BACHELOR

Rational binomial coefficients
behaviour and collisions

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Bachelor Final Project - Rational Binomial Coefficients: Behaviour and Collisions

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Abstract

In this paper we will analyze the behaviour of rational binomial coefficients as well as finding some ways to find rational binomial collisions. When analyzing the behaviour, we will state a concrete formula which can be worked with as well as an approximation using Stirling's approximation of factorials. Also, we will find an expression of the denominator, indirectly also giving us an expression for the numerator. The collision-problem will be tacked in three ways: trivial collisions, setting the numerator of the binomials and the utility of elliptic curves. The results in this article may contribute to a better understanding of rational binomial collisions as well as provide some nice properties which can be used in number theory.

1 Introduction

The binomial coefficient is an often seen mathematical object, most commonly known for its role in combinatorical problems. It also makes appearances outside of the field of combinatorics, for instance when speaking about the binomial expansion.
The coefficient can be interpreted in many different ways. One of them is the amount of different sequences consisting of k A's and $n-k$ B's. Likewise, it can be thought of as the number of different routes between two points in a grid of which k steps need to be made west and n need to be made to the north. These are just some examples, but far from all applications.
The ordinary binomial coefficient is defined by:

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n}{1} \cdot \frac{n-1}{2} \cdots \frac{n-k+1}{k} \ .$$

Where $n, k \in \mathbb{N}$. Many studies have already explored the many properties of the binomial coefficient. We will not do that here. Instead, we are interested in the behaviour of the coefficient if $n \in \mathbb{Q}$.
In this report we will start with analyzing the behaviour of the coefficient if $n \in \mathbb{Q}$. We will pay special attention to the overall number, but also to the denominator specifically. Indirectly, we thus also look at the numerator specifically.
Next, we are interested in so-called *binomial collisions.* These occur when the value of the coefficients are equal in value. Inspired by *binomial collisions and near collisions* [1], this report will explore the collisions of rational binomials.
A common method of exploration in this report will be the following approach: firstly a property is explored by adding a half to the n in the binomial coefficient above. Later, this will be generalized to any fraction.

2 Add a halve

We start by taking the following binomial coefficient where $n \in \mathbb{Z}$:

$$\binom{n+\frac{1}{2}}{k}.$$

These binomial coefficients have already been defined, namely as follows:

$$\binom{n+\frac{1}{2}}{k} = \frac{n+\frac{1}{2}}{1} \cdot \frac{n-1+\frac{1}{2}}{2} \cdots \frac{n-k+\frac{3}{2}}{k}\ .$$

In this report, we will sometimes use the notation $C[n,k]$ to denote the coefficient. Please do note that if the additional fraction is not $\frac{1}{2}$, but any fraction, we can find the value of the coefficient in a similar way. Our goal in this section is to derive some nice results, as well as get a feeling of how the binomial coefficient now behaves. In the next section, we will generalize the results we obtain in this section for any fraction in general if possible.

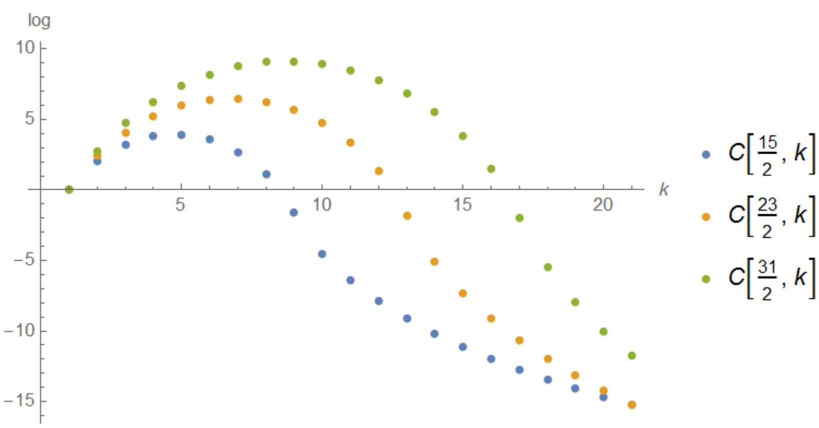


Figure 1: Plots for $n = 7$, $n = 11$ and $n = 15$

Note that the vertical axis displays a logarithmic value. In the figure above, we have plotted the absolute value of several coefficients with fixed n against several values of k. In the plot, $C[\frac{15}{2},k]$ represents the binomial coefficient $\binom{\frac{15}{2}}{k}$. Similarly, $C[\frac{23}{2},k]$ represents $\binom{\frac{23}{2}}{k}$ and $C[\frac{31}{2},k]$ represents $\binom{\frac{31}{2}}{k}$. The general behaviour of the coefficients seems the same. Firstly they increase, only to decrease and converge to 0 as the value of k increases (since x goes to 0 when $\log(x)$ goes to minus infinity).
This behavior is not odd or unexpected. Writing out a sequence shows that ultimately, we multiply with numbers less then 1, so naturally, the sequence is bound to decrease in absolute value.

2.1 A general formula

The definition given is hard to properly work with. That is why we firstly like to derive a formula instead of a definition to work with. We will derive a formula in two parts, one for the case where $n \geq k$, and one for when $n < k$. The reason why we do this will become apparent when we get there. Firstly the case $n \geq k$:

$$\begin{aligned}
\binom{n+\frac{1}{2}}{k} &= \frac{n+\frac{1}{2}}{1} \cdot \frac{n-1+\frac{1}{2}}{2} \cdots \frac{n-k+\frac{3}{2}}{k} \\
&= \frac{1}{k!} \cdot \left(n+\frac{1}{2}\right) \cdot \left(n-\frac{1}{2}\right) \cdots \left(n-k+\frac{3}{2}\right) \\
&= \frac{1}{2^k k!} \cdot (2n+1) \cdot (2n-1) \ldots (2n-2k+3) \\
&= \frac{1}{2^k k!} \cdot \frac{(2n+1) \cdot 2n \cdot (2n-1) \cdot (2n-2) \ldots (2n-2k+3) \cdot (2n-2k+2)}{2n \cdot (2n-2) \ldots (2n-2k+2)} \\
&= \frac{1}{2^k k!} \cdot \frac{(2n+1)!}{(2n-2k+1)!} \cdot \frac{1}{2n \cdot (2n-2) \ldots (2n-2k+2)} \\
&= \frac{1}{2^k k!} \cdot \frac{(2n+1)!}{(2n-2k+1)!} \cdot \frac{1}{2^k \cdot n \cdot (n-1) \ldots (n-k+1)} \\
&= \frac{1}{2^k k!} \cdot \frac{(2n+1)!}{(2n-2k+1)!} \cdot \frac{(n-k)!}{2^k \cdot n!} \\
&= \frac{1}{4^k k!} \cdot \frac{(2n+1)!}{(2n-2k+1)!} \cdot \frac{(n-k)!}{n!}
\end{aligned}$$

