c \in \mathbb{R}^{n_{w_c}}$ and $w_d \in \mathbb{R}^{n_{w_d}}$ denote the continuous and discrete generalised disturbance, respectively. The distinction in performance channels z_c and z_d is made, because the influence of the discrete generalised disturbance $w_d[k]$ cannot be evaluated by a continuous performance channel, which is generally not treated, see e.g., [5]–[8], [11]–[13], [15]–[17]. Furthermore, it can be noticed that a distinction is made between the continuous signal $w_c(t)$ and the discrete signal $w_d[k]$. Often when considering generalised continuous- or discrete-time plants, the same disturbance signals are considered on the plant and output.[1] However, in this hybrid plant they are separated in $w_c(t)$ and $w_d[k]$, due to their inherently different nature.

The plant is controlled by the discrete-time controller

$$\mathcal{K} : \begin{cases} x^c[k+1] = A^c x^c[k] + B^c y[k], \\ \hat{u}[k] = C^c x^c[k] + D^c y[k], \end{cases} \tag{3}$$

with the state vector $x^c \in \mathbb{R}^{n_c}$. The transfer $u \to y$, in (1), is strictly proper and can be assumed without loss of generality, since the controller $\mathcal{K}_D = \mathcal{K}(I + D_{yu}^p \mathcal{K})^{-1}$ gives the controller for $D_{yu}^p \neq 0$, see e.g., [3], [4]. For readability, the discrete- or continuous-time dependencies will be omitted in the remainder of this paper, except when more clarity is required.

Using the defined controller and plant, a closed-loop system can be formulated as a hybrid system by defining the

[1] In purely continuous- or discrete-time methods $D_{cc}^p \neq 0$ or $D_{dd}^p \neq 0$, respectively, can be handled by using loop shifting. Because of the hybrid nature of the system, we have decided to keep them in the problem formulation and address them directly, rather than using a generalised version of loop-shifting.

new state vector $\xi := \begin{bmatrix}(x^p)^\top & u^\top & (x^c)^\top\end{bmatrix}^\top$. Combining (1), (2) and (3) yields the closed-loop jump-flow system

$$\mathcal{S} : \begin{cases} \begin{bmatrix}\dot{\xi} \\ \dot{\tau}\end{bmatrix} &= \begin{bmatrix}A\xi + Bw_c \\ 1\end{bmatrix}, \quad \text{when } \tau \in [0, h), \\ \begin{bmatrix}\xi^+ \\ \tau^+\end{bmatrix} &= \begin{bmatrix}G\xi + Jw_d \\ 0\end{bmatrix}, \quad \text{when } \tau = h, \\ z_c &= C_c\xi + D_c w_c, \\ z_d &= C_d\xi + D_d w_d, \end{cases} \tag{4}$$

with $\xi \in \mathbb{R}^{n_\xi}$, such that $n_\xi = n_x + n_u + n_c$, in which

$$A := \begin{bmatrix}A^p & B_u^p & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}, B := \begin{bmatrix}B_c^p \\ 0 \\ 0\end{bmatrix}, \tag{5a}$$

$$G := \begin{bmatrix}I & 0 & 0 \\ D^c C_y^p & 0 & C^c \\ B^c C_y^p & 0 & A^c\end{bmatrix}, J := \begin{bmatrix}0 \\ D^c D_{yd}^p \\ B^c D_{yd}^p\end{bmatrix}, \tag{5b}$$

$$C_c := \begin{bmatrix}C_c^p & 0 & 0\end{bmatrix}, D_c := \begin{bmatrix}D_{cc}^p\end{bmatrix}, \tag{5c}$$

$$C_d := \begin{bmatrix}C_d^p & D_{du}^p & 0\end{bmatrix}, D_d := \begin{bmatrix}D_{dd}^p\end{bmatrix}, \tag{5d}$$

as is done in e.g., [9], [11]. It is assumed that the plant $\mathcal{P}$ is stabilisable and detectable and that the sample time h is non-pathological, which basically means that no sample times that lose the stabilisability and detectability properties of the plant are considered, see e.g., [6]. The problem that is treated is to synthesise controllers that stabilise system $\mathcal{S}$ and minimise the $\mathcal{H}_\infty$- or $\mathcal{H}_2$-norm. What is meant by stability is defined below.

Definition 1 *The closed-loop hybrid system $\mathcal{S}$, is said to be* globally exponentially stable *(GES), if there exist $c > 0$ and $0 \leq \rho < 1$ such that for all initial conditions $\xi(0) = \xi_0 \in \mathbb{R}^{n_\xi}$ with $w_c = 0$ and $w_d = 0$ for all $t \in \mathbb{R}$ and $k \in \mathbb{N}$, the corresponding solutions to* (4) *satisfy $\|\xi(t)\| \leq ce^{-\rho t}\|\xi_0\|$.*

To define the sampled-data $\mathcal{H}_\infty$ and $\mathcal{H}_2$ performance, we introduce the following signal norm, taken from [9], [10]:

$$\left\| \begin{bmatrix}x_c(t) \\ x_d[k]\end{bmatrix} \right\|_{\mathcal{L}_2 \times \ell_2} = (\|x_c\|_{\mathcal{L}_2}^2 + h\,\|x_d\|_{\ell_2}^2)^{\frac{1}{2}}. \tag{6}$$

Compared to [9], [10], a scaling with respect to h is introduced to make the generalised plant formulation (1) less dependent on the choice of h. The rationale behind it is that the discrete-time signal is interpreted as a continuous-time signal with ZOH, as in (2). This yields the following relation between the discrete-time ℓ_2- and $\mathcal{L}_2$-norm

$$\begin{aligned}\|x_d\|_{\mathcal{L}_2}^2 &= \int_0^\infty x_d(t)^\top x_d(t)dt = \sum_{k=0}^\infty \int_0^h x_d[k]^\top x_d[k]d\tau \\ &= \sum_{k=0}^\infty h x_d[k]^\top x_d[k] = h\,\|x_d\|_{\ell_2}^2,\end{aligned} \tag{7}$$

which explains the factor h in (6). This can be used to define the $\mathcal{H}_\infty$-norm, which is given below.

Definition 2 *The closed-loop sampled-data system $\mathcal{S}$ with $\xi_0 = 0$, is said to have an $\mathcal{H}_\infty$-norm defined as*

$$\|\mathcal{S}\|_\infty = \sup_{w_c \neq 0, w_d \neq 0} \frac{\left\| \begin{bmatrix}z_c(t) \\ z_d[k]\end{bmatrix} \right\|_{\mathcal{L}_2 \times \ell_2}}{\left\| \begin{bmatrix}w_c(t) \\ w_d[k]\end{bmatrix} \right\|_{\mathcal{L}_2 \times \ell_2}}. \tag{8}$$

The definition for the $\mathcal{H}_2$-norm is inspired by [5], [9], [15]. In this definition, with the necessary assumption that $D_c = D_{cc}^p = 0$. We use the causal state transition function of (4), such that the solution $x(t) = S(t, t_0)x_0$ with

$$S(t, t_0)x_0 = \begin{cases} 0 & t < t_0, \\ e^{A(t-t_0)}x_0 & t \geq t_0, t \in [0, h), \\ e^{A(t-h)}Ge^{A(h-t_0)}x_0 & t \geq t_0, t \in [h, 2h), \\ \vdots & \vdots, \end{cases} \tag{9}$$

which can be used to define the $\mathcal{H}_2$-norm, given below.

