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Model reduction for linear parameter varying systems through parameter projection

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***Abstract*—This paper deals with parameter reduction of linear parameter-varying systems as a means of reducing complex models. First, the notion of system norms in parameter varying systems is considered. The LPV norms are used to evaluate the performance of the reduction methods. We provide two methods to transform an affine LPV system into a parameter-ordered form and reduce the parameter space. One method is based on sensitivity functions of the transfer function and time evolution equations. Another method achieves reduction by finding affine upper bounds of the system gramians and is extended to time-varying parameters. With the affine gramians, a Hankel-norm approximation optimisation is formulated to find a parameter projection matrix. Both methods are applied to a illustrative example and a thermal model to show the achieved performance.**

I. Introduction

In the past 30 years, the framework of Linear Parameter-Varying (LPV) systems has been developed and approved as a reliable and efficient alternative for characterizing nonlinear/time-varying systems. The LPV framework has been successfully applied in a wide range of fields, from very-large-scale integration (VLSI) to aircraft [1]–[3]. However, the inherent nature of complex physics often results in high dimensional models in both state-space and parameter space. Such models are often applied to evaluate system performance over a range of the parameter values, such as the geometry in circuits [4] and MEMS devices [5]. To have reasonable computational complexity in terms of synthesis and simulation, parametrised model order reduction (pMOR) is in many circumstances required.

In the aforementioned applications, the high dimensional state-space is introduced by the spatialisation of partial differential equations (PDEs) to ordinary differential equations (ODEs), while maintaining high precision. The physical constraints and design parameters of such high-fidelity models lead to large parameter-space as well. Hence, we can categorise the pMOR problem as the following problems:

1) find a low-rank state while the parameter space is preserved. e.g., the projection-based parametric moment-matching reduction [6]
2) reduce both the size of the state and parameter space, e.g., two-step approach is introduced in [7], reduced rank regression is applied to parameter space and moment-matching is used to construct the reduced-order model.
3) only reduce the parameter space.

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In most pMOR work, the primary goal has been solving sub-problem one. A number of such methods have been proposed, which mainly focus on sampling methods [8]–[10]. On this topic, two survey papers [1], [11] are referred. However, the curse of dimensionality emerges in the construction of the parametrised reduced model. For example in [9], the dimension of the reduced model increases exponentially as the number of parameter interpolation points and moments increase. Even though it is not always stated, simple sampling schemes imply static dependence on the parameter. Techniques which explicitly deal with time-varying parameters are difficult to apply [12]. All these works are confined to state reduction without considering parameter reduction. The approaches which consider parameter reduction either require typical trajectories of the parameter [13] or lack interpretation and are limited in application [14]. Development of more general parameter reduction techniques can significantly improve the efficiency of simulation, without loss of generality, as shown in [7]. This leads to the second challenging topic, which has been left without enough attention, combined reduction of the dimension of the state and parameter spaces.

To our best knowledge, only a handful of research has touched upon this subject. In [7] a two-step approach is introduced. First, parameter reduction is employed to find a low-dimensional parameter space. The second step is constructing a reduced model via moment-matching. However, the reduced rank regression used for parameter reduction only quantifies the relation between the parameters and the outputs, which is limited by the type of the input excitation. Furthermore, the system dynamics of the parameters are not taken into account.

In this paper, the last sub-problem is addressed in two ways. Firstly by exploring the relationship between the parameter space and the system gramians, secondly by analysing the sensitivity of the evolution equations to parameter changes. In doing this a set definitions and naming convention for parametric system norms, aimed at characterising approximation errors in a consistent manner, are proposed. The methods in this paper are developed for time-invariant parameters, with an extension of the gramian based approach to the time-varying case. Especially with extension to time-varying parameters these approaches will be applicable to a wide variety of systems.

In section II the problem is formalised and the used notation is introduced. Section III and IV show the methods to achieve parameter reduction using Hankel-norm approximation and sensitivity analysis respectively. Both methods are applied to two systems and the results are analysed in section V.

II. Preliminaries and notation

Before developing techniques to achieve parameter reduction it is necessary to have a well defined system and notation. In general, the state-space description of a continuous-time linear parameter-varying (LPV) systems is given as

$$\dot{x}(t) = A(\theta)x(t) + B(\theta)u(t) \tag{1a}$$
$$y(t) = C(\theta)x(t) + D(\theta)u(t) \tag{1b}$$

where $x \in \mathbb{R}^{n_x}$ are states, $u \in \mathbb{R}^{n_u}$ inputs, $y \in \mathbb{R}^{n_y}$ outputs and $\theta \in \Theta \subseteq \mathbb{R}^{n_\theta}$ parameters. The state space matrices $A(\theta)$, $B(\theta)$, $C(\theta)$ and $D(\theta)$ may have any dependency on the parameter vector. In this class of systems, a distinction is to be made between systems where θ are time-varying or time-invariant. If $\dot{\theta} = 0$, the system can be viewed as a collection of LTI systems. Whereas for $\dot{\theta} \neq 0$, the dynamics of the system change over time. In Fig. 1 the relation between LTI and time-invariant LPV systems is illustrated.

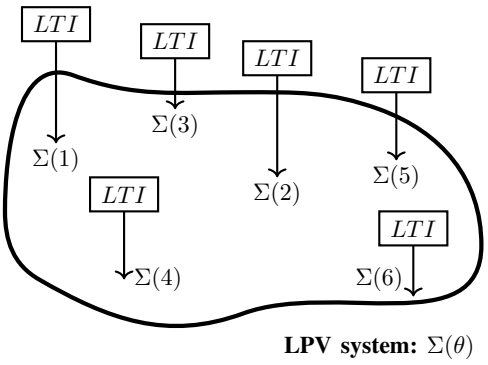

Figure 1: Relation between LPV systems and LTI systems for $\dot{\theta} = 0$.

In this work it is assumed that $\dot{\theta} = 0$ unless stated otherwise. Besides assuming $\dot{\theta} = 0$, the state space matrices are assumed to have affine dependency on θ. Furthermore the parameter space Θ constitutes a convex hull, $\text{Co}\{\omega_1, \omega_2, \dots \omega_q\}$, well representing the actual parameter space. Without loss of generality, the system is assumed to be normalised and scaled such that all $\theta \in [0\ 1]$. The final assumption is that a given LPV system is quadratically stable, observable and reachable $\forall \theta \in \Theta$. This immediately results in the reduced models being quadratically stable, observable and reachable as well. Which holds because reduction is achieved by setting a parameter equal to zero which is, by assumption, in Θ this holds.

