What are feasible stress scenarios and what is their impact on pension fund portfolios?

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What are feasible stress scenarios and what is their impact on pension fund portfolios?

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Abstract

The term stress scenario is a trending topic since the crisis in 2008. Also in the pension system, this term occurs more and more often. In this thesis stress scenarios, such as low interest rate or low inflation are constructed within a mathematical framework. This mathematical framework takes three risk factors into account, namely the return on investments, the inflation and the nominal interest rate. The framework consists of two parts, one part captures the dependency between the risk factors and the other part consists of the forecast models for each risk factor. The forecast of the first risk factor is studied by means of an autoregressive model, whereas the forecast of the latter two risk factors is studied by means of a short rate model. In this thesis both the Vasicek and the Black Karasinski short rate model are discussed in more detail and their performance based on historical data is compared. For the inflation and nominal interest rate also representative yield curve shocks are analysed based on historical data.

Besides the constructing of a mathematical framework to derive stress scenarios, this thesis also gives results on the impact of different stress scenarios on pension fund portfolios. This impact is measured with the funding ratio, which is the ratio of total asset value and liabilities. The stress scenarios can be applied to different pension fund portfolios and different asset allocations. In our example, where a specific pension fund portfolio and general asset allocation is chosen, the stress scenarios indicate the weak spot of the pension fund allocation. Also, the influence of the representative yield curve shocks to the funding ratio is analysed.

Management summary

Company and asset and liability management team

Aon Hewitt is part of the worldwide company Aon plc, which is a leading global professional services firm providing a broad range of risk, retirement and health solutions. They empower results for clients by using proprietary data and analytics to deliver insights that reduce volatility and improve performance [1]. Aon Hewitt is specialized in the pension system and provides services in the range from administrative work to consulting.

The problem treated in this thesis is relevant for the asset and liability management (ALM) team. This team regulates the total assets to meet current and future liabilities. For the total assets one can think of investing in global equity, real estate, stocks, government bonds, etc., whereas the liabilities denote the pension of the pensioners. The ratio of the assets and liabilities is called the funding ratio and the ALM team models this funding ratio over time. Also, it studies the influence of different asset allocations, e.g. 10 percent in real estate, 50 percent in global equity and 40 percent in government bonds, to the funding ratio.

Problem description

In the crisis of 2008 the average funding ratio of Dutch pension funds dropped significantly. A drop in the funding ratio is not wanted, since this means that the pension fund is not able the pay the liabilities, which is the pension of pensioners. As a consequence, the ALM team wants to model the influence of the different economic crashes on the funding ratio. These economic crashes are modeled using the so called stress scenarios and the main research question therefore becomes: *What are feasible stress scenarios for pension portfolios and what is their impact on pension fund portfolios in practice?* In our stress scenarios three risk factors are taken into account, namely the return on investments, the inflation rate and the nominal interest rate. The construction of the stress scenarios is further divided into two steps. The first step concerns the dependency between the risk factors and the second step the forecasts of the risk factors.

Model description

Our analysis relies on historical daily data from the Bloomberg program for the three aforementioned risk factors. First let us focus on the first subquestion, i.e. the dependency between the risk factors. This dependency is captured with the t-copula function, which captures more general dependency structures than simple linear dependency structures.

The second subquestion, i.e. the forecast of the risk factors, is studied per risk factor. Note that the forecast of the inflation is studied by forecasting the nominal interest rate and the real interest rate, since the inflation is the difference between those interest rates. For the forecast of the returns on investment an autoregressive model of order 1 is used which is given by

$$X_t = \psi X_{t-1} + \epsilon_t$$

where X_t denotes the distribution of the returns on the investment at time t, ψ is a real valued parameter that denotes the autocorrelation and ϵ_t captures white noise at time t. For the forecast of the real interest rate the two factor Vasicek short rate model is used, which is given by

$$\begin{cases} \mathrm{d}r_{\mathrm{real},t} = \alpha_1 \left(m_t - r_{\mathrm{real},t}\right) \mathrm{d}t + \sigma_1 \mathrm{d}W_{1,t} \\ \mathrm{d}m_t = \alpha_2 \left(\mu' - m_t\right) \mathrm{d}t + \sigma_2 \mathrm{d}W_{2,t}, \end{cases}$$

where $r_{\mathrm{real},t}$ denotes the real short rate at time t, α_1, α_2, σ_1, σ_2 and μ' are real valued parameters and $W_{1,t}$ and $W_{2,t}$ are two correlated Wiener processes at time t. Observe that here the real short rate is modeled, but with this short rate the real interest rate can be derived. For the forecast of the nominal interest rate the one factor Black Karasinski model is used, which is given by

$$\mathrm{d}\ln(r_{\mathrm{nom},t}) = \alpha_3 \left(\ln(\theta) - \ln(r_{\mathrm{nom},t}\right) \mathrm{d}t + \sigma_3 \mathrm{d}W_{3,t},$$

where $r_{\mathrm{nom},t}$ denotes the nominal short rate at time t, α_3, θ and σ_3 are real valued parameters and $W_{3,t}$ is a Wiener process at time t. Note that these parameters are not equal to the parameters of the two factor Vasicek model, since both short rate models are calibrated to the corresponding data. Again, with the nominal short rate the nominal interest rate can be derived.

Results

With both the dependency and the forecasts of the risk factors, stress scenarios with a certain belief can be constructed. As an example of the practical application of these stress scenarios the impact on the funding ratio of the following beliefs, which is the inputs of the model in terms of quantiles for at least one risk factor and at most three risk factors, is studied. Note that with the beliefs one could choose a specific economic crash together with the severity of that crash.

i) Low returns on global equity. Here the return on the global equity is assumed to be −30 percent for the first year. This corresponds to the quantile (5%) of the return distribution for the global equity and the prediction of the real interest rate follows the quantile (50%).
ii) Low inflation/Deflation. For the low inflation or deflation the quantile (20%) of the prediction of nominal interest rate and the quantile (80%) of the real interest rate is used.
iii) Low nominal interest rates. Here the quantile (10%) for the prediction of the nominal interest rate is observed together with the quantile (50%) of the real interest rate.

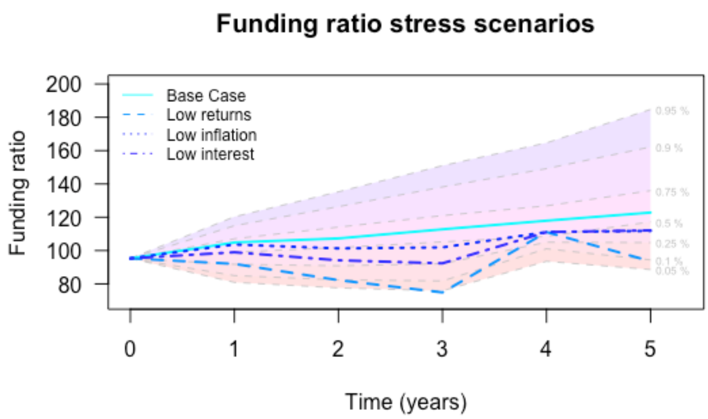


Figure 1: The impact of the stress scenarios on the funding ratio with a general asset allocation

From Figure 1 it can be seen that the low returns have the most negative effect on the funding ratio with this asset allocation.

Contents

Chapter 1

Introduction

"Since the global financial crisis and recession of 2007-2009, criticism of the economics profession has intensified. The failure of all but a few professional economists to forecast the episode - the aftereffects of which still linger - has led many to question whether the economics profession contributes anything significant to society." - Robert J. Shiller, Sterling professor of Economics at Yale university who received the Nobel Prize in Economics in 2013.

In the financial crisis of 2008 financial companies suffered a tremendous loss. One of the reasons for this tremendous loss was the failure to forecast the crisis. Forecasting a future crisis may be impossible, but being prepared for a crisis lessens the impact. To this purpose, stress testing, which analyses the ability of a financial company to deal with an economic crisis, is of importance. With stress testing the financial company hopes to get insight in the robustness to certain economic crashes. Observe that the stress testing of financial companies to the crisis of 2008, thus using this crisis as the stress scenario, is not sufficient because every crisis is different. Therefore, deriving stress scenarios is an important task within most financial companies and the exact mathematical definition of a stress scenario differs in each financial sector, i.e. banks, investment firms etc.

This thesis is written in collaboration with the asset and liability management team from Aon Hewitt. Aon Hewitt is part of the worldwide company Aon plc, which is a leading global professional services firm providing a broad range of risk, retirement and health solutions. They empower results for clients by using proprietary data and analytics to deliver insights that reduce volatility and improve performance [1]. Aon Hewitt specialize in the pension system and provides services ranging from administrative work to consulting.

For the motivation on why stress scenarios, for companies in the pension system, are of such importance, let us first illustrate the size of the pension fund system in the Netherlands compared to other countries. Beforehand, it is noteworthy that in every country the pension system is regulated differently. However, they all have the same common goal, namely to provide a basic income after retirement.

If we have a closer look on the size of the pension fund systems from the countries that are in the Organisation for Economic Co-operation and Development (OECD), already a difference between the countries is observed. Figure 1.1 shows the total assets value relative to the gross domestic product (GDP) of these countries. It can be seen that the Netherlands has one of the highest ratio of total asset value relative to the GDP, see [21] for a more detailed breakdown of the total asset value. Figure 1.1 also shows that for some countries the value of the total assets is more than 100 percent relative to the GDP. This gives a good illustration of the size of the pension fund system in the different countries. In [34] the value of all assets invested is estimated to be around 37 trillion for the 35 countries that are in the OECD, which means that pension funds own almost 70 percent of assets worldwide. In the Netherlands the total assets value is around 1.27 trillion Eur per 31-12-2016 according to the Dutch Central Bank (DNB). Investing this amount of money also comes with great responsibility. A pension fund company should therefore consider the outcome of the investments for different future scenarios. In these future scenarios also stress scenarios are of importance, since these scenarios give more insight on what happens in a financial market crisis. Note that the mathematical definition of a stress scenario is not yet provided, but this is discussed in the next sections.

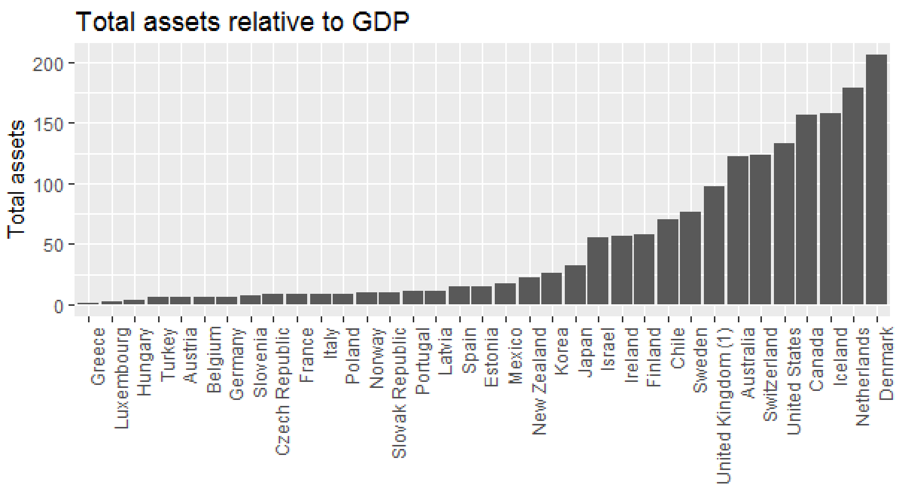


Figure 1.1: Total asset value of the countries in the Organisation for Economic Co-operation and Development relative to the gross domestic product per 31-12-2016, source [34]

For most people the pension system is a complex system and they only have a vague understanding of it. Since our focus is on the Dutch pension fund system, this system is described in more detail.

1.1 The Dutch pension fund system

The general framework of the Dutch pension fund system consists of three pillars, therefore this pension system is also referred to as the three pillar structure.

The first pillar is a state sponsored pay as you go (PAYG) pension which is called the old pension act (AOW), which was formulated in 1957. The PAYG means that the working population pays for the current pensioners. With this AOW the Dutch government ensures that all inhabitants get a basic income after retirement. Currently, the retirement age is 67 in the Netherlands. However, it was 65 and per 01-01-2018 the retirement age will increase to 68. This increase is mostly due to the increasing life expectancy. The basic income of the AOW is related to the minimum wage. For every year living in the Netherlands 2 percent of the right to AOW is built up and consequently after 50 years one has the full right to AOW. For more information on the AOW we refer to [36] and [23]. The latter gives an extensive study on alternatives for this first pillar and therefore mentions the benefits and distortions of this pillar.

The second pillar consists of occupational pensions. This pension is organized by the employer and is an extension to the first pillar in terms of income after retirement. The aim of this pillar is to have a pension equal to 75 percent of average salary. Note, that this aim was adapted from 70 percent of final salary to 75 percent of average salary in 2015 ("art 18a Wet LB Ouderdomspensioen") [38]. In the Netherlands, approximately 95 percent of employees participate in such a pension plan. Further, in [15] the authors study the risk based supervision of the Dutch pension system and the main focus is on this second pillar since it is the largest one in terms of number of participants. In the Netherlands it is obliged to regulate the pension of employees outside the company. This could be via a direct insurance by a life insurer or a pension fund.

The last pillar is more vaguely defined than the first and second pillar. In [10] the authors explicitly state that there is no unique definition. One possible definition is that this pillar covers the remaining pension plans and another possibility is that it covers the provision made on individual additional pension provisions, such as life annuities, lump sum insurances, etc. Furthermore, individuals can supplement their retirement benefits, e.g. by saving or reducing debts.

There are two types of funded pension schemes, the defined benefit (DB) and defined contribution (DC) scheme. First, let us discuss the defined benefit. In this pension scheme the benefit is derived based on either the final pay or the career average. The aim is to have an income after retirement of approximately 75 percent of average pay. In this scheme the benefits after retirement are fixed, hence the term defined benefit. In the Netherlands currently 94 percent of participants are enrolled in this type of pension scheme in the industry wide pension funds. Note that this number has been constant over the last few years. However, in the insured plans there is a strong trend from DB to DC due to the decreasing interest rate. The percentage DC plans has increased from 35 percent in 2005 to almost 60 percent in 2015. In the DC scheme the contribution is fixed instead of the benefits. In [40] the authors study investment strategies for the DC plan since the benefits are dependent on the investment returns. This paper states that many countries prefer DC over DB, which seems in contradiction with the numbers in the Netherlands.

In this thesis the focus is on the second pillar and in particular on the pension of employees that is regulated via a pension fund, since this is the money that is regulated by a pension fund company. This regulation is done by the asset and liability management team of such a company.

1.2 Asset and liability management

Asset and liability management (ALM) is managing or regulating the total assets to meet the current and future liabilities. In Figure 1.2 this structure is visually illustrated. The regulation consists of the precise asset allocation. An example of this allocation is: invest 10 percent in real estate, 50 percent in stocks and the remaining 40 percent in government bonds. This allocation is further optimized every month what implies that this allocation is a dynamic decision problem under uncertainty. This uncertainty comes from the interest rate, inflation and return on investments, which all are unknown quantities in the future. Another uncertainty comes from the participant's data. Participants could influence both the asset or liability side of the balance. For the asset side one example could be a participant who is hired. In this situation the participant of interest should pay pension to the pension fund of the new company that hired this participant and increases therefore the asset side.

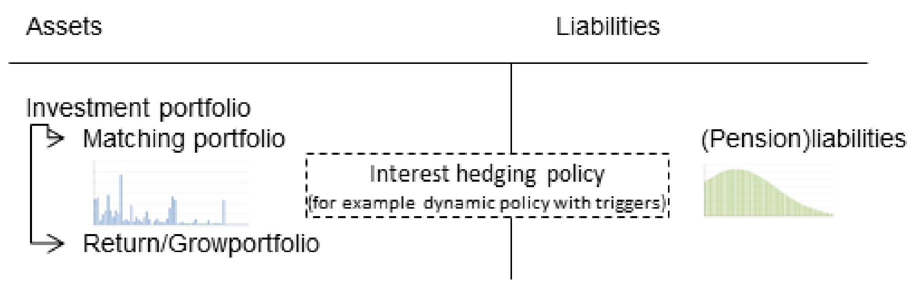


Figure 1.2: A visual illustration of the asset and liability management structure

The most common measure for the health of a pension fund is the funding ratio, which is the ratio of total asset value and liabilities. After the global financial crisis of 2008, the Dutch pension funds average funding ratio dropped drastically, see Figure 1.3.

Before 2007 the pension system was regulated in the Netherlands with the "Actuariele Principes Pensioenfondsen" (APP). In January 2007 a new pension agreement was introduced which was called "Financieel Toetsingskader" (FTK). The purpose of this new agreement was to combine the existing agreements. After the crisis of 2008 the FTK proved to be insufficient. Therefore, the FTK was changed into to the new FTK (nFTK) in January 2015. Another consequence of the crisis in 2008 was

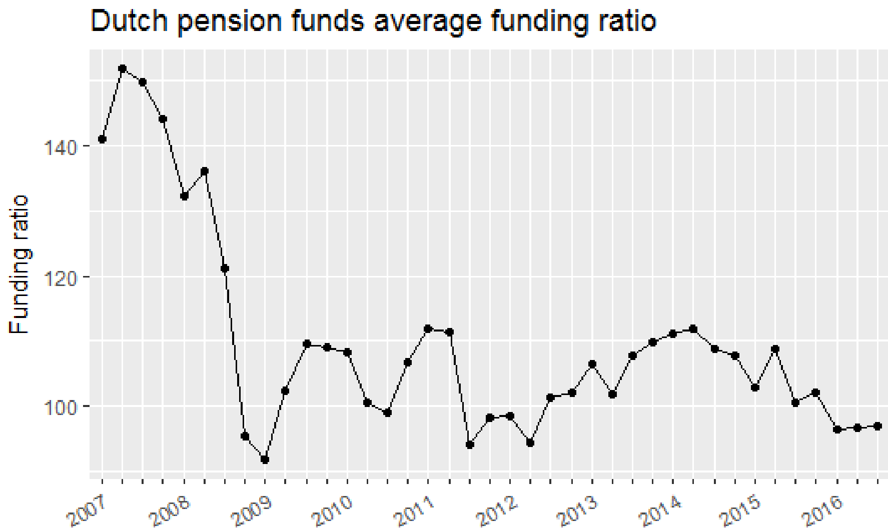


Figure 1.3: Dutch pension funds average funding ratio from 2007 to 2016 last updated on 31-1-2017, source [21]

that ALM models developed further and stochastic programming became of bigger importance, see [24] and [28] and the references therein. The main focus of these papers is on capturing the stochastic characteristics of these models in the best possible way. Most of these models heavily rely on the 95 percent confidence intervals or quantiles of the underlying distribution, an example for this underlying distribution can be the interest rate.

The increase of interest in ALM models also led to an increase of interest in stress scenarios. Here, stress scenarios can have different definitions. One definition is that stress scenarios define all the scenarios that lead to a specific threshold on the funding ratio, e.g. a funding ratio lower than 90 percent. Another definition is a certain quantile of the forecast. For this scenario one can think of the belief of low interest rate which corresponds in mathematical terms to the quantile (5%) of the predictions. This last definition is used throughout this thesis. As a consequence of this definition, observe that stress scenarios come with a certain probability. If a specific stress scenario leads to a severe outcome, thus a significant decrease in the funding ratio, the ALM team could decide to hedge against this risk involved by adapting the asset allocation. However this hedging also comes with a cost, therefore another important task of the ALM team is to determine if a stress scenario should be taken into account.

1.3 Problem description

On the one hand side it is illustrated that in the pension fund system the companies decide over a lot of money, which is mostly regulated by the ALM team. On the other hand, see Figure 1.3, it can be seen that not taking into account a stress scenario can have really severe effects on the funding ratio and thus the money of the participants. These two arguments are the basis of the main research question:

What are feasible stress scenarios for pension portfolios and what is their impact on pension fund portfolios in practice?

In this thesis the emphasis is on the first part of this question. The aim is to construct a model that can produce stress scenarios with a forecast of 5 years and a certain plausibility measure. The main challenge in stress scenarios is that they should be both plausible and consider extreme outcomes. Intuitively, it is clear that the most severe scenarios are less likely to happen. In the second part of the main research question, i.e. the impact on pension fund portfolios, the effect of the stress scenarios on the funding ratio is observed. For the ALM team this is of importance since it helps answering the question whether or not the team should take this stress scenario into account. Also, it gives more insight on what stress scenario influences the funding ratio the most. Here the importance of a realistic model is critical. If the ALM team decides to take this specific stress scenario into account, by means of hedging against this risk, but in practice the plausibility of this stress scenario is not correct or it is not able the capture the behavior of the economic market well, then the ALM team is basically wasting money by hedging against an unrealistic risk.

In our stress scenarios three risk factors that have a significant influence on the funding ratio of the pension portfolio are taken into account. The three risk factors are:

i) Return on investments

ii) Inflation rate

iii) Nominal interest rate

Here, the inflation rate denotes the general increase in services and goods in an economy and the nominal interest rate is the percentage charged to a borrower to use the cash of a lender. The first risk factor, which is a class of different categories such as global equity, bonds, etc., affects the asset side of the ALM balance. A return is a measure for the change in value of a category. Moreover, daily returns give the change between two consecutive days. The second and third risk factor affect the liability side. These risk factors are related by

$$\textit{Inflation rate} = \textit{Nominal interest rate} - \textit{Real interest rate}. \tag{1.1}$$

Thus in words, this means that the nominal interest rate is the interest rate before the inflation rate is taken into account, while the real interest is the interest rate while the inflation rate is taken into account. To avoid confusion on which interest rate is meant, in most cases the terms nominal and real interest rate are used. From a mathematical point of view, the real and nominal interest rates can be seen as stochastic processes over time.

Observe that not all the stochastic factors that influence the funding ratio of a pension fund portfolio are taken into account. The possible change of participant's data of a pension fund is not considered in our stress scenario. The first part of the main research question, thus the construction of the stress scenarios, is further divided into two sub questions:

i) What is the dependency between the risk factors?

ii) What is the forecast of each risk factor?

These two sub questions form the structure of this thesis, which is stated in more detail in Section 1.5.

Although, the plausibility of our stress scenarios is of importance, here already the stress scenarios, that are used to study the impact on the funding ratio, are stated. These stress scenarios correspond economic scenario where there is a crash of one risk factor.

i) Low returns on global equity

ii) Low inflation

iii) Low nominal interest rates

In the previous section it is already mentioned that after the crisis of 2008 there was an increase of interest in stress scenarios. However, most of this thesis is devoted to the derivation of stress scenarios, this suggests that the models known in the literature do not completely fit with our problem description. In the next section a literature overview is given and the difference between the literature models and our problem is explicitly stated.

1.4 Literature overview

In the literature there is an ongoing discussion regarding the distribution of risk factors in times of stress scenarios and whether or not this distribution is the same as in normal times. Throughout this thesis, and in the studied models from literature, the assumption is that the distribution of risk factors is the same as in normal times. Furthermore, the models in the literature are divided into two categories. The *direct stress scenarios* in which the models give as an output stress scenarios considering all the risk factors and the *indirect stress scenarios* in which the models do not give a stress scenario as output. Another distinction of the above mentioned scenarios is that the direct stress scenarios take into account all the risk factors, whereas the indirect stress scenarios only take into account one risk factor.

First three direct stress scenario selection methods are described, for a more detailed description we refer to Appendix A or the corresponding paper. Note that in these three methods the definition of a stress scenario differs from the definition given in the previous section, in every method, the definition is explicitly stated. In the literature the focus is more on selecting stress scenarios, whereas our model strives to construct stress scenarios. However, the different selecting methods give an overview what is known and will help to illustrate the difference and similarities of our model compared to the models stated here.