Definition 3 *The closed-loop sampled-data system $\mathcal{S}$ with $\xi_0 = 0$, is said to have an $\mathcal{H}_2$-norm defined as*

$$\|\mathcal{S}\|_2 = \left(\frac{1}{h} \int_0^h \operatorname{tr} \left\| \begin{bmatrix} C_c S(t, s)B \\ C_d S(kh, s)B \end{bmatrix} \right\|_{\mathcal{L}_2 \times \ell_2}^2 \mathrm{d}s + \operatorname{tr} \left(\left\| \begin{bmatrix} C_c S(t, 0^+)J \\ C_d S(kh, 0^+)J \end{bmatrix} \right\|_{\mathcal{L}_2 \times \ell_2}^2 + D_d^\top D_d \right) \right)^{\frac{1}{2}}. \tag{10}$$

The focus of this paper is solving the problem of synthesising controllers that satisfy stability and performance requirements following these definitions.

III. $\mathcal{H}_\infty$ Control

The method to synthesise discrete-time controllers that stabilise the sampled-data system $\mathcal{S}$ and guarantee a certain $\mathcal{H}_\infty$-norm, is presented in this section. First, a method to analyse the $\mathcal{H}_\infty$-norm is proposed. This method uses the solution to a Riccati differential equation to formulate LMIs, which analyse stability and the $\mathcal{H}_\infty$-norm of the sampled-data system. Then the analysis procedure is translated into a synthesis procedure, using a linearising congruence transformation, see e.g., [19].

A. $\mathcal{H}_\infty$ Analysis

The procedure for the $\mathcal{H}_\infty$-norm analysis is inspired by the impulsive system approach in [20]. Stability and $\mathcal{H}_\infty$-norm performance will be studied using a Lyapunov/storage function of the form

$$V(\xi, \tau) = \xi^\top P(\tau)\xi \tag{11}$$

for $\tau \in [0, h)$, where $P : [0, h] \to \mathbb{R}^{n_\xi \times n_\xi}$ with $P(\tau) \succ 0$. This function needs to satisfy several conditions to achieve the defined $\mathcal{H}_\infty$-norm, which are formulated in Lemma 4. This lemma states that the conditions for the Lyapunov function are separated in a flow and jump part, which allows us to solve the flow part explicitly and evaluate the satisfaction of the jump conditions with its solution.

Lemma 4 *Consider the closed-loop sampled-data system $\mathcal{S}$ and suppose that a valid Lyapunov/storage function $V(\xi,\tau)$, given by* (11)*, satisfies the conditions*

$$\dot{V}(\xi,\tau) \le \gamma^2 w_c^\top w_c - z_c^\top z_c, \tag{12a}$$

$$V(\xi^+,0) - V(\xi,h) \le \gamma^2 h w_d^\top w_d - h z_d^\top z_d. \tag{12b}$$

Then, the system $\mathcal{S}$ is GES and has a sampled-data $\mathcal{H}_\infty$-norm smaller than or equal to γ.

Proof: Stability is shown by setting $w_c = 0$ and $w_d = 0$, which yields a non-increasing Lyapunov/storage function. The remainder of the proof is inspired by [12], which shows that the combined hypotheses lead to the defined norm. To do so, we take (12a) and integrate over the flow, i.e., until right before the jump denoted by t_k^-, which yields

$$V(t_{k+1}^-) - V(t_k) \le \int_{t_k}^{t_{k+1}^-} \gamma^2 w_c^\top w_c - z_c^\top z_c dt. \tag{13}$$

Including the jump dynamics leads to

$$\begin{aligned} V(t_{k+1}) - V(t_k) \le &\int_{t_k}^{t_{k+1}^-} (\gamma^2 w_c^\top w_c - z_c^\top z_c)dt \\ &+ \gamma^2 h w_d^\top w_d - h z_d^\top z_d, \end{aligned} \tag{14}$$

which with $\xi_0 = 0$ and summed from $k = 0$ to $k \to \infty$ yields

$$0 \le \gamma^2 \left\| \begin{bmatrix} w_c(t) \\ w_d[k] \end{bmatrix} \right\|^2_{\mathcal{L}_2 \times \ell_2} - \left\| \begin{bmatrix} z_c(t) \\ z_d[k] \end{bmatrix} \right\|^2_{\mathcal{L}_2 \times \ell_2}, \tag{15}$$

since $V(0) = 0$ and $V(t_k) = 0$, which yields that the $\mathcal{H}_\infty$-norm is smaller than or equal to γ. This concludes the proof. ■

The results of Lemma 4 will first be used to derive tractable conditions for analysing stability and the $\mathcal{H}_\infty$-norm of $\mathcal{S}$, To do so, the function P is chosen to satisfy the Riccati differential inequality

$$\begin{aligned} \frac{d}{d\tau} P \le &- A^\top P - PA - C_c^\top C_c \\ &- (PB + C_c^\top D_c) M (B^\top P + D_c^\top C_c) \end{aligned} \tag{16}$$

with the matrix $M = (\gamma^2 I - D_c^\top D_c)^{-1}$. This follows easily by elaborating the condition (12a) and applying Schur's complement. The matrix M is assumed to exist and to be positive definite, which means that $\lambda_{\max}(D_c^\top D_c) < \gamma^2$.

Now, only condition (12b) still needs to be satisfied, which requires a relation between $P_h := P(h)$ and $P_0 := P(0)$. This relation can be established by finding the explicit solution to the Riccati differential equation using the Hamiltonian matrix

$$H = \begin{bmatrix} A + BMD_c^\top C_c & BMB^\top \\ -C_c^\top L C_c & -(A + BMD_c^\top C_c)^\top \end{bmatrix} \tag{17}$$

with $L := (I - \gamma^{-2} D_c D_c^\top)^{-1}$, which is positive definite if again $\lambda_{\max}(D_c D_c^\top) = \lambda_{\max}(D_c^\top D_c) < \gamma^2$. Then the matrix exponential

$$F(\tau) := e^{-H\tau} = \begin{bmatrix} F_{11}(\tau) & F_{12}(\tau) \\ F_{21}(\tau) & F_{22}(\tau) \end{bmatrix} \tag{18}$$

can be used to write the relation between P_0 and P_h as

$$P_0 = (F_{21}(h) + F_{22}(h)P_h)(F_{11}(h) + F_{12}(h)P_h)^{-1}, \tag{19}$$

which is well-defined on $[0, h]$ when the following assumption holds.

Assumption 5 *$F_{11}(\tau)$ is invertible for all $\tau \in [0, h]$.*

Since we are evaluating the matrix exponential at $\tau = h$ the time-dependencies for its partitions are henceforth omitted.

Using Assumption 5 the following matrices can be introduced

$$\hat{A} := F_{11}^{-1}, \tag{20a}$$

$$\hat{B}\hat{B}^\top := -F_{11}^{-1} F_{12}, \tag{20b}$$

$$\hat{C}^\top \hat{C} := F_{21} F_{11}^{-1}, \tag{20c}$$

for which it is shown in [20, Lemma A.1] that the products $-F_{11}^{-1}F_{12}$ and $F_{21}F_{11}^{-1}$ are positive semi-definite. These are conveniently denoted by $\hat{A}$, $\hat{B}$ and $\hat{C}$, since they are closely related to the equivalent discrete-time "lifted" system, see e.g., [21]. Using these matrices the function P_0 can, according to [20, Theorem III.2], be rewritten as

$$P_0 = \hat{C}^\top \hat{C} + \hat{A}^\top (P_h + P_h \hat{B}(I - \hat{B}^\top P_h \hat{B})^{-1} \hat{B}^\top P_h)\hat{A}, \tag{21}$$

and Theorem 6 can be established.

