BACHELOR

# A study on self similar fractals

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# A study on self similar fractals

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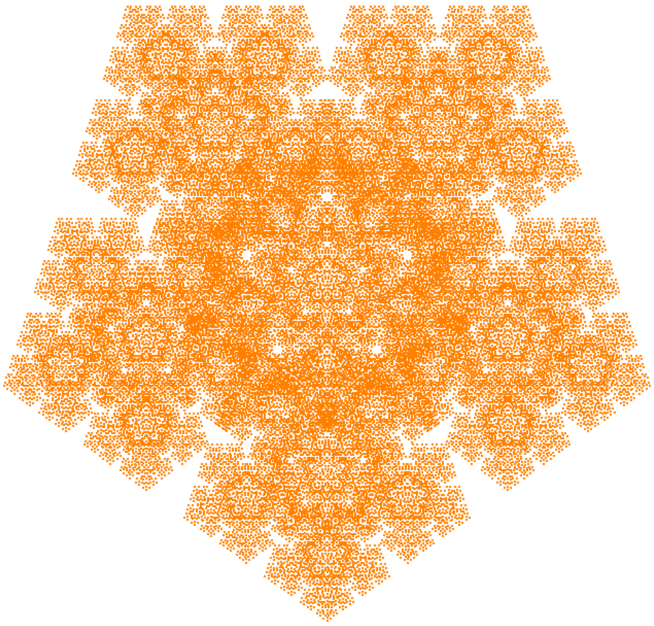

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# Abstract

This paper discusses mainly self-similar fractals and specifically the Normalized Hausdorff measure of the $\phi$-Koch curve. In order to do this some known definitions and theorems are explained to get a better understanding of fractals. For fractals in general, definitions and properties are given for the Hausdorff measure and Hausdorff dimension. There are different classes of fractals, for example the self-similar fractals. Self-similar fractals have properties that makes them easy to work with. It is for example possible to uniquely define self-similar fractals with contraction maps. Even more convenient is that it is easy to find the Hausdorff dimension, which is in general not the case. For self-similar fractals for which the open set condition holds it is known that the similarity dimension equals the Hausdorff dimension.

For self-similar fractals it is guaranteed that the Hausdorff measure is finite. The Hausdorff dimension of the $\phi$-Koch curve is known, but the Hausdorff measure is not. So far it seems that only an upper and lower bound is known when $\phi$ equals $\frac{\pi}{3}$. This paper presents five possibles covers for the $\phi$-Koch curve and determines the Normalized Hausdorff with respect to these covers. The result is within the known bounds, but it is unknown how close the results are to the real values.

# Contents

# Chapter 1

# Introduction

## 1.1 The origin of fractals

In the 19th century mathematicians invented strange structures to break loose from naturalism. These structures however turn out to be inherent in familiar objects around us. As Mandelbrot points out: *Clouds are not spheres, mountains are not cones, coastlines are not circles, nor does lightning travel in a straight line. [6]*

Since more and more of these strange structures were created the need for a term became increasingly apparent. Mandelbrot invented the world "fractal" to bring this large class of objects together. *I (Mandelbrot) coined fractal from the Latin adjective fractus. The corresponding Latin verb frangere means "to break": to create irregular fragments. It is therefore sensible (and how appropriate for our needs!) that, in addition to "fragmented" (as in fraction or refraction), fractus should also mean "irregular," both meanings being preserved in fragment. The proper pronunciation is frac'tal. the stress being placed as in frac'tion [6].*

A fractal is defined by Mandelbrot as *a set for which the Hausdorff dimension $D$* (which need not be integer, see section 2.3) *strictly exceeds the topological dimension $D_T$ [6].* However Mandelbrot points out that *I continue to believe that one would do better without a definition (my 1975 Essay included none). The immediate reason is that the present definition will be seen to exclude certain sets one would prefer to see included. More fundamentally, my definition involves $D$ and $D_T$, but it seems that the notion of fractal structure is more basic than either $D$ or $D_T$. Deep down, the importance of the notions of dimension is increased by their unexpected new use! In other words, one should be able to define fractal structures as being invariant under some suitable collection of smooth transformations. But this task is unlikely to be an easy one. To exemplify the difficulty in a standard context, let us recall that certain definitions of complex number fail to exclude the real numbers! At the present stage, the main need is to differentiate the basic fractals from the standard sets in Euclid. This is a need my definition does satisfy [6].*

In the next section of this chapter some useful definitions are given that may help to understand this paper. In chapter 2 definitions for dimension and measure are given for fractals in general. In chapter 3 self-similar fractals are discussed. Some important theorems for the dimension and the measure of such fractals are given. Also in this chapter the method and results are shown for the research on the $\phi$-Koch curve. Finally a summary of this report is given. The appendix discusses the volume of the n-dimensional unit ball (which will be referred to from chapter 2) and contains pieces of Mathematica codes.