1.4.1 Direct stress scenarios

First stress scenario

The first stress scenario selecting method studied in [27], describes a *reverse stress testing* approach to select stress scenarios by empirical likelihood. This approach is not specifically derived for pension fund portfolios, however it can be applied to a pension setting. The definition of a stress scenario used here, after applying this selection method to a pension system, is a scenario that leads to a specific threshold in the funding ratio. It is assumed that the joint distribution of the risk factors and the funding ratio is known, and that it is elliptical distributed. For the distribution of the risk factors it is assumed that they follow either an exponential regularly varying (ERV) distribution or a regularly varying (RV) distribution. The random vector $\boldsymbol{X}$ has an elliptical distribution if the characteristic function satisfies

$$\phi_{X-\mu}(t) = \phi(t'\Sigma t),$$

where μ is the location parameter and Σ a matrix. Furthermore, the random vector $\boldsymbol{X}$ is in the class RV(ν) if it holds that

$$\lim_{\ell \to \infty} \frac{\mathbb{P}\left(\boldsymbol{X} \geq \ell x\right)}{\mathbb{P}\left(\boldsymbol{X} \geq \ell\right)} = x^{-\nu}, \tag{1.2}$$

and $\boldsymbol{X}$ is in the class ERV(α, ν) if $\exp\left(\boldsymbol{X}^{\alpha}\right)$ is in the class RV(ν). The normal distribution and the Student's t-distribution are two examples of distributions that are both elliptical and regularly varying. In the paper the authors state that market data, i.e. the risk factors, are often approximated well by the Student's t-distribution. Under this assumptions the most likely loss scenario is explicitly derived.

Second stress scenario

The second stress scenario selecting method, studied in [35], uses a *forward stress testing* approach. Here a closed set of all scenarios is considered and the aim is to select the scenario that gives the lowest funding ratio, the so called least solvent likely event. The assumption is that the funding ratio can be derived by a simplistic model which captures the mutual dependency of the risk factors, i.e. $g(\boldsymbol{z}) = \boldsymbol{\mu} + \boldsymbol{A}z$ where $g(\boldsymbol{z})$ is the funding ratio, $\boldsymbol{A}$ the matrix that captures the dependency and $\boldsymbol{z}$ a specific scenario. Under both assumptions an explicit expression for this least solvent likely event is given. When one of more of the assumptions is violated the computation of the stress scenario can be complicated but it can still be derived with numerical optimization.

Both stress scenario selection models have restrictions on the distribution of the risk factors and the funding ratio. However, in practice the funding ratio of each scenario, under a specific allocation, can be derived with a complicated deterministic model. In our setting this model can be used, which means that both stress scenario selection methods can be applied but with the assumption and drawback of using a simplistic model for deriving the funding ratio. Furthermore, in these models from literature the aspect of time is not taken into account and remember that our aim is to construct stress scenarios with a forecast of 5 years.

Third stress scenario

In [31] the aspect of time is taken into account. Here the Mahalanobis distance is introduced as a measure for the probability of the stress scenarios. Let the forecast scenario path of the risk factors $\boldsymbol{Z}$ be a random vector with an elliptical distribution with mean $\boldsymbol{\mu} = \mathbb{E}[\boldsymbol{Z}]$ and covariance matrix $\boldsymbol{A} = \text{Cov}(\boldsymbol{Z})$, then the Mahalanobis distance of a realization of $\boldsymbol{Z}$ is given by

$$\text{Maha}(\boldsymbol{z}) := \sqrt{(\boldsymbol{\mu} - \boldsymbol{z})^T \boldsymbol{A}^{-1} (\boldsymbol{\mu} - \boldsymbol{z})}.$$

The selection of the worst case scenario is based on the total loss in terms of funding ratio over time, of all the scenarios that have a Mahalanobis distance less or equal than τ. In the case of multivariate normally distributed risk factors a specific expression for this worst case scenario is given. If the risk factors are not multivariate normally distributed, the authors sketch a Monte Carlo algorithm that is able to solve the selection problem. Although, in this paper no explicit assumptions on the loss function are made, it assumes that the loss function is known.

Discussion

In the literature, and these three papers in specific, (explicit) assumptions on the model to derive the funding ratio are made. In a pension fund portfolio also the asset allocation is of importance, whereas each different asset allocation gives a different funding ratio. If a simplistic model is assumed to derive the funding ratio, e.g. $g(\boldsymbol{z}) = \boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{z}$, this asset allocation is not taken into account. As a consequence, the selection methods in the literature cannot derive what the effect is of possible hedging against certain stress scenarios. Note, that extending the models in the literature to account for the asset allocation is possible but gathering the data, to determine the distribution for the funding ratio, per allocation can be difficult. Also, it is beneficial for us to use the model Aon Hewitt uses to calculate the funding ratio since this is already known.

Our approach, to tackle the problem of constructing stress scenarios, consists of two steps. The dependency between the risk factors is studied, see Chapter 3, together with the forecast of the three risk factors, Chapters 4, 5 and 6. Note that with both the real and nominal interest rate and the relation given in (1.1), the forecast of the inflation can be derived. When forecasting these interest rates, also stress testing procedures can be applied to the corresponding yield curve. These stress testing procedures, also called (representative) yield curve shocks, are indirect stress scenarios since only one risk factor is influenced and explicit stress scenarios are not given as an output. Before proceeding to the yield curve shock models let us explain what a yield curve is. Both the terms real and nominal interest rate on a time point do not correspond to one specific value, but have values for each maturity. All these values of the maturity together, on a specific time point, are called the yield curve. Note that the yield curve thus corresponds to the real or nominal interest rate.

1.4.2 Indirect stress scenarios

In the literature there are many papers on the study of yield curve shocks, e.g. [8], [2], [16], [20] and [30]. In [2] the authors give an overview of the most common methods used in practice with an extensive explanation of all the methods. Here the methods to derive the yield curve shocks that are of interest to us are briefly stated. If both the real and nominal yield curve can be used in the methodology no explicit distinction is mentioned.

In the *historical method*, the historical behavior of the yield curves is used to derive yield curves shocks. For each maturity the fitted distribution of the changes of the real and the nominal interest rate over time is derived. Once this distribution is known, a certain quantile shock for each maturity can be applied to the last day yield curve. Note that for this method it is not necessary to fit the distribution. The empirical distribution could also be used to compute a certain quantile. In this way no error is made when fitting the distribution.

One of the simplest methods to generate yield curve shock is the *standardized method*. In this method a parallel shift is applied to the yield curve. In [2] the authors propose a downward shift of 200 basis points, i.e. two percent. For banks it is obligatory to measure the impact on the economic value of equity of these yield curve shocks.

A similar method as the standardized method is the *standardized factor method*. In "Regeling Pensioenwet en Wet verplichte beroepspensioenregeling" [18], interest rate factors for the calculation of the private equity are given. For each maturity a decrease and increase percentage is given. Note that in contrast to the standardized model, here the shift does not have to be parallel. Moreover, the shift for the real and the nominal yield curve are different.

The *principal component analysis* (PCA) is based on the assumption that the yield curve can be described by a number of principal components. In our case, which is consistent with most of the literature, e.g. [7] and [29], these three most important components are the level, slope and curvature of the yield curve. For each maturity the factor loadings indicate the importance of the principal components. Moreover, with the factor loadings and the values of the principal components one can describe the yield curve on a specific time point. Note that the factor loading is constant over time while the values of the principal components change. To obtain the yield curve shock, a certain quantile of the changes in the value of the principal components is applied to the last day principal component values.

Observe that, in contrast to the direct stress scenarios, the indirect stress scenarios can be applied in our framework when deriving stress scenarios that satisfy the problem description. In the next section the overview of the thesis given where it can be found in which chapter these indirect stress scenarios are applied to historical data.

1.5 Overview of the thesis

The historical data for all the three risk factors, mentioned in Section 1.3, is introduced in Chapter 2. Note that also the subcategories within a risk factor are explicitly stated in this chapter. Using the historical data, the fitted distribution of the daily returns of these subcategories is studied, together with the dependency structure of the subcategories within the first risk factor.

After a preliminary statistical analysis of the data, the first part of the main research question given in the problem description, i.e. Section 1.3, is answered in two steps. Note that every chapter is devoted to a step or sub step. This helps the reader to understand the structure of this thesis, but as a consequence some chapters are lengthier than others.

Dependency structure step) The focus of the first step is on answering the first sub question stated in the problem description. Here the daily return data analysed in Chapter 2 are essential for capturing the dependency between the risk factors and the dependency of the different categories within the risk factors. This dependency is captured by means of a copula function. Such functions capture dependency structures permitting bigger flexibility than a simple linear correlation.

Marginal forecasting step) In the second step the second sub question is studied, i.e. the forecast models of each risk factor. More concretely, Chapter 4 studies the forecast of the return on investments risk factor. In this chapter an autoregressive model is used to forecast the daily returns of each subcategory. The parameters for the autoregressive model are estimated based on the data analysed in Chapter 2. For the forecast of the second risk factor, i.e. the inflation rate, observe that with Equation (1.1) forecasting this risk factor is equivalent to forecasting the real and the nominal interest rate. The forecast of the real interest rate at various maturities (i.e., the real interest rate yield curve) is studied in Chapter 5. In this chapter, a model based on stochastic differential equation (SDE), namely the Vasicek short rate model, see Section 5.1, is introduced and the performance measure, based on historical data, is derived. When comparing the performance measure different methods are used to calibrate the model to the data. Once the real interest rate yield curve is modelled using the Vasicek model, we apply the representative yield curve shocks, given in Section 1.4.2, to the historical data of the real interest rate. In Chapter 6 the forecast of the nominal interest rate is studied, again by means of a SDE short rate model. However, for the forecast of the nominal interest rate both the Vasicek and the Black Karasinski are considered. In this chapter the underlying assumptions in each model are discussed and the performance measure of both short rate models based on historical data are compared. Moreover, in line with Chapter 5, the representative yield curve shocks are analysed based on historical data.

With both the dependency and the marginal forecast models for each risk factor one can derive the forecast corresponding to the stress scenarios of their belief. After these two steps, the first part of the main research question given in the problem description, i.e. Section 1.3, is therefore answered. In Chapter 7 an example of the stress scenarios mentioned at the end of Section 1.3 is given and these stress scenarios are further studied. In particular, the impact of the stress scenarios on the funding ratio of a specific pension fund portfolio under a general allocation of the assets is studied. In this chapter the difference between the short rate models of Chapter 6, i.e. the Vasicek and Black Karasinski model, based on the funding ratio is studied. Moreover, also the stress scenarios corresponding to the representative yield curve shocks are compared.

Chapter 2

Data analysis

In this chapter the historical daily data obtained from Bloomberg is introduced for all the risk factors mentioned in Section 1.3. In particular, the distribution of the historical returns is studied, which is essential for capturing the dependency between and within the risk factors in Chapter 3. Moreover, the autocorrelation of the subcategories of the first risk factor, i.e. the returns on investment, is analyzed. Also, the hypothesis that large negative of positive returns are more correlated to the next day returns than mediocre returns is tested. A good understanding of the autocorrelation is of importance for the forecast of this risk factor performed in Chapter 4.

2.1 Descriptive statistics

The historical data that is used in this thesis is obtained from the Bloomberg model is introduced according to the three risk factors mentioned in Chapter 1. For the returns on the assets side we study benchmark investment indexes, which are commonly used by pension funds. One example is the global equity market, which is represented by the MSCI World. The MSCI World is one of the biggest indexes in global equity. Note that it is not possible to trade in this index, one can only approximate trading in this index by an Exchange Traded Fund (ETF), a so called index tracker. The aim of this thesis is not to model the whole economic market, only the indexes where a pension fund is mostly investing in are of interest to us. All these indexes are further discussed in Table 2.1.

All the data for these benchmark indexes comes from the Bloomberg program, sampled at daily basis, excluding non-trading days, from 31-12-2000 to 31-12-2016. In total this adds up to 4175 observations. The value is the closing value of that index on the specific day. However, not all indexes have 4175 observations due to specific holidays or the fact that some indexes are founded after 2001. Removing all the dates where an observation is missing from one or more indexes leads to 3553 remaining observations. Linear regression could be used to estimate the value of the missing data, but since we do not want to make the assumption that the data follows linear behavior between three consecutive points the 3553 observations available are used.

Assumption 1 *Historical data from 31-12-2000 to 31-12-2016 represents the behavior of the economic market well, especially in an economic crisis.*

In the data the economic crises in 2008 and 2011 are captured. However, these crises are different and less severe than the crisis of 1937. An argument that supports only looking at the financial market from 2001 onwards is that the financial market changed for instance due to the introduction of the Internet. Another argument is that one learns from every financial crisis which suggests that the crisis in 1937 is not that relevant in the contemporary economic market. One could also think of arguments against the decision of taking historical data from 2001 onward, since in stress scenarios only the tails of the return distribution are of importance. To get an understanding of the tail behavior, one needs sufficient data and only looking at data from 2001 onward could be insufficient.

Table 2.1: Return seeking asset indexes with a brief description

Index	Description
MSDEWIN	MSCI Daily Total Return Net World in EUR
MSDEEEMN	MSCI Emerging Markets Equity in EUR
M1WOMVOL	MSCI World Minimum Volatility
LET7TREU	Barclays 7-10 Year Euro Government Bond
LS06TREU	Barclays Bellwether Swap (EU): 10 years
LS08TREU	Barclays Bellwether Swap (EU): 30 years
LEC7TREU	Barclays EuroAgg Corporate 7-10 Year Total Return
LF98TRUU	Barclays US Corporate High Yield Total Return
BCOMTR	Bloomberg Commodity Index Total Return
JPEIDIVR	J.P. Morgan Emerging Market Bond Index (Diversified)
G4F0	French 7-10 Year Government bond
LECRTREU	Barclays Euro Aggregate Credit Total Return
G250NLEU	GPR 250 Index (Property Shares) Netherlands Euro Total
REIT	Dow Jones Equity Real Estate Investment Trust Total Return (Direct Real Estate US)
HFRXGLE	Hedge Fund Research HFRX Global Euro
BXIIBEU3	Barclays Benchmark 3months EUR Cash

With the index values of a specific index, the daily return at time t is derived by,

$$r_{i,t} = 100 \times \log\left(\frac{p_{i,t}}{p_{i,t-1}}\right),$$

where $p_{i,t}$ denotes the value of the i-th index at time t.

Table 2.2: Summary statistics of the returns per index

Index	Mean	SD	Kurtosis	Skewness	Min	Max
MSDEWIN	0.01	1.04	5.27	-0.16	-6.95	8.50
MSDEEEMN	0.03	1.22	5.86	-0.28	-8.48	10.08
M1WOMVOL	0.02	0.83	5.49	-0.17	-5.03	6.97
LET7TREU	0.02	0.29	4.39	0.02	-1.62	2.62
LS06TREU	0.03	0.38	8.68	0.09	-3.34	3.86
LS08TREU	0.04	0.88	8.41	0.04	-8.63	7.98
LEC7TREU	0.02	0.28	2.68	-0.58	-1.83	1.43
LF98TRUU	0.03	0.67	5.15	-0.35	-6.64	4.09
BCOMTR	0.00	1.04	1.42	-0.18	-5.16	5.39
JPEIDIVR	0.03	0.65	2.77	-0.21	-4.58	3.63
G4F0	0.02	0.30	2.86	-0.22	-2.12	1.81
LECRTREU	0.02	0.17	1.91	-0.50	-0.81	0.81
G250NLEU	0.03	1.26	4.94	-0.34	-7.31	7.39
REIT	0.04	1.94	17.40	-0.22	-20.85	16.93
HFRXGLE	0.00	0.24	9.57	-1.28	-2.11	2.04
BXIIBEU3	0.01	0.01	6.35	2.07	-0.03	0.10

Table 2.2 gives a simplistic overview of the returns for each index. From this table it can be concluded that some indexes have higher standard deviation than others and it seems that all the indexes have a mean approximately equal to zero. Also, it can be seen that the kurtosis and skewness differ from respectively 0 and 3, which is an indication that the returns are not normally distributed. Figure 2.1 shows the value and the cumulative density function of the daily returns for a specific index,

namely the MSDEWIN Index. There are three densities fitted on the returns, the normal distribution, Student's t-distribution and the skewed Student's t-distribution. The probability density function of the skewed Student's t-distribution is given by

$$f_{ST}(x;\mu,\sigma,\lambda,q) = \frac{\Gamma(\frac{1}{2}+q)}{\nu\sigma\sqrt{\pi q}\cdot\Gamma(q)\left(\frac{|x-\mu+m|^2}{q(\nu\sigma)^2(\lambda sign(x-\mu+m)+1)^2}+1\right)^{\frac{1}{2}+q}},$$

with

$$m = \frac{2\nu\sigma\lambda\sqrt{q}\cdot\Gamma\left(q-\frac{1}{2}\right)}{\sqrt{\pi}\cdot\Gamma\left(q+\frac{1}{2}\right)}, \quad \nu = \frac{1}{\sqrt{q\left(3\lambda^2+1\right)\left(\frac{1}{2q-2}\right)-\frac{4\lambda^2}{\pi}\left(\frac{\Gamma\left(q-\frac{1}{2}\right)}{\Gamma(q)}\right)^2}}.$$

Remark 1 *The skewed Student's t-distribution is an extension to the Student's t-distribution, i.e. $f_{ST}(x;\mu=0,\sigma=1,\lambda=0,q)$ is the probability density function of the Student's t-distribution with $2q$ degrees of freedom. Moreover, the skewed Student's t-distribution also satisfies condition* (1.2) *in Section 1.4 and therefore is in the class of regularly varying distributions. However, in contrast to the Student's t-distribution is the skewed Student's t-distribution not an elliptical distribution.*

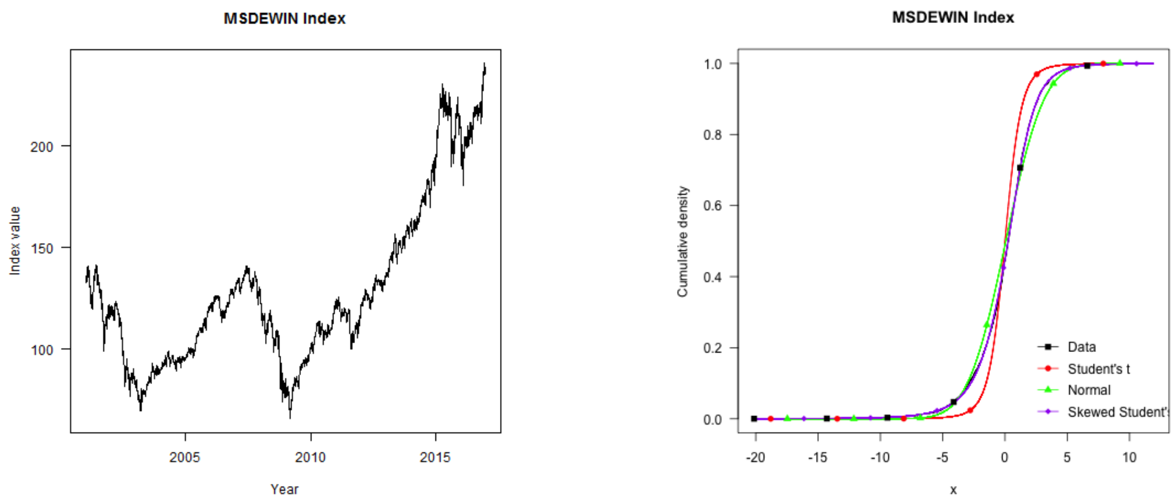


Figure 2.1: The value (left) and cumulative return distribution (right) of the MSDEWIN Index

In Figure 2.1 the characteristic fluctuation of the stock market is clearly visible. Also, one could observe the crisis in 2008, less clear is the crisis in 2011 but this crisis did especially affect the nominal interest rate. From the cumulative distribution function we can conclude that there are values less than -5 and greater than 5 but with very low probability, to be precise there are 12 observations outside the interval $[-5,5]$, from which 6 on the lower side and 6 on the upper side of this interval. Furthermore, from Figure 2.1 it seems that the skewed Student's t-distribution is the best fit out of these three distributions.

In Table 2.3 further investigation to the tail behavior of the historical data with respect to the three distributions is done. The probability below a certain return for the historical data and fitted distributions is stated and in parentheses the number of observations is given, whereas for the distribution this is the probability times the total number of observations. Note that in the table the focus is on the lower side of the returns of the MSDEWIN index, which are the returns that have a severe effect on the funding ratio of a pension portfolio.

Table 2.3: Probability and number of observations below a certain return percentage for the MSDEWIN index

Return (%)	History	Normal	Student's t	Skewed Student's t
-1	0.120 (511)	0.165 (701)	0.194 (821)	0.118 (501)
-2	0.032 (136)	0.027 (113)	0.067 (284)	0.032 (135)
-3	0.010 (43)	0.002 (8)	0.027 (113)	0.012 (49)
-4	0.003 (14)	0.000 (0)	0.013 (53)	0.005 (22)
-5	0.0014 (6)	0.0000 (0)	0.0067 (28)	0.0027 (11)
-6	0.0009 (4)	0.0000 (0)	0.0039 (17)	0.0016 (7)
-7	0 (0)	0.0000 (0)	0.0025 (10)	0.0010 (4)

Table 2.3 shows that the tails of the return distribution for this specific index are thicker than the tails of a normal distribution. Furthermore, if we calculate the p-value from the Kolmogorov-Smirnov (KS) test we get 0.0001 for the normal distribution, 0.0000 for the Student's t-distribution and 0.91565 for the skewed Student's t-distribution. We refer to Appendix B.2 for a more detailed discussion of this KS test and the p-value of the skewed Student's t-distribution for the other indexes. Throughout this thesis the KS is used since it is a nonparametric test. Note that here the p-value for the normal and Student's t-distribution is not included, since these p-values are smaller than 0.01 for all the indexes. Furthermore, note that in [27] the authors assumed Student's t-distributed returns, which does not give the best fit for the return distributions. Also, the analysis in this paper is not valid for skewed Student's t-distributed returns since it assumes an elliptical distribution.

Assumption 2 *The daily returns of the first risk factor (returns on investment) are skewed Student's t-distributed.*

For the inflation rate the inflation-linked bond with zero coupons is taken as a representative. A zero coupon bond is a debt service where no interest is paid but its full value at maturity. The index abbreviation of this bond in the Bloomberg program is EUSWI and historical daily data is available from 2004 up to and including 2016 excluding non-trading days. The Bloomberg program has data from this ticker with maturities 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 15, 20, 30, 40 and 50. However, for the dependency between risk factors only the inflation rates with a maturity of 5 or 15 years are considered.

For the derivation of the change in the inflation rate the same expression as for the returns of the benchmark indexes is used, namely

$$ri_{y_m,t} = 100 \times \log\left(\frac{i_{y_m,t}}{i_{y_m,t-1}}\right), \tag{2.1}$$

where $i_{y_m,t}$ denotes the inflation rate with a maturity of y_m years at time t. Note that a straightforward interpretation of the change $i_{y_m,t}$ is missing. Whereas for the index values of the investment assets only positive values are observed, for the inflation rate the historical data also contain negative values. Moreover, in the case of switching sign over time, thus going from $I_{y_m,t} = -0.1$ to $I_{y_m,t+1} = 0.1$ or vice versa, the expression given in (2.1) cannot give an outcome in the real numbers.

For further information on the value and cumulative distribution of the inflation rate with a maturity of 15 years the interest reader is refer to Figure B.1.

When performing the KS test on the inflation rate with maturity 15, where the null hypothesis assumes the Student's t-distribution, it gives a p-value of 0.01062. Note that with this p-value we reject the assumption that the observations $i_{15,t}$ are skewed Student's t-distributed. However, this distribution fits better when compared to the normal distribution and the Student's t-distribution. The p-value of both these distribution is < 0.001. For the maturity of 5 years the same conclusions hold.