Theorem 6 *Consider the closed-loop hybrid system $\mathcal{S}$ and let $\gamma > \sqrt{\lambda_{\max}(D_c^\top D_c)}$ and Assumption 5 hold. Suppose that there exists a matrix $P_h \succ 0$ such that*

$$\begin{bmatrix} P_h & \star & \star & \star & \star & \star \\ 0 & h\gamma^2 & \star & \star & \star & \star \\ 0 & 0 & I & \star & \star & \star \\ P_h\hat{A}G & P_h\hat{A}J & P_h\hat{B} & P_h & \star & \star \\ \hat{C}G & \hat{C}J & 0 & 0 & I & \star \\ C_d & D_d & 0 & 0 & 0 & h^{-1}I \end{bmatrix} \succcurlyeq 0. \tag{22}$$

Then, the sampled-data system $\mathcal{S}$ is GES *and has an $\mathcal{H}_\infty$-norm smaller than or equal to γ.*

Proof: The proof is based on showing that the conditions, given in Lemma 4, are satisfied and provide GES by the hypothesis of the theorem. For the proof that the storage function (11) is a well-defined function and the method to yield (21), the reader is referred to [20]. The first condition (12a) is satisfied by (16), for which the resulting relation between P_h and P_0 is given by (21). The second condition, given by (12b), where the Lyapunov function (11) and z_d from (4) are substituted yields

$$\begin{bmatrix} P_h & 0 \\ 0 & h\gamma^2 \end{bmatrix} - \begin{bmatrix} G^\top \\ J^\top \end{bmatrix} P_0 \begin{bmatrix} G & J \end{bmatrix} - \begin{bmatrix} C_d^\top \\ D_d^\top \end{bmatrix} h \begin{bmatrix} C_d & D_d \end{bmatrix} \succcurlyeq 0, \tag{23}$$

which after substitution of (21) and followed by repeated use of Schur's complement yields the LMI (22), concluding the proof. ∎

Remark 7 *This solution makes use of matrix exponentials, which impairs maintaining the direct parameters dependence of the system. However, polytopic embeddings that account for all bounded parameter variations, see e.g. [22], can still be used.*

B. $\mathcal{H}_\infty$ Synthesis

It is evident from (22) that the controller parameters, which are concealed in G and J, appear in products with P_h. To retrieve an optimal controller in the $\mathcal{H}_\infty$ sense, a linearising congruence transformation has to be performed. Due to the similar structure as the LTI case, the same congruence transformation can be done, see e.g., [19].

This transformation uses a partitioning of the matrix P_h as follows

$$P_h := \begin{bmatrix} Y & V \\ V^\top & \hat{Y} \end{bmatrix},\ P_h^{-1} := \begin{bmatrix} X & U \\ U^\top & \hat{X} \end{bmatrix}, \tag{24}$$

with $U = (I - XY)V^\top$, $\hat{Y} = V^\top(Y - X^{-1})V$ and $\hat{X} = V^{-1}(YXY - Y)V^{-\top}$. All partitions are of dimension $n_c \times n_c$ and the transformation matrix is given by

$$T := \begin{bmatrix} Y & I \\ V^\top & 0 \end{bmatrix}. \tag{25}$$

Let us then introduce the partitioning for the other matrices that adopt the following structure

$$G := \begin{bmatrix} \bar{Z} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \bar{Q} \\ I & 0 \end{bmatrix} \begin{bmatrix} A^c & B^c \\ C^c & D^c \end{bmatrix} \begin{bmatrix} 0 & I \\ \bar{W} & 0 \end{bmatrix}, \tag{26a}$$

$$J := \begin{bmatrix} 0 & \bar{Q} \\ I & 0 \end{bmatrix} \begin{bmatrix} A^c & B^c \\ C^c & D^c \end{bmatrix} \begin{bmatrix} 0 \\ \bar{R} \end{bmatrix}, \tag{26b}$$

$$\hat{A} := \begin{bmatrix} \bar{A} & 0 \\ 0 & I \end{bmatrix},\ \hat{B} := \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix},\ \hat{C} := \begin{bmatrix} \bar{C} & 0 \end{bmatrix}, \tag{26c}$$

$$C_d = \begin{bmatrix} \bar{C}_d & 0 \end{bmatrix},\ D_d = \bar{D}_d, \tag{26d}$$

which requires that the controller order is equal to $n_c = n_x + n_u$ in order to retrieve the controller parameters. The bar notation is used to indicate known matrices for the following results. These steps can be used to present Theorem 8.

Theorem 8 *Consider the closed-loop hybrid system $\mathcal{S}$ and let $\gamma > \sqrt{\lambda_{\max}(D_c^\top D_c)}$ and Assumption 5 hold. Suppose that there exist the matrices L, M, F, H and symmetric matrices $X, Y \succ 0$ with appropriate dimensions such that*

$$\begin{bmatrix} Y & \star & \star & \star & \star & \star & \star & \star \\ I & X & \star & \star & \star & \star & \star & \star \\ 0 & 0 & h\gamma^2 & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & I & \star & \star & \star & \star \\ Y\bar{A}\bar{Z}+M\bar{W} & L & M\bar{R} & Y\bar{B} & X & \star & \star & \star \\ \bar{A}(\bar{Z}+\bar{Q}H\bar{W}) & \bar{A}(\bar{Z}X+\bar{Q}F) & \bar{A}\bar{Q}H\bar{R} & \bar{B} & I & Y & \star & \star \\ \bar{C}(\bar{Z}+\bar{Q}H\bar{W}) & \bar{C}(\bar{Z}X+\bar{Q}F) & \bar{C}\bar{Q}H\bar{R} & 0 & 0 & 0 & I & \star \\ \bar{C}_d & \bar{C}_d X & \bar{D}_d & 0 & 0 & 0 & 0 & h^{-1}I \end{bmatrix} \succcurlyeq 0. \tag{27}$$

Then, the controller (3) *with matrices*

$$\begin{bmatrix} A^c & B^c \\ C^c & D^c \end{bmatrix} = \begin{bmatrix} V & Y\bar{A}\bar{Q} \\ 0 & I \end{bmatrix}^{-1} \left(\begin{bmatrix} L & M \\ F & H \end{bmatrix} - \begin{bmatrix} Y\bar{A}\bar{Z}X & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} U^\top & 0 \\ \bar{W}X & I \end{bmatrix}^{-1}, \tag{28}$$

where U and V are invertible and can be chosen, such that $UV^\top = I - XY$, renders the closed-loop sampled-data system $\mathcal{S}$ GES *and guarantees it to have an $\mathcal{H}_\infty$-norm smaller than or equal to γ.*

Proof: Applying the congruence transformation $\mathrm{diag}(P_h^{-1}T, I, I, P_h^{-1}T, I, I)$ to (22) yields

$$\begin{bmatrix} T^\top P_h^{-1} T & \star & \star & \star & \star & \star \\ 0 & h\gamma^2 & \star & \star & \star & \star \\ 0 & 0 & I & \star & \star & \star \\ T^\top \hat{A} G P_h^{-1} T & T^\top \hat{A} J & T^\top \hat{B} & T^\top P_h^{-1} T & \star & \star \\ \hat{C} G P_h^{-1} T & \hat{C} J & 0 & 0 & I & \star \\ C_d P_h^{-1} T & D_d & 0 & 0 & 0 & h^{-1} I \end{bmatrix} \succcurlyeq 0. \tag{29}$$

Then, using the matrix partitionings (24), (25) and (26) and the controller parametrisation (28), the matrix inequality (29) can be rewritten to (27). This concludes the proof. ∎

IV. $\mathcal{H}_2$ Control

Controllers that stabilise the sampled-data system $\mathcal{S}$ and guarantee a certain $\mathcal{H}_2$-norm can be done using approximately the same procedure as for the $\mathcal{H}_\infty$ case. Similar to the previous section, first the $\mathcal{H}_2$-norm analysis method is presented, using LMIs that are formulated in terms of the explicit solution to the emerging Riccati differential equation to find a valid sampled-data observability Gramian. Then, the analysis procedure is translated into a synthesis procedure, using a linearising congruence transformation, see e.g., [19]. At last a computational approximation is given to yield tractable results.

