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Polling systems with and without glue periods

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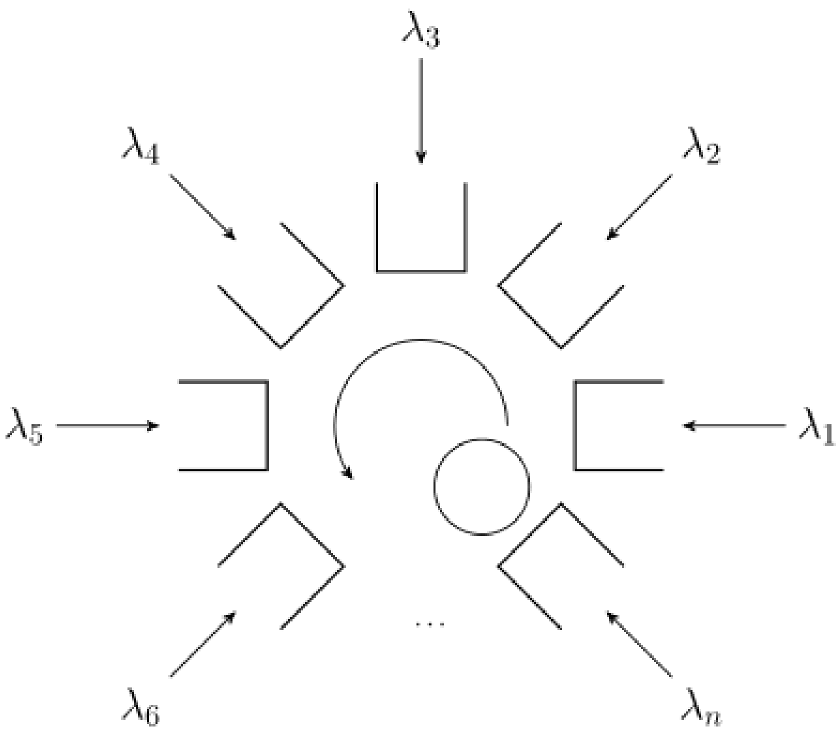


Abstract

A polling system is a queuing system that consists of N stations, each with a corresponding queue, which are served by a single server, in a fixed order. First we study a system with exhaustive and gated service. From this the mean queue length, its variance and the covariance between queue lengths of different stations were simulated. The mean queue length was also compared to exact values. Then, for a system with glue periods, a system with deterministic glue periods and exponential switch-over times and a system with exponential glue periods and deterministic switch-over times were studied. For both systems the mean waiting times were simulated and compared to exact values. Then, with a system with exponential glue periods and deterministic switch-over times, the mean glue time was varied. Here the mean queue length, the squared coefficient and correlation between queue lengths were simulated and compared to figures from exact calculations. Also, for several systems used earlier, the density of the waiting times were simulated. Lastly systems with a Uniform and Pareto glue distribution were studied. It was observed that changes in the arrival rate and average service time result in similar changes in queue length at all stations, while changes in mean retrial time and mean glue period time result in mostly station specific changes in queue length.

Contents

Chapter 1

Introduction

A polling system is a queuing system that consists of N stations, each with a corresponding queue, which are served by a single server, in a fixed order. There are many real-life examples of polling systems.

One of the first polling systems modeled was about a patrolling repairman. This repairman would walk rounds and during these rounds visited machines that might need repairs. Here the stations are the machines and the parts of these machines needed to be fixed are the customers that are waiting to be 'served'.

Another example of a polling system is a one-lane bridge, with on both sides a traffic signal. Here there are two queues, namely the cars that want to go to the other side on each side of the bridge. The server is the traffic light, which, if green, is serving a customer. Polling systems are also used in data-networks where one server might need to receive and/or send data to multiple other servers. Here the customers are the data-packages and the server is the cable that transmits the signal from one server to the other.

Optical networks are another example. Polling system with glue periods, which will be studied in this report, were inspired by this example. In an optical network light signals are used to send messages between different locations. When one of this light messages arrives at a location, this message either has to be received by the location within a short time or send away to try again later. This because light cannot be queued, though it can be slowed down a little. This means that if a message does not arrive shortly before the server starts serving, the message will be send away to try again later. If it does arrive shortly before the start of service, it will be 'glued' to the station queue, by slowing the customer down, and served.

Polling systems exist in many shapes and sizes. There are several parts that determine the shape and size of the system.

First we discuss the service discipline. The service discipline decides which customers the server is going to serve when it arrives at a queue. The discipline has rules to determine who to serve. The two service disciplines that will be discussed in this report, are the exhaustive and gated service disciplines. With exhaustive service the server serves all customers in the queue of the station it is at and when the queue is empty the server moves on to the next station. If the service discipline is gated, the server will serve only the customers that were already present the moment the server arrived at the station. This means if a customer arrives at a station where the server is serving, he will have to wait for the server to return to the queue before he is served. Other service disciplines can have different rules or combinations of rules. For example the server can only stay a certain time at a station (time-limited), or

can only serve a certain maximum number of customers (k-limited).
Second, there are different types of customer behavior. The most basic is that the customer arrives at a queue and waits till he is served, which is what will be studied at in this report. It is also possible that if a queue is too long the customer will leave to try again later.
A system can also have something called 'glue periods'. Only customers who arrive during a glue period for the station they arrive for will go in the queue. If the customer does not arrive during the glue period of the station they arrive at, he will leave to try again later. In this report, the glue periods are located right after the switch-over and before the start of serving.
Last, there are the probability distributions determining the inter-arrival, switch-over, glue and service times. Different types of distributions describe different behavior and thus can describe different situations.
In Chapter 2 the polling model will be discussed. First a closer explanation of the studied polling systems will be given, followed by the notation. Next the determination of the sation sizes is discussed on. We continue by providing some information of branching processes and lastly about the Pseudo-Conservation law.
In the next chapter, it is explained how the simulation is built. In Chapter 4 the results from the simulation are given, starting with a system with gated or exhaustive service. After that the result of a system with glue periods will be given, followed by the conclusion.

Chapter 2

Model description

In this chapter the variations of a polling system with and without glue period will be described. As mentioned in the introduction, a polling system is a system where one server serves N different stations. Each station has its own queue, arrival rate and service time. There are several important parts in a polling system. These are the service discipline, customer behavior, the arrival, service and switch-over process and the server routing.
The service discipline determines which customers will be served and when the server will move on to the next station. In this report we discuss the exhaustive and gated service discipline. Exhaustive service simply means that the server will stay at a station serving customers till the queue is empty, then the server will move to the next station. If the service discipline is gated, then the server will only serve the customers that are at the station at the moment the server arrives at the station. If the server is serving at a station, and a customer arrives at this station, this customer will have to wait till the server returns to this station in the next cycle.
The customer behavior is another important part of the system. Customer behavior determines how customers behave in certain situations. For example they might leave if they have to wait longer than a certain time. In this report the customers will enter the queue the moment they arrive, if this is possible. If there are glue periods entering the queue might not be possible and the customer will try again later.
Next we have the arrival, service and switch-over processes. The duration time for each of these processes are random variables. The arrival process is normally given by a Poisson distribution. Thi means that the inter-arrival times are exponentially distributed. The service times are independent random variables, that may vary per station. In this report exponential service times are used, that may have a different mean at different stations. The switch-over times in this report are independent random variables with either an exponential distribution or simply constants.
Later in the report systems with glue periods will be studied. When a customer arrives at station i and at the moment of arrival it is a glue period of station i, the customer will be 'glued' to a temporary queue and wait to be served. If there is no glue period at the station i, then this customer will leave and retry later. The glue period is located between the end of a switch-over and the start of service.
The last part of the model is the server routing. In this report the server will follow the stations in cyclic order, which can be seen in Figure 2.1.

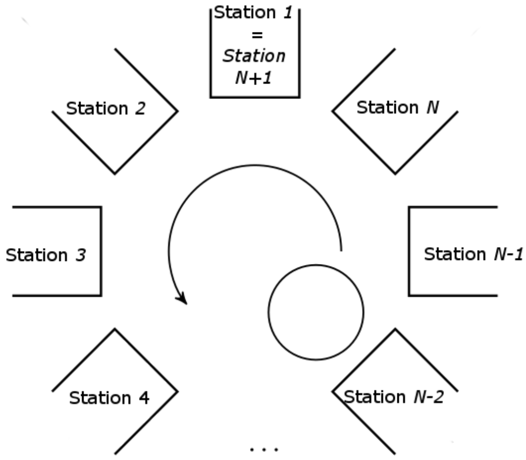


Figure 2.1: cyclic order

2.1 Notation

In this report the inter-arrival and service times are exponentially distributed. The arrival process has an arrival rate λ_i for station i. The random variable for the service times is B_i for station i. The random variable S_i gives the switch-over time from station i to station $i+1$, where station $N+1$ is station 1.
If the system has glue periods, there are two extra i.i.d. random variables G_i, the duration of a glue period for station i, and R_i, the time between an arrival outside of the glue period and the retrial. This return time is exponentially distributed with mean ν_i^{-1}.
The utilization of the server at station i is $\rho_i = \lambda_i \mathbb{E}[B_i]$, where $\rho = \sum_{i=1}^{N} \rho_i$. Lastly there is the cycle length, which is the time it takes for the server to complete a round past all stations. The cycle length is here defined as the time between two successive arrivals of the server at a station. To determine the mean cycle length, the mean idle time per cycle is determined, which is $\sum_{i=1}^{N}(\mathbb{E}[G_i] + \mathbb{E}[S_i])$. This is the time that the server is not in a service period and thus is in either glue or switch-over period. Since ρ is the part of the mean time the server is serving, $1-\rho$ is the part of the mean time it is a glue or service period. This indicates thjat the mean idle time per cycle can also be given by $(1-\rho)\mathbb{E}[C]$ and that the mean cycle time is as follows: $\mathbb{E}[C] = \frac{1}{1-\rho}\sum_{i=1}^{N}(\mathbb{E}[G_i] + \mathbb{E}[S_i])$.

2.2 Station size of a system with glue periods

The results from the simulation with glue periods will be compared with results found in article [1]. First we will describe how the authors of article [1] determined the waiting times for systems with exponential glue periods.
First some extra notation, consistent with the notation in article [1]. The glue periods are exponentially distributed, with $\mathbb{E}[G_i] = 1/\gamma_i$. $\beta_i(\mathbf{z})$ is defined as follows: $\beta_i(\mathbf{z}) := \tilde{B}_i(\sum_{j=1}^N \lambda_j(1-z_j))$, with $\tilde{B}_i(x)$ the Laplace Stieltjes transform of the service time distribution at station i, where $\mathbf{z} = (z_1, z_2, \ldots, z_n)$. We also define $\sigma_i(\mathbf{z}) := \tilde{S}_i(\sum_{j=1}^N \lambda_j(1-z_j))$, with $\tilde{S}_i(x)$ the Laplace Stieltjes transform of the switch-over time distribution from station i to station $i+1$. For a vector $\mathbf{l}$ of length N, $|\mathbf{l}| = l_1 + l_2 + \ldots + l_N$. The inequality $\mathbf{l}' \leq \mathbf{l}$ is interpreted component-wise. Lastly $\Gamma_{i,m}^{(\mathbf{l})} = \frac{1}{\mathbf{l}!}\frac{\delta^{|\mathbf{l}|}}{\delta \mathbf{z}^{\mathbf{l}}}((\beta_i(\mathbf{z})-1)^m \sigma_i(\mathbf{z}))|_{\mathbf{z}=\mathbf{1}-}$.
First the station size at arbitrary time points was determined. To do this the function $\phi_i(\mathbf{z}; w)$ was defined. Let $M_i^o(t)$ be the number of customers in the orbit of station i at time t, for all i and $\Upsilon(t)$ be the number of customers in the glue queue at time t. Then $\phi_i(\mathbf{z}; w)$ is defined as follows:

$$\phi_i(\mathbf{z}; w) = \int_0^\infty \phi_i(\mathbf{z}; w; t)dt,$$

with $\phi_i(\mathbf{z}; w; t)$ as follows:

$$\phi_i(\mathbf{z}; w; t) = \mathbb{E}[z_1^{M_1^o(\tau_i+t)} \cdots z_N^{M_N^o(\tau_i+t)} \omega^{\Upsilon(\tau_i+t)} \mathbb{1}_{\{G_i>t\}}].$$

Theorem 1 of article [1] gives that generating function $\phi_i(\mathbf{z}; w)$ satisfies a differential equation, Equation (2.1), for $i = 1, \ldots, N$.

$$\nu_i(w - z_i)\frac{\delta}{\delta z_i}\phi_i(\mathbf{z}; w) - \left(\sum_{j=1, j\neq i}^{N}(\lambda_j(1-z_j)) + \lambda_i(1-w) + \gamma_i\right)\phi_i(\mathbf{z}; w)$$
$$+\gamma_{i-1}\phi_{i-1}(\mathbf{z}; \beta_{i-1}(\mathbf{z}))\sigma_{i-1}(\mathbf{z}) = 0 \qquad (2.1)$$

Next, let $R_{v,i}(\mathbf{z}, w)$, $R_{s,i}(\mathbf{z})$ and $R_{g,i}(\mathbf{z}, w)$ be the generating function of the number of customers in each station at an arbitrary time point in respectively a service (visit), switch-over and glue period, where $\mathbf{z} = (z_1, ..., z_N)$. Then $R_{v,i}(\mathbf{z}, w)$, $R_{s,i}(\mathbf{z})$ and $R_{g,i}(\mathbf{z}, w)$ can be expressed in terms of $\phi_i(\mathbf{z}; w)$.