The assumed affine nature of the system allows to write the state-space equations using the tensor product $(\theta \otimes I_n)$. Using this notation the system matrices are expressed in the following manner

$$A_\theta = \left(\underbrace{\begin{bmatrix} 1 & \theta_1 & \dots & \theta_{n_\theta} \end{bmatrix}}_{\theta} \otimes I_{n_x} \right) \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{n_\theta} \end{bmatrix} = \bar{\theta}_{n_x} \bar{A}^\theta. \tag{2}$$

In the presented notation, subscript θ represents a parameter dependency and superscript θ a stacked state space matrix. In the remainder of this paper the short hand notation $\bar{\theta}_{n_\bullet}$ will be used for the tensor product with an identity matrix, where the subscript indicates the size of the identity matrix. Using this short hand notation, the affine LPV system is rewritten as

$$\dot{x}(t) = \bar{\theta}_{n_x} \bar{A}^\theta x(t) + \bar{\theta}_{n_x} \bar{B}^\theta u(t) \tag{3a}$$
$$y(t) = \bar{\theta}_{n_y} \bar{C}^\theta x(t) + \bar{\theta}_{n_y} \bar{D}^\theta u(t) \tag{3b}$$

With the system defined, transformations of the system will be defined next. The parameter reduction methods proposed in this work are defined as linear transformations of the original parameter space. To have a consistent transformation of the parameters it needs to be invertible if $n_r = n_\theta$. Furthermore an ordering property is required such that the transformed system parameters are in order of contribution to the system dynamics. The measure of contribution depends on the method used. The parameter transformation is defined in (4), where T satisfies the orthonormality property (5). The parameter space is also transformed using the same transformation matrix $\tilde{\Theta} = \Theta T$. In the case of having redundant parameters, T need not be full rank and thus not satisfy (5). However, due to the redundancy the original system can be reduced without loss, decreasing n_θ.

$$\tilde{\theta} := \theta \cdot \begin{bmatrix} T_0 & T_1 & T_2 & \dots & T_{n_\theta} \end{bmatrix} = \theta T \tag{4}$$
$$T T^T = T^T T = I_{n_\theta} \tag{5}$$

In this work we take T_0 as being 1 in the first element and 0 elsewhere to keep the zero system equal. In general this is not required but the transformed parameters may not include 0 and thus a different approach is needed to achieve reduction, e.g. setting the removed parameters to the nominal value.

To keep input-output relation equivalent for $n_r = n_\theta$, the system matrices are transformed according to (6). The system theoretic equivalence is easily shown as $(\theta T \otimes I)(T^T \otimes I) = (\theta \otimes I)(T T^T \otimes I)$, resulting in $(\theta \otimes I)$. Therefore it holds that $\tilde{A}_\theta = A_\theta$, and similar for the other matrices.

$$\tilde{A}_\theta = \tilde{\theta}_{n_x} \tilde{A}^\theta = (\theta T \otimes I_{n_x})(T^T \otimes I_{n_x}) A^\theta \tag{6a}$$
$$\tilde{B}_\theta = \tilde{\theta}_{n_x} \tilde{B}^\theta = (\theta T \otimes I_{n_x})(T^T \otimes I_{n_x}) B^\theta \tag{6b}$$
$$\tilde{C}_\theta = \tilde{\theta}_{n_y} \tilde{C}^\theta = (\theta T \otimes I_{n_y})(T^T \otimes I_{n_y}) C^\theta \tag{6c}$$
$$\tilde{D}_\theta = \tilde{\theta}_{n_y} \tilde{D}^\theta = (\theta T \otimes I_{n_y})(T^T \otimes I_{n_y}) D^\theta \tag{6d}$$

After transformation the system is in parameter ordered form. Reduction to a parameter order n_r is then achieved by setting all parameters $\theta_{r+1} \ \dots \ \theta_{n_\theta}$ to zero. This results effectively in a system with n_r parameters. The effective parameters are calculated as

$$\theta_r := \theta \begin{bmatrix} T_0 & T_1 & T_2 & \dots & T_{n_r} \end{bmatrix} = \theta T_r. \tag{7}$$

To evaluate the performance of the reduced system, an error system is defined (8), being the difference between the original and reduced systems. The interconnection of the error system is shown in Fig. 2. The resulting error system of two affine systems will remain affinely dependent on the parameter.

$$\Sigma_e = \left[\begin{array}{c|c} A_\theta & B_\theta \\ \hline C_\theta & D_\theta \end{array}\right] - \left[\begin{array}{c|c} A_{\theta r} & B_{\theta r} \\ \hline C_{\theta r} & D_{\theta r} \end{array}\right] \tag{8}$$

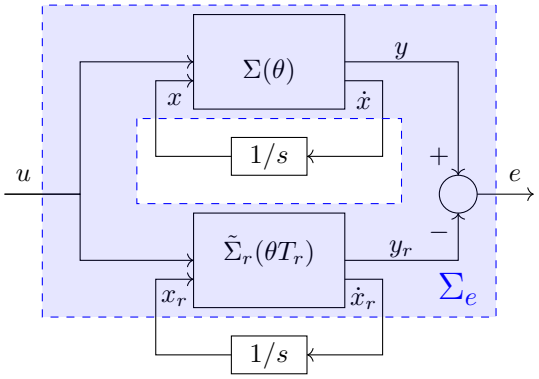

Figure 2: Interconnection of the parameter reduced error system. The blue area represents the error LFR.

