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Structure Preserving Discretization of port-Hamiltonian Distributed Parameter Systems

B.C. van Huijgevoort

Abstract
Many methods in scientific computing disregard the underlying physical properties of the systems they simulate. In this paper a new way of spatially discretizing distributed parameter systems is presented. This method is applicable to port-Hamiltonian systems and preserves the Dirac structure and energy balance underlying these kind of systems. The dynamics of the system is described for all of the elements in the mesh, which are interconnected to describe the behaviour over the whole geometry. This method leads to more accurate solutions and gives full insight into the spatial distribution of energy.

1. Introduction

In this paper a specific class of dynamical systems, which evolve both over time and space is considered. These systems are typically described by partial differential equations and they have an infinite-dimensional state space. Infinite-dimensional systems impose challenges in for example finding analytical solutions and in the numerical simulation of these systems. To get insight in these systems, and to analyze their properties, it is typically necessary to resort to numerical tools. Then it is possible to compute their solutions, to determine system responses and to test multiple scenarios while varying parameters, initial conditions or applying different system excitations. Scientific computing is the field that provides numerical methods to simulate infinite-dimensional systems. Most of these methods require discretization in the temporal and spatial domain. Since controlling an infinite-dimensional system implies controlling an infinite number of states with a finite number of control parameters, controlling these systems usually requires discretization in the temporal and spatial domain as well. The result of a discretization will be a finite set of ordinary differential equations, which is easier to simulate than partial differential equations. Most conventional methods model the behavior of infinite-dimensional systems by approximation, while disregarding the physical structure underlying the system. Examples of such spatial discretization methods are the finite difference method (FDM), the finite volume method (FVM) and the finite element method (FEM) [**?**]. However, partial differential equations are mostly inferred from first principle physical laws that represent conservation laws, such as energy-, momentum-, charge- and mass conservation. Some famous examples of distributed parameter systems are the Burgers's equation in 1D [**?**], the wave equation and the Navier-Stokes equations in 2D or 3D [**?**, **?**] and the Maxwell equations in 3D [**?**, **?**]. Most conventional discretization methods numerically solve partial differential equations and some incorporate adjustments a posteriori to comply to physical laws, however, in principle they ignore these conservation laws. This is due to the fact that most discretization methods in scientific computing substitute the spatial derivative operator by a discrete approximation on a set of sample points (the mesh points). By doing so, the approximate operator is a mathematical construct that does not take system properties into account, since it disregards the system and its properties completely.

In this paper only distributed parameter systems that can be represented as a port-Hamiltonian system will be considered, since they have specific structural properties, such as stability and passivity. More precisely, port-Hamiltonian systems are suitable to model systems, which have an inherent loss of energetic content (dissipation), when they are not excited by their environment. In addition, their energy level is fully determined by the energy that is delivered to the system through its external ports. These properties make them very interesting for many applications, such as control design in mechatronic systems, electrical engineering, robotics, etc. The port-Hamiltonian framework is an extension of the Hamiltonian framework, where ports are

defined as the input and output variables whose product is power. The Hamiltonian is a function that represents the total energy in the energy-storing elements of a system and it plays a significant role in our new spatial discretization approach. The behavior of a system is defined by the Hamiltonian and a geometric structure, i.e., a Dirac structure. For this specific structure power-conservation holds, which guarantees passivity [**?**, **?**]. This means that at any time interval the change of energy will never exceed the energy supplied through external ports. Many physical systems can be written as a Hamiltonian system. Examples can be found in various domains; LRC circuits and transmission lines from the electrical domain [**?**, **?**, **?**], a magnetically levitated ball [**?**] and Maxwell equations [**?**, **?**, **?**] from the electromagnetical domain, spring-damper systems, e.g. vibrating mechanical structures, from the mechanical domain [**?**], fluid dynamics [**?**, **?**], and many more.

By extending a Hamiltonian system to a port-Hamiltonian system description, multiple (even infinite) systems, can be interconnected using the ports. A port is used to describe the interaction of the port-Hamiltonian system with the environment. Besides that they can be used to described the exchange of energy between two systems. So, in general ports represent power exchanges and are very suitable to make interconnections among multiple port-Hamiltonian systems. This is due to the fact that any interconnection of multiple port-Hamiltonian systems is port-Hamiltonian as well. The modular behaviour of these models makes it easy to guarantee important properties, such as stability, passivity and conservation of quantities. Conventional discretization methods destroy the port-Hamiltonian structure, while it is very beneficial to maintain it. The main goal of this paper is to develop a discretization method for infinite-dimensional port-Hamiltonian systems that preserves the structure of port-Hamiltonian systems in which the Hamiltonian function represents energy. Besides that, passivity and modularity of interconnections should also be preserved, with the highest accuracy of the approximation possible.

Some simple examples of partial differential equations are the heat equation

$$\frac{\partial w}{\partial t} = -\alpha \nabla^2 w, \tag{1}$$

and the wave equation

$$\frac{\partial^2 w}{\partial t^2} = -c^2 \nabla^2 w, \tag{2}$$

where $w(z,t)$ represents temperature or position respectively. Note that $w(z,t)$ is a function of space $z \in Z \subset \mathbb{R}^n$ and time $t \in T \subseteq \mathbb{R}$. Here Z is the spatial domain and T the temporal domain. Typically, Z is assumed bounded and T is unilateral and unbounded. The parameters α and c^2 are physical constants and ∇ is the Laplace operator which in a three-dimensional spatial domain is given by

$$\nabla = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_3}. \tag{3}$$

Most partial differential equations can be written in port-Hamiltonian form using physical relations. Consider the displacement of a vibrating string/plate, which is described by the wave equation (2). This can be written in port-Hamiltonian form as

$$\begin{bmatrix} \dot{\varepsilon} \\ \dot{\rho} \end{bmatrix} = \begin{bmatrix} 0 & \nabla \\ -\nabla & 0 \end{bmatrix} \begin{bmatrix} \sigma \\ v \end{bmatrix}, \tag{4}$$

where ε is strain, ρ is kinetic momentum, σ is stress and $v = \frac{\partial w}{\partial t} = \dot{w}$ is velocity. The derivation is given in Appendix B. Note that this example illustrates that (2) can be written in port-Hamiltonian form, where details on the port-Hamiltonian form are given in the next section. Therefore this example is continued in subsection 2.4. For now, notice that (2) and (4) are equivalent.

Structure preserving spatial discretization has already been carried out for one-dimensional systems [**?**, **?**, **?**, **?**], and two-dimensional systems [**?**, **?**]. The two-dimensional framework however, has not been applied to a model of a real system yet, nor is its convergence proven. The extension to three-dimensional systems has not been done either. In this paper, the two-dimensional approach developed in [**?**] will be expanded to three-dimensional geometries. There exist also some completely different approaches, that are more mathematically oriented instead of engineering oriented. They are developed for n-dimensional manifolds [**?**, **?**, **?**].

The paper is organized as follows. The next section gives some background of the port-Hamiltonian framework. The problem definition of this project is given in section 3. In section 4, the discretization approach and its interconnection structure are described. Section 5 discusses the approach for three-dimensional manifolds. It shows that a mistake has been made in the approach and that there is not enough freedom to finalize with a finite-dimensional port-Hamiltonian system. A solution is designed for one-dimensional geometries in section 6. This paper ends with the conclusion and recommendations that are given in section 7. Some important mathematical concepts used in this paper are given in Appendix A. A way to find expressions for the shape functions for three- and one-dimensional manifolds is given in Appendix F and J respectively. The remaining appendices contain mathematical proofs or detailed derivations.

2. Port-Hamiltonian systems

Since this paper considers distributed parameter systems that can be described in port-Hamiltonian form, before formally describing the problem addressed in this paper, some insight into the port-Hamiltonian framework is necessary. First of all a distinction has to be made between finite-dimensional port-Hamiltonian systems and infinite-dimensional port-Hamiltonian systems. Infinite-dimensional port-Hamiltonian systems admit equivalent descriptions as partial differential equations, while finite-dimensional port-Hamiltonian systems admit equivalent descriptions as ordinary differential equations.

2.1. Finite-dimensional systems

Finite-dimensional port-Hamiltonian systems without feedthrough are generally given by

$$\begin{cases} \dot{x} = (J-R)\frac{\partial H(x)}{\partial x} + Bu \\ y = B^T \frac{\partial H(x)}{\partial x}, \end{cases} \tag{5}$$

where $x(t) \in \mathbb{R}^N$ is the N-dimensional state variable, $u(t)$ and $y(t)$ the input and output variables, respectively and $H : \mathbb{R}^N \to \mathbb{R}$ is the Hamiltonian or energy function. Every port consists of an input/output pair $(u(t), y(t))$, with $\dim(u)$=$\dim(y)$. Furthermore, J and R are real-valued $N \times N$ matrices that are skew symmetric and positive semi-definite, respectively. J represents the exchange of energy between the energy storing elements and R represents the losses in the system [**?**, **?**].

A property of (port-)Hamiltonian systems is that they are dissipative, which means that they satisfy the dissipation inequality

$$H(x(t_1)) \leq H(x(t_0)) + \int_{t_0}^{t_1} y(t)^T u(t)\, \mathrm{d}t \tag{6}$$

for all $t_0 < t_1$ and all system trajectories (u, x, y) that satisfy (5). The dissipation inequality shows that the total change of power in the system can never be bigger than the power delivered through its ports. By taking an infinitesimal time step between t_0 and t_1, (6) can equivalently be written as

$$\frac{d}{dt} H(x(t)) \leq y(t)^T u(t), \tag{7}$$

for all time $t \in T$. This clearly shows that the variation in internal stored energy never exceeds the power supplied through the input and output. If (6) and (7) are equalities, the system is conservative, which is the case if there are no losses in the system ($R = 0$). Assume a system with multiple inputs u and outputs y, where a port of the system is defined as any pair (u_i, y_i), with its product being power. Then each port delivers a power of

$$P_i(t) = y_i^T(t) u_i(t) \tag{8}$$

to the system. If the system has K ports, the total power delivered to the system equals

$$P(t) = \sum_{i=1}^{K} y_i^T(t) u_i(t). \tag{9}$$

Effort and flow

Two important concepts in port-Hamiltonian systems are flow f and effort e. The flow variable can be seen as the variable that establishes an equilibrium and describes the rate of change of the energy variables. Whereas the effort determines the equilibrium. Electrical current is for example a flow variable and voltage an effort variable [**?**].
The effort is chosen as:

$$e := \frac{\partial H}{\partial x}. \tag{10}$$

Besides that, the flows are chosen as the rate of change of the state:

$$f = \dot{x}. \tag{11}$$

These choices result in an equivalent representation of a finite-dimensional port-Hamiltonian system (5) based on the effort and flow given by

$$\begin{cases} f = (J-R)e + Bu \\ y = B^T e. \end{cases} \tag{12}$$

Typically, the state of the system, consists of the energy variables p and q; $x(t) = \mathrm{col}(p(t), q(t))$. The effort (10) can thus equivalently be written as the vector of co-energy variables:

$$e^p := \tfrac{\partial H(p,q)}{\partial p}, \quad e^q := \tfrac{\partial H(p,q)}{\partial q}. \tag{13}$$

Similarly, the flow (11) can equivalently be written as the vector of the rate of change of the energy variables:

$$f^p := \tfrac{dp}{dt}, \quad f^q := \tfrac{dq}{dt}. \tag{14}$$

Power balance

The Hamiltonian represents the energy in the system, so the derivative of the Hamiltonian with respect to time represents the change of energy. Using the definition of e (10) and f (11), the following can be found

$$\frac{\mathrm{d}H(x)}{\mathrm{d}t} = \frac{\partial H(x)}{\partial x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t} = e^T f. \tag{15}$$

Substituting the expression for the flow and for y^T (12), while exploiting the skew-symmetric property for J leads to

$$e^T f = y^T u - e^T R e. \tag{16}$$

This can be seen as the power balance of a finite-dimensional port-Hamiltonian system, where $e^T f$ is the change of power in the system, $y^T u$ is power flowing over the boundary and $e^T R e$ is the dissipated power. This can be equivalently written as a finite-dimensional bi-linear pairing

$$\langle e \mid f \rangle := e^T f - y^T u + e^T R e, \tag{17}$$

which by (12) always equals zero.

2.2. Infinite-dimensional systems

Some partial differential equations can be written as an infinite-dimensional port-Hamiltonian system, which on a n-dimensional manifold $Z \subset \mathbb{R}^n$ can generally be written as

$$\begin{aligned} \begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} &= (J-R) \begin{bmatrix} \delta_p H \\ \delta_q H \end{bmatrix} \\ \begin{bmatrix} u \\ y \end{bmatrix} &= \begin{bmatrix} B \\ C \end{bmatrix} \begin{bmatrix} \delta_p H|_{\partial Z} \\ \delta_q H|_{\partial Z} \end{bmatrix}. \end{aligned} \tag{18}$$

In the infinite-dimensional case, $p(z,t)$ and $q(z,t)$ do not only depend on time t but also depend on space $z \in Z$. Similarly, J and R become operators on (Hilbert) spaces of functions on Z. Furthermore, δ_p and δ_q are the variational derivatives with respect to energy variable p and q respectively. The variables $u(t)$ and $y(t)$ are the input and output and are assumed to depend only on time.

The Hamiltonian

The Hamiltonian $H(p,q)$ or energy function describes the total energy in a physical system and is given by [**?**, **?**]

$$H(p,q) = \int_Z \mathscr{H}(p,q), \tag{19}$$

where $\mathscr{H}(p,q)$ is the energy density and Z is the manifold. The energy density considered in this paper is quadratic and is defined as

$$\mathscr{H}(p,q) := \frac{1}{2}[c^p(p \wedge \star p) + c^q(q \wedge \star q)], \tag{20}$$

where c^p and c^q are physical material properties; $c^p, c^q \in \mathbb{R}$. An explanation of the notation used in this paper, such as the $\wedge$- and $\star$-operator, is given in Appendix A. Substituting (20) into (19) shows that the Hamiltonian is

$$H(p,q) = \frac{1}{2}\int_Z [c^p(p \wedge \star p) + c^q(q \wedge \star q)]. \tag{21}$$

Effort and flow

The efforts are chosen as the vector of co-energy variables:

$$\begin{aligned} e^p &:= \delta_p H = \tfrac{\partial \mathscr{H}(p,q)}{\partial p} = c^p \star p, \\ e^q &:= \delta_q H = \tfrac{\partial \mathscr{H}(p,q)}{\partial q} = c^q \star q. \end{aligned} \tag{22}$$

Since p and q depend on time t and space z, the efforts are also functions of time and space. Besides that, the flows are chosen as the rate of change of the energy variables:

$$f^p := \tfrac{\partial p}{\partial t}, \quad f^q := \tfrac{\partial q}{\partial t}. \tag{23}$$

Note that this is the same choice as for the finite-dimensional case (14), however, now the partial derivative with respect to time is taken since the energy variables depend on time and space.

In (18), some terms in J can be the Laplace operator, as shown in the example of the wave equation (4). This operator creates difficulties, however, an equivalent representation exists that does not have this drawback. This representation is based on differential geometry, where Appendix A explains its notation. Besides that, using the definitions for effort and flow, the system in (18) can equivalently be written as

$$\begin{aligned} \begin{bmatrix} f^p \\ f^q \end{bmatrix} &= \begin{bmatrix} -\sigma\star & (-1)^r \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e^p \\ e^q \end{bmatrix} \\ \begin{bmatrix} f^b \\ e^b \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-n^q} \end{bmatrix} \begin{bmatrix} e^p|_{\partial Z} \\ e^q|_{\partial Z} \end{bmatrix}. \end{aligned} \tag{24}$$

Here n^p and n^q define the form of p and q respectively and $r = n^p n^q + 1$. The term *form* refers to a differential form, which is the core of differential geometry and is introduced in Appendix A. Furthermore, $|_{\partial Z}$ denotes restriction to the boundary of the n-dimensional manifold Z. The dissipation in the system is represented by σ. Here, σ can be a scalar if the dissipation is homogeneous in all directions, a diagonal matrix if the dissipation is inhomogenous and a non-symmetric matrix if the dissipation is anisotropic. In (24) f^p and f^q represent the flow of energy variable p and the flow of energy variable q respectively. Similarly e^p and e^q represent the effort of energy variable p and the effort of energy variable q respectively. Furthermore, f^b and e^b model the interaction of the system at the boundary of the spatial domain, ∂Z. Comparing (24) with (18) shows that the role of J is taken over by the exterior derivatives d and the role of R is given by σ. Besides that the input and output in (18) are given by the flow and effort over the boundary, which are formally described as the evaluation of the variational derivative of H at the boundary

$$\begin{aligned} e^p\,|_{\partial Z} &= \delta_p H\,|_{\partial Z} \\ e^q\,|_{\partial Z} &= \delta_q H\,|_{\partial Z}\,. \end{aligned} \tag{25}$$

Power balance

Proposition 2.1. *The power balance for an infinite-dimensional port-Hamiltonian system is given by*

$$\begin{aligned} &\int_Z [f^p \wedge e^p + f^q \wedge e^q] = \\ &- \int_{\partial Z} [e^b \wedge f^b] - \int_Z [e^p \wedge \sigma \star e^p]. \end{aligned} \tag{26}$$

This expresses that the change of power on the manifold Z equals the power flowing over the boundary minus the losses in the system. This is derived by using the derivative of the Hamiltonian with respect to time. The proof can be found in Appendix C. The resemblance between this energy balance and the one of the finite-dimensional case (16), will be used later in this approach.

2.3. Dirac structure

As mentioned in the introduction, the behaviour of a system can be defined by the Hamiltonian and a geometric structure, i.e. a Dirac structure. A finite-dimensional Dirac structure is defined as follows [**?**]

Definition 1. *A* Dirac structure *on $\mathscr{F} \times \mathscr{E}$ is a subspace $\mathscr{D} \subset \mathscr{F} \times \mathscr{E}$ equipped with a bilinear-paring $\langle \cdot \mid \cdot \rangle$ on $\mathscr{F} \times \mathscr{E}$, such that*

i $\langle e \mid f \rangle = 0$, *for all* $(f,e) \in \mathscr{D}$,

ii $dim\mathscr{D} = dim\mathscr{F}$.

Here, $\mathscr{E}$ and $\mathscr{F}$ are finite-dimensional subspaces of a vector space. In line with (14) and (13), let $\mathscr{F} = \mathbb{R}^{n_p} \times \mathbb{R}^{n_q} \times \mathbb{R}^{n_b}$ and let $\mathscr{E} = \mathbb{R}^{n_p} \times \mathbb{R}^{n_q} \times \mathbb{R}^{n_b}$ be the spaces in which flows $f = col(f^p, f^q, f^b)$ and efforts $e = \mathrm{col}(e^p, e^q, e^b)$ assume their values. Here $\langle e \mid f \rangle$ denotes the dual product of e and f. For $e, f \in \mathbb{R}^N$, this is the same as $e^T f$. This product of effort and flow represents power, so property *i* in the definition of a Dirac structure corresponds to power conservation. Property *ii* shows that a Dirac structure is a maximally dimensional subspace. Details on this definition can be found in [**?**].

Note that for the finite-dimensional case, (12) is a Dirac structure;

$$\begin{aligned} &\mathscr{D} := \{(e^p, e^q, e^b), (f^p, f^q, f^b) \in \mathscr{E} \times \mathscr{F} \mid \\ &\begin{bmatrix} f \\ y \end{bmatrix} = \begin{bmatrix} J-R & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} e \\ u \end{bmatrix} \}, \end{aligned} \tag{27}$$

with respect to the following bi-linear pairing

$$\langle e \mid f \rangle := e^T f - y^T u + e^T R e. \tag{28}$$

This can be proven to be a Dirac structure by defining a new flow and effort as

$$\hat{f} = \begin{bmatrix} f \\ -y \end{bmatrix}, \quad \hat{e} = \begin{bmatrix} e \\ u \end{bmatrix}. \tag{29}$$

1D	2D	3D
$n^p = 1$	$n^p = 2$	$n^p = 3$
$n^q = 1$	$n^q = 1$	$n^q = 1$
	$n^p = 1$	$n^p = 1$
	$n^q = 2$	$n^q = 3$
		$n^p = 2$
		$n^q = 2$

Table 1: Possible forms of energy variables for one-dimensional (1D), two dimensional (2D) and three-dimensional (3D) geometries

These new flows and efforts can be used to rewrite (27) to

$$\begin{aligned} &\mathscr{D} := \{(e^p, e^q, e^b), (f^p, f^q, f^b) \in \mathscr{E} \times \mathscr{F} \mid \\ &\hat{f} = \begin{bmatrix} J-R & B \\ -B^T & 0 \end{bmatrix} \hat{e} \}. \end{aligned} \tag{30}$$

Proposition 2.2. *$\mathscr{D}$ as given in* (30) *is a finite-dimensional Dirac structure if $R = 0$.*

Proof. The proof is given in Appendix D. ■

Infinite-dimensional Dirac structures

In the infinite-dimensional port-Hamiltonian framework [**?**], the linear space of flows is defined as

$$\mathscr{F}_{p,q} := \Lambda^{n^p}(Z) \times \Lambda^{n^q}(Z) \times \Lambda^{n-n^p}(\partial Z) \tag{31}$$

and the linear space of efforts is defined as

$$\mathscr{E}_{p,q} := \Lambda^{n-n^p}(Z) \times \Lambda^{n-n^q}(Z) \times \Lambda^{n-n^q}(\partial Z). \tag{32}$$

Here,

$$n^p + n^q = n + 1 \tag{33}$$

should hold, [**?**, **?**], where n^p and n^q define the form of p and q respectively and n is the dimension of the manifold Z. The possible choices for n^p and n^q for various n are given in Table 1.

For infinite-dimensional systems the definition of a Dirac structure can also be written as [**?**]

Definition 2. *A (constant) Dirac structure on $\mathscr{F}_{p,q} \times \mathscr{E}_{p,q}$ is a subspace $\mathscr{D} \subset \mathscr{F}_{p,q} \times \mathscr{E}_{p,q}$ such that*

$$\mathscr{D} = \mathscr{D}^{\perp}, \tag{34}$$

where $\perp$ denotes the orthogonal complement with respect to the bi-linear pairing $\langle \cdot \mid \cdot \rangle$.

An indefinite symmetric bi-linear pairing $\langle \cdot \mid \cdot \rangle$ is defined as

$$\langle (f^p, e^p), (f^q, e^q) \rangle := \langle e^p \mid f^q \rangle + \langle e^q \mid f^p \rangle, \tag{35}$$

with $(f^p,e^p),(f^q,e^q) \in \mathscr{F}_{p,q} \times \mathscr{E}_{p,q}$. This bi-linear pairing is also non-degenerate, meaning that if $\langle (f^p,e^p),(f^q,e^q)\rangle = 0$ for all (f^q,e^q), this implies that $(f^p,e^p) = 0$ [**?**].

The Dirac structure corresponding to (24) is written as [**?**]

$$\begin{aligned}\mathscr{D} := \{&(e^p,e^q,e^b),(f^p,f^q,f^b) \in \mathscr{E}_{p,q} \times \mathscr{F}_{p,q} \mid \\ &\begin{bmatrix} f^p \\ f^q \end{bmatrix} = \begin{bmatrix} -\sigma\star & (-1)^r \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e^p \\ e^q \end{bmatrix} \\ &\begin{bmatrix} f^b \\ e^b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-n^q} \end{bmatrix} \begin{bmatrix} e^p|_{\partial Z} \\ e^q|_{\partial Z} \end{bmatrix}\},\end{aligned} \tag{36}$$

with respect to the following non-degenerate bi-linear pairing $\langle e \mid f \rangle \in \mathbb{R}$ with $e \in \mathscr{E}_{p,q}$ and $f \in \mathscr{F}_{p,q}$:

$$\begin{aligned}&\langle (e^p,e^q,e^b),(f^p,f^q,f^b)\rangle := \\ &\textstyle\int_Z [e^p \wedge f^p + e^q \wedge f^q] + \int_{\partial Z} [e^b \wedge f^b] \\ &\textstyle + \int_Z [e^p \wedge \sigma \star e^p].\end{aligned} \tag{37}$$

Proposition 2.3. *$\mathscr{D}$ from* (36) *defines a (constant) Dirac structure for the infinite dimensional port-Hamiltonian system.*

Proof. This Proposition is proven for $(n,n^p,n^q) = (3,3,1)$ in Appendix E. ■

2.4. Examples

This subsection discusses some examples of partial differential equations or ordinary differential equations written as infinite-dimensional port-Hamiltonian or finite-dimensional port-Hamiltonian system respectively.