This is true only if $n \geq k$, as $(n-k)!$ would otherwise give rise to problems, though if we were to replace any of such 'problematic' terms by an approximation by using e.g. Stirling's formula, we might still be able to utilize this function for the case $n < k$. Still, we handle this problem as a separate case and try to derive a separate result. Now we try to find a similar result for $n < k$:

$$\begin{aligned}
\binom{n+\frac{1}{2}}{k} &= \binom{n+\frac{1}{2}}{n} \cdot \frac{\frac{1}{2}}{n+1} \cdot \frac{-\frac{1}{2}}{n+2} \cdots \frac{n-k+\frac{3}{2}}{k} \\
&= \binom{n+\frac{1}{2}}{n} \cdot \frac{1}{2n+2} \cdot \frac{(n+1)!}{k!} \cdot \left(-\frac{1}{2} \cdot -\frac{3}{2} \cdots \left(n-k+\frac{3}{2}\right)\right) \\
&= \binom{n+\frac{1}{2}}{n} \cdot \frac{(-1)^{n-k-1}}{2n+2} \cdot \frac{(n+1)!}{k!} \cdot \frac{1}{2^{k-n-1}} \cdot (1 \cdot 3 \ldots (2k-2n-3)) \\
&= \binom{n+\frac{1}{2}}{n} \cdot \frac{(-1)^{n-k-1}}{2n+2} \cdot \frac{(n+1)!}{k!} \cdot \frac{1}{2^{k-n-1}} \cdot \frac{(2k-2n-2)!}{2 \cdot 4 \ldots (2k-2n-2)} \\
&= \binom{n+\frac{1}{2}}{n} \cdot \frac{(-1)^{n-k-1}}{2n+2} \cdot \frac{(n+1)!}{k!} \cdot \frac{1}{2^{k-n-1}} \cdot \frac{(2k-2n-2)!}{2^{k-n-1} \cdot (k-n-1)!} \\
&= \frac{1}{4^n n!} \cdot \frac{(2n+1)!}{n!} \cdot \frac{(-1)^{n-k-1}}{2n+2} \cdot \frac{(n+1)!}{k!} \cdot \frac{1}{4^{k-n-1}} \cdot \frac{(2k-2n-2)!}{(k-n-1)!} \\
&= \frac{(2n+1)!}{4^{k-1} \cdot 2 \cdot n! \cdot k!} \cdot \frac{(-1)^{k-n-1} \cdot (2k-2n-2)!}{(k-n-1)!}
\end{aligned}$$

So ultimately, we receive the following result:

$$\binom{n+\frac{1}{2}}{k} = \begin{cases} \dfrac{1}{4^k k!} \cdot \dfrac{(2n+1)! \cdot (n-k)!}{(2n-2k+1)! \cdot n!} & \text{if } n \geq k \\[2ex] \dfrac{(-1)^{k-n-1} \cdot (2n+1)! \cdot (2k-2n-2)!}{4^{k-1} \cdot 2 \cdot n! \cdot k! \cdot (k-n-1)!} & \text{if } n < k \end{cases} . \tag{1}$$

2.2 An approximate formula

Now that we have constructed a more usable formula with which we can calculate the value of the binomial coefficients, we want to implement it to the observations stated above. However, regardless which of n or k is larger, we need to first calculate 5 factorials, as both parts of the formula contain a factorial. This is a lot of work. Therefore, we will first use *Stirling's Theorem* to approximate the value of the coefficient using the formula above.

Theorem (Stirling's approximation): $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

The proof can be found in many books on analysis and will be left out for the moment. We now implement it in the formula found above. Firstly for $n \geq k$:

$$\begin{aligned}
\binom{n+\frac{1}{2}}{k} &= \frac{1}{4^k k!}\cdot\frac{(2n+1)!\cdot(n-k)!}{(2n-2k+1)!\cdot n!}\\
&\approx \frac{1}{4^k}\cdot\frac{e^k}{k^k}\cdot\frac{(2n+1)^{2n+1}}{e^{2n+1}}\cdot\frac{(n-k)^{n-k}}{e^{n-k}}\cdot\frac{e^{2n-2k+1}}{(2n-2k+1)^{2n-2k+1}}\cdot\frac{e^n}{n^n}\cdot\sqrt{\frac{2\pi(2n+1)\cdot 2\pi(n-k)}{2\pi k\cdot 2\pi(2n-2k+1)\cdot 2\pi n}}\\
&= \frac{1}{4^k k^k}\cdot\frac{(2n+1)^{2n+1}}{n^n}\cdot\frac{(n-k)^{n-k}}{(2n-2k+1)^{2n-2k+1}}\cdot\frac{e^{k+(2n-2k+1)+n}}{e^{2n+1+(n-k)}}\cdot\sqrt{\frac{(2n+1)(n-k)}{2\pi k\cdot n(2n-2k+1)}}\\
&= \frac{1}{4^k k^k}\cdot\frac{2n+1}{2n-2k+1}\cdot\frac{(2n+1)^{2n}}{n^n}\cdot\frac{(n-k)^{n-k}}{(2n-2k+1)^{2n-2k}}\cdot\sqrt{\frac{(2n+1)(n-k)}{2\pi k\cdot n(2n-2k+1)}}\\
&\approx \frac{1}{4^k k^k}\cdot\frac{2n+1}{2n-2k+1}\cdot\frac{(4n+4)^n}{(4n-4k+4)^{n-k}}\cdot\sqrt{\frac{(2n+1)(n-k)}{2\pi k\cdot n(2n-2k+1)}}\\
&= \frac{1}{k^k}\cdot\frac{2n+1}{2n-2k+1}\cdot\frac{(n+1)^n}{(n-k+1)^{n-k}}\cdot\sqrt{\frac{(2n+1)(n-k)}{2\pi k\cdot n(2n-2k+1)}}
\end{aligned}$$

Similarly, for the other case:

$$\begin{aligned}
\binom{n+\frac{1}{2}}{k} &= \frac{(2n+1)!\cdot(-1)^{k-n-1}\cdot(2k-2n-2)!}{4^{k-1}\cdot 2\cdot n!\cdot k!\cdot(k-n-1)!}\\
&\approx \frac{(-1)^{k-n-1}\cdot(2n+1)^{2n+1}\cdot(2k-2n-2)^{2k-2n-2}\cdot e^n\cdot e^k\cdot e^{k-n-1}}{4^{k-1}\cdot 2\cdot n^n\cdot k^k\cdot(k-n-1)^{k-n-1}\cdot e^{2n+1}\cdot e^{2k-2n-2}}\cdot\sqrt{\frac{2\pi(2n+1)\cdot 2\pi(2k-2n-2)}{2\pi n\cdot 2\pi k\cdot 2\pi(k-n-1)}}\\
&= \frac{(-1)^{k-n-1}\cdot(2n+1)^{2n+1}\cdot(2k-2n-2)^{2k-2n-2}}{4^{k-1}\cdot 2\cdot n^n\cdot k^k\cdot(k-n-1)^{k-n-1}}\cdot\sqrt{\frac{2n+1}{\pi nk}}\\
&\approx \frac{(-1)^{k-n-1}}{4^{k-n-1}k^k}\cdot n(4n+4)^n\cdot 2^{2k-2n-2}\cdot(k-n-1)^{k-n-1}\cdot\sqrt{\frac{2n+1}{\pi nk}}\\
&= \frac{(-1)^{k-n-1}\cdot n(4n+4)^n\cdot(k-n-1)^{k-n-1}}{4^n\cdot k^k}\cdot\sqrt{\frac{2n+1}{\pi nk}}
\end{aligned}$$

So ultimately, we receive the following formula to approximate the value of a rational binomial coefficient:

$$\binom{n+\frac{1}{2}}{k} \approx \begin{cases} \frac{1}{k^k}\cdot\frac{2n+1}{2n-2k+1}\cdot\frac{(n+1)^n}{(n-k+1)^{n-k}}\cdot\sqrt{\frac{(2n+1)(n-k)}{2\pi k\cdot n(2n-2k+1)}} & \text{if } n \geq k \\ \frac{(-1)^{k-n-1}\cdot n(4n+4)^n\cdot(k-n-1)^{k-n-1}}{4^n\cdot k^k}\cdot\sqrt{\frac{2n+1}{\pi nk}} & \text{if } n < k \end{cases}. \qquad (2)$$