In the LTI setting, error systems are evaluated in any of the well established and computable norms. Of these, the H_∞-, H_2- and Hankel-norm are commonly used in model reduction. For LPV systems the framework of norms has not had a lot of attention. In this paper we establish a notation and definition which may be applied to extend LTI norms to LPV norms. We define a notation convention for, average and peak parameter norms. To keep notation similar with standard LTI norms, the average is denoted as $p_{2,\bullet}$ and the peak by $p_{\infty,\bullet}$. The system norm is then used as the second subscript, e.g. the average H_2 norm over the parameter space is defined as the $p_{2,2}$-norm. In this work we define parametric norms of the H_∞, H_2 and H (Hankel) LTI norms as follows

$$\|\Sigma_e(\theta)\|_{p_{\infty,\infty}} := \max_{\theta\in\Theta} \|\Sigma_e(\theta)\|_{H_\infty} \tag{9a}$$

$$\|\Sigma_e(\theta)\|_{p_{\infty,2}} := \max_{\theta\in\Theta} \|\Sigma_e(\theta)\|_{H_2} \tag{9b}$$

$$\|\Sigma_e(\theta)\|_{p_{\infty,H}} := \max_{\theta\in\Theta} \|\Sigma_e(\theta)\|_{H} \tag{9c}$$

$$\|\Sigma_e(\theta)\|_{p_{2,\infty}} := \sqrt{\left(\frac{1}{V_\Theta}\int_\Theta \|\Sigma_e(\theta)\|_{H_\infty}\, d\theta\right)^2} \tag{9d}$$

$$\|\Sigma_e(\theta)\|_{p_{2,2}} := \sqrt{\left(\frac{1}{V_\Theta}\int_\Theta \|\Sigma_e(\theta)\|_{H_2}\, d\theta\right)^2} \tag{9e}$$

$$\|\Sigma_e(\theta)\|_{p_{2,H}} := \sqrt{\left(\frac{1}{V_\Theta}\int_\Theta \|\Sigma_e(\theta)\|_{H}\, d\theta\right)^2} \tag{9f}$$

The $p_{\infty,\infty}$- and $p_{\infty,H}$-norm are of particular interest in this work. The first results in the peak gain of the system over all parameters and therefore may be used to express the worst-case error of a reduced system. As stated in [16] the bounded real lemma inequality can be evaluated on the vertices of Θ for affine LPV systems with time-invariant θ. The $p_{\infty,\infty}$-norm, can therefore be evaluated over a finite number of points as

$$\|\Sigma_e(\theta)\|_{p_{\infty,\infty}} = \max_{w_i} \|\Sigma_e(w_i)\|_{H_\infty} \tag{10}$$

In section III-A a detailed discussion of the $p_{\infty,H}$-norm is given. The other norms will not be applied in this research and no algorithm for the computation of such norms are given, as they are not trivially found. Developing ways to evaluate such norms may show to be useful, particularly the $p_{2,2}$-norm as it is closely related to the interpretation of the H_2-norm, being the average gain of the system.

III. Hankel-norm reduction

In model reduction, gramian based approaches are widely applied as they are well understood and error bounds are easily obtained. For LTI systems the reachability $\mathcal{P}$ and observability $\mathcal{Q}$ gramians are found as the positive definite solutions to

$$A\mathcal{P} + \mathcal{P}A^T + BB^T = 0 \tag{11a}$$

$$A^T\mathcal{Q} + \mathcal{Q}A + C^TC = 0 \tag{11b}$$

The system gramians are then used to find optimal Hankel-norm approximations by balanced truncation. In the LPV framework one may define similar notions to find Hankel-norm approximations.

Theorem 1: For a time invariant LPV systems which is reachable, observable and stable for all parameter realisations, the reachability gramian is found as the solution $\mathcal{P}(\theta)$ to the Lyapunov equation

$$A(\theta)\mathcal{P}(\theta) + \mathcal{P}(\theta)A^T(\theta) + B(\theta)B^T(\theta) = 0 \tag{12a}$$

$$\mathcal{P}(\theta) \succ 0 \tag{12b}$$

Furthermore there exist a solution $P(\theta)$, from a family of solutions, satisfying the Lyapunov inequality

$$A(\theta)P(\theta) + P(\theta)A^T(\theta) + B(\theta)B^T(\theta) = 0 \tag{13a}$$

$$P(\theta) \succ 0 \tag{13b}$$

Moreover, all solutions of the inequality $P(\theta)$, constitute an upper bound on the solution of the equality $\mathcal{P}(\theta)$. That is, $P(\theta) \succeq \mathcal{P}(\theta) \succ 0$.
The proof is included in Appendix A. □

The preceding theorem is required because finding an exact solution to the Lyapunov equation is neither trivial nor tractable. In other work the gramian is often replaced by a static matrix [17]. This is possible since parameter dependence is not a requirement for making statements on stability. For parameter reduction, there must be parameter dependence, expressing changes in the system due to parameter variations. Therefore an affine dependence of the gramian (14) is proposed, resulting in the Lyapunov equation as given in (15).

$$\mathcal{P}(\theta) \preceq P^T\bar{\theta}_{n_x}^T = \bar{\theta}_{n_x}P \tag{14}$$

$$\bar{\theta}_{n_x}\left[\bar{A}^\theta P^T + P\bar{A}^{\theta^T} + \bar{B}^\theta\bar{B}^{\theta^T}\right]\bar{\theta}_{n_x}^T \preceq 0 \tag{15a}$$

$$\bar{\theta}_{n_x}P \succ 0 \tag{15b}$$

Due to the assumed affine structure of the system, $\mathcal{P}(\theta)$ is a convex function. Therefore the proposed affine gramian satisfies the inequality for all values of θ if and only if it is satisfied at the vertices. A formal proof of which is given in [18]. Note that this does not hold for general parametric dependence of the system or of the gramian. Furthermore we point out that an affine solution always exist. Proving this is trivial as a there exists a static solution for stable systems and static dependence is a subset of the proposed affine dependence.

Finding a solution to this Lyapunov inequality is done using linear matrix inequalities (LMI). Every vertex of the parameter

space results in two LMIs, (15a) and (15b). In case of simple bounding constraints on the parameters, the result is a total of $2^{n_\theta+1}$ LMIs. Like many problems in the LPV setting, this approach becomes intractable with growing parameter space. The difference with other problems is that there is no tuning involved in this process. Therefore there is a one time cost of calculating the gramian where, for example, in control synthesis multiple iterations are required to satisfy design constraints.