A. $\mathcal{H}_2$ Analysis

The main difference for the $\mathcal{H}_2$ procedure is that observability Gramians are used instead of a Lyapunov/storage function. This is also done in [9], which leads to conditions given in Lemma 9. Using these conditions a similar procedure as for the $\mathcal{H}_\infty$-norm can be performed.

Lemma 9 *Consider the closed-loop hybrid system $\mathcal{S}$ and suppose that a valid sampled-data observability Gramian $P(\tau) \succ 0$ satisfies the conditions*

$$\frac{d}{d\tau} P(\tau) \le -A^\top P(\tau) - P(\tau) A - C_c^\top C_C, \tag{30a}$$

$$P(h) \le G^\top P(0) G + h C_d^\top C_d. \tag{30b}$$

Then, the sampled-data system $\mathcal{S}$ is GES and has the $\mathcal{H}_2$-norm:

$$\|\mathcal{S}\|_2 = \left(\frac{1}{h} \int_0^h \mathrm{tr}\left(B^\top P(s) B \right) ds + \mathrm{tr}(J^\top P(0) J + D_d^\top D_d) \right)^{\frac{1}{2}}. \tag{31}$$

Proof: The proof can be found in [9, Th. 4.1]. ∎

First, the continuous-time part of the observability Gramian, given by (30a), is calculated. For this Riccati differential inequality, another Hamiltonian matrix can be formulated as follows

$$H := \begin{bmatrix} A & 0 \\ -C_c^\top C_c & -A^\top \end{bmatrix}, \tag{32}$$

for which the matrix exponential

$$F(\tau) := e^{-H\tau} = \begin{bmatrix} F_{11}(\tau) & 0 \\ F_{21}(\tau) & F_{22}(\tau) \end{bmatrix}, \tag{33}$$

where again Assumption 5 holds and thus can be used to write the relation between P_0 and P_h as

$$\begin{aligned} P(h-\tau) &= (F_{21}(\tau) + F_{22}(\tau)P_h)F_{11}^{-1}(\tau), \\ &= \hat{C}^\top(\tau)\hat{C}(\tau) + \hat{A}^\top(\tau)P_h\hat{A}(\tau), \end{aligned} \tag{34}$$

for $\tau \in [0, h)$ with matrices $\hat{A}$ and $\hat{C}$ described by (26). For the sake of clarity, the following notation is introduced $\hat{A}_0 = \hat{A}(0)$ and $\hat{C}_0 = \hat{C}(0)$, which is also applied to its partitions $\bar{A}_0$ and $\bar{C}_0$, defined in (26).

Theorem 10 *Consider the closed-loop hybrid-system $\mathcal{S}$, and let Assumption 5 hold. Suppose that the matrix $P_h > 0$ exists such that*

$$\begin{bmatrix} P_h & \star & \star & \star \\ P_h\hat{A}_0G & P_h & \star & \star \\ \hat{C}_0G & 0 & I & \star \\ C_d & 0 & 0 & h^{-1}I \end{bmatrix} \succcurlyeq 0. \tag{35}$$

Then, the sampled-data system has an $\mathcal{H}_2$-norm given by (31).

Proof: The proof is similar to the proof of Theorem 6 with slightly different conditions, given by Lemma 9. ∎

B. $\mathcal{H}_2$ Synthesis

The same procedure as in Section III-B is applied. Therefore, the partitioning and congruence transformation, given by (24), (25) and (26), is used. By taking these steps with respect to Theorem 10, Theorem 11 can be established.

Theorem 11 *Consider the closed-loop hybrid-system $\mathcal{S}$, and let Assumption 5 hold. Suppose that the matrices L, M, F, H and symmetric matrices $X, Y \succ 0$ and S exist such that*

$$\begin{bmatrix} Y & \star & \star & \star & \star & \star \\ I & X & \star & \star & \star & \star \\ Y\bar{A}_0\bar{Z} + M\bar{W} & L & Y & \star & \star & \star \\ \bar{A}_0(\bar{Z} + \bar{Q}H\bar{W}) & \bar{A}_0(\bar{Z}X + \bar{Q}F) & I & X & \star & \star \\ \bar{C}_0(\bar{Z} + \bar{Q}H\bar{W}) & \bar{C}_0(\bar{Z}X + \bar{Q}F) & 0 & 0 & I & \star \\ C_d & C_dX & 0 & 0 & 0 & h^{-1}I \end{bmatrix} \succcurlyeq 0, \tag{36}$$

$$\begin{bmatrix} S & \star & \star & \star \\ \bar{C}_0(\bar{J} + \bar{Q}H\bar{R}) & I & \star & \star \\ M\bar{R} & 0 & Y & \star \\ \bar{A}_0\bar{Q}H\bar{R} & 0 & I & X \end{bmatrix} \succcurlyeq 0. \tag{37}$$

Then, the controller (3) *with matrices*

$$\begin{aligned} \begin{bmatrix} A^c & B^c \\ C^c & D^c \end{bmatrix} = \begin{bmatrix} V & Y\bar{A}\bar{Q} \\ 0 & I \end{bmatrix}^{-1} \left(\begin{bmatrix} L & M \\ F & H \end{bmatrix} - \right. \\ \left. \begin{bmatrix} Y\bar{A}\bar{Z}X & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} U^\top & 0 \\ \bar{W}X & I \end{bmatrix}^{-1}, \end{aligned} \tag{38}$$

where U and V are invertible and can be chosen, such that $UV^\top = I - XY$, renders the closed-loop sampled-data system $\mathcal{S}$ and guarantees it to have an upper bounded $\mathcal{H}_2$-norm given by

$$\begin{aligned} \|\mathcal{S}\|_2 \le \left(\frac{1}{h} \int_0^h \operatorname{tr}\left(B^\top P(s) B\right)\, ds + \right. \\ \left. \operatorname{tr}(S + D_d^\top D_d), \right)^{\frac{1}{2}}. \end{aligned} \tag{39}$$

Proof: The proof for this theorem is similar to the proof of Theorem 8, where the difference comes from the extra condition (37). This matrix inequality arises from

$$J^\top P(0) J \le S, \tag{40}$$

where S functions as an upper bound. This concludes the proof. ∎

C. $\mathcal{H}_2$ Approximation

In order to get to tractable results for the sampled-data $\mathcal{H}_2$-norm, an approximation is required, which will be explained in this section. The presented $\mathcal{H}_2$-norm includes the computation of an integral over the trace of the sampled-data observability Gramian, which cannot be calculated directly. Existing approximations for this $\mathcal{H}_2$-norm definition only consider impulses right before the jump, see e.g., [11], [12]. An approximation that closer represents the definition is proposed. The method consists of taking finite grid points

$$\begin{aligned} \|\mathcal{S}\|_2 \approx \left(\frac{1}{h} \sum_{i=0}^{N} \operatorname{tr}\left(B^\top P(\tau_i) B\right) + \right. \\ \left. \operatorname{tr}(J^\top P(0) J + D_d^\top D_d) \right)^{\frac{1}{2}}, \end{aligned} \tag{41}$$

where $P(\tau)$ is given by (34) and $\tau_i = \frac{i}{N}h$ with $i \in \{0, \ldots, N\}, N \in \mathbb{N}$ and $\tau_0 = 0$ if $N = 0$. By taking a sufficiently large N, a close approximation of the sampled-data $\mathcal{H}_2$-norm can be found.