Assumption 3 *The daily returns of the second risk factor (inflation rate) are skewed Student's t-distributed.*

The European Swap rate, which has the index abbreviation EUSA, is used as a representative of the nominal interest rate. Historical data is available from 2002 up to and including 2016, since 2002 was the start of the EUR currency. For this ticker the Bloomberg program contains data with the same maturities as for the inflation rate. In comply with the inflation rate, for the nominal interest rate only the maturities of 5 or 15 years are used when capturing the dependency between the risk factors. Also, the same expression for a change in the nominal interest rate is used, i.e. Equation (2.1).

For further information on the value and cumulative distribution of the nominal interest rate with a maturity of 15 years the interest reader is refer to Figure B.2.

When performing the KS test, with the null hypothesis assuming the skewed Student's t-distribution, on the nominal interest rate with a maturity of 15 years the p-value of this test is 0.9389. This p-value does not reject the null hypothesis, which suggests that the underlying distribution of the returns is skewed Student's t-distributed. This conclusion also holds for the nominal interest rate with a maturity of 5 years.

Assumption 4 *The daily returns on the third risk factor (nominal interest rate) are skewed Student's t-distributed.*

Remark 2 *From the daily index value also weekly and monthly returns of the risk factors can be derived. On these weekly or monthly returns one can again perform the KS test from which it follows that the returns of the two risk factors returns on investment and nominal interest rate are still skewed Student's t-distributed but with other parameters. Although the objective is to stress scenarios with a time period of 5 years, using daily gives us more information about the dependency.*

2.2 Autocorrelation of the returns on investment

In this section the autocorrelation function (ACF) of the returns for each index is observed, see Figure 2.2 for the ACF of the MSDEWIN index.

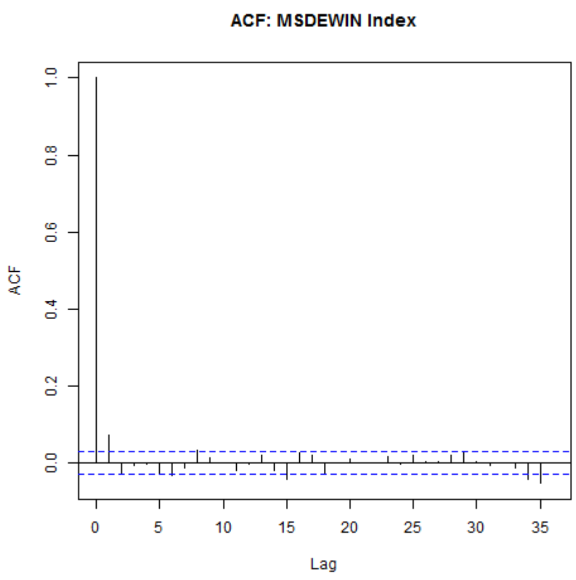


Figure 2.2: The autocorrelation function for the historical returns of the MSDEWIN index

Although, the autocorrelation coefficient of lag 1 is significant, namely $\rho = 0.08$, it is extremely low. The phenomena of low correlation of stock prices is extensively studied in the literature, see [5] for an overview of all the relevant literature. Most of the papers mentioned in this review paper study the predictability conditioned on large prior changes.

In [6] the authors study low autocorrelation in short term stock returns, where short term means lag 1. The idea of this paper is to divide the returns over multiple bands based on the value. The assumption is that large returns have higher autocorrelation than small returns. For testing the correlation we use the Spearman's rank correlation coefficient ρ which is non parametric. The null and alternative hypotheses of this test are given by,

$$\begin{aligned} H_0 &: \rho_{X,Y} = 0 \\ H_1 &: \rho_{X,Y} \neq 0, \end{aligned} \tag{2.2}$$

where X represents the returns in the specific band and Y the next day's returns with respect to X.

Table 2.4: Statistics for correlation for the historical returns of the MSDEWIN Index when dividing the returns into bands

Band	Observations	Autocorrelation	p-value
returns ≤ -2.56	50	-0.08	0.557
$-2.56 <$ returns ≤ -1.97	44	-0.037	0.811
$-1.97 <$ returns ≤ -1.37	111	-0.186	0.051
$-1.37 <$ returns ≤ -0.78	235	0.069	0.290
$-0.78 <$ returns ≤ -0.18	670	-0.040	0.300
$-0.18 <$ returns ≤ 0.41	997	0.016	0.617
$0.41 <$ returns ≤ 1.01	634	0.141	0.000
$1.01 <$ returns ≤ 1.60	192	-0.027	0.715
$1.60 <$ returns ≤ 2.20	64	-0.139	0.274
returns > 2.20	54	0.089	0.521

From Table 2.4 it can be seen that only the band $0.41 <$ returns ≤ 1.01 is rejecting the null hypothesis stated in (2.2). Note that this outcome thus does not comply with the results stated in [6]. One difference is the number of returns, in the paper the authors study the daily returns of 30 stocks traded from 1987 to 2007, which results in approximate 150.000 data points. Another difference is the data. In the paper daily returns of certain stocks are observed while in our case benchmark indexes are observed, for example of the benchmark for the global equity.

Remark 3 *In this analysis the bands are chosen in such a way that we have approximately* 50 *observations in the lower and upper band. One discussion point can be the number of observations in these boundary bands, therefore the same analysis is performed for the case of* 100 *observations in the lower and upper band. The conclusion made from Table 2.4 does not change when analyzing more observations in a band.*

2.3 Summary of the data analysis

In this chapter the marginal distributions of the historical returns are studied. Based on the KS test, the skewed Student's t-distribution describes these returns the best, when compared to the normal and Student's t-distribution. For the underlying fitted distribution of the first and third risk factor, i.e. the returns on investment and the nominal interest rate, the KS test gave a significant p-value, but for the second risk factor, i.e. the inflation rate, this was not the case. Furthermore, the autocorrelation of the first risk factor is studied. It is observed that there is a low autocorrelation for each index within this risk factor. One solution to this low autocorrelation, studied in the literature, is to divide the historical returns into bands, with the assumption that large positive and negative returns are more correlated to the next day return. However, this remark was not validated based on the data at hand.

Chapter 3

Dependency between risk factors

The aim of this chapter is to derive a framework that generates daily returns with the same characteristic as historical returns, but without any forecasting assumption. Observe that in Chapter 2 we performed the data analysis on the historical data obtained from Bloomberg and it is showed that the skewed Student's t-distribution fits the return distributions of all the categories within the three risk factors the best. This return distribution is essential for capturing the dependency. As our analysis illustrates, the underlying dependency between and within the risk factors is not linear, and for this reason we have investigated more general dependency structures captured by copula functions, instead of simple linear dependency structures captured by the correlation coefficient.

3.1 Capturing dependency

After introducing the historical data and studying the marginal distribution of the returns in Chapter 2, the focus is on capturing the dependency between risk factors and the dependency within a risk factor.

First, let us visually show the dependency between the MSDEWIN and BCOMTR index. This is achieved by plotting the corresponding scatter plot of these two indexes, cf. Figure 3.1. From Figure 3.1 it seems that the two benchmark indexes are not linear dependent. Therefore, the correlation coefficient, which is defined as

$$\rho_{X,Y} := \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}},$$

where $\text{Cov}(X,Y)$ denotes the covariance between the two random variables X and Y, is not an appropriate measure for the dependency. The reason for this is that one limitation of the correlation coefficient is that it can only capture linear dependency, see [11] for an extensive discussion on all the limitations. As an improvement to this dependency measure, copulas, which can capture more than only linearly dependency, are introduced.

Before proceeding let us first give the definition of a copula function.

Definition 1 *Let C be a function from $[0,1]^2 \to [0,1]$ such that $C(u,0) = C(0,v) = 0$, $C(u,1) = u$ and $C(1,v) = v$. Further, for every u_1, u_2, v_1, v_2 in $[0,1]$ with $u_1 \leq u_2$ and $v_1 \leq v_2$,*

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0,$$

then the function C is called a copula function.

Further, Sklar's Theorem given in [33, Theorem 2.3.3] states that for any distribution F there exists a copula C such that

$$F(x,y) = C\left(F_1(x), F_2(y)\right),$$

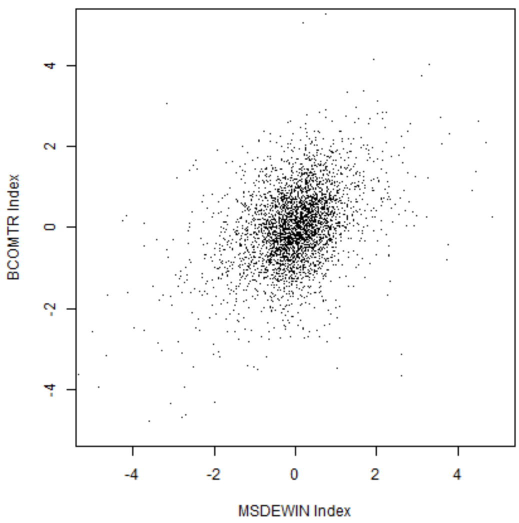


Figure 3.1: Historical data of the returns that show the dependency between the MSDEWIN and BCOMTR index

which is unique for continuous distributions F_1 and F_2.

For a more visual understanding on the different copula functions and the importance of an improvement of the standard correlation coefficient the interested reader is referred to the Section B.3 and Figure B.3.

In our model the goal is to find a copula that can capture the dependency between the risk factors and the dependency within each risk factor. This means that the copula describes a joint distribution of dimension equal to the total number of all the indexes. For visual presentation the analysis is showed for the dependency between the MSDEWIN and MSDEEEMN index, which then is extended to the desired copula with a higher number of dimensions.

For the historical data of our example the log-likelihood of different copulas, such as the independence copula, Gaussian copula, Student's t-copula, Clayton copula and Gumbel copula are compared. According to the log-likelihood based on historical data, the Student's t-copula gives the best fit.

Assumption 5 *The dependency between and within the risk factor is captured with the Student's t-copula function.*

Definition 2 *Let ν denote the degrees of freedom and P the correlation matrix with elements ρ then the Student's t-copula of dimension d is given by*

$$C^t_{\nu,P}(\boldsymbol{u}) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_d)} \frac{\Gamma(\frac{\nu+d}{2})}{\Gamma(\frac{\nu}{2})\sqrt{(\pi\nu)^d|P|}} \left(1 + \frac{\boldsymbol{x'}P^{-1}\boldsymbol{x}}{\nu}\right)^{-\frac{\nu+d}{2}} d\boldsymbol{x},$$

where t_ν^{-1} denotes the quantile function of the Student's t-distribution, $\boldsymbol{x'}$ the transposed random vector of $\boldsymbol{x}$ and P^{-1} the inverse of matrix P.

With the Student's t-copula and the fitted skewed Student's t-distribution the historical data can be replicated. Figure 3.2 shows the historical returns together with the simulated returns according to the Student's t-copula. In Figure 3.2 it can be seen that both the dependency and the marginal distributions of the historical data are captured well. For these specific indexes historical returns can therefore be replicated. The method to replicate the historical returns is further extended to all the risk factors. However, the copula that captures the dependency between risk factors cannot visually be represented. Therefore, in the next section, remarkable or interesting dependency between risk factors or within risk factors is highlighted.

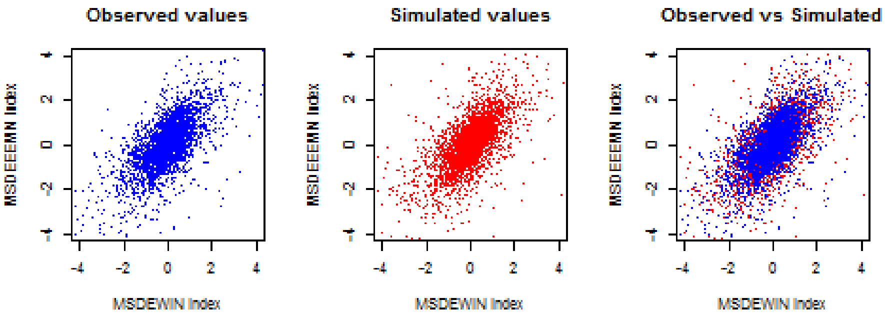


Figure 3.2: Observed (blue) and simulated via the Student's t-copula (red) returns of MSDEWIN against MSDEEEMN

3.2 Highlighted dependency between risk factors

In this section we highlight and discuss some remarkable observations, regarding the copula used to describe the dependency between or within the risk factors.

When looking at the dependency measure, ρ, of the Student's t-copula we observe that this measure between the EUSA15 and the EUSWI15 is $\rho = 0.025$. This indicates that the daily change in the inflation rate and the nominal interest rate is not highly dependent. For further research one could investigate the dependency between those risk factors in more detail and even use another definition for the change, e.g. the absolute difference.

The other observation is that the EUSWI is not highly dependent to all other indexes mentioned in Table 2.1. All the dependency measures are lower than $\rho = 0.3$. For the EUSA it holds that it is highly dependent on some indexes mentioned in the table. The dependency measure between EUSA and the 10 or 30 years Barclays Bellwether Swap are respectively $\rho = -0.83$ and $\rho = -0.86$.

Another dependency that deserves some more discussion is the dependency between the HFRXGLE index and the MSDEWIN or MSDEEEMN index. In words, this is the correlation between hedge funds and the global- or emerging market equity. One would expect that investing in hedge funds should be more beneficial in a stress scenario compared to a normal scenario. However, we observe that the correlation is respectively, $\rho = 0.66$ and $\rho = 0.55$. This means that a drop in the global equity also results in a drop in the hedge funds however the dependency is not as strong as for example the dependency between global- and emerging markets equity.

3.3 Discussion

A possible drawback of this model is the low dependency between the inflation and the other risk factors. This can be due to the specific measure that is used to derive the change in inflation rate, see (2.1) or a characteristic of the historical data. Note that, with the dependency of the risk factors, the dependency is studied on a specific time point and not over time. However, in stress scenarios one should give a forecast of the risk factors. An option is to forecast one risk factor and use the copula to derive the forecast of the other risk factors. As a consequence of the low dependency between the inflation and the other risk factors, forecast models for all the risk factors are studied, see Chapters 4, 5 and 6. This approach comes with both advantages and disadvantages. One advantage is that the stress scenarios vary less than in the case where one risk factor is assumed to be known. However a disadvantage is that the whole model depends more on the forecasting models. Noteworthy is that in both options a forecast model of at least one risk factors is used.

Chapter 4

Forecasting the returns on investments

In this chapter the forecasting of the returns on investments is studied, by means of an autoregressive model. This model is already heavily studied in the context of finance, e.g. [39] where the author studies this model to forecast the returns of the Dow Jones and in [13] the forecast of the US, UK and Japanese stock index futures markets is studied.

The autoregressive model of order p, denoted by AR(p), is defined as,

$$X_t = \sum_{i=1}^{p} \psi_i X_{t-i} + \epsilon_t,$$

where X_t is in our case the distribution of the returns at time t, ψ is a parameter of the model and ϵ_t captures white noise and thus is standard normal distributed at time t. The order p of the model corresponds to the number of previous historical returns that have a significant effect on the return of interest.

Assumption 6 *The AR(p) model assumes a non-moving average.*

The assumption of a non-moving average is checked visually when looking at the returns over time and is consistent with the identical assumptions of the distribution in Chapter 2.

Remark 4 *For the forecast of the returns on investments also the ARMA(p,q) model is studied. This model does not assume a non-moving average. Both the AR(p) and ARMA(p,q) are compared based on the AIC and BIC. From this study it turned out that the AR(p) model gave the best fit.*

In Section 2.2, see Figure 2.2, it is observed that only the first order is significant for all the categories within the returns on investment risk factor. However, it is also observed that the autocorrelation parameter is low and dividing the returns into bands does not increase this autocorrelation coefficient. To conclude, this means that the AR(1) model seems to be the best model to forecast the first risk factor, i.e., returns on investment.

Figure 4.1 depicts the historical value together with the forecast of 10 sample paths. From the sample paths that forecast the value of the MSDEWIN index, see Figure 4.1, the conclusion can be drawn that there is a large deviation between the paths. Another drawback of our model is not considering the volatility clustering of the returns. In Figure 4.2 it can be seen that in times of a crisis, e.g. in 2008, the returns have a higher volatility than normal and that therefore this volatility is clustered over time.

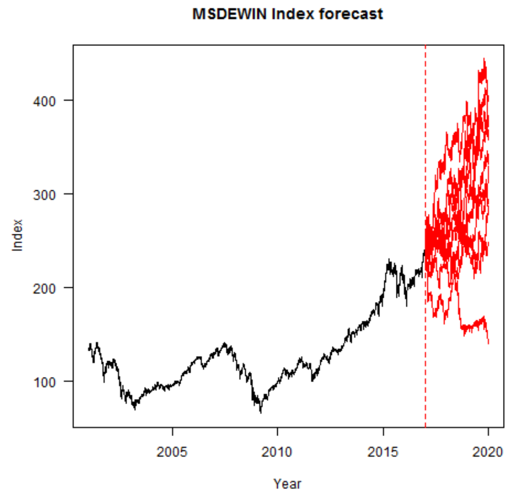


Figure 4.1: The historical index value together with the forecast of this index value of the MSDEWIN index

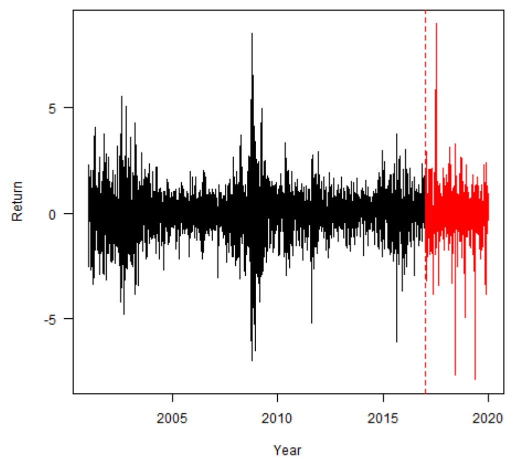


Figure 4.2: The historical returns (black) together with the forecast of these returns (red) of the MSDEWIN index

Figure 4.2 shows that the returns in our forecast have neither the characteristics of a real crises or low volatile returns.

4.1 Discussion

To forecast the returns on investments the AR model is used. One drawback when applying this model to our data is the low autocorrelation. A low autocorrelation means high deviation in the forecasts. One solution to this low correlation, studied in the literature, is to divide the historical returns into bands, with the assumption that large positive and negative returns are more correlated to the next day return. However, for our data this assumption was not valid, see Chapter 2. To conclude, forecasting the returns on investment is a difficult problem and with the ACF no strong correlation between subsequent returns is found. Especially when giving a forecast for the risk factors over a period of 5 years by daily returns, this means high deviation between different forecasts. Although this seems a major drawback of the model, concluding that there is a lot of room for improvement is not entirely correct. One could argue that forecasting the returns is difficult since there is too much randomness involved. Moreover, when a mathematical model can capture this randomness and gives an accurate forecast of the returns, one could argue that the inventor of this model is able to make infinite profit.

From this chapter it is clear that forecasting the returns on investments is difficult, but our model can still rely on the forecast of the other risk factors, i.e. the real interest rate and the nominal interest rate. These forecasts are studied in the next Chapters.

Chapter 5

Forecasting the real interest rate

To model the future real interest rate a so called short rate model is used. Short rate models are heavily studied in the literature, see e.g. [12] and [32]. One of the most important properties of short rate models is the mean-reversion. There is still an ongoing discussion on which method one should use to model the real interest rates. Models that are often used are Merton's, Cox-Ingersoll-Ross and Vasicek model. In this chapter the real interest rate is modeled by the Vasicek model. This decision is based on both literature, e.g. see [25] and the references therein and the expertise of Aon Hewitt.

Assumption 7 *The Vasicek short rate model is the best model for the real interest rate.*

Furthermore, different calibration methods are given to obtain the parameters of the model and the performance of this model with respect to historical data is studied.

5.1 Vasicek model

Within the class of the Vasicek model one could decide between the one factor and the two factor model. An ongoing discussion in the field of mathematics is the trade off between simple and simplified models to complex and realistic models, e.g. [19]. For this reason both the one factor and two factor Vasicek models are introduced and compared.

The one factor Vasicek model is given by the stochastic differential equation

$$\mathrm{d}R_{\text{real},t} = \alpha_1 \left(\theta - R_{\text{real},t}\right) \mathrm{d}t + \sigma_1 \mathrm{d}W_{1,t},$$

where α_1, θ and σ are real valued parameters, $W_{1,t}$ denotes a Wiener process at time t and $R_{\text{real},t}$ denotes the real short rate at time t. The parameter θ corresponds to the long term interest rate, α_1 denotes the speed of convergence to this long term real interest rate, where a value closer to 0 corresponds to a slower convergence and a value closer to 1 to a faster convergence, and σ_1 is a measure for the volatility (i.e., a measure for the fluctuation of the real short rate over time) in the model.

The solution to this stochastic differential equation is given by

$$R_{\text{real},t} = R_{\text{real},s} e^{-\alpha_1 (t-s)} + \theta \left(1 - e^{-\alpha_1 (t-s)}\right) + \sigma_1 \int_s^t e^{-\alpha_1 (t-v)} \mathrm{d}W_{1,v}.$$

Conditionally on the filtration at time s, denoted with $\mathcal{F}_s$, the short real interest rate is normally distributed with mean

$$\mathbb{E}[R_{\text{real},t} | \mathcal{F}_s] = r_{\text{real},s} e^{-\alpha_1 (t-s)} + \theta \left(1 - e^{-\alpha_1 (t-s)}\right), \tag{5.1}$$

and variance

$$\mathbb{V}\text{ar}\left(R_{\text{real},t} | \mathcal{F}_s\right) = \frac{\sigma_1^2}{2\alpha_1} \left(1 - e^{-2\alpha_1 (t-s)}\right). \tag{5.2}$$

The conditional mean and variance are of importance for the performance measure that is derived in Section 5.5 and are showed in the visualization of the one factor short Vasicek model.

With the solution for the short real interest rate one can calculate the zero-coupon bond price with maturity T, where T is in years, with the formula

$$b(0,T) = \mathbb{E}[e^{-\int_0^T R_{\text{real},v}\mathrm{d}v}]. \tag{5.3}$$

For the one factor Vasicek model the zero-coupon bond price is given by

$$b(0,T) = e^{a_1(0,T) - r_{\text{real},0}\cdot a_2(0,T)}, \tag{5.4}$$

where,

$$\begin{aligned} a_1(0,T) &= (\theta - \frac{\sigma_1^2}{2\alpha_1^2})(T + a_2(0,T)) - \frac{\sigma_1^2}{4\alpha_1}\left(a_2(0,T)\right)^2, \\ a_2(0,T) &= \frac{1 - e^{-\alpha_1 T}}{\alpha_1}. \end{aligned}$$

Further, if we assume that there are no-arbitrage opportunities (i.e., it is assumed that one cannot make profit without facing a certain risk) the yield curve at maturity T of the real interest rate that corresponds to this one factor Vasicek model is given by

$$y_T = \frac{-\ln(b(0,T))}{T}. \tag{5.5}$$

This expression is in Section 5.2 used to calibrate the corresponding model to the historical data.