The previously presented method assumed static parameter dependence $\dot{\theta} = 0$. In case the parameter is dynamic but has limited rate of change, $\underline{\dot{\theta}} < \dot{\theta} < \bar{\dot{\theta}}$, the Lyapunov inequality may be adjusted according to (16). The resulting LMIs are again quadratic in θ, now with a constant offset. For affine LPV systems it still holds that the inequality is satisfied if it is satisfied on the vertices of the parameter space. The formal proof is again found in [18].

$$\dot{\mathcal{P}}(\theta) + \bar{A}_\theta \mathcal{P}(\theta) + \mathcal{P}(\theta) A_\theta^T + \bar{B}_\theta \bar{B}_\theta^T \tag{16a}$$

$$\Downarrow \text{ Affine gramian}$$

$$P\dot{\theta} + \bar{\theta}_{n_x} \left[\bar{A}^\theta P^T + P \bar{A^\theta}^T + \bar{B}^\theta \bar{B^\theta}^T \right] \bar{\theta}_{n_x}^T \tag{16b}$$

$$\Downarrow \text{ Bounded parameter velocity}$$

$$P\underline{\dot{\theta}}_{n_x} + \bar{\theta}_{n_x} \left[\bar{A}^\theta P^T + P \bar{A^\theta}^T + \bar{B}^\theta \bar{B^\theta}^T \right] \bar{\theta}_{n_x}^T \tag{16c}$$

$$P\bar{\dot{\theta}}_{n_x} + \bar{\theta}_{n_x} \left[\bar{A}^\theta P^T + P \bar{A^\theta}^T + \bar{B}^\theta \bar{B^\theta}^T \right] \bar{\theta}_{n_x}^T \tag{16d}$$

Extending the approach to parameters bounded in velocity increases the number of LMIs to $3^{n_\theta+1}$. By imposing an additional constraint $P_i \succ 0$ results in (16c)$\prec$(16d) and thus (16c) can be dropped. This reduces the number of LMIs to $2^{n_\theta+1} + n_\theta$. The previously presented methods to obtain an affine reachability gramian can be applied for finding an affine observability gramian Q as well.

A. Hankel-norm approximation

The gramians found using the method described in the preceding, can be used to find Hankel-norm approximations of an LPV system. Before going to the details of Hankel-norm approximation for LPV systems, we briefly explain the Hankel operator (convolution operator) for LTI systems and Hankel-norm. The Hankel operator of a linear system Σ is defined by

$$\mathcal{H} : \mathcal{L}_2(\mathbb{R}_-^{n_u}) \longmapsto \mathcal{L}_2(\mathbb{R}_+^{n_y}), u_- \longmapsto y_+$$

$$\text{where} \quad \mathcal{H}(u_-)(t) = \int_{-\infty}^{0} H(t-\tau)u(\tau)d\tau, t \in \mathbb{R}_+, \tag{17}$$

it maps the past inputs u_- into future outputs y_+. Here, H is the impulse response of Σ. The ℓ_2-induced norm of $\mathcal{H}$ is defined as

$$||\mathcal{H}||_{\mathcal{L}_2-\text{ind}} = \sup_{||u_-||_2<\infty} \frac{||y_+||_2}{||u_-||_2} = ||\Sigma||_H. \tag{18}$$

The quantity $||\Sigma||_H$ is named the Hankel-norm of the system Σ. Furthermore, the Hankel-norm is equal to the largest Hankel singular-value of $\mathcal{H}$, if Σ is stable: $||\Sigma||_H = \sigma_{\max}(\mathcal{H})$.

Lemma 1: Given a reachable, observable and stable system Σ of dimension n, the Hankel singular-values are equal to positive square roots the eigenvalues of the product of $\mathcal{PQ}$

$$\sigma_i(\Sigma) = \sqrt{\lambda_i(\mathcal{PQ})}, i = 1, ..., n \tag{19}$$

where $\mathcal{P}$ and $\mathcal{Q}$ are the infinite gramians of Σ.

For a time-invariant LPV system the Hankel operator may be defined for every $\theta \in \Theta$. It now maps the past inputs and parameters into future outputs.

$$\mathcal{H}_\theta : \mathcal{L}_2(\mathbb{R}_-^{n_u} \times \Theta) \longmapsto \mathcal{L}_2(\mathbb{R}_+^{n_y}),\ u_-, \theta \longmapsto y_+$$

$$\text{where} \tag{20}$$

$$\mathcal{H}_\theta(\theta, u_-)(t) = \int_{-\infty}^{0} H(\theta, t-\tau)u(\tau)d\tau, t \in \mathbb{R}_+, \theta \in \Theta,$$

It follows, because y_+ is dependent on θ, that the ℓ_2-induced norm of $\mathcal{H}_\theta$ is now also parameter dependent. By assumption Σ has been defined to be stable $\forall \theta \in \Theta$ and therefore the Hankel-norm can be expressed for every θ as

$$\|\Sigma(\theta)\|_H = \sigma_{\max}(\mathcal{H}_\theta) \tag{21}$$

Theorem 2: Given a reachable, observable and stable linear parameter-varying system $\Sigma(\theta)$ of dimension n and parameter realisation θ, the Hankel singular-values are upper bounded by the positive square roots of the eigenvalues of the product of $\bar{\theta}_{n_x} P Q^T \bar{\theta}_{n_x}^T$

$$\sigma_i(\Sigma(\theta)) \leq \sqrt{\lambda_i(\bar{\theta}_{n_x} P Q^T \bar{\theta}_{n_x}^T)}, i = 1, ..., n \tag{22}$$

where P and Q are the upper bounding affine gramians of $\Sigma(\theta)$.

The proof is given in Appendix B. □

In [19] it has been shown that balanced truncation can achieve optimal Hankel-norm approximation. The aim for LPV Hankel-norm approximation remains, minimising the loss between the full and reduced order models. To formalise this into an optimisation problem we first note that $\tilde{P} = (T^T \otimes I)P$ satisfies (15) for the transformed system (6). Therefore we can state that the Hankel singular-values are upper bounded by (23a), or equivalently (23b).

$$\tilde{\theta}_{n_x} \tilde{P} \tilde{Q}^T \tilde{\theta}_{n_x}^T \tag{23a}$$

$$\bar{\theta}_{n_x} (TT^T \otimes I) P Q^T (TT^T \otimes I) \bar{\theta}_{n_x}^T \tag{23b}$$

Expressing the Hankel singular-values in terms of original Gramians and parameter project matrix allows to derive a loss function, quadratic in θ. This loss function is then used to give a min-max optimisation as

$$\min_T \max_{\theta \in \Theta} \left\| \bar{\theta}_{n_x} \left[PQ^T - (TT^T \otimes I) P Q^T (TT^T \otimes I) \right] \bar{\theta}_{n_x}^T \right\|_2^2 \tag{24}$$

Due to the quadratic θ-dependency, we evaluate the parameters on the vertices of Θ only. Changing the optimisation to a finite optimisation over θ.

$$\min_T \max_{\omega_i} \left\| \bar{\theta}_{n_x} \left[PQ^T - (TT^T \otimes I) P Q^T (TT^T \otimes I) \right] \bar{\theta}_{n_x}^T \right\|_2^2 \tag{25}$$

From this equation it is clear that for full order approximations any orthonormal T results in a zero loss approximation.

