BACHELOR

Roots of chromatic polynomials of graphs

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Award date:
2017

Roots of Chromatic Polynomials of Graphs

Bachelor Thesis

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Date

17-08-2017

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Abstract

We begin with explaining graphs, graph coloring and the chromatic polynomial which arises from graph coloring. After that we continue with the deletion-contraction theorem and some specific examples of graphs and their chromatic polynomial. The last section is about Hubai's improvement of Sokal's theorem about bounding complex roots, where we address Hubai's proof in greater detail.

Acknowledgements

I would like to thank my supervisor Aart Blokhuis for all his assistance in the creation of this Bachelor Thesis.

1 Introduction

In 1736 Euler introduced graphs in his paper about the Seven Bridges of Königsberg Problem. Graphs G consist of a set of vertices $V(G)$ with a set of edges $E(G)$, who are drawn between some of the vertices. Graph Theory concerns itself with the problems which can be expressed with the help of graphs. One of the areas in Graph Theory concerns the coloring of the vertices or the edges of graphs, which is called graph coloring. Starting with the Four Color Problem in 1852, graph coloring has become a relevant help in solving problems related to scheduling, register allocation and other limited resource divisions.

Graph coloring in its origins searches for the smallest k for which a given graph G can be properly colored with k colors. A natural next choice for research is in how many ways this given graph can be properly colored with k colors. This results in a function $\mathsf{chr}(G, k)$ which outputs in how many different ways the graph G can be properly colored with k colors. This function turns out to be a polynomial, which makes it easier to work with. Finding this polynomial for any given graph is too difficult, but for a few specific examples we can give the explicit polynomial.

One of the interesting parts of polynomials are its roots. Even so much that these roots define the polynomial up to multiplicative constant factor. Therefore it is an interesting research topic. Tamás Hubai[1] improved the bound for the roots of the chromatic polynomial related to the maximum degree of the graph. In this paper we will recreate that proof.

2 Preliminaries

2.1 Basic Definitions and Introductions into Graphs

Every advanced interesting subject has its roots in a simple object. For roots of chromatic polynomials of graphs this is, unsurprisingly, graphs. So we will start with some basic definitions and other introductions into graphs.

2.1.1 Basic definitions

Definitions are the best if everybody uses the same ones for the same named objects. Thus we will use the basic definitions of Graph Theory, as described by Robin J. Wilson in his book *Introduction to Graph Theory* [3]:

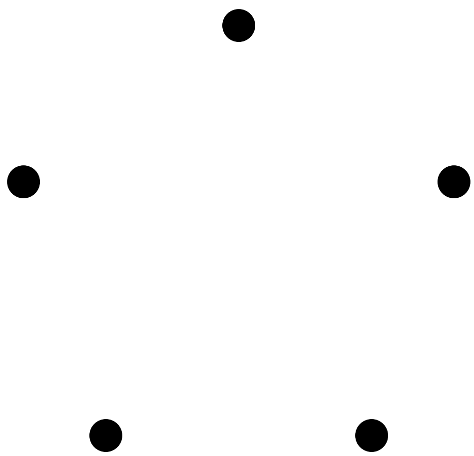
- A **graph** G consists of a non-empty finite set $V(G)$ of elements called **vertices**, and a finite family $E(G)$ of unordered pairs of (not necessarily distinct) elements of $V(G)$ called **edges**.

- $V(G)$ is called the **vertex set** and $E(G)$ is called the **edge family** of G.
- If an edge is the unordered pair v, w, the edge is said to **join** the vertices v and w and is labeled vw.
- Two vertices v and w of a graph G are said to be **adjacent** if there is an edge vw joining them; the vertices v and w are also said to be **incident** with the edge vw.
- The **degree** of a vertex v of a graph G is the number of edges incident with v.
- A **simple graph** is one in which there is at most one edge joining a given pair of vertices and there are no **loops**, or edges joining a given vertex with itself.
- A graph is **connected** if for each pair of vertices u, v there is a sequence of vertices $v_0, v_1, v_2, \ldots v_n$, where $v_0 = u$ and $v_n = v$, such that $v_i v_{i+1}$ is an edge, for all i with $0 \leq i \leq n - 1$.

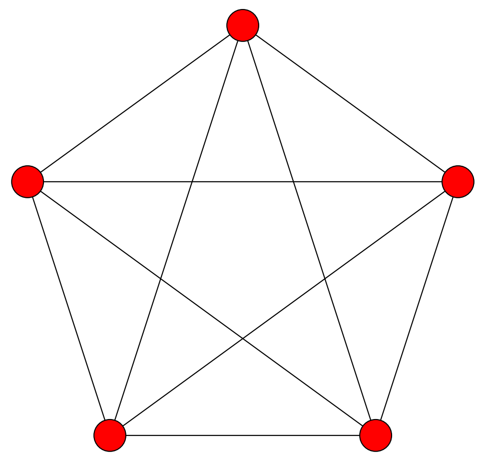
With these definitions, we can now describe some specific interesting types of graphs.