$$R_{v,i}(\mathbf{z}, w) = \frac{\gamma_i}{\rho_i \mathbb{E}[C]}\frac{\phi_i(\mathbf{z}, w) - \phi_i(\mathbf{z}, \beta_i(\mathbf{z}))}{w - \beta_i(\mathbf{z})}\frac{1-\beta_i(\mathbf{z})}{\sum_{j=1}^N \lambda_j(1-z_j)}, \qquad (2.2)$$

$$R_{s,i}(\mathbf{z}) = \frac{\gamma_i}{\mathbb{E}[S_i]}\phi_i(\mathbf{z}, \beta_i(\mathbf{z}))\frac{1-\sigma_i(\mathbf{z})}{\sum_{j=1}^N \lambda_j(1-z_j)}, \qquad (2.3)$$

$$R_{g,i}(\mathbf{z}, w) = \gamma_i \phi_i(\mathbf{z}, w). \qquad (2.4)$$

These functions were determined with the generating functions of the number of customers in each station at the *start* of each period type: $\tilde{R}_v^{(i)}(\mathbf{z}, w)$ for the visit (service) period of station i, $\tilde{R}_s^{(i)}(\mathbf{z})$ for the switch-over period from station i to station $i+1$ and lastly $\tilde{R}_g^{(i)}(\mathbf{z}, w)$ for the glue period of station i. We refer to Proposition 1 of article [1] for the equations of these

generating functions in terms of $\phi_i(\mathbf{z}; w)$.
The number of customers at an arbitrary time point in a service period consists of two parts: the number of customers who arrived between the start of the current service period and the current time, and the number of customers at the start of the service period without the ones already served, denoted by $\check{r}_1(\mathbf{z})$ and $\check{r}_2(\mathbf{z}, w)$. The formulas for these two are given respectively in Equations (2.5) and (2.6). In Equation (2.7) $R_{v,i}(\mathbf{z}, w)$ is given.

$$\check{r}_1(\mathbf{z}) = \frac{1 - \beta_i(\mathbf{z})}{\mathbb{E}[B_i](\sum_{j=1}^{N} \lambda_j(1 - z_j))} \tag{2.5}$$

$$\check{r}_2(\mathbf{z}, w) = \frac{\tilde{R}_v^{(i)}(\mathbf{z}, w) - \tilde{R}_v^{(i)}(\mathbf{z}, \beta_i(\mathbf{z}))}{\mathbb{E}[Y_i^{(iq)}](w - \beta_i(\mathbf{z}))} = \frac{\gamma_i}{\mathbb{E}[Y_i^{(iq)}]} \frac{\phi_i(\mathbf{z}, w) - \phi_i(\mathbf{z}, \beta_i(\mathbf{z}))}{w - \beta_i(\mathbf{z})} \tag{2.6}$$

$$R_{v,i}(\mathbf{z}, w) = \check{r}_1(\mathbf{z}) \cdot \check{r}_2(\mathbf{z}, w) \tag{2.7}$$

Equation (2.3) gives the generating function of the number of customers at an arbitrary time moment during the switchover. This number of customers also consists of two parts: customers that had already arrived before the start of the switchover period and the customers that arrived during the part of the switchover period that already passed. The generating function of the number of customers that arrived during the part of the switchover period that already passed is given by $\frac{1-\sigma_i(\mathbf{z})}{\mathbb{E}[S_i](\sum_{j=1}^{N} \lambda_j(1-z_j))}$. The generating function of the customers that arrived before the switchover period can be given by $\tilde{R}_s^{(i)}(\mathbf{z})$. Since both numbers of customers are independent the generating function $R_{s,i}(\mathbf{z})$ is found by multiplication of the two generating functions.
The generating function of the number of customers at an arbitrary time point in a glue period can be found in Equation (2.4). This was calculated by the theory of Markov regenerative processes.

$$R_g^{(i)}(\mathbf{z}, w) = \gamma_i \int_0^\infty \phi_i(\mathbf{z}, w; t) dt \tag{2.8}$$

To determine the moments, the generating functions $R_{v,i}(\mathbf{z}, w)$, $R_{s,i}(\mathbf{z})$ and $R_{g,i}(\mathbf{z}, w)$ are first scaled, as seen respectively in Equations (2.9), (2.10) and (2.11).

$$\psi_{v,i}^{(\mathbf{l},m)} = \frac{1}{\mathbf{l}!m!} \frac{\delta^{|\mathbf{l}|+m}}{\delta \mathbf{z}^{\mathbf{l}} \delta w^m} R_{v,i}(\mathbf{z}, w)|_{\mathbf{z}=\mathbf{1}-, w=1-'} \tag{2.9}$$

$$\psi_{s,i}^{(\mathbf{l})} = \frac{1}{\mathbf{l}!} \frac{\delta^{|\mathbf{l}|}}{\delta \mathbf{z}^{\mathbf{l}}} R_{s,i}(\mathbf{z})|_{\mathbf{z}=\mathbf{1}-, w=1-'} \tag{2.10}$$

$$\psi_{g,i}^{(\mathbf{l},m)} = \frac{1}{\mathbf{l}!m!} \frac{\delta^{|\mathbf{l}|+m}}{\delta \mathbf{z}^{\mathbf{l}} \delta w^m} R_{g,i}(\mathbf{z}, w)|_{\mathbf{z}=\mathbf{1}-, w=1-'} \tag{2.11}$$

Here for vectors $\mathbf{z}$ and $\mathbf{l}$ of length N, $\mathbf{l}! = l_1! l_2! \cdots l_N!$, $\mathbf{z}^{\mathbf{l}} = z_1^{l_1} z_2^{l_2} \cdots z_N^{l_N}$ and $\delta \mathbf{z}^{\mathbf{l}} = \delta z_1^{l_1} \delta z_2^{l_2} \cdots \delta z_N^{l_N}$.
Now let M_i^o be the number of customers in orbit of station i in steady state, M_i^{oq} the number of customers glued in the queue and orbit of station i in steady state and M_i the total number of customers for station i in steady state. Let Υ be the number of glued customers in steady state. Define $I_{v,i}$, $I_{g,i}$ and $I_{s,i}$ as indicator random variables, where $I_{v,i}$ is 1 if the server is at station i serving in steady state and else zero, $I_{g,i}$ is 1 if station i is in glue period and else

zero in steady state and lastly $I_{s,i}$ is 1 if the server is switching from station i to station $i+1$ in steady state and else zero. From this we get for $i = 1, \ldots, N$,

$$
\begin{aligned}
M_i^o =& \sum_{k=1}^{N} M_i^o (I_{v,k} + I_{g,k} + I_{s,k}), \\
M_i^{oq} =& M_i^o + \Upsilon (I_{v,i} + I_{g,i}), \\
M_i =& M_i^{oq} + I_{v,i}.
\end{aligned}
$$

Where M_i^o equals the sum of customers in orbit during the serving, glue and switch-over period of all stations. M_i^{oq} is the number of customers in orbit for station i and the number of customers that are glued to the queue of station i. And lastly M_i is the number of customers in orbit of station i and the number of customers glued to station i and the customer being served, if it is a serving period for station i. The expectation of M_i can be calculated by using the following formulas:

$$
\begin{aligned}
\mathbb{E}[M_i^o I_{v,k}] =& \rho_k \psi_{v,k}^{(\mathbf{1_i},0)}, \\
\mathbb{E}[M_i^o I_{g,k}] =& \frac{\mathbb{E}[G_k]}{\mathbb{E}[C]} \psi_{g,k}^{(\mathbf{1_i},0)}, \\
\mathbb{E}[M_i^o I_{s,k}] =& \frac{\mathbb{E}[S_k]}{\mathbb{E}[C]} \psi_{s,k}^{(\mathbf{1_i})}, \\
\mathbb{E}[\Upsilon I_{v,i}] =& \rho_i \psi_{v,i}^{(\mathbf{0},1)}, \\
\mathbb{E}[\Upsilon I_{g,i}] =& \frac{\mathbb{E}[G_i]}{\mathbb{E}[C]} \psi_{g,i}^{(\mathbf{0},1)}, \\
\mathbb{E}[I_{v,i}] =& \rho_i.
\end{aligned}
$$

Higher moments can be calculated similarly.

2.3 Branching process

Branching processes are also called Galton-Watson processes and are discrete time processes X_n [3]. Here n denotes the generation and X_n the number of individuals in generation n. In a branching process each individual produces a random number of offspring, according to a distribution that is the same for each individual. The number of offspring generated by two different individuals are independent. This is for two individuals of either the same or different generations. Branching processes are discrete-time Markov chains, with only nonnegative values for X_n.
In polling systems with exhaustive and gated service disciplines, the queue lengths at the start and end of a service period are branching processes. Polling systems with this type of service discipline satisfy the branching property and can be analyzed using theory from branching processes. The branching property given below is defined by [4]. The Q_i means station i and k_i denotes the number of customers in queue at station i. *PGF* is the probability generating function.

Branching Property *If the server arrives at Q_i to find k_i customers there, then during the course of the server's visit, each of these k_i customers will effectively be replaced in an i.i.d.*

manner by a random population having PGF $h_i(z_1, ..., z_N)$, which can be any N-dimensional PGF.

Since polling systems with exhaustive and gated service satisfy this property, there are probability generating functions $h_{i,e}$ and $h_{i,g}$ for the random population that replace the customers at a station for respectively exhaustive and gated service. For a system with the gated service holds $h_{i,g}(\mathbf{z}) = \tilde{B}_i(\sum_{j=1}^{N} \lambda_j(1 - z_j))$. In a system with gated service, customers that arrive during service will not be served till the next cycle starts. This means that if a customer is served, this customer will be replaced by the customers that arrive during its service time. In this report the service distribution is exponential, thus $\tilde{B}_i(s) = \frac{1}{1+\mathbb{E}[B_i]s}$
For exhaustive service $h_{i,e}(\mathbf{z}) = \tilde{BP}_i(\sum_{j\neq i} \lambda_j(1 - z_j))$ holds, where $\tilde{BP}_i$ is the Laplace-Stieltjes transform of a busy period distribution of a system with only type i customers and $\tilde{BP}_i(s) = \tilde{B}_i(\lambda_i + s + \lambda_i \tilde{BP}_i(s))$. Here all customers that arrive at station i during the service period of station i will be served during this service period. This means that the customers that were present at the begin of the service period will be replaced with the customers that arrive during the whole service period (or busy-period) at the end of the service period, except at station i where the queue will be empty at the end of service period.

2.4 The Pseudo-Conservation law

Determining the exact mean waiting time for each queue can be very difficult and complicated. An example is the calculation of the mean waiting time for a system with glue periods at Section 2.2. Here it can be seen that to calculate the mean some difficult calculation have to be made, and as the number of stations grows the difficulty of the calculations will increase. A better approach is to determine the weighted sum of the mean waiting times, otherwise known as the pseudo-conservation law. Because the weighted sum of mean waiting times is easier to determine, is it an important basis for approximations of the mean waiting times.
In article [1] the pseudo conservation law for a polling system with general glue period is given:

$$\sum_{i=1}^{N} \rho_i \mathbb{E}[W_i] = \rho \frac{\sum_{i=1}^{N} \lambda_i \mathbb{E}[B_i^2]}{2(1-\rho)} + \rho \frac{\mathbb{E}[(\sum_{i=1}^{N}(S_i + G_i))^2]}{2\mathbb{E}[\sum_{i=1}^{N}(S_i + G_i)]} + \frac{\mathbb{E}[\sum_{i=1}^{N}(S_i + G_i)]}{2(1-\rho)}(\rho^2 - \sum_{i=1}^{N} \rho_i^2) + \sum_{i=1}^{N} \mathbb{E}[F_i] \tag{2.12}$$

This equation was found with the help of [2]. There it is also explained that the steady state workload is in distribution the same as the sum of two independent quantities. The first quantity is the steady state workload of the same queueing system, but then without the switch-over and glue periods. The second quantity is the steady state workload at an arbitrary time in a switch-over or glue period.
In Equation (2.12), $\mathbb{E}[F_i]$ is the amount of work left in station i at the end of a visit period to station i. This is the only quantity that cannot directly be determined out of the distributions of the inter-arrival, switch-over, service, glue or retrial time. To determine the work left at this moment, first it is important to know that $\mathbb{E}[F_i] = \mathbb{E}[Z_i^{(i)}]\mathbb{E}[B_i]$, where $\mathbb{E}[Z_i^{(i)}]$ is the mean number of customers for station i present in the system at the end of a visit to station i.
Now $\mathbb{E}[Z_i^{(i)}]$ needs to be determined. In [1] this is done by deriving the relation between

$\mathbb{E}[Z_i^{(i)}]$ and $\mathbb{E}[Y_i^{(iq)}]$, where $\mathbb{E}[Y_i^{(iq)}]$ is the number of customers in the queue of station i at the start of a visit period. $\mathbb{E}[Y_i^{(iq)}]$ consists of three parts. First there are the customers that were in the system before the last visit period to station i and who retry during this glue period, $(1-\tilde{G}_i(\nu_i))\mathbb{E}[Z_i^{(i)}]$. As mentioned before $\mathbb{E}[Z_i^{(i)}]$ is the mean number of customers for station i present at the end of a visit to station i. This is multiplied by the fraction of the number of customers that will retry during the glue period of station i, $(1-\tilde{G}_i(\nu_i))$. Here $\tilde{G}_i(s)$ is the Laplace-Stieltjes transform of the glue period distribution at station i. $\tilde{G}_i(\nu_i)$ is the probability that the retrial time is larger than the glue period time.
The second part of $\mathbb{E}[Y_i^{(iq)}]$ is the mean number of customers that arrived in between the end of the last visit to station i and the start of the glue period of this station and retry during the glue period. Since $\lambda_i((1-\rho_i)\mathbb{E}[C]-\mathbb{E}[G_i])$ is the mean number of customers that arrive from the end of the last visit to station i to the beginning of the glue period of station i, the expression $(1-\tilde{G}_i(\nu_i))\lambda_i((1-\rho_i)\mathbb{E}[C]-\mathbb{E}[G_i])$ gives the number of people that retry during this glue period.
The last part are the customers that arrive during this glue period, which equals $\lambda_i\mathbb{E}[G_i]$. Combining these three parts gave Equation (2.13) for $\mathbb{E}[Y_i^{(iq)}]$. It is known that $\rho_i\mathbb{E}[C] = \mathbb{E}[Y_i^{(iq)}]\mathbb{E}[B_i]$, and thus $\mathbb{E}[Y_i^{(iq)}] = \lambda_i\mathbb{E}[C]$. From this $\mathbb{E}[Z_i^{(i)}]$ and $\mathbb{E}[F_i]$ can be determined.