For reduced models, where $\text{rank}(TT^T) = n_r < n_\theta$, loss occurs for systems that are observable, reachable and have no redundant parameters.

In LTI approximation the loss function is related to the energy of the error system since the Hankel singular-values are a measure of energy in each state receptively. For the proposed method the interpretation is more nuanced as the affine gramians constitute an upper bound on the actual gramians. If this upper bound is tight, the loss function is a good indication of the error. The tightness of this upper bound is heavily dependent on the system, and determining the tightness is not trivial. The insight into the $p_{\infty,\mathcal{H}}$-norm error determined from the affine gramian approximation is therefore questionable. For this reason we asses the approximation error in the relative $p_{\infty,\infty}$-norm (26) because it can be computed exactly.

$$\frac{\left\|\Sigma(\theta) - \tilde{\Sigma}_r(\theta T_r)\right\|_{p_{\infty,\infty}}}{\|\Sigma(\theta)\|_{p_{\infty,\infty}}} \tag{26}$$

IV. Sensitivity analysis

Another method for parameter reduction is sensitivity analysis, which is similar to principal component analysis [20], [21]. This method evaluates how sensitive the outputs are to small changes in the parameter values. It can be viewed as principal component analysis without the requirement of needing a typical parameter trajectory. Evaluation can be done in either the time or the frequency domain.

A. Frequency domain

Since it is assumed that $\dot{\theta} = 0$ the notion of transfer functions is still applicable, if $\dot{\theta} \neq 0$ this approach is will not be applicable. The transfer function of an affine static LPV system is given in (27). Here q is either z for discrete-time or s for continuous-time systems. The transfer sensitivity is defined as the Jacobian of the transfer function towards the parameters, as shown in (28).

$$H(\theta, s) = D_\theta + C_\theta(Iq - A_\theta q)^{-1}B_\theta \tag{27}$$

$$\begin{aligned} {}_\theta H(\theta, q) =& \frac{dD_\theta}{d\theta} + \frac{dC_\theta}{d\theta}(Iq - A_\theta q)^{-1}B_\theta \\ &+ C_\theta \frac{d(Iq - A_\theta q)^{-1}}{d\theta} B_\theta \\ &+ C_\theta(Iq - A_\theta q)^{-1}\frac{dB_\theta}{d\theta} \end{aligned} \tag{28}$$

The i^{th} element of the Jacobian can be represented as the system given in (29), where the colours correspond to separate elements of the Jacobian as in (28). It is easily observed that the resulting system again has affine dependency on the parameters.

$${}^i_\theta H(q, \theta) = \left[\begin{array}{cccc|c} A_\theta & 0 & 0 & 0 & B_\theta \\ 0 & A_\theta & A_i & 0 & 0 \\ 0 & 0 & A_\theta & 0 & B_\theta \\ 0 & 0 & 0 & A_\theta & B_i \\ \hline C_i & C_\theta & 0 & C_\theta & D_i \end{array}\right] = \left[\begin{array}{c|c} \mathcal{A}_{\theta,i} & \mathcal{B}_{\theta,i} \\ \hline \mathcal{C}_{\theta,i} & D_i \end{array}\right] \tag{29}$$

From the Jacobian an ordering of the parameters could be determined from the $p_{\infty,\infty}$-norm of each respective element. However, the transfer-sensitivity function does not take into account cross-correlations between parameters. Multiplying the transfer-sensitivity by its complex conjugate will result in a transfer-sensitivity covariance matrix (TSCM). The elements of this matrix are defined as

$$\Pi_{ij} := \left\| {}^i_\theta H(\theta, q)^* {}^j_\theta H(\theta, q) \right\|_{p_{\infty,\infty}} \tag{30}$$

This product of systems gives element i, j of the TSCM, and is equal to

$$\Pi_{i,j} = \left\| \left[\begin{array}{cc|c} \mathcal{A}^*_{\theta,i} & \mathcal{C}^*_{\theta,i}\mathcal{C}_{\theta,j} & \mathcal{C}^*_{\theta,i}D_j \\ 0 & \mathcal{A}_{\theta,j} & \mathcal{B}_{\theta,j} \\ \hline \mathcal{B}^*_{\theta,i} & D_i^*\mathcal{C}_{\theta,j} & D_i^*D_j \end{array}\right] \right\|_{p_{\infty,\infty}} \tag{31}$$

All elements of this system have affine dependency on θ, except for $\mathcal{C}^*_{\theta,i}\mathcal{C}_{\theta,j}$ which in general does not posses such a property. Still the $p_{\infty,\infty}$-norm can be evaluated over a finite set of points. This can be shown by extending the parameter space with all quadratic elements of θ. The system will then have affine dependence on the extended parameter space and thus can be evaluated over an extended convex hull. If the extended convex hull is not properly chosen it may lead to conservatism.

The resulting TSCM will be a symmetric $n_\theta \times n_\theta$ matrix. By taking the singular value decomposition (SVD) of this matrix, $\Pi = TST^*$, the parameter transformation matrix is found. This transformation orders the parameter directions in terms of transfer sensitivity covariance and is orthonormal, constituting a valid transformation as defined in (4).

B. Time domain

The second sensitivity analysis approach is based in the time domains. The proposed time-domain approach, unlike principal component analysis, does not require a typical trajectory of the scheduling variables. Instead the solution is written as time dependent equation. In discrete time the output evolution equation is given as

$$y(k) = C_\theta A_\theta^k x_0 + \left[\sum_{i=1}^k C_\theta A_\theta^{i-1} B_\theta u(k-i)\right] + D_\theta u(k) \tag{32}$$

The sensitivity function at time k is simply calculated as the Jacobian of this equation towards θ. For affine parameter dependency (33) can be calculated in terms of the stacked matrices $\bullet^\theta$ and state space matrices $\bullet_\theta$.

$$\begin{aligned} {}_\theta y(k) =& \frac{dC_\theta A_\theta^k}{d\theta} x_0 + \frac{dD_\theta}{d\theta} u(k) + \\ & \left[\sum_{i=1}^k \frac{dC_\theta A_\theta^{i-1} B_\theta}{d\theta} u(k-i)\right] \end{aligned} \tag{33}$$

In discrete time the sensitivity can be written as a row vector multiplied by a column vector stacking all inputs. Stacking all outputs, ${}_\theta y_0$, ${}_\theta y_1$ $\dots$ ${}_\theta y_{k_{\max}}$, into a column vector $Y =$. Taking the derivative of Y towards the i^{th} parameter gives a matrix multiplied by the input vector ${}^i M(\theta) U$. In general this matrix is parameter dependent and difficult to evaluate.