- A **null graph** is one in which the edge family, $E(G)$ is empty. A null graph of n vertices is denoted by N_n.
- A **complete graph** is a simple graph in which each pair of distinct vertices are adjacent. Complete graphs on n vertices are denoted by K_n.
- A **cycle graph** is a connected graph in which the degree of each vertex is 2. A cycle graph of n vertices is denoted by C_n.
- A **path graph** on n vertices is the graph obtained when an edge is removed from the cycle graph C_n. A path graph of n vertices is denoted P_n.

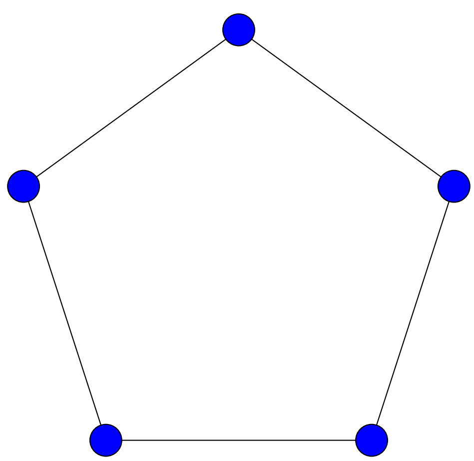
In the pictures below examples of aforementioned graphs are given.



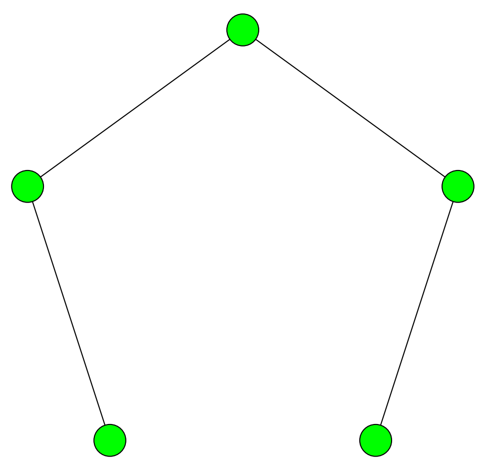
Null graph N_5



Complete graph K_5



Cycle graph C_5



Path graph P_5

2.2 Graph Coloring

Graph coloring, or more specifically *vertex coloring* means the assignment of colors to the vertices of a graph in such a way that no two adjacent vertices share the same color.

It is easy to see that every simple graph can be colored by giving every vertex a different color. So, to make it mathematically interesting the different colors have to be restricted to a fixed finite set $\mathcal{S}$. It is easy to see that the choice of actual colors is irrelevant, and therefore any graph property related to coloring may and will only depend on the cardinality $|\mathcal{S}| = k$. Instead of colors we may as well label the nodes using the numbers $1, 2, \ldots, k$.

Formally, a *k-coloring* of a graph G is a function $\sigma : V(G) \to \{1, 2, \ldots, k\}$ which satisfies $\sigma(i) \neq \sigma(j)$ for any edge $ij = e \in E$. Note that it is not required to use all the colors. The graph is said to be *k-colorable* if such a function exists. The *chromatic number* $\chi(G)$ is the minimal k for which the graph G is *k-colorable*, and we say that G is *k-chromatic* if $\chi(G) = k$.

Clearly, a graph containing a loop cannot be properly colored while multiple edges don't add any additional restriction on the coloring. The extra components of multi graphs versus simple graphs are, for our purposes, thus uninteresting or unnecessary. Therefore we'll assume that the graphs being examined are simple.

2.2.1 Applications for Graph Coloring

Graph coloring has its origins in the Four Color Conjecture. The conjecture (now theorem) states that given a plane divided in contiguous regions, the regions can be colored by at most 4 different colors such that no 2 adjacent regions have the same color. The theorem was first proposed by Francis Guthrie in 1852 and remained unsolved until 1976.

Graph coloring is particularly useful in problems where there are limited resources which are shared by various entities. For example register allocation. In most programming languages you allocate many variables and expect them to behave all in the same way. The resources used to store the values of these variables are registers and RAM. The registers are the fast option and RAM is the slow option of the two. In the usual setup of computers there are not a lot of registers, so it can be the case that when there are a lot variables not each variables can get its own registers.

That is where compilers and register allocation comes in. Some variables are not used at the same time in the program and can use in the same registers for a faster execution of the program. The compiler uses a graph where each vertex represents a variable and there is an edge between two vertices, when the two variables must be used at the same time. Then the colors represent the registers,

or RAM if not enough registers are available, and by graph coloring the compiler decides which variable gets which registers or RAM.