$$\mathbb{E}[Y_i^{(iq)}] = (1-\tilde{G}_i(\nu_i))\mathbb{E}[Z_i^{(i)}] + (1-\tilde{G}_i(\nu_i))(1-\rho_i)\lambda_i\mathbb{E}[C] + \tilde{G}_i(\nu_i)\mathbb{E}[G_i]\lambda_i, \tag{2.13}$$

$$\mathbb{E}[Z_i^{(i)}] = \lambda_i\rho_i\mathbb{E}[C] + \frac{\lambda_i\tilde{G}_i(\nu_i)}{1-\tilde{G}_i(\nu_i)}(\mathbb{E}[C]-\mathbb{E}[G_i]), \tag{2.14}$$

$$\mathbb{E}[F_i] = \rho_i^2\mathbb{E}[C] + \frac{\rho_i\tilde{G}_i(\nu_i)}{1-\tilde{G}_i(\nu_i)}(\mathbb{E}[C]-\mathbb{E}[G_i]). \tag{2.15}$$

In Equation (2.15) there are two terms. The first term, $\rho_i^2\mathbb{E}[C]$ is the amount of work of customers who arrive at station i during its own visit period. The second part of the equation equals the amount of work of customers for station i who were waiting to retry at station i at the beginning of the visit period of station i. This can be found by rewriting the fraction:

$$\frac{\rho_i\tilde{G}_i(\nu_i)}{1-\tilde{G}_i(\nu_i)}(\mathbb{E}[C]-\mathbb{E}[G_i]) = \sum_{k=1}^{\infty}(\tilde{G}_i(\nu_i))^k\rho_i(\mathbb{E}[C]-\mathbb{E}[G_i]). \tag{2.16}$$

The mean number of customers arriving during a cycle, but not the glue period of a cycle, is $\lambda_i(\mathbb{E}[C]-\mathbb{E}[G_i])$. $(\tilde{G}_i(\nu_i))^k\rho_i(\mathbb{E}[C]-\mathbb{E}[G_i])$ is the amount of work of customers who arrived during the k^{th} previous cycle, but not during its glue period.
Now since $\mathbb{E}[C] = \frac{1}{1-\rho}\sum_{i=1}^N(\mathbb{E}[G_i]+\mathbb{E}[S_i])$, Equation (2.17) can be obtained from (2.12) and (2.15).

$$\begin{aligned}\sum_{i=1}^N \rho_i\mathbb{E}[W_i] =& \rho\left(\frac{\sum_{i=1}^N\lambda_i\mathbb{E}[B_i^2]}{2(1-\rho)} + \frac{\mathbb{E}[(\sum_{i=1}^N(S_i+G_i))^2]}{2\mathbb{E}[\sum_{i=1}^N(S_i+G_i)]}\right) + \frac{\mathbb{E}[\sum_{i=1}^N(S_i+G_i)]}{2(1-\rho)}\left(\rho^2+\sum_{i=1}^N\rho_i^2\right)\\ &+ \sum_{i=1}^N\frac{\rho_i\tilde{G}_i(\nu_i)}{1-\tilde{G}_i(\nu_i)}\left(\frac{\mathbb{E}[\sum_{i=1}^N G_i+S_i]}{1-\rho}-\mathbb{E}[G_i]\right).\end{aligned} \tag{2.17}$$

Since our glue period distribution is either exponentially distributed or deterministic, the Laplace Stieltjes transform for these can be determined. If the distribution is exponential:

$$\tilde{G}_i(\nu_i) = \frac{1/\mathbb{E}[G_i]}{1/\mathbb{E}[G_i] + \nu_i} = \frac{1}{1 + \mathbb{E}[G_i]\nu_i}. \tag{2.18}$$

If the distribution is deterministic, G_i is a real number and $\tilde{G}_i(\nu_i) = e^{-\nu_i \cdot G_i}$.

Chapter 3

Simulation

In this chapter we will explain how the simulation works. The simulation is made in Java.

3.1 Polling system without glue periods

First the simulation without the glue periods will be explained. The simulation is made so that it has either exhaustive or gated service. In this simulation there are three possible events, the arrival of a customer, the end of a switch-over and the end of service of a customer. The explanation of these events can be found below.

3.1.1 Arrival of a customer

At the start of the simulation the first arrival for each station is determined. From then on the arrival of the next customer is planned when a customer arrives. When a customer for station i arrives, the simulation determines the time the next customer for station i arrives. The algorithm for the arrival of a customer is given in Algorithm 1.

Algorithm 1 Arrival of a customer

Require: time: current time, Qn: integer that gives which queue the customer arrives in, Queues: list with the Arraylist of customers for each station, Eventlist: list of future events, arrivalDistribution: list with the distributions for the inter-arrival times for each queue

Ensure: Updated Queues and Eventlist

$c \leftarrow newCustomer(time)$

$Queues.get(Qn).addCustomer(c)$ ▷ adds customer to queue

$t \leftarrow arrivalDistribution.get(Qn).nextRandom()$ ▷ determines the inter-arrival time between the current and next customer

$e \leftarrow newEvent(Arrival, time + t)$ ▷ Make next arrival event

$e.setQueue(QN)$ ▷ set the queue the customer will arrive in

$Eventlist.add(e)$ ▷ add event to list of events in time order

3.1.2 End of switch-over

At the end of a switch-over the server just finished going from station $i-1$ to station i. First the length of the queue of station i will be checked, if there are no customers in this queue, the next END OF SWITCH-OVER event will be planned. What happens when there are customers in the queue will differ for exhaustive and gated service. In case of exhaustive service the only thing that will happen is that the end of service will be planned for the first customer in the queue. In the case of gated service, a new temporary queue will be made. All the customers, that are at queue i the moment the server arrives, will be transferred to this temporary queue: $gatedQ$. After this THE END OF SERVICE for the first customer in this temporary queue will be determined. The algorithms for the exhaustive and gated service can be seen respectively in Algorithms 2 and 3.

Algorithm 2 End of switch-over, Exhaustive service

Require: time: current time, Qn: integer that gives which queue server is at, N: total number of queues, Ncycles: number of rounds the server has made, Queues: list with the Arraylist of customers for each station, Eventlist: list of future events, Result: object that saves wanted information, serviceDistribution: list of distributions for service time of a customer, switchDistribution: list of distributions of the switch-over times

Ensure: Updated Qn and Eventlist

if $Qn \geq N-1$ **then** ▷ update the queue the server is at
  $Qn \leftarrow 0$
  $Ncycles \leftarrow Ncycles + 1$
else
  $Qn \leftarrow Qn + 1$
end if
$Result.updateStart(Queues, Qn)$ ▷ update result at start of a service period
if $Size(Queues.get(Qn)) > 0$ **then** ▷ check if there are customers in the queue
  $t \leftarrow serviceDistribution.get(Qn).nextRandom()$
  $e \leftarrow newEvent(EndOfService, time + t)$
  $Eventlist.add(e)$ ▷ create first end of service

  $c \leftarrow Queues.get(Qn).getFirstCustomer()$
  $Result.updateRandom.(c, time, currentQ)$ ▷ update result for random time
else
  $Result.updateEnd(Queues, Qn)$ ▷ update results at end of service period
  $t \leftarrow switchDistribution.get(Qn).nextRandom()$
  $e \leftarrow newEvent(EndOfSwitchover, time + t)$
  $Eventlist.add(e)$ ▷ create the end of the next switch-over
end if

Algorithm 3 End of switch-over, gated service

Require: time: current time, Qn: integer that gives which queue server is at, N: total number of queues, Ncycles: number of rounds the server has made, Queues: list with the Arraylist of customers for each station, Eventlist: list of future events, Result: object that saves wanted information, serviceDistribution: list of distributions for service time of a customer, switchDistribution: list of distributions of the switch-over times

Ensure: Updated Qn and Eventlist, makes temporary gatedQueue which stores to be served customers

```
if Qn ≥ N − 1 then                                      ▷ update the queue the server is at
    Qn ← 0
    Ncycles ← Ncycles + 1
else
    Qn ← Qn + 1
end if
Result.updateStart(Queues, Qn)                ▷ update result at start of a service period
gatedQueue ← Queues.get(Qn).copieQ()
Queues.get(Qn).emptyQueue()  ▷ move all customer from station Qn to temporary queue
if Size(gatedQueue) > 0 then                  ▷ check if there are customers in the queue
    t ← serviceDistribution.get(Qn).nextRandom()
    e ← newEvent(EndOfService, time + t)
    Eventlist.add(e)                                        ▷ create first end of service

    c ← gatedQueue.getFirstCustomer()
    Result.updateRandom.(c, time, currentQ)             ▷ update result for random time
else
    Result.updateEnd(Queues, Qn)               ▷ update results at end of service period
    t ← switchDistribution.get(Qn).nextRandom()
    e ← newEvent(EndOfSwitchover, time + t)
    Eventlist.add(e)                               ▷ create the end of the next switchover
end if
```

3.1.3 End of service

The END OF SERVICE consists of two steps. First the customer that was served will be removed from the queue that is being served, then the next event will be planned. If there are still customers in the queue being served, the END OF SERVICE of the next customer in the queue will be planned. If the queue that is being served is empty the end of switch-over will be planned. This is for both exhaustive and gated service, the only difference is which queue is being served. If the service discipline is exhaustive, then the queue of the station the server is at will be served. But if the service discipline is gated, then the temporary queue *gatedQ* will be served instead. The algorithm for the END OF SERVICE can be seen below. In case of gated service, the *servQueue* will be the queue that was created at the end of switch-over, *gatedQueue*. Otherwise it will be $Queues.get(Qn)$, the queue of the station the server is currently at.

Algorithm 4 End of service

Require: time: current time, Qn: integer that gives which queue the server is at, servQueue: Queue of customers that need to be served, Eventlist: list of future events, serverDistribution: list of distributions for service time of a customer, switchDistribution: list of distributions of the switchovertimes
Ensure: Updated Queues, servQueue and Eventlist

Customer $c \leftarrow servQueue.getFirstCustomer()$ ▷ get next customer to be served
$Result.updateRandom(c, time, currentQ)$ ▷ update result for random time
if $size(servQueue) > 0$ **then** ▷ there are still customers that need to be served
  $t \leftarrow serviceDistribution.get(Qn).nextRandom()$
  $e \leftarrow newEvent(EndOfService, time + t)$
  $Eventlist.add(e)$ ▷ create next end of sevice

  $c \leftarrow gatedQueue.getFirstCustomer()$
  $Result.updateRandom.(c, time, currentQ)$ ▷ update result for random time
else ▷ no more customers to be served
  $Result.updateEnd(Queues, Qn)$ ▷ update results at end of service period
  $t \leftarrow switchDistribution.get(Qn).nextRandom()$
  $e \leftarrow newEvent(EndOfSwitchover, time + t)$
  $Eventlist.add(e)$ ▷ create the end of the next switchover
end if

3.2 Polling system with glue periods

After the simulation of a polling system without glue periods, the simulation with glue periods will be explained. As mentioned before, if a customer arrives at a station, and it is not a glue period for this station, then the customer will leave and return after a certain time to try again. If it is a glue period for the station the customer arrived at, then the customer will enter a temporary queue. At the end of the glue period, all the customers that are in the temporary queue will be served.
The simulation without glue periods has three events: arrival of a customer, the end of a switch-over and the end of the service of a customer. The simulation of a system with glue periods has these three events as well, and two more events are added to the simulation: the return of a customer and the end of a glue period. The explanation of the five events can be found below.

3.2.1 Arrival of a customer

At the ARRIVAL OF A CUSTOMER, first-time arriving customers of the system arrive. If a customer for queue i arrives, and it is a glue period for queue i, then the customer will be put in the 'glue' queue, where he will be waiting for service. If a customer arrives at station i and it is not a glue period for station i, then the customer will be put in the 'return' queue for station i. Also the event return of a customer will be planned for this customer.

Algorithm 5 Arrival of a customer

Require: time: current time, Qn: integer that gives which queue the customer arrives in, Queues: list with the Arraylist of customers for each station who are retrying, glueQueue: queue of glued customer, Eventlist: list of future events, arrivalDistribution: list with the distributions for the inter-arrival times for each queue, retryDistribution: list with the distributions for the retry time, glue(x): TRUE if it is the glue period for queue x else FALSE

Ensure: Updated Queues, glueQueue and Eventlist

Customer $c \leftarrow newCustomer(time)$

if glue(Qn) **then** ▷ Check if customer can join glue queue

  $glueQueue.addCustomer(c)$ ▷ adds customer to glue queue

else ▷ Customer will try again later

  $t \leftarrow retryDistribution.get(Qn).nextRandom()$ ▷ determines time till the customer tries again (returns)

  $e \leftarrow newEvent(Return, time + t)$ ▷ Make Return event

  $e.setQueue(QN)$ ▷ set the queue the customer will arrive in

  $Eventlist.add(e)$ ▷ add event to list of events in time order

  $c.setReturnTime(time + t)$

  $Queues.get(Qn).addCustomer(c)$ ▷ add customer in queue in return time order

end if

$t \leftarrow arrivalDistribution.get(Qn).nextRandom()$ ▷ determines the inter-arrival time between the current and next customer

$e \leftarrow newEvent(Arrival, time + t)$ ▷ Make next arrival event

$e.setQueue(QN)$ ▷ set the queue the customer will arrive in

$Eventlist.add(e)$ ▷ add event to list of events in time order

3.2.2 Return of a customer

This event is when a customer arrived at the system before, but had to leave because it was not a glue period for the station. Here the same thing happens as at the first arrival of a customer. If there is a glue period for the queue the customer arrived for, then he will be put in the 'glue' queue. In case it is not a glue period for the station the customer arrived at, then the customer will be placed back in the 'return' queue, and another RETURN OF CUSTOMER event for this customer will be planned.