The stochastic differential equation that describes the two factor Vasicek model is given by

$$\begin{aligned} \mathrm{d}R_{\text{real},t} &= \alpha_1\left(M_t - R_{\text{real},t}\right)\mathrm{d}t + \sigma_1 \mathrm{d}W_{1,t} \\ \mathrm{d}M_t &= \alpha_2\left(\mu' - M_t\right)\mathrm{d}t + \sigma_2 \mathrm{d}W_{2,t}, \end{aligned} \tag{5.6}$$

with α_1, α_2, σ_1, σ_2 and μ' real valued parameters and $W_{1,t}$ and $W_{2,t}$ denote two correlated Wiener processes at time t, with correlation coefficient ρ. The solution to Equation (5.6) is given by

$$\begin{aligned} R_{\text{real},t} = {} & R_{\text{real},s}e^{-\alpha_1(t-s)} + \alpha_1 M_s \frac{e^{-\alpha_2(t-s)} - e^{-\alpha_1(t-s)}}{\alpha_1 - \alpha_2} \\ & + \mu'\left(1 - e^{-\alpha_1(t-s)} - \alpha_1\frac{e^{-\alpha_2(t-s)} - e^{-\alpha_1(t-s)}}{\alpha_1 - \alpha_2}\right) \\ & + \alpha_1\sigma_2\int_s^t \frac{e^{-\alpha_2(t-v)} - e^{-\alpha_1(t-v)}}{\alpha_1 - \alpha_2}\mathrm{d}W_{2,v} \\ & + \sigma_1\int_s^t e^{-\alpha_1(t-v)}\mathrm{d}W_{1,v}. \end{aligned}$$

Again, the real short rate is, conditionally on the filtration $\mathcal{F}_s$, normally distributed. The mean is given by

$$\begin{aligned} \mathbb{E}[R_{\text{real},t}|\mathcal{F}_s] = {} & r_{\text{real},s}e^{-\alpha_1(t-s)} + \alpha_1 m_s \frac{e^{-\alpha_2(t-s)} - e^{-\alpha_1(t-s)}}{\alpha_1 - \alpha_2} \\ & + \mu'\left(1 - e^{-\alpha_1(t-s)} - \alpha_1\frac{e^{-\alpha_2(t-s)} - e^{-\alpha_1(t-s)}}{\alpha_1 - \alpha_2}\right) \end{aligned}$$

and variance

$$\begin{aligned}
\mathbb{V}\text{ar}\left(R_{\text{real},t}|\mathcal{F}_s\right) = & \frac{\alpha_1\sigma_2^2}{2(\alpha_1-\alpha_2)^2}\left(1-e^{-2\alpha_1(t-s)}\right) \\
& + \frac{\alpha_1^2\sigma_2^2}{(\alpha_1-\alpha_2)^2}\frac{1}{2\alpha_2}\left(1-e^{-2\alpha_2(t-s)}\right) \\
& + \frac{\sigma_1^2}{2\alpha_1}\left(1-e^{-2\alpha_1(t-s)}\right) \\
& + \frac{2\alpha_1\sigma_1\sigma_2\rho}{(\alpha_1-\alpha_2)(\alpha_1+\alpha_2)}\left(1-e^{-(\alpha_1+\alpha_2)(t-s)}\right) \\
& - \frac{\sigma_1\sigma_2\rho}{(\alpha_1-\alpha_2)}\left(1-e^{-2\alpha_1(t-s)}\right) \\
& - \frac{2\alpha_1^2\sigma_2^2}{(\alpha_1-\alpha_2)^2(\alpha_1+\alpha_2)}\left(1-e^{-(\alpha_1+\alpha_2)(t-s)}\right).
\end{aligned} \tag{5.7}$$

The bond price of this two factor Vasicek model is given by

$$\begin{aligned}
b(0,T) = \exp\Bigg\{ & \frac{r_{\text{real},0}}{\alpha_1}\left(e^{-\alpha_1 T}-1\right) + \frac{\alpha_1 m_0}{\alpha_1-\alpha_2}\left(\frac{e^{-\alpha_2 T}-1}{\alpha_2} - \frac{e^{-\alpha_1 T}-1}{\alpha_1}\right) \\
& + \mu'\left(-T - \frac{e^{-\alpha_1 T}-1}{\alpha_1} - \frac{\alpha_1}{\alpha_1-\alpha_2}\left(\frac{e^{-\alpha_2 T}-1}{\alpha_2} - \frac{e^{-\alpha_1 T}-1}{\alpha_1}\right)\right) \\
& + \frac{1}{2}\left(\frac{\sigma_2\alpha_1}{\alpha_1-\alpha_2}\right)^2\left(\frac{1}{\alpha_2^2}\left(T - \frac{1-e^{-\alpha_2 T}}{\alpha_2} - \frac{\left(1-e^{-\alpha_2 T}\right)^2}{2\alpha_2}\right)\right) \\
& + \frac{1}{2}\left(\frac{\sigma_2\alpha_1}{\alpha_1-\alpha_2}\right)^2\left(\frac{1}{\alpha_1^2}\left(T - \frac{1-e^{-\alpha_1 T}}{\alpha_1} - \frac{\left(1-e^{-\alpha_1 T}\right)^2}{2\alpha_1}\right)\right) \\
& + \left(\frac{\sigma_2\alpha_1}{\alpha_1-\alpha_2}\right)^2\left(\frac{1}{\alpha_1\alpha_2}\left(T - \frac{1-e^{-\alpha_1 T}}{\alpha_1} + \frac{1-e^{-(\alpha_1+\alpha_2)T}}{\alpha_1+\alpha_2}\right)\right) \\
& + \frac{1}{2}\left(\frac{\sigma_1}{\alpha_1}\right)^2\left(T - \frac{1-e^{-\alpha_1 T}}{\alpha_1} - \frac{\left(1-e^{-\alpha_1 T}\right)^2}{2\alpha_1}\right)\Bigg\},
\end{aligned} \tag{5.8}$$

and the yield can then be derived with the expression given in (5.3).

Remark 5 *In the one factor and the two factor Vasicek model the underlying process is assumed to be a Brownian motion. For further research one could extend the stochastic differential equations to also consider a jump component, which can be modeled by a Lévy process, see [9] for an extensive study on these stochastic differential equations. In the case of the one factor model this extended stochastic differential equation becomes*

$$dR_{real,t} = \alpha_1\left(\theta - R_{real,t}\right)dt + \sigma_1 dW_{1,t} + R_{real,t}dZ_{1,t},$$

where $dZ_{1,t}$ denotes a Lévy process. It is still possible to derive a solution of this stochastic differential equation and therefore derive the yield.

5.2 Yield curve calibration

There are different methods to calibrate our models to historical data. The two methods that are most commonly used in the literature and practice are a calibration method that is based on the last day and a calibration method that is based on more than one day which also uses time-varying characteristics of the underlying model.

5.2.1 Method 1: Last day optimization

The first method is an optimization problem that depends only on the last day of the historical data. With the expressions given in (5.4) and (5.8) we have derived explicit expressions for the bond price and thus the yield. The expression for the yield curve can be used to estimate the parameters by ordinary least squared error. Note, that in this method the parameters are calibrated to the yield curve of only one day, which is in most of the cases the last day of data available.

Remark 6 *Initially, the yield curve consists of real interest rates with a certain maturity corresponding to the EUSWI and EUSA data from Bloomberg. The DNB published an article in which it is stated that for interpolation of the yield curve pension funds should use the method proposed by the DNB, for more details we refer to [22]. With this method one could interpolate the yield curve and derive the interest rate with an arbitrary maturity.*

Figure 5.1 shows the yield curve of 31-12-2016 together with the one and two factor Vasicek yield curve calibrated according to Method 1.

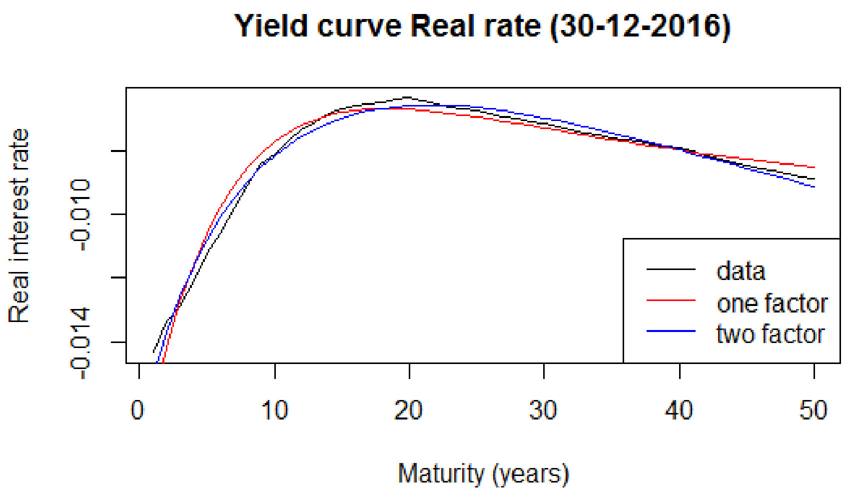
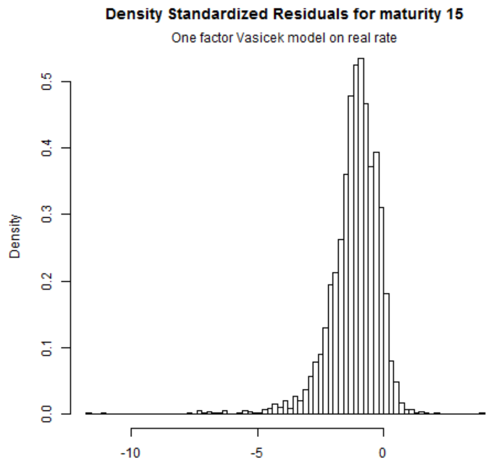


Figure 5.1: Yield curve of the one factor and two factor Vasicek model for the real interest rate calibrated according to Method 1

Figure 5.1 depicts that both the one factor and two factor Vasicek model can be calibrated well to this yield curve. However, this is not the case for all yield curve shapes, see Appendix B.4 for the different yield curve shapes with the calibrated yield curves. Since, the yield curve corresponding to the two factor model has more parameters than the one factor model and the observation that the yield curve of the one factor model is embedded in the two factor model, the two factor model should always describe the yield curve better.

Assumption 8 *Method 1 can calibrate to the yield curve well and thus the errors are white noise, i.e. standard normal distributed.*

In Figure 5.2 the density of the standardized error, that is made when calibrating the one and two factor Vasicek model according to Method 1, is shown for maturity 15. Note that standardizing in this case means dividing the error by the standard deviation, which can be done since it is visually checked that the deviation is constant over time. Moreover, with the KS test the null hypothesis that the errors are from a skewed Student's t-distribution is not rejected. The figure shows that for maturity



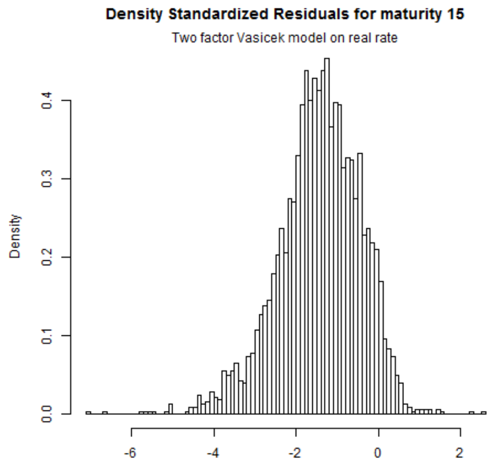


Figure 5.2: Residuals for the one factor (left) and two factor (right) Vasicek model, maturity 15 years and calibrated according to Method 1

15 years the yield is underestimated however, this is not the case for all the maturities. To conclude, Assumption 8 is violated but there is not a straightforward extension of this model.

Remark 7 *The error of the yield curve calibration is skewed Student's t-distributed and therefore the yield calibration can be written as*

$$y_t = f(X_t) + e_t,$$

where X_t is the entire yield curve, y_t the historical observed yield curve and e_t the error at time t. However, it is not clear how this error influences the outcome of our model. The aim of our model is to give a prediction of the real interest rate over a time period of 5 years. For future research one can study the effect of this error term on the outcome of our model.

The sensitivity to the last day can be a disadvantage of this calibration method. If one observes a shock in the yield curve over time but on the next day this shock already canceled out, this calibration method would give very different results.

Remark 8 *An extension to Method 1 could be to optimize the parameters using more than one day of historical data. The underlying assumption that one makes in this extension is that the parameters do not change over the time period that is evaluated. Especially for the parameter $r_{real,0}$, this assumption is restrictive since it is known that the short real interest rate differs over time.*

5.2.2 Method 2: Log-likelihood approach

The second method, which is studied in [26] and [4], uses a log-likelihood approach. In this method the real interest rate for a certain maturity is assumed to be exact. With this assumption the short rate can be derived once we know the parameters of the model. Another key observation is that conditioned on the short rate at time s we know the probability density function of the short rate at time t, namely a normal distribution with mean and variance given for the one factor model in respectively (5.1) and (5.2). If the conditional probability density is denoted by $f_{r_{t+s}}(r_{t+s}|r_s)$, the log-likelihood can be written as

$$\log \mathbb{Q} = \sum_{t=1}^{D} \log \left(f_{r_{t+s}}(r_{t+s}|r_s) \right), \tag{5.9}$$

where D denotes the number of days taken into account. This log-likelihood only looks at the real interest rate on a specific maturity. To account for this fact also the log-likelihood of other maturities is considered.

Assumption 9 *The real interest rates with these other maturities are not exactly observed but with an error, which is identical independent normally distributed with mean zero.*

The log-likelihood of this part can be written as

$$\log \mathbb{P} = -\frac{D}{2}\log(2\pi) - \frac{D}{2}\log(\det\Sigma) - \frac{1}{2}\sum_{t=1}^{D}(\hat{\boldsymbol{y}}_t - \boldsymbol{y}_t)'\Sigma^{-1}(\hat{\boldsymbol{y}}_t - \boldsymbol{y}_t), \tag{5.10}$$

where Σ is the covariance matrix, $\boldsymbol{y}_t$ is the observed yield and $\hat{\boldsymbol{y}}_t$ is the estimated yield.

To estimate the parameters of the model one should maximize the total log-likelihood which is given by

$$\log L = \alpha \log \mathbb{Q} + (1-\alpha)\log \mathbb{P},$$

where the α weighs the different log-likelihoods given by Equations (5.9) and (5.10). In words, the parameter α gives a weight of the importance of the exact yield curve on a certain maturity to the approximate yield curve on other maturities.

Assumption 10 *The parameters, except the short real rate $r_{real,0}$, do not change over the time period of interest.*

Note that this assumption, in contrary to Remark 8, does not assume that the short real rate is constant over time. Now, consider the setting where the real interest rate with maturity 5 is exact and the interest rate with maturities 1, 2, 10, 15 and 40 are approximately correct. Furthermore Table 5.1 shows the evaluation of the total log-likelihood, for $D = 5$ and $D = 10$ consecutive days. As a performance measure the Akaike information criterion (AIC) [3] and the Bayesian information criterion (BIC) [37] are used, respectively given by

$$\text{AIC} = 2k - 2\log(L),$$

and

$$\text{BIC} = \log(D)\,k - 2\log(L),$$

where k denotes the number of parameters. Note that here the aim is to have the lowest AIC of BIC.

Table 5.1: Performance of Method 2 for $D = 5$ (left) and $D = 10$ (right) when considering the one factor Vasicek model on the real interest rate

α	$\log L$	AIC_L	BIC_L
0.1	135.685	6.18	3.05
0.3	99.669	6.8	3.67
0.5	91.257	6.97	3.85
0.7	63.304	7.7	4.58
0.9	45.334	8.37	5.25

α	$\log L$	AIC_L	BIC_L
0.1	296.326	4.62	7.04
0.3	195.973	5.44	7.86
0.5	152.362	5.95	8.37
0.7	136.654	6.17	8.59
0.9	86.679	7.08	9.5

Table 5.1 shows that in both cases, $D = 5$ and $D = 10$, $\alpha = 0.1$ gives the highest log-likelihood. Note that taking α equal to zero is not possible for this method. Furthermore, in this table it can be found that the log-likelihood increases when the parameter D is increased. As a consequence the AIC in the case $D = 10$ is lower than for the case $D = 5$. However, since the BIC penalizes for the value of this

parameter we observe that the BIC is lower for the case $D = 5$. In Appendix C.1 Table C.1 the same execution is done for the two factor Vasicek model. Note that for this model the underlying process of the $m(t)$, in Equation (5.6), is unknown which introduces extra uncertainty. When comparing the one factor and two factor Vasicek model we can conclude that the one factor models gives higher log-likelihood and therefore also a lower AIC and BIC. A drawback to this performance is that only last day data is considered.

For the case $\alpha = 0.1$ both methods are compared according to the root mean squared deviation (RMSD), which takes into account more historical yield curves. The RMSD for maturity y_m is given by

$$\text{RMSD}(y_m) = \sqrt{\frac{\sum_{t=1}^{T}(\hat{y}_{t,m} - y_{t,m})^2}{D}},$$

where $y_{t,m}$ denotes the yield at time t with maturity m and $\hat{y}_{t,m}$ the estimated yield.

In Table 5.2 we compare the one factor with the two factor Vasicek model and the two methods to calibrate the yield curve explained in this section. In the case of Method 2 the weight parameter is $\alpha = 0.1$.

Table 5.2: RMSD comparison for the one and two factor Vasicek model with different calibration methods based on the last 50 observations and $\alpha = 0.1$ for Method 2

	One factor			Two factor		
	Method 1	Method 2		Method 1	Method 2	
Maturity		$D = 5$	$D = 10$		$D = 5$	$D = 10$
RMSD(1)	0.0006	0.0019	0.0019	0.0004	0.0038	0.0054
RMSD(5)	0.0005	0	0	0.0006	0.0011	0.0011
RMSD(10)	0.0002	0.0017	0.0018	0.0006	0.0030	0.0056
RMSD(15)	0.0006	0.0031	0.0033	0.0012	0.0055	0.0115
RMSD(40)	0.0002	0.0055	0.0062	0.0009	0.0195	0.0383

In Table 5.2 one can observe that Method 1 has in the most cases the lowest RMSD for both the one factor and two factor Vasicek model. However, this indicates not directly that this method has a better performance when forecasting the short real interest rate. The robustness to the yield curve can be an advantage and in Section 5.5 further analysis on the performance of the model regarding these methods is studied. When comparing the cases $D = 5$ and $D = 10$ of Method 2, we can conclude that the case $D = 5$ gives a lower RMSD for both the one and two factor Vasicek model. Note that this is consistent with the conclusion made from Table 5.1.

Assumption 11 *Method 2 can calibrate to the yield curve well and thus the errors are white noise, i.e. standard normal distributed.*

Again, the errors made when calibrating this method to the yield curve are analysed. In line with Method 1, the errors are skewed Student's t-distributed and therefore violate Assumption 11.

Another method to fit the yield curve is with polynomials or splines. Splines are mostly used to interpolate curves, but can also be used to fit curves. In Figure 5.3 a historical yield curve is shown together with polynomials of different degrees of freedom (df), fitted to this yield curve. It is known that increasing the degrees of freedom will improve the fitting which is also observed in the figure. Note that the yield curve for which the one factor and two factor had difficulty calibrating is chosen for this analysis.

For the one factor and two factor yield curve the calibration methods are compared based on the RMSD. In Table 5.3 the RMSD of the polynomial fitting are stated for different maturities and degrees of freedom.

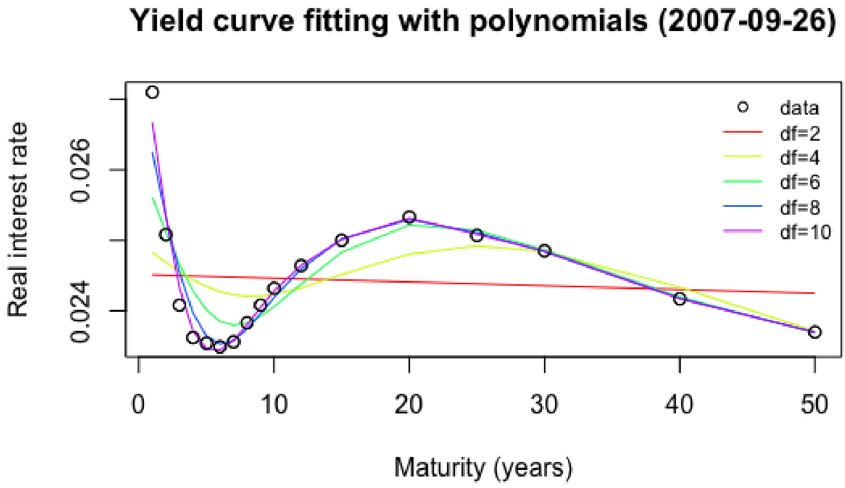


Figure 5.3: Yield curve polynomial fitting for different degrees of freedom

Table 5.3: RMSD comparison for the polynomial fitting for different degrees of freedom based on the last 50 observations

	df = 2	df = 4	df = 6
RMSD(1)	0.0028	0.0006	0.0002
RMSD(5)	0.0008	0.0002	0.0001
RMSD(10)	0.0015	0.0005	0.0001
RMSD(15)	0.0025	0.0005	0.0001
RMSD(40)	0.0010	0.0002	0

When comparing Table 5.3 with Table 5.2 it can be seen that the error made when taking df = 4 is approximately of the same order as the error made when calibrating according to the yield curve of the one factor Vasicek model. However, in Table 5.3 one can clearly see that the error made for df = 6 is smaller than the error made when calibrating according to the two factor model.

Note that with the one factor and two factor Vasicek model already a forecast model for the real short interest rate is assumed, whereas the polynomial fitting only can give the real short interest on a specific time from the historical yield curve. Despite of this difference, the estimation for the real short interest rate can be compared. For the polynomial fitting the short rate is equal to $r_{\text{real},0}$=0.025 for df = 2 and increases almost linearly to $r_{\text{real},0}$=0.028 for df = 10. Comparing these values to the values of the one factor $r_{\text{real},0}$=0.025 and the two factor $r_{\text{real},0}$=0.041 model it can be observed that in the case of the two factor model this real short rate is much higher.

5.3 Representative real yield curve shocks

In Section 1.4 the indirect stress scenarios are already introduced. In this section these different yield curve shock methods are applied to the historical data for the real interest rate, with the exception of the standardized method since this method only shifts the yield curve by two percent.

For the historical method certain quantiles of the empirical distribution of the changes in the real interest rate are applied to the last day yield curve. In this way no error is made when fitting the distribution.

Figure 5.4 shows different quantile increases of the returns applied to the yield curve of 31-12-2016. One major drawback of this yield curve shock method is indicated by the yield curve shock on the

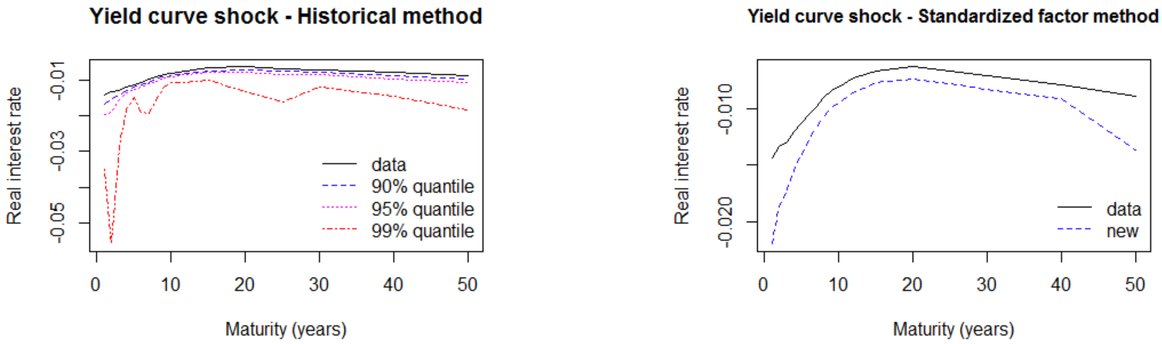


Figure 5.4: Yield curve shock according the historical method (left) and the standardized factor method (right) on the real interest rate yield curve

real interest rate. In the quantile (99%) the yield curve has fluctuations which are not observed in practice. This figure also shows the standardized factor method. One remarkable observation is that in [18] an increase or decrease in percentage is stated. However, when the yield curve is negative the decrease in percentage becomes an increase in yield and vice versa. When the whole yield lies below or above zero, this is not a problem since one can still take the increase or decrease that leads to a decrease in the yield, see Figure 5.4. When comparing both methods the conclusion can be made that the standardized factor method considers a significant change and keeps the continuity property.