Besides register allocation, graph coloring is used in various scheduling jobs, pattern matching and sudoku solving.

2.3 The Chromatic Polynomial

Consider the number of different k-colorings of a given graph G as a function of k, and denote it by $\mathsf{chr}(G, k)$.

Theorem 2.1 $\mathsf{chr}(G, k)$ *is a polynomial in* k*.*

Proof. For any coloring of G the nonempty color classes constitute a partition of $V(G)$ where each part is a vertex set. We may count those colorings that give a certain partition and add them up for all such partitions to find the total number of colorings. Since $V(G)$ is a finite set, it has a finite number of partitions, so it is sufficient to show that the number of colorings for a single partition is a polynomial in k.

Fix a partition with p parts, each of them being a stable set. By assigning a different color to each part, we get all the colorings belonging to the partition. We may pick the first color in k possible ways, the second in $k-1$ ways, etc. so there are $k(k-1)\dots(k-p+1)$ colorings, which is obviously a polynomial. Note that this also works when $k < p$. □

Corollary 2.2 $\mathsf{chr}(G, k)$ *is of degree* $n = |V(G)|$*.*

Proof. There is no partition with more than n parts and only a single partition with exactly n parts, the one where each part consists of a different vertex. For this partition, the number of colorings is a polynomial of degree n while for all other partitions it has a degree smaller than n. Since the number of the polynomials is finite, as mentioned in the previous proof, the sum of such polynomials is a polynomial of degree n. □

Having established this fact, we may call $\mathsf{chr}(G, k)$ the *chromatic polynomial* of G.

The fact the function $\mathsf{chr}(G, k)$ is a polynomial gives us a few useful properties. For instance, we are no longer bound to positive integer values for k. Any $q \in \mathbb{C}$ can be substituted for k. Then $\mathsf{chr}(G, q)$ of course doesn't give the number of possible q-colorings anymore, but the result can still give us valuable information about $\mathsf{chr}(G, k)$ and G itself. We'll see some of these results later on. Another advantage is that we can examine the polynomial's coefficients and roots and connect them to graph properties and invariants.

2.4 Deletion-contraction property

This property is one of the most important properties in graph-coloring theory, since it's used in proving many theorems and answering questions on graph-theory. One such proof concerns finding the chromatic polynomial for any given graph G.

The "dumb" approach would be to go through all k-colorings of G and count the valid ones. Then use Lagrange-interpolation to find the polynomial. As can be expected from a "dumb" approach this is quite labor intensive and even NP-complete. It shouldn't come as a surprise that we want do a little better than this brute force method. Sadly we can't do a lot better, since deciding upon 3-colorability is NP-complete.

The idea is to do a sort of induction on the vertices and the edges. Select two vertices i and j from $V(G)$ with no edge between them. We can now distinguish two different classes of colorings of G:

1. Colorings where i and j are colored with different colors.
2. Colorings where i and j are colored with the same color.

The first class corresponds to all the colorings of $G + ij$. This is true, since the extra edge ij ensures that i and j are colored differently. The second class corresponds with all the colorings of G where the vertices i and j are merged. This means that i and j merge into 1 vertex which is connected to all vertices in G which at least one of i and j was connected to. Because the two vertices where merged into 1 vertex, they must also have had the same color. This is merging of vertices is called the contraction of i, j and denoted by G/ij.

We now have found a sum which equals all the colorings of G which can be expressed as follows:

$$\mathsf{chr}(G, k) = \mathsf{chr}(G + ij, k) + \mathsf{chr}(G/ij, k)$$

The first term gains an edge and the second term loses a vertex and can lose some edges. So we want to change the formula, so that in both terms the edges plus the vertices are decreasing. For this, substitute H into $G + ij$ and bring the term with contraction to the other side:

$$\mathsf{chr}(H, k) = \mathsf{chr}(H \backslash ij, k) - \mathsf{chr}(H/ij, k).$$

Note that in the current form we have a relation between these chromatic polynomials evaluated at some positive integer k. However, since two degree n polynomials agreeing on $n + 1$ points are identical, the same expression also holds for the polynomial itself. We'll omit this kind of reasoning in the future.

Since both $H \backslash ij$ and H/ij have fewer edges than H, we may apply this observation to facilitate induction or recursion for statements about the chromatic polynomial. As stated in the beginning, this method will prove quite powerful to be used several times, so we'll refer to it as the *deletion-contraction argument*.