When considering the synthesis problem where the partitioning of (24) and (26) is used, the function needs to be rewritten in terms of the optimisation variables. To do so, the following realisation is made

$$\begin{aligned} B^\top P(h-\tau) B &= B^\top \left(\hat{C}^\top(\tau)\hat{C}(\tau) + \hat{A}^\top(\tau) P_h \hat{A}(\tau) \right) B \\ &= \begin{bmatrix} B_w^p \\ 0 \end{bmatrix}^\top \left(\bar{C}^\top(\tau)\bar{C}(\tau) + \bar{A}^\top(\tau) Y \bar{A}(\tau) \right) \begin{bmatrix} B_w^p \\ 0 \end{bmatrix}, \end{aligned} \tag{42}$$

for which the new partitioned function $\bar{P}(\tau)$ can be defined by

$$\bar{P}(h-\tau) = \bar{C}^\top(\tau)\bar{C}(\tau) + \bar{A}^\top(\tau) Y \bar{A}(\tau). \tag{43}$$

This yields the upper bound of the synthesis objective function, which yields a close approximation for $\mathrm{tr}(S) \to \mathrm{tr}(J^\top P(0)J)$, given by

$$\|\mathcal{S}\|_2 \approx \left(\frac{1}{h} \sum_{i=0}^{N} \mathrm{tr} \left(\begin{bmatrix} B_w^p \\ 0 \end{bmatrix}^\top \bar{P}(\tau_i) \begin{bmatrix} B_w^p \\ 0 \end{bmatrix} \right) + \mathrm{tr}(S + D_d^\top D_d) \right)^{\frac{1}{2}}, \tag{44}$$

where $\tau_i = \frac{i}{N}h$ and $\tau_i = \frac{i}{N}h$ with $i \in \{0, \ldots, N\}, N \in \mathbb{N}$ and $\tau_0 = 0$ if $N = 0$.

V. $\mathcal{H}_\infty$ Loop Shaping

It has been shown that the sampled-data $\mathcal{H}_\infty$-norm controller synthesis problem can be solved. However, applying synthesis methods for practical applications requires additional steps. Traditional $\mathcal{H}_\infty$ loop shaping appends weighting filters to the plant on which the same synthesis techniques can be applied, see e.g., [3]. However, due to the hybrid nature of the plant, some modifications of this method have to be made. Therefore, this section focusses on formulating a so-called generalised plant for sampled-data systems, for which the procedure described in Section III-B can be applied. This formulation that is developed for continuous-time systems uses weighting filters to describe the behaviour of the loop transfers of the closed-loop systems, see e.g., [3], [4] and references therein.

A. Generalised Plant Formulation

The main issue that is encountered in the generalised plant formulation for the hybrid system is dealing with the discrete- and continuous-time part. This means that the weighting filters cannot be applied directly to a plant to obtain the generalised plant on which the synthesis procedure can be performed. Instead, the weighting filters that are designed for the discrete-time signals need to be applied on the discrete part of the hybrid system and the converse for the continuous part. The issue is that in the synthesis procedure the discrete-time state transition of the plant (1) is assumed to be

$$x^{p+} = Ix^p, \tag{45}$$

which does not hold when weighting filters are applied, since the filter dynamics are included in the plant. This requires some changes that will be illustrated in this section.

Recall the continuous-time dynamics of the plant (1) and the discrete-time dynamics (45). For this hybrid plant, we introduce the hybrid weighting filters for the corresponding signals W_{w_c}, W_{w_d}, W_{z_c} and W_{z_d}, that normalise and shape the signals with the following relation $w_c = W_{w_c}\tilde{w}_c$, $w_d = W_{w_d}\tilde{w}_d$, $\tilde{z}_c = W_{z_c}z_c$ and $\tilde{z}_d = W_{z_d}z_d$, such as shown in Fig. 2. The normalised signals $\tilde{w}_c$ and $\tilde{w}_d$ are defined such

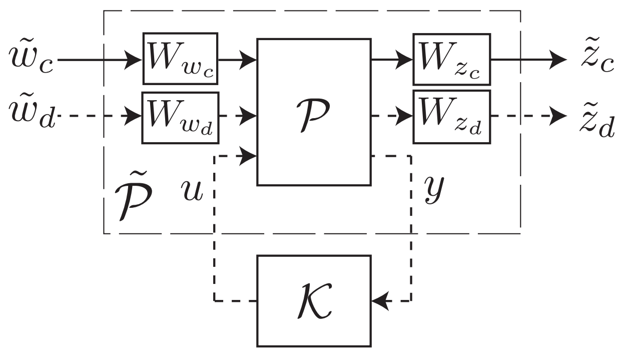


Fig. 2: The generalised plant with the appended filters, where the dashed lines indicate discrete signals.

that $\|\tilde{w}_c\|_{\mathcal{L}_2} \leq 1$ and $\|\tilde{w}_d\|_{\ell_2} \leq 1$. Then, these hybrid filters are given by

$$W_{w_c} : \begin{cases} \dot{x}^{w_c} = A^{w_c}x^{w_c} + B^{w_c}\tilde{w}_c, \\ x^{w_c+} = Ix^{w_c}, \\ w_c = C^{w_c}x^{w_c} + D^{w_c}\tilde{w}_c, \end{cases} \tag{46a}$$

$$W_{w_d} : \begin{cases} \dot{x}^{w_d} = 0, \\ x^{w_d+} = A^{w_d}x^{w_d} + B^{w_d}\tilde{w}_d, \\ w_d = C^{w_d}x^{w_d} + D^{w_d}\tilde{w}_d, \end{cases} \tag{46b}$$

$$W_{z_c} : \begin{cases} \dot{x}^{z_c} = A^{z_c}x^{z_c} + B^{z_c}z_c, \\ x^{z_c+} = Ix^{z_c}, \\ \tilde{z}_c = C^{z_c}x^{z_c} + D^{z_c}z_c, \end{cases} \tag{46c}$$

$$W_{z_d} : \begin{cases} \dot{x}^{z_d} = 0, \\ x^{z_d+} = A^{z_d}x^{z_d} + B^{z_d}z_d, \\ \tilde{z}_d = C^{z_d}x^{z_d} + D^{z_d}z_d. \end{cases} \tag{46d}$$