Algorithm 6 Return of a customer

Require: time: current time, Qn: integer that gives which queue the customer arrives in, servQueue: Queue of customers that need to be served, Eventlist: list of future events, retryDistribution: list with the distributions for the retry time, glue(x): TRUE if it is the glue period for queue x else FALSE

Ensure: Updated Queues, glueQueue and Eventlist

```
Customer c ← Queues.getFirstCustomer(Qn)
if glue(Qn) then                                    ▷ Check if customer can join glue queue
    glueQueue.addCustomer(c)                        ▷ adds customer to glue queue
else                                                ▷ Customer will try again later
    t ← retryDistribution.get(Qn).nextRandom()      ▷ determines time till the customer
tries again (returns)
    e ← newEvent(Return, time + t)                  ▷ Make Return event
    e.setQueue(QN)                                  ▷ set the queue the customer will arrive in
    Eventlist.add(e)                                ▷ add event to list of events in time order

    c.setReturnTime(time + t)
    Queues.get(Qn).addCustomer(c)                   ▷ add customer in queue in return time order
end if
```

3.2.3 End of switch-over

At the end of a switch-over, the event END OF GLUE PERIOD will be planned. Also, the variable for at which queue the server is and the variable for indicating if it is a glue period for the current queue, is updated.

Algorithm 7 End of switch-over

Require: time: current time, Qn: integer that gives which queue the customer arrives in, servQueue: Queue of customers that need to be served, Eventlist: list of future events, glue(x): TRUE if it is the glue period for queue x else FALSE, glueDistribution: list of distributions for servertime of a customer

Ensure: Updated Qn, glue(x) and Eventlist

```
if Qn ≥ N − 1 then                                  ▷ update the queue the server is at
    Qn ← 0
    Ncycles ← Ncycles + 1
else
    Qn ← Qn + 1
end if
glue(Qn) ← TRUE
t ← glueDistribution.get(Qn).nextRandom()
e ← newEvent(EndOfGlue, time + t)
Eventlist.add(e)                                    ▷ create the end of the glue period
```

3.2.4 End of glue period

In comparison to the basic simulation, the service of customers now starts in the end of a glue period. In case the 'glue' queue is not empty the event END OF SERVICE of a customer is planned. If the 'glue' queue is empty, the next END OF SWITCH-OVER will be planned.

Algorithm 8 End of glue period

Require: <u>time:</u> current time, <u>Qn:</u> integer that gives which queue server is at, <u>N:</u> total number of queues, <u>Ncycles:</u> number of rounds the server has made, <u>Queues:</u> list with the Arraylist of customers for each station, <u>glueQueue:</u> Queue with customers that arrived during glue period, <u>glue(x):</u> TRUE if it is the glue period for queue x else FALSE, <u>Eventlist:</u> list of future events, <u>Result:</u> object that saves wanted information, <u>serviceDistribution:</u> list of distributions for service time of a customer, <u>switchDistribution:</u> list of distributions of the switch-over times
Ensure: Updated Qn, glueQueue, glue(x) and Eventlist

```
glue(Qn) ← FALSE
Result.updateStart(Queues, Qn)              ▷ update result at start of a service period
if Size(glueQueue) > 0 then                 ▷ check if there are customers in the queue
    t ← serviceDistribution.get(Qn).nextRandom()
    e ← newEvent(EndOfService, time + t)
    Eventlist.add(e)                        ▷ create first end of service

    c ← gatedQueue.getFirstCustomer()
    Result.updateRandom.(c, time, currentQ) ▷ update result for random time
else
    Result.updateEnd(Queues, Qn)            ▷ update results at end of service period
    t ← switchDistribution.get(Qn).nextRandom()
    e ← newEvent(EndOfSwitchover, time + t)
    Eventlist.add(e)                        ▷ create the end of the next switchover
end if
```

3.2.5 End of service

At the END OF SERVICE of a customer, it is determined if the 'glue' queue is empty or not. If it is empty then the event END OF SWITCH-OVER will be planned, otherwise, the END OF SERVICE for the next customer in the 'glue' queue will be determined and planned. This means that the algortihm is the same as Algorithm 4, with the *servQueue* the *glueQueue*.

Chapter 4

Results

In this chapter the results of the simulation will be showed and discussed. The results of the simulation will be compared with results found in other research. We will start with the system without glue periods, and later continue with the system with glue periods.

4.1 Results of the system without glue periods

First the simulation will be run with the same parameters as in examples 3.1 and 3.3 in article [5]. The outcomes of the simulation can be compared with the examples. In these examples both exhaustive and gated have two stations, where both the service and switch-over times follow an exponential distribution with mean 1. The arrivals take place according to Poisson processes with $\lambda_1 = 0.6$ and $\lambda_2 = 0.2$.
The mean waiting time determined in [5] can be seen in Table 4.1. The mean waiting time for station i is $\mathbb{E}[W_i]$. In this Table the mean queue length can also be seen, determined with the following formula: $\mathbb{E}[L_i] = \lambda\mathbb{E}[W_i] + \lambda\mathbb{E}[B_i]$, here $\mathbb{E}[L_i]$ stands for the average number of customers in the queue at station i and $\mathbb{E}[B_i]$ is the mean service time for a customer.
The simulation has the same parameters as the example. To calculate the mean waiting time and its confidence interval the simulation will run 100 times, where in each run the server makes 10.000 rounds.

Service discipline	$\mathbb{E}[W_1]$	$\mathbb{E}[W_2]$	$\mathbb{E}[L_1]$	$\mathbb{E}[L_2]$
Exhaustive	5.2	11.5	3.9	2.5
Gated	12.7701	9.6898	8.2320	2.1280

Table 4.1: Mean waiting time for the settings in Winands [5]

The results for the simulation can be found in Table 4.2. In this Table the 95% confidence intervals for the queue lengths can be found as well. The exact values of the queue lengths lies in the confidence interval, so the simulation gives a good approximation of the queue lengths.

			Confidence Interval	
Service discipline	$\mathbb{E}[L_1]$	$\mathbb{E}[L_2]$	$\mathbb{E}[L_1]$	$\mathbb{E}[L_2]$
Exhaustive	3.8899	2.4939	(3.8702; 3.9096)	(2.4762; 2.5116)
Gated	8.2595	2.1359	(8.2104; 8.3085)	(2.1234; 2.1483)

Table 4.2: Values found for mean queue length through simulating

The queue length at specific moments in time is also determined, at the start and the end of a service period. The random variables for the queue length at the start or at the end of a service period are respectively L_i^s and L_i^e. In Table 4.3 the mean queue lengths and the variations of the queue lengths at specific moments for both the exhaustive and gated service discipline can be seen.
Table 4.3 shows that the average queue length of station i at the end of service of station i is zero. This is because the end of service with an exhaustive service discipline will only come when the queue of that station is empty.
This means that the average queue length at station i at the end of service of station j gives the average number of arrivals during the switchover time from station i to station j and service of station j. It is interesting that with exhaustive service both of these are relatively close (compared to their different arrival rates), which might be because the service period length at station 1 is longer than at station 2, since there are more customers to serve (on average) at station 1. During this longer service time, there is more time for station 2 customers to arrive, leading to a relatively high end of service of station 1 queue length average. With gated service the ratio between the average queue length of station i at the end of service of station j is relatively the same as ratio between the arrival rates. This is because the customers in the queue of station i at the end of the service period of the station j had a switch-over period and the service periods of both station 1 and 2 to arrive. For which the average duration is the same, no matter which station is started from, since the average switch-over time between the stations is the same.
Some of the variations of the queue lengths are relatively high as compared to the average queue length, it means that even though the average queue length is low it is possible (and even likely) that a queue length occurs that is much higher than the mean.

	Location of server	At start of service				At end of service			
		$\mathbb{E}[L_1^s]$	$\mathbb{E}[L_2^s]$	$\sigma^2[L_1^s]$	$\sigma^2[L_2^s]$	$\mathbb{E}[L_1^e]$	$\mathbb{E}[L_2^e]$	$\sigma^2[L_1^e]$	$\sigma^2[L_2^e]$
Exhaustive	Queue 1	2.40	0.20	8.16	0.24	0	1.40	0	6.00
	Queue 2	0.60	1.60	0.96	6.24	1.80	0	7.20	0
Gated	Queue 1	6.00	0.60	27.46	0.90	3.60	1.80	15.64	4.22
	Queue 2	4.20	2.00	16.60	4.46	5.40	0.40	26.50	0.66

Table 4.3: Average queue length and variation at start and end of service

The covariance between the queue lengths at a random and specific time can be found in Table 4.4.
The at end of service covariance of the queue length of the system with exhaustive service is zero, which means that the correlation between the queue length of station 1 and station 2 also is zero. This is understandable since the queue length of one of the two stations is always

zero at the end of a service period. The other covariances are positive, indicating that if the queue length of one station are longer the queue length of the other station are also longer.

	Location server	Exhaustive	Gated
$Cov(L_1, L_2)$	–	2.25	5.71
$Cov(L_1^s, L_2^s)$	Queue 1	0.12	1.96
$Cov(L_1^s, L_2^s)$	Queue 2	0.12	5.30
$Cov(L_1^e, L_2^e)$	Queue 1	0	5.19
$Cov(L_1^e, L_2^e)$	Queue 2	0	1.84

Table 4.4: The co-variance for the length of the queues

4.2 Results of the system with glue periods

Next, the system with glue periods will be studied. The comparison in this part of the results will be done with several numerical examples from the article [1].

4.2.1 Deterministic glue periods

First a polling system with two queues will be considered. The arrival process is a Poisson process. The return, switch-over and service times times are also exponentially distributed, but the glue periods are deterministic.
The parameters of queue 2 will be varied and the parameters of queue 1 will be fixed, $\lambda_1 = 1$, $\mathbb{E}[B_1] = 0.45$, $\mathbb{E}[S_1] = 1$, $G_1 = 0.5$ and $\nu_1 = 1$. The values of the parameters of queue 2 can be found in Table 4.5. The exact values for the waiting time can be found in article [1]. The simulation runs 500 times, where the server makes 7500 full rounds each. The confidence intervals are 95% confidence intervals. The results of this simulation, as well as the exact results can be seen in Table 4.6.

Parameter	parameters of the second station				
set number	λ_2	$\mathbb{E}[B_2]$	$\mathbb{E}[S_2]$	G_2	ν_2
$D1$	1	0.45	1	0.5	1
$D2$	0.5	0.45	1	0.5	1
$D3$	0.5	0.2	1	0.5	1
$D4$	0.5	0.2	2	0.5	1
$D5$	0.5	0.2	2	1	1
$D6$	0.5	0.2	2	1	0.5

Table 4.5: The parameter sets for which the polling system will be studied

Parameter set number	exact values $\mathbb{E}[W_1]$	$\mathbb{E}[W_2]$	simulated values $\mathbb{E}[W_1]$	$\mathbb{E}[W_2]$	confidence intervals $\mathbb{E}[W_1]$	$\mathbb{E}[W_2]$
$D1$	71.61	71.61	71.3702	71.3671	$(71.2370, 71.5033)$	$(71.2345, 71.4997)$
$D2$	21.44	20.34	21.4016	20.3190	$(21.3787, 21.4245)$	$(20.2958, 20.3421)$
$D3$	15.18	13.96	15.1719	13.9483	$(15.1582, 15.1857)$	$(13.9340, 13.9626)$
$D4$	20.52	18.82	20.4970	18.9086	$(20.4784, 20.5156)$	$(18.8907, 18.9265)$
$D5$	23.01	11.48	22.9653	11.5756	$(22.9460, 22.9847)$	$(11.5654, 11.5859)$
$D6$	22.97	20.31	22.9298	20.3858	$(22.9111, 22.9485)$	$(20.3678, 20.4038)$

Table 4.6: Exact and simulated mean waiting times with deterministic glue periods,

In Table 4.6 it can be seen that halving the arrival intensity of one of the two stations more than halves the average waiting time for both stations. The influence does not differ much for the different stations, just as with lowering the average service time. Lowering service time does not have as much influence as lowering the arrival rate did to the waiting times. Only changing the glue time and the ν_2 seem to result in a big difference in the average waiting time for different stations ($D5$ and $D6$ in the table)

4.2.2 Exponential glue periods

Next a polling system with exponential glue periods and three stations will be considered. Here the switch-over times are deterministic. The inter-arrival, return and service times are exponentially distributed. The parameters of queue 1 stay fixed, as well as the switch-over times and retrial rates. The parameters are $\lambda_1 = 1$, $\mathbb{E}[B_1] = 0.3$, $\mathbb{E}[G_1] = 0.5$, $S_1 = S_2 = S_3 = 1$ and $\nu_1 = \nu_2 = \nu_3 = 1$. The rest of parameters are as shown in Table 4.7.
In Table 4.8 the results of the simulation are shown and in Table 4.9 the 95% confidence intervals are given. As seen before the results seem to approximate the exact values closely. Only with parameter set $E4$ the simulated values differ slightly more from the exact values. Since the utilization rate is $\rho = 0.95$ with parameter set $E4$, the system is a very busy system. Thus the waiting times will vary more than in other systems. Here the simulation runs 500 times with 6000 cycles each run.