For example, it gives us an alternative proof for $\mathsf{chr}(G, k)$'s polynomicity. Perform induction by the number of edges and vertices. For an edgeless graph all the k^n colorings are permissible and thus the claim is true. Otherwise $\mathsf{chr}(G, k)$ can be written as the difference of two terms which by induction are polynomials. Therefore $\mathsf{chr}(G, k)$ is also a polynomial.

When we use the *deletion-contraction argument* in an algorithm for finding the chromatic number, we can use the expression for finding the runtime for the worst case scenario. In the first term on the right rand side the number of edges reduces by 1 and the number of vertices stays the same when compared to the left hand side. In the second term the number of edges reduces by at least 1 and the number of vertices reduces by 1. The worst case scenario for the runtime is when in the second term the number of edges reduces by 1. So the expression changes to recursion formula:

$$a_{v+e} \geq a_{v+e-1} + a_{v+e-2}$$

with $v = |V(G)|$ and $e = |E(G)|$
We can substitute the solution ω^{v+e} for a_{v+e}. The recursion than reduces to $\omega^2 = \omega + 1$, which is the same formula as the one that is used for the Fibonacci sequence. This is a quadratic equation with solution $\omega_{1,2} = \frac{1 \pm \sqrt{5}}{2}$. So $a_{v+e} = A * (\frac{1+\sqrt{5}}{2})^{v+e} + B * (\frac{1-\sqrt{5}}{2})^{v+e}$. Note that $\frac{1-\sqrt{5}}{2} < 1 < \frac{1+\sqrt{5}}{2}$ so that $O(A*(\frac{1+\sqrt{5}}{2})^{v+e} + B*(\frac{1-\sqrt{5}}{2})^{v+e}) = O((\frac{1+\sqrt{5}}{2})^{v+e}) \approx O(1.6180^{v+e})$.

2.5 Chromatic Polynomials for specific graph families

Even though we can't calculate the chromatic polynomial for any graph in polynomial time, we can however find the polynomials for some specific types of graphs. We will discuss a few of those types and prove that the chromatic polynomial we give is correct.

We start off easy with the Null graph.

Claim 2.3 *The chromatic polynomial of the Null graph on n vertices is*

$$\mathsf{chr}(N_n, k) = k^n$$

Proof. Let $k \in \mathbb{N}$ be arbitrary. Then each of the n vertices can be colored with any of the k available colors in all colorings, since there are no edges. Thus $\mathsf{chr}(N_n, k) = k^n$. □

Next we have the complete graph.

Claim 2.4 *The chromatic polynomial of the complete graph on n vertices is*

$$\mathsf{chr}(K_n, k) = k(k-1)(k-2)\cdots(k-n+1)$$

Proof. Let $k \in \mathbb{N}$ be arbitrary. For the first vertex we have k colors to choose from. For the second vertex we can choose any color except for the color chosen for the first vertex, because all vertices are connected. For the third vertex we cannot choose the colors from

the first and the second vertex, since they are connected. Following this pattern we color all vertices, where we eventually have $k-(n-1)=k-n+1$ colors left to choose from. So when we count all the colorings we get the choices for the first vertex times the second times ... times the n^{th} vertex. This results in $k(k-1)(k-2)\cdots(k-n+1)=\mathsf{chr}(K_n,k)$.
□

Similar claims can be proven for trees, cycles and wheels. These results can be found in [1].

2.5.1 Example of Composite graph families

Royle [4] studies another interesting graph family and gives the corresponding chromatic polynomial. We will recreate his work here.

Let A and B be two graphs each with a distinguished 4-cycle, $a_1a_2a_3a_4$ and $b_1b_2b_3b_4$ respectively. Then we construct a graph G, which is the graph obtained by *gluing together* A and B at their aforementioned 4-cycles. Thus we identify a_i with b_i for each $1 \leq i \leq 4$. Then, for each colouring of G we get a corresponding coloring of A and B, where $a_1a_2a_3a_4$ and $b_1b_2b_3b_4$ are colored in the same way.
To construct the entire chromatic polynomial of G out of information from A and B is a bit more difficult. For this we are first going to look at the proper colorings of A. These proper colorings can be divided in 4 different types based on the different ways the 4-cycle $a_1a_2a_3a_4$ can be colored. The proper colorings of A are than of course the sum of the proper colorings of those 4 types. The types are:

1. $\sigma(a_1)=\sigma(a_3)$ and $\sigma(a_2)=\sigma(a_4)$
2. $\sigma(a_1)=\sigma(a_3)$ and $\sigma(a_2)\neq\sigma(a_4)$
3. $\sigma(a_1)\neq\sigma(a_3)$ and $\sigma(a_2)=\sigma(a_4)$
4. $\sigma(a_1)\neq\sigma(a_3)$ and $\sigma(a_2)\neq\sigma(a_4)$

For each type the number of colorings is equal to the chromatic polynomial of a certain auxiliary graph. This graph is created by taking A and identifying the vertices in $a_1a_2a_3a_4$ who have the same color and adding an edge between the vertices who have a different color. So for example for type 2, we would identify a_1 and a_3 and add an edge between a_2 and a_4. Name the 4 chromatic polynomials $\mathsf{chr}_1(A,k),\mathsf{chr}_2(A,k),\mathsf{chr}_3(A,k)$ and $\mathsf{chr}_4(A,k)$. As stated before the sum of these polynomials equals $\mathsf{chr}(A,k)$.