The application of these filters to the plant (1) and (45) yields a new state vector $\tilde{x}^p = \begin{bmatrix} x^{p\top} & x^{w_c\top} & x^{z_c\top} & x^{w_d\top} & x^{z_d\top} \end{bmatrix}^\top$, due to the filter dynamics that are now also included in the plant. The generalised plant that includes all these filters can be rewritten to the sampled-data generalised plant

$$\tilde{\mathcal{P}} : \begin{cases} \dot{\tilde{x}}^p(t) = \tilde{A}^p\tilde{x}^p(t) + \tilde{B}_c^p\tilde{w}_c(t) + B_u^p u(t), \\ \tilde{x}^{p+}[k] = \tilde{G}^p\tilde{x}^p[k] + \tilde{J}^p\tilde{w}_d[k], \\ \tilde{z}_c(t) = \tilde{C}_c^p\tilde{x}^p(t) + \tilde{D}_{cc}^p\tilde{w}_c(t), \\ \tilde{z}_d[k] = \tilde{C}_d^p\tilde{x}^p[k] + \tilde{D}_{dd}^p\tilde{w}_d[k] + D_{du}^p\hat{u}[k], \\ y[k] = C_y^p\tilde{x}^p[k] + \tilde{D}_{yd}^p\tilde{w}_d[k], \end{cases} \tag{47}$$

with

$$\tilde{A}^p := \begin{bmatrix} A^p & B^p_c C^{w_c} & 0 & 0 & 0 \\ 0 & A^{w_c} & 0 & 0 & 0 \\ B^{z_c} C^p_c & B^{z_c} D^p_{cc} C^{w_c} & A^{w_c} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tilde{B}^p_c := \begin{bmatrix} B^p_c D^{w_c} \\ B^{w_c} \\ B^{z_c} D^p_{cc} D^{w_c} \\ 0 \\ 0 \end{bmatrix}, \tag{48a}$$

$$\tilde{G}^p := \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & A^{w_d} & 0 \\ B^{z_d} C^p_d & 0 & 0 & B^{z_d} D^p_{dd} C^{w_d} & A^{z_d} \end{bmatrix}, \tilde{J}^p := \begin{bmatrix} 0 \\ 0 \\ 0 \\ B^{z_d} D^p_{dd} D^{w_d} \\ D^{z_d} D^p_{dd} D^{w_d} \end{bmatrix}, \tag{48b}$$

$$\tilde{C}^p_c := \begin{bmatrix} B^{z_c} C^p_c & D^{z_c} D^p_{cc} C^{w_c} & C^{z_c} & 0 & 0 \end{bmatrix}, \tilde{D}^p_{cc} := \begin{bmatrix} D^{z_c} D^p_{cc} D^{w_c} \end{bmatrix}, \tag{48c}$$

$$\tilde{C}^p_d := \begin{bmatrix} D^{z_d} C^p_d & 0 & 0 & D^{z_d} D^p_{dd} C^{w_d} & C^{z_d} \end{bmatrix}, \tilde{D}^p_{dd} := \begin{bmatrix} D^{z_d} D^p_{dd} D^{w_d} \end{bmatrix}, \tag{48d}$$

$$\tilde{C}^p_y := \begin{bmatrix} C^p_y & 0 & 0 & D^p_{yd} C^{w_d} & 0 \end{bmatrix}, \tilde{D}^p_{yd} := \begin{bmatrix} D^p_{yd} D^{w_d} \end{bmatrix}. \tag{48e}$$

These filters do not only increase the order of the plant, but also alter the matrices G and J. Although the changes of

$$\tilde{G} := \begin{bmatrix} \tilde{G}^p & 0 & 0 \\ D^c \tilde{C}^p_y & 0 & C^c \\ B^c \tilde{C}^p_y & 0 & A^c \end{bmatrix}, \tilde{J} := \begin{bmatrix} \tilde{J}^p \\ D^c \tilde{D}^p_{yu} \\ B^c \tilde{D}^p_{yu} \end{bmatrix}, \tag{49}$$

seem minor, more modifications are required for synthesis, since partitioning (26) is no longer valid. Despite these new matrices, the analysis of Theorem 6 can still be applied.

B. Synthesis Procedure for $\mathcal{H}_\infty$ Loop Shaping

The modification described in (47), (48) and (49) is applied, which introduces the new matrices $\breve{A}$, $\breve{B}$ and $\breve{C}$, which are the resulting matrices from (20). The Hamiltonian, as in (17), is constructed using the new continuous-time plant matrices from (48a) and (48c). The required modification to partitioning (26) yields

$$\tilde{G} := \begin{bmatrix} \tilde{Z} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \tilde{Q} \\ I & 0 \end{bmatrix} \begin{bmatrix} A^c & B^c \\ C^c & D^c \end{bmatrix} \begin{bmatrix} 0 & I \\ \tilde{W} & 0 \end{bmatrix}, \tag{50a}$$

$$\tilde{J} := \begin{bmatrix} \tilde{E} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \tilde{Q} \\ I & 0 \end{bmatrix} \begin{bmatrix} A^c & B^c \\ C^c & D^c \end{bmatrix} \begin{bmatrix} 0 \\ \tilde{R} \end{bmatrix}, \tag{50b}$$

$$\breve{A} := \begin{bmatrix} \tilde{A} & 0 \\ 0 & I \end{bmatrix}, \breve{B} := \begin{bmatrix} \tilde{B} \\ 0 \end{bmatrix}, \breve{C} := \begin{bmatrix} \tilde{C} & 0 \end{bmatrix}, \tag{50c}$$

$$C_d = \begin{bmatrix} \tilde{C}_d & 0 \end{bmatrix}, D_d = \tilde{D}_d, \tag{50d}$$

for which all the affected known matrices are now denoted with a tilde instead of a bar. Furthermore, is the generalised closed-loop system $\tilde{\mathcal{S}}$ formulated in the same way as (4)-(5), but now using the generalised plant $\tilde{\mathcal{P}}$ with (49).

Theorem 12 *Consider the closed-loop hybrid system $\tilde{\mathcal{S}}$ and let $\gamma > \sqrt{\lambda_{\max}(\tilde{D}_c^\top \tilde{D}_c)}$ and Assumption 5 hold. Suppose that there exist the matrices L, M, F, H and symmetric matrices $X, Y \succ 0$ with appropriate dimensions such that*

$$\begin{bmatrix} Y & \star & \star & \star & \star & \star & \star & \star \\ I & X & \star & \star & \star & \star & \star & \star \\ 0 & 0 & h\gamma^2 & \star & \star & \star & \star & \star \\ 0 & 0 & 0 & I & \star & \star & \star & \star \\ Y\tilde{A}\tilde{Z}+M\tilde{W} & L & Y\tilde{A}\tilde{E}+M\tilde{R} & Y\tilde{B} & X & \star & \star & \star \\ \tilde{A}(\tilde{Z}+\tilde{Q}H\tilde{W}) & \tilde{A}(\tilde{Z}X+\tilde{Q}F) & \tilde{A}(\tilde{E}+\tilde{Q}H\tilde{R}) & \tilde{B} & I & Y & \star & \star \\ \tilde{C}(\tilde{Z}+\tilde{Q}H\tilde{W}) & \tilde{C}(\tilde{Z}X+\tilde{Q}F) & \tilde{C}(\tilde{E}+\tilde{Q}H\tilde{R}) & 0 & 0 & 0 & I & \star \\ \tilde{C}_d & \tilde{C}_d X & \tilde{D}_d & 0 & 0 & 0 & 0 & h^{-1}I \end{bmatrix} \succcurlyeq 0, \tag{51}$$

Then, the controller (3) *with matrices*

$$\begin{bmatrix} A^c & B^c \\ C^c & D^c \end{bmatrix} = \begin{bmatrix} V & Y\tilde{A}\tilde{Q} \\ 0 & I \end{bmatrix}^{-1} \left(\begin{bmatrix} L & M \\ F & H \end{bmatrix} - \begin{bmatrix} Y\tilde{A}\tilde{Z}X & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} U^\top & 0 \\ \tilde{W}X & I \end{bmatrix}^{-1}, \tag{52}$$

where U and V are invertible and can be chosen, such that $UV^\top = I - XY$, renders the closed-loop sampled-data system $\tilde{\mathcal{S}}$ GES and guarantees it to have an $\mathcal{H}_\infty$-norm smaller than or equal to γ.