Another possible yield curve shock method is the principal component analysis (PCA). Remember that this method is based on the assumption that the yield curve can be described by three principal components, which correspond to the level, slope and curvature.

In Figure 5.5 the loadings of these three components are shown for each maturity. For the real interest rate the first three principal components explain respectively 60, 20 and 8 percent of the variation in the yield, which means that these three components together explain 88 percent of the variation in the yield. With the factor loadings of each maturity and the values of the principal components one can describe the yield curve on a specific time point. Note that the factor loading is constant over time while the values of the principal components change. A certain quantile of the changes in the value of the principal components is applied to the principal component values of 31-12-2016.

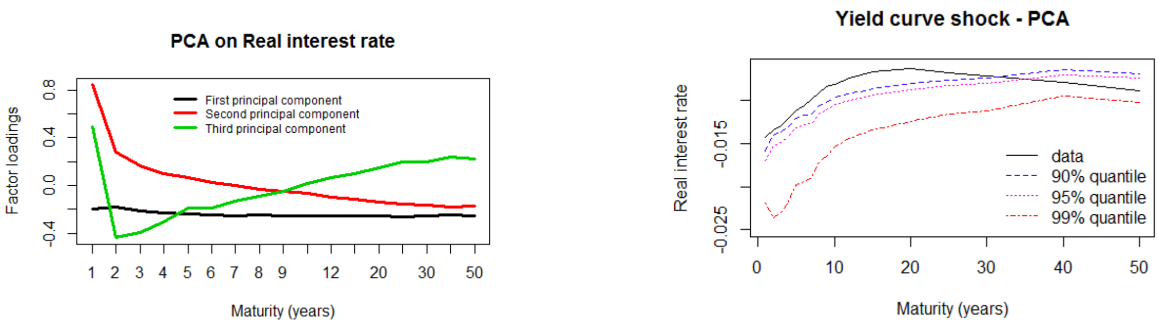


Figure 5.5: The factor loadings (left) and the yield curve shock (right) on the real yield curve corresponding to the PCA method

Note that for all these yield curve shocks methods only calibration method 1 described in Section 5.2 can be used to calibrate the model to these yield curves. This is due to the fact that all these shocks methods give one new yield curve as output.

5.4 Visualization Vasicek model

After getting the parameters via one of the two calibration methods described in Section 5.2, applied on the real yield curve or the yield curve shocks, the solution to the stochastic differential equation via the Euler discretization method can be simulated. For this simulation the step size and number of simulated paths are of huge importance and are therefore studied at the end of this section.

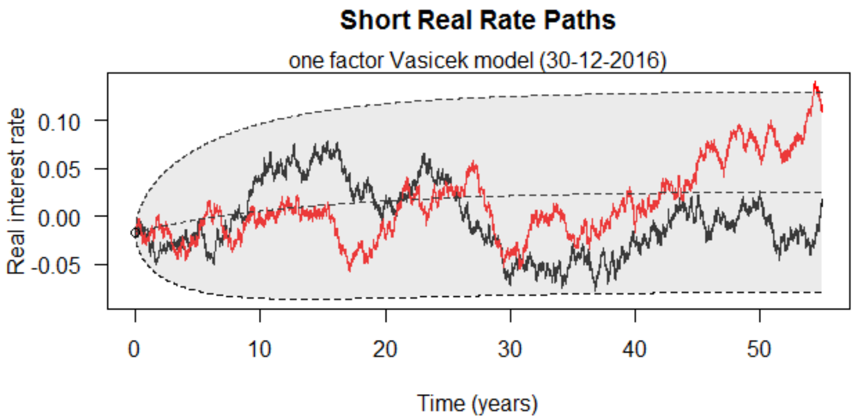


Figure 5.6: Two sample paths and the 95 percent prediction intervals of the one factor Vasicek model with parameters calibrated to the yield curve with method 1

Figure 5.6 shows two sample paths for the real interest rate for a period of 55 years together with the expected value and the 95 percent prediction bounds. These prediction intervals do not imply that a certain path happens with 95 percent but that the real interest rate on a certain time point in the future is predicted to fall between these bounds 95 percent of the times. Observe, that the Vasicek model allows the real rate to become negative.

With n real short rate paths we can derive an estimation for the real interest rate with a specific maturity y_m

$$\hat{r}^{y_m}_{\text{real}} = \frac{-\log\left(\frac{1}{n}\sum_{i=1}^{n}\exp\left(-\sum_{j=0}^{s_u}\tilde{r}_{\text{real},j,i}\text{d}t\right)\right)}{y_m},$$

where $s_u = \frac{y_m}{\text{d}t}$, $\hat{r}$ denotes the estimated value for the real interest rate with maturity y_m, $\tilde{r}_{\text{real},t,i}$ the short rate at time t for path i and $\text{d}t$ the step interval chosen in the simulation.

Furthermore, the real interest rate with a specific maturity from a specific path i is given by

$$\tilde{r}^{y_m}_{\text{real},y_t,i} = \frac{-\log\left(\exp\left(-\sum_{j=s_l}^{s_u}\tilde{r}_{\text{real},j,i}\text{d}t\right)\right)}{y_m}, \tag{5.11}$$

where $s_l = \frac{y_t}{\text{d}t}$ and $s_u = \frac{y_t+y_m}{\text{d}t}$.

In the case of using the Euler discretization method it follows that the predicted real interest rate with maturity y_m in y_t years from now is normally distributed with mean

$$\begin{aligned}\bar{r}^{y_m}_{\text{real},y_t} &= \frac{1}{n}\sum_{i=1}^{n} \tilde{r}^{y_m}_{\text{real},y_t,i} \\ &= -\frac{\sum_{j=s_l}^{s_u}\left(\sum_{i=s_l}^{j-1}\binom{j-2}{i-1}(-\alpha_1 \mathrm{d}t)^i\,(\theta - r_{\text{real},0})\right)\mathrm{d}t}{y_m} \\ &\quad -\frac{\sum_{j=s_l}^{s_u}\left(\sum_{i=s_l}^{j-2}\binom{j-2}{i}(-\alpha_1 \mathrm{d}t)^i\theta\right)\mathrm{d}t}{y_m},\end{aligned} \tag{5.12}$$

and variance

$$\begin{aligned}s^{2,y_m}_{\text{real},y_t} &= \frac{1}{n}\sum_{i=1}^{n}\left(\tilde{r}^{y_m}_{\text{real},y_t,i} - \bar{r}^{y_m}_{\text{real},y_t}\right)^2 \\ &= \frac{\left(\sum_{j=s_l}^{s_u}\left(\sum_{i=s_l}^{j-2}\binom{j-1}{i}(-\alpha_1 \mathrm{d}t)^{i-1}\beta\sqrt{\mathrm{d}t}\right)\mathrm{d}t\right)^2}{y_m^2} \\ &\quad + \frac{\left(\sum_{j=s_l}^{s_u}\left((-\alpha_1 \mathrm{d}t)^{j-3}\beta\sqrt{\mathrm{d}t}\mathrm{d}t\right)\mathrm{d}t\right)^2}{y_m^2}.\end{aligned}$$

For the two factor Vasicek model one can still derive an explicit expression of the bond price and thus the yield curve. Again the same two methods can be used for the calibration of the yield curve. In Figure 5.7 the visualization of the two factor model is given when calibrated to the last day yield curve, thus Method 1.

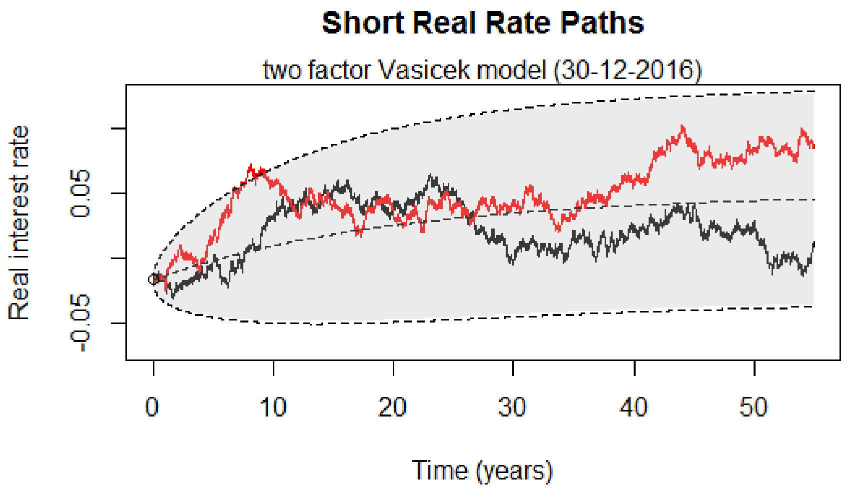


Figure 5.7: Two Sample paths and the 95 percent prediction intervals of the two factor Vasicek model with parameters calibrated to the yield curve with method 1

Note that although the paths in Figure 5.7 show a similar behavior as in Figure 5.6, there is a significant difference between the two figures. For the two factor model the volatility of the short rate is smaller than the variance, see Equations (5.2) and (5.7), of the short rate in the one factor model. This shows that although the yield curves were quite similar, see Figure 5.1, the outcome of the two models are different.

In Appendix C.3 the visualization of the one factor and two factor Vasicek model when calibrating to Method 2 are given. From both Figures C.1 and C.2 it can be seen that especially in the beginning of the forecast the behavior differs. The prediction interval is larger in the case of the one factor model than in the case of the two factor model.

5.4.1 Yield curve via simulation

In this section the influence of the step size (dt) and number of simulations ($nsim$) when simulating the real short rate using the Euler discretization is studied. Since it is known that the yield curve of the simulations should converge to the analytic yield curve, the number of simulations gives a measure for the speed of convergence. The convergence is defined as the number of simulations for which the simulated yield curve, with a specific maturity, is for 10 consecutive observations within a bandwidth of 2.5 percent of the analytic yield. The minimal threshold for this number of simulations is 4000. Note that in (5.12) the number of intermediate steps within one year is of importance, since the corresponding sum approximates the integral from (5.3).

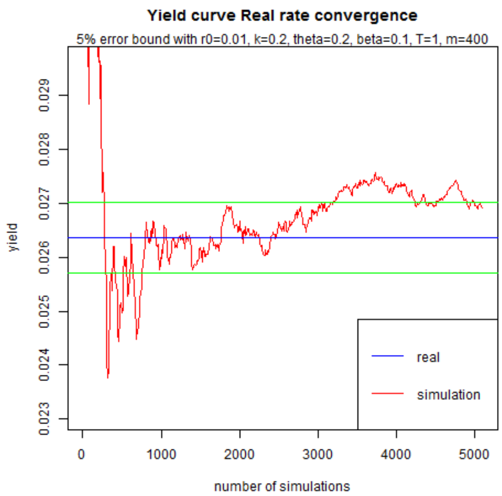


Figure 5.8: Yield curve convergence for the base case

Figure 5.8 shows the simulated yield and the real yield together with the bandwidth. Table 5.4 gives the number of simulations per setting. Note, that the minimal number of simulations is 4000, but in some cases it is clearly visible that the yield curve did already converge before this number, which is indicated with a star.

Table 5.4: Number of simulations per setting for $T = 1$ and $dt = 0.025$

Parameter	value	$nsim$	value	$nsim$	value	$nsim$	value	$nsim$
$r_{\text{real},0}$	0	12250	0.01	5000	0.02	3000*		
α_1	0.1	4500	0.2	5000	0.4	4250	0.8	4000
θ	0.1	5000	0.2	5000	0.4	5000	0.8	2000*
σ_1	0.05	2000*	0.1	5000	0.2	15000		

From Table 5.4 and Tables C.2 and C.3 in Appendix C.2 we conclude that there are certain rules of thumb for the number of simulations. Increasing the parameter β leads to a significant increase in the number of simulations. This also holds for the parameter k with the exception of the lowest value of k. Here an $1/x$ relation to the number of simulations seems appropriate. Also the influence of the time is observable when comparing the three tables. When the maturity is increased also the number of simulations until convergence increases.

5.5 Performance Vasicek model

In the previous sections we indirectly assumed that the Vasicek short rate model, models the reality well. In this section the performance of both the one factor and two factor model under both calibration methods is studied. The quantification of the performance is based on the prediction of $r^{y_m}_{\text{real},y_t}$. Furthermore, let $q^{y_m}_{\text{real},y_t}(\alpha)$ denote the quantile (α%) of the prediction of the real interest rate when deriving the real interest rate per short rate path, i.e. $q^{y_m}_{\text{real},y_t}(\alpha) = \bar{r}^{y_m}_{\text{real},y_t} \pm z_\alpha s^{2,y_m}_{\text{real},y_t}$ with z_α the quantile (α%) of the standard normal distribution.

Table 5.5 shows the performance of the Vasicek model when forecasting the real interest rate with maturity 1. This maturity fluctuates the most and is therefore the most difficult interest rate to predict. Bound$_{50}$ represents the number of observations that lie between $q_{0.25}$ and $q_{0.75}$. With the same reasoning Bound$_{95}$ denotes the number of observations that lie between $q_{0.025}$ and $q_{0.975}$.

Table 5.5: Performance one factor Vasicek model

		Method 1		Method 2 ($T = 5$)	
	Observations	Bound$_{95}$	Bound$_{50}$	Bound$_{95}$	Bound$_{50}$
$r^1_{\text{real},1}$	2389	2218	2014	2385	2366
$r^1_{\text{real},2}$	1632	1447	1262	1630	1608
$r^1_{\text{real},3}$	1476	1343	1056	1472	1433
$r^1_{\text{real},4}$	1323	1193	880	1317	1252
$r^1_{\text{real},5}$	1171	1036	739	1166	1113

In Table 5.5 one can observe that Method 2 performs better than Method 1. However, this is also due to the higher volatility in the short rate when calibrating according to Method 2. In Appendix C.3 both the one factor and two factor Vasicek model is visualized under calibration Method 2, see respectively Figures C.1 and C.2. When calibrating to Method 2 it can be observed that the real short rate has a larger prediction interval. As a consequence also the prediction interval for the forecast of the real interest rate is larger. Since our performance measure does not penalize for this deviation it is clear that this measure suggests that calibration Method 2 performs the best.

Table 5.6: Performance two factor Vasicek model

		Method 1		Method 2 ($T = 5$)	
	Observations	Bound$_{95}$	Bound$_{50}$	Bound$_{95}$	Bound$_{50}$
$r^1_{\text{real},1}$	2389	1894	1357	2222	1480
$r^1_{\text{real},2}$	1632	1222	736	1419	822
$r^1_{\text{real},3}$	1476	1013	523	1286	678
$r^1_{\text{real},4}$	1323	843	356	1171	717
$r^1_{\text{real},5}$	1171	691	178	1040	679

When comparing the performance of the one factor Vasicek model, Table 5.5, to the performance of the two factor Vasicek model, Table 5.6, we observe remarkable differences. The two factor Vasicek model performs worse than the one factor Vasicek model according to the performance measure we used. One explanation of this remarkable result can be the fact when calibrating according the Method 1, the statistical program R that is used experiences difficulty in minimizing the least squared errors. Based on historical data, the value of the least squared error of the one factor Vasicek model is in almost 30 percent of the cases lower than the value of the two factor Vasicek. Another possibility is that the one factor Vasicek model gives a better prediction. This result has been oftentimes observed in the literature in the context of revenue management as a result of overfitting the data before performing optimization, see [19] and the references therein. Furthermore, in [19] the authors note that simple models can produce more accurate results than complected models. This might exactly explain why the simpler (one factor model) can be more accurate than the rather more complicated model (two factor model).

5.5.1 Sensitivity analysis Vasicek model

After the performance of the Vasicek model also the sensitivity of the one factor model is studied. The sensitivity analysis measures the influence of the parameters to the outcome of the model. Here, the parameters are decreased or increased by 5 percent and then again the performance is studied.

Table 5.7: Sensitivity performance one factor Vasicek model for parameter $r_{\text{real},0}$ when calibrating according to Method 1

Index	Number	-5%		5%	
		Bound_{95}	Bound_{50}	Bound_{95}	Bound_{50}
$r^1_{\text{real},1}$	2391	2217	2028	2216	2013
$r^1_{\text{real},2}$	1634	1447	1263	1450	1261
$r^1_{\text{real},3}$	1476	1338	1058	1342	1064
$r^1_{\text{real},4}$	1323	1191	879	1200	882
$r^1_{\text{real},5}$	1171	1031	728	1039	739

If we compare Table 5.7 to Table 5.5, the conclusion can be made that changing the estimate of the parameter $r_{\text{real},0}$ does not significantly change the outcome of the performance measure. However this can be due to the characteristic of the model. If the mean reversion parameter k is close to 1, the initial short rate $r_{\text{real},0}$ is not important since the short rate will converge to the long term interest rate θ. In Appendix C.4 the sensitivity analysis is given for the other parameters α_1, θ and σ_1. When comparing Tables C.4, C.5 and C.6 it can be seen that the parameter σ_1 influences the performance the most. This is to be expected since this parameter influences the width of the prediction interval.

5.6 Conclusions

In this chapter the one and two factor Vasicek short rate model are studied together with two different yield curve calibration methods. All these models and calibration methods are compared via historical data. From this analysis, the conclusion can be made that the yield curve calibration that takes into account more days gives a better performance. Also from this performance measure, it is observed that the one factor predicts the future real interest rate correctly with a higher probability. Noteworthy is that a better performance does not immediately suggest that one should use the corresponding model or calibration method. A disadvantage of the one factor model when calibrating according to Method 2 is for example that the historical real interest rate, even in the case of the prediction interval with a width of 50 percent, lies in almost all the cases within this interval. This is due to the large deviation of the predicted real interest rate.

Another drawback of the Vasicek short rate model is that the direct outcome, the real short rate, cannot be compared to historical data since this term is a fictive term from which there is no data. In our performance measure the forecast of the real interest rate, which follows from the real short rate, is compared to historical data. However, one could also come up with other performance measures, which include the deviation of this real interest rate prediction.

Since the forecast of the inflation is described by both the real interest rate and the nominal interest rate, in the next chapter the forecast of the nominal interest rate is studied.

Chapter 6

Forecasting the nominal interest rate

Again, in line with Chapter 5 a short rate model is used to forecast the nominal interest rates. Before 2014 the belief was that the nominal interest rate could not be negative. Under this assumption the Black Karasinski, sometimes referred to as the exponential Vasicek model, is often used since this model cannot output negative short rates and thus negative interest rates, see e.g. [12] and [32]. However, in the last two years this assumption of non-negative nominal interest rates is violated for the short term, i.e. maturities under 5 year, interest rate. Therefore, it could be the case that the Vasicek model predicts the nominal interest rate better. In this chapter the Black Karasinski model is introduced and compared to the Vasicek model based on historical performance.

6.1 Black Karasinski model

Within the class of the Black Karasinski model one could decide between the one factor and the two factor model. Both models are described and the difference is studied. The one factor Black Karasinski model is given by

$$\mathrm{d}\ln(R_{\text{nom},t} = \alpha_3\left(\ln(\theta) - \ln(R_{\text{nom},t})\right)\mathrm{d}t + \sigma_3 \mathrm{d}W_{3,t},$$

where α_3, θ and σ_3 are real valued parameters, $W_{3,t}$ denotes a Wiener process at time t and $R_{\text{nom},t}$ denotes the nominal short rate at time t. In this model,in contrast to the one factor Vasicek short rate model, the parameters do not have an intuitive interpretation. The solution to this stochastic differential equation is given by

$$\begin{aligned} R_{\text{nom},t} = \exp\Big\{&\ln(R_{\text{nom},s})e^{-\alpha_3(t-s)} + \theta\left(1 - e^{-\alpha_3(t-s)}\right) \\ &+\sigma_3 \int_s^t e^{-\alpha_3(t-v)}\mathrm{d}W_{3,v}\Big\}. \end{aligned} \tag{6.1}$$

In the solution (6.1), the characteristic of non-negative nominal short rates can be observed since there is the exponential function. Conditionally on the filtration $\mathcal{F}_s$ the nominal short rate is lognormal distributed with mean

$$\begin{aligned} \mathbb{E}[R_{\text{nom},t}|\mathcal{F}_s] = \exp\Big\{&\ln(r_{\text{nom},s})e^{-\alpha_3(t-s)} + \theta\left(1 - e^{-\alpha_3(t-s)}\right) \\ &+\frac{\sigma_3^2}{4\alpha_3}\left(1 - e^{-2\alpha_3(t-s)}\right)\Big\} \end{aligned}$$

and variance

$$\begin{aligned}\mathbb{V}\mathrm{ar}\left(R_{\mathrm{nom},t}|\mathcal{F}_s\right) = \exp\Big\{&2\ln(r_{\mathrm{nom},s})e^{-\alpha_3(t-s)} + 2\theta\left(1-e^{-\alpha_3(t-s)}\right)\\ &+\frac{\sigma_3^2}{2\alpha_3}\left(1-e^{-2\alpha_3(t-s)}\right)\Big\}\\ &\cdot\left(\exp\left\{\frac{\sigma_3^2}{2\alpha_3}\left(1-e^{-2\alpha_3(t-s)}\right)\right\}-1\right).\end{aligned}$$

For the Black Karasinski model the relation in Equation (5.3) still holds. In contrast to the Vasicek model where an explicit expression for the bond price could be obtained, in the Black Karasinski model there is no explicit expression for the bond price. This would mean that the Black Karasinski model cannot be calibrated to the yield curve. As an alternative, the bond price is computed via an approximation given in [17]. Here the approximation is stated, but for further proofs the interested reader is referred to the paper.

$$b(0,T) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left\{-\int_0^T \bar{r}_{T,t}g_{0,T}(t)h_{0,T}(z,t)\mathrm{d}t\right\}e^{-\frac{1}{2}z^2}\mathrm{d}z, \tag{6.2}$$

with

$$\bar{r}_{T,t} = r_{\mathrm{nom},0}^{e^{-\alpha_3 t}}\exp\left\{\int_0^t e^{-\alpha_3(t-s)}\alpha_3\log\left(\theta\right)\mathrm{d}t\right\},$$

$$g_{0,T}(t) = \exp\left\{\frac{\sigma_3^2}{2}\left(\frac{1}{2\alpha_3}\left(1-e^{-2\alpha_3 t}\right)-\lambda_0(T)f_{0,T}(t)^2\right)\right\}$$

and

$$h_{0,T}(z,t) = \exp\left\{\sigma_3\sqrt{\lambda_0(T)}f_{0,T}(t)z\right\},$$

$$f_{0,T}(t) = \sqrt{\frac{2}{T+\alpha_3\lambda_0(T)}}\sin\left(\omega(T)t\right) \quad \text{and} \quad \lambda_0(T) = \frac{1}{\alpha_3^2+\omega(T)^2},$$

where $\omega(T)$ is the solution of the equation given by

$$\omega\cdot\mathrm{cotangent}\left(\omega\cdot T\right) = -\alpha_3.$$

Note that again once the bond price is derived, the yield can be calculate via Equation (5.5). The two factor Black Karasinski model is given by

$$\begin{aligned}\mathrm{d}\ln(R_{\mathrm{nom},t}) &= \alpha_3\left(\ln(m_t)-\ln(R_{\mathrm{nom},t})\right)\mathrm{d}t + \sigma_3\mathrm{d}W_{3,t}\\ \mathrm{d}\ln(m_t) &= \alpha_4\left(\mu'-\ln(m_t)\right)\mathrm{d}t + \sigma_4\mathrm{d}W_{4,t},\end{aligned}$$

where α_3, α_4, σ_3, σ_4 and μ' are real valued parameters and $W_{3,t}$ and $W_{4,t}$ denote two correlated Wiener processes at time t. After calculation the solution to this stochastic differential equation is obtained

$$\begin{aligned}R_{\mathrm{nom},t} = \exp\Bigg\{&\ln(R_{\mathrm{nom},s})e^{-\alpha_3(t-s)} + \alpha_3\ln(M_s)\frac{e^{-\alpha_4(t-s)}-e^{-\alpha_3(t-s)}}{\alpha_3-\alpha_4}\\ &+\mu'\left(1-e^{-\alpha_3(t-s)}-\alpha_4\frac{e^{-\alpha_4(t-s)}-e^{-\alpha_3(t-s)}}{\alpha_3-\alpha_4}\right)\\ &+\alpha_3\sigma_4\int_s^t\frac{e^{-\alpha_4(t-v)}-e^{-\alpha_3(t-v)}}{\alpha_3-\alpha_4}\mathrm{d}W_{4,v}\\ &+\sigma_3\int_s^t e^{-\alpha_3(t-v)}\mathrm{d}W_{3,v}\Bigg\},\end{aligned}$$

with expectation

$$\begin{aligned}\mathbb{E}\left[R_{\text{nom},t}\middle|\,\mathcal{F}_s\right] = \exp\Bigg\{ & \ln(r_{\text{nom},s})e^{-\alpha_3(t-s)} + \alpha_3\ln(m_s)\frac{e^{-\alpha_4(t-s)} - e^{-\alpha_3(t-s)}}{\alpha_3-\alpha_4} \\ & + \mu'\left(1 - e^{-\alpha_3(t-s)} - \alpha_3\frac{e^{-\alpha_4(t-s)} - e^{-\alpha_3(t-s)}}{\alpha_3-\alpha_4}\right) \\ & + \left(\frac{\alpha_3\sigma_4}{\alpha_3-\alpha_4}\right)^2\left(\frac{1}{4\alpha_4}\left(1-e^{-2\alpha_4(t-s)}\right) + \frac{1}{4\alpha_3}\left(1-e^{-2\alpha_3(t-s)}\right)\right) \\ & + \frac{\sigma_3^2}{4\alpha_3}\left(1-e^{-2\alpha_3(t-s)}\right)\Bigg\}.\end{aligned}$$

For the bond price of the two factor model an approximation by the same reasoning as for the one factor Black Karasinski model is derived. However, when analyzing the error made by this approximation we observed that this expression is numerically unstable. As a consequence, the performance of the two factor method cannot be studied and is therefore not used to forecast the nominal interest rate. For further research one could improve the approximation that is given and compare the performance of the one factor and two factor model.