We now have dissected the chromatic polynomial of A in 4 parts, which we can do with B in the same way. Our goal is now to join these parts in such a way that the result will be the chromatic polynomial of G. For this it will be convenient to write these 4 parts in a different way, such that we can do matrix multiplication with them. That is why we define the *partitioned chromatic polynomial* $P(A, k)$ in the following way:

$$P(A,k) = \begin{pmatrix} \mathsf{chr}_1(A,k) \\ \mathsf{chr}_2(A,k) \\ \mathsf{chr}_3(A,k) \\ \mathsf{chr}_4(A,k) \end{pmatrix}$$

$P(A, k)$ is a vector in $\mathbb{Z}[k]^4$. Note that $P(A, k)$ is also dependent on which 4-cycle you choose if there are more present in A.

Claim 2.5 *Let A and B be two graphs with distinguished 4-cycles and corresponding partitioned chromatic polynomials $P(A, k)$ and $P(B, k)$. Then the chromatic polynomial of the graph G obtained by gluing together A and B is the sole entry of the 1×1 matrix $P(A, k)^T DP(B, k)$ where*

$$D = \begin{pmatrix} \frac{1}{\langle k \rangle_2} & 0 & 0 & 0 \\ 0 & \frac{1}{\langle k \rangle_3} & 0 & 0 \\ 0 & 0 & \frac{1}{\langle k \rangle_3} & 0 \\ 0 & 0 & 0 & \frac{1}{\langle k \rangle_4} \end{pmatrix}$$

where $\langle k \rangle_i$ denotes the i-th falling factorial $k(k-1)\cdots(k-i+1)$.

Proof. Note that the sole entry of $P(A, k)^T DP(B, k)$ equals:

$$\frac{\mathsf{chr}_1(A,k)\mathsf{chr}_1(B,k)}{\langle k \rangle_2} + \frac{\mathsf{chr}_2(A,k)\mathsf{chr}_2(B,k)}{\langle k \rangle_3} + \frac{\mathsf{chr}_3(A,k)\mathsf{chr}_3(B,k)}{\langle k \rangle_3} + \frac{\mathsf{chr}_4(A,k)\mathsf{chr}_4(B,k)}{\langle k \rangle_4}.$$

For any positive integer k, the product $\mathsf{chr}_i(A, k)\mathsf{chr}_i(B, k)$ is equal to the number of pairs (σ, ρ), where σ is a proper coloring of $\mathsf{chr}_i(A, k)$ and ρ is a proper coloring of $\mathsf{chr}_i(B, k)$. For our graph G only the pairs (σ, ρ) which use the same ordered sets of colors make a proper coloring. Otherwise σ and ρ would color $a_1a_2a_3a_4$ differently. For any coloring σ which uses s colors on $a_1a_2a_3a_4$ only 1 in $\langle k \rangle_s$ colorings ρ uses the same ordered sets of colors. Since all σ of the same type use the same number of colors on $a_1a_2a_3a_4$, we can just divide the product $\mathsf{chr}_i(A, k)\mathsf{chr}_i(B, k)$ by $\langle k \rangle_s$, with the appropriate s, to get the number of pairs (σ, ρ) from which we can make a coloring of G. Since every coloring on G induces a coloring on A and B, the aforementioned colorings are all colorings on G of that type. When we sum the results for all i we get the claim. □

3 Bounding complex roots

3.1 Introduction

Sokal [2] has proved that there is a bound for the absolute value of the complex roots of the chromatic polynomial. Fernández and Procacci improved this bound and Dong and Koh did this as well. In this section we will prove a slightly improved version of this bound. We will do this in the same way Tamás Hubai [1] did.

3.2 Motivation

Before we start to prove our bound, we will give some motivation why such bounds are interesting. When the roots are nonnegative integers, they describe the noncolorability of the associated graph with a certain number of colors. A bound for these roots is easily found. If the graph has a degree of D, then the graph can be colored with $D+1$ colors. Note that this is a bound and not the actual chromatic number of the graph.