3 Any fraction

The first thing we want to do now that we have found a nice formula for the case $\binom{n+\frac{1}{2}}{k}$, is to generalize this. Can we find a nice formula for $\binom{n+\frac{p}{q}}{k}$, or is it merely luck that we were able to derive results when adding $\dfrac{1}{2}$?
Pick any $n, p, q, k \in \mathbb{N}$, such that q does not divide p and $p < q$. We make these constraints such that the fraction cannot be simplified and therefore gives unique results.
Like we did for the case $\frac{1}{2}$, we start from the definition:

$$\begin{aligned}\binom{n+\frac{p}{q}}{k} &= \frac{n+\frac{p}{q}}{1} \cdot \frac{n-1+\frac{p}{q}}{2} \cdots \frac{n-k+1+\frac{p}{q}}{k} \\ &= \frac{1}{k!} \cdot \left(n+\frac{p}{q}\right) \cdot \left(n-1+\frac{p}{q}\right) \cdots \left(n-k+1+\frac{p}{q}\right) \\ &= \frac{1}{k! \cdot q^k} \cdot (nq+p) \cdot (nq-q+p) \dots (nq-(k-1)q+p)\end{aligned}$$

Now substitute $nq + p$ by x:

$$= \frac{1}{k! \cdot q^k} \cdot x \cdot (x-q) \dots (x-(k-1)q)$$

We note that this multiplication is similar to the falling factorial [5]. Therefore we can rewrite this to:

$$\begin{aligned}&= \frac{1}{k! \cdot q^k} \sum_{i=0}^{k} s(k,i) x^i q^{k-i} \\ &= \frac{1}{k! \cdot q^k} \sum_{i=0}^{k} s(k,i)(nq+p)^i q^{k-i}\end{aligned}$$

Here, $s(k, i)$ denotes the *Stirling number of the first kind*. The equality holds as a property of the falling factorial. This result cannot easily be converted into an easier to use formula and will therefore be left as it is. The result itself is nice, but may be hard to work with. Nevertheless, a computer will undoubtedly know the Stirling numbers for many of the binomials we can ask and therefore, this formula is reasonably easy to implement. A table containing the first Stirling numbers can be found in appendix A.

4 Behavior of the numerator and denominator

When working with rational binomials, we have seen seen rational results. As a result, we are interested in the size of the numerator and denominator. If we can find an expression for one, we also have the other. This is due to the fact that we already have a formula for the entire coefficient. Multiplying the whole thing by the denominator results in the numerator, and dividing the numerator by the whole thing results in the denominator.

It is a known result that the denominator of $\binom{n+\frac{1}{2}}{k}$ is $2^{2k-[k]_2}$, where $[k]_2$ is the binary weight of k.

Proof: $\binom{n+\frac{1}{2}}{k} = \frac{n+\frac{1}{2}}{1} \cdot \frac{n-\frac{1}{2}}{2} \cdots \frac{n-k+\frac{3}{2}}{k} = \frac{(2n+1)(2n-1)\dots(2n-2k+3)}{2^k k!}$.
It may be obvious that the numerator is not divisible by 2, as we only multiply odd numbers there. Therefore the 2^k will remain in the denominator. Most of $k!$ will divide away, as is the case too with the integer-version of the coefficient. Only the factors 2 will not. We are now left with the question how many factors 2 are in $k!$. Every second number of $k!$ has a factor two. If we write k in base 2, then $k = k_0 + 2k_1 + 4k_2 + 8k_3 + 16k_4 + \dots$, with $k_i \in \{0, 1\}$. Then:

$$\left\lfloor \frac{k}{2} \right\rfloor = k_1 + 2k_2 + 4k_3 + 8k_4 + \dots$$

Likewise, every fourth number in $k!$ contains an additional 2, and every eighth again contain an extra. We find:

$$\left\lfloor \frac{k}{4} \right\rfloor = k_2 + 2k_3 + 4k_4 + \dots$$

$$\left\lfloor \frac{k}{8} \right\rfloor = k_3 + 2k_4 + \dots$$

$$\left\lfloor \frac{k}{16} \right\rfloor = k_4 + \dots$$

Now, the amount of factors 2 in $k!$ thus equals the sum of these all, so

$$\begin{aligned}
\left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{k}{8} \right\rfloor + \left\lfloor \frac{k}{16} \right\rfloor + \dots &= k_1 + 3k_2 + 7k_3 + 15k_4 + \dots \\
&= (1-1)k_0 + (2-1)k_1 + (4-1)k_2 + (8-1)k_3 + (16-1)k_4 + \dots \\
&= k_0 + 2k_1 + 4k_2 + 8k_3 + 16k_4 + \dots - k_0 - k_1 - k_2 - k_3 - k_4 - \dots \\
&= k - [k]_2
\end{aligned}$$

Thus, we have 2^k and $2^{k-[k]_2}$ in the denominator, resulting in $2^{2k-[k]_2}$. ∎

It can actually be shown that similar results are true for any prime p, such that

$$\binom{n+\frac{q}{p}}{k} = \frac{(np+q)\cdot((n-1)p+q)\cdot\ldots\cdot((n-k+1)p+q)}{p^k k!} .$$

After we have proven that comparable expressions can be derived for any prime, we will generalize to any given number, albeit prime or not.
Similarly to the case where $p = 2$, we are left asking ourselves how many factors p there are in $k!$.
The approach is the same also:
We first begin by defining $[k]_p$. Write k in base p, so $k = k_0 + pk_1 + p^2k_2 + \dots$, then $[k]_p = k_0 + k_1 + k_2 + \dots$ is the analogue of the binary weight used above.

$$k = k_0 + pk_1 + p^2k_2 + p^3k_3 + p^4k_4 + \dots$$
$$\left\lfloor \frac{k}{p} \right\rfloor = k_1 + pk_2 + p^2k_3 + p^3k_4 + \dots$$
$$\left\lfloor \frac{k}{p^2} \right\rfloor = k_2 + pk_3 + p^2k_4 + \dots$$
$$\left\lfloor \frac{k}{p^3} \right\rfloor = k_3 + pk_4 + \dots$$
$$\left\lfloor \frac{k}{p^4} \right\rfloor = k_4 + \dots$$

Then adding these all results in

$$\begin{aligned}
&\left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{k}{p^2} \right\rfloor + \left\lfloor \frac{k}{p^3} \right\rfloor + \left\lfloor \frac{k}{p^4} \right\rfloor + \dots \\
&\qquad = k_1 + (p+1)k_2 + (p^2+p+1)k_3 + (p^3+p^2+p+1)k_4 + \dots \\
&\qquad = \frac{1-1}{p-1}k_0 + \frac{p-1}{p-1}k_1 + \frac{p^2-1}{p-1}k_2 + \frac{p^3-1}{p-1}k_3 + \frac{p^4-1}{p-1}k_4 + \dots \\
&\qquad = \frac{1}{p-1}(k_0 + pk_1 + p^2k_2 + p^3k_3 + p^4k_4 + \dots - k_0 - k_1 - k_2 - k_3 - k_4 - \dots) \\
&\qquad = \frac{1}{p-1}(k - [k]_p)
\end{aligned}$$

Therefore, the denominator would now be $p^k \cdot p^{\frac{1}{p-1}(k-[k]_p)}$.

From this result, the step towards all other added fractions is easily made. Say we pick a non-prime number r, as the value of the denominator of the fraction we add in the coefficient. Then we can decompose it into its prime factorization: $r = p_1^{a_1} p_2^{a_2} \dots p_n^{a_n}$, $p_1 \neq p_2 \neq \dots \neq p_n$, $a_1, a_2, \dots, a_n \in \mathbb{N}$. The denominator would contain r^k, as we multiply k times by a fraction with a factor r in the denominator. We now ask ourselves the question how many times each different prime of p_1, p_2, $\dots$, p_n appears in $k!$. This is the result we derived above. Thus, the denominator is

$$r^k \cdot \prod_{i=1}^{n} p_i^{\alpha_i}, \text{ with } \alpha_i = \frac{1}{p_i - 1}(k - [k]_{p_i}) \ . \tag{3}$$

Checking these results with Wolfram Alpha gives us a confirmation that these results are correct. The code is added in the appendix.

As we stated at the beginning of this section, we now not only have the denominator, but also the numerator. Multiplying the entire coefficient by its denominator results in the numerator.

5 Analysis of the minimum and maximum

As stated earlier, we are interested in the minimum and maximum among other things. The approximation-formula may be rather hard to derive nice results from. It is because of that reason that we will first use a somewhat rougher version of Stirling's approximation, namely, the one forgetting about the root-term.