The wave equation

The wave equation (without losses) written as (4) can now be recognized to be in infinite-dimensional port-Hamiltonian form with Hamiltonian

$$\begin{aligned} H(\varepsilon,\rho) &= \tfrac{1}{2} \textstyle\int_Z [\varepsilon \wedge \sigma + \rho \wedge v] \\ &= \tfrac{1}{2} \textstyle\int_Z [\varepsilon \wedge E \star \varepsilon + \rho \wedge \frac{1}{\mu} \star \rho], \end{aligned} \tag{38}$$

which is of the same form as defined in (21). Its Dirac structure can be written as

$$\begin{aligned}\mathscr{D} := \{&(e^p,e^q,e^b),(f^p,f^q,f^b) \in \mathscr{E}_{p,q} \times \mathscr{F}_{p,q} \mid \\ &\begin{bmatrix} f^p \\ f^q \end{bmatrix} = \begin{bmatrix} 0 & (-1)^r \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e^p \\ e^q \end{bmatrix} \\ &\begin{bmatrix} f^b \\ e^b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-n^q} \end{bmatrix} \begin{bmatrix} e^p|_{\partial Z} \\ e^q|_{\partial Z} \end{bmatrix}\},\end{aligned} \tag{39}$$

with

$$\begin{aligned} f^p &= \dot{\varepsilon} & e^p &= \sigma \\ f^q &= \dot{\rho} & e^q &= v. \end{aligned} \tag{40}$$

Lossless transmission line

A lossless transmission line of length L is represented by the telegraph equations [**?**]

$$\begin{aligned} \tfrac{\partial \phi}{\partial t} &= -\tfrac{\partial V}{\partial z} \\ \tfrac{\partial q}{\partial t} &= -\tfrac{\partial I}{\partial z}. \end{aligned} \tag{41}$$

Here $q(t,z)$ is the charge density and $\phi(t,z)$ is the flux density. $I(t,z)$ is the current and $V(t,z)$ the voltage, which equal

$$\begin{aligned} e^q(t,z) = V(t,z) &= \tfrac{q(t,z)}{C(z)} \\ e^\phi(t,z) = I(t,z) &= \tfrac{\phi(t,z)}{L(z)} \end{aligned} \tag{42}$$

respectively. The total energy stored in the transmission line at time t is equal to

$$H(\phi,q) = \frac{1}{2} \int_0^L [\frac{\phi^2(t,z)}{L(z)} + \frac{q^2(t,z)}{C(z)}], \tag{43}$$

which can be written in the same form as in (21) using differential geometry. The result is

$$H(\phi,q) = \frac{1}{2} \int_0^L [\frac{1}{C(z)} q \wedge \star q + \frac{1}{L(z)} \phi \wedge \star \phi]. \tag{44}$$

Therefore according to (22), the efforts of the infinite-dimensional port-Hamiltonian system are current I and voltage V. Besides that, the energy variables (p and q) are the flux density ϕ and the charge density q. So, according to (23), the flows of the port-Hamiltonian system are $\dot{\phi}$ and $\dot{q}$. The partial differential equations (41) can then be represented as an infinite-dimensional port-Hamiltonian system as follows

$$\begin{bmatrix} \dot{\phi} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{d} \\ -\mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} I \\ V \end{bmatrix}. \tag{45}$$

The input and output of a transmission line will be the voltage at point 0 and the current at point L, or the other way around. Without defining whether f^b or e^b is the input, this can be written as

$$\begin{bmatrix} f^b \\ e^b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} I|_{\partial Z} \\ V|_{\partial Z} \end{bmatrix}, \tag{46}$$

with $\partial Z = \{0,L\}$. Together (45) and (46) complete the infinite-dimensional port-Hamiltonian description of the lossless transmission line. Note that this is the exact same form as in (24), for $(n,n^p,n^q) = (1,1,1)$ and $\sigma = 0$.

LC circuit

Previously a lossless transmission line was written in infinite-dimensional port-Hamiltonian form. A

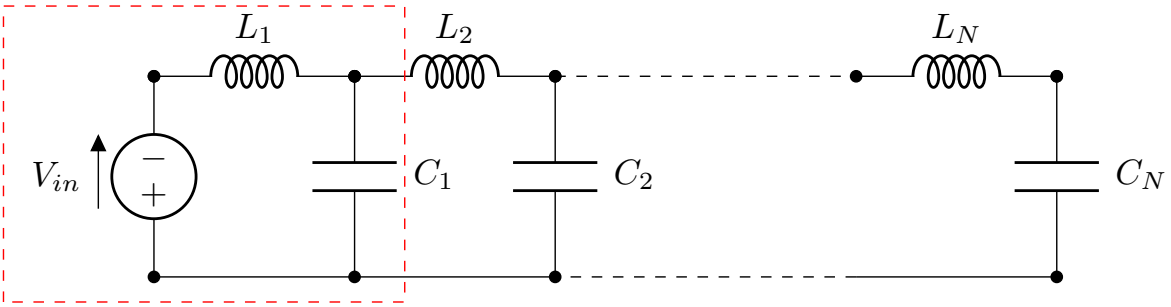


Figure 1: Model of a transmission line

transmission line is typically modeled/approximated by a finite number of LC-circuits in series, as shown in Fig. 1. Here one LC circuit is given in the red box. This LC-circuit can be written in finite-dimensional port-Hamiltonian form.

The Hamiltonian of this circuit is similar as the Hamiltonian for the transmission line and is given by

$$H(\phi,q) = \frac{1}{2}[\frac{\phi^2(t)}{L} + \frac{q^2(t)}{C}]. \tag{47}$$

Therefore the energy variables are the flux of the inductor (ϕ) and the charge of the capacitor (q). The effort is defined as in (13), so the efforts are the current of the inductor I_L and the voltage over the capacitor V_C. Writing the circuit as a finite-dimensional port-Hamiltonian system then boils down to writing $\dot{\phi}$ and $\dot{q}$ as a function of I_L and V_C. The distributed voltage and the distributed current are given in (42). From these equations a similar relation can be derived for the space-independent flux through an inductor and the space-independent charge through a capacitor. The result is

$$\begin{aligned} \phi(t) &= LI_L(t) \\ q(t) &= CV_C(t). \end{aligned} \tag{48}$$

Taking the time derivative and using Kirchoff's laws leads to the following description of this LC-circuit

$$\begin{aligned} \begin{bmatrix} \dot{\phi} \\ \dot{q} \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} I_L \\ V_C \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} V_{in} \\ I_C &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} I_L \\ V_C \end{bmatrix}, \end{aligned} \tag{49}$$

which is in finite-dimensional port-Hamiltonian form as can be seen by comparing it with (12), where

$$\begin{aligned} f^\phi &= \tfrac{d\phi}{dt}, \quad e^p = \tfrac{\partial H}{\partial \phi} = I_L, \quad u = V_{in} \\ f^q &= \tfrac{dq}{dt}, \quad e^q = \tfrac{\partial H}{\partial q} = V_C, \quad y = I_C. \end{aligned} \tag{50}$$

3. Problem definition

The problem considered in this paper is the approximate modeling of distributed parameter systems in port-Hamiltonian form. The goal is to find a finite-dimensional Dirac structured approximation of the infinite-dimensional Dirac structure, generally given by (36). To do so, the Dirac structured approximation contains multiple finite dimensional Dirac structures, generally given in (27), interconnected in a power-conserving way. Besides that, the conserved quantities should also be preserved after the discretization in order to obtain a physically realistic model. Besides inheriting the physical structure of the original system, after interconnection, the balance equations underlying the original system should be satisfied and it is desired to retain full insight in the spatial distribution of energy, represented by the Hamiltonian, along the complete geometry. Some corresponding sub problems are: What is a good 3D geometrical geometry for this method? What is a good 3D shape to use for the elements in the mesh? What is the minimal state dimension of such an element? What is an appropriate way to mesh the 3D geometrical structure? Is it possible to infer a discretization method that results in a sparse finite dimensional model? How should the elements be interconnected, such that power is preserved? What is the accuracy of the approximation? How can this method be implemented on real systems? What is the accuracy of this method compared to a conventional method?

The previously developed method [**?**] is designed for an arbitrary *two-dimensional* geometry, in this paper it will be expanded to *three-dimensional* geometries.

4. Discretization in 3D

This section describes an approach to discretize three-dimensional port-Hamiltonian distributed parameter systems, generally given by (24). It is based on the two-dimensional approach presented in [**?**]. In this case the spatial domain is defined as a three-dimensional manifold Z with a two-dimensional boundary ∂Z. Here $Z \subset \mathbb{R}^3$ which is assumed to be bounded and closed as a subset of $\mathbb{R}^3$.

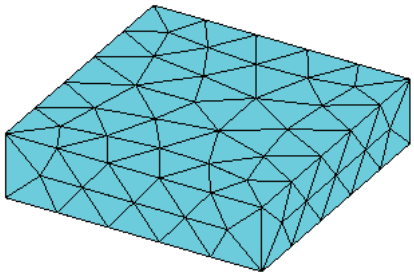


Figure 2: Tetrahedral mesh of a plate

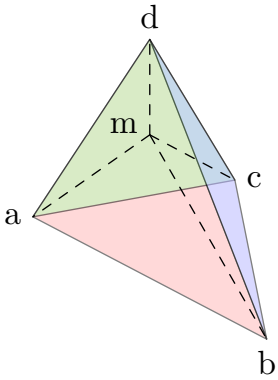


Figure 3: Element Z_{abcd} with $m \in Z_{abcd}$

4.1. Three-dimensional tetrahedral meshing

The three-dimensional geometry is meshed into tetrahedra, which is the simplest three-dimensional shape with non-zero volume [**?**]. This results in the least complex derivation of the finite-dimensional port-Hamiltonian systems, since the derivation only has to be performed for one element instead of multiple. However, it also results in a large number of elements in the geometry. After showing that this simple structure is rich enough, other structures that may lead to a decrease of the number of elements in the mesh, can be chosen as well. The idea is that after interconnecting the tetrahedra in the mesh, the behavior on the whole spatial domain Z is described in port-Hamiltonian form.

Tetrahedral meshing is used very often and as long as the geometry is not too complicated the geometry can be covered by a mesh of tetrahedral elements (of different volumes) where neighboring tetrahedral share full faces. [**?**, **?**]. There exist geometries that can not be covered exactly by finite meshes of tetrahedra. Spheres and ellipsoids being examples of these. However, it is possible for i.a. rectangles, L-shapes and pyramids [**?**]. A tetrahedral mesh of a three-dimensional plate is given in Fig. 2.

A single tetrahedral element Z_{abcd} is given in Fig. 3. The *external faces* of the element are Z_w, with $w \in \{acb, abd, bcd, dca\}$ and the *external edges* are Z_{ij}, with $ij \in \{ab, bc, ca, ad, bd, cd\}$. By defining a point m in the interior of Z_{abcd}, multiple internal faces and edges are defined. The *internal faces* are Z_{lmn}, with $lmn \in \{abm, bcm, cam, adm, bdm, cdm\}$ and the *internal edges* are Z_k, with $k \in \{am, bm, cm, dm\}$. The point m is added to the tetrahedral structure to create additional degrees of freedom in the spatial discretization approach, such that it is possible to meet all requirements. [**?**].

4.2. Three dimensional manifolds

Table 1 shows that for three-dimensional manifolds both $(n^p, n^q) = (2,2)$ and $(n^p, n^q) = (3,1)$ are valid options. $(n^p, n^q) = (1,3)$ is also a valid option, however, these systems are equivalent to a system with $(n^p, n^q) = (3,1)$ and will therefore not be considered here. For now, it is chosen to focus only on systems with $(n^p, n^q) = (3,1)$.

For three-dimensional manifolds with this choice for n^p and n^q, the space of flows and the space of efforts can be derived from (31) and (32). They become

$$\begin{array}{lcl} \mathscr{F}_{p,q} & := & \Lambda^3(Z) \times \Lambda^1(Z) \times \Lambda^0(\partial Z) \\ \mathscr{E}_{p,q} & := & \Lambda^0(Z) \times \Lambda^2(Z) \times \Lambda^2(\partial Z). \end{array} \tag{51}$$

Since $(f^p, f^q, f^b) \in \mathscr{F}_{p,q}$ and $(e^p, e^q, e^b) \in \mathscr{E}_{p,q}$, this is equivalent to

$$\begin{array}{lcllcl} f^p & \in & \Lambda^3(Z) & e^p & \in & \Lambda^0(Z) \\ f^q & \in & \Lambda^1(Z) & e^q & \in & \Lambda^2(Z) \\ f^b & \in & \Lambda^0(\partial Z) & e^b & \in & \Lambda^2(\partial Z). \end{array} \tag{52}$$

The infinite-dimensional port-Hamiltonian system described in terms of effort and flow (24), for a three-dimensional manifold with $(n^p, n^q) = (3,1)$ is a Dirac structure

$$\begin{array}{l} \mathscr{D} := \{(e^p, e^q, e^b), (f^p, f^q, f^b) \in \mathscr{E}_{p,q} \times \mathscr{F}_{p,q} \mid \\ \begin{bmatrix} f^p \\ f^q \end{bmatrix} = \begin{bmatrix} -\sigma\star & \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e^p \\ e^q \end{bmatrix} \\ \begin{bmatrix} f^b \\ e^b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^p|_{\partial Z} \\ e^q|_{\partial Z} \end{bmatrix} \}, \end{array} \tag{53}$$

with respect to the following non-degenerate bi-linear pairing $\langle e \mid f \rangle \in \mathbb{R}$ on $\mathscr{E}_{p,q} \times \mathscr{F}_{p,q}$ with $e \in \mathscr{E}_{p,q}$ and $f \in \mathscr{F}_{p,q}$:

$$\begin{array}{l} \langle (e^p, e^q, e^b), (f^p, f^q, f^b) \rangle := \\ \int_Z [e^p \wedge f^p + e^q \wedge f^q] + \int_{\partial Z} [e^b \wedge f^b] \\ + \int_Z [e^p \wedge \sigma \star e^p]. \end{array} \tag{54}$$

In (53) $|_{\partial Z}$ denotes restriction to the boundary. This means that this effort only exists on the boundary of the geometry Z and not inside. Note that this does not change its form. A proof that (53) subject to (54) is a Dirac structure is given in Appendix E.

It is desired to obtain an approximate finite-dimensional Dirac structure for $\mathscr{D}$ in (53) with the following requirements.

Requirement 1:
The dynamics of the interconnection of the finite-dimensional port-Hamiltonian systems approximates the dynamics of the infinite-dimensional port-Hamiltonian system (53).

Requirement 2:
Every tetrahedral element satisfies a lossless power balance with its neighboring elements.

Requirement 3:
The Hamiltonian of every tetrahedral element represents the Hamiltonian in the distribution when restricted to (integrated over) the tetrahedron.

Requirement 4:
The interconnection structure should be power-preserving.

The goal of the next sections is to give an approximation of the behaviour of the system over one tetrahedron in port-Hamiltonian form (12), while satisfying these requirements.

4.3. Requirement 1: Approximation of dynamics

The spaces of the different flows are given in (51) as $f^p \in \Lambda^3(Z)$ and $f^q \in \Lambda^1(Z)$, which means that f^p is a three-form and f^q is a one-form. These flows are approximated using spatial temporal expansions and are defined as

$$\begin{aligned} f^p(t,z) :=& f^p_{abdm}(t)\omega^p_{abdm}(z) + f^p_{bcdm}(t)\omega^p_{bcdm}(z) \\ &+ f^p_{dcam}(t)\omega^p_{dcam}(z) + f^p_{acbm}(t)\omega^p_{acbm}(z) \\ f^q(t,z) :=& f^q_{am}(t)\omega^q_{am}(z) + f^q_{bm}(t)\omega^q_{bm}(z) \\ &+ f^q_{cm}(t)\omega^q_{cm}(z) + f^q_{dm}(t)\omega^q_{dm}(z). \end{aligned} \tag{55}$$

Here the three-form shape functions satisfy

$$\int_{Z_{s_1}} \omega^p_{s_2} = \begin{cases} 1 \text{ for } s_1 = s_2 \\ 0 \text{ for } s_1 \neq s_2, \end{cases} \tag{56}$$

where $s_1, s_2 \in \{abdm, bcdm, dcam, acbm\}$ are tetrahedra. Equivalently the one-form shape functions satisfy

$$\int_{Z_{k_1}} \omega^q_{k_2} = \begin{cases} 1 \text{ for } k_1 = k_2 \\ 0 \text{ for } k_1 \neq k_2, \end{cases} \tag{57}$$

where $k_1, k_2 \in \{am, bm, cm, dm\}$ are line segments inside the tetrahedron. This choice decouples the different flows per internal tetrahedron (f^p) or per internal line segment (f^q).

The efforts $e^p \in \Lambda^0(Z)$ and $e^q \in \Lambda^2(Z)$ are approximated using the spatial temporal expansions

$$\begin{aligned} e^p_{abcd}(t,z) :=& e^p_a(t)\omega^p_a(z) + e^p_b(t)\omega^p_b(z) \\ &+ e^p_c(t)\omega^p_c(z) + e^p_d(t)\omega^p_d(z) \\ e^q_{abcd}(t,z) :=& e^q_{acb}(t)\omega^q_{acb}(z) + e^q_{abd}(t)\omega^q_{abd}(z) \\ &+ e^q_{bcd}(t)\omega^q_{bcd}(z) + e^q_{dca}(t)\omega^q_{dca}(z). \end{aligned} \tag{58}$$

Here the zero-form shape functions satisfy

$$\omega^p_{l_1}(l_2) = \begin{cases} 1 \text{ for } l_1 = l_2 \\ 0 \text{ for } l_1 \neq l_2, \end{cases} \tag{59}$$

where $l_1, l_2 \in \{a, b, c, d\}$ are points. The two-form shape functions satisfy

$$\int_{Z_{w_1}} \omega^q_{w_2} = \begin{cases} 1 \text{ for } w_1 = w_2 \\ 0 \text{ for } w_1 \neq w_2, \end{cases} \tag{60}$$

where $w_1, w_2 \in \{acb, abd, bcd, dca\}$ are triangles.

To satisfy the first Requirement, the approximate flows (55) and efforts (58) should satisfy the original partial differential equation (53). However, in general this is not possible pointwise in $(z,t) \in Z \times T$, but it may be possible after integration over the spatial geometry Z_{abcd}.

Proposition 4.1. *The first Requirement is satisfied for arbitrary shape functions satisfying* (56)*,* (57)*,* (59)*,* (60) *if*

$$\begin{bmatrix} f^p_{abdm}(t) \\ f^p_{bcdm}(t) \\ f^p_{dcam}(t) \\ f^p_{acbm}(t) \end{bmatrix} = -M_1 \begin{bmatrix} e^p_a(t) \\ e^p_b(t) \\ e^p_c(t) \\ e^p_d(t) \end{bmatrix} + M_2 \begin{bmatrix} e^q_{acb}(t) \\ e^q_{bcd}(t) \\ e^q_{abd}(t) \\ e^q_{dca}(t) \end{bmatrix} \tag{61}$$

and

$$\begin{bmatrix} f^q_{am}(t) \\ f^q_{bm}(t) \\ f^q_{cm}(t) \\ f^q_{dm}(t) \end{bmatrix} = M_3 \begin{bmatrix} e^p_a(t) \\ e^p_b(t) \\ e^p_c(t) \\ e^p_d(t) \end{bmatrix}, \tag{62}$$

where

$$M_1 = \begin{bmatrix} \int_{Z_{T_1}} \sigma \star \omega^p_a & \int_{Z_{T_1}} \sigma \star \omega^p_b & \int_{Z_{T_1}} \sigma \star \omega^p_c & \int_{Z_{T_1}} \sigma \star \omega^p_d \\ \int_{Z_{T_2}} \sigma \star \omega^p_a & \int_{Z_{T_2}} \sigma \star \omega^p_b & \int_{Z_{T_2}} \sigma \star \omega^p_c & \int_{Z_{T_2}} \sigma \star \omega^p_d \\ \int_{Z_{T_3}} \sigma \star \omega^p_a & \int_{Z_{T_3}} \sigma \star \omega^p_b & \int_{Z_{T_3}} \sigma \star \omega^p_c & \int_{Z_{T_3}} \sigma \star \omega^p_d \\ \int_{Z_{T_4}} \sigma \star \omega^p_a & \int_{Z_{T_4}} \sigma \star \omega^p_b & \int_{Z_{T_4}} \sigma \star \omega^p_c & \int_{Z_{T_4}} \sigma \star \omega^p_d \end{bmatrix} \tag{63}$$

and

$$M_2 = \begin{bmatrix} \int_{\partial Z_{T_1}} \omega^q_{acb} & \int_{\partial Z_{T_1}} \omega^q_{bcd} & \int_{\partial Z_{T_1}} \omega^q_{abd} & \int_{\partial Z_{T_1}} \omega^q_{dca} \\ \int_{\partial Z_{T_2}} \omega^q_{acb} & \int_{\partial Z_{T_2}} \omega^q_{bcd} & \int_{\partial Z_{T_2}} \omega^q_{abd} & \int_{\partial Z_{T_2}} \omega^q_{dca} \\ \int_{\partial Z_{T_3}} \omega^q_{acb} & \int_{\partial Z_{T_3}} \omega^q_{bcd} & \int_{\partial Z_{T_3}} \omega^q_{abd} & \int_{\partial Z_{T_3}} \omega^q_{dca} \\ \int_{\partial Z_{T_4}} \omega^q_{acb} & \int_{\partial Z_{T_4}} \omega^p_{bcd} & \int_{\partial Z_{T_4}} \omega^q_{abd} & \int_{\partial Z_{T_4}} \omega^q_{dca} \end{bmatrix} \tag{64}$$

with

$$\begin{array}{ll} Z_{T_1} = Z_{abdm} & Z_{T_3} = Z_{dcam} \\ Z_{T_2} = Z_{bcdm} & Z_{T_4} = Z_{acbm}. \end{array} \tag{65}$$

Furthermore,

$$M_3 = \begin{bmatrix} \int_{\partial Z_{am}} \omega^p_a & \int_{\partial Z_{am}} \omega^p_b & \int_{\partial Z_{am}} \omega^p_c & \int_{\partial Z_{am}} \omega^p_d \\ \int_{\partial Z_{bm}} \omega^p_a & \int_{\partial Z_{bm}} \omega^p_b & \int_{\partial Z_{bm}} \omega^p_c & \int_{\partial Z_{bm}} \omega^p_d \\ \int_{\partial Z_{cm}} \omega^p_a & \int_{\partial Z_{cm}} \omega^p_b & \int_{\partial Z_{cm}} \omega^p_c & \int_{\partial Z_{cm}} \omega^p_d \\ \int_{\partial Z_{dm}} \omega^p_a & \int_{\partial Z_{dm}} \omega^p_b & \int_{\partial Z_{dm}} \omega^p_c & \int_{\partial Z_{dm}} \omega^p_d \end{bmatrix}. \tag{66}$$

Remark. *The dependency of the shape functions ω on the position z has been left out of these (and most of the following) equations for compactness of notation.*

Proof. Proposition 4.1 can be derived by substituting the approximate flows (55) and efforts (58) into the infinite-dimensional port-Hamiltonian system description (53). This gives an approximation of the dynamics of the system on the element Z_{abcd}. The substitution results in

$$\begin{aligned} &f^p_{abdm}(t)\omega^p_{abdm}(z) + f^p_{bcdm}(t)\omega^p_{bcdm}(z) \\ &+f^p_{dcam}(t)\omega^p_{dcam}(z) + f^p_{acbm}(t)\omega^p_{acbm}(z) = \\ &-\sigma \star [e^p_a(t)\omega^p_a(z) + e^p_b(t)\omega^p_b(z) \\ &\qquad +e^p_c(t)\omega^p_c(z) + e^p_d(t)\omega^p_d(z)] \\ &+\mathrm{d}[e^q_{acb}(t)\omega^q_{acb}(z) + e^q_{abd}(t)\omega^q_{abd}(z) \\ &\qquad +e^q_{bcd}(t)\omega^q_{bcd}(z) + e^q_{dca}(t)\omega^q_{dca}(z)] \end{aligned} \tag{67}$$

and

$$\begin{aligned} &f^q_{am}(t)\omega^q_{am}(z) + f^q_{bm}(t)\omega^q_{bm}(z) \\ &+f^q_{cm}(t)\omega^q_{cm}(z) + f^q_{dm}(t)\omega^q_{dm}(z) = \\ &\mathrm{d}[e^p_a(t)\omega^p_a(z) + e^p_b(t)\omega^p_b(z) \\ &\qquad +e^p_c(t)\omega^p_c(z) + e^p_d(t)\omega^p_d(z)]. \end{aligned} \tag{68}$$

Ideally (67) and (68) are satisfied exactly, however, this is not possible, therefore (67) is integrated over the volumes Z_s for $s \in \{abdm, bcdm, dcam, acbm\}$. While using Stokes' theorem (A.3) and (56) this yields (61) with M_1 given in (63) and M_2 given in (64).