Parameter set number	parameters of the second and third station λ_2	$\mathbb{E}[B_2]$	$\mathbb{E}[G_2]$	λ_3	$\mathbb{E}[B_3]$	$\mathbb{E}[G_3]$
$E1$	1	0.3	0.5	1	0.3	0.5
$E2$	1	0.3	0.5	0.5	0.3	0.5
$E3$	1	0.3	0.5	0.5	0.1	0.5
$E4$	2	0.3	0.5	0.5	0.1	0.5
$E5$	2	0.15	0.5	0.5	0.1	0.5
$E6$	2	0.15	2	0.5	0.1	0.5
$E7$	2	0.15	2	0.5	0.1	1

Table 4.7: The parameter sets for which the polling system will be studied

Parameter set number	exact values $\mathbb{E}[W_1]$	$\mathbb{E}[W_2]$	$\mathbb{E}[W_3]$	simulated values $\mathbb{E}[W_1]$	$\mathbb{E}[W_2]$	$\mathbb{E}[W_3]$
$E1$	121.0	121.0	121.0	120.7477	120.8516	120.8550
$E2$	47.59	47.58	46.74	47.5016	47.5177	46.6693
$E3$	33.65	33.64	32.54	33.6512	33.6049	32.5098
$E4$	246.8	246.6	242.3	245.5609	252.5763	240.2830
$E5$	33.52	33.51	32.42	33.5127	33.4857	32.3607
$E6$	44.88	19.71	43.64	44.8606	19.7055	43.6039
$E7$	48.66	21.42	28.75	48.5913	21.4026	28.7138

Table 4.8: Exact and simulated mean waiting times with deterministic glue periods

parameter set number	confidence interval $\mathbb{E}[W_1]$	$\mathbb{E}[W_2]$	$\mathbb{E}[W_3]$
$E1$	$(120.5674, 120.9280)$	$(120.6648, 121.0384)$	$(120.6737, 121.0363)$
$E2$	$(47.4477, 47.5554)$	$(47.4628, 47.5727)$	$(46.6086, 46.7300)$
$E3$	$(33.6151, 33.6873)$	$(33.5684, 33.6414)$	$(32.4710, 32.5486)$
$E4$	$(245.0357, 246.0861)$	$(252.0456, 253.1071)$	$(239.7559, 240.8101)$
$E5$	$(33.4791, 33.5463)$	$(33.4530, 33.5184)$	$(32.3245, 32.3969)$
$E6$	$(44.8116, 44.9097)$	$(19.6917, 19.7193)$	$(43.5535, 43.6543)$
$E7$	$(48.5389, 48.6437)$	$(21.3881, 21.4172)$	$(28.6840, 28.7436)$

Table 4.9: Confidence interval for mean waiting time, exponential glue periods

In Table 4.8 it can be seen, similarly as with deterministic glue periods, that changes in arrival rate and mean service time result into similar changes to all three stations. This while, by changes in mean glue time and ν (retry time variable), the changes in queue length seem to be opposite for the station the variables changed for and the other stations.

4.2.3 Queue lengths for varying mean glue time

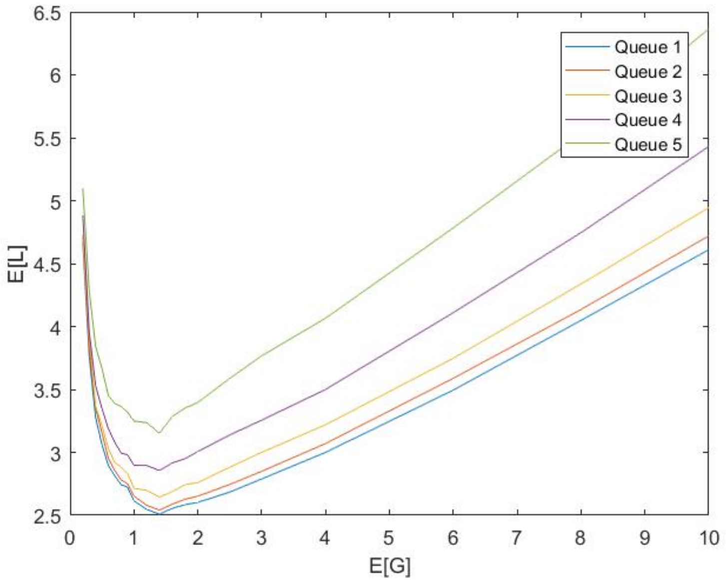
Next the behavior of the queue length for several different mean glue times will be studied. Here the queue length means the station size, thus all customers in orbit and glue queue for a station. Not only the mean queue length will be determined but also the squared coefficient of variation for each queue length and the correlation between queue lengths of different stations. In Equations (4.1) and (4.2) the formulas for the squared coefficient of variation (SCV) and the correlation (Cor) are given.

$$SCV(L_i) = \frac{Var(L_i)}{\mathbb{E}[L_i]^2}. \tag{4.1}$$

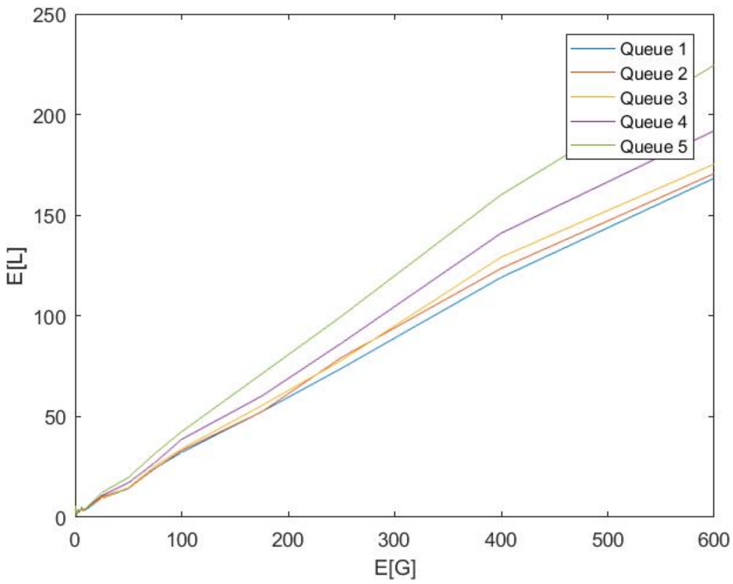
$$Cor(L_i, L_j) = \frac{\mathbb{E}[L_i L_j] - \mathbb{E}[L_i]\mathbb{E}[L_j]}{\sqrt{Var(L_i)Var(L_j)}}. \tag{4.2}$$

In this simulation the switch-over times are deterministic, the glue periods are exponentially distributed and there are five stations. The arrival rate of customers for station i is $\lambda_i = 0.025$ for all i. The mean service times, $\mathbb{E}[B_1]$, $\mathbb{E}[B_2]$, $\mathbb{E}[B_3]$, $\mathbb{E}[B_4]$ and $\mathbb{E}[B_5]$, are respectively 1, 2,

4, 8 and 16. The mean switch-over times, $\mathbb{E}[S_i]$ are 1 for all i. The retrial rate is $\nu_i = 1$, also for all i. Lastly $\mathbb{E}[G_{i]}] = \mathbb{E}[G]$, thus the mean glue length is the same for each queue.
In Figures 4.1, 4.2 and 4.3 the results are given, for the mean queue length, the squared coefficient of variation and the correlation, respectively.



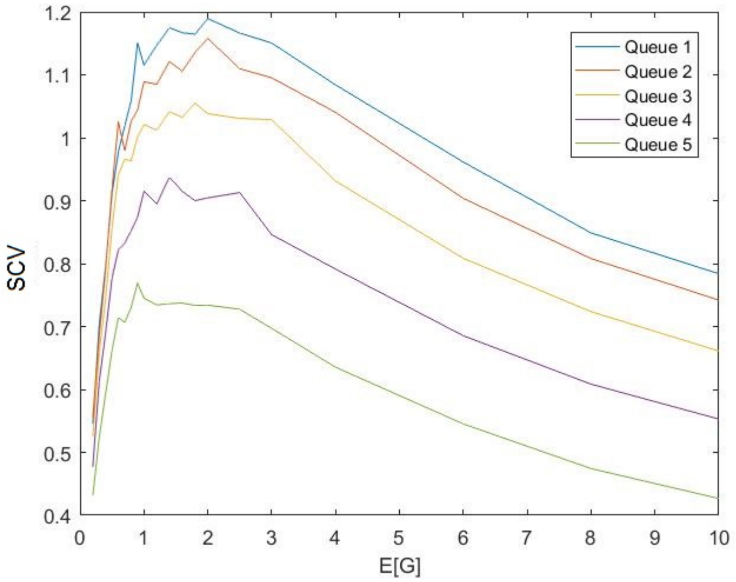
(a) $\mathbb{E}[L_i]$, $\mathbb{E}[G] \in (0, 10)$



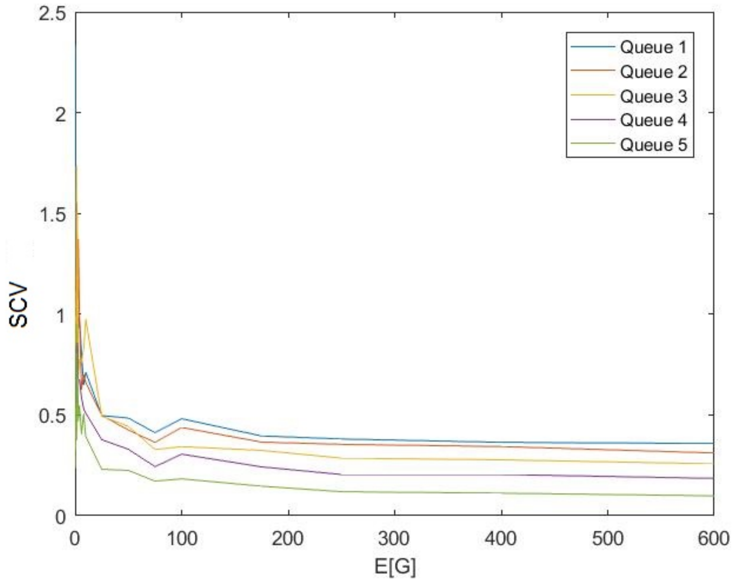
(b) $\mathbb{E}[L_i]$, $\mathbb{E}[G] \in (0, 600)$

Figure 4.1: Mean queue length for varying $\mathbb{E}[G]$

In Figure 4.1 the plots of the mean queue length can be seen. It is visible that if the mean glue time is larger than 3 the mean queue length grows linear. You can also see that there is a minimum queue length, which takes place where the value of the mean glue time is around 1.3.



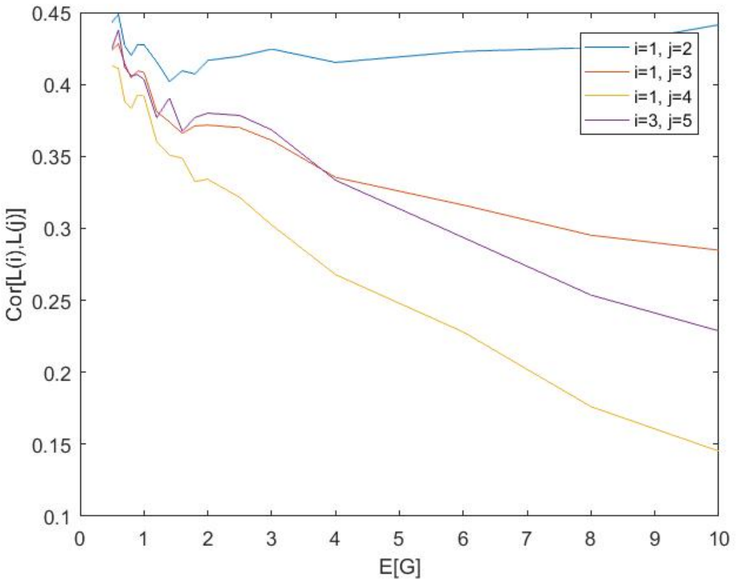
(a) $\mathrm{SCV}(L_i)$, $\mathbb{E}[G] \in (0, 10)$



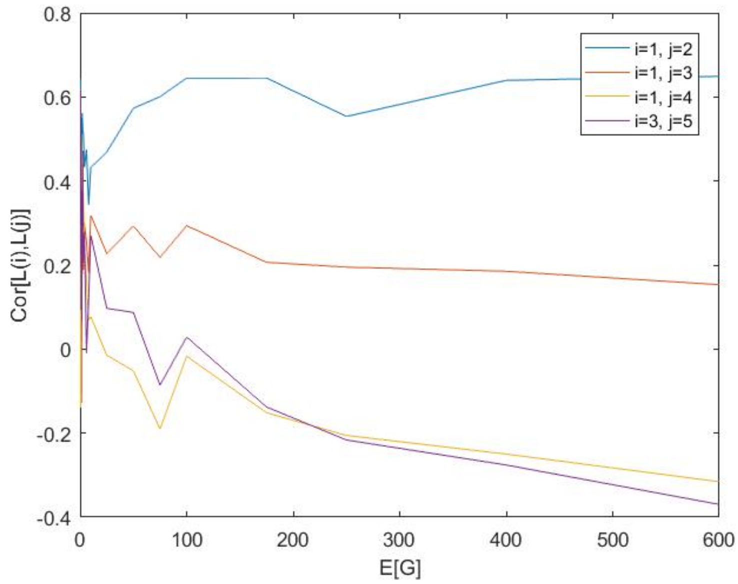
(b) $\mathrm{SCV}(L_i)$, $\mathbb{E}[G] \in (0, 600)$

Figure 4.2: Squared coefficient of variation for varying $\mathbb{E}[G]$

In Figure 4.2 the squared coefficient of variation can be seen. It shows that at a similar location as the minimum waiting time, there is a maximum in squared coefficient of variation. The SCV takes a value of higher than 1 for several queues. This means that here the variation of the queue length is bigger than the mean queue length. Indicating that some queue lengths occur that are very high in comparison with the mean queue length. The squared coefficient of variation seems to converge to a constant for a higher mean glue time.



(a) Cor(L_i,L_j), $\mathbb{E}[G] \in (0, 10)$



(b) Cor(L_i,L_j), $\mathbb{E}[G] \in (0, 600)$

Figure 4.3: Correlation between queue lengths for varying $\mathbb{E}[G]$

In Figures 4.3a and 4.3b the correlation between several queue lengths can be seen, for values of the mean glue time in respectively the intervals $(0, 10)$ and $(0, 600)$. The correlation between the queue lengths for stations 1 and 2, 1 and 3, 1 and 4 and lastly 3 and 5 are given. For low mean glue times it can be seen that the correlations seem to have similar values.