Proof: The proof is similar to the proof of Theorem 8 with the slight change introduced by the different partitioning (50b). ■

VI. Examples

In this section, we will illustrate the presented results using numerical examples. This section is split in two parts, where first an academic example, complying to the literature, is considered. Then, a simple practical loop shaping design problem is presented to illustrate the generalised plant formulation proposed in this paper.

A. Academic Example

This example is taken from [11] and the plant is given by

$$\begin{cases} \dot{x}^p(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x^p(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_c(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\ z_c(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x^p(t), \\ z_d[k] = d\hat{u}[k], \\ y[k] = \begin{bmatrix} 1 & 0 \end{bmatrix} x^p[k] + dw_d[k], \end{cases} \tag{53}$$

where $d \in \{0, 1\}$ is a parameter that can be chosen $d = 0$ to compare with results based on continuous lifting, see e.g., [5], [8], [17], while $d = 1$ includes discrete noise, but can just be compared to results of [11]. The sampling time is chosen to be $h = 1$. In this section the notation γ_d is used to indicate the $\mathcal{H}_\infty$-norm of the system for the corresponding parameter d.

Let us start with the case when $d = 0$, for which our results can be compared with the existing MATLAB routine *sdhinfsyn*, which is based on the lifting approach of [5]. This routine returns $\gamma_0 = 0.84$, while the approach of [11] yields $\gamma_0 = 0.88$. The conservatism of this result is due to the linearising approximation of the Riccati differential equation that has been used to obtain a tractable solution.

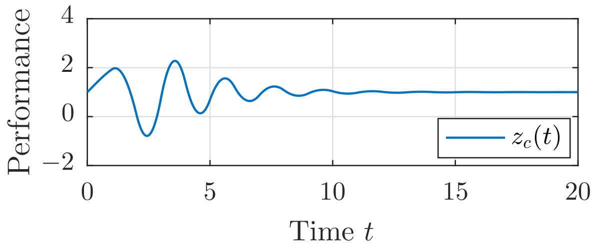


Fig. 3: Response for $d = 0$ with $w_c(t) = \frac{1}{\gamma_0}$.

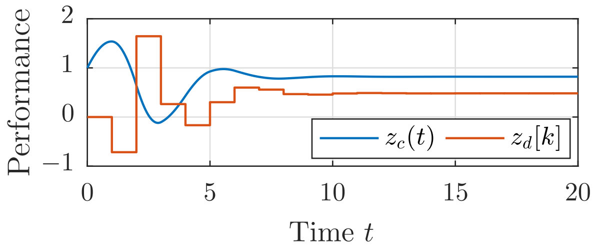
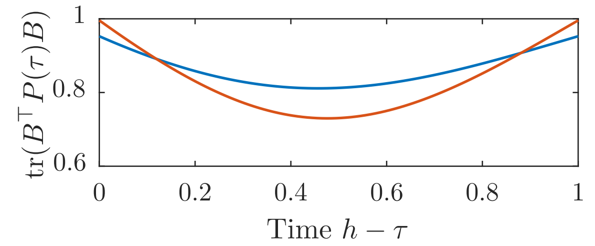


Fig. 4: Response for $d = 1$ with $w_c(t) = w_d[k] = \frac{1}{\gamma_1 \sqrt{2}}$.

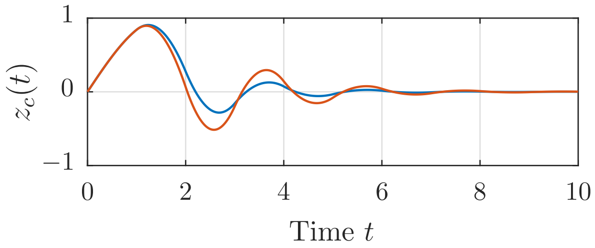
Employing Theorem 8 and reducing the controller with balanced truncation, see e.g., [3], to compare controllers of the same order, results in $\gamma_0 = 0.84$, which indicates for this case that no conservatism is introduced. This means that, according to the $\mathcal{H}_\infty$-norm definition used in this paper, the relation $\|z_c\|_{\mathcal{L}_2} \leq \gamma_0 \|w_c\|_{\mathcal{L}_2}$ holds. Therefore, if the plant is simulated with disturbance $w_c(t) = \frac{1}{\gamma_0}$, the steady state performance output should yield $z_c(t) \leq 1$. The simulation results are presented in Fig. 3 and indeed yield this relation.

Moving on to the case that $d = 1$, which the MATLAB routine *sdhinfsyn* cannot solve. The method proposed in [11] yields $\gamma_1 = 2.16$, while $\gamma_1 = 2.09$ is obtained by applying Theorem 8, which is less conservative, due to the reasons explained before. In this case, when the noise input $w_c(t) = w_d[k] = \frac{1}{\gamma_1 \sqrt{2}}$ is chosen, the $\mathcal{H}_\infty$-norm definition (8) should require the steady state response $\sqrt{z_c(t_k)^2 + z_d[k]^2} \leq 1$. The simulation results for this case are presented in Fig. 4, for which the result $\sqrt{z_c(t_k)^2 + z_d[k]^2} \approx 0.95 < 1$, for which it can be concluded that the $\mathcal{H}_\infty$-norm is indeed satisfied.

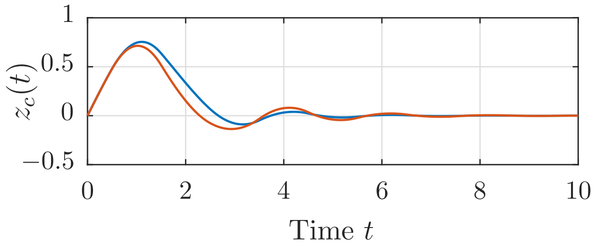
To illustrate the $\mathcal{H}_2$ results, the same example is used as the previous section, which is described by the matrices in (53). The results of the impulse response right before the jump of the hybrid system, denoted by $\delta(h)$, from [11], yields an $\mathcal{H}_2$-norm of 1.10. This method, with a single grid point, can also be used for the proposed LMIs in this paper. This choice of approximation can then be compared with using more grid points, conform to the defined method in this paper. This comparison is made in Fig. 5 where next to $N = 0$ also $N = 1000$ is used. It is visible that for $N = 0$ the response to impulses around the sample moment are damped faster, but not for pulses at other times during the intersample-time. The resulting $\mathcal{H}_2$-norm for various choices of N are shown in Fig. 6, which shows that for this case not that many points need to be taken. Furthermore, it can be seen that Theorem (11) results in a smaller value of the $\mathcal{H}_2$-norm for $N = 0$ than in [11], while using the same approximation method, which indicates that the proposed LMIs introduce less conservatism.



(a) $\mathcal{H}_2$ objective functions



(b) $w_c(t) = \delta(t - h)$



(c) $w_c(t) = \delta(t - \frac{1}{2}h)$

Fig. 5: The difference in objective functions and the resulting time-domain impulse responses at different moments for the case $d = 0$ with $N = 0$ (blue) and $N = 1000$ (orange).