Similarly integration of (68) over the line segments Z_k, for $k \in \{am, bm, cm, dm\}$ while using Stokes' theorem (A.3) and (57) yields (62) where M_3 is given in (66). ■

4.4. Requirement 2: Approximating the power balance

The bi-linear pairing (54) holds for the whole system, but it should also hold for each element Z_{abcd}. This is mentioned by Requirement 2. From the bi-linear pairing (54) it can be seen that the power balance over one element can be written as

$$\begin{aligned} &\textstyle\int_{Z_{abcd}} [e^p \wedge f^p + e^q \wedge f^q] := \\ &-\textstyle\int_{\partial Z_{abcd}} [e^b \wedge f^b] - \int_{Z_{abcd}} [e^p \wedge \sigma \star e^p]. \end{aligned} \tag{69}$$

Substituting the definition of f^b and e^b from (53) leads to

$$\begin{aligned} &\textstyle\int_{Z_{abcd}} [e^p \wedge f^p + e^q \wedge f^q] := \\ &\textstyle\int_{\partial Z_{abcd}} [e^q \mid_{\partial Z} \wedge e^p \mid_{\partial Z}] - \int_{Z_{abcd}} [e^p \wedge \sigma \star e^p]. \end{aligned} \tag{70}$$

Therefore the power balance can also be written as

$$P^p(t) + P^q(t) = P^b(t) - P^\sigma(t), \tag{71}$$

with

$$\begin{array}{lcl} P^p(t) & := & \int_{Z_{abcd}} [e^p \wedge f^p] \\ P^q(t) & := & \int_{Z_{abcd}} [e^q \wedge f^q] \\ P^b(t) & := & \int_{\partial Z_{abcd}} e^q \mid_{\partial Z} \wedge e^p \mid_{\partial Z} \\ P^\sigma(t) & := & \int_{Z_{abcd}} e^p \wedge \sigma \star e^p. \end{array} \tag{72}$$

Here $P^p(t)$ and $P^q(t)$ is the power corresponding to the variables indexed with p and q respectively, $P^b(t)$ is the power supplied through the boundary of the element and P^σ is the power dissipation over the element.

In (16) the power balance of a finite-dimensional port-Hamiltonian system is shown, which equals

$$e^T f = y^T u - e^T R e. \tag{73}$$

This balance has the same form as the power balance of the infinite-dimensional system given in (71) and will be used in the following derivation.

Proposition 4.2 (Approximations of P^p and P^q)**.** *Let e be defined as*

$$e(t) = \begin{bmatrix} e^p(t) \\ e^q(t) \end{bmatrix} \tag{74}$$

with

$$e^p(t) = \begin{bmatrix} e^p_{abdm}(t) \\ e^p_{bcdm}(t) \\ e^p_{dcam}(t) \\ e^p_{acbm}(t) \end{bmatrix}, \; e^q(t) = \begin{bmatrix} e^q_{am}(t) \\ e^q_{bm}(t) \\ e^q_{cm}(t) \\ e^q_{dm}(t) \end{bmatrix} \tag{75}$$

and f as

$$f(t) = \begin{bmatrix} f^p(t) \\ f^q(t) \end{bmatrix} \tag{76}$$

with

$$f^p(t) = \begin{bmatrix} f^p_{abdm}(t) \\ f^p_{bcdm}(t) \\ f^p_{dcam}(t) \\ f^p_{acbm}(t) \end{bmatrix}, \quad f^q(t) = \begin{bmatrix} f^q_{am}(t) \\ f^q_{bm}(t) \\ f^q_{cm}(t) \\ f^q_{dm}(t) \end{bmatrix}. \tag{77}$$

Then

$$\langle e, f \rangle = \langle e^p, f^p \rangle + \langle e^q, f^q \rangle = P^p_{abcd} + P^q_{abcd}, \tag{78}$$

where $P^p(t) \approx P^p_{abcd}(t)$ and $P^q(t) \approx P^q_{abcd}(t)$, provided that the efforts in (75) *are defined for arbitrary shape functions satisfying* (56), (57), (59), (60) *as*

$$\begin{bmatrix} e^p_{abdm}(t) \\ e^p_{bcdm}(t) \\ e^p_{dcam}(t) \\ e^p_{acbm}(t) \end{bmatrix} := M_4 \begin{bmatrix} e^p_a(t) \\ e^p_b(t) \\ e^p_c(t) \\ e^p_d(t) \end{bmatrix} \tag{79}$$

and

$$\begin{bmatrix} e^q_{am}(t) \\ e^q_{bm}(t) \\ e^q_{cm}(t) \\ e^q_{dm}(t) \end{bmatrix} := M_5 \begin{bmatrix} e^q_{acb}(t) \\ e^q_{abd}(t) \\ e^q_{bcd}(t) \\ e^q_{dca}(t) \end{bmatrix}. \tag{80}$$

Here

$$M_4^T = \int_{Z_{abcd}} \begin{bmatrix} \omega^p_a \\ \omega^p_b \\ \omega^p_c \\ \omega^p_d \end{bmatrix} \wedge \begin{bmatrix} \omega^p_{abdm} & \omega^p_{bcdm} & \omega^p_{dcam} & \omega^p_{acbm} \end{bmatrix} \tag{81}$$

and

$$M_5^T = \int_{Z_{abcd}} \begin{bmatrix} \omega^q_{acb} \\ \omega^q_{abd} \\ \omega^q_{bcd} \\ \omega^q_{dca} \end{bmatrix} \wedge \begin{bmatrix} \omega^q_{am} & \omega^q_{bm} & \omega^q_{cm} & \omega^q_{dm} \end{bmatrix}. \tag{82}$$

Note that the flows from (77) *are already defined, either in its original definition* (55) *or in* (61) *and* (62)*.*

Remark. *In this Proposition and in Proposition 4.3, the approximations denoted with $\approx$ are exact if $f(t,z)$ and $e(t,z)$ satisfy the expansions in* (55) *and* (58) *respectively, which is guaranteed by these constructions. This also means that the accuracy of the local solutions of the expansions are the same as the accuracy of the power balance.*

Proof. The expression for the approximation of P^p and P^q can be derived by substituting the approximations of the efforts (58) and flows (55) into (72). For $P^p(t)$ this results in

$$\begin{aligned} P^p(t) \approx\ & P^p_{abcd}(t) \\ :=\ & \textstyle\int_{Z_{abcd}} e^p \wedge f^p \\ =\ & \begin{bmatrix} e^p_{abdm}(t) & e^p_{bcdm}(t) & e^p_{dcam}(t) & e^p_{acbm}(t) \end{bmatrix} \cdot \\ & \begin{bmatrix} f^p_{abdm}(t) & f^p_{bcdm}(t) & f^p_{dcam}(t) & f^p_{acbm}(t) \end{bmatrix}^T, \end{aligned} \tag{83}$$

where four new efforts are defined as (79).
Similarly $P^q(t)$ is approximated as

$$\begin{aligned} P^q(t) \approx\ & P^q_{abcd}(t) \\ :=\ & \textstyle\int_{Z_{abcd}} e^q \wedge f^q \\ =\ & \begin{bmatrix} e^q_{am}(t) & e^q_{bm}(t) & e^q_{cm}(t) & e^q_{dm}(t) \end{bmatrix} \cdot \\ & \begin{bmatrix} f^q_{am}(t) & f^q_{bm}(t) & f^q_{cm}(t) & f^q_{dm}(t) \end{bmatrix}^T, \end{aligned} \tag{84}$$

where four new efforts are defined as (80).
By defining the effort of the port-Hamiltonian system (12) as

$$e(t) = \begin{bmatrix} e^p(t) \\ e^q(t) \end{bmatrix} = \begin{bmatrix} e^p_{abdm}(t) \\ e^p_{bcdm}(t) \\ e^p_{dcam}(t) \\ e^p_{acbm}(t) \\ e^q_{am}(t) \\ e^q_{bm}(t) \\ e^q_{cm}(t) \\ e^q_{dm}(t) \end{bmatrix} \tag{85}$$

and the flow of the port-Hamiltonian system as

$$f(t) = \begin{bmatrix} f^p(t) \\ f^q(t) \end{bmatrix} = \begin{bmatrix} f^p_{abdm}(t) \\ f^p_{bcdm}(t) \\ f^p_{dcam}(t) \\ f^p_{acbm}(t) \\ f^q_{am}(t) \\ f^q_{bm}(t) \\ f^q_{cm}(t) \\ f^q_{dm}(t) \end{bmatrix} \tag{86}$$

their multiplication, $e^T f$, still represents the change of energy in the system. However, it is now approximated as (78). ■

Proposition 4.3 (Approximation of P^b)**.** *Define four new efforts at the boundary of Z as*

$$\begin{bmatrix} e^p_{acb}(t) \\ e^p_{abd}(t) \\ e^p_{bcd}(t) \\ e^p_{dca}(t) \end{bmatrix} := M_6 \begin{bmatrix} e^p_a(t) \\ e^p_b(t) \\ e^p_c(t) \\ e^p_d(t) \end{bmatrix}, \tag{87}$$

with

$$M_6 = \int_{\partial Z_{abcd}} \begin{bmatrix} \omega^q_{acb} \\ \omega^q_{abd} \\ \omega^q_{bcd} \\ \omega^q_{dca} \end{bmatrix} \wedge \begin{bmatrix} \omega^p_a & \omega^p_b & \omega^p_c & \omega^p_d \end{bmatrix}, \tag{88}$$

The power over the boundary $P^b(t)$ is approximated as

$$\begin{aligned} P^b(t) &\approx y^T u \\ &= e^p_{acb}(t)e^q_{acb}(t) + e^p_{abd}(t)e^q_{abd}(t) \\ &\quad + e^p_{bcd}(t)e^q_{bcd}(t) + e^p_{dca}(t)e^q_{dca}(t), \end{aligned} \tag{89}$$

where the input and output should be defined appropriately.

Proof. $P^b(t)$ can be approximated in a similar way as $P^p(t)$ and $P^q(t)$. The approximations of the efforts (58) and flows (55) are substituted into the definition of $P^b(t)$ (72). The results is

$$\begin{aligned} P^b(t) &\approx P^b_{abcd}(t) \\ &:= \textstyle\int_{\partial Z_{abcd}} [e^q\,|_{\partial Z} \wedge e^p\,|_{\partial Z}] \\ &= \begin{bmatrix} e^q_{acb}(t) & e^q_{abd}(t) & e^q_{bcd}(t) & e^q_{dca}(t) \end{bmatrix} \cdot \\ &\quad \begin{bmatrix} e^p_{acb}(t) & e^p_{abd}(t) & e^p_{bcd}(t) & e^p_{dca}(t) \end{bmatrix}^T, \end{aligned} \tag{90}$$

where four new efforts are defined as (87) with

$$M_6 = \int_{\partial Z_{abcd}} \begin{bmatrix} \omega^q_{acb}\,|_{\partial Z} \\ \omega^q_{abd}\,|_{\partial Z} \\ \omega^q_{bcd}\,|_{\partial Z} \\ \omega^q_{dca}\,|_{\partial Z} \end{bmatrix} \wedge \begin{bmatrix} \omega^p_a\,|_{\partial Z} & \omega^p_b\,|_{\partial Z} & \omega^p_c\,|_{\partial Z} & \omega^p_d\,|_{\partial Z} \end{bmatrix}. \tag{91}$$

This expression for M_6 is equivalent to the one in (88), since Z_w, with $w \in \{acb, abd, bcd, dca\}$ are external faces and Z_l, with $l \in \{a,b,c,d\}$ are external points and therefore they all lie on the boundary of the geometry ∂Z. This means that the restriction to the boundary is satisfied and does not impose any additional restrictions on the shape functions.

The power over the boundary is approximated by the multiplication of the input and output of the system, so the input and output of the finite-dimensional system are chosen such that $y^T u$ is given by (89).

However, the terms $e^p_w(t)e^q_w(t)$, with $w \in \{acb, abd, bcd, dca\}$ do not represent the energy flowing over boundary w, since

$$e^p_w(t)e^q_w(t) \neq \int_{Z_w} e^p \wedge e^q, \text{ for } w \in \{acb, abd, bcd, dca\}. \tag{92}$$

In [**?**, **?**], this equality is assumed to hold and since this assumption is useful for interconnecting the elements and power-conservation still applies, it is assumed here as well. This assumption acts as an approximation of the power on the boundaries of the element Z_{abcd}. The approximation is defined as

$$\bar{e}^p(t,z)|_{\partial Z_{abcd}} := \begin{cases} e^p_{acb}(t) \text{ for } z \in Z_{acb} \backslash \{ab, bc, ac, b, c\} \\ e^p_{abd}(t) \text{ for } z \in Z_{abd} \backslash \{bd, a, b\} \\ e^p_{bcd}(t) \text{ for } z \in Z_{bcd} \backslash \{cd, c, d\} \\ e^p_{dca}(t) \text{ for } z \in Z_{dca} \backslash \{ad, a, d\}. \end{cases} \tag{93}$$

The power over the boundary is thus approximated by

$$\begin{aligned} P^b_{abcd}(t) &:= \textstyle\int_{\partial Z_{abcd}} \bar{e}^p\,|_{\partial Z} \wedge e^q\,|_{\partial Z} \\ &= e^p_{acb} \textstyle\int_{Z_{acb}} e^q + e^p_{abd} \int_{Z_{abd}} e^q \\ &\quad + e^p_{bcd} \textstyle\int_{Z_{bcd}} e^q + e^p_{dca} \int_{Z_{dca}} e^q \\ &= e^p_{acb}(t)e^q_{acb}(t) + e^p_{abd}(t)e^q_{abd}(t) \\ &\quad + e^p_{bcd}(t)e^q_{bcd}(t) + e^p_{dca}(t)e^q_{dca}(t). \end{aligned} \tag{94}$$

This shows that the total power over the boundary remains the same, so power conservation still holds and assumption (93) can be used. ■

Furthermore, the dissipated power is approximated by

$$P^\sigma(t) \approx P^\sigma_{abcd}(t) := e(t)^T R e(t). \tag{95}$$

An approximation for the power dissipation P^σ can be derived from the resemblance between (71) and (73). It can then be approximated by (95). Dissipation matrix R will be defined later when setting up the finite-dimensional model in port-Hamiltonian form. The expression for R can be found in equation (118) below.

4.5. Requirement 3: Approximating the Hamiltonian

The flows are defined as (23), which means that the approximated flows should satisfy

$$\begin{aligned} f^p_{abcd}(t,z) &:= \tfrac{\partial p_{abcd}}{\partial t} \\ f^q_{abcd}(t,z) &:= \tfrac{\partial q_{abcd}}{\partial t} \end{aligned} \tag{96}$$

The approximated energy variables can then be derived from (55) as

$$\begin{aligned} p_{abcd}(t,z) &= p_{abdm}(t)\omega^p_{abdm}(z) + p_{bcdm}(t)\omega^p_{bcdm}(z) \\ &\quad + p_{dcam}(t)\omega^p_{dcam}(z) + p_{acbm}(t)\omega^p_{acbm}(z) \\ q_{abcd}(t,z) &= q_{am}(t)\omega^q_{am}(z) + q_{bm}(t)\omega^q_{bm}(z) \\ &\quad + q_{cm}(t)\omega^q_{cm}(z) + q_{dm}(t)\omega^q_{dm}(z), \end{aligned} \tag{97}$$

where $p_{abcd} \in \Lambda^{n^p} = \Lambda^3$ and $q_{abcd} \in \Lambda^{n^q} = \Lambda^1$. From (96) it is inferred that

$$\begin{aligned} f^p_s(t) &= \tfrac{dp_s(t)}{dt} \quad \text{for } s \in \{abdm, bcdm, dcam, acbm\} \\ f^q_k(t) &= \tfrac{dq_s(t)}{dt} \quad \text{for } k \in \{am, bm, cm, dm\}. \end{aligned} \tag{98}$$

Proposition 4.4. *The third Requirement is met if the Hamiltonian is approximated by*

$$\begin{aligned} H_{abcd}(p(t),q(t)) &= H^p_{abcd}(p(t)) + H^q_{abcd}(q(t)) \\ &= \tfrac{1}{2}x(t)^T \begin{bmatrix} M_7 & 0 \\ 0 & M_8 \end{bmatrix} x(t), \end{aligned} \tag{99}$$

where $p(t)$ and $q(t)$ are defined as

$$\begin{aligned} p(t) &= \mathrm{col}(p_{abdm}, p_{bcdm}, p_{dcam}, p_{acbm}) \\ q(t) &= \mathrm{col}(q_{am}, q_{bm}, q_{cm}, q_{dm}). \end{aligned} \tag{100}$$

The state per element has dimension eight and is equal to

$$x(t) = \mathrm{col}(p(t), q(t)). \tag{101}$$

Furthermore, the M-matrices are defined as

$$M_7 = \int_{Z_{abcd}} c^p \begin{bmatrix} \omega^p_{abdm} \\ \omega^p_{bcdm} \\ \omega^p_{dcam} \\ \omega^p_{acbm} \end{bmatrix} \wedge\star \begin{bmatrix} \omega^p_{abdm} & \omega^p_{bcdm} & \omega^p_{dcam} & \omega^p_{acbm} \end{bmatrix} \tag{102}$$

and

$$M_8 = \int_{Z_{abcd}} c^q \begin{bmatrix} \omega^q_{am} \\ \omega^q_{bm} \\ \omega^q_{cm} \\ \omega^q_{dm} \end{bmatrix} \wedge\star \begin{bmatrix} \omega^q_{am} & \omega^q_{bm} & \omega^q_{cm} & \omega^q_{dm} \end{bmatrix}. \tag{103}$$

Remark. *In subsection 4.2 it is chosen to only focus on systems with $(n^p, n^q) = (3,1)$, even though systems with $(n^p, n^q) = (2,2)$ also exist on three-dimensional manifolds (See Table 1). The state is defined as $x(t) := \mathrm{col}(p(t), q(t))$. If $n^p = 3$, $p(t)$ is a three-form and will be approximated by partitioning each element, i.e. a tetrahedron abcd, into four smaller tetrehedra. This eventually leads to $p(t)$ in* (100)*. If p is a three-form, q is a one-form ($n^q = 1$) and four internal edges are identified, which leads to $q(t)$ in* (100)*. The state thus has dimension eight. Similarly, if $(n^p, n^q) = (2,2)$, the number of internal faces will determine the state dimension. There are six internal faces Z_{lmn} with $lmn \in \{abm, bcm, cam, adm, bdm, cdm\}$, so the state dimension will be twelve in this case (six for p and six for q).*

Proof. The expression for the approximated Hamiltonian is derived by substituting the approximate energy variables into (21). The result is

$$H_{abcd}(x) = \tfrac{1}{2}\int_{Z_{abcd}} [c^p(p_{abcd} \wedge\star p_{abcd}) + c^q(q_{abcd} \wedge\star q_{abcd})], \tag{104}$$

which can be decomposed as $H_{abcd}(x) = H^p_{abcd}(x) + H^q_{abcd}(x)$. These two parts can be approximated by

$$\begin{aligned} H^p_{abcd}(x) &= \tfrac{1}{2}\int_{Z_{abcd}} c^p(p_{abcd} \wedge\star p_{abcd}) \\ H^q_{abcd}(x) &= \tfrac{1}{2}\int_{Z_{abcd}} c^q(q_{abcd} \wedge\star q_{abcd}). \end{aligned} \tag{105}$$

By substituting the approximate energy variables (97), $H^p_{abcd}(x)$ can be written as

$$\begin{aligned} H^p_{abcd}(x) &= \tfrac{1}{2}\int_{Z_{abcd}} c^p(p_{abcd} \wedge\star p_{abcd}) \\ &= \tfrac{1}{2} \begin{bmatrix} p_{abdm}(t) \\ p_{bcdm}(t) \\ p_{dcam}(t) \\ p_{acbm}(t) \end{bmatrix}^T M_7 \begin{bmatrix} p_{abdm}(t) \\ p_{bcdm}(t) \\ p_{dcam}(t) \\ p_{acbm}(t) \end{bmatrix}, \end{aligned} \tag{106}$$

with M_7 given in (102).
Similarly

$$\begin{aligned} H^q_{abcd}(x) &= \tfrac{1}{2}\int_{Z_{abcd}} c^q(q_{abcd} \wedge\star q_{abcd}) \\ &= \tfrac{1}{2} \begin{bmatrix} q_{am}(t) \\ q_{bm}(t) \\ q_{cm}(t) \\ q_{dm}(t) \end{bmatrix}^T M_8 \begin{bmatrix} q_{am}(t) \\ q_{bm}(t) \\ q_{cm}(t) \\ q_{dm}(t) \end{bmatrix}, \end{aligned} \tag{107}$$

with M_8 given in (103).

Adding these two approximations leads to the expression for the approximation of the Hamiltonian as given in (99). ■

The effort of the finite-dimensional model can be calculated using the state $x(t)$ by using the definition

$$e := \frac{\partial H_{abcd}(x)}{\partial x}. \tag{108}$$

Since M_7 and M_8 are symmetrical, it holds that

$$e = \underbrace{\begin{bmatrix} M_7 & 0 \\ 0 & M_8 \end{bmatrix}}_{Q} x, \tag{109}$$

where Q is the energy density matrix. This expression for the effort should equal the effort defined in (85), which is enforced in the next subsection by (123). Here the effort of the port-Hamiltonian system is defined as in (111).

4.6. Finite-dimensional model

The goal of this section is to write a finite-dimensional approximation of the system (53) in port-Hamiltonian form. However, first it has to be written as (an image representation of) a finite-dimensional Dirac structure.

For the derivation of this Dirac structure, a coefficient vector is defined as

$$v(t) := \begin{bmatrix} e^p_a(t) \\ e^p_b(t) \\ e^p_c(t) \\ e^p_d(t) \\ e^q_{acb}(t) \\ e^q_{abd}(t) \\ e^q_{bcd}(t) \\ e^q_{dca}(t) \end{bmatrix} \in \mathbb{R}^8. \tag{110}$$

Then the flow and effort can be expressed in terms of $v(t)$ using (61), (62) and (79), (80). The result is

$$f(t) = \begin{bmatrix} -M_1 & M_2 \\ M_3 & 0 \end{bmatrix} v(t), \quad e(t) = \begin{bmatrix} M_4 & 0 \\ 0 & M_5 \end{bmatrix} v(t). \tag{111}$$

The input u and output y can also be expressed in terms of v, because their coefficients can be expressed in terms of $v(t)$ using (87). This results in

$$\begin{bmatrix} e^p_{acb}(t) \\ e^p_{abd}(t) \\ e^p_{bcd}(t) \\ e^p_{dca}(t) \\ e^q_{acb}(t) \\ e^q_{abd}(t) \\ e^q_{bcd}(t) \\ e^q_{dca}(t) \end{bmatrix} = \begin{bmatrix} M_6 & 0 \\ 0 & I_4 \end{bmatrix} v(t). \tag{112}$$

In this equation $I_4 \in \mathbb{R}^{4\times 4}$ is the identity matrix. Similar as in [**?**] u and y are then defined as,

$$\begin{aligned} u &:= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} M_6 & 0 \\ 0 & I_4 \end{bmatrix} v(t), \\ y &:= \begin{bmatrix} U_2 & U_1 \end{bmatrix} \begin{bmatrix} M_6 & 0 \\ 0 & I_4 \end{bmatrix} v(t), \end{aligned} \tag{113}$$

where $U_1, U_2 \in \mathbb{R}^{4x4}$ are both diagonal matrices. The same as in [**?**], "if e^p is causal on the ith face, the ith diagonal element of U_1 is set to 1 and the ith diagonal element of U_2 is set to 0. If e^q is causal on the ith face, the ith diagonal element of U_1 is set to 0 and the ith diagonal element of U_2 is set to 1." This construction leads to

$$U_1^T U_2 = 0. \tag{114}$$

Note that this construction for u and y, makes sure that $y^T u$ satisfies (89).