Note that for the yield calibration the same methods as in Section 5.2 can be used. However, using the approximation for the nominal yield curve has as a consequence that the second calibration method, i.e. the log-likelihood approach, is time consuming. Deriving the parameters for the Black Karasinski model according to this method takes more than 5 hours which means that even when simulating in parallel the simulati for the performance exceeds the time for this thesis. Since the performance of this calibration method is not performed it is left as an open research problem.

Figure 6.1 shows the yield curve according to the approximation and the simulated yield curve, i.e. (5.12) applied to the nominal interest rate. It can be seen that the approximation given by (6.2) is representing the actual yield curve well for the case of $r_{\text{nom},0} = 0.014$, $\alpha_1 = 0.3$, $\theta = 0.03$ and $\sigma_1 = 0.3$, which are the parameters when calibrating the yield curve to 31-12-2016. However, it was already mentioned that this accurate approximation comes with the cost of time.

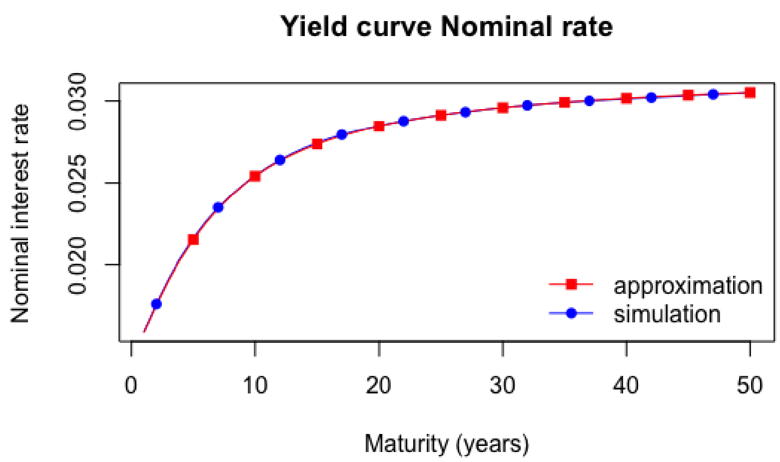


Figure 6.1: Approximation and the simulation value of the nominal yield curve according to the Black Karasinski model, where for the simulation 5000 paths are simulated with 250 days in one year

6.2 Representative nominal yield curve shocks

In Section 5.3 the yield curve shocks are applied to the real yield curve. In this section the same methods are applied to the nominal yield curve.

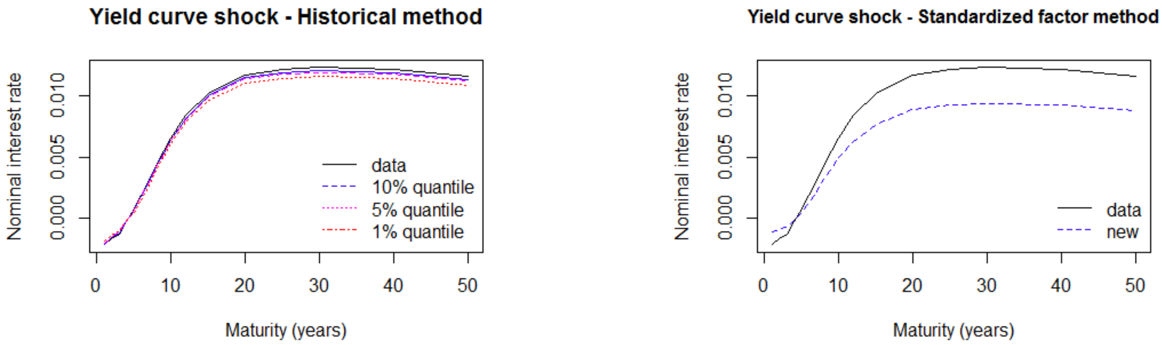


Figure 6.2: Yield curve shock according the historical method (left) and the standardize factor method (right) on the nominal yield curve

Figure 6.2 shows the historical yield curve shock method, which applies a certain quantile of the historical changes to the nominal yield curve of 31-12-2016. In this figure it can be seen that the change in the yield curve according to the historical method is almost negligible for all quantiles.

For the standardized factor method a remarkable shock can be observed. In Section 5.3 it was already mentioned that in this shock method whenever the yield curve is negative the decrease becomes an increase in the yield curve and vice versa. However, for the nominal interest rate it can be observed that the short term is negative and the long term positive. With this yield curve shock method one does not get a new yield curve that lies completely above or below the yield curve of the data.

The loadings of the three components (level, slope and curvature) are shown for each maturity are shown in Figure 6.3. For the nominal interest rate the first three principal components explain respectively 85, 10 and 2 percent of the variation in the nominal yield curve, which means that the three components together explain 97 percent of the variation. When comparing this to the PCA of the real interest rate, the conclusion can be made that the first three components capture the behavior of the nominal yield curve better.

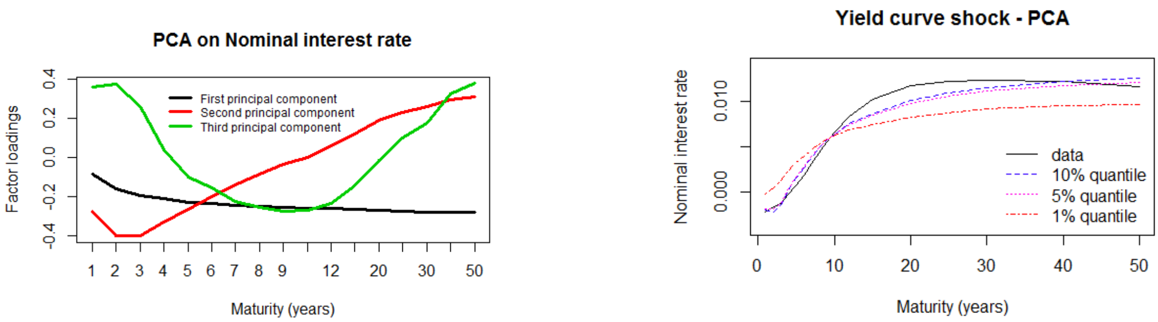


Figure 6.3: The factor loadings (left) and the yield curve shock (right) on the nominal yield curve corresponding to the PCA method

With the factor loadings of each maturity and the values of the principal components one can describe the yield curve on a specific time point. Note, that the factor loading is constant over time while the values of the principal components change. A certain quantile of the changes in the value of the principal components is applied to the principal component values of 31-12-2016, see Figure 6.3

that depicts different quantiles changes.

Remark 9 *The yield curve according to the quantile (1%) of the PCA, seems similar to the yield curve of the standardized factor method. Although not stated in [18], it could be the case that this method uses underlying properties of the PCA method to determine the standardized factors.*

6.3 Visualization Black Karasinski model

With the approximation of the yield curve we can again calibrate the yield curve of this model to the yield curve of 31-12-2016 or the yield curves generated by the yield curve shocks. Note, that in practice it can be seen that nominal interest rate with maturities lower than five years are negative. However, the Black Karasinski model cannot model negative interest rates. To tackle this problem a shift to the yield curve is applied. For the visualization in Figure 6.4 a constant shift of 0.02 is used. This means that the indirect assumption is that the nominal interest cannot go below 2 percent.

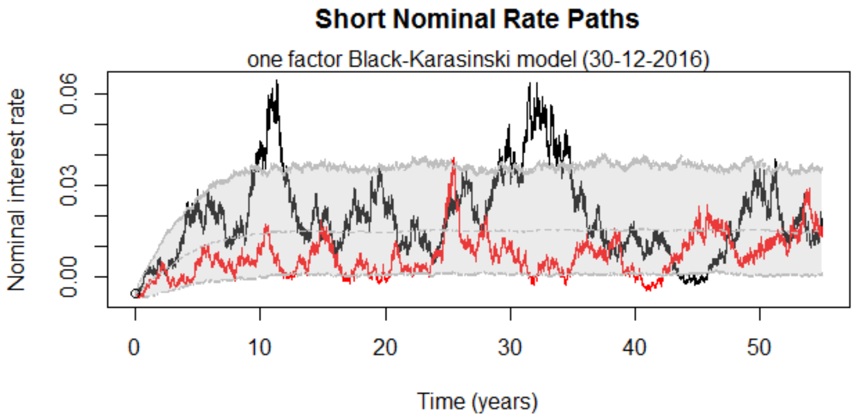


Figure 6.4: Sample paths and the quantiles (5%, 95%) of the one factor Black Karasinski model with parameters calibrated to the yield curve

Figure 6.4 shows samples paths according to the one factor Black Karasinski model calibrated according to method 1. An already mentioned drawback of this model is that it cannot have negative nominal interest rates, but from the figure and the simulation it can be observed that the 5 percent and even the quantile (1%) is not close to -0.02.

Remark 10 *Although it is hard to see directly from the approximation, for the yield curve in the one factor Black Karasinski model, it can be derived from simulation that changing the value of the parameter* $r_{nom}(0)$ *does not only change the starting point of the yield curve. For the solution of the one factor Black Karasinski model, see* (6.1)*, changing the parameter* $r_{nom}(0)$ *only influences the level of the paths. Note that this is not consistent which means that shifting the yield curve can have a negative effect on the performance of this model. A major drawback is that this error that is possibly made, cannot be explicitly derived. Both the error and the negative short term nominal interest rates could be an argument to change to the Vasicek model.*

Figure 6.5 shows two paths of the short nominal interest rate when using the two factor Vasicek with calibration Method 1. When comparing Figures 6.4 and 6.5, it can be seen that the two factor Vasicek model generates nominal short rates which are completely below zero, whereas in the one factor Black Karasinski model the long term forecast is for most of the paths above zero.

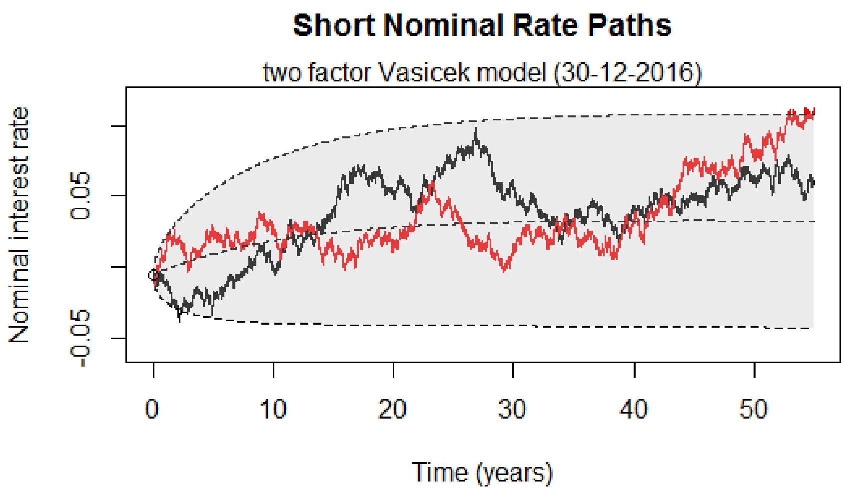


Figure 6.5: Sample paths and the 5 and 95 percent prediction intervals of the two factor Vasicek model with parameters calibrated to the nominal yield curve with method 1

6.4 Performance Black Karasinski model

Since it is not entirely clear which model we should consider to forecast the nominal interest rate, the performance of both the Black Karasinski and the Vasicek model are compared. This can also give a first indication whether shifting the nominal yield curve, in the case of the Black Karasinski model, has a negative effect on the performance. For this performance test the same procedure as in Section 5.5 is used.

Table 6.1: Performance one factor Vasicek model on nominal interest rate

		Method 1		Method 2 ($D = 5$)	
	Observations	Bound$_{95}$	Bound$_{50}$	Bound$_{95}$	Bound$_{50}$
$r^1_{\text{nom},1}$	2389	2203	2044	2386	2288
$r^1_{\text{nom},2}$	1632	1426	1217	1629	1535
$r^1_{\text{nom},3}$	1476	1426	931	1472	1337
$r^1_{\text{nom},4}$	1323	1181	564	1315	1063
$r^1_{\text{nom},5}$	1171	993	144	1164	790

In Table 6.1 it can be seen that again, for the one factor Vasicek model, the calibration method 2 can predict the nominal interest rate in most of the cases well. When comparing the forecast of the real interest rate, i.e. Table 5.5, with the nominal interest rate, i.e. Table 6.1, it can be observed that in general the one factor Vasicek model can predict the nominal interest rate better. There is not an immediate explanation for this, but it can be due to the fact that the real interest rate has slightly more fluctuation.

In Table 6.2 the performance of the one factor Black Karasinski model on the nominal interest rate is given. It can be seen that the performance is poor in comparison to the performance of the Vasicek model. To get a better understanding of this poor performance the prediction of the real interest with maturity 1 in 4 years is studied in more detail. Figure 6.6 shows the forecast and the observed value of this nominal interest rate. From this figure the conclusion can be made that the forecast based on the Black Karasinski model is systematically overestimating the actual nominal interest rate according to the historical data. Here the assumption of non-negative nominal interest rate can play a major role

Table 6.2: Performance one factor Black Karasinski model on nominal interest rate

		Method 1	
	Observations	Bound$_{95}$	Bound$_{50}$
$r^1_{\text{nom.1}}$	2388	1070	589
$r^1_{\text{nom,2}}$	1631	494	207
$r^1_{\text{nom,3}}$	1473	303	162
$r^1_{\text{nom,4}}$	1320	115	74
$r^1_{\text{nom,5}}$	1168	93	0

in this overestimating, since with this assumption the quantile (5%) lies higher. Another argument is that within the yield curve, which implicitly gives the expected future behavior of the nominal interest rate, this assumption was also embedded what then carried through to our models.

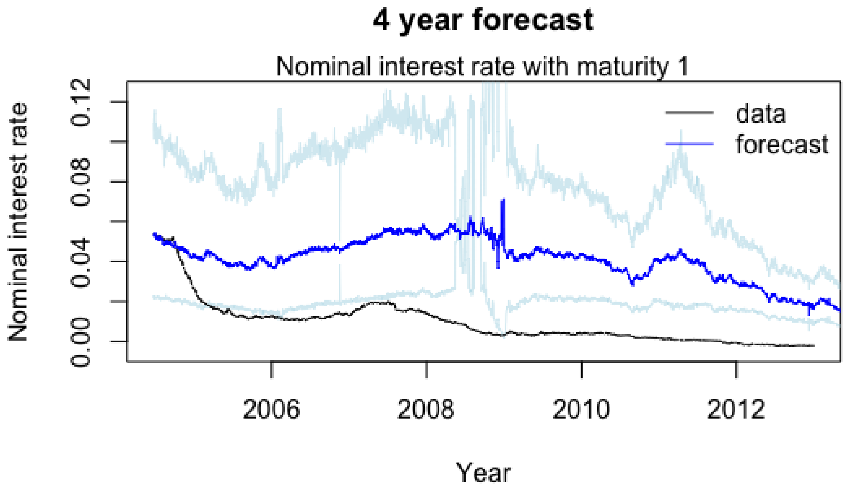


Figure 6.6: 4 Year forecast together with the 95 percent prediction interval of the nominal interest rate with maturity 1 based on the Black Karasinski model with calibration Method 1

Remark 11 *In Figure 6.6 the crisis of 2008 can be observed. It can be seen that both the expected forecast and the prediction intervals of this forecast fluctuate more in this time period.*

6.5 Conclusions

In this chapter the Black Karasinski model is introduced to forecast the nominal interest rate. Although it belongs to the same short rate model class as the Vasicek model, the analytical expressions for this model are more difficult and sometimes it is not even possible to derive such explicit expression, e.g. an expression for the yield curve. The approximation for the yield curve that is used approximates the yield curve almost perfectly, but when calibrating this model to the yield curve according to Method 2 this method is so time consuming that it was not possible to perform the performance on historical data.

From the comparison between the Vasicek and the Black Karasinski model the conclusion can be made that the Vasicek models can predict the nominal interest rate more accurately in most of the cases. Again, the prediction according the calibration Method 2 gives more deviation in the yield and as a consequence this model almost predicts the nominal interest rate perfectly, which can be a disadvantage of this calibration method. One aspect that is studied in the next chapter is the impact on the funding ratio of a general pension fund portfolio when using the Vasicek or the Black Karasinski model to forecast the nominal interest rate.

Chapter 7

Stress scenarios in practice

In the previous chapters the dependency and the forecast of each risk factor are studied. All together this forms a framework from which stress scenarios can be derived. The plausibility of the stress scenarios are measured according to the quantiles. In the problem description, cf. Section 1.3, already a brief outline of the different scenarios is given. In this chapter the scenarios are further described in more detail and the impact of the stress scenarios on the pension fund portfolio in terms of funding ratio is studied. Also, the influence of a yield curve shock on the funding ratio and the difference between modeling the nominal interest rate with the Vasicek or Black Karasinski model is studied.

7.1 Derivation of stress scenarios

In our stress scenario framework different quantiles of the prediction of the risk factors can be derived. In this framework one of the inputs is a belief in terms of quantiles for at least one risk factor and at most all the three risk factors. Note that with the input of a belief of one risk factor, e.g. the nominal interest rate follows the quantile (5%), the stress scenarios have larger confidence intervals than with two or even three risk factors as an input. For the practical application of the impact on the funding ratio, the beliefs of two risk factors are taken into account and the following stress scenarios are studied.

i) Low returns on global equity. Here the return on the global equity is assumed to be -30 percent for the first year. This corresponds to the quantile (5%) of the return distribution for the global equity and the prediction of the real interest rate follows the quantile (50%).

ii) Low inflation. For the low inflation or deflation the quantile (20%) of the prediction of nominal interest rate and the quantile (80%) of the real interest rate is used.

iii) Low nominal interest rates. Here the quantile (10%) for the prediction of the nominal interest rate is observed together with the quantile (50%) of the real interest rate.

When deriving the funding ratio of a pension fund portfolio one needs to give an asset allocation of this portfolio. The asset allocation that is used in the remaining of this section is shown in Figure 7.1.

7.2 Funding ratio in stress scenarios

When studying the impact of the stress scenarios on the funding ratio also the base case is included, which has the belief that the real and nominal interest rate both follow the quantile (50%) of our forecasts. In the case of the representative yield curve shocks this base case helps us to better understand the impact of such a yield curve shock on the funding ratio. Observe that the yield curve shock method is applied to both real and nominal interest rate simultaneously. One could also study the

Real estate		7%
Assets		31%
Global markets	80%	
Emerging markets	20%	
Fixed income		57%
Bonds	60%	
Credits	20%	
High yield	10%	
Emerging markets debs	10%	
Alternative assets		5%
Hedge funds	60%	
Commodities	40%	
Total		100%

Figure 7.1: General allocation pension fund

impact on the funding ratio when coupling the yield curve shocks, e.g. when the historical method is applied to the real interest rate and the PCA method is applied to the nominal interest rate. However, this is not studied here.

Figure 7.2 shows the funding ratio when applying the different yield curve shocks. It can be seen that the parallel shift of 2 percent gives the lowest initial funding ratio. The base case without applying any yield curve shocks gives the highest initial funding ratio. Note that for both the standardize factor and the PCA method this was beforehand not clear since the nominal short term interest rate increases in these methods. From this figure it can be concluded that besides the parallel shift the other two methods give reasonable yield curve shocks. Moreover, the base case scenario for these methods gives a slightly increasing forecast of the funding ratio over time.

Also it can be seen that at year 4 an increase in funding ratio is observed, this is due to a pension reduction rule from the nFTK. The pension rule states that every pension fund that is for 5 consecutive years below the funding ratio of 105 percent needs to decrease the pension of the pensioners, thus decreasing the liability side, in such a way that the funding ratio increases to 105 percent.

After studying the representative yield curve shocks, the impact of the stress scenarios when calibrated to the two different methods is studied. Figure 7.3 depicts the funding ratio over time of the base case when deriving stress scenarios according to Method 1 (last day optimization) and Method 2 (log-likelihood approach). Again, in both of these methods the forecast of the nominal interest rate is done by both the Vasicek and the Black Karasinski model.

Figure 7.3 depicts the funding ratio over time for the base case when calibrating the Vasicek model according to the different methods and forecasting the nominal interest rate with both the Vasicek and Black Karasinski model. When comparing the calibration methods it can be observed that for method 1 a steep increase in the first year is observed but this flattens already after the first year. This can be due to the specific yield curve structure of the last day, since this behavior is not observed in the cases when calibrating according to method 2. When comparing the Vasicek and the Black Karasinski model to forecast the nominal interest rate not a consistent behavior can be seen. It is remarkable that the base case where the forecast nominal interest rate is done by the Vasicek model and the calibration according to method 1 and the complete opposite case the same tail behavior can be observed, namely a decreasing funding ratio.