We firstly note that we know the coefficient converges to zero. This means that ultimately the denominator becomes significantly larger than the numerator. We therefore want the numerator only. As always, we start by adding the fraction $\frac{1}{2}$ to the upper number of the binomial coefficient and analyze the behavior of the numerator. We firstly start with the maximum it attains when $n > k$:

$$\begin{aligned}
\binom{n+\frac{1}{2}}{k} \cdot 2^{2k-[k]_2} &= \frac{1}{4^k k!} \cdot \frac{(2n+1)! \cdot (n-k)!}{(2n-2k+1)! \cdot n!} \cdot 2^{2k-[k]_2} \\
&\approx \frac{1}{k^k} \cdot \frac{(2n+1)^{2n+1}(n-k)^{n-k} \cdot e^{2n-2k+1} \cdot e^n \cdot e^k}{(2n-2k+1)^{2n-2k+1} n^n \cdot e^{2n+1} \cdot e^{n-k}} \\
&= \frac{1}{k^k} \cdot \frac{(2n+1)^{2n+1}(n-k)^{n-k}}{(2n-2k+1)^{2n-2k+1} n^n} \\
&\approx \frac{(2n)^{2n}(n-k)^{n-k}}{k^k (2n-2k)^{2n-2k} n^n} \\
&= \frac{4^n \cdot n^{2n} \cdot (n-k)^{n-k}}{k^k \cdot 4^{n-k} \cdot (n-k)^{2n-2k} \cdot n^n} \\
&= \frac{4^k \cdot n^n}{k^k \cdot (n-k)^{n-k}} \ .
\end{aligned}$$

Lets assume that the position of the maximum is only dependent on the size of k compared the the value of n, as we have seen some general behavior of the coefficients. Assume $k = \kappa n$:

$$\binom{n+\frac{1}{2}}{k} \cdot 2^{2k-[k]_2} \approx \frac{4^{(\kappa n)} \cdot n^n}{(\kappa n)^{\kappa n} \cdot (n-\kappa n)^{n-\kappa n}} \ .$$

Since every term has a power n, we can take the n-th root. This will not influence the results we want to derive due to the fact that the n-th power is a strictly increasing function:

$$\begin{aligned}
\sqrt[n]{\binom{n+\frac{1}{2}}{k} \cdot 2^{2k-[k]_2}} &\approx \frac{4^\kappa \cdot n}{(\kappa n)^\kappa \cdot (n-\kappa n)^{1-\kappa}} \\
&= \frac{4^\kappa \cdot n}{\kappa^\kappa \cdot n^\kappa \cdot n^{1-\kappa} \cdot (1-\kappa)^{1-\kappa}} \\
&= \frac{4^\kappa}{\kappa^\kappa \cdot (1-\kappa)^{1-\kappa}} \ .
\end{aligned}$$

Maximizing this function with Mathematica gives $\kappa = \frac{4}{5}$, so choosing $k = \frac{4}{5}n$ gives a good indication of the maximum of the denominator. It should be noted that there is, of course, a small error in this value for κ. As we used Stirling's approximation, we have made an error that asymptotically goes to zero. Also, we made a little error at the fourth line of the above derivation, neglecting some $+1$. This too is an error which becomes smaller as our binomials increase. In the end, the errors do not have much significance but they should be kept in mind.
The fact that κ has such a nice value suggests to us that finding the optimal value for κ can easily

be done by hand. This is indeed the case, as we will show. Maximizing a functions requires taking the derivative and equation it to zero. We firstly rewrite the found expression.

$$\frac{4^\kappa}{\kappa^\kappa \cdot (1-\kappa)^(1-\kappa)} = e^{\log(4^\kappa)} e^{-\log(\kappa^\kappa)} e^{-\log((1-\kappa)^{1-\kappa})} = e^{\log(4^\kappa)-\kappa\log(\kappa)-(1-\kappa)\log(1-\kappa)} \,.$$

Optimizing this comes down to optimizing the exponent, so we find:

$$\begin{aligned}
\frac{d}{d\kappa}\big(\kappa\log(4) - \kappa\log(\kappa) - (1-\kappa)\log(1-\kappa)\big) &= 0\\
\log(4) - \log(\kappa) - 1 + \log(1-\kappa) + 1 &= 0\\
\log\left(\frac{4}{\kappa}\right) = -\log(1-\kappa) &= \log\left(\frac{1}{1-\kappa}\right)\\
\Rightarrow 4(1-\kappa) = \kappa \Rightarrow 5\kappa = 4 \to \kappa &= \frac{4}{5}\,.
\end{aligned}$$

Indeed, this wasn't a hard exercise.

Similarly, we may be interested in the minimum when $n < k$. In that case, we apply the same tactic as we did above, namely: use a rough approximation of Stirling's formula. Since these value's can become negative, we take absolute values.

$$\begin{aligned}
\left|\binom{n+\frac{1}{2}}{k}\cdot 2^{2k-[2]_2}\right| &= \frac{(2n+1)!\cdot(2k-2k-2)!}{4^{k-1}\cdot 2\cdot n!\cdot k!\cdot(k-n-1)!}\cdot 2^{2k-[k]_2}\\
&\approx \frac{(2n+1)^{2n+1}\cdot(2k-2n-2)^{2k-2n-2}\cdot e^n\cdot e^k\cdot e^{k-n-1}\cdot 2^{2k-[k]_2}}{2\cdot 4^{k-1}\cdot n^n\cdot k^k\cdot(k-n-1)^{k-n-1}\cdot e^{2n+1}\cdot e^{2k-2n-2}}\\
&= \frac{(2n+1)^{2n+1}\cdot(2k-2n-2)^{2k-2n-2}\cdot 2^{2k-[k]_2}}{2\cdot 4^{k-1}\cdot n^n\cdot k^k\cdot(k-n-1)^{k-n-1}}\\
&\approx \frac{2^{1-[k]_2}\cdot(2n)^{2n}\cdot 2^{2k-2n}\cdot(k-n)^{2(k-n)}}{n^n\cdot k^k\cdot(k-n)^{k-n}}\\
&= \frac{2^{1-[k]_2}\cdot n^n\cdot 2^{2k}\cdot(k-n)^{k-n}}{k^k}\,.
\end{aligned}$$

Again assume $k = \kappa n$, then:

$$\left|\binom{n+\frac{1}{2}}{k}\cdot 2^{2k-[2]_2}\right| \approx \frac{2^{1-[k]_2}\cdot n^n\cdot 2^{2\kappa n}\cdot((\kappa-1)n)^{(\kappa-1)n}}{(\kappa n)^{\kappa n}}\,.$$

We now again that the n^{th} root, with similar intentions as before:

$$\begin{aligned}
\sqrt[n]{\left|\binom{n+\frac{1}{2}}{k}\cdot 2^{2k-[2]_2}\right|} &\approx \frac{n\cdot 2^{2\kappa}\cdot((\kappa-1)n)^{\kappa-1}}{(\kappa n)^\kappa}\\
&= \frac{4^\kappa\cdot(\kappa-1)^{\kappa-1}}{\kappa^\kappa}\,.
\end{aligned}$$

Minimizing this result gives $\kappa = \frac{4}{3}$, so by choosing $k = \frac{4}{3}n$, we can get an estimate for the minimum of the coefficient. Here too we note that there is a small error due to the usage of Stirling's approximation and neglecting some $+1$'s along the way. Also in this case, asymptotically this error is marginal.