In the particular case where $A_\theta = A_0$ and either $C_\theta = C_0$ or $B_\theta = B_0$, the matrix ${}^iM(\theta)$ is parameter independent with a similar structure to

$$ {}^iM(\theta) = \begin{bmatrix} D_i & 0 & \dots & 0 \\ C_0A_0B_i & D_i & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_0A_0^{k_{\max}-1}B_i & C_0A_0^{k_{\max}-2}B_i & \dots & D_i \end{bmatrix} \tag{34} $$

To take cross correlation into account ${}^i_\theta Y$ is pre multiplied by ${}^j_\theta Y^T$, being equal to ${}^j_\theta Y^T {}^i_\theta Y = U^T {}^jM^T {}^iMU$. The largest singular values of ${}^jM^T {}^iM$ result in the peak gain between the sensitivity functions of the i^{th} and j^{th} parameter. The collection matrix of all the maximum singular values $\bar{\sigma}_{ij}$ gives an $n_p \times n_p$ matrix, the sensitivity covariance matrix (SCM).

$$ \bar{S} = \begin{bmatrix} \bar{\sigma}_{11} & \bar{\sigma}_{12} & \dots & \bar{\sigma}_{1n_\theta} \\ \bar{\sigma}_{21} & \bar{\sigma}_{22} & \dots & \bar{\sigma}_{2n_\theta} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\sigma}_{n_\theta 1} & \bar{\sigma}_{n_\theta 2} & \dots & \bar{\sigma}_{n_\theta n_\theta} \end{bmatrix} \tag{35} $$

From the SCM a similar approach is taken to derive the transformation matrix as with the TSCM, by taking the SVD. One note on this approach is the choice of $k_{\max}$ which should be chosen at least as large as the largest time constant of the system. Choosing $k_{\max}$ small will exclude dynamics and may lead to bad approximations.

V. Results

In this paper, two ways of achieving model reduction through parameter projections are proposed. In doing this a notation is proposed to express LPV system norms. Off the proposed norms, (36) are the most important for this work.

$$ \|\Sigma_e(\theta)\|_{p_{\infty,\infty}} := \max_{\theta\in\Theta} \|\Sigma_e(\theta)\|_{H_\infty} \tag{36a} $$

$$ \|\Sigma_e(\theta)\|_{p_{\infty,H}} := \max_{\theta\in\Theta} \|\Sigma_e(\theta)\|_{H} \tag{36b} $$

The parametric Hankel-norm (36b) is used to find a Hankel-norm approximation similar to LTI balanced truncation. It has been shown that the parametric Hankel-norm can be upper bounded by

$$ \sigma_{\max}(\Sigma(\theta)) \leq \sqrt{\lambda_{\max}(\bar{\theta}_{n_x} PQ^T \bar{\theta}_{n_x}^T)} \tag{37} $$

Using this upper bound, a min-max optimisation is constructed to find an optimal Hankel-norm approximation as

$$ \min_T \max_{\omega_i} \left\| \bar{\theta}_{n_x} \left[PQ^T - (TT^* \otimes I) PQ^T (TT^* \otimes I) \right] \bar{\theta}_{n_x}^T \right\|_2^2 \tag{38} $$

The parametric H_∞-norm (36a) is used for another parameter reduction method, sensitivity analysis as well as error evaluation. In sensitivity analysis, the derivative towards the parameters is used to identify parameter directions in which the output is most sensitive to changes. In the frequency domain it is expressed as the transfer-sensitivity covariance matrix (TSCM), of which the elements are defined as

$$ \Pi_{ij} := \left\| \frac{d}{d\theta_i} H(\theta, q)^* \frac{d}{d\theta_j} H(\theta, q) \right\|_{p_{\infty,\infty}} \tag{39} $$

To illustrate these methods of parameter reduction through sensitivity analysis and Hankel-norm approximation, two examples systems are used. The first system is an illustrative LPV system and the second a thermal model simulation.

A. Illustrative LPV system

For the illustrative system it is assumed that the parameters are between zero and one. To ensure stability of the system, all A_i are diagonal matrices having eigenvalues < 0. The nominal system is generated such that it is reachable and observable. The other matrices are generated randomly, which may lead to loss of reachability and observability. In practice the latter did not prove to be a problem. The generated system is of the form

$$ \dot{x}(t) = A_0x(t) + B_0u(t) + \sum_{i=1}^{5} A_ix(t) + B_iu(t) \tag{40a} $$

$$ y(t) = C_0x(t) + D_0u(t) + \sum_{i=1}^{5} C_ix(t) + D_iu(t) \tag{40b} $$

Having generated a random system as specified above, a Hankel-norm and TSCM approximation are determined. For these approximations, Fig. 3 shows the output evolution for inputs which is constant for $t \in [0\ 25]$ and zero elsewhere. The light blue lines are the full order model evolutions. To illustrate the accuracy of the reduced models, the errors are shown in Fig. 4. Determining the best approximation from these figures is difficult as it will change for a different parameter realisation. Fig. 5 shows the relative $p_{\infty,\infty}$-error of the reductions to be non-increasing with growing parameter order for both methods. It also shows that for this example, the sensitivity analysis outperforms the gramian based method.