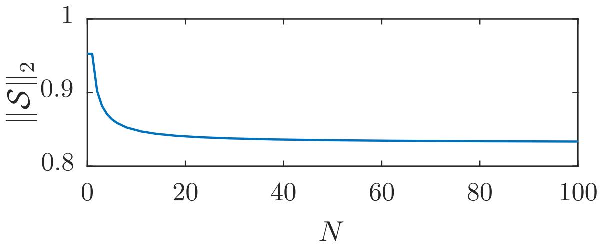


Fig. 6: The $\mathcal{H}_2$-norm for several choices of N with $d = 0$.

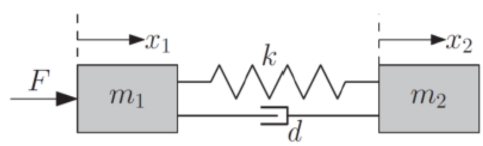


Fig. 7: The considered plant, where we want to control the velocity of the non-collocated mass $\dot{x}_2$.

B. $\mathcal{H}_\infty$ Loop Shaping of a Mass-spring-damper System

Generally, reference tracking is one of the main goals in control. Therefore, design problems for sampled-data systems will be illustrated on a mass-spring-damper-mass system, which is shown in Fig. 7. For this example, we want to have the velocity of the non-collocated mass $\dot{x}_2$ track a user-defined reference r, while rejecting the disturbance signal w. The model for this system is given by

$$\begin{cases} \begin{bmatrix} \ddot{x}_1 \\ \dot{x}_1 \\ \ddot{x}_2 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -\frac{d}{m} & -\frac{k}{m} & \frac{d}{m} & \frac{k}{m} \\ 1 & 0 & 0 & 0 \\ \frac{d}{m} & \frac{k}{m} & -\frac{d}{m} & -\frac{k}{m} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ x_1 \\ \dot{x}_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{m} \\ 0 \\ 0 \\ 0 \end{bmatrix} w + \begin{bmatrix} \frac{1}{m} \\ 0 \\ 0 \\ 0 \end{bmatrix} u, \\ z_d = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x^p + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}, \\ y = \begin{bmatrix} 0 & 0 & -1 & 0 \end{bmatrix} x^p + r, \end{cases} \tag{54}$$

with sampling time $h = 0.1$ s and where the parameters are given in Table I. These parameters are chosen such that the low-frequent dynamics allow a low sampling frequency to better illustrate the sampled output in the response. The Bode diagram of the system can be found in Fig. 8.

TABLE I: The considered parameters for the example system, described in (54).

Parameter	Value	Unit
m	0.1	kg
d	0.01	Ns/m
k	0.01	N/m

Then, weighting filters are designed to reject disturbances until 0.01 Hz and achieve a bandwidth of around 0.6 Hz for references up to 10 m/s. Next to that, the input should not exceed 2000 N. However, evaluating if the performance criteria are met and tuning these weighting filters is generally done using Bode diagrams, which are not available for sampled-data systems. Therefore, we chose to discretise the plant using Tustin's discretisation method, see e.g., [1], yielding the closed-loop transfers shown in Fig. 9. This shows that the design criteria are met and are validated by the step response shown in Fig. 10.

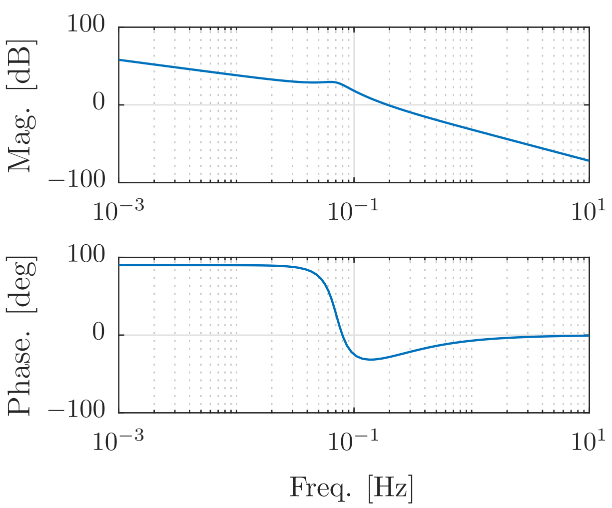
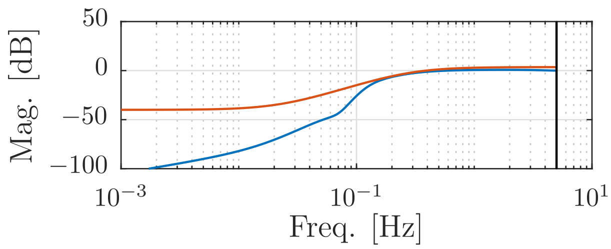


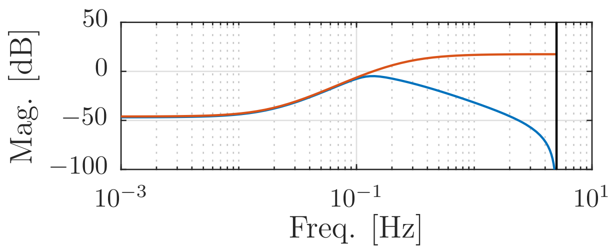
Fig. 8: The Bode diagram of the considered plant.

VII. Conclusion

Digital platforms are often employed to control continuous-time systems, while in controller design

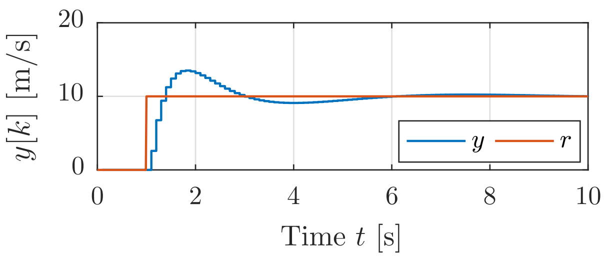


(a) $r \to z_d$

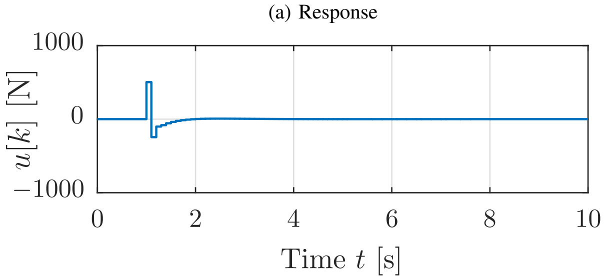


(b) $w \to z_d$

Fig. 9: The closed-loop transfers of the discretised plant with the synthesised controller (blue) with their corresponding weighting filters (orange) that were used for design.



(a) Response



(b) Input

Fig. 10: The step response of the sampled-data system.

the effect of the introduced sampling is often neglected by selecting a sufficiently high sampling/update frequency. In this paper, a tractable method is proposed to synthesise optimal controllers for these sampled-data systems with respect to the $\mathcal{H}_\infty$- and $\mathcal{H}_2$-norm. This is done by formulating LMIs using the explicit solution to a Riccati differential equation. This solution does incorporate matrix exponentials, complicating extensions towards parameter dependencies.

This methodology is extended to perform continuous-time design techniques, such as $\mathcal{H}_\infty$ loop shaping, to sampled-data systems. This is done by applying hybrid filters and adjusting certain assumptions during the synthesis procedure. These methods are illustrated using an academic example from the literature and a practical design problem for a simple mass-spring-damper system. The presented results show that controllers with a good approximation of the optimal $\mathcal{H}_\infty$- and $\mathcal{H}_2$-norm are synthesised. Furthermore, it is shown that $\mathcal{H}_\infty$ loop shaping can be applied to more practical systems, while taking the effect of sampling into account.

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