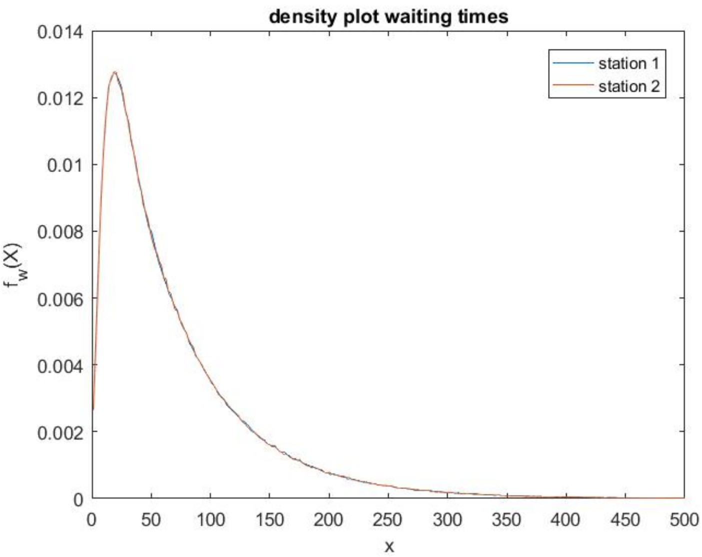
The correlation between two queue lengths indicates how one queue length changes if the other queue length changes. For low mean glue times the correlations are positive, This means that the growth of the queue length of one station results in the growth of the queue length of another station. For large mean glue time, there are two correlations that are negative, meaning that if the queue length in one of the two stations grows, the queue length of the other station decreases. Station sizes are continuously growing, except during the service period of that station. Since the service periods of two successive station are close to one another there is a long interval after the second service period for which the station sizes of both stations are growing. Indicating that for a big part of the time the two stations behave similarly, which might be the cause for the positive correlation. For two stations that are not successive there are two shorter intervals in between the service periods, which might be the reason the correlation is negative.

4.2.4 Waiting time distribution

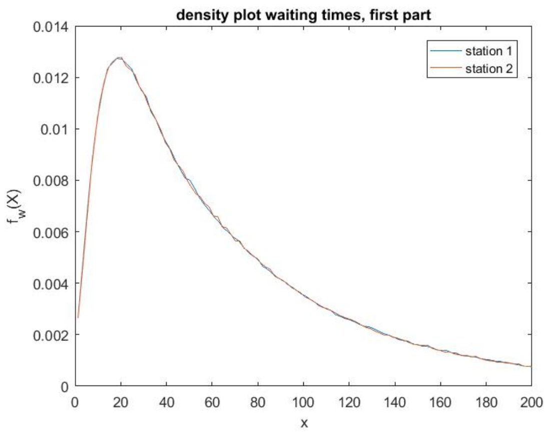
Next the distribution of the waiting time will be studied. In Figures 4.4 - 4.10, the density and cumulative distribution function for the waiting times at different stations are shown. In the second plot there is zoomed in on the first part of the density plot. Each figure is made with parameters from subsection 4.2.1 and 4.2.2, which can be found in Table 4.5 and 4.7. The figures named with a 'D' have deterministic glue periods and exponential switch-over times, and the ones named with an 'E' have exponential glue periods and deterministic switch-over times. In Table 4.10 the approximated locations of the peaks are given.
The waiting time distributions for different parameters follow the same pattern, only the locations and heights of the peaks differ. As the waiting time approaches zero it seems that the probability of that waiting time occuring also approaches zero. The only time when it is possible that the waiting times are zero is if the server can start serving the customer the moment they arrive. Here the server does not start serving until the end of the glue period, where customers that will be served arrive in. This means that a customer always has to wait at least until the glue period ends, even if he arrived almost at the end of a glue period.
Another thing that can be noticed is that the probability for a high waiting time is low. This might be because this is only possible if the retries of a customer fall outside of the glue period multiple times. The probability that the waiting time is longer than the waiting time at the location of the peak of the graph is big because most of the area of below the graph is on the right side of the peak.
In Figure 4.10 the waiting density of a polling system with parameter set $E7$ can be seen. Here it clearly can be seen that different stations can have very different locations for the peak, while the peaks for the other parameter sets seem to be at similar points, as can be seen in Table 4.16. The average waiting times for different stations differ quite a bit as well, with station 1 having the biggest waiting time of 48.66, and station 2 and 3 respectively having average waiting times of 21.42 and 28.75. It is interesting that for location of the peaks in the graph the peak for station 2 comes first, then station 1 and lastly station 3. And station 1 seems to have the biggest tail out of the three stations. This means that the steepness of the graph after the peak can slightly differ depending on the parameters.
Figure 4.11 shows the density plot with different cycle lengths. Here it is seen that if the simulation runs more cycles, the peak height is similar. The only difference is that if the cycle length is longer, the graph is more smooth. Meaning that density is properly approximated by the simulation.

parameter set number	Location of peak W_1	W_2	W_3
$D1$	20	20	-
$D3$	3	1	-
$D5$	5	1	-
$E1$	31	30	33
$E3$	8	8	8
$E5$	9	10	10
$E7$	13	10	16

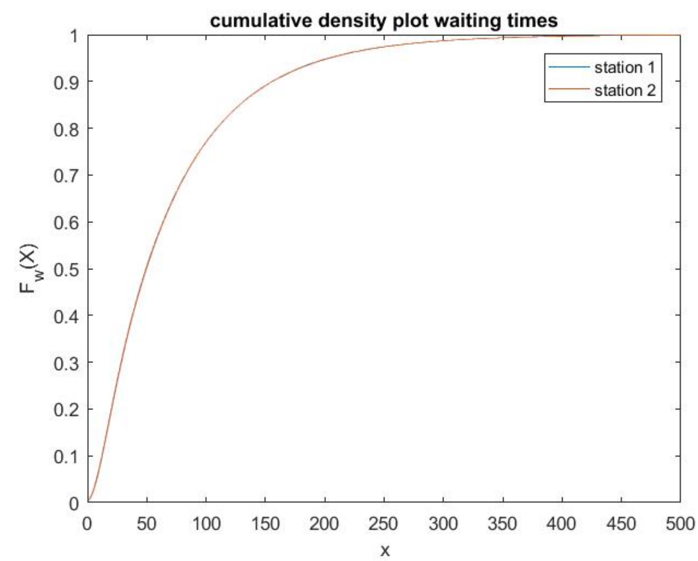
Table 4.10: Approximate values for locations of the density peak



(a) density of the waiting times

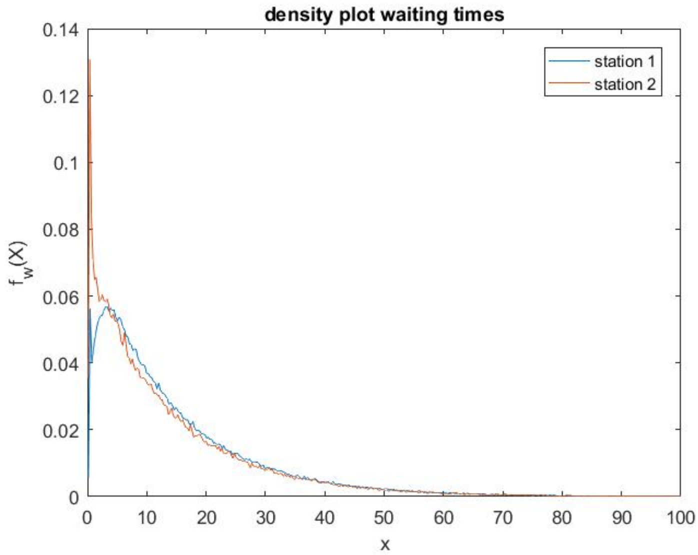


(b) first part of density of the waiting times

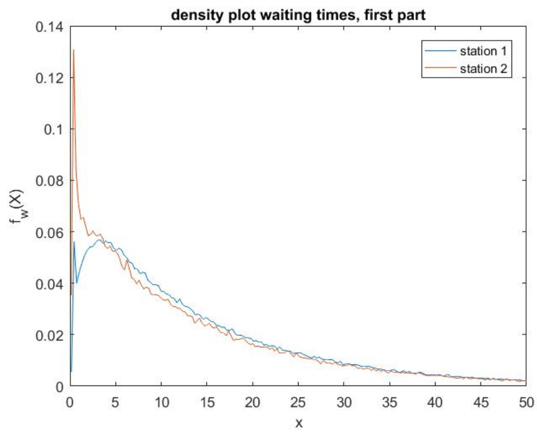


(c) cumulative distribution, waiting times

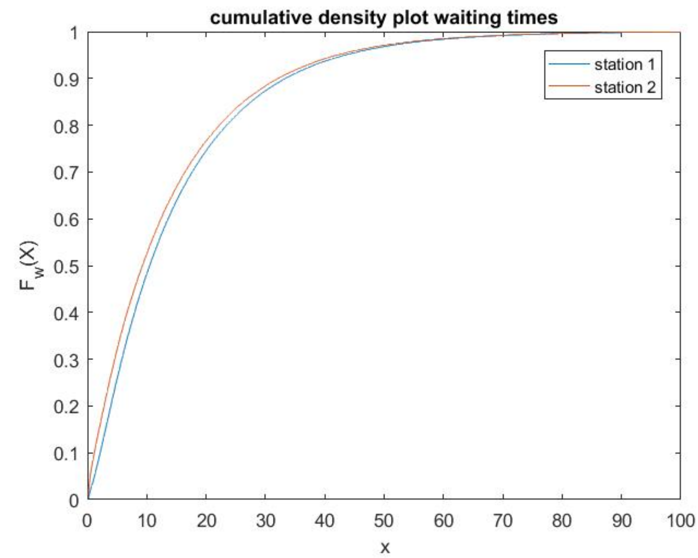
Figure 4.4: Waiting time distributions for parameters $D1$ from Table 4.7



(a) density of the waiting times

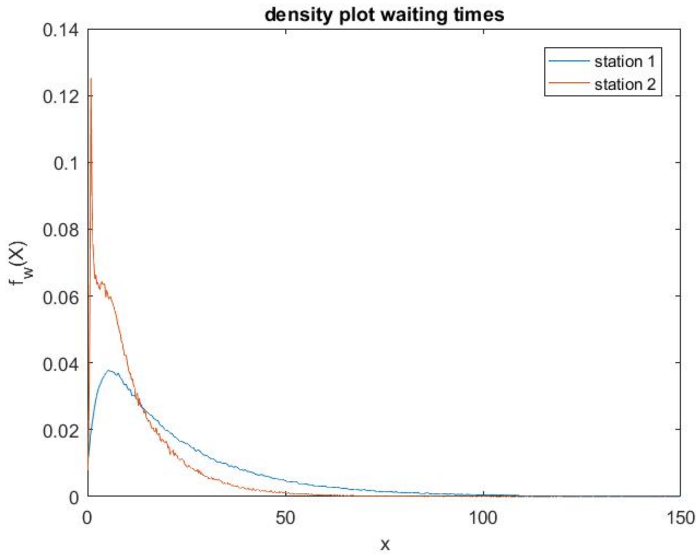


(b) first part of density of the waiting times

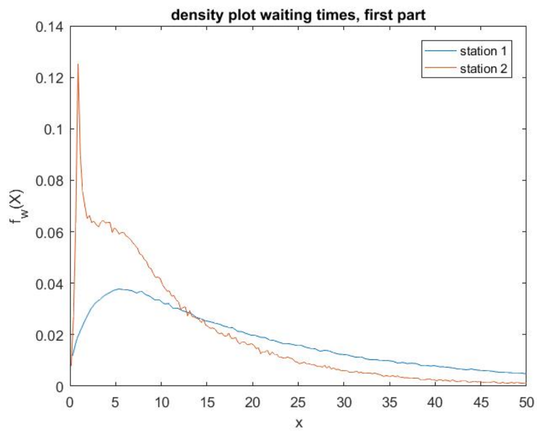


(c) cumulative distribution, waiting times

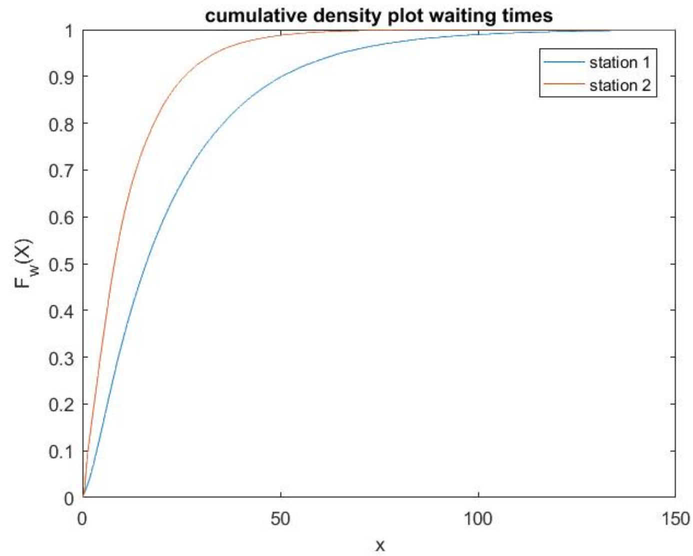
Figure 4.5: Waiting time distributions for parameters $D3$ from Table 4.7



(a) density of the waiting times

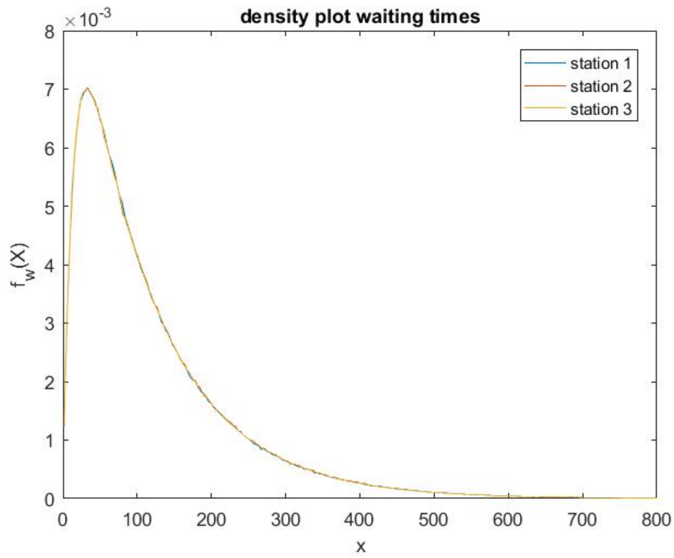


(b) first part of density of the waiting times

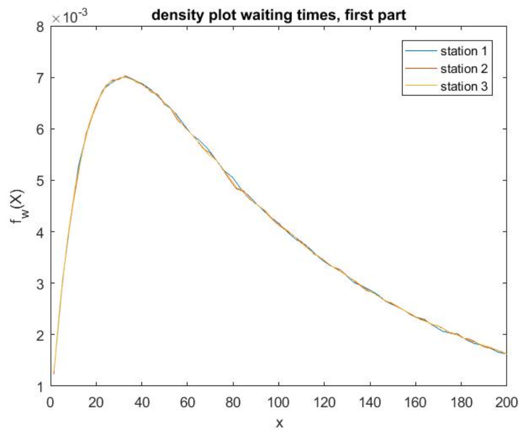


(c) cumulative distribution, waiting times

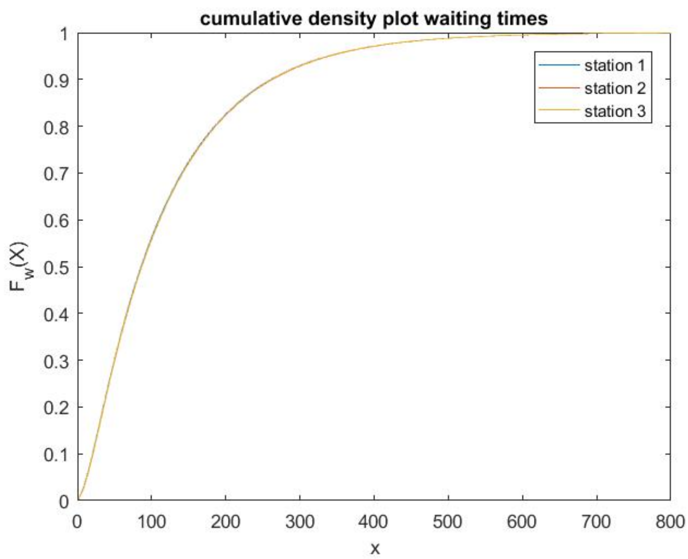
Figure 4.6: Waiting time distributions for parameters $D5$ from Table 4.7