Again we suspect we could have done this by hand. We take the same approach by first rewriting the found expression and then equating the derivative of the exponent to zero.

$$\frac{4^\kappa(\kappa-1)^{\kappa-1}}{\kappa^\kappa} = e^{\log(4^\kappa)}e^{\log((\kappa-1)^{\kappa-1})}e^{-\log(\kappa^\kappa)} = e^{\kappa\log(4)+(\kappa-1)\log(\kappa-1)-\kappa\log(\kappa)}$$

Now for the optimization part:

$$\begin{aligned}\frac{d}{d\kappa}\big(\kappa\log(4)+(\kappa-1)\log(\kappa-1)-\kappa\log(\kappa)\big) &= 0\\ \log(4)+\log(\kappa-1)+1-\log(\kappa)-1 &= 0\\ \log(4\kappa-4) &= \log(\kappa)\\ \Rightarrow 4\kappa-4=\kappa\Rightarrow 3\kappa=4\Rightarrow\kappa &= \frac{4}{3}\end{aligned}$$

Indeed this formula too was easily optimizable.

Now, as the general formula for a binomial coefficient with other fractions was dependent on the Stirling numbers of the first kind, it may become rather difficult to retrieve a nice estimate for the minimum and the maximum value of the numerator. However, we do know the exact value of the denominator, so ultimately combining these two will result in formula which may be minimizable and maximizable.

6 Collisions

As noted in the introduction, we are interested in binomial collisions. Some research has already been done on this topic. In [1], a definition is given for binomial collisions with integer coefficients. In order to use this concept for rational coefficients, we define a binomial collision by the following relation: let $n, m \in \mathbb{Q}\backslash\mathbb{N}, n \neq m$ and $k, l \in \mathbb{N}$. A *rational binomial collision* occurs when

$$\binom{n}{k} = \binom{m}{l}.$$

In this section, we will explore the various ways a binomial collision can occur for different values of $n, m \in \mathbb{Q}\backslash\mathbb{N}, k, l \in \mathbb{N}$. For an analysis of cases $n, m, k, l \in \mathbb{N}$, I would like to refer to the paper by Blokhuis, Brouwer and De Weger [1].
We will start with some obvious collisions. The most obvious being the case $n = m$ and $k = l$, though by the way we defined a collision, this cannot be.

So what if $n \neq m$, but $k = l$? Like we did for previous results, we first explore the case when we add a halve to the upper number. We get:

$$\binom{n+\frac{1}{2}}{k} = \binom{m+\frac{1}{2}}{k}$$
$$\frac{\left(n+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)\ldots\left(n-k+\frac{3}{2}\right)}{k!} = \frac{\left(m+\frac{1}{2}\right)\left(m-\frac{1}{2}\right)\ldots\left(m-k+\frac{3}{2}\right)}{k!}$$
$$\frac{(2n+1)(2n-1)\ldots(2n-2k+3)}{2^k k!} = \frac{(2m+1)(2m-1)\ldots(2m-2k+3)}{2^k k!}$$

For a collision to occur, the numerators too have the be the same. This is hard to realize, but not totally impossible. Observe, for instance, that

$$\begin{aligned}\binom{8\frac{1}{2}}{4} &= \frac{\frac{17}{2}\cdot\frac{15}{2}\cdot\frac{13}{2}\cdot\frac{11}{2}}{4!} \\ &= \frac{(-1)^4\cdot\frac{17}{2}\cdot\frac{15}{2}\cdot\frac{13}{2}\cdot\frac{11}{2}}{4!} \\ &= \frac{\frac{-17}{2}\cdot\frac{-15}{2}\cdot\frac{-13}{2}\cdot\frac{-11}{2}}{4!} \\ &= \frac{\frac{-11}{2}\cdot\frac{-13}{2}\cdot\frac{-15}{2}\cdot\frac{-17}{2}}{4!} = \binom{-5\frac{1}{2}}{4}.\end{aligned}$$

This reminds us of a kind of symmetry that occurs when $n, k \in \mathbb{N}$, given by the relation $\binom{n}{k} = \binom{n}{n-k}$. Though not completely the same case, such a relation might be playing a role here. So we have the suspicion that $\binom{n+\frac{1}{2}}{k} = \binom{k-n-\frac{3}{2}}{k}$. That is, as long as k is even, as we would otherwise be left with a factor -1.
Lets make a formal proof out of this. Assume k is even, $n, k \in \mathbb{N}$.

$$\begin{aligned}\binom{n+\frac{1}{2}}{k} &= \frac{(2n+1)(2n-1)\ldots(2n-2k+5)(2n-2k+3)}{2^k k!} \\ &= \frac{(-2n-1)(-2n+1)\ldots(-2n+2k-5)(-2n+2k-3)}{2^k k!} \\ &= \frac{(2k-2n-3)(2k-2n-5)\ldots(-2n+1)(2n-1)}{2^k k!} = \binom{k-n-\frac{3}{2}}{k}\end{aligned}$$

This is already a neat result! However, as before, we now want to generalize to any added fraction. Again, lets assume k to be even, as in the case it is not we have a difference of a factor -1:

$$\binom{n+\frac{q}{p}}{k} = \frac{(np+q)\cdot((n-1)p+q)\ldots((n-k+2)p+q)\cdot((n-k+1)p+q)}{p^k k!}$$
$$= \frac{((k-n-1)p-q)\cdot((k-n-2)p+q)\ldots((-n+1)p-q)\cdot(-np-q)}{p^k k!} = \binom{k-n-1-\frac{q}{p}}{k}$$

Luckily, when choosing $\frac{q}{p} = \frac{1}{2}$, we obtain the same result as we had previously. Once more we note: this result is only true if k is even. If this is not the case, there is only a factor -1 difference.

Until now, we have always used the condition that k must be the same. However, this is very limiting. So when can we have a whole binomial collision when this number may not be the same? The first thing that comes to mind is that the denominator of the binomial coefficient is known, as we have proven earlier. It consists of a certain amount of the same prime number(s), depending on the fraction we add. So at the moment, we assume that if we want to find a simple collision, we must carefully choose a correct fraction. As an example, when can $\binom{n+\frac{1}{2}}{k} = \binom{m+\frac{1}{4}}{l}$? We can calculate, for each value of k and l, what the denominators will be.

k / l	**denominator** $\binom{n+\frac{1}{2}}{k}$	**denominator** $\binom{m+\frac{1}{4}}{l}$
1	2	4
2	8	32
3	16	128
4	128	2048
5	256	8192
6	1024	65536
7	2048	262144

Table 1: Values of various rational binomials

So according to this table, we can have a collision by choosing $k = 4$ and $l = 3$, as this would result in the same denominator (being 128 when simplified). Although we do not know what values we must pick for n and m, we can now know whether or not we are doing work for nothing. Also note that by choosing $k = 7$ and $l = 4$ we may get a collision because of similar reasons.
To continue the example, what values for n and m must we choose? If we write out what the coefficients would be, we get:

$$\binom{n+\frac{1}{2}}{4} = \binom{m+\frac{1}{4}}{3}$$
$$\frac{(2n+1)(2n-1)(2n-3)(2n-5)}{4!\cdot 2^4} = \frac{(4m+1)(4m-3)(4m-7)}{3!\cdot 4^3}$$
$$\frac{(2n+1)(2n-1)(2n-3)(2n-5)}{384} = \frac{(4m+1)(4m-3)(4m-7)}{384}$$
$$(2n+1)(2n-1)(2n-3)(2n-5) = (4m+1)(4m-3)(4m-7)$$

Using the computer program Mathematica to solve this equation (type: Solve[$(2n+1)(2n-1)(2n-3)(2n-5) == (4m+1)(4m-3)(4m-7)$, $\{n,m\}$, Integers]), we get two pairs of n and m being an integer solution, namely $n = 0 \wedge m = 1$ and $n = 2 \wedge m = 1$. Checking what the binomial values are, indeed, we see: $\binom{\frac{1}{2}}{4} = \binom{1\frac{1}{4}}{3} = \frac{-5}{128}$ and $\binom{2\frac{1}{2}}{4} = \binom{1\frac{1}{4}}{3} = \frac{-5}{128}$ These are thus indeed binomial collisions. Perhaps by pure coincidence, we have found a triple collision, where three coefficients are the same. The fact that $\binom{\frac{1}{2}}{4} = \binom{2\frac{1}{2}}{4}$ is due to the symmetric property found above.