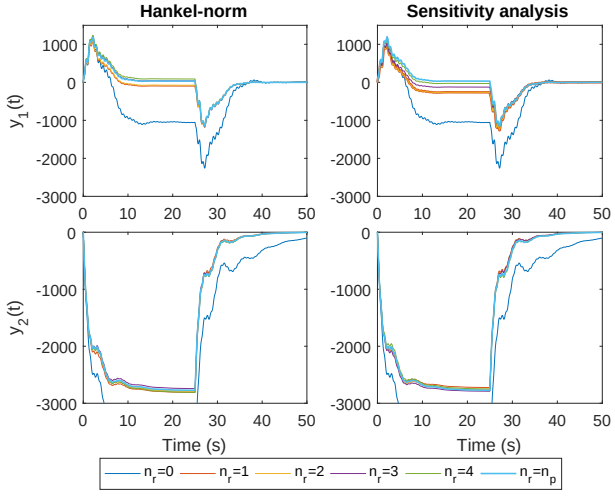

Figure 3: System evolution of the randomised LPV system, at a random parameter realisation, for different reduction orders and methods.

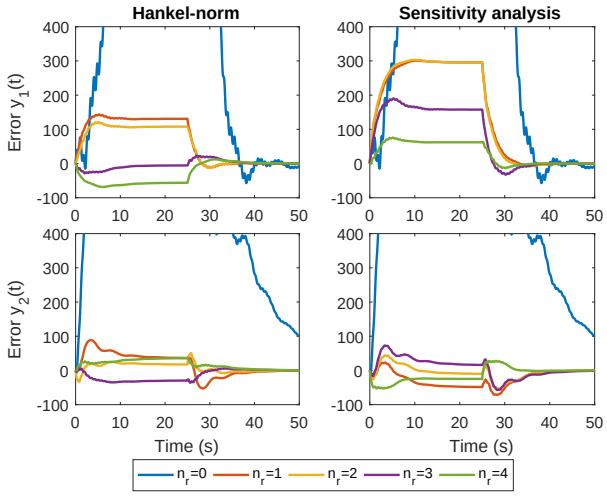

Figure 4: Output error of the randomised LPV system, at a random parameter realisation, for different reduction orders

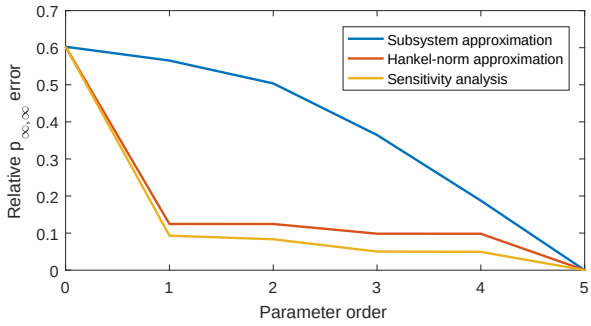

Figure 5: Relative $p_{\infty,\infty}$-error of the randomised LPV system for different parameter orders and approximation methods.

B. Thermal simulation

The second system is a thermal simulation consisting of five coupled blocks, all having parameter dependent heat capacity. The system is illustrated in Fig. 6 and a state-space representation is generated of the form (41) is generated using COMSOL, having 45 states, 2 inputs & outputs and 5 parameters.

$$\dot{x} = A_0 x + B_0 u + \sum_{i=1}^{5} (A_i \theta_i x + B_i \theta_i u) \tag{41a}$$

$$y = C_0 x \tag{41b}$$

To evaluate the error of the reduced model, a simulation with constant inputs from $t \in [0\ 250]$ and zero afterwards. For a particular realisation of the parameter, the error between the reduced models and full order model is shown in Fig. 7. Clearly the non-parametric model $n_r = 0$ performs the worst of all models showing that the zero model is not a good approximation of the system. The best performing model is the $n_r = 4$ model as the error is almost zero. However, it cannot be concluded from one parameter realisation what the performance of the reduction is. Therefore a global error bound is used to conclude performance.

In Fig. 8 the relative $p_{\infty,\infty}$-error is plotted for different reduction orders. As a comparison subsystem Hankel-norm

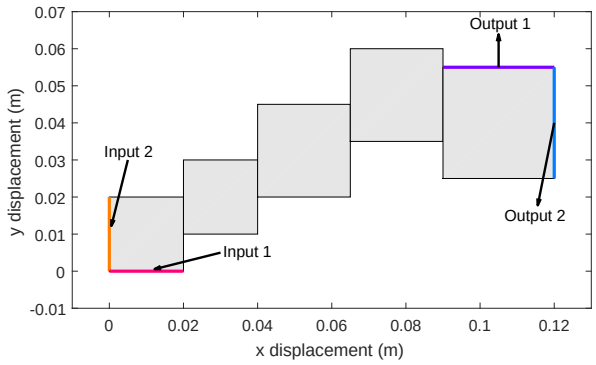

Figure 6: Thermal model with five different material blocks. Orange and pink are heat inputs. Purple and blue are outputs.

approximation is used where reduction is based singular values of $P_i Q_i$. The $n_p - n_r$ parameters having the smallest 'subsystem' Hankel-norm are removed. This figure illustrates that reducing the system using sensitivity analysis is comparable to reduction in subsystem Hankel-norm. It is also clear that approximation in the $p_{\infty,\mathcal{H}}$-norm using transformation optimisation yields improved results for $n_r < 4$. For approximation of $n_r = 4$ the Hankel-norm optimisation method performs worse in $p_{\infty,\infty}$ error. This is due to a combination of issues, the non-convexity of the optimisation (25) and the approximation loss of affine gramian (14).

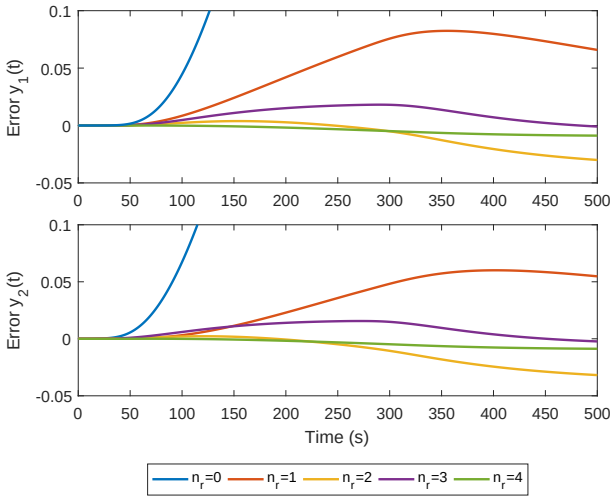

Figure 7: Simulation error of the thermal system for different parameter orders at a randomly selected parameter in the parameter space.