(a) density of the waiting times

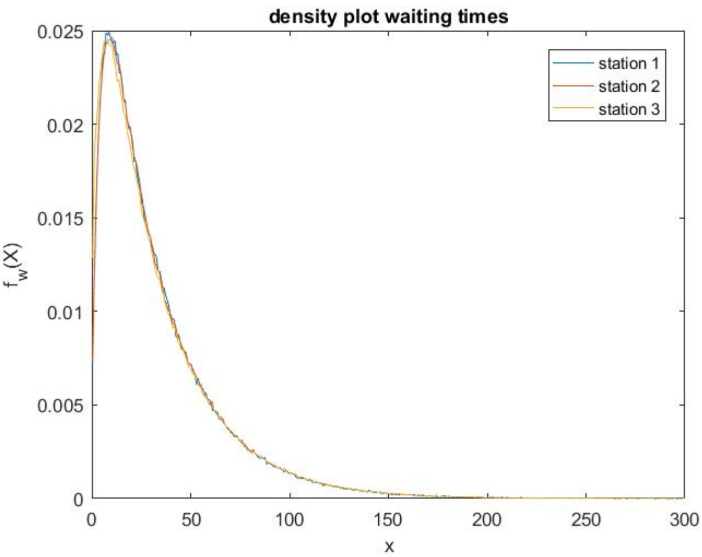


(b) first part of density of the waiting times

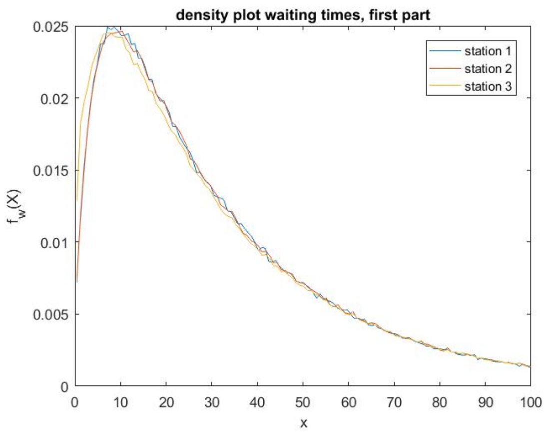


(c) cumulative distribution, waiting times

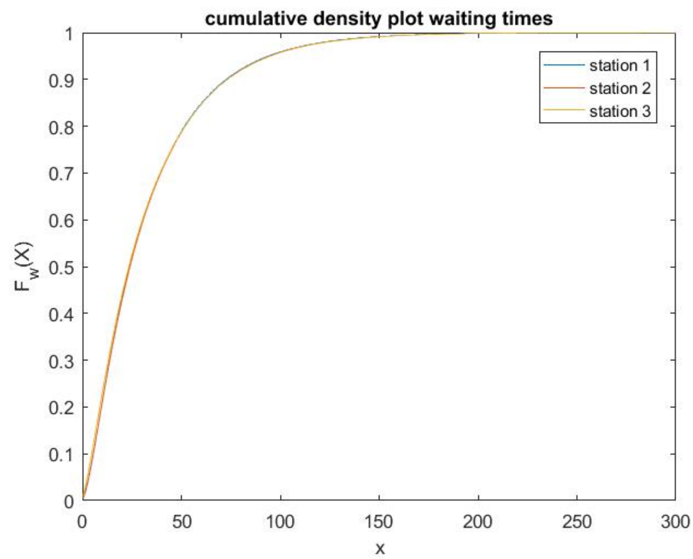
Figure 4.7: Waiting time distributions for parameters $E1$ from Table 4.7



(a) density of the waiting times

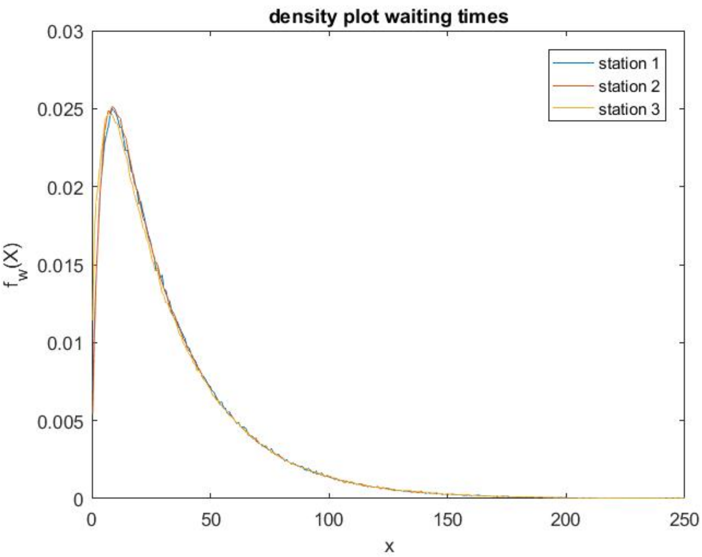


(b) first part of density of the waiting times

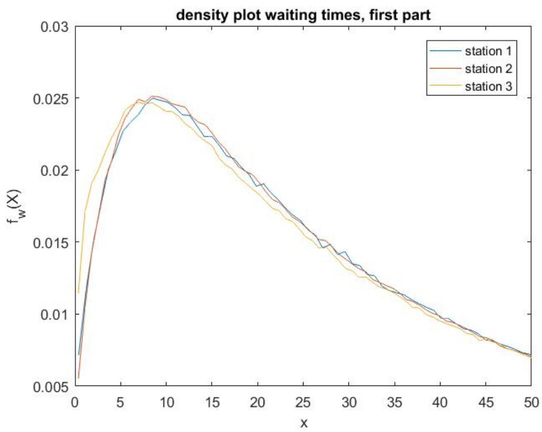


(c) cumulative distribution, waiting times

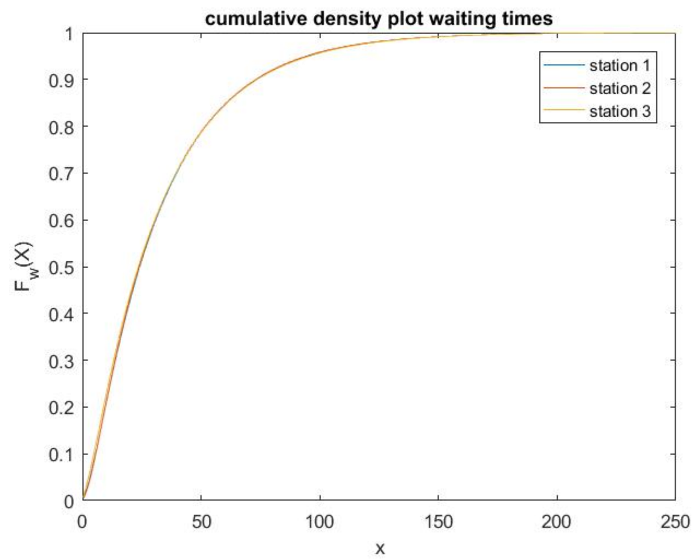
Figure 4.8: Waiting time distributions for parameters $E3$ from Table 4.7



(a) density of the waiting times

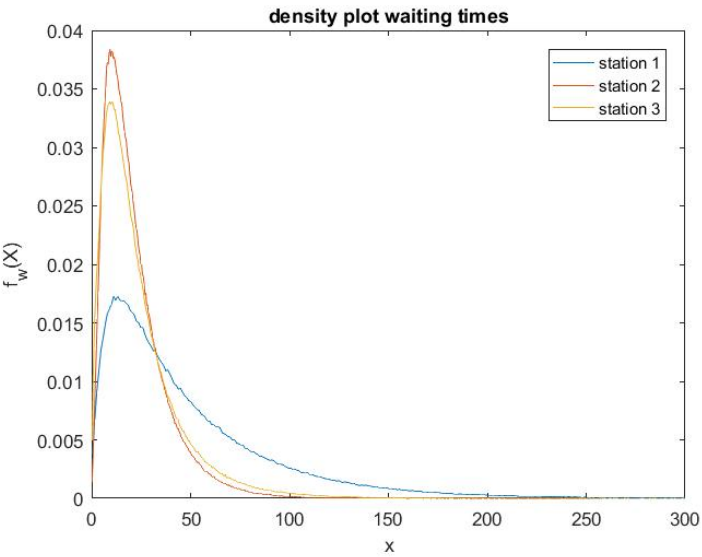


(b) first part of density of the waiting times

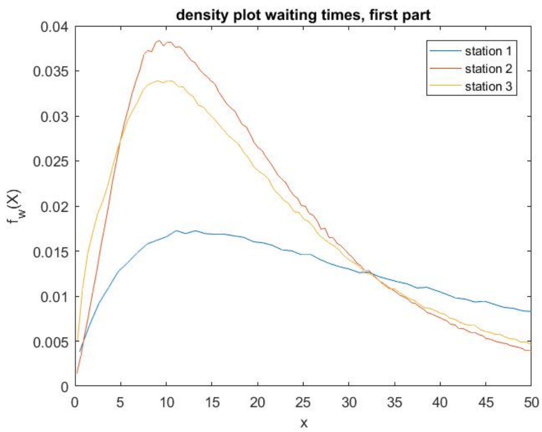


(c) cumulative distribution, waiting times

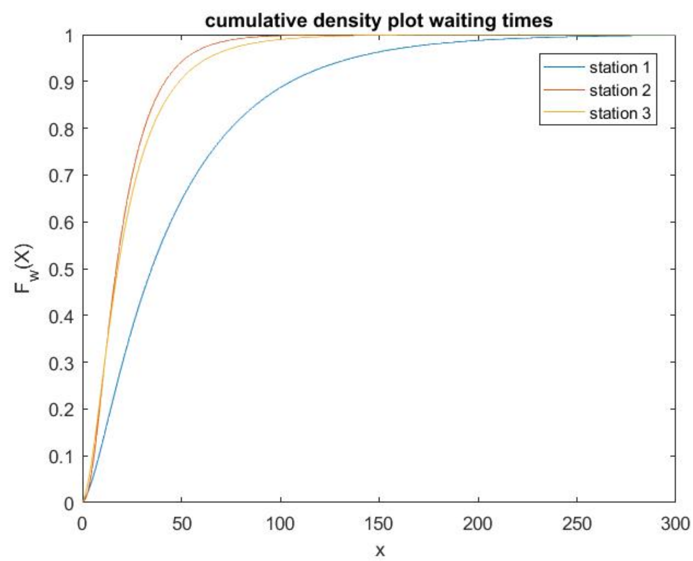
Figure 4.9: Waiting time distributions for parameters $E5$ from Table 4.7



(a) density of the waiting times



(b) first part of density of the waiting times



(c) cumulative distribution, waiting times

Figure 4.10: Waiting time distributions for parameters $E7$ from Table 4.7

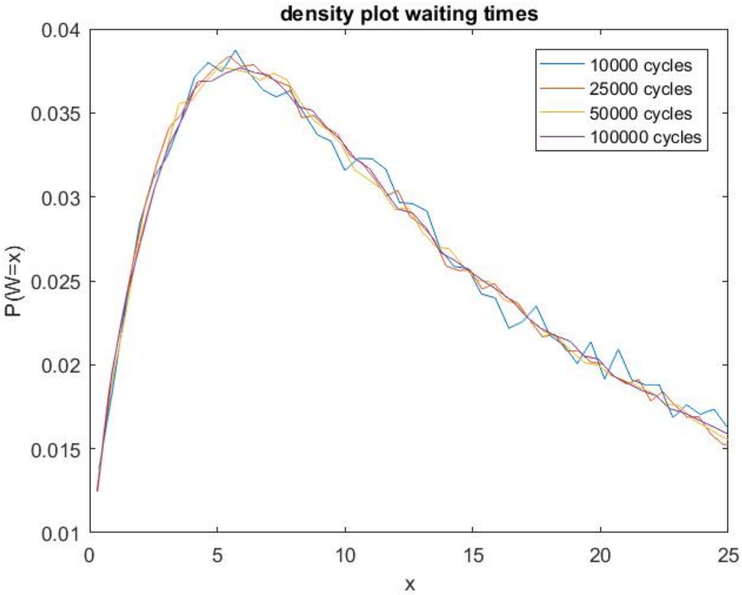


Figure 4.11: Density $E1$ for different number of cycles

4.3 Different glue distributions

In this section, different distributions for the glue period will be studied. Here the switch-over time distributions will be constant and the service distributions exponential. The same mean values as used for the simulation of a system with exponential glue periods and constant switch-over time will be used, which can be seen in Table 4.7. For both Uniform and Pareto glue period the simulation runs 500 times with 2000 cycles each run.