Proposition 4.5. *Define E^T and F^T as*

$$\begin{aligned} \begin{bmatrix} f \\ -y \end{bmatrix} &= \underbrace{\begin{bmatrix} -M_1 & M_2 \\ M_3 & 0 \\ -U_2M_6 & -U_1 \end{bmatrix}}_{E^T \in \mathbb{R}^{12\times 8}} v = E^T v \\ \begin{bmatrix} e \\ u \end{bmatrix} &= \underbrace{\begin{bmatrix} M_4 & 0 \\ 0 & M_5 \\ U_1M_6 & U_2 \end{bmatrix}}_{F^T \in \mathbb{R}^{12\times 8}} v = F^T v, \end{aligned} \tag{115}$$

then

$$\mathscr{D} := Im \begin{bmatrix} E^T \\ F^T \end{bmatrix} \tag{116}$$

is not *a Dirac structure.*

Proof. The proof is given in Appendix G. Here the system is even expanded and reduced to get square matrices E and F. However, this changes the dynamics of the systems and therefore does not meet all the requirements. ■

Remark. *The fact that this is not a finite-dimensional Dirac structure is a huge issue and means that something went wrong in this approach. More precisely, the dimension of v is not high enough. The dimension of v should be dim(v) =dim(u)+dim(e), which is twelve in this case. Therefore there is not a sufficient number of degrees of freedom in the structure to meet all requirements of a Dirac structure. However, it is assumed that this is solvable and therefore the approach will continue. The remainder of this approach gives insight in writing a system in port-Hamiltonian form and in the interconnection structure of multiple finite-dimensional port-Hamiltonian systems. A detailed discussion on what went wrong here (and also in the two-dimensional case from [**?**]) is discussed in the next section.*

Proposition 4.6. *Assume that* (116) *is a finite-dimensional Dirac structure, then the finite-dimensional port-Hamiltonian system can be represented as*

$$\begin{cases} \dot{x} = (J-R)Qx + Bu \\ y = B^T Qx, \end{cases} \tag{117}$$

with J and R equal to

$$\begin{aligned} J &= \begin{bmatrix} 0 & V_{1,2} \\ V_{2,1} & 0 \end{bmatrix} \in \mathbb{R}^{8\times 8} \\ R &= \begin{bmatrix} -V_{1,1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{8\times 8} \end{aligned} \tag{118}$$

where

$$\begin{array}{lcl} V_{1,1} & = & -M_1 M_4^{-1} \\ V_{1,2} & = & (M_2 - M_4^{-T} M_6^T U_2^T U_2) M_5^{-1} \\ V_{2,1} & = & (M_3 - M_5^{-T} U_1^T U_1 M_6) M_4^{-1} \\ V_{2,2} & = & 0. \end{array} \tag{119}$$

In (117)*, Q equals* (109)*, B equals*

$$B^T = \begin{bmatrix} U_2 & U_1 \end{bmatrix} \begin{bmatrix} M_6 & 0 \\ 0 & I_4 \end{bmatrix} \begin{bmatrix} M_4 & 0 \\ 0 & M_5 \end{bmatrix}^{-1}, \tag{120}$$

and the state is given in (101)*.*

Proof. The finite-dimensional port-Hamiltonian system can be described in terms of $v(t)$ by substituting (111) and (113) into (12). This results in

$$\begin{bmatrix} -M_1 & M_2 \\ M_3 & 0 \end{bmatrix} v(t) = (J-R) \begin{bmatrix} M_4 & 0 \\ 0 & M_5 \end{bmatrix} v(t) + B \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} M_6 & 0 \\ 0 & I_4 \end{bmatrix} v(t) \tag{121a}$$

$$\begin{bmatrix} U_2 & U_1 \end{bmatrix} \begin{bmatrix} M_6 & 0 \\ 0 & I_4 \end{bmatrix} v(t) = B^T \begin{bmatrix} M_4 & 0 \\ 0 & M_5 \end{bmatrix} v(t) \tag{121b}$$

From the second equation an expression for B can be derived as (120), provided M_4 and M_5 are invertible. This can be guaranteed by choosing the shape functions in M_4 and M_5 wisely.
From (121a), after substituting the expression for B, a solution for J and R can be found as

$$(J-R) = \begin{bmatrix} V_{1,1} & V_{1,2} \\ V_{2,1} & V_{2,2}, \end{bmatrix} \tag{122}$$

where the components of V are given in (119).
However, it is preferred to have an expression for J and R separately. An expression for R can be found by setting the dissipation term σ to 0, since R will be the zero matrix in that case. Setting σ to 0 leads to M_1 becoming the zero matrix, which means $V_{1,1}$ will become zero as well. Therefore the expressions for J and R are (118).

In Appendix H this proof is continued by proving that J and R from (117) are skew-symmetric and positive semi-definite respectively. ■

The coefficient vector $v(t)$ can be computed from the states using

$$v(t) = \begin{bmatrix} M_4 & 0 \\ 0 & M_5 \end{bmatrix}^{-1} \begin{bmatrix} M_7 & 0 \\ 0 & M_8 \end{bmatrix} x(t). \tag{123}$$

This relation quickly shows that the effort defined in (109) indeed equals the effort of the port-Hamiltonian system defined in (85).

4.7. Requirement 4: Interconnection structure

The dynamics over the different elements is now described in port-Hamiltonian form by (117). By interconnecting the elements, the dynamics over the whole spatial domain is described. An important part of the interconnection structure has already been mentioned in the previous subsection.

There are two different types of boundaries in the system; internal boundaries on Z and external boundaries on ∂Z. Internal boundaries connect two elements and external boundaries connect the system with its environment or with a different port-Hamiltonian system. On internal boundaries the causal directions have to match, so the output of element i is the input of element j and vice verse. This is guaranteed by defining the matrices U_1 and U_2 as in subsection 4.6. First interconnecting elements on internal boundaries is explained in more detail, since this imposes more challenges than on the external boundaries due to power-conservation. Next it is explained why it is always possible to satisfy the causal directions.

Internal boundaries

The elements are interconnected via the faces Z_w, with $w \in \{acb, abd, bcd, dca\}$. Two elements i and j can be interconnected as shown in Fig. 4. By definition efforts going out of the elements (outputs) are negative and efforts going into the elements (inputs) are positive. As illustrated by the circular arrows, the order of integration on element i and j is opposite. For any k-form $\alpha(z)$, it holds that

$$\int_Z \alpha(z) = -\int_{-Z} \alpha(z), \tag{124}$$

where $-Z$ denotes the manifold with orientation opposite to that of Z [**?**]. Therefore $e_i^q(t)$ and $e_j^q(t)$ have opposite directions caused by integration over opposite oriented manifolds. However, as seen in (93), this does not automatically hold for $e_i^p(t)$ and $e_j^p(t)$, since there is no integration involved anymore. This is solved by adding a -1-term between $e_i^p(t)$ and $e_j^p(t)$, as can be seen in Fig. 4.

The complete interconnection structure as given in Fig. 4 is a 0-junction, which is a power-conserving element [**?**], so power conservation through the interconnection still holds. The power over the boundary of element i is $y_i^T u_i$ and over the boundary of element j is $y_j^T u_j$. The total power through the interconnection should be zero, so

$$y_i^T u_i + y_j^T u_j = 0 \tag{125}$$

should hold. Since the output and input of the elements are the effort and flow over the boundary, this can be

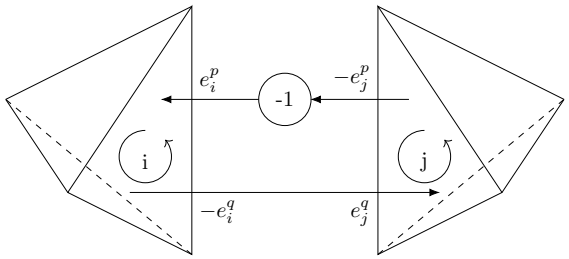


Figure 4: Interconnection between element i and j

equivalently written as

$$-e_i^q e_i^p - e_j^p e_j^q = 0. \tag{126}$$

The interconnection structure from Fig. 4 can be written as the following two equalities.

$$\begin{cases} -e_i^q(t) &= e_j^q(t) \\ e_i^p(t) &= e_j^p(t) \end{cases} \tag{127}$$

Substituting these expressions into (126) shows that power-conservation indeed holds, since (125) holds.

Causal directions

As mentioned before, either e^b is the input of the element and f^b is the output or the other way around. Assume there are N elements, then there are $M = 4N$ external faces. Define a vector $x \in X \subset \mathbb{Z}^M$, where each element of the vector is either zero or one. If e^b is causal on the face, element $x_i = 1$ and if f^b is causal, element $x_i = 0$. There are two types of constraints; one for internal faces and one for external faces. For every internal face connecting face i with face j, the following should hold

$$x_i + x_j = 1. \tag{128}$$

This guarantees that the causal directions match. For the external faces the causal direction should be specified, but it is less constrained. The causal direction can be $s_k \in \{0,1\}$, resulting in

$$x_i = s_k. \tag{129}$$

These constraints define a solution space X, where every $x \in X$. Every face is either an internal face or an external face, so every face either has to satisfy (128) or (129). Every internal face is connected to exactly one other internal face, so for every pair i, j, (128) has two possible solutions. Therefore a solution for (128) can always be found. The choice of causality for the external faces is completely free, so a solution to (129) always exists. This means that solution space X is never empty.

5. Discussion 3D approach

This section discusses two things. First it discusses what goes wrong when setting up the finite-dimensional port-Hamiltonian system is done as in subsection 4.6. Next it discusses what causes the fact that (116) is not a Dirac structure.

5.1. port-Hamiltonian system

Even tough taking the inverses lead to a port-Hamiltonian system, the steps starting with writing the port-Hamiltonian system in terms of v (121) makes a compromise that eventually leads to an incorrect result. More precisely, in (121) the port-Hamiltonian system is split in two parts; the $\dot{x}$ part (121a) and the y part (121b). Next B^T is computed from the second part and substituted in the first part. This is where it goes wrong. By doing so, the input u satisfying the requirements is not free. This means that the port-Hamiltonian system in (117) is different from the system described in terms of v by (111) and (113), while the last system approximates the dynamics of the infinite-dimensional port-Hamiltonian system correctly.

This can be seen by looking more closely at the underlying Dirac structure. The same derivation as in subsection 4.6 can be done using the Dirac structure of the system expressed in terms of $v(t)$ as shown in Appendix G. Here it becomes clear that $u(t)$ is not free, but should be equal to

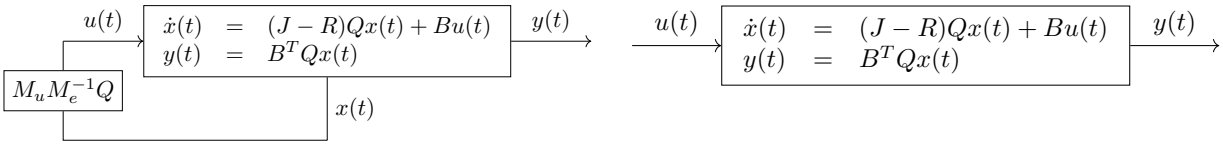
$$\begin{aligned} u(t) &= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} M_6 & 0 \\ 0 & I_4 \end{bmatrix} \begin{bmatrix} M_4 & 0 \\ 0 & M_5 \end{bmatrix}^{-1} e(t) \\ &= \begin{bmatrix} U_1 M_6 M_4^{-1} & U_2 M_5^{-1} \end{bmatrix} e(t). \end{aligned} \tag{130}$$

In other words; $u(t)$ given in (113) is not the same as $u(t)$ in (117) and therefore the port-Hamiltonian system is different than the system described in terms of $v(t)$.

Besides that, in (109) a relation between $e(t)$ and $x(t)$ is stated, meaning that the input of the port-Hamiltonian system (130) is not free, but depends on the state. Therefore the system described in (111) and (113) is an autonomous system and not a port-Hamiltonian system. This also means that the interconnection structure from subsection 4.7 cannot be realized, since there is no input port that can be used to interconnect multiple elements.

This autonomous system can be represented as depicted in Fig. 5a, where

$$\begin{aligned} M_u &= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} M_6 & 0 \\ 0 & I_4 \end{bmatrix} \\ M_e &= \begin{bmatrix} M_4 & 0 \\ 0 & M_5 \end{bmatrix}. \end{aligned} \tag{131}$$



(a) Actual system

(b) port-Hamiltonian system (117)

Figure 5: Difference between the port-Hamiltonian system (117) and the actual system

The port-Hamiltonian system (117) as depicted in Fig. 5b is the same as the actual system however, the relation between the input $u(t)$ and the state $x(t)$ is omitted. The matrices J, R, Q and B however, are the same. Therefore the behavior of the simulated system is different than the behavior of the actual system.

5.2. Dirac structure

The current derivation in the previous section leads to a projection of e,f,u and y onto coefficient vector v, which is not of maximal dimension and therefore does not satisfy the definition of a finite-dimensional Dirac structure on $\mathscr{E} \times \mathscr{F}$ according to Definition 1. This means that with the current e,f,u and y, v should be of dimension twelve, leading to square matrices E^T and F^T and a different derivation of the finite-dimensional port-Hamiltonian system. Since there is not enough freedom, the dimension of v is too small to guarantee a free input u, which leads to a compromise, namely, an autonomous port-Hamiltonian system as explained before.

6. Discretization in 1D

The problems in the three-dimensional case are caused by a lack of freedom in the free choices of input and outputs over the boundary of the geometries. This section illustrates a possible solution for one-dimensional manifolds. The reason to go back to one-dimensional manifolds is that it is desired to first get a working method for, and full understanding of, one-dimensional manifolds before even trying two- and three- dimensional manifolds. This method can then be expanded to two-dimensional manifolds and eventually to three-dimensional manifolds. The existing one-dimensional methods either do not include the point m [**?**, **?**] or only use one fixed input u [**?**]. The existing two-dimensional methods also either do not use the point m [**?**] or do not end with a suitable finite-dimensional port-Hamiltonian system that after interconnection approximates the dynamics of the original system [**?**]. Therefore it is really necessary to start with developing a suitable one-dimensional method. For this method, the proposed shape functions can be found in Appendix J.

6.1. Meshing

In the one-dimensional case the spatial domain is defined as a one-dimensional manifold Z with a zero-dimensional boundary ∂Z. Here $Z \subset \mathbb{R}^1$ which is assumed to be bounded and closed as a subset of $\mathbb{R}^1$. The simplest finite covering of Z consists of line segments, which are of the form $Z_{ab} := [a,b]$, with $a < b$. Each line segment is partitioned in two smaller line segments using the intermediate point m, with $a < m < b$, so the elements Z_{ab} look like the one given in Fig. 6.

6.2. One dimensional manifolds

From Table 1 it can be concluded that for one-dimensional manifolds, $(n^p, n^q) = (1,1)$. The linear space of flows can be derived from (31) and is defined as

$$\mathscr{F}_{p,q} := \Lambda^1(Z) \times \Lambda^1(Z) \times \Lambda^0(\partial Z). \tag{132}$$

The linear space of efforts can be derived from (32) and is defined as

$$\mathscr{E}_{p,q} := \Lambda^0(Z) \times \Lambda^0(Z) \times \Lambda^0(\partial Z). \tag{133}$$

For $(n^p, n^q) = (1,1)$, the infinite-dimensional port-Hamiltonian system from (24) becomes

$$\begin{aligned} \begin{bmatrix} f^p \\ f^q \end{bmatrix} &= \begin{bmatrix} -\sigma\star & \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e^p \\ e^q \end{bmatrix} \\ \begin{bmatrix} f^b \\ e^b \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^p|_{\partial Z} \\ e^q|_{\partial Z} \end{bmatrix}. \end{aligned} \tag{134}$$

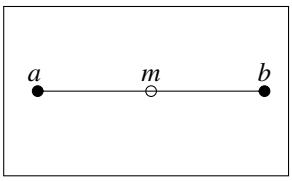


Figure 6: Element Z_{ab} with $m \in Z_{ab}$

This is a Dirac structure

$$\begin{aligned}&\mathscr{D} := \{(e^p, e^q, e^b), (f^p, f^q, f^b) \in \mathscr{E}_{p,q} \times \mathscr{F}_{p,q} \mid \\ &\begin{bmatrix} f^p \\ f^q \end{bmatrix} = \begin{bmatrix} -\sigma\star & \mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} e^p \\ e^q \end{bmatrix} \\ &\begin{bmatrix} f^b \\ e^b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} e^p|_{\partial Z} \\ e^q|_{\partial Z} \end{bmatrix}\},\end{aligned} \tag{135}$$

with respect to the following non-degenerate bi-linear pairing $\langle e \mid f \rangle \in \mathbb{R}$ with $e \in \mathscr{E}_{p,q}$ and $f \in \mathscr{F}_{p,q}$:

$$\begin{aligned}&\langle (e^p, e^q, e^b), (f^p, f^q, f^b) \rangle := \\ &\textstyle\int_Z [e^p \wedge f^p + e^q \wedge f^q] + \int_{\partial Z} [e^b \wedge f^b] \\ &\textstyle + \int_Z [e^p \wedge \sigma \star e^p].\end{aligned} \tag{136}$$

The Hamiltonian is chosen quadratic and is defined as in (21). The requirements for one-dimensional manifolds are the same as for three-dimensional manifolds and are given at the end of subsection 4.2.

6.3. Requirement 1: Approximation of the dynamics

The spaces of the different flows are given in (132) as $f^p \in \Lambda^1$ and $f^q \in \Lambda^1$. These flows are approximated using the following spatial temporal expansions

$$\begin{aligned}f^p(t,z) &:= f^p_{am}(t)\omega^p_{am}(z) + f^p_{mb}(t)\omega^p_{mb}(z) \\ f^q(t,z) &:= f^q_{am}(t)\omega^q_{am}(z) + f^q_{mb}(t)\omega^q_{mb}(z).\end{aligned} \tag{137}$$

The one-form shape functions ω^p_s and ω^q_s for line segments $s \in \{am, mb\}$ satisfy

$$\int_{Z_{s_1}} \omega_{s_2} = \begin{cases} 1 \text{ for } s_1 = s_2 \\ 0 \text{ for } s_1 \neq s_2. \end{cases} \tag{138}$$

So far this approach is similar to the three-dimensional approach, however, the point m will be included in the approximation of the effort, such that the dimensions in the end will be correct. The efforts $e^p \in \Lambda^0(Z)$ and $e^q \in \Lambda^0(Z)$ are approximated using the spatial temporal expansions

$$\begin{aligned}e^p_{ab}(t,z) &= e^p_a(t)\omega^p_a(z) + e^p_m(t)\omega^p_m(z) + e^p_b(t)\omega^p_b(z) \\ e^q_{ab}(t,z) &= e^q_a(t)\omega^q_a(z) + e^q_m(t)\omega^q_m(z) + e^q_b(t)\omega^q_b(z).\end{aligned} \tag{139}$$

Here the zero-form shape functions satisfy

$$\omega_{l_1}(l_2) = \begin{cases} 1 \text{ for } l_1 = l_2 \\ 0 \text{ for } l_1 \neq l_2, \end{cases} \tag{140}$$

where $l_1, l_2 \in \{a, b\}$ are points and

$$\omega_m(l) = \begin{cases} 1 \text{ for } l = m \\ 0 \text{ for } l \neq m \end{cases} \tag{141}$$

where $l \in \{a, m, b\}$ are points.

To satisfy the first Requirement, the approximate flows (137) and efforts (139) should satisfy the original partial differential equation (135). However, in general this is not possible pointwise in $(z,t) \in Z \times T$, but it may be possible after integration over the spatial geometry Z_{ab}.

Proposition 6.1. *The first Requirement is satisfied for arbitrary shape functions satisfying* (138)*,* (140)*,* (141) *if*

$$\begin{bmatrix} f^p_{am}(t) \\ f^p_{mb}(t) \end{bmatrix} = -M_1 \begin{bmatrix} e^p_a(t) \\ e^p_m(t) \\ e^p_b(t) \end{bmatrix} + M_2 \begin{bmatrix} e^q_a(t) \\ e^q_m(t) \\ e^q_b(t) \end{bmatrix} \tag{142}$$

and

$$\begin{bmatrix} f^q_{am}(t) \\ f^q_{mb}(t) \end{bmatrix} = M_3 \begin{bmatrix} e^p_a(t) \\ e^p_m(t) \\ e^p_b(t) \end{bmatrix}. \tag{143}$$

Here

$$M_1 = \begin{bmatrix} \int_{Z_{am}} \sigma \star \omega^p_a & \int_{Z_{am}} \sigma \star \omega^p_m & \int_{Z_{am}} \sigma \star \omega^p_b \\ \int_{Z_{mb}} \sigma \star \omega^p_a & \int_{Z_{mb}} \sigma \star \omega^p_m & \int_{Z_{mb}} \sigma \star \omega^p_b \end{bmatrix}, \tag{144}$$

$$M_2 = \begin{bmatrix} \int_{\partial Z_{am}} \omega^q_a & \int_{\partial Z_{am}} \omega^q_m & \int_{\partial Z_{am}} \omega^q_b \\ \int_{\partial Z_{mb}} \omega^q_a & \int_{\partial Z_{mb}} \omega^q_m & \int_{\partial Z_{mb}} \omega^q_b \end{bmatrix} \tag{145}$$

and

$$M_3 = \begin{bmatrix} \int_{\partial Z_{am}} \omega^p_a & \int_{\partial Z_{am}} \omega^p_m & \int_{\partial Z_{am}} \omega^p_b \\ \int_{\partial Z_{mb}} \omega^p_a & \int_{\partial Z_{mb}} \omega^p_m & \int_{\partial Z_{mb}} \omega^p_b \end{bmatrix}. \tag{146}$$

Proof. This can be derived by substituting the approximate flows (137) and efforts (139) into the Dirac structure (135), which gives an approximation of the behaviour of the system on the element Z_{ab}. The result is

$$\begin{aligned}&f^p_{am}(t)\omega^p_{am}(z) + f^p_{mb}(t)\omega^p_{mb}(z) = \\ &-\sigma\star[e^p_a(t)\omega^p_a(z) + e^p_m(t)\omega^p_m(z) + e^p_b(t)\omega^p_b(z)] \\ &+\mathrm{d}[e^q_a(t)\omega^q_a(z) + e^q_m(t)\omega^q_m(z) + e^q_b(t)\omega^q_b(z)]\end{aligned} \tag{147}$$

and

$$\begin{aligned}&f^q_{am}(t)\omega^q_{am}(z) + f^q_{mb}(t)\omega^q_{mb}(z) = \\ &\mathrm{d}[e^p_a(t)\omega^p_a(z) + e^p_m(t)\omega^p_m(z) + e^p_b(t)\omega^p_b(z)].\end{aligned} \tag{148}$$

Integration of (147) over the line segments Z_{am} and Z_{mb} while using Stokes' theorem (A.3) and (138) yields (142), where M_1 and M_2 are given in (144) and (145) respectively.

Similarly integration of (148) over the lines Z_k, for $k \in \{am, mb\}$ while using Stokes' theorem (A.3) and (138) yields (143), where M_3 is given in (146). ■

6.4. Requirement 2: Approximating the power balance

The power balance for infinite-dimensional port-Hamiltonian systems is given in (26). This holds for the whole system, but it should also hold for each element Z_{abcd}. Therefore, it can also be written as

$$P^p(t) + p^q(t) = P^b(t) - P^\sigma(t) \tag{149}$$

with

$$\begin{array}{lcl} P^p(t) & := & \int_{Z_{ab}} [e^p \wedge f^p] \\ P^q(t) & := & \int_{Z_{ab}} [e^q \wedge f^q] \\ P^b(t) & := & \int_{\partial Z_{ab}} e^q \mid_{\partial Z} \wedge e^p \mid_{\partial Z} \\ P^\sigma(t) & := & \int_{Z_{ab}} e^p \wedge \sigma \star e^p. \end{array} \tag{150}$$

In (16) the power balance of a finite-dimensional port-Hamiltonian system is shown, which equals

$$e^T f = y^T u - e^T R e. \tag{151}$$

This balance has the same form as the power balance of the infinite-dimensional system given in (149) and will be used in the following derivation.