Now, since both the impact of the representative yield curve shocks and the different calibration methods on the funding ratio of a general pension fund portfolio are studied, let us proceed to study the funding ratio of the stress scenarios mentioned in Section 7.1. For these stress scenarios the two

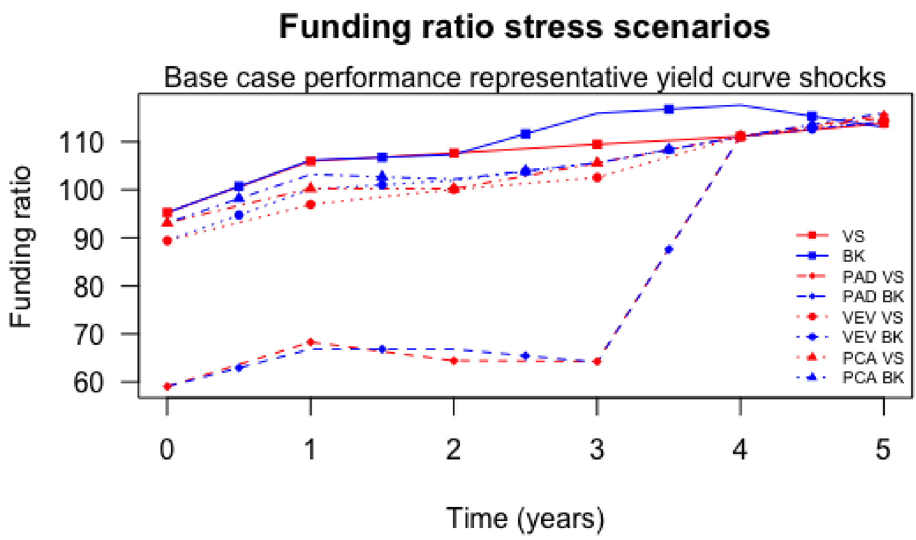


Figure 7.2: Impact of the yield curve shocks on the funding ratio, where VS denotes that the Vasicek model is used for the forecast of the nominal interest rate and BK the Black Karasinski model, PAD=standardized method, VEV=standardized factor method and PCA=principal component analysis

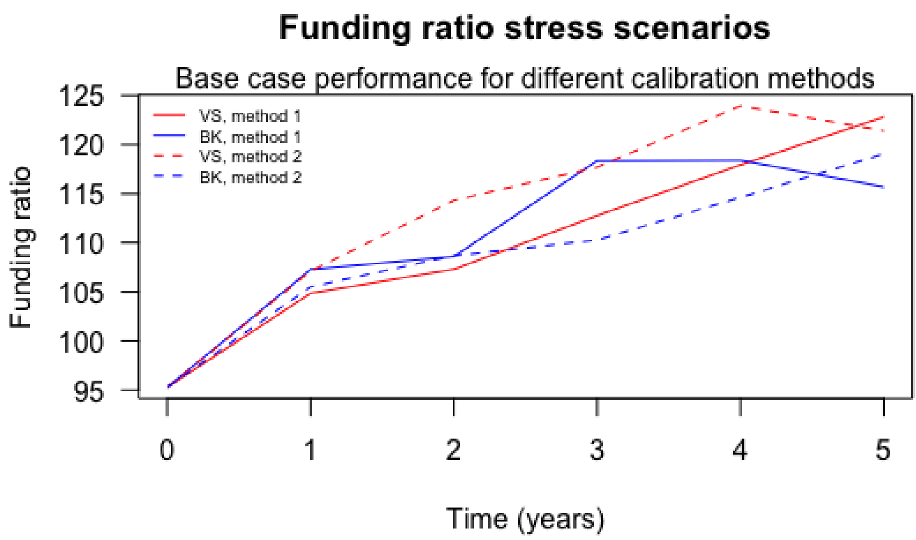


Figure 7.3: Stress scenarios comparison between using the Vasicek and the Black Karasinski model for the nominal interest rate and calibrating the Vasicek model according to the different calibration methods

factor Vasicek model is used to forecast the real interest rate and the one factor Black Karasinski model is used to forecast the nominal interest rate, both calibrated to the yield curve according the method 1. This does not imply that this setting is the best overall setting. These stress scenarios are also compared to the standard scenarios Aon Hewitt used to calculate the funding ratio. Not all these scenarios are given, only the quantiles (5%, 10%, 25%, 50%, 75%, 90%, 95%) are shown.

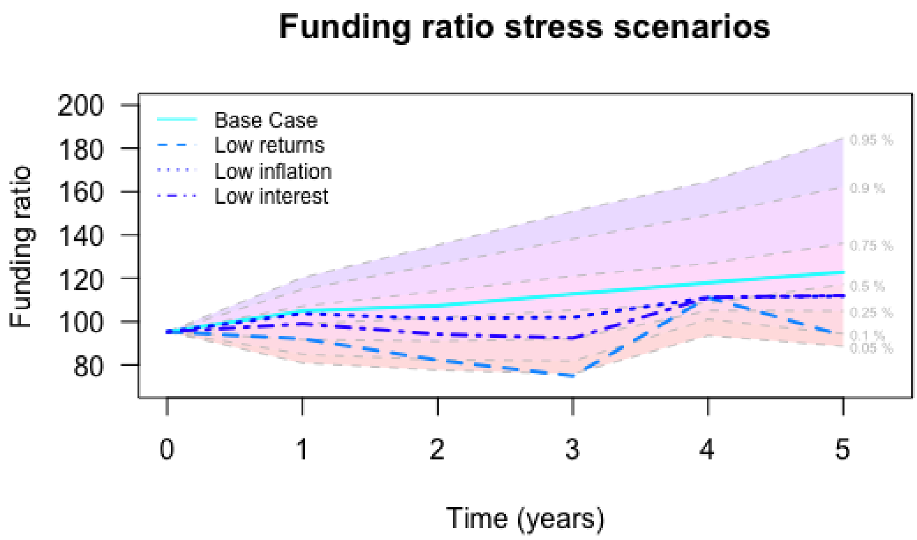


Figure 7.4: Stress scenarios in practice where the real interest rate is modelled with the Vasicek model and the nominal interest with the Black Karasinski model

Figure 7.4 shows the stress scenarios over time. In this figure it can be seen that the low returns on investment have the worst outcome regarding the funding ratio. Moreover, it can be observed that this stress scenario for the inflation does not have a severe impact on the funding ratio. When comparing our stress scenarios with the already known scenarios from Aon Hewitt, it can be seen that our base case scenario lies almost perfectly on the median and that the other stress scenario lie below the median. Note that here certain quantiles are taken for the stress scenario, see Section 7.1, but one can adjust the severity of the stress scenarios by taking a lower quantile.

Chapter 8

Conclusions and discussion

In this thesis we constructed a framework that derives stress scenarios for pension fund portfolios and studied the impact of these stress scenarios on the funding ratio. In our approach historical daily data, for different categories within a risk factor, from 2002 onward is obtained from Bloomberg to study the distribution of the returns. This distribution is essential for capturing the dependency between and within the risk factors. With both the distribution of the returns and the dependency, the predicted daily returns share the same characteristics as the historical returns. Here, no forecast assumptions were made. The most crucial assumption is that in times of a crisis the distribution of the returns of the risk factors is the same as in normal times. To avoid making this assumption one possibility is to only look at historical data from a crisis. However, in this case one should define a crisis and in this thesis it was shown, in Chapter 3, that not all risk factors were affected in the same manner by a crisis. For the distribution of the returns the skewed Student's t-distribution was used. This assumption was based on the Kolmogorov Smirnov test. However, for the daily returns of the inflation this test gave an insignificant p-value. Observe, that it is not immediately clear what the consequences are to the stress scenarios when taking another distribution for the returns. However, our two step approach and the corresponding models should still be valid, but maybe with another copula function. To capture the dependency, under the skewed Student's t-distribution assumption of the returns, the t-copula was used, which gave the highest log-likelihood out of several copula functions.

For the forecast of the first risk factor, i.e. the returns on investment, the autoregressive model with lag 1 was used, see Chapter 4. Although this lag was significant for all categories within this risk factor, the autocorrelation was low, i.e. approximately 0.1. As a consequence, there is a high deviation when simulating different forecasts. Therefore, also the forecast of the other two risk factors was studied.

The forecast of the second risk factor, i.e. inflation, was derived from the forecast of the real and nominal interest rate. For the forecast of the real interest rate the one and two factor Vasicek short rate model were studied. When calibrating both models to the historical yield curves it was observed that the two factor model gave in most of the cases a better fit. When comparing the performance, i.e. comparing the prediction of the model to the historical data, of both models it was observed that the one factor predicted the real interest rate better. However, this was due to the larger deviation in this prediction.

For the forecast of the nominal interest rate the Black Karasinski short rate model was studied. In this model an analytic expression for the yield curve is impossible to derive, therefore an approximation was used. Furthermore, the assumption was that this interest rate is non-negative which is also a property of the Black Karasinski model. However, for the last two years negative short term nominal interest rate were observed. Therefore, also the performance of the Vasicek model was studied. When comparing the performance of both models it was observed that the Vasicek model predicted the nominal interest rate better. However, in this model the deviation of the prediction was larger which also resulted in highly unlikely predictions.

To conclude, with both the dependency and the forecast stress scenarios with a certain belief can be derived. As an example the stress scenarios corresponding to the beliefs stated in Section 7.1 were derived, and the impact on the funding ratio was studied. It was observed that the low returns on the investments had the most severe impact. However, one could have different beliefs, which result in different stress scenarios and thus a different impact on funding ratio.

When comparing our model to the direct stress scenario models from the literature, i.e. Section 1.4, a few differences can be observed. In our framework the funding ratio corresponding to the stress scenario is not explicitly given but can be derived, see Chapter 7. This means that for a specific scenario the impact of different asset allocations can be studied, which was not possible in the models from literature.

One of the most important advantages of our model is that only stress scenarios, without an explicit funding ratio, are derived. This means that the impact on the funding ratio for different asset allocations can be studied. For Aon Hewitt this is of importance since now both the different allocations as well as the impact of different stress scenario on the funding ratio of a pension fund portfolio can be studied. Another advantage is that as an input one can give their own beliefs on the risk factors, which come with certain plausibility. A disadvantage of our framework is that the stress scenarios are highly dependent on the forecast models that are used. Due to the complexity of both our stress scenario derivation model and the model Aon Hewitt uses to calculate the funding ratio corresponding to these stress scenarios, it is not immediately clear what the influence on the funding ratio is when violating one or more assumption stated throughout this thesis.

8.1 Further research

For further research different aspects of our framework, that could improve the stress scenarios, can be studied. The first aspect is the distribution of the returns of the second risk factor, i.e. the inflation. Here, the Skewed Student's t-distribution was used, but the Kolmogorov Smirnov test gave an insignificant p-value. For further research the fit of other distribution can be investigated. Note that the definition of the returns, i.e. (2.1), influences this distribution, which means that taking a different definition, e.g. absolute difference, could also improve the p-value.

Another aspect that could be improved is the forecast of the first risk factor, i.e. the return on investment. In our model this forecast is done by means of an autoregressive model. Due to the low autocorrelation in this model, there is high deviation between the forecasts. In future research one could study other time series models, which give better forecasts with lower deviation. However, it can also be the case that this risk factor is difficult to predict and that this is the autoregressive model is best forecast model.

When comparing the yield curve calibration of the one factor and two factor Vasicek model it was observed that the two factor model gave in almost all the cases the best fit. However, observe that the one factor model is embedded in the two factor model. Therefore, it should be the case that this two factor model gives always the best fit, which does not imply that this model also gives the better prediction. In future research one could improve the optimization of the ordinary least squares, so that the two factor model always calibrates better to the yield curve than the one factor model. Note that this can also influence the performance of the two factor model.

In our current framework stress scenario without the confidence intervals are given. When deriving the stress scenarios of interest to us multiple times, it was observed that the deviation of the risk factors within the stress scenarios was not large. However, the influence of the difference to the funding ratio is not studied. For further research one could extend our model to give multiple stress scenarios with the same belief, from which the funding ratio could then be derived. With the funding ratio of all these stress scenarios the confidence interval on the funding ratio can be derived.

8.2 Recommendation to Aon Hewitt

The recommendations for Aon Hewitt are divided into two parts. First let us conclude the study of the derivation of our stress scenarios. It was shown that the one factor Vasicek model could predict the real interest rate with greater probability due to the higher deviation than the two factor Vasicek model. The advantage of the two factor Vasicek on the other hand was that it could describe the yield curve in most of the cases better.

For the forecast of the nominal interest rate the properties of the Black Karasinski model were discussed. Especially the non-negative assumption raised the question if this model gives still the best prediction, since nowadays negative short term nominal interest rates are observed. From our performance study it became clear that the Black Karasinski overestimated the historical interest rate and as a consequence the Vasicek model could predict this interest rate in more cases better. One recommendation for improving the methods used at Aon Hewitt to determine the forecast of the nominal interest rate in the scenarios is to further investigate the difference and performance of both models.

For the recommendation of the impact of the stress scenarios on the funding ratio, note that it is not possible to make one clear statement, since every stress scenario has a different impact on different pension fund portfolios. However, one could view this whole thesis as a recommendation. With our framework it is possible to identify the stress scenarios that have a weak or severe impact on the funding ratio. Knowing this one could decide to change the asset allocation, looking at the correlation in Table B.2, and make the pension fund portfolio more robust against this stress scenario. In this case the stress scenarios give a recommendation for the allocation of a pension portfolio.

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Appendix A

Detailed overview literature

In this appendix the direct stress scenario selection method stated in Section 1.4.1 are further explained. Note that in contrary to the corresponding paper, here the models are explicitly applied to a pension fund setting. Also the stress scenarios that are used at Aon Hewitt in the UK are given.

A.1 Stress scenario selected by empirical likelihood

Here the stress scenario selection studied in [27] is applied to a pension setting. The goal of this stress scenario is to quantify the likelihood of the scenarios that lead to a specific threshold. This is called *reverse* stress testing. From a pension viewpoint this could be the funding ratio that falls beneath the funding deficit or shortfall. i.e. a funding ratio below 105 percent.

First let us introduce the same setting as in the paper. Let $\boldsymbol{Z}$ be a random d-dimensional vector with probability density f on $\mathbb{R}^d$. Here $\boldsymbol{Z}$ represents the market factors influencing a pension portfolio, for example interest and inflation rate. Further, let $(\boldsymbol{Z}, F_R)$ be the joint distribution of the market factors and the pension funding ratio. Note that here the funding ratio can be deterministic function of the market factors. Denote the conditional density of the market factors given the funding ratio falling beneath fr by $f(\boldsymbol{z}|F_R < fr)$. The goal is to find the most likely loss scenario, $\boldsymbol{z}^*(fr)$, that lead to this threshold,

$$\boldsymbol{z}^*(fr) = \arg\max_{\boldsymbol{z}\in\mathbb{R}^d} f(\boldsymbol{z}|F_R < fr).$$

The first step to achieve this goal is to estimate the conditional mean $\mathbb{E}[\boldsymbol{Z}|F_R < fr]$. Observe that this is equivalent to estimating the unconditional mean of $\boldsymbol{Z}$ when only taking into account the observations where $F_R < fr$. Note that here the number of observations is of high importance when estimating this unconditional mean. It is known that these observations come from a distribution so simply taking the sample mean as an estimate would introduce an error. The authors use the empirical likelihood (EL) method studied in [35] to overcome this problem. The optimization problem boils down to,

$$\max_{w_1,w_2,\ldots,w_n} \sum_{i=1}^{n} \log(w_i) \qquad \text{subject to} \qquad \sum_{i=1}^{n} w_i = 1,\ \sum_{i=1}^{n} w_i \boldsymbol{z_i} = \boldsymbol{x},$$

where the candidate value $\boldsymbol{x}$ is given by,

$$\mathcal{R}(\boldsymbol{x}) = \max\left\{\prod_{i=1}^{n} nw_i : \sum_{i=1}^{n} w_i \boldsymbol{z_i} = \boldsymbol{x}, \sum_{i=1}^{n} w_i = 1, w_i \geq 0, i = 1, 2, ..., n\right\}.$$

With the set $\mathcal{R}(\boldsymbol{x})$, one can derive $\bar{\boldsymbol{z}}(fr)$ which equals the $\boldsymbol{x}$ that maximizes this set. Observe, that this method does not rely on the underlying distribution and therefore is non parametric. Another option is to fit the joint distribution $(\boldsymbol{Z}, F_R)$ and then derive the conditional mean analytically. The downside of this method is that one needs to fit the joint distribution and there is not a straightforward method in general for this.

The second step is to perform a scaling correction for $\mathbb{E}[\boldsymbol{Z}|F_R < fr]$ to obtain $\boldsymbol{z}^*(fr)$. In [27, Proposition 1] this scaling correction is given for different cases. The proposition is stated here for completeness of this method.

Proposition 1 *Suppose the distribution of* $\boldsymbol{Y} = (\boldsymbol{Z}, F_R)$ *is elliptical, i.e.* $\boldsymbol{Y} = \boldsymbol{\mu}+\boldsymbol{A}\boldsymbol{X}$ *with* $\boldsymbol{\mu}$ *a vector and* $\boldsymbol{A}$ *a matrix, with* $\boldsymbol{X}$ *either ERV*(α, ν)*, for some* $\alpha, \nu > 0$*, or RV*(ν)*, with* $\nu > 1$*. Let* $\boldsymbol{z}^*(fr) \in \mathbb{R}^d$ *be the most likely loss scenario and let* $\bar{\boldsymbol{z}}(fr) \in \mathbb{R}^d$ *denote the conditional mean* $\mathbb{E}[\boldsymbol{Z}|F_R < fr]$*. Then there exists a positive scalar sequence* k_{fr} *such that*

$$\boldsymbol{z}^*(fr) = k_{fr}\bar{\boldsymbol{z}}(fr), \quad \textit{and} \quad k_{fr} \to k \textit{ as } fr \to \infty,$$

where

- $k = 1$ *for all ERV*(α, ν) *distributions;*
- $k = (\nu - 1)/\nu$ *for all RV*(ν) *distributions,* $\nu > 1$*.*

Note that here RV(ν) refers to regularly varying and ERV(α, ν) to exponential regularly varying. The random variable $\boldsymbol{X}$ is in the class RV(ν) if it holds that

$$\lim_{\ell\to\infty} \frac{\mathbb{P}\left(\boldsymbol{X} \geq \ell x\right)}{\mathbb{P}\left(\boldsymbol{X} \geq \ell\right)} = x^{-\nu},$$

and $\boldsymbol{X}$ is in the class ERV(α, ν) if $\exp\left(\boldsymbol{X}^\alpha\right)$ is in the class RV(ν).

As a result of this model one gets the most likely loss scenario. However, this still does not give any information about the actual probability of this event happening. A first implication of the plausibility of this specific scenario could be one divided by the total number of observations. However, a drawback is that here all the scenarios are assumed to be equally likely to happen. With this in mind we could use this ratio as a lower bound and the ratio of the observations that satisfy $F_R < fr$ to the total number of observations can function as an upper bound. Note that these bounds can differ in each case and the length of the set that satisfy $F_R < fr$ is strongly dependent on the threshold fr.

Another disadvantage is the coverage studied in Table 1 in the paper of this model when using the multivariate t distribution. Especially with small sample sizes, the coverage is low, which is exactly the situation of interest.

Although, this model uses a non parametric method to estimate the unconditional mean, it does assume the distribution to find the most likely loss scenario. This assumption restricts the method to only be valid for the RV(ν) and ERV(α, ν) distribution. According to the authors from this paper, the Student t-distribution with ν degrees of freedom often approximates the market data well, where $5 < \nu < 7$. The Student t-distribution belongs to the family of regularly varying distribution and thus if the market data is indeed approximated well with this distribution, the restriction is not a disadvantage of the model.

A.2 Multivariate stress scenarios

This stress scenario is based on [31], where a stochastic risk model is considered, i.e. $F_R = g(\boldsymbol{Z})$, where $\boldsymbol{Z}$ denotes the random vector of the market risk factors. The goal of this paper is to find the least solvent likely event (LSLE), which is defined as

$$\boldsymbol{z}_{LSLE} = \arg\min \{g(\boldsymbol{z}) : \boldsymbol{z} \in S\},$$

where S is the closed set of all scenarios. This is called *forward* stress testing. The assumption of a closed set of stress scenarios seems quite restrictive, but one can simply generate this set with an arbitrary number of realizations of $\boldsymbol{Z}$ or use historical data to construct this set. As an intermediate step let the half-space $H_{\boldsymbol{y},\boldsymbol{\mu}}$, for any $\boldsymbol{y} \in \mathbb{R}^d$ and vector $\boldsymbol{\mu}$, be given by,

$$H_{\boldsymbol{y},\boldsymbol{\mu}} = \left\{\boldsymbol{z} \in \mathbb{R}^d : g(\boldsymbol{z}) \leq \boldsymbol{\mu}'\boldsymbol{y}\right\}.$$

Now, the probability that Z lies in the half-space $H_{\boldsymbol{y},\boldsymbol{\mu}}$ equals,

$$\mathbb{P}_{\boldsymbol{Z}}\left(H_{\boldsymbol{y},\boldsymbol{\mu}}\right) = \mathbb{P}\left(g(\boldsymbol{Z}) \leq \boldsymbol{\mu}'\boldsymbol{y}\right).$$

With this probability we can define the scenario set as,

$$Q_\alpha = \bigcap \{H_{\boldsymbol{y},\boldsymbol{\mu}} : \mathbb{P}_Z\left(H_{\boldsymbol{y},\boldsymbol{\mu}}\right) \geq \alpha\}.$$

In [31] the relation to depth sets is explained, and it is proved that the the set Q_α and the depth set D_α are equal if $\boldsymbol{Z}$ has a continuous probability density. Let the depth of the realization, $\boldsymbol{z}$, of $\boldsymbol{Z}$ be given by,

$$\text{depth}(\boldsymbol{z}) = \inf_{\boldsymbol{\mu}:\mu\neq 0} \mathbb{P}_{\boldsymbol{Z}}(H_{\boldsymbol{z},\boldsymbol{\mu}}),$$

then it follows that the depth set is

$$\begin{aligned} D_\alpha &= \left\{\boldsymbol{z} \in \mathbb{R}^d : \text{depth}(\boldsymbol{z}) \geq 1-\alpha\right\} \\ &= \bigcap \{H_{\boldsymbol{y},\boldsymbol{\mu}} : \mathbb{P}_{\boldsymbol{Z}}(H_{\boldsymbol{y},\boldsymbol{\mu}} > \alpha)\}. \end{aligned}$$

For an additive function $q_\alpha(\boldsymbol{\mu})$ the value for $\boldsymbol{z}$ that maximizes $g(\boldsymbol{z})$ on the set Q_α is equal to $\boldsymbol{z}_{LSLE}$ and it holds that $g(\boldsymbol{z}_{LSLE}) = q_\alpha(\boldsymbol{\mu})$.

Under some mild conditions we can find an explicit expression for the LSLE. Let for example $g(\boldsymbol{z}) = \boldsymbol{\mu}'\boldsymbol{z}$ and $\boldsymbol{Z} \sim E_d(\boldsymbol{\mu}, \Sigma, \psi)$, then it follows that

$$\boldsymbol{z}_{LSLE} = \arg\max \left\{\boldsymbol{\mu}'\boldsymbol{z} : (\boldsymbol{z}-\boldsymbol{\mu})'\Sigma^{-1}(\boldsymbol{z}-\boldsymbol{\mu}) \leq k_\alpha^2\right\},$$

where k_α^2 corresponds to the α quantile of the distribution $\boldsymbol{Z}$. This optimization problem can be solved with the Kuhn-Tucker approach, which gives

$$\boldsymbol{z}_{LSLE} = \boldsymbol{\mu} + \frac{\Sigma\boldsymbol{\mu}}{\sqrt{\boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu}}} k_\alpha.$$

In this example we assumed that the risk factors only have a linear relation impact on the model outcome. However, in reality it is known that all risk factors are mutual dependent. The most simplistic model that captures this behavior is $g(\boldsymbol{z}) = \boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{z}$.

Using the half-space $H_{\boldsymbol{y},\boldsymbol{\mu}}$ this method gives a quantitative measure for the plausibility. However, when the assumptions of linear impacts and elliptical distribution of the risk factors is not made, the computation of the $\boldsymbol{z}_{LSLE}$ can be complicated. In this case numerical optimization can be a good alternative.