Theorem 3.1 (Sokal) *There exists a universal constant c such that for all simple graphs with maximum degree $\leq D$ the roots of the chromatic polynomial lie in the disc $|q| \leq cD$. Also, if the second-largest degree is $\leq D$, all complex roots have $|q| \leq cD+1$. Furthermore, $c \leq 7.963907$.*

In 1912 Birkhoff introduced the chromatic polynomial. His reason to introduce this structure was to find an analytic proof of the 4-color problem by investigating the real and complex roots of the chromatic polynomial. Up to this date, this proof has not been found, but many other interesting properties of these roots have. The roots of a polynomial define the polynomial uniquely up to multiplicative constant.

3.3 Preliminaries

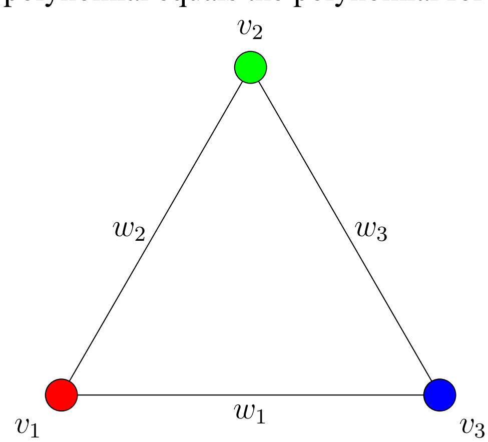
We are going to apply the ideas of linear relaxation on this problem. For this, we need to change a discrete structure of graphs into a continuous structure. We've chosen the edges for this. Instead of that an edge is or isn't there, each edge now is there and has a weight between 0 and 1 associated with it. You could say that when an edge has weight 0 it is completely not there, with a weight of 1 it is completely there and in between those values it is partially there. To work with this model, we first have to extend the definition of the chromatic polynomial and some related concepts to weighted graphs.
Let $G=(V,E)$ be a simple graph with edge weights $0 \leq w_e \leq 1$ for every edge e. It would be helpful if there was an edge between every pair of points. We can easily

realize this by making the graph complete and attach weight 0 to every edge added in this process.
Now we can redefine the chromatic polynomial as follows:
For a nonnegative integer q, define the chromatic polynomial as

$$\mathsf{chr}(G,q) = \sum \prod (1-w_e)$$

Here the sum enumerates all possible q-colorings of G while the product iterates over edges connecting vertices of the same color.
We can prove that we indeed get a polynomial in a similar way as in the unweighted case.

In the picture below we can see an example of a triangle graph with weighted edges. For q colors there are q^3 different colorings of which there are $q(q-1)(q-2)$ proper. $3q(q-1)$ are with 2 vertices with the same color and q are with all vertices with the same color. The proper colorings each contribute 1 to the sum so in total contribute $q(q-1)(q-2)$. The colorings with all vertices colored the same contribute each $(1-w_1)(1-w_2)(1-w_3)$ so in total $q(1-w_1)(1-w_2)(1-w_3)$. Of the last group a third of the colorings contributes $(1-w_1)$, a third $(1-w_2)$ and a third $(1-w_3)$, which brings the total contribution to $q(q-1)(3-w_1-w_2-w_3)$. Then $q(q-1)(q-2)+q(q-1)(3-w_1-w_2-w_3)+q(1-w_1)(1-w_2)(1-w_3) = q^3-(w_1+w_2+w_3)q^2+(w_1w_2+w_1w_3+w_2w_3-w_1w_2w_3)q$ is the total sum. So we can clearly see that when w_1, w_2 and w_3 are all equal to 1 the polynomial equals the polynomial for the nonweighted version of this graph.



The *degree* of a vertex v is modified to mean the sum of weights on all edges incident to v. Edge *deletion* $G\backslash e$ is handled by zeroing the corresponding weight. Edge contraction with weighted edges is a bit more complicated. Suppose we want to do an edge *contraction* G/e with $e = uv$ and let's call the contracted point $\tilde{u}$. Then the new edge from w to $\tilde{u}$ will have a weight of $w_1 \oplus w_2 := w_1+w_2-w_1w_2 = (w_1-1)(w_2-1)-1$, where w_1 is the weight of uw and w_2 is the weight of vw. This is defined like this in such a way that the product $(1-w_1)(1-w_2)$ does not change in it's new form $(1-w_1 \oplus w_2)$.

And thus each part, which corresponded with u, v and w all having the same color, doesn't change in the chromatic polynomial. The usual reduction changes to:

$$\mathsf{chr}(G,q) = \mathsf{chr}(G\backslash e,q) - w_e * \mathsf{chr}(G/e,q).$$

The extra w_e is there, since not all the contribution where u and v are the same color needs to be subtracted from $\mathsf{chr}(G\backslash e,q)$. Only for the part that the edge e was there: it's weight w_e. When w_e equals 1, the equation reduces to the simple deletion-contraction equation.