For both systems computation of the gramians takes in the order of 1000 seconds, on a dual core machine. This indicates that the LMI implementation using YALMIP [22] is not efficient but acceptable for small systems. As a practical note, it should be taken care of to have A_0, B_0 and C_0 of similar magnitude to avoid numerical trouble. Since the optimisation in (24) has not shown to be convex, it is ran using different initial condition to find a close approximation of the Hankel singular-values. Because of this non-convexity, the resulting transformation matrix might not be the global optimal approximation.

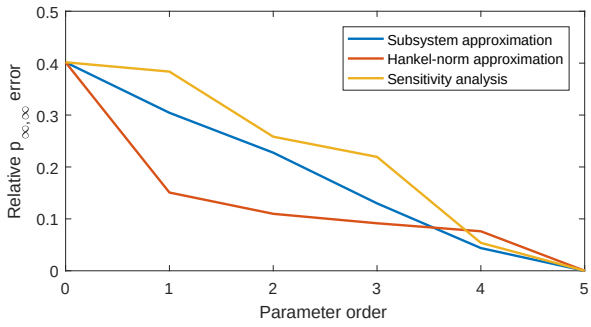

Figure 8: Relative $p_{\infty,\infty}$-error of the thermal system for different parameter orders and different reduction methods.

VI. Conclusion

The methods developed in this paper show that model reduction for LPV systems using parameter reduction can result in accurate models using a parameter space of lower dimension. The first method of sensitivity analysis is simple to apply but it is based on notions which may be difficult to apply for systems which have time varying parameters. The more general approach is to extend LTI Hankel-norm approximation, which is well understood, to LPV systems. An affine upper bound on the gramian is used to make it computationally tractable while including parameter dependency. Using the affine gramian to approximate the system in terms of maximal Hankel-norm loss is made. Both methods achieve reduced models well approximating example systems. From the results it cannot be concluded which of the methods results in better approximations, as it is dependent on the system

To evaluate the performance and error bounds of the models, this work introduced multiple parametric norms. The question which norm to use is dependent on the system and application, but computation is not always trivial. The peak gain over all parameters and frequencies ($p_{\infty,\infty}$-norm) is used in the developed methods as it shows worst case performance loss.

Although these methods are complete in the sense of achieving parameter reduction, and evaluation thereof, there are several improvements to be made. To apply the gramian method to large scale systems the computation of the gramians must be improved in terms of computational cost and efficiency. For systems where computation of the gramians is possible, it still remains to show the tightness of the bound it imposes on the Hankel singular-values. The optimisation of the parametric Hankel singular-value reduction remains open for improvement due to the non-convexity of the problem. Finally evaluation of other parametric norms yields interesting questions to be answered.

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APPENDIX A
PROOF OF THE LYAPUNOV INEQUALITY AS AN UPPER BOUND TO THE LYAPUNOV EQUALITY

Consider a stable LTI system with stable matrix A. Lyapunov equation theorem shows that there is a unique solution $P \succ 0$ to

$$A\mathcal{P} + \mathcal{P}A^T + Q = 0 \tag{42}$$

where $Q = BB^T \succ 0$. The same system admits a solution space $P \in \tilde{\mathcal{P}}$, being all solutions that satisfy

$$AP + PA^T + Q \preceq 0 \tag{43}$$

The equality can be rewritten and substituted into the inequality to give

$$A(P - \mathcal{P}) + (P - \mathcal{P})A^T \preceq 0 \tag{44}$$

From Lyapunov equation theorem and stable A, we have $P - \mathcal{P} \succeq 0$ as the solution of the above inequality. Extending to stable, observable and reachable time-invariant LPV systems, it suffices to observe that for every $\theta \in \Theta$ a solution can be found to the Lyapunov (in)equality concluding the proof. □

APPENDIX B
PROOF OF THE UPPER BOUND ON THE HANKEL SINGULAR-VALUES FROM THE UPPER BOUNDING GRAMIANS

From [23] we take two properties of eigenvalue arithmetic of symmetric matrices $A, B \in \mathbb{R}^{n_x \times n_x}$,

$$\begin{aligned} \lambda_i(A + B) \geq 0 \text{ if } \lambda_i(A) \geq 0 \text{ and } \lambda_i(B) \geq 0, \\ \lambda_i(AB) \geq 0 \text{ if } \lambda_i(A) \geq 0 \text{ and } \lambda_i(B) \geq 0, \end{aligned} \tag{45}$$

for $i = 1, ..., n$. For clarity we drop θ, and define a function F below:

$$\begin{aligned} F &:= PQ - \mathcal{P}\mathcal{Q}, \\ &= (P - \mathcal{P})(Q - \mathcal{Q}) + \mathcal{P}(Q - \mathcal{Q}) + (P - \mathcal{P})\mathcal{Q}. \end{aligned} \tag{46}$$

Given $P(\theta) \succeq \mathcal{P}(\theta) \succ 0$ and $Q(\theta) \succeq \mathcal{Q}(\theta) \succ 0$, applying properties (45) and *Theorem 1* to (46) yields $\lambda_i(F) \geq 0$, $i = 1, \ldots, n$. With the properties of Hankel matrix $PQ = Q^{\frac{1}{2}}PQ^{\frac{1}{2}}$, for the symmetric $F = (Q^{1/2}PQ^{1/2} - \mathcal{Q}^{1/2}P\mathcal{Q}^{1/2}) = F^T$ it is holds that the eigenvalues of F are positive. Therefore, $PQ \succeq \mathcal{P}\mathcal{Q}$ is concluded. Next consider the eigenvector x_1 associated to the largest eigenvalue of $\mathcal{P}\mathcal{Q}$. Then the following holds

$$x_1^T\mathcal{P}\mathcal{Q}x_1 \leq x_1^T(PQ)x_1 \Rightarrow \lambda_1(\mathcal{P}\mathcal{Q}) \leq \frac{x_1^T(PQ)x_1}{||x_1||_2^2}. \tag{47}$$

By the definition of eigenvalue decomposition, we have

$$\frac{x_1^T(PQ)x_1}{||x_1||_2^2} \leq \sup_{y_1} \frac{y_1^T(PQ)y_1}{||y_1||_2^2} = \lambda_{\max}(PQ), \tag{48}$$

here y_1 is the eigenvector associated to the largest eigenvalue of PQ. Thus, the $\lambda_{\max}(\mathcal{P}\mathcal{Q}) \leq \lambda_{\max}(PQ)$ is proved. □