The equation $(2n+1)(2n-1)(2n-3)(2n-5) = (4m+1)(4m-3)(4m-7)$ derived above looks rather hard to solve by hand. If we look closely, we might recognize the form of an elliptic curve in it. We first rewrite the equation:

$$\begin{aligned}
(2n+1)(2n-1)(2n-3)(2n-5) &= (4m+1)(4m-3)(4m-7) \\
((2n-2)^2-3^2)((2n-2)^2-1^2) &= ((4m-3)^2-4^2)(4m-3) \\
((2n-2)^2-9)((2n-2)^2-1) &= ((4m-3)^2-16)(4m-3) \\
((2n-2)^2-5)^2-4^2 &= ((4m-3)^2-16)(4m-3) \\
((2n-2)^2-5)^2 &= (4m-3)^3-16(4m-3)+16
\end{aligned}$$

Now we substitute $y = (2n-2)^2 - 5$ and $x = 4m-3$ to get

$$y^2 = x^3 - 16x + 16 \ .$$

This is an elliptic curve of the so-called Weierstrass form, which is an equation in the form $y^2 = x^3 + ax + b$ where a and b are integers. Although we will not dive too deeply into the theory of elliptic curves, we do want to address the topic a little.
The idea of using elliptic curves to find binomial collisions of rational binomials has been used before [4]. Using elliptic curves, we can find binomial collisions without firstly setting the lower number of the binomial coefficients. This option is limited to a few nice cases in which the equation we get when writing out the coefficients is reducible to an elliptic curve.
One of this nice cases is $k = 2 \wedge l = 3$. What do we get? Now, let $n, m \in \mathbb{Q}$:

$$\begin{aligned}
\binom{n}{2} &= \binom{m}{3} \\
\frac{n(n-1)}{2} &= \frac{m(m-1)(m-2)}{6} \\
6n(n-1) &= 2m(m-1)(m-2) \\
2^6 \cdot 3^4 n(n-1) &= 2^6 \cdot 3^3 m(m-1)(m-2) \\
72n(72n-72) &= 12m(12m-12)(12m-24) \\
(72n-36)^2 - 36^2 &= (12m-12)\big((12m-12)^2 - 12^2\big)
\end{aligned}$$

Now substitute $y = 72n - 36$ and $x = 12m - 12$. We get:

$$\begin{aligned}
y^2 - 1296 &= x(x^2 - 144) \\
y^2 &= x^3 - 144x + 1296
\end{aligned}$$

This again is an elliptic curve of the so-called Weierstrass form. Solving this equation with a solver like Mathematica will probably result in y being equal to the root of the right-hand-side. This is not what we want. We want to have integer solutions to the problem, or perhaps rational. For this, MAGMA is a useful solver.
Solving this problem using MAGMA gives us $x = -15$ and $y = -9$. Using that $y = 72n - 36$ and $x = 12m - 12$, we get $n = \dfrac{3}{8}$ and $m = \dfrac{-1}{4}$. Checking this indeed shows a collision: $\binom{\frac{3}{8}}{2} = \binom{\frac{-1}{4}}{3} = \dfrac{-15}{128}$.

But this is far from the only solution we get! MAGMA provides us with a bunch of solutions to the equation we received. In the table below, many of the results will be presented.

x	**y**	**n**	**m**	$\binom{n}{2} = \binom{m}{3}$
-15	-9	$\frac{3}{8}$	$\frac{-3}{12}$	$\frac{-15}{128}$
-12	36	1	0	0
-8	-44	$\frac{-1}{9}$	$\frac{1}{3}$	$\frac{5}{81}$
0	-36	0	1	0
4	28	$\frac{8}{9}$	$\frac{4}{3}$	$\frac{-4}{81}$
9	27	$\frac{7}{8}$	$\frac{7}{4}$	$\frac{-7}{128}$
12	-36	0	2	0
24	-108	-1	3	1
40	-244	$\frac{-26}{9}$	$\frac{13}{3}$	$\frac{455}{81}$
48	324	5	5	10
108	-1116	-15	10	120
252	-3996	-55	22	1540
420	8604	120	36	7140
4953	-348579	$\frac{-38727}{8}$	$\frac{1655}{4}$	$\frac{1500090345}{128}$

Table 2: Collisions found by elliptic curves

So what do we see? We see some integer collisions, which are interesting enough on their own. Furthermore, we see some rational binomials occurring.
These numbers are the case when we want x and y, the substitute variables from the elliptic curve, to be in $\mathbb{Z}$. If we allow them to be in $\mathbb{Q}$ as well, then in MAGMA we can preset the prime factors dividing them. Other results can then be: $x = \frac{-527}{36} \Rightarrow m = \frac{-95}{432}$, $y = \frac{-3529}{216} \Rightarrow n = \frac{4247}{15552}$ and indeed the binomials are then the same, both having the value $\frac{-48012335}{483729408}$, so the results in the table are not the only results possible.

We want to note one more nice result we get by using elliptic curves, but firstly, we need to tell a little something about how the curves work. In the image below, we can see an elliptic curve with two points P and Q. The line through these points intersect the elliptic curve in a third point, which we will name $P * Q$. Then, $P + Q$ is defined to be $-(P * Q)$, which is the projection through the x-axis.
In the case where you do not have two points, but only one, you can take the tangent line to that point to find another intersection point. This method adds a point P to itself P. So, given any point or two points, we can add them.
What is so nice now, is that there is a theorem called Mordell's theorem, stating if a non-singular rational cubic curve, then there is a set of rational points which is finite, such that we can get all other rational points by the addition of several points with each other, as we described above. In other words; If we have some particular solutions, we know we have them all. Before we prove

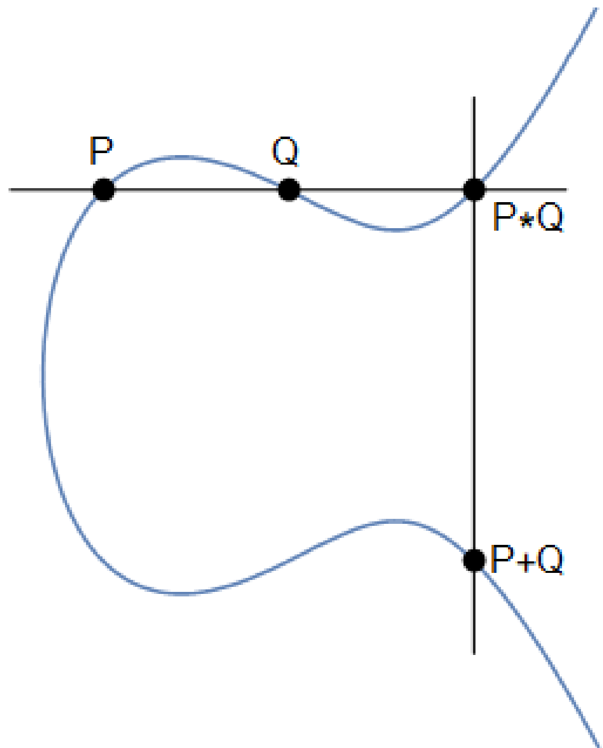


Figure 2: Addition of points on elliptic curve

Mordell's theorem, we want to point out that the previously mentioned solver MAGMA can find these particular solutions. We will give an example after the proof.
We will demonstrate part of the proof of this theorem as is done in *Rational points on elliptic curves* by Silverman and Tate [2]. Firstly we need to accept that the zero element is the point at infinity. Now we can proof that the set of solutions to the curve is a group. We need to show the following four properties:

1. If P and Q lie on the curve, then so does $P + Q$;
2. Associativity: $P + (Q + R) = (P + Q) + R$;
3. $0 + P = P$;
4. $P + -P = 0$, i.e. the existence of an inverse.