Proposition 6.2 (Approximations of P^p and P^q)**.** *Let e be defined as*

$$e(t) = \begin{bmatrix} e^p(t) \\ e^q(t) \end{bmatrix} \tag{152}$$

with

$$e^p(t) = \begin{bmatrix} e^p_{am}(t) \\ e^p_{mb}(t) \end{bmatrix}, \quad e^q(t) = \begin{bmatrix} e^q_{am}(t) \\ e^q_{mb}(t) \end{bmatrix} \tag{153}$$

and f as

$$f(t) = \begin{bmatrix} f^p(t) \\ f^q(t) \end{bmatrix} \tag{154}$$

with

$$f^p(t) = \begin{bmatrix} f^p_{am}(t) \\ f^p_{mb}(t) \end{bmatrix}, \quad f^q(t) = \begin{bmatrix} f^q_{am}(t) \\ f^q_{mb}(t) \end{bmatrix}. \tag{155}$$

Then

$$\langle e, f \rangle = \langle e^p, f^p \rangle + \langle e^q, f^q \rangle = P^p_{ab} + P^q_{ab}, \tag{156}$$

where $P^p(t) \approx P^p_{ab}(t)$ and $P^q(t) \approx P^q_{ab}(t)$. The new efforts in (153) *are defined for arbitrary shape functions satisfying* (138)*,* (140)*,* (141) *as*

$$\begin{bmatrix} e^p_{am}(t) \\ e^p_{mb}(t) \end{bmatrix} := M_4 \begin{bmatrix} e^p_a(t) \\ e^p_m(t) \\ e^p_b(t) \end{bmatrix}, \tag{157}$$

and

$$\begin{bmatrix} e^q_{am}(t) \\ e^q_{mb}(t) \end{bmatrix} := M_5 \begin{bmatrix} e^q_a(t) \\ e^q_m(t) \\ e^q_b(t) \end{bmatrix}. \tag{158}$$

Here

$$M_4 = \int_{Z_{ab}} \begin{bmatrix} \omega^p_a \wedge \omega^p_{am} & \omega^p_m \wedge \omega^p_{am} & \omega^p_b \wedge \omega^p_{am} \\ \omega^p_a \wedge \omega^p_{mb} & \omega^p_m \wedge \omega^p_{mb} & \omega^p_b \wedge \omega^p_{mb} \end{bmatrix} \tag{159}$$

and

$$M_5 = \int_{Z_{ab}} \begin{bmatrix} \omega^q_a \wedge \omega^q_{am} & \omega^q_m \wedge \omega^q_{am} & \omega^q_b \wedge \omega^q_{am} \\ \omega^q_a \wedge \omega^q_{mb} & \omega^q_m \wedge \omega^q_{mb} & \omega^q_b \wedge \omega^q_{mb} \end{bmatrix}. \tag{160}$$

Note that the flows from (155) *are already defined, either in its original definition* (137) *or in* (142) *and* (143)*.*

Remark. *In this Proposition and in Proposition 6.3 the approximations denoted with $\approx$ are exact if $f(t,z)$ and $e(t,z)$ satisfy the expansions in* (137) *and* (139) *respectively, which is guaranteed by this construction.*

Proof. The expressions for the approximations of P^p and P^q can be derived by substituting the approximations of the flows (137) and the efforts (139) into (150), which leads to an approximated power balance. For $P^p(t)$ this results in

$$\begin{array}{lcl} P^p(t) & \approx & P^p_{ab}(t) \\ & := & \int_{Z_{ab}} e^p \wedge f^p \\ & = & \begin{bmatrix} e^p_{am}(t) & e^p_{mb}(t) \end{bmatrix} \cdot \\ & & \begin{bmatrix} f^p_{am}(t) & f^p_{mb}(t) \end{bmatrix}^T, \end{array} \tag{161}$$

where two new efforts are defined as (157).
Similarly $P^q(t)$ is approximated as

$$\begin{array}{lcl} P^q(t) & \approx & P^q_{ab}(t) \\ & := & \int_{Z_{ab}} e^q \wedge f^q \\ & = & \begin{bmatrix} e^q_{am}(t) & e^q_{mb}(t) \end{bmatrix} \cdot \\ & & \begin{bmatrix} f^q_{am}(t) & f^q_{mb}(t) \end{bmatrix}^T, \end{array} \tag{162}$$

where two new efforts are defined as (158).

By defining the effort and flow of the port-Hamiltonian system as

$$e(t) = \begin{bmatrix} e^p_{am}(t) \\ e^p_{mb}(t) \\ e^q_{am}(t) \\ e^q_{mb}(t) \end{bmatrix}, \quad f(t) = \begin{bmatrix} f^p_{am}(t) \\ f^p_{mb}(t) \\ f^q_{am}(t) \\ f^q_{mb}(t) \end{bmatrix}, \tag{163}$$

their multiplication, $e^T f$, still represents the change of energy in the system. However, it is now approximated as

$$e^T f = P^p_{ab} + P^q_{ab}. \tag{164}$$

■

Proposition 6.3 (Approximation of P^b)**.** *Define two new efforts at the boundary of Z as*

$$\begin{bmatrix} \hat{e}_a^p(t) \\ \hat{e}_b^p(t) \end{bmatrix} = \hat{M}_6 \begin{bmatrix} e_a^p \\ e_m^p \\ e_b^p \end{bmatrix}, \tag{165}$$

with

$$\hat{M}_6 = \int_{\partial Z_{ab}} \begin{bmatrix} \omega_a^q \\ \omega_b^q \end{bmatrix} \wedge \begin{bmatrix} \omega_a^p & \omega_m^p & \omega_b^p \end{bmatrix}. \tag{166}$$

The power over the boundary $P^b(t)$ is then approximated as

$$\begin{array}{rcl} P^b(t) & \approx & y^T u \\ & = & \hat{e}_a^p(t) e_a^q(t) + \hat{e}_b^p(t) e_b^q(t). \end{array} \tag{167}$$

where the input and output should be defined appropriately.

Proof. $P^b(t)$ can be approximated in a similar way as $P^p(t)$ and $P^q(t)$. The approximations of the efforts (139) and flows (137) are substituted into the definition of P^b in (150). This results in

$$\begin{array}{rcl} P^b(t) & \approx & P_{ab}^b(t) \\ & = & \int_{\partial Z_{ab}} [e^q \mid_{\partial Z} \wedge e^p \mid_{\partial Z}] \\ & = & \begin{bmatrix} e_a^q(t) & e_m^q(t) & e_b^q(t) \end{bmatrix} \cdot \\ & & \begin{bmatrix} \hat{e}_a^p(t) & \hat{e}_m^p(t) & \hat{e}_b^p(t) \end{bmatrix}^T, \end{array} \tag{168}$$

where three new efforts are defined as

$$\begin{bmatrix} \hat{e}_a^p(t) \\ \hat{e}_m^p(t) \\ \hat{e}_b^p(t) \end{bmatrix} = M_6 \begin{bmatrix} e_a^p \\ e_m^p \\ e_b^p \end{bmatrix}, \tag{169}$$

with

$$M_6 = \int_{\partial Z_{ab}} \begin{bmatrix} \omega_a^q \mid_{\partial Z} \\ \omega_m^q \mid_{\partial Z} \\ \omega_b^q \mid_{\partial Z} \end{bmatrix} \wedge \begin{bmatrix} \omega_a^p \mid_{\partial Z} & \omega_m^p \mid_{\partial Z} & \omega_b^p \mid_{\partial Z} \end{bmatrix}. \tag{170}$$

Points a and b lie on the boundary of the element, so $\omega_a \mid_{\partial Z} = \omega_a$ and $\omega_b \mid_{\partial Z} = \omega_b$. On the other hand, m does not lie on the boundary of the element, so $\omega_m \mid_{\partial Z} = 0$, which is satisfied if ω_m satisfies (141). Combining (169) and (170), shows that (141) implies that, $\hat{e}_m^p$ always equals zero. Therefore $P_{ab}^b(t)$ defined in (168) implies that

$$P_{ab}^b(t) = \begin{bmatrix} e_a^q(t) & e_b^q(t) \end{bmatrix} \cdot \begin{bmatrix} \hat{e}_a^p(t) & \hat{e}_b^p(t) \end{bmatrix}^T, \tag{171}$$

where two new efforts are defined as in (165). The power over the boundary is approximated by the multiplication of the input and output of the system, so the input and output of the finite-dimensional system are chosen such that $y^T u$ is given by

$$\begin{array}{rcl} P^b(t) & \approx & y^T u \\ & = & \hat{e}_a^p(t) e_a^q(t) + \hat{e}_m^p(t) e_m^q(t) + \hat{e}_b^p(t) e_b^q(t). \end{array} \tag{172}$$

However, since $\hat{e}_m^p(t)$ always equals 0, this is equivalent to (167). ■

This makes more sense, since now the input and output are defined on the boundary of the element, instead of also at the point m.

The terms $e_w^p(t) e_w^q(t)$, with $w \in \{a,b\}$ represent the energy flowing over boundary w, since

$$e_w^p(t) e_w^q(t) = \int_{Z_w} e^p \wedge e^q, \text{ for } w \in \{a,b\}. \tag{173}$$

An approximation for the power dissipation P^σ can be derived from the resemblance between (71) and (73). It can then be approximated by:

$$P^\sigma(t) \approx P_{ab}^\sigma(t) = e(t)^T R e(t). \tag{174}$$

6.5. Requirement 3: Approximating the Hamiltonian

The flows are defined as (23), which means that the approximated flows should satisfy

$$\begin{array}{rcl} f_{ab}^p(t,z) & := & \frac{\partial p_{ab}}{\partial t} \\ f_{ab}^q(t,z) & := & \frac{\partial q_{ab}}{\partial t} \end{array} \tag{175}$$

for suitable definitions of p_{ab} and q_{ab}. The approximated energy variables can then be derived from (137) as

$$\begin{array}{l} p_{ab}(t,z) = p_{am}(t)\omega_{am}^p(z) + p_{mb}(t)\omega_{mb}^p(z) \\ q_{ab}(t,z) = q_{am}(t)\omega_{am}^q(z) + q_{mb}(t)\omega_{mb}^q(z) \end{array} \tag{176}$$

where $p_{ab} \in \Lambda^{n^p} = \Lambda^1$ and $q_{ab} \in \Lambda^{n^q} = \Lambda^1$. From (175) it is inferred that

$$\begin{array}{l} f_s^p(t) = \frac{dp_s(t)}{dt} \quad \text{for } s \in \{am, mb\} \\ f_k^q(t) = \frac{dq_s(t)}{dt} \quad \text{for } k \in \{am, mb\}. \end{array} \tag{177}$$

Proposition 6.4. *The third Requirement is met if the Hamiltonian is approximated by*

$$\begin{array}{rcl} H_{ab}(p_{ab}, q_{ab}) & = & H_{ab}^p(p_{ab}) + H_{ab}^q(q_{ab}) \\ & = & \frac{1}{2} x(t)^T \begin{bmatrix} M_7 & 0 \\ 0 & M_8 \end{bmatrix} x(t), \end{array} \tag{178}$$

where $p(t)$ and $q(t)$ are defined as

$$\begin{array}{l} p(t) = \mathrm{col}(p_{am}, p_{mb}) \\ q(t) = \mathrm{col}(q_{am}, q_{mb}). \end{array} \tag{179}$$

The state per element has dimension four and is equal to

$$x(t) = \mathrm{col}(p(t), q(t)). \tag{180}$$

Furthermore, the M-matrices are defined as

$$M_7 = \int_{Z_{ab}} c^p \begin{bmatrix} \omega^p_{am} \\ \omega^p_{mb} \end{bmatrix} \wedge\star \begin{bmatrix} \omega^p_{am} & \omega^p_{mb} \end{bmatrix} \tag{181}$$

and

$$M_8 = \int_{Z_{ab}} c^q \begin{bmatrix} \omega^q_{am} \\ \omega^q_{mb} \end{bmatrix} \wedge\star \begin{bmatrix} \omega^q_{am} & \omega^q_{mb} \end{bmatrix}. \tag{182}$$

Proof. The expression for the approximated Hamiltonian is derived by substituting the approximate energy variables into (21). The result is

$$H_{ab}(p_{ab}, q_{ab}) := \frac{1}{2} \int_{Z_{ab}} [c^p(p_{ab} \wedge\star p_{ab}) + c^q(q_{ab} \wedge\star q_{ab})], \tag{183}$$

which can be decomposed as $H_{ab}(p_{ab}) = H^p_{ab}(p_{ab}) + H^q_{ab}(q_{ab})$. These two parts can be approximated as

$$\begin{array}{l} H^p_{ab}(p_{ab}) = \frac{1}{2} \int_{Z_{ab}} c^p(p_{ab} \wedge\star p_{ab}) \\ H^q_{ab}(q_{ab}) = \frac{1}{2} \int_{Z_{ab}} c^q(q_{ab} \wedge\star q_{ab}). \end{array} \tag{184}$$

By substituting the approximate energy variables, $H^p_{ab}(x)$ can be written as

$$\begin{array}{rcl} H^p_{ab}(p_{ab}) & = & \frac{1}{2} \int_{Z_{ab}} c^p(p_{ab} \wedge\star p_{ab}) \\ & = & \frac{1}{2} \begin{bmatrix} p_{am}(t) \\ p_{mb}(t) \end{bmatrix}^T M_7 \begin{bmatrix} p_{am}(t) \\ p_{mb}(t) \end{bmatrix} \end{array}, \tag{185}$$

with M_7 given in (181).
Similarly

$$\begin{array}{rcl} H^q_{ab}(q_{ab}) & = & \frac{1}{2} \int_{Z_{ab}} c^q(q_{ab} \wedge\star q_{ab}) \\ & = & \frac{1}{2} \begin{bmatrix} q_{am}(t) \\ q_{mb}(t) \end{bmatrix}^T M_8 \begin{bmatrix} q_{am}(t) \\ q_{mb}(t) \end{bmatrix} \end{array}, \tag{186}$$

with M_8 given in (182).

Adding these two approximations leads to the expression for the approximation of the Hamiltonian as in (178). ■

The effort of the finite-dimensional model can be calculated from the state $x(t)$ using the definition

$$e := \frac{\partial H_{ab}(x)}{\partial x}. \tag{187}$$

Since M_7 and M_8 are symmetrical, it holds that

$$e = \underbrace{\begin{bmatrix} M_7 & 0 \\ 0 & M_8 \end{bmatrix}}_{Q} x, \tag{188}$$

where Q is the energy density matrix. This expression for the effort should equal the effort defined in (163), which is enforced in the next subsection by (200). Here the effort of the port-Hamiltonian system is defined as in (190).

6.6. Finite-dimensional model

The goal of this section is to write a finite-dimensional approximation of the system (135) in port-Hamiltonian form. However, first it has to be written as (an image representation of) a finite-dimensional Dirac structure.

For the derivation of this Dirac structure, a coefficient vector is defined as

$$v(t) = \begin{bmatrix} e^p_a(t) \\ e^p_m(t) \\ e^p_b(t) \\ e^q_a(t) \\ e^q_m(t) \\ e^q_b(t) \end{bmatrix} \in \mathbb{R}^6. \tag{189}$$

Then the flow and effort can be expressed in terms of $v(t)$ using (142), (143) and (157), (158):

$$f(t) = \begin{bmatrix} -M_1 & M_2 \\ M_3 & 0 \end{bmatrix} v(t), \quad e(t) = \begin{bmatrix} M_4 & 0 \\ 0 & M_5 \end{bmatrix} v(t). \tag{190}$$

The coefficients from u and y can also be expressed in terms of $v(t)$:

$$\begin{bmatrix} \hat{e}^p_a(t) \\ \hat{e}^p_b(t) \\ e^q_a(t) \\ e^q_b(t) \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{M}_6 & 0 \\ 0 & \hat{I} \end{bmatrix}}_{M_{uy} \in \mathbb{R}^{4\times 6}} v(t), \tag{191}$$

so u and y can also be expressed in terms of $v(t)$. In this equation $\hat{I}$ equals (198).
Define u and y similar as in [**?**],

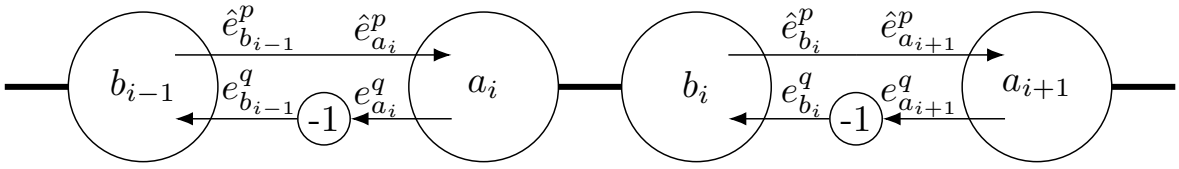
$$\begin{array}{l} u = \begin{bmatrix} U_1 & U_2 \end{bmatrix} M_{uy} v(t), \\ y = \begin{bmatrix} U_2 & U_1 \end{bmatrix} M_{uy} v(t), \end{array} \tag{192}$$

where $U_1, U_2 \in \mathbb{R}^{2\times 2}$ are both diagonal matrices. The same as in [**?**], "if e^p is causal on the ith edge, the ith diagonal element of U_1 is set to 1 and the ith diagonal element of U_2 is set to 0. If e^q is causal on the ith edge, the ith diagonal element of U_1 is set to 0 and the ith diagonal element of U_2 is set to 1." This construction then always leads to

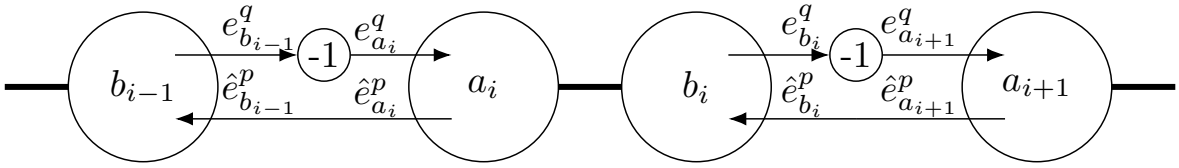
$$U_1^T U_2 = 0. \tag{193}$$

In the one-dimensional case, there are four possibilities for U_1 and U_2, leading to the following inputs and outputs.

$$\begin{array}{rcl} U_1 & = & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ u & = & \begin{bmatrix} \hat{e}^p_a(t) \\ \hat{e}^p_b(t) \end{bmatrix}, \quad y = \begin{bmatrix} e^q_a(t) \\ e^q_b(t) \end{bmatrix} \end{array} \tag{194a}$$



(a) Option *a*



(b) Option *b*

Figure 7: Possible interconnection structures in 1D

or

$$\begin{aligned} U_1 &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ u &= \begin{bmatrix} \hat{e}^p_a(t) \\ e^q_b(t) \end{bmatrix}, \quad y = \begin{bmatrix} e^q_a(t) \\ \hat{e}^p_b(t) \end{bmatrix} \end{aligned} \tag{194b}$$

or

$$\begin{aligned} U_1 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ u &= \begin{bmatrix} e^q_a(t) \\ \hat{e}^p_b(t) \end{bmatrix}, \quad y = \begin{bmatrix} \hat{e}^p_a(t) \\ e^q_b(t) \end{bmatrix} \end{aligned} \tag{194c}$$

or

$$\begin{aligned} U_1 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ u &= \begin{bmatrix} e^q_a(t) \\ e^q_b(t) \end{bmatrix}, \quad y = \begin{bmatrix} \hat{e}^p_a(t) \\ \hat{e}^q_b(t) \end{bmatrix} \end{aligned} \tag{194d}$$

Note that this defines all the partitions of (191) and that this construction for u and y, makes sure that $y^T u$ satisfies (167).

Proposition 6.5. *Define E^T and F^T as*

$$\begin{aligned} \begin{bmatrix} f \\ y \end{bmatrix} &= \underbrace{\begin{bmatrix} -M_1 & M_2 \\ M_3 & 0 \\ U_2\hat{M}_6 & U_1\hat{I} \end{bmatrix}}_{E^T \in \mathbb{R}^{6\times 6}} v = E^T v \\ \begin{bmatrix} e \\ u \end{bmatrix} &= \underbrace{\begin{bmatrix} M_4 & 0 \\ 0 & M_5 \\ U_1\hat{M}_6 & U_2\hat{I} \end{bmatrix}}_{F^T \in \mathbb{R}^{6\times 6}} v = F^T v, \end{aligned} \tag{195}$$

then

$$\mathscr{D} := Im \begin{bmatrix} E^T \\ F^T \end{bmatrix} \tag{196}$$

is a finite-dimensional Dirac structure in the sense of Definition 1.

Proof. The proof is given in Appendix I. ■

Proposition 6.6. *The finite-dimensional port-Hamiltonian system can now be represented by*

$$\begin{bmatrix} f \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} -M_1 & M_2 \\ M_3 & 0 \\ U_2\hat{M}_6 & U_1\hat{I} \end{bmatrix}}_{E^T} \underbrace{\begin{bmatrix} M_4 & 0 \\ 0 & M_5 \\ U_1\hat{M}_6 & U_2\hat{I} \end{bmatrix}^{-1}}_{F^{-T}} \begin{bmatrix} e \\ u \end{bmatrix}, \tag{197}$$

where

$$\hat{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{198}$$

Proof. A finite-dimensional port-Hamiltonian system (12) with feedthrough can be written as

$$\begin{bmatrix} f \\ y \end{bmatrix} = \begin{bmatrix} J-R & B \\ B^T & D \end{bmatrix} \begin{bmatrix} e \\ u \end{bmatrix}. \tag{199}$$

The same can be done for the system described in terms of coefficient vector v ((190) and (192)) by eliminating v. This results in (197). ■

In order to make sure that (188) holds, v is computed from the state using

$$v(t) = \underbrace{\begin{bmatrix} M_4 & 0 \\ 0 & M_5 \\ [U_1 & U_2]M_{uy} \end{bmatrix}^{-1}}_{F^{-T}} \begin{bmatrix} M_7 & 0 & 0 \\ 0 & M_8 & 0 \\ 0 & 0 & I_2 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}, \tag{200}$$

where $I_2 \in \mathbb{R}^{2\times 2}$ denotes the identity matrix.

In (197) it is necessary for F^T to be invertible. Matrix F^T is square; $F^T \in \mathbb{R}^{6\times 6}$, so it is invertible iff it has full rank. Since $M_4 \in \mathbb{R}^{2\times 3}$, $U_1\hat{M}_6$ should have one non-zero columns for F^T to be full rank. The same holds for $U_2\hat{I}$, which is caused by $M_5 \in \mathbb{R}^{2\times 3}$. Therefore the only possible inputs and outputs are given in

(194b) and (194c). In other words, there are two possible input/output pairs

$$(u,y) = \begin{cases} (\mathrm{col}(\hat{e}_a^p, e_b^q), \mathrm{col}(e_a^q, \hat{e}_b^p)) & \text{option a} \\ (\mathrm{col}(e_a^q, \hat{e}_b^p), \mathrm{col}(\hat{e}_a^p, e_b^q)) & \text{option b.} \end{cases} \tag{201}$$

6.7. Requirement 4: Interconnection structure

The elements will be interconnected in a similar way as described in section 4.7. Similar as for three-dimensional manifolds, a minus-sign is introduced to guarantee power-conservation. The minus-sign acts on the efforts of energy variable q. The two possibilities to connect element i to its neighboring elements are given in Fig. 7.

Proposition 6.7. *Both interconnection structures given in Fig.7 are power-preserving.*

Proof. In order for an interconnection structure to be power-preserving,

$$y_i^T u_i + y_{i+1}^T u_{i+1} = 0 \tag{202}$$

should be true. For option a, as depicted in 7a, this means that

$$\hat{e}_{b_i}^p e_{b_i}^q + e_{a_{i+1}}^q \hat{e}_{a_{i+1}}^p = 0 \tag{203}$$

should hold. The interconnection structure from option a is given by

$$\begin{cases} -e_{a_{i+1}}^q &= e_{b_i}^q \\ \hat{e}_{a_{i+1}}^p &= \hat{e}_{b_i}^p. \end{cases} \tag{204}$$

Substituting these expression into (203) leads to

$$\hat{e}_{b_i}^p e_{b_i}^q + e_{a_{i+1}}^q \hat{e}_{a_{i+1}}^p = \hat{e}_{b_i}^p e_{b_i}^q - e_{b_i}^q \hat{e}_{b_i}^p = 0, \tag{205}$$

which shows that interconnection structure a indeed is power-preserving. Interconnection structure option b can be proven to be power-preserving in the exact same way. ■

Therefore, even though the inputs and outputs are more restricted than in the three-dimensional case, there is enough freedom for interconnection. However, this interconnection structure restricts the choice of boundary ports, as can also be seen in Fig. 7. The choice of the boundary port at a_0 determines the interconnection structure for *all* of the other elements and therefore also the boundary ports at b_N, where the N-th element is the last one in the mesh.

6.8. Conclusion

This approach for one-dimensional manifolds is promising, since it solves the lack of freedom in the free choices of input and outputs over the boundary of the geometries. Besides that there is still enough freedom to interconnect multiple elements.