4.3.1 Uniform glue periods

First the glue distribution will be the Uniform distribution on the interval $[\frac{1}{2}\mathbb{E}[G_i], \frac{3}{2}\mathbb{E}[G_i]]$. The results can be seen in Table 4.11, with the 95% confidence intervals in Table 4.12.
When you compare the results with the system with exponential glue period in Table 4.8, it can be seen that the average waiting times are less with an uniform glue period. The difference between the waiting times for different stations for the same parameter set are similar for both exponential and uniform glue periods. For example for both exponential and uniform glue periods with parameter set $E2$, the average waiting time for station 1 and 2 is almost the same and the average waiting time for station 3 follows closely behind.
In Figure 4.12 the waiting time density of the parameter sets $E1$, $E3$ and $E5$ can be seen. Here it can be seen that with Uniform distributed glue period the shape of the waiting time density stays the same. Though it looks like that the graph is a bit wider at the peak.

parameter set number	simulated values $\mathbb{E}[W_1]$	$\mathbb{E}[W_2]$	$\mathbb{E}[W_3]$
$E1$	101.6685	101.6946	101.6716
$E2$	40.1807	40.2166	38.8924
$E3$	28.3874	28.4129	26.8236
$E4$	203.2859	216.2194	192.6411
$E5$	28.2297	28.2319	26.6744
$E6$	37.8618	14.3608	35.8125
$E7$	41.1010	15.5802	21.0856

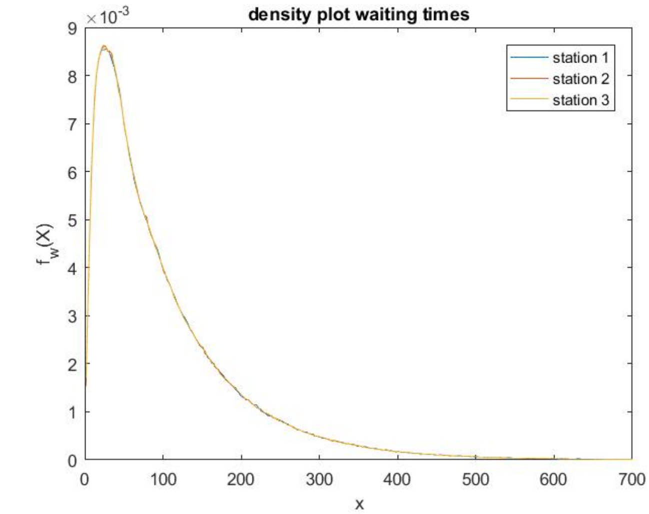
Table 4.11: Simulated values for Uniform distribution of glue periods

parameter set number	confidence interval $\mathbb{E}[W_1]$	$\mathbb{E}[W_2]$	$\mathbb{E}[W_3]$
$E1$	$(101.4370, 101.9001)$	$(101.4646, 101.9247)$	$(101.4427, 101.9004)$
$E2$	$(40.1243, 40.2371)$	$(40.1583, 40.2748)$	$(38.8340, 38.9508)$
$E3$	$(28.3504, 28.4244)$	$(28.3778, 28.4480)$	$(26.7851, 26.8621)$
$E4$	$(202.6275, 203.9443)$	$(215.5296, 216.9091)$	$(192.0324, 193.2498)$
$E5$	$(28.1975, 28.2619)$	$(28.2004, 28.2633)$	$(26.6413, 26.7074)$
$E6$	$(37.8236, 37.9001)$	$(14.3489, 14.3728)$	$(35.7723, 35.8528)$
$E7$	$(41.0589, 41.1431)$	$(15.5675, 15.5929)$	$(21.0640, 21.1071)$

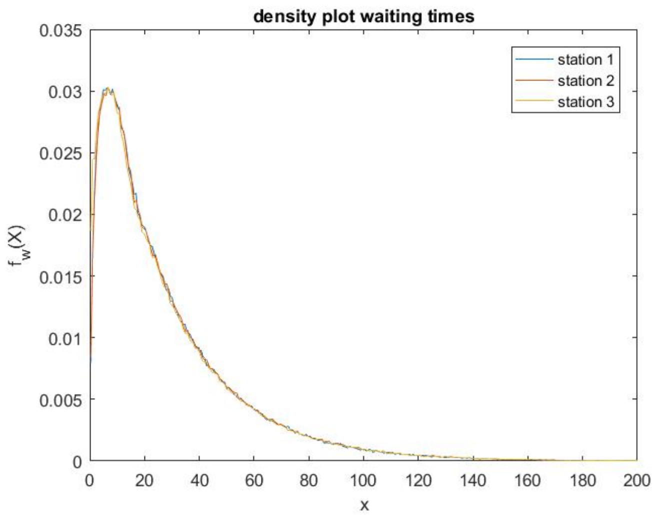
Table 4.12: Confidence intervals for Uniform distribution of glue periods

parameter set number	Location of peak W_1	W_2	W_3
$E1$	28	23	23
$E3$	8	6	6
$E5$	8	6	5

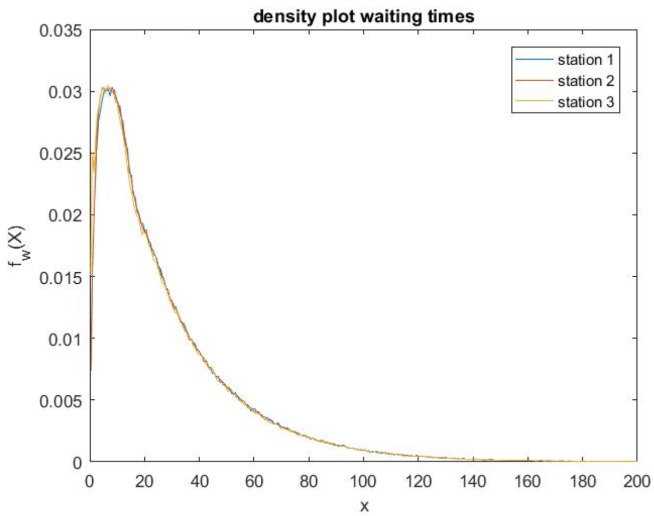
Table 4.13: Approximate values for locations of the density peak, Uniform glue period



(a) Parameter set $E1$



(b) Parameter set $E3$



(c) Parameter set $E5$

Figure 4.12: Waiting time density for different parameter sets, Uniform glue period

4.3.2 Pareto glue periods

Lastly a system with Pareto distributed glue periods will be studied. The parameters will be the same as in the system with exponential glue periods. The Pareto distribution has two parameters α and x_m. First $\alpha = 3$ is chosen, to ensure that the mean and variance of the glue period is not infinite. Since we know the mean glue period time from the parameters and $\mathbb{E}[G_i] = \frac{x_m \cdot \alpha}{\alpha - 1}$, x_m can be determined.
In Table 4.14 and 4.15 the results and 95% confidence intervals can be found. Here it can be seen that the means are slightly higher than with a Uniform glue period, but still lower than with exponential glue periods.
Again the locations of the peak is almost the same as with the Uniform and Exponential glue periods and the difference is mostly in the tail. Meaning that the probability on a higher waiting time is bigger, even though the location of the peak is the same.

parameter	simulated values		
set number	$\mathbb{E}[W_1]$	$\mathbb{E}[W_2]$	$\mathbb{E}[W_3]$
$E1$	104.3872	104.3621	104.4195
$E2$	41.2567	41.2708	39.9721
$E3$	29.1419	29.1081	27.5801
$E4$	207.9534	220.3041	220.3041
$E5$	29.0297	28.9987	27.4688
$E6$	38.9888	14.7163	37.0066
$E7$	42.2162	15.9454	21.8885

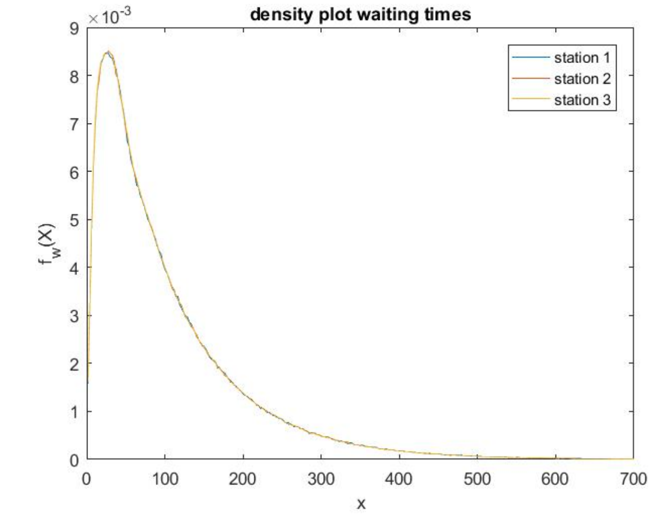
Table 4.14: Simulated values for Pareto distribution of glue periods

parameter	confidence interval		
set number	$\mathbb{E}[W_1]$	$\mathbb{E}[W_2]$	$\mathbb{E}[W_3]$
$E1$	$(104.1263, 104.6480)$	$(104.1064104.6178)$	$(104.1626104.6765)$
$E2$	$(41.1923, 41.3210)$	$(41.2067, 41.3349)$	$(39.9088, 40.0354)$
$E3$	$(29.1038, 29.1801)$	$(29.0703, 29.1459)$	$(27.5401, 27.6202)$
$E4$	$(207.2572, 208.6496)$	$(219.5529, 221.0553)$	$(196.8924, 198.2161)$
$E5$	$(28.9950, 29.0645)$	$(28.9950, 29.0645)$	$(27.4329, 27.5047)$
$E6$	$(38.9396, 39.0381)$	$(14.7019, 14.7306)$	$(36.9576, 37.0556)$
$E7$	$(42.1682, 42.2643)$	$(15.9302, 15.9607)$	$(21.8643, 21.9127)$

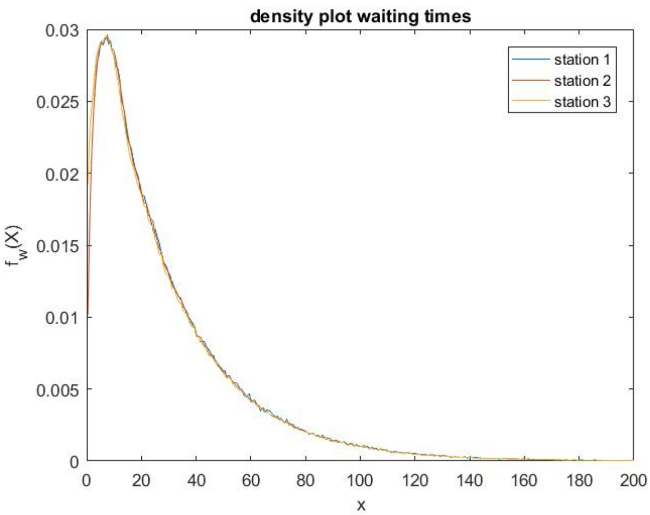
Table 4.15: Confidence intervals for Pareto distribution of glue periods

parameter	Location of peak		
set number	W_1	W_2	W_3
$E1$	25	28	32
$E3$	7	7	7
$E5$	6	7	9

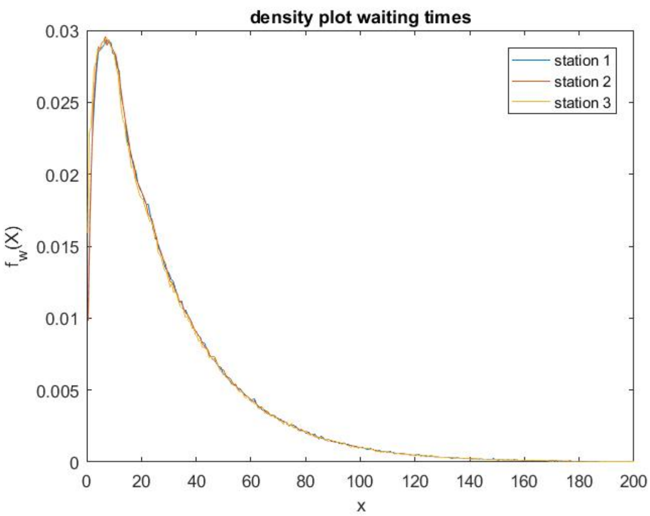
Table 4.16: Approximate values for locations of the density peak, Paret glue period



(a) Parameter set $E1$



(b) Parameter set $E3$



(c) Parameter set $E5$

Figure 4.13: Waiting time density for different parameter sets, Pareto glue period

Chapter 5

Conclusion and Reflection

In this report, polling systems, with and without glue periods, were studied and simulations of these systems were made. The simulation results were checked against known results of polling systems and it was found that the simulated values approximated the exact values properly. This means that the simulation accurately simulates the behavior of the systems.
What was noticed is that the changes in the arrival rate and mean service time resulted in similar changes in the average waiting times for all stations. Changes in the mean glue time and the retrial time caused opposite changes for the station the parameter changed for and the other stations.
For glue periods of different types of distributions, similar waiting time densities and average waiting times were found. The average waiting times for Uniform glue periods were the lowest, with Pareto glue periods coming in second. Exponential glue periods have the longest average waiting times.
A lot more can be learned of the behavior of polling systems, through simulation, when calculating the exact values is too difficult. Different types of distributions for the inter-arrival, switch-over times, service, glue and retrial times can be examined. The waiting time density can also be studied more closely. For example the effects of changes in certain parameters can be examined.
Another thing that can be studied is simulating polling systems with different types of serving disciplines, for example time-limited or k-limited. Different type of customer behaviors can also be integrated in the simulation, for example customers leave and try again later if the queue is of a certain length, or they leave and try again later after waiting a certain time.

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Appendix A

Simulation

1. **Polling1**: Java,*Simulation without glue periods, with exhaustive and gated service*
2. **Polling2**: Java, *Simulation with glue periods*
3. Mathlab files:
   (a) **runPolling1.m**: *gives results for simulation with either exhaustive or gated service*
   (b) **runPolling2.m**: *gives results for simulation with glue periods*
   (c) **tableRun.m**: function *to determine mean waiting times and corresponding confidense intervals*
   (d) **plotWaitDist.m**: function *function to make graphs of density of the waiting times*