The chromatic polynomial of the triangle graph can also be determined using the deletion-contraction equation. Via multiple application of this equation we will get a sum of chromatic polynomials of graphs with all edges 0 times some constants. The chromatic polynomial of graphs with all edges 0 is according to claim 3.1 equal to q^v, where v is the number of vertices. For this particular graph the result is:

$$q^3 - w_2q^2 - w_1q^2 + w_1w_2q - w_3 + w_3 * w_2 \oplus w_1q$$

Oddly this result looks unsymmetrical and not equal to our earlier result, but when we replace the $\oplus$ with it's definition we do indeed find the same result.

$$q^3 - (w_1 + w_2 + w_3)q^2 + (w_1w_2 + w_1w_3 + w_2w_3 - w_1w_2w_3)q$$

In what follows the variable always be q, so to make our formulas a bit more readable we'll write $[G] := \mathsf{chr}(G,q)$, so the formula above now reads

$$[G] = [G\backslash e] - w_e * [G/e].$$

Changing the weight of a single edge will be marked as $G[e : w_e]$.
During the proof, we want to assume that the set of vertices does not change. Thus for the proof edge contraction can not be used, so let's define *contraction with compensation* as the contraction of an edge followed by the addition of a new isolated vertex, using the notion of $G \nearrow e$. We may think of it as moving all edges from a given vertex to another one. Obviously $[G \nearrow e] = q * [G/e]$, so the reduction can be written as

$$[G] = [G\backslash e] - \frac{w_e}{q} * [G \nearrow e]. \tag{1}$$

3.4 Proving the theorem

Lemma 3.2 *Let* $0 \le s,t \le 1$ *and* $0 \le x < 1$. *Recall the definition* $s \oplus t = s + t - st$. *Then*

$$\log(1 - sx) - \log(1 - (s \oplus t)x) \le -t \log(1 - x)$$

Proof. We are going to prove the lemma with the use of convexity of the left hand side and linearity of the right hand side, while considered as functions of t. If the inequality then holds when $t = 0$ and $t = 1$ the inequality holds for $0 \leq t \leq 1$. Thus consider both sides of the inequality as a function of t. Then for $t = 0$ the left hand side equals:

$$\log(1 - sx) - \log(1 - (s \oplus 0)x) = \log(1 - sx) - \log(1 - (s)x) = 0$$

The right hand side gives the following:

$$-0 \log(1 - x) = 0$$

So there is equality when $t = 0$.
For $t = 1$:

$$\log(1-sx)-\log(1-(s\oplus 1)x) = \log(1-sx)-\log(1-x) < -\log(1-x) = -1*\log(1-x)$$

Since $1 - sx < 1$ within the range of the allowed s and x. So $\log(1 - sx) < 0$.
So the inequality holds when $t = 1$.
Now we only need to prove that the left-hand side is convex. We know that $\log t$ is concave on its entire domain and thus $\log(a + bt)$ is also concave for any real a and b. Substituting $a = 1 - sx$ and $b = sx - x$ yields that $\log(1 - (s \oplus t)x)$ is concave too. This proves the proposition because $\log(1 - sx)$ is constant. □

Theorem 3.3 *Let G be a simple weighted graph as defined above with an edge $e = ij$ selected. Suppose that all vertices, possible except one of i or j, have a degree of at most D. We call this the degree criterion.*
Suppose q is restricted in the following way:

$$|q| \geq (1 + \frac{1}{2D})^{2D}(2D + 1) \tag{2}$$

Let $0 \leq w_1, w_2 \leq 1$ be some weights. Then for any $q \in \mathbb{C}$ which satisfies 2 we claim:

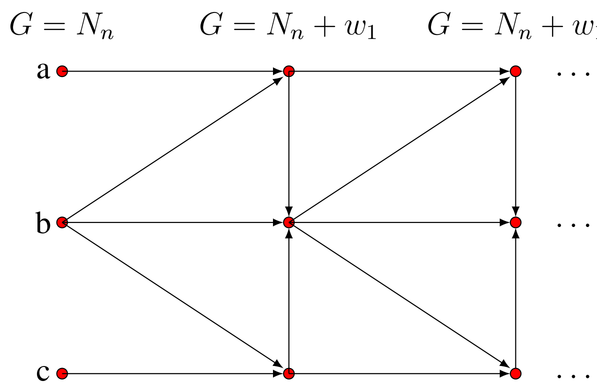
1. $$\left|\log \frac{[G \nearrow e]}{[G \backslash e]}\right| \leq 2D \log(1 + \frac{1}{2D})$$

2. $$\left|\log \frac{[G[e : w_1]]}{[G[e : w_1 \oplus w_2]]}\right| \leq w_2 \log(1 + \frac{1}{2D})$$