7. Conclusions and recommendations

This paper addresses the problem of how to approximate distributed parameter systems in infinite-dimensional port-Hamiltonian form, by interconnecting a finite number of finite-dimensional port-Hamiltonian systems. Using the underlying Dirac structure, the research problem can be translated to finding a finite-dimensional Dirac structure that approximates the infinite dimensional Dirac structure. The finite-dimensional port-Hamiltonian systems (the elements) should satisfy four requirements. First of all they should approximate the dynamics of the distributed parameter system. Besides that they should approximate its power balance and its energy distribution, which is given by the Hamiltonian. After satisfying these requirements they should be interconnected in a power-conserving way.

The implications of each of these requirements are developed in a differential-geometric setting and the requirements are characterized separately as demands on the spatial temporal expansion of the flow and effort. The first Requirement imposes conditions directly on the coefficients of the expansion of the flow. For the second Requirement the resemblance between the power balance of an infinite-dimensional Dirac structure and a finite-dimensional Dirac structure is used to define the effort and flow of the finite-dimensional port-Hamiltonian system. Besides that, the power over the boundary is also approximated, which defines the product between the input and output. The power over the boundary is not exact per external face, but per tetrahedron. More precisely, the sum of the power over the four external faces is exact, but the power per external face is not. This is a compromise in this approach. The approximations used in this construction are exact if the expansions of the space-dependent flow and effort are exactly satisfied, which is guaranteed in this approach. The third Requirement approximates the energy distribution per element and defines a relation between the approximated energy variables, whose coefficients are defined as the state of the finite-dimensional port-Hamiltonian system. Proving a Dirac structure and setting up the finite-dimensional model uses a coefficient vector and shows that the flow, effort, input and output can all be expressed in terms of these coefficients. Unfortunately there is not enough freedom to satisfy all of the requirements and have a free input u.

For three-dimensional manifolds, the shape of the elements is chosen as a tetrahedron. Even though this is the simplest three-dimensional shape with non-zero volume, it seems to be rich enough to satisfy all of the requirements after adding a point m inside. This is not

proven in this paper, however, it is assumed that the shape of the element is not causing the problems faced in the current approach. Further research is necessary to verify this. Currently, the minimal state dimension of the finite-dimensional system is eight. This holds for $(n^p, n^q) = (3,1)$. Spatially discretizing a system where $(n^p, n^q) = (2,2)$, leads to more states, namely a state dimension of twelve. A possible interconnection structure that preserves power is developed for three-dimensional manifolds. Since the three-dimensional approach is not correctly developed in this paper it is impossible to conclude anything about the sparsity of the finite-dimensional models and the implementation of such a method on real systems.

In this approach shape functions are used for interpolation in the spatial domain. Multiple requirements are specified for these shape functions and they are generally difficult to find.

In previous work [**?**], there were not sufficient degrees of freedom obtained to properly define the input signals over the boundary of the geometric structure (line-segments in 1D, triangles in 2D, tetrahedron in 3D). More precisely, the current derivation leads to a projection of the effort, flow, input and output onto a coefficient vector v, which is not of maximal dimension and does therefore not lead to a finite-dimensional Dirac structure. This lack of freedom leads to a compromise, namely an autonomous port-Hamiltonian system, which cannot be connected to other port-Hamiltonian systems.

Therefore the approach has been re-developed for one-dimensional geometries. The main difference is that the effort at the point m is included in the expansion of the efforts. The result is a Dirac structure with sufficient degrees of freedom, which leaves sufficient freedom to interconnect elements. Similar as for the three-dimensional approach, shape functions are generally difficult to find.

The next step will be to verify that the approach indeed works for one-dimensional manifolds. This can be done by simulating a partial differential equation, e.g. the wave equation, using the approach mentioned in section 6. When the dynamics are correctly approximated, the power balance has to be checked carefully and it is wise to see whether the inputs and outputs of neighboring elements are connected correctly. Next the one-dimensional method can be expanded to two- and three-dimensional manifolds. Finally these methods should be compared to conventional methods such as FEM. They could be compared on e.g. accuracy and time it takes to perform the discretization.

The method itself can be improved as well. In this approach the shape function are determined immediately (at least their requirements are), however, the requirements such as (59) and (60) make sense, but are not necessary for the approach. They can be left out and the freedom this creates can be used to satisfy the invertibility of F^T or the skew-symmetric property of J for example. Besides that, the current approach makes a compromise by approximating the energy flowing over the boundary per face, while the total power over the boundary is correct. However, it would be better if the energy flowing over each external face of the tetrahedra is exact as well. This can be imposed as a condition on the shape functions, leading to a non-linear requirement. This cannot be satisfied with the proposed shape functions, due to a lack of freedom, but may be possible for different shape functions or a different three-dimensional approach.

Once the two- and three-dimensional method are developed correctly, it is advised to design them for a mesh with differently shaped elements, such as a mesh containing both tetrahedra and boxes. This could decrease the number of elements in the mesh, which will speed up the spatial discretization. Besides that, it is also interesting to develop this approach for time-varying meshes.

Acknowledgment

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Appendices

A. Mathematical concepts

This appendix discusses the most important mathematical concepts of differential-geometry used in this paper [**?**, **?**].

On a three-dimensional manifold $Z \subset \mathbb{R}^3$, there exist zero-forms $\Lambda^0(Z)$, one-forms $\Lambda^1(Z)$, two-forms $\Lambda^2(Z)$ and three-forms $\Lambda^3(Z)$. A zero-form can be evaluated at points on the manifold $z \in Z$ and behave as functions $f(z) \in \Lambda^0(Z)$, assigning numbers to points $f(z) : Z \to \mathbb{R}$. A one-form can be integrated over a line segment on the manifold Z and consists of a function assigning numbers to line segments. Generally a one-form $g \in \Lambda^1(Z)$ is given by $g(z) = g_1(z)\mathrm{d}z_1 + g_2(z)\mathrm{d}z_2 + g_3(z)\mathrm{d}z_3$, with functions $g_1(z), g_2(z), g_3(z) : Z \to \mathbb{R}$. A two-form $h \in \Lambda^2$ can be integrated over a surface inside the manifold and is generally given by $h(z) = h_1\mathrm{d}z_1\mathrm{d}z_2 + h_2\mathrm{d}z_2\mathrm{d}z_3 + h_3\mathrm{d}z_3\mathrm{d}z_1$, with functions $h_1(z), h_2(z), h_3(z) : Z \to \mathbb{R}$. A three-form $k \in \Lambda^3$ can be integrated over a sub-volume of the domain and is generally given by $k(z) = k_1(z)\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3$, with function $k_1(z) : Z \to \mathbb{R}$.

In differential geometry, spatial differentation is performed by the exterior derivative d, which maps a k-form to a $(k+1)$-form;

$$\mathrm{d} : \Lambda^k(Z) \to \Lambda^{k+1}(Z). \tag{A.1}$$

On a three-dimensional manifold, the exterior derivative is defined as

$$\mathrm{d} = \frac{\partial}{\partial z_1}\mathrm{d}z_1 + \frac{\partial}{\partial z_2}\mathrm{d}z_2 + \frac{\partial}{\partial z_3}\mathrm{d}z_3. \tag{A.2}$$

Besides that, the exterior derivative of all k-forms with $k > 2$ are defined to be zero, so for $\omega \in \Lambda^3(Z)$ the exterior derivative equals zero: $\mathrm{d}\omega(z) = 0$.

Stokes' theorem is used in differential geometry and makes a useful statement about integration of differential forms on a manifold. It is given by

$$\int_Z \mathrm{d}\omega = \int_{\partial Z} \omega, \tag{A.3}$$

which holds for any k-form ω on any p-dimensional manifold Z, with $(p-1)$-dimensional boundary ∂Z.

A different important concept in differential geometry is the wedge product $\wedge$. This operator combines a k-form ω_1 with a l-form ω_2 into a $(k+l)$-form $\omega_1 \wedge \omega_2$. The following properties hold for a wedge product:

$$\omega_1 \wedge \omega_2 = (-1)^{kl}\omega_2 \wedge \omega_1 \tag{A.4a}$$

$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge \mathrm{d}\omega_2. \tag{A.4b}$$

Besides that if c is a constant, the following holds as well

$$(c\omega_1) \wedge \omega_2 = c(\omega_1 \wedge \omega_2) = \omega_1 \wedge (c\omega_2). \tag{A.5}$$

Very often the wedge product is denoted as a product, for example $\mathrm{d}z_1 \wedge \mathrm{d}z_2$ is usually denoted as the two-form $\mathrm{d}z_1\mathrm{d}z_2$. Some properties of the exterior derivative combined with the wedge product are

$$\begin{array}{lcll} \mathrm{d}z_i \wedge \mathrm{d}z_i & = & 0, \quad i \in \{1,2,3\} & \\ \mathrm{d}\mathrm{d}\omega & = & 0, \quad \omega \in \Lambda^k(Z), & \forall k \in \mathbb{N}. \end{array} \tag{A.6}$$

Another import concept in differential geometry is the Hodge star $\star$, which on a n-dimensional manifold converts a k-form ω into a $(n-k)$-form $\star\omega$. The Hodge star is defined as

$$\omega_1 \wedge \star\omega_2 = \langle \omega_1, \omega_2 \rangle \mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3, \tag{A.7}$$

where ω_1 and ω_2 are k-forms and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product.
The behavior of the Hodge star in common situations is given by

$$\begin{array}{llll} \star 1 & = \mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3 & \star(\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3) & = 1 \\ \star\mathrm{d}z_1 & = \mathrm{d}z_2 \wedge \mathrm{d}z_3 & \star(\mathrm{d}z_2 \wedge \mathrm{d}z_3) & = \mathrm{d}z_1 \\ \star\mathrm{d}z_2 & = \mathrm{d}z_3 \wedge \mathrm{d}z_1 & \star(\mathrm{d}z_3 \wedge \mathrm{d}z_1) & = \mathrm{d}z_2 \\ \star\mathrm{d}z_3 & = \mathrm{d}z_1 \wedge \mathrm{d}z_2 & \star(\mathrm{d}z_1 \wedge \mathrm{d}z_2) & = \mathrm{d}z_3. \end{array} \tag{A.8}$$

If a Hodge star is applied twice on $\omega \in \Lambda^k$, the following holds:

$$\star\star\omega = (-1)^{k(n-k)}\omega. \tag{A.9}$$

This means that on a n-dimensional manifold where n is odd, the inverse of the Hodge star $\star^{-1}$ is the Hodge star itself. Note that on a three-dimensional manifold $k(n-k) = k(3-k)$ can never be odd for $k \in \{0,1,2,3\}$, so

$$\star\star\omega = \omega \tag{A.10}$$

and

$$\omega_1 = \star\omega_2 \tag{A.11}$$

is the same as

$$\star\omega_1 = \star\star\omega_2 = \omega_2. \tag{A.12}$$

B. Wave equation in port-Hamiltonian form

In order to write the wave equations (2) in port-Hamiltonian form, Hooke's law

$$\sigma = E\varepsilon \tag{B.13}$$

is used, where E is the modules of elasticity. Besides that, the formula for kinetic momentum is also used

$$\rho = mv = m\dot{w}, \tag{B.14}$$

where m is the mass of the string/plate. In the case of a vibrating string/plate,

$$\varepsilon = \nabla w \tag{B.15}$$

and therefore $\dot{\varepsilon}$ can be written as

$$\dot{\varepsilon} = \frac{\partial}{\partial t}(\nabla w) = \nabla \frac{\partial w}{\partial t} = \nabla v. \tag{B.16}$$

The derivative of the kinetic momentum is given by

$$\dot{\rho} = \frac{\partial}{\partial t}(mv) = m\frac{\partial v}{\partial t} = m\frac{\partial^2 w}{\partial t^2}. \tag{B.17}$$

Substituting the wave equation (2) leads to

$$\dot{\rho} = -mc^2\nabla^2 w, \tag{B.18}$$

which using Hooke's law (B.13) and (B.15) can be written as

$$\dot{\rho} = -mc^2\frac{1}{E}\nabla\sigma. \tag{B.19}$$

Constant c^2 equals

$$c^2 = \frac{E}{m}, \tag{B.20}$$

so the derivative of the kinetic momentum actually equals

$$\dot{\rho} = -\nabla\sigma. \tag{B.21}$$

Combining (B.16) and (B.21) yields (4) and concludes writing the wave equations in port-Hamiltonian form.

C. Proof of power balance

This appendix proves the expression for the power balance of an infinite-dimensional port-Hamiltonian system, which is given in (26). In order to achieve an expression for the power balance similar as in (16), it is desired to find an expression for the change of energy. This is given by the derivative of the Hamiltonian with respect to time. The Hamiltonian is given in (21), which using the choices for the efforts (22) can be written as

$$H(p,q) = \frac{1}{2}\int_Z [e^p \wedge p + e^q \wedge q] \tag{C.22}$$

Using the choice for the flows from (23), the change of energy is [**?**]

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{1}{2}\int_Z [e^p \wedge f^p + e^q \wedge f^q]. \tag{C.23}$$

Substituting the expressions for f^p and f^q from (24) leads to

$$\begin{array}{l}\frac{1}{2}\int_Z [e^p \wedge f^p + e^q \wedge f^q] = \frac{1}{2}\int_Z [e^p \wedge -\sigma \star e^p] + \\ \frac{1}{2}\int_Z [e^p \wedge (-1)^r \mathrm{d}e^q + e^q \wedge \mathrm{d}e^p].\end{array} \tag{C.24}$$

The last term can be rewritten using (A.4a):

$$e^p \wedge \mathrm{d}e^q = (-1)^{(n-n^p)(n-n^q+1)}\mathrm{d}e^q \wedge e^p. \tag{C.25}$$

By also using the homogeneity of the wedge product (A.5) the last term in (C.24) can be rewritten to

$$\begin{array}{l}\frac{1}{2}\int_Z [e^p \wedge (-1)^r \mathrm{d}e^q + e^q \wedge \mathrm{d}e^p] = \\ \frac{1}{2}\int_Z [(-1)^{r+(n-n^p)(n-n^q+1)}\mathrm{d}e^q \wedge e^p + e^q \wedge \mathrm{d}e^p].\end{array} \tag{C.26}$$

Since $n^p + n^q = n+1$ and $r = n^p n^q + 1$, $r + (n-n^p)(n-n^q+1) = n^p n^q + 1 + (n-n^p)(n^p) = n^p n^q + 1 + (n^q - 1)(n^p) = 2n^p n^q + 1 - n^p = 2n^p n^q + n^q - n$. So, the term $(-1)^{r+(n-n^p)(n-n^q+1)}$ equals $(-1)^{2n^p n^q + n^q - n}$, which can also be written as $(-1)^{2n^p n^q}(-1)^{n^q - n}$. $2n^p n^q$ is always even so $(-1)^{2n^p n^q}(-1)^{n^q-n} = (-1)^{n^q-n}$. Since $(-1)^{-x} = \frac{1}{(-1)^x} = (-1)^x$, $(-1)^{n^q-n} = (-1)^{n-n^q}$. Therefore the last term of (C.24) can be written as

$$\begin{array}{l}\frac{1}{2}\int_Z [e^p \wedge (-1)^r \mathrm{d}e^q + e^q \wedge \mathrm{d}e^p] = \\ \frac{1}{2}\int_Z [(-1)^{n-n^q}\mathrm{d}e^q \wedge e^p + e^q \wedge \mathrm{d}e^p].\end{array} \tag{C.27}$$

Now the following property of the wedge product will be used, (A.4b):

$$\mathrm{d}(e^q \wedge e^p) = \mathrm{d}e^q \wedge e^p + (-1)^{n-n^q} e^q \wedge \mathrm{d}e^p. \tag{C.28}$$

Multiplying (C.28) with $(-1)^{n-n^q}$ leads to

$$\begin{array}{l}(-1)^{n-n^q}\mathrm{d}(e^q \wedge e^p) = \\ (-1)^{n-n^q}\mathrm{d}e^q \wedge e^p + (-1)^{n-n^q}(-1)^{n-n^q} e^q \wedge \mathrm{d}e^p.\end{array} \tag{C.29}$$

This is equivalent to

$$\begin{aligned}&(-1)^{n-n^q}\mathrm{d}(e^q\wedge e^p)=\\&(-1)^{n-n^q}\mathrm{d}e^q\wedge e^p+(-1)^{2n-2n^q}e^q\wedge\mathrm{d}e^p.\end{aligned}\tag{C.30}$$

Since $2n-2n^q$ is always even, $(-1)^{2n-2n^q}$ equals 1 and

$$\begin{aligned}&(-1)^{n-n^q}\mathrm{d}(e^q\wedge e^p)=\\&(-1)^{n-n^q}\mathrm{d}e^q\wedge e^p+e^q\wedge\mathrm{d}e^p.\end{aligned}\tag{C.31}$$

Substituting this into (C.27) shows that the last term of (C.24) can be written as

$$\begin{aligned}&\tfrac{1}{2}\textstyle\int_Z[e^p\wedge(-1)^r\mathrm{d}e^q+e^q\wedge\mathrm{d}e^p]=\\&\tfrac{1}{2}\textstyle\int_Z[(-1)^{n-n^q}\mathrm{d}(e^q\wedge e^p)].\end{aligned}\tag{C.32}$$

Applying Stokes' theorem (A.3) yields

$$\begin{aligned}&\tfrac{1}{2}\textstyle\int_Z[e^p\wedge(-1)^r\mathrm{d}e^q+e^q\wedge\mathrm{d}e^p]=\\&\tfrac{1}{2}\textstyle\int_{\partial Z}[(-1)^{n-n^q}e^q\mid_{\partial Z}\wedge e^p\mid_{\partial Z}].\end{aligned}\tag{C.33}$$

Substituting the expression for f^b and e^b from (24) yields

$$\begin{aligned}&\tfrac{1}{2}\textstyle\int_Z(-1)^r\mathrm{d}e^q\wedge e^p+\mathrm{d}e^p\wedge e^q]=\\&\tfrac{1}{2}\textstyle\int_{\partial Z}[-e^b\wedge f^b].\end{aligned}\tag{C.34}$$

Substituting this expression for the last term in (C.24) leads to the power balance given in (26).

D. Proof of finite-dimensional Dirac structure

This appendix proves Proposition 2.2. According to Definition 1, this is a Dirac structure iff

i $\langle e\mid f\rangle=0$, for all $(f,e)\in\mathscr{D}$,

ii $\dim\mathscr{D}=\dim\mathscr{F}$.

In order to prove a Dirac structure power should be conserved, so a lossless system is assumed ($R=0$). For $\hat{e},\hat{f}\in\mathbb{R}^N$, the first property is the same as $\hat{e}^T\hat{f}=0$. Substituting the expression for $\hat{f}$ leads to

$$\hat{e}^T\hat{f}=\begin{bmatrix}e\\u\end{bmatrix}^T\begin{bmatrix}f\\-y\end{bmatrix}=\begin{bmatrix}e\\u\end{bmatrix}^T\begin{bmatrix}J&B\\-B^T&0\end{bmatrix}\begin{bmatrix}e\\u\end{bmatrix}.\tag{D.35}$$

In a port-Hamiltonian system J is skew symmetric, so

$$\begin{bmatrix}J&B\\-B^T&0\end{bmatrix}\tag{D.36}$$

is also skew-symmetric. A property of a skew-symmetric matrix A is that

$$v^TAv=-v^TA^Tv=-v^TAv=0\quad\forall v\in\mathbb{R}^N.\tag{D.37}$$

Therefore

$$\hat{e}^T\hat{f}=0\tag{D.38}$$

and the first property of a Dirac structure is satisfied.

The second property of a Dirac structure is satisfied if

$$\begin{bmatrix}J&B\\-B^T&0\end{bmatrix}\tag{D.39}$$

has full rank, which is true if B has full rank, which is no loss of generality.

E. Proof of infinite-dimensional Dirac structure

This appendix proves that (53) with bi-linear pairing (54) is a Dirac structure. The proof is based on the proofs given in [**?**] and [**?**]. The first step is to proof that $\mathscr{D} \subset \mathscr{D}^\perp$ and the second step is to proof that $\mathscr{D}^\perp \subset \mathscr{D}$. If both of these conditions are true, $\mathscr{D}$ is a Dirac structure [**?**]. Besides that, the pairing given in (54) can be symmetrized, such that a symmetric bi-linear pairing $\langle\langle \cdot, \cdot \rangle\rangle$ is found for any $(e_1, f_1), (e_2, f_2) \in \mathscr{E} \times \mathscr{F}$, given by

$$\langle\langle (e_1, f_1), (e_2, f_2) \rangle\rangle := \langle e_1 \mid f_2 \rangle + \langle e_2 \mid f_1 \rangle. \tag{E.1}$$

Step 1: Proof that $\mathscr{D} \subset \mathscr{D}^\perp$. Let $(e_1, f_1) \in \mathscr{D}$ and consider any $(e_2, f_2) \in \mathscr{D}$. Substitution of (53) into the first term of the symmetrization (E.1) of the bi-linear pairing (54) results in

$$\begin{array}{l} \int_Z [e_1^p \wedge f_2^p + e_1^q \wedge f_2^q + e_2^p \wedge f_1^p + e_2^q \wedge f_1^q] = \\ \int_Z [e_1^p \wedge (-\sigma \star e_2^p + \mathrm{d}e_2^q) + e_1^q \wedge (\mathrm{d}e_2^p) \\ + e_2^p \wedge (-\sigma \star e_1^p + \mathrm{d}e_1^q) + e_2^q \wedge (\mathrm{d}e_1^p)]. \end{array} \tag{E.2}$$

Using the properties (A.4a) and (A.4b) with $(n^p, n^q) = (3,1)$:

$$\begin{array}{rcl} e^q \wedge \mathrm{d}e^p & = & \mathrm{d}e^p \wedge e^q \\ \mathrm{d}(e^p \wedge e^q) & = & \mathrm{d}e^p \wedge e^q + e^p \wedge \mathrm{d}e^q \end{array} \tag{E.3}$$

and Stokes' theorem (A.3), this term can be rewritten to

$$\begin{array}{l} \int_Z [e_1^p \wedge f_2^p + e_1^q \wedge f_2^q + e_2^p \wedge f_1^p + e_2^q \wedge f_1^q] = \\ \int_{\partial Z} [e_1^p \wedge e_2^q + e_2^p \wedge e_1^q] \\ - \int_Z [e_1^p \wedge \sigma \star e_2^p + e_2^p \wedge \sigma e_1^p]. \end{array} \tag{E.4}$$

Substitution of (53) into the second and last term of the symmetrization (E.1) of the bi-linear pairing (54) results in

$$\begin{array}{l} \int_{\partial Z} [e_1^b \wedge f_2^b + e_2^b \wedge f_1^b] \\ + \int_Z [e_1^p \wedge \sigma \star e_2^p + e_2^p \wedge \sigma \star e_1^p] = \\ \int_{\partial Z} [-e_1^q \wedge e_2^p - e_2^q \wedge e_1^p] \\ + \int_Z [e_1^p \wedge \sigma \star e_2^p + e_2^p \wedge \sigma \star e_1^p]. \end{array} \tag{E.5}$$

Adding these terms to the first term (E.4) and using the property $e^q \wedge e^p = e^p \wedge e^q$, shows that $\langle\langle (e_1, f_1), (e_2, f_2) \rangle\rangle = 0$. Therefore, $\mathscr{D} \subset \mathscr{D}^\perp$.

Step 2: Proof that $\mathscr{D}^\perp \subset \mathscr{D}$. Let $(e_1, f_1) \in \mathscr{D}$. As shown before, the right hand side of the bi-linear form (E.1) equals zero for any $(e_2, f_2) \in \mathscr{D}$.