By definition, $P + Q$ is again an element on the curve, so this property holds. Also, if we believe infinity is the zero-element and we say $P = (x_P, y_P)$, then $P * 0$ is the intersection point of the curve with the vertical line through P, so the point $(x_P, -y_P)$, as the curve is symmetric in the x-axis. But then $P + 0 = (x_P, - - y_P) = (x_P, y_P) = P$.
Also, picking $-P$ to be $(x_P, -y_P)$, we can say $P + -P$ becomes the zero element, as the line through P and $-P$ is a vertical line.
Associativity is a rather complicated proof as it requires a bit more knowledge of elliptic curves and will for the sake of this paper be emitted, though it is proven in the book. In the end, Mordell's theorem holds and thus we know we have all solutions to the problem!
Since we don't go into elliptic curves too much, we would like to refer to Silverman and Tate [2] or Tzanakis [3], which are good sources when one may want to have a better understanding of this subject.

As an example we take the equation we found earlier: $y^2 = x^3 - 144x + 1296$, where $y = 72n - 36$ and $x = 12m - 12$. We will show how to use MAGMA to find the generators of the group, thus finding all solutions to the problem. Using the MAGMA code below we find pairs (x, y) which suffice the equation by using the generators of the group. In this case there are two, but if there had been any other amount we could use any linear combination of them to get our answers. The list of pairs is not complete, as we only printed the combinations $aP + bQ$ where P and Q are the two generators and a and b are $0, 1, ..., 4$. We do know there are more but at the moment we only want to show how to implement the theory. This way, we get 25 points.

```
E:=EllipticCurve([0,0,0,-144,1296]);
G:=Generators(E);
P:=G[1];
Q:=G[2];
for x in [0 .. 4] do
    for y in [0 .. 4] do
        N:= x*P + y*Q;
        N;
    end for;
end for;
```

Figure 3: MAGMA code for generating solutions (x, y)

```
(0 : 1 : 0)
(0 : 36 : 1)
(4 : -28 : 1)
(252 : 3996 : 1)
(-248/49 : 14932/343 : 1)
(24 : -108 : 1)
(12 : 36 : 1)
(-12 : -36 : 1)
(48 : -324 : 1)
(33/4 : 207/8 : 1)
(52/9 : -692/27 : 1)
(108 : 1116 : 1)
(-8 : 44 : 1)
(9 : -27 : 1)
(40 : 244 : 1)
(-15708/1681 : -2944836/68921 : 1)
(3936/49 : -244476/343 : 1)
(105/16 : 1611/64 : 1)
(-96/25 : -5292/125 : 1)
(420 : -8604 : 1)
(-2915480/269361 : 5568454036/139798359 : 1)
(144801/13225 : -48855951/1520875 : 1)
(26680/961 : 4073516/29791 : 1)
(-12276/841 : 413748/24389 : 1)
(1972/121 : -76204/1331 : 1)
```

Figure 4: MAGMA results

In the code we define the elliptic curve, find the generators, give the first generator the symbol P and the second Q. Next we make a for-loop in which the 25 combinations of P and Q are calculated. The results have been processed in the table below.

x	**y**	**n**	**m**	$\binom{n}{2} = \binom{m}{3}$
0	36	1	1	0
4	-28	$\frac{1}{9}$	$\frac{4}{3}$	$\frac{-4}{81}$
252	3996	56	22	1540
$\frac{-248}{49}$	$\frac{14932}{343}$	$\frac{3410}{3087}$	$\frac{85}{147}$	$\frac{550715}{9529569}$
24	-108	-1	3	1
12	36	1	2	0
-12	-36	0	0	0
48	-324	-4	5	10
$\frac{33}{4}$	$\frac{207}{8}$	$\frac{55}{64}$	$\frac{27}{16}$	$\frac{-495}{8192}$
$\frac{52}{9}$	$\frac{-692}{27}$	$\frac{35}{243}$	$\frac{40}{27}$	$\frac{3640}{59049}$
108	1116	16	10	120
-8	44	$\frac{10}{9}$	$\frac{1}{3}$	$\frac{5}{81}$
9	-27	$\frac{1}{8}$	$\frac{7}{4}$	$\frac{-7}{128}$
40	244	$\frac{35}{9}$	$\frac{13}{3}$	$\frac{455}{81}$
$\frac{-15708}{1681}$	$\frac{2944836}{68921}$	$\frac{-6440}{68921}$	$\frac{372}{1681}$	$\frac{242662420}{4750104241}$
$\frac{3936}{49}$	$\frac{244476}{343}$	$\frac{-3224}{343}$	$\frac{377}{49}$	$\frac{5750004}{117649}$
$\frac{105}{16}$	$\frac{1611}{64}$	$\frac{435}{512}$	$\frac{99}{64}$	$\frac{33495}{524288}$
$\frac{-96}{25}$	$\frac{-5292}{125}$	$\frac{-11}{125}$	$\frac{17}{25}$	$\frac{748}{15625}$
420	-8604	-119	36	7140
$\frac{-2915480}{269361}$	$\frac{5568454036}{139798359}$	$\frac{1325149370}{1258185231}$	$\frac{79213}{808083}$	$\frac{44368743304221215}{1583030075506523361}$
$\frac{144801}{13225}$	$\frac{-48855951}{1520875}$	$\frac{655061}{12167000}$	$\frac{101167}{52900}$	$\frac{7541022273279}{296071778000000}$
$\frac{26680}{961}$	$\frac{4073516}{29791}$	$\frac{643249}{268119}$	$\frac{9553}{2883}$	$\frac{120650998685}{71887798161}$
$\frac{105}{16}$	$\frac{1611}{64}$	$\frac{435}{512}$	$\frac{99}{64}$	$\frac{33495}{524288}$
$\frac{-12276}{841}$	$\frac{413748}{24389}$	$\frac{17941}{24389}$	$\frac{-182}{841}$	$\frac{57841784}{594823321}$
$\frac{1972}{121}$	$\frac{-76204}{1331}$	$\frac{-3536}{11979}$	$\frac{856}{363}$	$\frac{27430520}{143496441}$

Table 3: Collisions found by the generators of elliptic curves

Comparing the results we found to the results we found earlier, one may note that we get many new collisions. It is striking that there does not seem to be many similar collisions. This may be the case because we only added P and Q to each other a maximum of four times each. Perhaps, if we had added P to itself more times than four, we may have gotten other results. Nonetheless, adjusting the code by increasing the for-loops does result in more collisions, thus giving an idea of how to find them.

In short we can conclude that there are various approaches to finding collisions. A few of them we showed, being the somewhat trivial symmetric property, the comparing of the denominators and the use of elliptic curves.

7 Summary

In this article we analyzed $\binom{n+\frac{p}{q}}{k} = \frac{n+\frac{p}{q}}{1} \cdot \frac{n-1+\frac{p}{q}}{2} \cdots \frac{n-k+1+\frac{p}{q}}{k}$.