3. $[G] \neq 0$

where $\log$ *is the principal branch of the complex logarithm function.*

Proof. The Theorem consists of three parts: a, b and c. So naturally we are going to prove these parts separately. However, these parts are intricately connected with each other. The picture below was added for a little more insight in those connections. Apply induction based on the number of nonzero weighted edges, including the edge e if $w_1 > 0$ or $w_2 > 0$. If all weights are zero, the claims are trivial.



a. We morph the graph $G \backslash e$ to $G \nearrow e$ by a set of successive edge deletions and additions. Possibly swapping i and j, we may assure that the degree of j is at most D. For each edge jk, reset its original weight w_{jk} to zero and add it to the edge ik so that its weight becomes $w'_{ik} = w_{ik} \oplus w_{jk}$. Intermediate graphs have fewer nonzero edges than G, and the degree criterion also holds for them, so we can apply part b. Since in our induction prove we already have proven b. for graphs G' with fewer nonzero edges than G. Then we obtain that each step results in a difference of no more than $w*_{jk} \log(1 + \frac{1}{2D})$ in the logarithm of the chromatic polynomial. Adding up yields that the total difference is at most $2D \log(1 + \frac{1}{2D})$.

b. Let w_e denote the current weight of the edge e and consider the partial logarithmic derivative of $[G]$ with respect to w_e, expanding its absolute value using the reduction formula (1):

$$\left|\frac{\partial}{\partial w_e} \log[G]\right| = \left|\frac{\frac{\partial}{\partial w_e}[G]}{[G]}\right| \overset{(1)}{=} \left|\frac{\frac{\partial}{\partial w_e}([G\backslash e] - \frac{w_e}{q} * [G \nearrow e])}{[G\backslash e] - \frac{w_e}{q} * [G \nearrow e]}\right| =$$

$$\left|\frac{\frac{1}{q} * [G \nearrow e]}{[G\backslash e] - \frac{w_e}{q} * [G \nearrow e]}\right| = \left|\frac{K}{1 + w_e K}\right|$$

where

$$K = \frac{-[G \nearrow e]}{q * [G\backslash e]}$$

Note that here implicitly part c. is used, otherwise we couldn't divide by $[G]$.
From part a. we have

$$\left|\frac{[G \nearrow e]}{[G \backslash e]}\right| = e^{Re\left(\log \frac{[G \nearrow e]}{[G \backslash e]}\right)} \leq e^{\left|\log \frac{[G \nearrow e]}{[G \backslash e]}\right|} \leq$$

$$\leq e^{2D \log(1+\frac{1}{2D})} = \left(1 + \frac{1}{2D}\right)^{2D}$$

and therefore

$$|K| \leq \frac{(1+\frac{1}{2D})^{2D}}{|q|} \leq \frac{1}{2D+1}$$

according to the degree criterion for q. Note that $|K| < 1$ and therefore $|w_e K| < 1$, which we'll need shortly. Now we may bound the multiplicative change in the chromatic polynomial:

$$\left|\log \frac{[G[e:w_1]]}{[G[e:w_1 \oplus w_2]]}\right| = \left|\int_{w_1}^{w_1 \oplus w_2} \frac{\partial}{\partial w_e} \log[G]\, dw_e\right| \leq \int_{w_1}^{w_1 \oplus w_2} \left|\frac{\partial}{\partial w_e} \log[G]\right|\, dw_e \leq$$

$$\leq \int_{w_1}^{w_1 \oplus w_2} \frac{|K|}{1 - w_e|K|} = \log(1 - w_1|K|) - \log(1 - (w_1 \oplus w_2)|K|) \leq$$

$$\leq -w_2 \log(1 - |K|) \leq -w_2 \log(1 - \frac{1}{2D+1}) = w_2 \log(1 + \frac{1}{2D})$$

c. The claim is trivial for the empty graph. For an arbitrary graph, we may subsequently change each edge weight of the empty graph to the desired value using b, causing only a bounded multiplicative change to $[G]$ in each step. It follows that the result cannot be zero either. □

Theorem 3.4 *If D denotes the second-largest degree in G, then all roots of the chromatic polynomial lie within the disc $|q| < (2D+1)e$.*

Proof. Suppose the contrary. A root that violates the claim satisfies

$$|q| \geq (2D+1)e > (2D+1)(1 + \frac{1}{2D})^{2D}$$

so we may apply the previous theorem, resulting in a contradiction. □

Remark 3.5 *We proved an asymptotic factor of $2e \approx 5.436564$, which, despite the slightly larger additive constant, gives a stronger result than Sokal's for any positive integer D. This is also an improvement compared to the articles mentioned in the introduction 3.1, and as to our knowledge, is the sharpest bound known at the moment.*

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