Substituting (53) into the symmetrisation (E.1) of the bi-linear pairing (54) for (e_2, f_2) results in

$$\begin{array}{l} \int_Z [e_1^p \wedge f_2^p + e_1^q \wedge f_2^q + e_2^p \wedge f_1^p + e_2^q \wedge f_1^q] \\ + \int_{\partial Z} [e_1^b \wedge f_2^b + e_2^b \wedge f_1^b] \\ + \int_Z [e_1^p \wedge \sigma \star e_2^p + e_2^p \wedge \sigma \star e_1^p] \\ = \int_Z [e_1^p \wedge (-\sigma \star e_2^p + \mathrm{d}e_2^q) + e_1^q \wedge \mathrm{d}e_2^p + e_2^p \wedge f_1^p + e_2^q \wedge f_1^q] \\ + \int_{\partial Z} [e_1^b \wedge e_2^p - e_2^q \wedge f_1^b] \\ + \int_Z [e_1^p \wedge \sigma \star e_2^p + e_2^p \wedge \sigma \star e_1^p] \\ = 0. \end{array} \tag{E.6}$$

Using Stokes' theorem, the boundary integral can be rewritten and the expression becomes

$$\begin{array}{l} \int_Z [e_1^p \wedge (-\sigma \star e_2^p + \mathrm{d}e_2^q) + e_1^q \wedge \mathrm{d}e_2^p + e_2^p \wedge f_1^p + e_2^q \wedge f_1^q] \\ + \int_Z [+\mathrm{d}e_1^b \wedge e_2^p + e_1^b \wedge \mathrm{d}e_2^p - \mathrm{d}e_2^q \wedge f_1^b - e_2^q \wedge \mathrm{d}f_1^b] \\ + \int_Z [e_1^p \wedge \sigma \star e_2^p + e_2^p \wedge \sigma \star e_1^p] \\ = 0. \end{array} \tag{E.7}$$

All of the terms that are an element of $\mathscr{D}$ (the elements with superscript 2) are brought to the left hand side of the wedge product, resulting in

$$\begin{array}{l} \int_Z [e_2^p \wedge (f_1^p + \mathrm{d}e_1^b + \sigma \star e_1^p) + \mathrm{d}e_2^p \wedge (e_1^q + e_1^b)] \\ + \int_Z [e_2^q \wedge (f_1^q - \mathrm{d}f_1^b) + \mathrm{d}e_2^q \wedge (e_1^p - f_1^b)] \\ + \int_Z [\sigma \star e_2^p \wedge (-e_1^p + e_1^p)] = 0. \end{array} \tag{E.8}$$

Next, Stokes' theorem will be applied and (A.4b) with $(n^p, n^q) = (3,1)$ will be used:

$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2, \tag{E.9}$$

where k is the order of the form of ω_1, which is zero in this case. This step removes the $\mathrm{d}e_2^p$ and $\mathrm{d}e_2^q$ terms and leads to

$$\begin{array}{l} \int_Z [e_2^p \wedge (f_1^p + \sigma \star e_1^p - \mathrm{d}e_1^q))] + \int_{\partial Z} [e_2^p \wedge (e_1^q + e_1^b)] \\ + \int_Z [e_2^q \wedge (f_1^q - \mathrm{d}e_1^p)] + \int_{\partial Z} [e_2^q \wedge (e_1^p - f_1^b)] \\ = 0 \end{array} \tag{E.10}$$

for any $(e_2, f_2) \in \mathscr{D}$. This yields the following equalities

$$\begin{array}{rcl} f_1^p & = & -\sigma \star e_1^p + \mathrm{d}e_1^q \\ f_1^q & = & \mathrm{d}e_1^p \\ f_1^b & = & e_1^p \mid_{\partial Z} \\ e_1^b & = & -e_1^q \mid_{\partial Z}. \end{array} \tag{E.11}$$

This means that since (53) holds, $(e_1, f_1) \in \mathscr{D}$ also holds and therefore $\mathscr{D}^\perp \subset \mathscr{D}$.

Since $\mathscr{D} \subset \mathscr{D}^\perp$ and $\mathscr{D}^\perp \subset \mathscr{D}$, $\mathscr{D} = \mathscr{D}^\perp$ and therefore $\mathscr{D}$ is a Dirac structure.

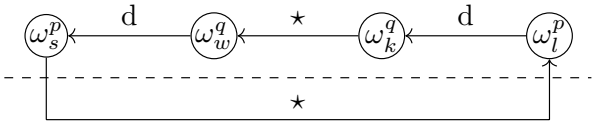

Figure 8: Dependencies

F. Shape functions

Throughout the discretization approach, shape functions are used to interpolate the efforts and flows on the spatial domain. Some properties of the shape functions are already given in this paper, e.g. in equations (56) and (57). This section shows how to find expressions for the different shape functions. Besides satisfying the requirements already given, the physical relations between the efforts and flows are maintained as good as possible as well. Finding shape functions is generally difficult and a solution is not always guaranteed, also in this approach one physical relation cannot be met.

The efforts and flows depend on each other and therefore the shape functions also depend on each other. The dependencies between shape functions is derived in the same way as explained in [**?**] and [**?**]. The result is shown in Fig. 8. This section will show how to find shape functions that satisfy these relations, assuming they exist. The relation shown under the dashed line in Fig. 8 will not be met.

The shape functions will be chosen as (piecewise) polynomials, since they are linear in their coefficients. Therefore the shape functions can be found by solving a set of linear equations, which is numerically efficient.

Similar as in [**?**] and [**?**], $\omega_s^p(z)$ for $s \in \{abdm, bcdm, dcam, acbm\}$ will be chosen as piecewise constant. The top left arrow in Fig. 8 implies choosing $\omega_w^q(z)$ for $w \in \{acb, abd, bcd, dca\}$ as affine two-forms. The Hodge star above the dashed line implies choosing $\omega_k^q(z)$ for $k \in \{am, bm, cm, dm\}$ in the same class as $\omega_w^q(z)$, however, they should be one-forms. Therefore $\omega_k^q(z)$ for $k \in \{am, bm, cm, dm\}$ are chosen affine one-forms. Finally, the top right arrow implies choosing $\omega_l^p(z)$ for $l \in \{a, b, c, d\}$ as second order polynomials. Clearly the bottom arrow is not satisfied by these choices.

Define $\omega_s^p(z)$

$\omega_s^p(z)$ for $s \in \{abdm, bcdm, dcam, acbm\}$ should satisfy the following four requirements

$$\int_{Z_{s_1}} \omega_{s_2}^p = \begin{cases} 1 \text{ for } s_1 = s_2 \\ 0 \text{ for } s_1 \neq s_2, \end{cases} \tag{F.1}$$

where $s_1, s_2 \in \{abdm, bcdm, dcam, acbm\}$ are tetrahedra.

It is chosen to be a piecewise constant and it should be a three-form ($\omega_s^p(z) \in \Lambda^3(Z)$). The following expression fulfills all of these conditions:

$$\omega_s^p(z) = \begin{cases} \frac{\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3}{\int_{Z_s} \mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3} & \text{for } z \in Z_s \\ 0 & \text{for } z \notin Z_s \end{cases} \tag{F.2}$$

with $s \in \{abdm, bcdm, dcam, acbm\}$.

The volume of a tetrehedron is given by

$$\begin{aligned} V_{abcd} &= \int_{Z_{abcd}} dz_1 dz_2 dz_3 \\ &= \tfrac{1}{6} |\det \begin{bmatrix} (a-d) & (b-d) & (c-d) \end{bmatrix}|. \end{aligned} \tag{F.3}$$

The point m will be chosen as the center of the tetrahedron, given by

$$m := \frac{1}{4}(a+b+c+d). \tag{F.4}$$

This choice of m means that

$$V_{abdm} = V_{bcdm} = V_{dcam} = V_{acbm} = \frac{1}{4} V_{abcd}. \tag{F.5}$$

Therefore, $\omega_s^p(z)$ (F.2) can be written as

$$\omega_s^p(z) = \begin{cases} \frac{4}{V_{abcd}} \mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3 & \text{for } z \in Z_s \\ 0 & \text{for } z \notin Z_s \end{cases} \tag{F.6}$$

with $s \in \{abdm, bcdm, dcam, acbm\}$ and V_{abcd} given in (F.3).

Define $\omega_w^q(z)$

$\omega_w^q(z)$ for $w \in \{acb, abd, bcd, dca\}$ is chosen as affine two-forms and they should satisfy the following four requirements

$$\int_{Z_{w_1}} \omega_{w_2}^q = \begin{cases} 1 \text{ for } w_1 = w_2 \\ 0 \text{ for } w_1 \neq w_2, \end{cases} \tag{F.7}$$

where $w_1, w_2 \in \{acb, abd, bcd, dca\}$.

The general form of an affine two-form is

$$\alpha(z) = \begin{bmatrix} z^T P + Q^T \end{bmatrix} \begin{bmatrix} \mathrm{d}z_1\mathrm{d}z_2 \\ \mathrm{d}z_2\mathrm{d}z_3 \\ \mathrm{d}z_3\mathrm{d}z_1 \end{bmatrix}, \tag{F.8}$$

with $P \in \mathbb{R}^{3\times 3}$ and $Q \in \mathbb{R}^3$. So there are twelve parameters to be determined. Since there are twelve parameters and four requirements, this system is underdetermined and many solutions exist.

Using (F.7) and Stokes' theorem (A.3), the following is found

$$\begin{aligned}\int_{Z_{abcd}} \mathrm{d}\omega_w^q &= \int_{\partial Z_{abcd}} \omega_w^q \\ &= \int_{Z_{acb}} \omega_w^q + \int_{Z_{abd}} \omega_w^q + \int_{Z_{bcd}} \omega_w^q + \int_{Z_{dca}} \omega_w^q \\ &= 1 \quad \forall w \in \{acb, abd, bcd, dca\}.\end{aligned} \tag{F.9}$$

The exterior derivative of $\omega_w^q(z)$ are constant three-forms, so constants can be taken out of integral as follows

$$\begin{aligned}1 &= \int_{Z_{abcd}} \mathrm{d}\omega_w^q \\ &= \star \mathrm{d}\omega_w^q \int_{Z_{abcd}} \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 \\ &= \star \mathrm{d}\omega_w^q V_{abcd}.\end{aligned} \tag{F.10}$$

Therefore, the exterior derivative of $\omega_w^q(z)$ is given by

$$\mathrm{d}\omega_w^q(z) = \tfrac{1}{V_{abcd}} \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 \quad \forall w \in \{acb, abd, bcd, dca\}. \tag{F.11}$$

The exterior derivative of the general form of an affine two form (F.8) is given by

$$\mathrm{d}\alpha(z) = (P_{12} + P_{23} + P_{31}) \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3. \tag{F.12}$$

Setting

$$\begin{cases} P_{23} := -P_{31} \\ Q := 0 \end{cases} \tag{F.13}$$

does not void (F.11) and leads to nine parameters and five equations.

Define $\omega_k^q(z)$

$\omega_k^q(z)$ for $k \in \{am, bm, cm, dm\}$ is chosen as an affine one-form and they should satisfy the following four requirements

$$\int_{Z_{k_1}} \omega_{k_2}^q = \begin{cases} 1 \text{ for } k_1 = k_2 \\ 0 \text{ for } k_1 \neq k_2, \end{cases} \tag{F.14}$$

where $k_1, k_2 \in \{am, bm, cm, dm\}$ are line segments.

The general form of an affine one-form is

$$\beta(z) = \begin{bmatrix} z^T P + Q^T \end{bmatrix} \begin{bmatrix} \mathrm{d}z_1 \\ \mathrm{d}z_2 \\ \mathrm{d}z_3 \end{bmatrix}, \tag{F.15}$$

with $P \in \mathbb{R}^{3\times 3}$ and $Q \in \mathbb{R}^3$. So there are twelve parameters to be determined and four requirements. Thus this system is underdetermined and many solutions exist.

$\omega_k^q(z)$ should however, also satisfy

$$\mathrm{d}\omega_k^q = -\mathrm{d}\mathrm{d}\omega_l^p = 0. \tag{F.16}$$

The exterior derivative of the general form of an affine one form is

$$\begin{aligned}\mathrm{d}\beta(z) = &(P_{21} - P_{12}) \mathrm{d}z_1 \mathrm{d}z_2 + (P_{31} - P_{13}) \mathrm{d}z_1 \mathrm{d}z_3 \\ &+ (P_{32} - P_{23}) \mathrm{d}z_2 \mathrm{d}z_3.\end{aligned} \tag{F.17}$$

Adding additional requirements

$$\begin{cases} Q &:= 0 \\ P_{21} &:= P_{12} \\ P_{31} &:= P_{13} \\ P_{32} &:= P_{23} \end{cases} \qquad \forall k \in \{am, bm, cm, dm\} \tag{F.18}$$

results in nine parameters and seven requirements and guarantees that (F.16) is satisfied.

Define $\omega_l^p(z)$

$\omega_l^p(z)$ for points $l \in \{a, b, c, d\}$ are second order polynomials and should satisfy

$$\omega_{l_1}^p(l_2) = \begin{cases} 1 \text{ for } l_1 = l_2 \\ 0 \text{ for } l_1 \neq l_2, \end{cases} \tag{F.19}$$

where $l_1, l_2 \in \{a, b, c, d\}$.

The general form of second order polynomial functions is

$$\gamma(z) = z^T P z + Q^T z + R, \tag{F.20}$$

where $P = P^T \in \mathbb{R}^{3\times 3}$, $Q \in \mathbb{R}^3$ and $R \in \mathbb{R}$. So, there are ten parameters to be determined. Since there are four requirements, this system is underdetermined.
In addition to these four requirements, the following choices are made

$$\begin{cases} \omega_l^p(m) &:= \frac{1}{4} \quad \forall l \in \{a, b, c, d\} \\ Q &:= 0 \end{cases} \tag{F.21}$$

resulting in seven parameters and five equations.
Besides that, the property

$$\omega_a^p(y) + \omega_b^p(y) + \omega_c^p(y) + \omega_d^p(y) = 1 \quad \forall y \in \{a, b, c, d, m\} \tag{F.22}$$

also holds.

G. Proof Dirac structure 2

This section proves that the system expressed in terms of v by (111) and (113) or by (116) is *not* a Dirac structure.
In order to prove a Dirac structure, a lossless system is assumed. In that case, the power balance from (73) becomes

$$e^T f = y^T u, \tag{G.1}$$

where e, f, y and u are given in (111) and (113) respectively. This power balance also gives a relation between the different M-matrices. Substituting (111) and (113) into (G.1) and using (114) leads to the following equality

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T \begin{bmatrix} 0 & M_4^T M_2 \\ M_5^T M_3 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T \begin{bmatrix} 0 & M_6^T U_2^T U_2 \\ U_1^T U_1 M_6 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{G.2}$$

This is the same as

$$\begin{aligned} &v_1^T M_4^T M_2 v_2 + v_2^T M_5^T M_3 v_1 = \\ &v_1^T M_6^T U_2^T U_2 v_2 + v_2^T U_1^T U_1 M_6 v_1. \end{aligned} \tag{G.3}$$

Next the parts with v_2^T in front and v_1 at the back will be transposed. Besides that the following relations can be derived from the construction of U_1 and U_2:

- $U_1^T U_1 = U_1^T = U_1$
- $U_2^T U_2 = U_2^T = U_2$
- $U_2 = I_4 - U_1$.

These relations are used to derive that $U_1^T U_1 + U_2^T U_2 = U_1 + U_2 = I_4$. Therefore (G.3) can also be written as

$$v_1^T M_4^T M_2 v_2 + v_1^T M_3^T M_5 v_2 = v_1^T M_6^T v_2. \tag{G.4}$$

Since this should hold for any v_1 and v_2, this is the same as

$$M_4^T M_2 + M_3^T M_5 = M_6^T. \tag{G.5}$$

For a Dirac structure it is necessary to have $\langle e \mid f \rangle = 0$, with $R = 0$, this equals (G.1). By defining a new flow $\hat{f}$ and a new effort $\hat{e}$ as

$$\hat{f} = \begin{bmatrix} f \\ \hline -y \end{bmatrix} \in \mathbb{R}^{12\times 1}, \quad \hat{e} = \begin{bmatrix} e \\ \hline u \end{bmatrix} \in \mathbb{R}^{12\times 1}, \tag{G.6}$$

$\langle e \mid f \rangle = \hat{e}^T \hat{f}$.
For these new flows and efforts, $\hat{e}^T \hat{f} = 0$ holds. Using equations (111) and (113), this new effort and flow can be written in terms of coefficient vector v (110) as follows:

$$\begin{aligned} \hat{f} &= \begin{bmatrix} f \\ \hline -y \end{bmatrix} = \underbrace{\begin{bmatrix} -M_1 & M_2 \\ M_3 & 0 \\ \hline -U_2 M_6 & -U_1 \end{bmatrix}}_{E^T \in \mathbb{R}^{12\times 8}} v \\ \hat{e} &= \begin{bmatrix} e \\ \hline u \end{bmatrix} = \underbrace{\begin{bmatrix} M_4 & 0 \\ 0 & M_5 \\ \hline U_1 M_6 & U_2 \end{bmatrix}}_{F^T \in \mathbb{R}^{12\times 8}} v. \end{aligned} \tag{G.7}$$

However, a Dirac structure is maximally dimensional, so E^T and F^T should be square matrices and the length of v should be the same as the length of $\hat{f}$ [**?**]. Therefore this system is not a Dirac structure. However, maybe this can be achieved by expanding v to $v \in \mathbb{R}^{12}$ or by reducing v to $v \in \mathbb{R}^8$.

Solution 1: Expansion

The first option is to expand the system in (G.7) to a Dirac structure of dimension twelve. This means that v should have length twelve as well. Since it is also desired to keep u free, a new vector $\hat{v}$ is chosen to be

$$\hat{v} = \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} e^p_a \\ e^p_b \\ e^p_c \\ e^p_d \\ e^q_{acb} \\ e^q_{abd} \\ e^q_{bcd} \\ e^q_{dca} \\ u \end{bmatrix} \in \mathbb{R}^{12}. \tag{G.8}$$

Next an expansion on (G.7) is made. In order to prevent losing rank, f is described in terms of v and u. Here a lossless system is assumed, so M_1 equals zero. This means that according to (G.7)

$$f = \begin{bmatrix} 0 & M_2 \\ M_3 & 0 \end{bmatrix} v. \tag{G.9}$$

From (G.7), an expression for v can be found equal to

$$v = \begin{bmatrix} M_4^{-1} & 0 \\ 0 & M_5^{-1} \end{bmatrix} e. \tag{G.10}$$

So

$$f = \begin{bmatrix} 0 & M_2 \\ M_3 & 0 \end{bmatrix} \begin{bmatrix} M_4^{-1} & 0 \\ 0 & M_5^{-1} \end{bmatrix} e = \begin{bmatrix} 0 & M_2 M_5^{-1} \\ M_3 M_4^{-1} & 0 \end{bmatrix} e. \tag{G.11}$$

Now f is described as a function of e, but it should also be a function of u. Eventually this Dirac structure

should be rewritten to port-Hamiltonian form, which can be written as

$$\begin{bmatrix} f \\ -y \end{bmatrix} = \begin{bmatrix} J & B \\ -B^T & 0 \end{bmatrix} \begin{bmatrix} e \\ u \end{bmatrix}. \tag{G.12}$$

From this equation matrix $-B^T$ can be calculated using the expression for y and v from (G.7):

$$\begin{aligned} -y &= \begin{bmatrix} -U_2M_6 & -U_1 \end{bmatrix} v \\ &= \begin{bmatrix} -U_2M_6 & -U_1 \end{bmatrix} \begin{bmatrix} M_4^{-1} & 0 \\ 0 & M_5^{-1} \end{bmatrix} e \\ &= \begin{bmatrix} -U_2M_6M_4^{-1} & -U_1M_5^{-1} \end{bmatrix} e, \end{aligned} \tag{G.13}$$

so

$$-B^T = \begin{bmatrix} -U_2M_6M_4^{-1} & -U_1M_5^{-1} \end{bmatrix}. \tag{G.14}$$

Besides that f can be written as

$$f = \begin{bmatrix} 0 & M_2M_5^{-1} \\ M_3M_4^{-1} & 0 \end{bmatrix} e - Bu + Bu \tag{G.15}$$

and u can be written as a function of e as follows

$$\begin{aligned} u &= \begin{bmatrix} U_1M_6 & U_2 \end{bmatrix} v \\ &= \begin{bmatrix} U_1M_6 & U_2 \end{bmatrix} \begin{bmatrix} M_4^{-1} & 0 \\ 0 & M_5^{-1} \end{bmatrix} e \\ &= \begin{bmatrix} U_1M_6M_4^{-1} & U_2M_5^{-1} \end{bmatrix} e \end{aligned} \tag{G.16}$$

Substituting this expression for u into the first u in (G.15), but not for the second u leads to

$$f = \begin{bmatrix} 0 & V_{1,2} \\ V_{2,1} & 0 \end{bmatrix} e + Bu, \tag{G.17}$$

with

$$\begin{aligned} V_{1,2} &= (M_2 - M_4^{-T}M_6^TU_2^TU_2)M_5^{-1} \\ V_{2,1} &= (M_3 - M_5^{-T}U_1^TU_1M_6)M_4^{-1}. \end{aligned} \tag{G.18}$$

This can be written back into terms of $\hat{v}$ using (G.7) and leads to

$$\begin{aligned} \tilde{f} = \begin{bmatrix} f_e \\ -y \end{bmatrix} &= \underbrace{\left[\begin{array}{cc:c} 0 & Jv_1 & M_4^{-T}M_6^TU_2^T \\ Jv_2 & 0 & M_5^{-T}U_1^T \\ \hdashline U_2M_6 & U_1 & 0 \end{array}\right]}_{\hat{E}^T \in \mathbb{R}^{12\times 12}} \hat{v} \\ \tilde{e} = \begin{bmatrix} e \\ u_e \end{bmatrix} &= \underbrace{\left[\begin{array}{cc:c} M_4 & 0 & 0 \\ 0 & M_5 & 0 \\ \hdashline 0 & 0 & I_4 \end{array}\right]}_{\hat{F}^T \in \mathbb{R}^{12\times 12}} \hat{v} \end{aligned} \tag{G.19}$$

with

$$\begin{aligned} Jv_1 &= M_2 - M_4^{-T}M_6^TU_2^TU_2 \\ Jv_2 &= M_3 - M_5^{-T}U_1^TU_1M_6 \end{aligned} \tag{G.20}$$

An advantage of this expansion is that the input u_e can now be chosen freely, however, the dynamics of the system are not the same anymore. This can be seen by comparing f in (G.7) with f_e in (G.19), while noticing that u in (G.7) is not the same as u_e in (G.19).

Then by choosing the Dirac structure to be the image of the defined matrices $\hat{E}^T$ and $\hat{F}^T$;

$$\mathscr{D} = \text{Im} \underbrace{\begin{bmatrix} \hat{E}^T \\ \hat{F}^T \end{bmatrix}}_{\in \mathbb{R}^{24\times 12}}, \tag{G.21}$$

the Dirac structure can be written in image representation [**?**];

$$\mathscr{D} = \{(\tilde{f}, \tilde{e}) \in \mathscr{F} \times \mathscr{F}^* \mid \tilde{f} = \hat{E}^*\hat{v}, \tilde{e} = \hat{F}^*\hat{v}, \hat{v} \in \mathbb{R}^{12}\}. \tag{G.22}$$

To prove that this is a Dirac structure, the following two properties should hold [**?**]:

- $\hat{F}\hat{E}^T + \hat{E}\hat{F}^T = 0$
- $\dim \mathscr{D} = \dim \mathscr{F}$.

If there are no losses in the system, $\hat{F}\hat{E}^T + \hat{E}\hat{F}^T$ equals

$$\hat{F}\hat{E}^T + \hat{E}\hat{F}^T = \begin{bmatrix} 0 & M_3^TM_5 + M_4^TM_2 - M_6^T & 0 \\ M_5^TM_3 + M_2^TM_4 - M_6 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{G.23}$$

Substituting (G.5) and its transpose into (G.23) indeed leads to

$$\hat{F}\hat{E}^T + \hat{E}\hat{F}^T = 0. \tag{G.24}$$

However, a Dirac structure should also be of maximal dimension [**?**], meaning that

$$\dim \mathscr{D} = \dim \mathscr{F}. \tag{G.25}$$

In this case $\dim \mathscr{D} = 12$ and $\dim \mathscr{F} = 12$. Therefore this subspace is also of maximal dimension and the system is a Dirac structure. However, the system dynamics have changed, so this does not prove that the original system as described in (116) is a Dirac structure.

Solution 2: Reduction

The previous expanded Dirac structure is useful for proving a Dirac structure, however, not for simulating. This is due to the fact that the input u has changed; u_e in (G.19) is a different u then in (G.7). Therefore the Dirac structure in (G.7) will now be reduced to have dimension eight.