A.3 Multi-period stress testing

A major point of discussion is the plausibility of the stress scenarios. In [14] the Mahalanobis distance is introduced as a measure to compare different scenarios on plausibility. Let the future scenario path of the risk factors $\boldsymbol{Z}$ be a random vector with an elliptical distribution with mean $\boldsymbol{\mu} = \mathbb{E}[\boldsymbol{Z}]$ and covariance matrix $\boldsymbol{A} = \mathrm{Cov}(\boldsymbol{Z})$, then the Mahalanobis distance is given by,

$$\mathrm{Maha}(\boldsymbol{z}) := \sqrt{(\boldsymbol{\mu} - \boldsymbol{z})^T \boldsymbol{A}^{-1} (\boldsymbol{\mu} - \boldsymbol{z})}.$$

The general objective of this paper is to find the worst case scenario that has Mahalanobis distance less or equal than τ. The set of these possible worst case scenarios is given by

$$\mathrm{Ell}_\alpha := \{\boldsymbol{z} \in \boldsymbol{Z} : \mathrm{Maha}(\boldsymbol{z}) \leq \tau\}.$$

To find the worst case scenario one needs to have a measure that indicates the severity of this scenario. In this paper the authors use the maximum loss over a finite horizon which equals

$$\mathrm{MaxLoss}_S(L) := \max_{\boldsymbol{z} \in S} \mathbb{E}\left[\sum_{s=t+1}^{t+m} \sum_i L_{i,t}(\boldsymbol{u}_i, \boldsymbol{z}) \,|I_t, \boldsymbol{z}\right],$$

where $L_{i,t}$ is the loss function. Note that this loss function can differ depending on the setting that is studied.

Next, there are two methods described in [14] to obtain the worst case scenario, denoted by $\bar{\boldsymbol{z}}$. The first method uses a Monte Carlo algorithm to solve the optimization problem given by,

$$\bar{\boldsymbol{z}} := \mathrm{MaxLoss}_{\mathrm{Ell}_\tau}(L) = \max_{\boldsymbol{z} \in \mathrm{Ell}_\tau} \mathbb{E}\left[\sum_{s=t+1}^{t+m} \sum_i L_{i,t}(\boldsymbol{u}_i, \boldsymbol{z}) \,|I_t, \boldsymbol{z}\right].$$

In the second method a linear approximation for the loss function is used to calculate the worst case scenario analytically, from which follows that

$$\bar{\boldsymbol{z}} = \boldsymbol{\mu} - \frac{\tau}{\sqrt{\boldsymbol{l}^T \boldsymbol{A} \boldsymbol{l}}} \boldsymbol{A}\boldsymbol{l}.$$

However, the second method is only valid when the risk factor is multivariate normal distributed, i.e. $\boldsymbol{Z} \sim N(\boldsymbol{\mu}, \boldsymbol{A})$.

A major drawback to this paper is that it requires the distribution of the future scenario path of the risk factors as input. In reality this is often not a known distribution. Also, the performance of stress scenarios is strongly dependent on the performance of these future scenario path distribution. If this distribution does not represent the future of the risk factors, the stress scenarios are useless.

An advantage is that the Mahalanobis distance is used to quantify the plausibility of the scenarios that are considered. Furthermore, in this method not only one time moment is considered but a finite horizon.

Appendix B

Descriptive statistics of the data

B.1 Distribution inflation and nominal interest rate

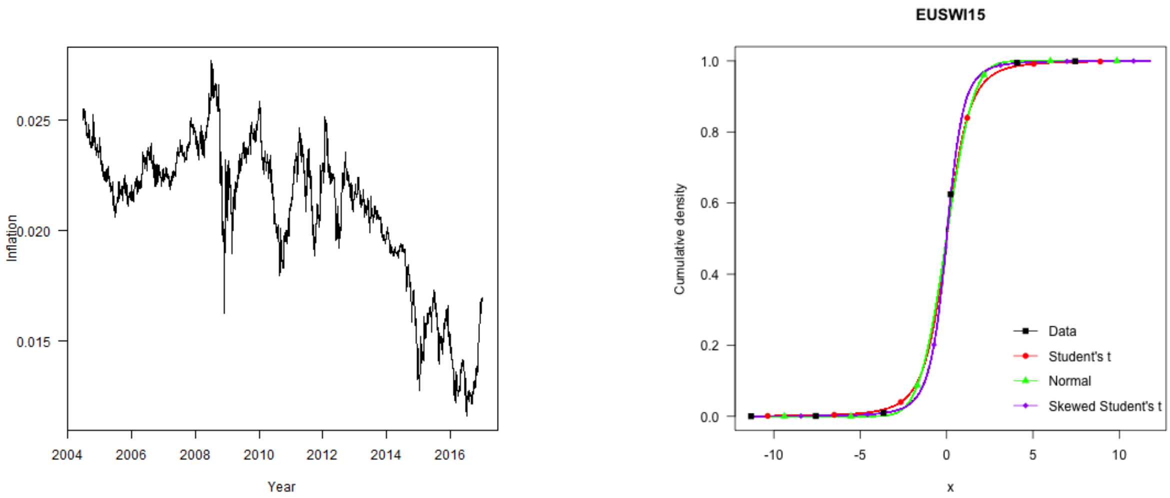

Figure B.1: The index value (left) and cumulative density (right) of the EUSWI15

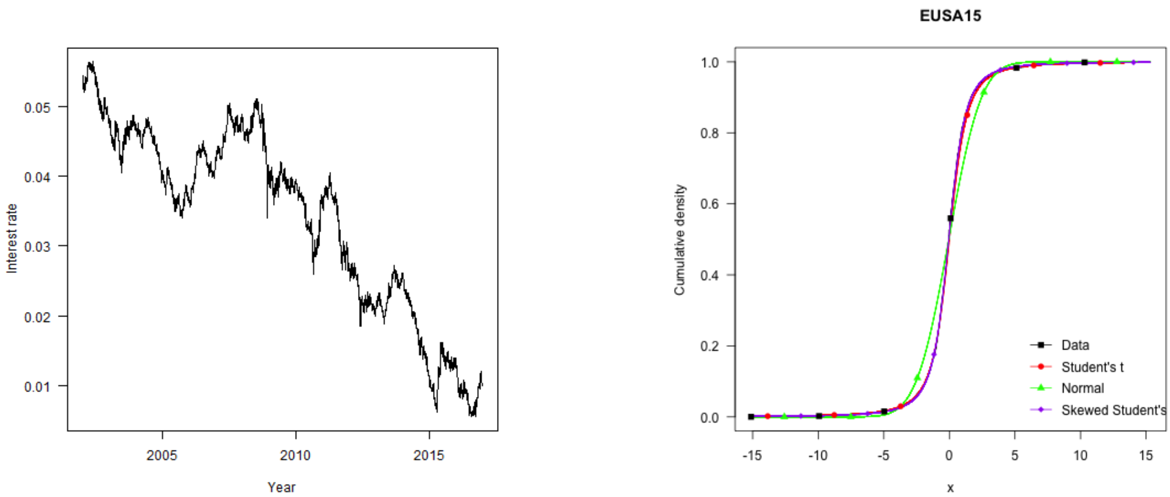

Figure B.2: The index value (left) and cumulative distribution (right) of the EUSA15

B.2 Kolmogorov-Smirnov test

To measure the goodness of fit of the skewed Student's t-distribution we performed the Kolmogorov-Smirnov (KS) test and used a bootstrapping algorithm. Observe that both methods do not test the same hypothesis. When using the KS test the null hypothesis is $H_0 : F_{\text{data}} = F_{ST}$. Note that we test that the data comes from a specific distribution. The test statistic, D_n, for this test equals

$$D_n = \sup_x |F_{\text{data}}(x) - F_{ST}(x)|.$$

The p-value is calculated via the Kolmogorov distribution

$$1 - \alpha = \mathbb{P}(K < \sqrt{n} D_n) = 1 - 2 \sum_{k=1}^{\infty} (-1)^{k-1} e^{-2k^2 n D_n^2}.$$

In the bootstrap algorithm we do not test that the data is from a specific distribution but account for the fact that the parameters have some uncertainty. In this bootstrap algorithm we use 1000 simulations.

Table B.1: Statistics for skewed Student's t-distribution

	Kolmogorov-Smirnov		Bootstrap
Index	Test statistic	p-value	p-value
MSDEWIN	0.0086	0.91565	0.311
MSDEEEMN	0.0062	0.99685	0.852
M1WOMVOL	0.0079	0.95548	0.429
LET7TREU	0.0073	0.95536	0.676
LS06TREU	0.0112	0.74681	0.094
LS08TREU	0.0105	0.81666	0.109
LEC7TREU	0.0134	0.45829	0.014
LF98TRUU	0.0113	0.68436	0.049
BCOMTR	0.0108	0.73433	0.076
JPEIDIVR	0.0173	0.18421	0
G4F0	0.0141	0.38016	0.003
LECRTREU	0.0103	0.78606	0.139
G250NLEU	0.0185	0.11429	0.009
REIT	0.0089	0.9	0.230
BXIIBEU3	-	-	

B.3 Copula descriptives

In Table B.2 the correlation parameter corresponding to the t-copula function is given for each risk factor.

Table B.2: Correlation copula matrix

Index	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
MSDEWIN (1)																	
MSDEEEMN (2)	0.68																
M1WOMVOL (3)	0.86	0.58															
LET7TREU (4)	-0.16	-0.12	-0.07														
LS06TREU (5)	-0.25	-0.17	-0.09	0.82													
LS08TREU (6)	-0.24	-0.17	-0.08	0.70	0.94												
LEC7TREU (7)	-0.13	-0.01	-0.02	0.80	0.88	0.77											
LF98TRUU (8)	0.42	0.40	0.62	-0.03	0.05	0.04	0.13										
BCOMTR (9)	0.38	0.38	0.32	-0.09	-0.10	-0.11	-0.05	0.26									
JPEIDIVR (10)	0.41	0.42	0.63	0.08	0.15	0.13	0.20	0.91	0.27								
G4F0 (11)	-0.21	-0.16	-0.09	0.91	0.88	0.77	0.83	0.01	-0.10	0.12							
LECRTREU (12)	-0.15	-0.04	-0.03	0.81	0.89	0.77	0.98	0.12	-0.05	0.20	0.86						
G250NLEU (13)	0.50	0.43	0.37	-0.03	-0.20	-0.19	-0.07	0.05	0.17	0.04	-0.12	-0.09					
REIT (14)	0.68	0.36	0.73	-0.06	-0.09	-0.08	-0.05	0.33	0.20	0.37	-0.08	-0.06	0.29				
HFRXGLE (15)	0.66	0.56	0.47	-0.08	-0.20	-0.21	-0.04	0.04	0.34	0.04	-0.16	-0.07	0.47	0.37			
BXIIBEU3 (16)	-0.03	0.01	-0.02	0.02	0.04	0.01	0.01	-0.01	-0.01	0.00	0.03	0.04	-0.01	-0.01	-0.01		
EUSWI (17)	0.11	0.12	0.03	-0.10	-0.24	-0.27	-0.15	-0.02	0.09	-0.05	-0.18	-0.16	0.14	0.03	0.14	-0.01	
EUSA (18)	0.23	0.14	0.07	-0.63	-0.83	-0.86	-0.70	-0.06	0.11	-0.14	-0.70	-0.71	0.16	0.06	0.18	-0.00	0.26

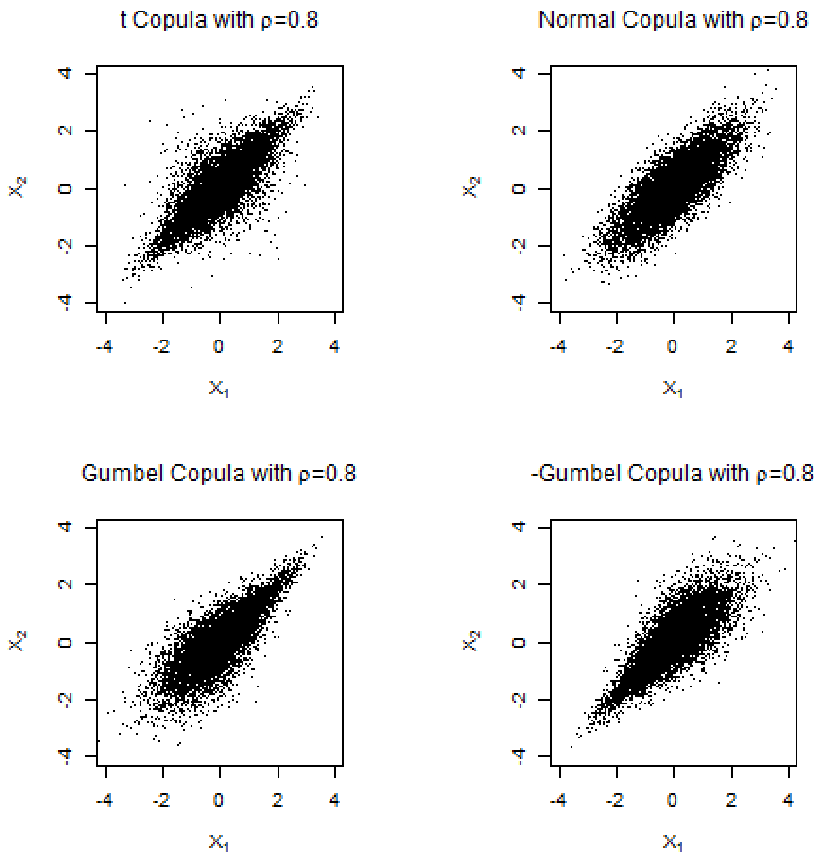

Figure B.3: Different forms of dependency captured with copulas all with the same correlation coefficient $\rho = 0.8$ and marginals which are standard normal distributed

B.4 Different yield curve shapes

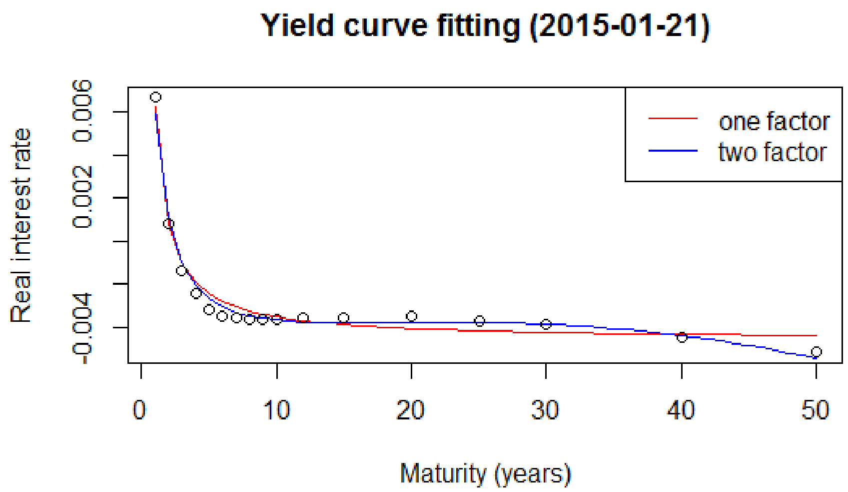

Figure B.4: Yield curve of the one factor and two factor Vasicek model for the real interest rate

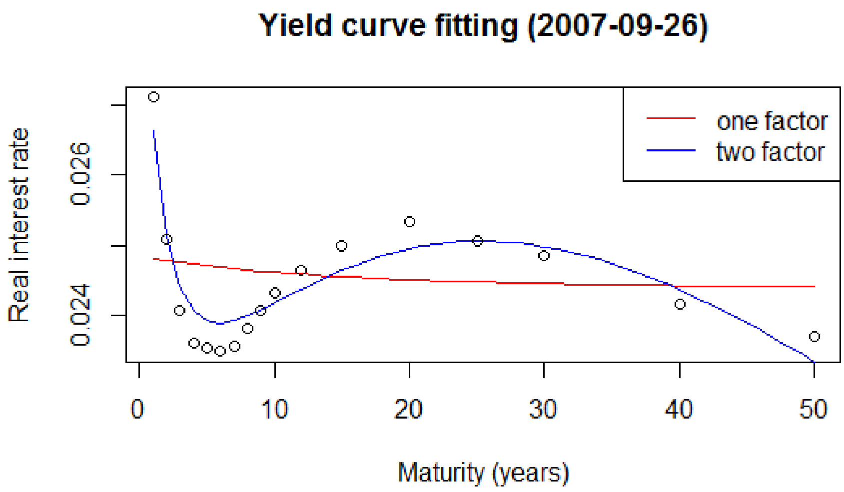

Figure B.5: Yield curve of the one factor and two factor Vasicek model for the real interest rate

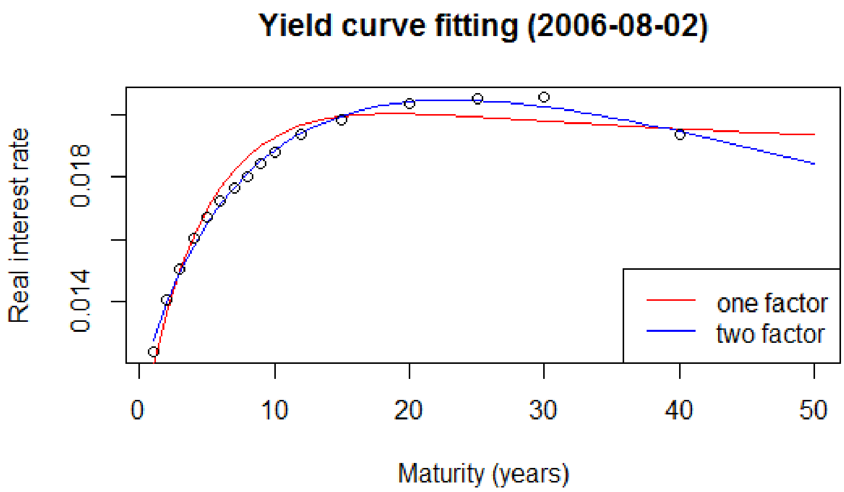

Figure B.6: Yield curve of the one factor and two factor Vasicek model for the real interest rate

Appendix C

Performance of the short rate models

C.1 Yield curve performance

Table C.1: Performance yield curve calibration on two factor model with Method 2 for $D = 5$ (left) and $D = 10$ (right)

α	$\log L$	AIC_L	BIC_L
0.1	97.262	12.85	8.55
0.3	83.753	13.14	8.85
0.5	70.285	13.49	9.2
0.7	56.039	13.95	9.65
0.9	44.053	14.43	10.13

α	$\log L$	AIC_L	BIC_L
0.1	195.138	11.45	14.78
0.3	183.383	11.58	14.91
0.5	129.911	12.27	15.59
0.7	108.153	12.63	15.96
0.9	84.571	13.12	16.45

C.2 Yield curve convergence

Table C.2: Number of simulations per setting for $T = 5$ and $dt = 0.025$

Parameter	value	*nsim*	value	*nsim*	value	*nsim*	value	*nsim*
			base case					
$r_{\text{real},0}$	0	4500	0.1	7500	0.2	4500		
α_1	0.1	7000	0.2	7500	0.4	5250	0.8	2000
θ	0.1	4000	0.2	7500	0.4	3000*	0.8	2000*
σ_1	0.05	3000*	0.1	7500	0.2	>25000		

Table C.3: Number of simulations per setting for $T = 10$ and $dt = 0.025$

Parameter	value	*nsim*	value	*nsim*	value	*nsim*	value	*nsim*
			base case					
$r_{\text{real},0}$	0	2000*	0.1	4000	0.2	11000		
α_1	0.1	>25000	0.2	4000	0.4	1000*	0.8	1000*
θ	0.1	9000	0.2	4000	0.4	1000*	0.8	500*
σ_1	0.05	1000*	0.1	4000	0.2	20000		

C.3 Visualization Vasicek model

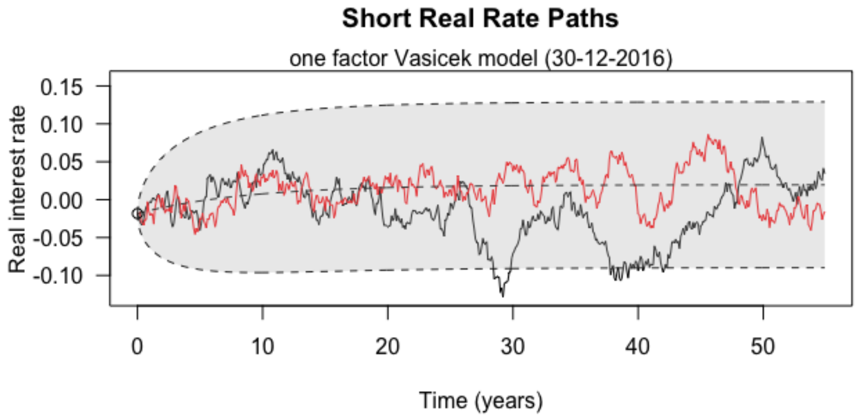

Figure C.1: Sample paths and the 95 percent prediction intervals of the one factor Vasicek model with parameters calibrated to the yield curve with method 2

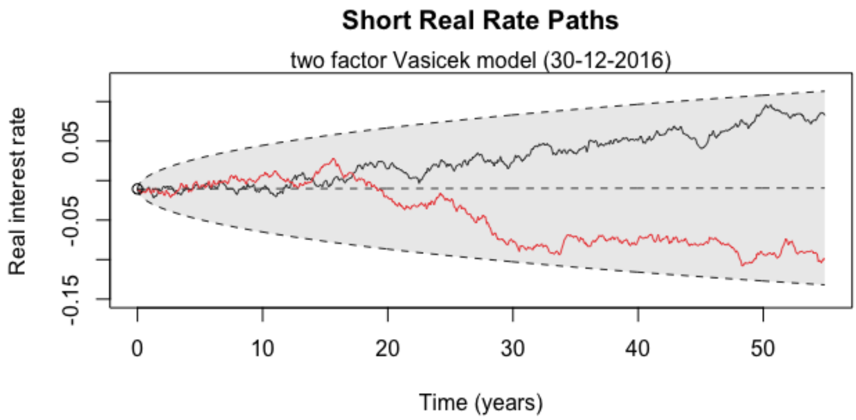

Figure C.2: Sample paths and the 95 percent prediction intervals of the two factor Vasicek model with parameters calibrated to the yield curve with method 2

C.4 Sensitivity analysis

Table C.4: Sensitivity performance one factor Vasicek model for parameter k when calibrating according to Method 1

Index	Number	-5%		5%	
		Bound_{95}	Bound_{50}	Bound_{95}	Bound_{50}
$r^1_{\text{real},1}$	2391	2218	2019	2215	2019
$r^1_{\text{real},2}$	1634	1452	1267	1448	1261
$r^1_{\text{real},3}$	1476	1346	1068	1340	1052
$r^1_{\text{real},4}$	1323	1194	894	1193	861
$r^1_{\text{real},5}$	1171	1038	743	1032	707

Table C.5: Sensitivity performance one factor Vasicek model for parameter θ when calibrating according to Method 1

Index	Number	-5%		5%	
		Bound_{95}	Bound_{50}	Bound_{95}	Bound_{50}
$r^1_{\text{real},1}$	2391	2218	2024	2217	2022
$r^1_{\text{real},2}$	1634	1448	1261	1447	1262
$r^1_{\text{real},3}$	1476	1343	1068	1340	1058
$r^1_{\text{real},4}$	1323	1196	892	1189	863
$r^1_{\text{real},5}$	1171	1036	749	1031	719

Table C.6: Sensitivity performance one factor Vasicek model for parameter σ when calibrating according to Method 1

Index	Number	-5%		5%	
		Bound_{95}	Bound_{50}	Bound_{95}	Bound_{50}
$r^1_{\text{real}}(1)$	2391	2210	2007	2223	2039
$r^1_{\text{real}}(2)$	1634	1439	1252	1458	1270
$r^1_{\text{real}}(3)$	1476	1334	1042	1354	1078
$r^1_{\text{real}}(4)$	1323	1186	860	1204	905
$r^1_{\text{real}}(5)$	1171	1023	701	1045	760