We found that

$$\binom{n+\frac{1}{2}}{k} = \begin{cases} \frac{1}{4^k k!} \cdot \frac{(2n+1)! \cdot (n-k)!}{(2n-2k+1)! \cdot n!} & \text{if } n \geq k \\ \frac{(-1)^{k-n-1} \cdot (2n+1)! \cdot (2k-2n-2)!}{4^{k-1} \cdot 2 \cdot n! \cdot k! \cdot (k-n-1)!} & \text{if } n < k \end{cases}.$$

and more generally

$$\binom{n+\frac{p}{q}}{k} = \frac{1}{k! \cdot q^k} \sum_{i=0}^{k} s(k,i)(nq+p)^i q^{k-i}$$

where $s(k,i)$ are the Stirling numbers of the first kind.
We found an approximation for $\binom{n+\frac{1}{2}}{k}$ given by

$$\binom{n+\frac{1}{2}}{k} \approx \begin{cases} \frac{1}{k^k} \cdot \frac{2n+1}{2n-2k+1} \cdot \frac{(n+1)^n}{(n-k+1)^{n-k}} \cdot \sqrt{\frac{(2n+1)(n-k)}{2\pi k \cdot n(2n-2k+1)}} & \text{if } n \geq k \\ \frac{(-1)^{k-n-1} \cdot n(4n+4)^n \cdot (k-n-1)^{k-n-1}}{4^n \cdot k^k} \cdot \sqrt{\frac{2n+1}{\pi n k}} & \text{if } n < k \end{cases}.$$

Next we found a formula for the value of the denominator of a $\binom{n+\frac{p}{q}}{k}$ given by

$$denominator = q^k \cdot \prod_{i=1}^{n} p_i^{\alpha_i}, \text{ with} \alpha_i = \frac{1}{p_i - 1}(k - [k]_{p_i}) \ .$$

Using this result we were able to find the minimal and maximal value of the numerator. The maximum (given $n > k$) occurs when $k \approx \frac{4}{5}n$, and the minimum (given $n < k$) occurs when $k \approx \frac{4}{3}n$. We were interested in the minimum and maximum of the numerator specifically as the value of the whole coefficient converges to zero.

Then we moved to collisions. We found a triplet of ways to find these. First of all

$$\binom{n+\frac{q}{p}}{k} = \binom{k-n-1-\frac{q}{p}}{k}$$

if k is even. Secondly we can set use the formula for the denominator to find two numbers which can result in the same numerator and then derive an equation by using the definition of the binomial coefficient. Thirdly, we can make use of elliptic curves by rewriting certain nice cases of coefficients. This last method was not explored as thoroughly as one may prefer, as the field of elliptic curves is rather large and not entirely relevant to this paper. Though, when having more knowledge on this topic, undoubtly some great results can be found.

8 Suggestions for further research

Firstly I would like to note that we did not thoroughly explore the last topic addressed in this paper, being the application of elliptic curves. When diving more deeply into this topic there might be some surprising results that may make it easier to find collisions, that can find multiple collisions or even provide us with a family of collisions.

In this article we have added fractions, but what also could be interesting is a binomial of the form $\binom{n+a\sqrt{b}}{k}$. It is not hard to see that this will result in a similar number $\frac{\tilde{n}+\tilde{a}\sqrt{b}}{\tilde{k}}$. Can we find nice formula's or collisions with these kind of binomial coefficients? We have restricted ourselves to the rational binomial coefficients because we wanted a better understanding of how these work. It could be that these can be generalized, however, that was not what we were interested in.

Similarly, we could take a look at complex binomials. Imagine working with $\binom{n+bi}{k}$. Can we describe something about the behaviour of such numbers, and can we find a way to find collisions?

References

[1] Blokhuis, A., Brouwer A., De Weger, B. (2017). Binomial collisions and near collisions

[2] Silverman, J., Tate, J. (1992). Rational Points on Elliptic Curves, Published by Springer-Verlag

[3] Tzanakis, N. (2013). Elliptic Diophantine Equations, Published by De Gruyter

[4] De Weger, B. (1997). Equal Binomial Coefficients: Some elementary Considerations. *Journal of number theory 63, 373-386*

[5] Wikipedia. *Stirling numbers of the first kind*. Consulted may 15, 2018, at https://en.wikipedia.org/wiki/Stirling_numbers_of_the_first_kind

Appendix

Appendix A: Stirling numbers

The following table of *unsigned Stirling numbers* can be found at [5], or in books containing common sets of numbers.

s(k,i)	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	1									
1	0	1								
2	0	1	1							
3	0	2	3	1						
4	0	6	11	6	1					
5	0	24	50	35	10	1				
6	0	120	274	225	85	15	1			
7	0	720	1764	1624	735	175	21	1		
8	0	5040	13068	13132	6769	1960	322	28	1	
9	0	40320	109584	118124	67284	22449	4536	546	36	1

Appendix B: Some mathematica code

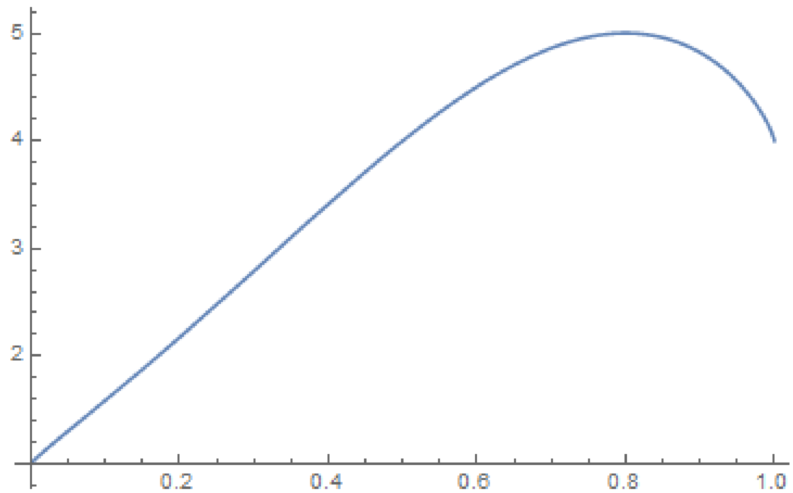
In this paper I often note that checks with Wolfram Mathematica suggest that results we found are right. Some of that code will be presented here for the reader.

```
In[1]:= (* Factorize the number q *)
        FactorFunction[q_] := FactorInteger[q][[All, 1]]

In[2]:= (* Calculate the denominator of a binomial coefficient with
         added fraction with denominator q, Choose k. *)
        CalcDenominator[q_, k_] := (t = q^k; FactorFunction[q];
          Do[
           t = t * FactorFunction[q][[i]]^
              (1 / (FactorFunction[q][[i]] - 1) *
                (k - Total[IntegerDigits[k, FactorFunction[q, [[i]]]]]])),
           {i, Length[FactorFunction[q]]}]; t)
```

Figure 5: Code to calculating the denominator

```
In[28]:= MaxFormula[k_] := (4^k) / (k^k * (1 - k) ^ (1 - k))

In[29]:= Plot[MaxFormula[k], {k, 0, 1}]
```

Out[29]=



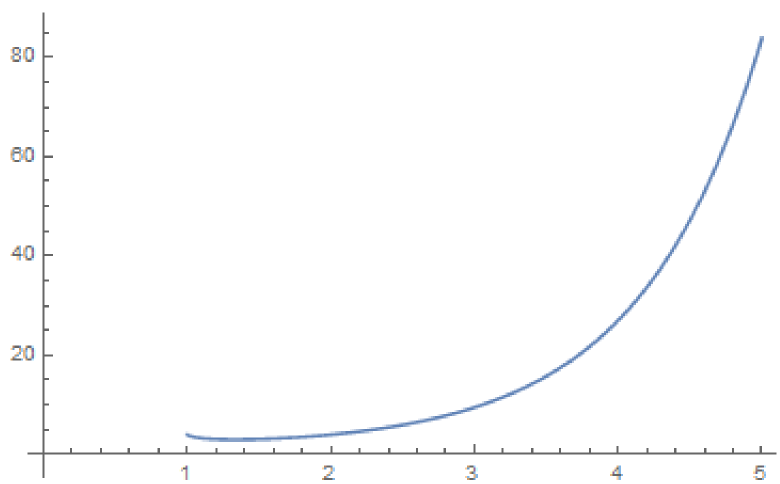

```
In[30]:= Maximize[MaxFormula[k], k]
```

Out[30]= $\{5, \{k \to \frac{4}{5}\}\}$

Figure 6: Code to approximate the maximum

```
In[31]:= MinFormula[k_] := (4^k * (k - 1) ^ (k - 1)) / k^k

In[32]:= Plot[MinFormula[k], {k, 0, 5}]
```

Out[32]=




```
In[33]:= Minimize[MinFormula[k], k]
```

Out[33]= $\{3, \{k \to \frac{4}{3}\}\}$

Figure 7: Code to approximate the minimum