For shortness of notation, system (G.7) is described

by

$$\begin{aligned}\hat{f} &= \left[\begin{array}{c} f \\ \hline -y \end{array}\right] = \underbrace{\left[\begin{array}{c} M_f \\ \hline M_y \end{array}\right]}_{E^T \in \mathbb{R}^{12\times 8}} v \\ \hat{e} &= \left[\begin{array}{c} e \\ \hline u \end{array}\right] = \underbrace{\left[\begin{array}{c} M_e \\ \hline M_u \end{array}\right]}_{F^T \in \mathbb{R}^{12\times 8}} v.\end{aligned} \tag{G.26}$$

A new vector $\tilde{v}$ is chosen to satisfy

$$v = P\tilde{v} \text{ s.t. } M_u P = \begin{bmatrix} I & 0 \end{bmatrix} \tag{G.27}$$

with P invertible. This new vector $\tilde{v}$ will make sure that one of the components of $\tilde{v}$ is equal to the input u, which is necessary for this reduction method. Assuming M_6 is invertible, a suitable matrix P satisfying these requirements is

$$P = \begin{bmatrix} M_6^{-1}U_1 & M_6^{-1}U_2 \\ U_2 & U_1 \end{bmatrix}. \tag{G.28}$$

Matrix P can also be written as

$$P = \begin{bmatrix} M_6^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} U_1 & U_2 \\ U_2 & U_1 \end{bmatrix} \tag{G.29}$$

and its inverse equals

$$P^{-1} = \begin{bmatrix} U_1 & U_2 \\ U_2 & U_1 \end{bmatrix}^{-1} \begin{bmatrix} M_6 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \\ U_2 & U_1 \end{bmatrix} \begin{bmatrix} M_6 & 0 \\ 0 & I \end{bmatrix} \tag{G.30}$$

In that case (G.26) can be described in terms of $\tilde{v}$ in the following manner

$$\begin{aligned}\hat{f} &= \left[\begin{array}{c} f \\ \hline -y \end{array}\right] = \underbrace{\left[\begin{array}{c} M_f P \\ \hline M_y P \end{array}\right]}_{E^T \in \mathbb{R}^{12\times 8}} \tilde{v} \\ \hat{e} &= \left[\begin{array}{c} e \\ \hline u \end{array}\right] = \underbrace{\left[\begin{array}{c} M_e P \\ \hline \begin{bmatrix} I & 0 \end{bmatrix} \end{array}\right]}_{F^T \in \mathbb{R}^{12\times 8}} \tilde{v}.\end{aligned} \tag{G.31}$$

By using the P given in (G.28), $\tilde{v}$ equals

$$\tilde{v} = \begin{bmatrix} u \\ \tilde{v}_2 \end{bmatrix}. \tag{G.32}$$

Next a transformation matrix T will be chosen such that

$$TM_eP = \begin{bmatrix} \tilde{M}_1 & 0 \\ 0 & \tilde{M}_2 \end{bmatrix}, \tag{G.33}$$

where T should be invertible. A possible T is

$$T = \begin{bmatrix} U_1^T M_6 M_4^{-1} & U_2^T M_5^{-1} \\ U_2^T M_6 M_4^{-1} & U_1^T M_5^{-1} \end{bmatrix}, \tag{G.34}$$

where M_6 is assumed to be invertible.

Define

$$\tilde{e} = Te \qquad \tilde{f} = T^{-T} f, \tag{G.35}$$

then

$$\tilde{e}^T \tilde{f} = e^T T^T T^{-T} f = e^T f. \tag{G.36}$$

This product can be split up into

$$\tilde{e}^T \tilde{f} = \tilde{e}_1^T \tilde{f}_1 + \tilde{e}_2^T \tilde{f}_2, \tag{G.37}$$

where

$$\tilde{e}_1^T \tilde{f}_1 = u^T \tilde{M}_1^T f_1. \tag{G.38}$$

This means that

$$e^T f = \tilde{e}^T \tilde{f} = \tilde{e}_1^T \tilde{f}_1 + \tilde{e}_2^T \tilde{f}_2 = y^T u \tag{G.39}$$

can be rewritten to

$$\tilde{e}_2^T \tilde{f}_2 = y^T u - u^T \tilde{M}_1 \tilde{f}_1. \tag{G.40}$$

So the power balance (and therefore also $FE^T + EF^T = 0$) is still satisfied if

$$\begin{bmatrix} \tilde{e}_2 \\ u \end{bmatrix} = \begin{bmatrix} 0 & \tilde{M}_2 \\ I & 0 \end{bmatrix} \tilde{v} \tag{G.41}$$

and

$$y_{new} = y + \tilde{M}_1^T \tilde{f}_1 \tag{G.42}$$

If P and T are chosen as in (G.28) and (G.34), then

$$\begin{aligned}\tilde{e} = Te &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \tilde{v} \\ \tilde{f} = T^{-T} f &= \begin{bmatrix} 0 & F_{1,2} \\ F_{2,1} & 0 \end{bmatrix} \tilde{v}\end{aligned} \tag{G.43}$$

where

$$\begin{aligned} F_{1,2} &= U_1^T M_6^{-T} M_4^T M_2 U_1 + U_2^T M_5^T M_3 M_6^{-1} U_2 \\ F_{2,1} &= U_2^T M_6^{-T} M_4^T M_2 U_2 + U_1^T M_5^T M_3 M_6^{-1} U_1. \end{aligned} \tag{G.44}$$

The new reduced Dirac structure of dimension eight is then given by

$$\begin{aligned}\begin{bmatrix} \tilde{e}_2 \\ u \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}}_{F^T} \tilde{v} \\ \begin{bmatrix} \tilde{f}_2 \\ -y_{new} \end{bmatrix} &= \underbrace{\begin{bmatrix} F_{2,1} & 0 \\ 0 & I + F_{1,2} \end{bmatrix}}_{E^T} \tilde{v}.\end{aligned} \tag{G.45}$$

This reduction is useful, however, not for simulating, since it has the disadvantage that it is impossible to implement initial conditions on $\tilde{e}_1$. Therefore it cannot capture the complete dynamics of the original system and does not prove that (116) is a Dirac structure.

H. Proof port-Hamiltonian structure

Equation (117) is a port-Hamiltonian system iff J is skew symmetric; $J = -J^T$ and R is positive semi-definite; $R \geq 0$.

J is skew symmetric

Equations (119) and (118) show that J is given by

$$J = \begin{bmatrix} 0 & V_{1,2} \\ V_{2,1} & 0 \end{bmatrix} \in \mathbb{R}^{8x8}, \tag{H.1}$$

where

$$\begin{array}{rcl} V_{1,2} & = & (M_2 - M_4^{-T} M_6^T U_2^T U_2) M_5^{-1} \\ V_{2,1} & = & (M_3 - M_5^{-T} U_1^T U_1 M_6) M_4^{-1}. \end{array} \tag{H.2}$$

J is skew-symmetric iff $J + J^T = 0$. In this case, $J + J^T$ equals

$$J + J^T = \begin{bmatrix} 0 & V_{1,2} + V_{2,1}^T \\ V_{2,1} + V_{1,2}^T & 0 \end{bmatrix}. \tag{H.3}$$

Here

$$\begin{array}{l} V_{1,2} + V_{2,1}^T = \\ M_2 M_5^{-1} + M_4^{-T} M_3^T - \\ M_4^{-T} M_6^T U_2^T U_2 M_5^{-1} - M_4^{-T} M_6^T U_1^T U_1 M_5^{-1}, \end{array} \tag{H.4}$$

which should be equal to zero. Using the fact that $U_2^T U_2 + U_1^T U_1 = I_4$, it can be written as

$$M_2 M_5^{-1} + M_4^{-T} M_3^T - M_4^{-T} M_6^T M_5^{-1} = 0. \tag{H.5}$$

Pre-multiplying with M_4^T and post-multiplying with M_5 leads to

$$M_4^T M_2 + M_3^T M_5 - M_6^T = 0. \tag{H.6}$$

This is the same equality as in equation (G.23) and using the equalities from (G.5) it can be proven to be true. Since $V_{2,1} + V_{1,2}^T = (V_{1,2} + V_{2,1}^T)^T$, (H.3) equals the zero matrix and therefore J is skew-symmetric.

R is positive semi-definite

Combining equations (119) and (118) shows that R is given by

$$R = \begin{bmatrix} M_1 M_4^{-1} & 0 \\ 0 & 0 \end{bmatrix}. \tag{H.7}$$

M_4 is given in (81) and equals

$$M_4 = \int_{Z_{abcd}} \begin{bmatrix} \omega_a^p \wedge \omega_{abdm}^p & \dots & \omega_d^p \wedge \omega_{abdm}^p \\ \omega_a^p \wedge \omega_{bcdm}^p & \dots & \omega_d^p \wedge \omega_{bcdm}^p \\ \omega_a^p \wedge \omega_{dcam}^p & \dots & \omega_d^p \wedge \omega_{dcam}^p \\ \omega_a^p \wedge \omega_{acbm}^p & \dots & \omega_d^p \wedge \omega_{acbm}^p \end{bmatrix}, \tag{H.8}$$

which can also be written as

$$M_4 = \begin{bmatrix} \star\omega_{abdm}^p \int_{Z_{abdm}} \star\omega_a^p & \dots & \star\omega_{abdm}^p \int_{Z_{abdm}} \star\omega_d^p \\ \star\omega_{bcdm}^p \int_{Z_{bcdm}} \star\omega_a^p & \dots & \star\omega_{bcdm}^p \int_{Z_{bcdm}} \star\omega_d^p \\ \star\omega_{dcam}^p \int_{Z_{dcam}} \star\omega_a^p & \dots & \star\omega_{dcam}^p \int_{Z_{dcam}} \star\omega_d^p \\ \star\omega_{acbm}^p \int_{Z_{acbm}} \star\omega_a^p & \dots & \star\omega_{acbm}^p \int_{Z_{acbm}} \star\omega_d^p \end{bmatrix}, \tag{H.9}$$

since w_s^p, for tetrahedra $s \in \{abdm, bcdm, dcam, acbm\}$ is piecewise constant. Here the properties (A.4a) and (A.7) are used. The inverse of M_4 is easier to calculate if M_4 is written as

$$M_4 = \begin{bmatrix} \star\omega_{abdm}^p & 0 & 0 & 0 \\ 0 & \star\omega_{bcdm}^p & 0 & 0 \\ 0 & 0 & \star\omega_{dcam}^p & 0 \\ 0 & 0 & 0 & \star\omega_{acbm}^p \end{bmatrix} G, \tag{H.10}$$

where

$$G = \begin{bmatrix} \int_{Z_{abdm}} \star\omega_a^p & \int_{Z_{abdm}} \star\omega_b^p & \int_{Z_{abdm}} \star\omega_c^p & \int_{Z_{abdm}} \star\omega_d^p \\ \int_{Z_{bcdm}} \star\omega_a^p & \int_{Z_{bcdm}} \star\omega_b^p & \int_{Z_{bcdm}} \star\omega_c^p & \int_{Z_{bcdm}} \star\omega_d^p \\ \int_{Z_{dcam}} \star\omega_a^p & \int_{Z_{dcam}} \star\omega_b^p & \int_{Z_{dcam}} \star\omega_c^p & \int_{Z_{dcam}} \star\omega_d^p \\ \int_{Z_{acbm}} \star\omega_a^p & \int_{Z_{acbm}} \star\omega_b^p & \int_{Z_{acbm}} \star\omega_c^p & \int_{Z_{acbm}} \star\omega_d^p \end{bmatrix}. \tag{H.11}$$

Since the damping σ is constant on the tetrahedra Z_s with $s \in \{abdm, bcdm, dcam, acbm\}$, M_1 (63) can also be written as a function of matrix G:

$$\begin{array}{rcl} M_1 & = & \begin{bmatrix} \int_{Z_{abdm}} \sigma \star \omega_a^p & \dots & \int_{Z_{abdm}} \sigma \star \omega_d^p \\ \int_{Z_{bcdm}} \sigma \star \omega_a^p & \dots & \int_{Z_{bcdm}} \sigma \star \omega_d^p \\ \int_{Z_{dcam}} \sigma \star \omega_a^p & \dots & \int_{Z_{dcam}} \sigma \star \omega_d^p \\ \int_{Z_{acbm}} \sigma \star \omega_a^p & \dots & \int_{Z_{acbm}} \sigma \star \omega_d^p \end{bmatrix} \\ & = & \underbrace{\begin{bmatrix} \sigma_{abdm} & 0 & 0 & 0 \\ 0 & \sigma_{bcdm} & 0 & 0 \\ 0 & 0 & \sigma_{dcam} & 0 \\ 0 & 0 & 0 & \sigma_{acbm} \end{bmatrix}}_{\sigma_t} G \end{array} \tag{H.12}$$

Note that σ on an element scale is chosen to be piecewise constant

$$\sigma(z) = \begin{cases} \sigma_s \text{ for } z \in Z_s \\ 0 \text{ for } z \notin Z_s, \end{cases} \tag{H.13}$$

with tetrahedra $s \in \{abdm, bcdm, dcam, acbm\}$.

The product $M_1 M_4^{-1}$ then becomes

$$
\begin{aligned}
&M_1 M_4^{-1} \\
&= \sigma_t G G^{-1} \begin{bmatrix} \frac{1}{\star\omega^p_{abdm}} & 0 & 0 & 0 \\ 0 & \frac{1}{\star\omega^p_{bcdm}} & 0 & 0 \\ 0 & 0 & \frac{1}{\star\omega^p_{dcam}} & 0 \\ 0 & 0 & 0 & \frac{1}{\star\omega^p_{acbm}} \end{bmatrix} \\
&= \sigma_t \begin{bmatrix} \frac{1}{\star\omega^p_{abdm}} & 0 & 0 & 0 \\ 0 & \frac{1}{\star\omega^p_{bcdm}} & 0 & 0 \\ 0 & 0 & \frac{1}{\star\omega^p_{dcam}} & 0 \\ 0 & 0 & 0 & \frac{1}{\star\omega^p_{acbm}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\sigma_{abdm}}{\star\omega^p_{abdm}} & 0 & 0 & 0 \\ 0 & \frac{\sigma_{bcdm}}{\star\omega^p_{bcdm}} & 0 & 0 \\ 0 & 0 & \frac{\sigma_{dcam}}{\star\omega^p_{dcam}} & 0 \\ 0 & 0 & 0 & \frac{\sigma_{acbm}}{\star\omega^p_{acbm}} \end{bmatrix} \\
&= \begin{bmatrix} \sigma_{abdm}\frac{1}{\star\omega^p_{abdm}} & & & 0 \\ & \ddots & & \\ & & & \\ 0 & & & \sigma_{acbm}\frac{1}{\star\omega^p_{acbm}} \end{bmatrix}.
\end{aligned}
\tag{H.14}
$$

This means that R is a diagonal matrix, which is positive semi-definite if all of the diagonal term (its eigenvalues) are zero or positive. The damping terms σ_s for tetrahedra $s \in \{abdm, bcdm, dcam, acbm\}$ are positive by definition. ω^p_s, for tetrahedra $s \in \{abdm, bcdm, dcam, acbm\}$ is given by

$$
\omega^p_s(z) = \begin{cases} \frac{4}{V_{abcd}} \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 & \text{for } z \in Z_s \\ 0 & \text{for } z \notin Z_s \end{cases}, \tag{H.15}
$$

so $\star\omega^p_s$ equals

$$
\star\omega^p_s(z) = \frac{4}{V_{abcd}} \text{ for } z \in Z_s. \tag{H.16}
$$

Here, V_{abcd} is the volume of tetrahedron $abcd$, which is always positive and therefore the eigenvalues of $M_1 M_4^{-1}$ are all positive. This means that the eigenvalues of R are all positive or zero and R is positive semi-definite.

Since J is skew-symmetric and R is positive semi-definite, (117) is a port-Hamiltonian system.

I. Dirac structure 1D

This section proves that the system described in terms of v by (190) and (192) is an image representation of a Dirac structure. In order to prove a Dirac structure, a lossless system is assumed, so the power balance from (73) becomes

$$
e^T f = y^T u. \tag{I.1}
$$

By renaming the matrices, the system in terms of v can be written as

$$
\begin{aligned}
f &= M_f v, \quad e = M_e v \\
y &= M_y v, \quad u = M_u v.
\end{aligned} \tag{I.2}
$$

Substituting this into the lossless power balance as mentioned before leads to a relation between the M-matrices. The result is

$$
\begin{aligned}
&e^T f - y^T u = 0 \\
&v^T M_e^T M_f v + v^T M_y^T M_u v = 0.
\end{aligned} \tag{I.3}
$$

For a Dirac structure it is necessary to have $\langle e \mid f \rangle = 0$, so a new effort and flow is defined as

$$
\hat{f} = \begin{bmatrix} f \\ -y \end{bmatrix} \in \mathbb{R}^{6\times 1}, \quad \hat{e} = \begin{bmatrix} e \\ u \end{bmatrix} \in \mathbb{R}^{6\times 1}. \tag{I.4}
$$

Now $\langle e \mid f \rangle = \hat{e}^T \hat{f}$ and it should be proven that $\hat{e}^T \hat{f} = 0$. This new effort and flow still satisfy the power balance. The system can then be described in terms of v (189) by

$$
\begin{aligned}
\hat{f} &= \begin{bmatrix} f \\ -y \end{bmatrix} = \underbrace{\begin{bmatrix} M_f \\ -M_y \end{bmatrix}}_{E^T \in \mathbb{R}^{6\times 6}} v \\
\hat{e} &= \begin{bmatrix} e \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} M_e \\ M_u \end{bmatrix}}_{F^T \in \mathbb{R}^{6\times 6}} v.
\end{aligned} \tag{I.5}
$$

This is an image representation of a Dirac structure if it satisfies [**?**]

- $FE^T + EF^T = 0$
- $\operatorname{rank}\begin{bmatrix} F \mid E \end{bmatrix} = \dim \mathscr{F}$.

The first requirement is satisfied since

$$
\begin{aligned}
&FE^T + EF^T = \\
&v^T \begin{bmatrix} M_e^T & M_u^T \end{bmatrix} \begin{bmatrix} M_f \\ M_y \end{bmatrix} v + v^T \begin{bmatrix} M_f^T & M_y^T \end{bmatrix} \begin{bmatrix} M_e \\ M_u \end{bmatrix} v \\
&= v^T M_e^T M_f v + v^T M_u^T M_y v + v^T M_f^T M_e v + v^T M_y^T M_u v
\end{aligned} \tag{I.6}
$$

Substituting (I.3) and its transpose indeed shows that $FE^T + EF^T = 0$.

The second requirement is true iff F has full rank, which is imposed by the fact that F^T should be invertible in order to rewrite the system to a port-Hamiltonian

system. This means that the rank of $\begin{bmatrix} F \mid E \end{bmatrix}$ equals six and the dimension of $\mathscr{F}$ equals six as well.

Both the requirements are met, so the system represented by (190) and (192) is a Dirac structure in image representation.

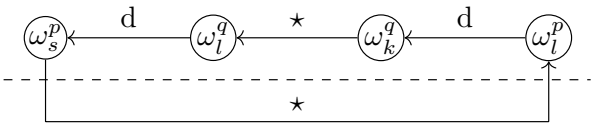

Figure 9: Dependencies

J. Shape functions 1D

This section shows how to find expressions for the different shape functions used in the 1D approach. Besides satisfying the requirements already given, the physical relations between the efforts and flows are maintained as good as possible as well. Finding shape functions is generally difficult and a solution is not always guaranteed, also in this approach one physical relation cannot be met.

The efforts and flows depend on each other and therefore the shape functions also depend on each other. The dependencies between shape functions is derived in the same way as explained in [**?**] and [**?**]. The result is shown in Fig. 9. This section will show how to find shape functions that satisfy these relations, assuming they exist. The relation shown under the dashed line in Fig. 9 will not be met.

The shape functions will be chosen as (piecewise) polynomials, since they are linear in their coefficients. Therefore the shape functions can be found by solving a set of linear equations, which is numerically efficient.

Similar as in [**?**], $\omega_s^p(z)$ for line segments $s \in \{am, mb\}$ will be chosen as piecewise constant. The top left arrow in Fig. 9 implies choosing $\omega_l^q(z)$ for points $l \in \{a, m, b\}$ as affine zero-forms. The Hodge star above the dashed line implies choosing $\omega_s^q(z)$ for $s \in \{am, mb\}$ in the same class as $\omega_l^q(z)$, however, they should be one-forms. Therefore $\omega_s^q(z)$ for line segments $s \in \{am, mb\}$ are chosen affine one-forms. Finally, the top right arrow implies choosing $\omega_l^p(z)$ for points $l \in \{a, m, b\}$ as second order polynomials. Clearly the bottom arrow is not satisfied by these choices.

Define $\omega_s^p(z)$

$\omega_s^p(z)$ for line segments $s \in \{am, mb\}$ should satisfy the following requirements

$$\int_{Z_{s_1}} \omega_{s_2}^p = \begin{cases} 1 \text{ for } s_1 = s_2 \\ 0 \text{ for } s_1 \neq s_2, \end{cases} . \tag{J.1}$$

It is chosen to be a piecewise constant and it should be a one-form ($\omega_s^p(z) \in \Lambda^1(Z)$). The following expres-

sion fulfills all of these conditions:

$$\omega_s^p(z) = \begin{cases} \frac{\mathrm{d}z_1}{\int_{Z_s} \mathrm{d}z_1} & \text{for } z \in Z_s \\ 0 & \text{for } z \notin Z_s, \end{cases} \tag{J.2}$$

where $s \in \{am, mb\}$. The length of a line is given by

$$L_{ab} = \int_{Z_{ab}} \mathrm{d}z_1 = b - a. \tag{J.3}$$

By choosing the point m in the middle of the element; $m := \frac{1}{2}(a+b)$,

$$L_{am} = L_{mb} = \frac{1}{2} L_{ab}. \tag{J.4}$$

Therefore, $\omega_s^p(z)$ from (J.2) can be written as

$$\omega_s^p(z) = \begin{cases} \frac{2}{b-a}\,\mathrm{d}z_1 & \text{for } z \in Z_s \\ 0 & \text{for } z \notin Z_s, \end{cases} \tag{J.5}$$

where $s \in \{am, mb\}$ are line segments.

Define $\omega_l^q(z)$
$\omega_l^q(z)$ for points $l \in \{a, m, b\}$ is chosen as affine zero-forms and they should satisfy the following requirements

$$\omega_{l_1}^q(l_2) = \begin{cases} 1 \text{ for } l_1 = l_2 \\ 0 \text{ for } l_1 \neq l_2, \end{cases} \tag{J.6}$$

where $l_1, l_2 \in \{a, b\}$ and

$$\omega_m^q(l) = \begin{cases} 1 \text{ for } l = m \\ 0 \text{ for } l \neq m \end{cases} \tag{J.7}$$

where $l \in \{a, m, b\}$.

The general form of an affine zero-form is

$$f(z) = \alpha z + \beta \tag{J.8}$$

There are two parameters to be determined; α and β. ω_a^q and ω_b^q should each satisfy two requirements, so they can be determined uniquely. However, ω_m^q should satisfy three requirements. Therefore it is chosen to be a non-smooth, or more precisely non-differentiable, zero-form, which has a certain α and β on the interval Z_{am}, α_{am} and β_{am}, and a different α and β on the interval Z_{mb}, α_{mb} and β_{mb}. In that case, ω_m^q has four parameters and four requirements leading to an unique solution.

Define $\omega_s^q(z)$
$\omega_s^q(z)$ for line segments $s \in \{am, mb\}$ is chosen as an affine one-form and they should satisfy the following four requirements

$$\int_{Z_{s_1}} \omega_{s_2}^q = \begin{cases} 1 \text{ for } s_1 = s_2 \\ 0 \text{ for } s_1 \neq s_2, \end{cases} \tag{J.9}$$

where $s_1, s_2 \in \{am, mb\}$.

The general form of an affine one-form is

$$f(z) = \alpha z + \beta\, \mathrm{d}z_1. \tag{J.10}$$

There are two parameters to be determined and two requirements, thus a solution can be uniquely determined.

Define $\omega_l^p(z)$
$\omega_l^p(z)$ for points $l \in \{a, m, b\}$ are second order polynomials and should satisfy

$$\omega_{l_1}^p(l_2) = \begin{cases} 1 \text{ for } l_1 = l_2 \\ 0 \text{ for } l_1 \neq l_2, \end{cases} \tag{J.11}$$

where $l_1, l_2 \in \{a, b\}$ and

$$\omega_m^p(l) = \begin{cases} 1 \text{ for } l = m \\ 0 \text{ for } l \neq 0, \end{cases} \tag{J.12}$$

where $l \in \{a, m, b\}$

The general form of second order polynomial functions is

$$f(z) = \alpha z^2 + \beta z + \gamma. \tag{J.13}$$

There are three parameters to be determined; α, β and γ. ω_a^q and ω_b^q should each satisfy two requirements, so this is an underdetermined set of equations. In addition to the two requirements another requirement can be added, such as

$$\omega_l^p(m) := \frac{1}{2} \quad \forall l \in \{a, b\}. \tag{J.14}$$

However, it is also possible to set

$$\beta := 0, \tag{J.15}$$

leading to a unique solution as well.

ω_m^p has to satisfy three requirements, so it is not necessary to add an additional requirement or set one of the parameters to zero.