In this paper some pictures can be found of fractals. Pictures are created with Mathematica as described in chapter 3, unless denoted otherwise.

## 1.2 Useful definitions

In this section a list of useful definitions in alphabetic order is presented.

**Cauchy sequence:**

A sequence of points $x_n$ in a metric space $(X, d)$ is a Cauchy sequence if, $\forall \epsilon > 0$, $\exists n_0 \in \mathbb{N} : \forall n > n_0, m > n_0 \in \mathbb{N}, d(x_n, x_m) < \epsilon$.

**compact:**

A subset $K \subset (X, d)$ (meaning that $K \subset X$ with respect to distance $d$) is compact if it satisfies one of the following two equivalent conditions,

- every cover of K with open sets has a finite subcover.
- every infinite sequence $X_n$ contains a subsequence with a limitpoint in K.

Special case, $(X, d)$ is complete: $K$ is compact iff $K$ is closed and bounded. A property of a compact set $K$: Let $X_i \subset K$ be a sequence of closed sets, $X_i \subset X_j$, $i > j$. Then $\bigcap X_i \neq \emptyset$.

**complete:**

A metric space $(X, d)$ is complete when every Cauchy-sequence is convergent. For example the metric space $(\mathbb{R}^n, d)$ is complete, while $(\mathbb{Q}^n, d)$ is not.

**contraction map:**

Let $c < 1$ be a nonnegative real number, the function $f$ on the metric space $(X, d)$ is a contraction map with **contraction factor** $c$ if $d(f(x), f(y)) \leq cd(x, y)$, $\forall x, y \in X$.

**convergent sequence:**

A sequence $X_n$ in $(X, d)$ is convergent with limit $x$ if, $\forall \epsilon > 0 \exists n_0 \in \mathbb{N} : \forall n > n_0 \in \mathbb{N}, d(X_n, x) < \epsilon$. Note that every convergent sequence is a Cauchy sequence. Beware that Cauchy sequence need not to be convergent. For example in $\mathbb{Q}$ the decimal expansion of $\sqrt{2}$ gives the sequence $(1, 1.4, 1.41, 1.414, 1.4142, ...)$ is a Cauchy sequence with no limit in $\mathbb{Q}$. A Cauchy sequence is convergent iff the $(X, d)$ is complete.

**$\Gamma$ function:**

The following properties hold for the Gamma function.

- $\Gamma(1/2) = \sqrt{\pi}$
- $\Gamma(n) = (n-1)\Gamma(n-1)$, $n \in \mathbb{N}$
- $\Gamma(n) = (n-1)!$, $n \in \mathbb{N}$
- $\Gamma(s) = \int_0^\infty t^{s-1} \exp(-t) dt$, $Re(s) > 0$.

**metric space:**

The metric space $(X, d)$ is a space with metric $d : X \times X \to \mathbb{R}$ (the distance) in $X$. For a metric the following must hold,

**(i)** $\forall x \in X$, $d(x, x) = 0$.

**(ii)** $\forall x, y \in X, x \neq y$, $0 < d(x, y) < \infty$.

**(iii)** $\forall x, y \in X$, $d(x, y) = d(y, x)$.

**(iv)** $\forall x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$.

**Similarities:**

Let $c < 1$ be a nonnegative real number, the function $f$ is a similarity with **ratio** $c$ if $d(f(x), f(y)) = cd(x, y)$, $\forall x, y \in X$.

**Transformation:**

A transformation is also known as a function, but imparts a geometrical flavor. A transformation is **affine** if $f(x)$ is a linear translation.

# Chapter 2

# Fractals in General

## 2.1 Introduction

Recall that according to Mandelbrot a fractal can be defined as a set for which the Hausdorff dimension (which need not be integer, see section 2.3) strictly exceeds the topological dimension. To understand what fractals are and how they behave it is useful to look at the mathematical space they live in. Furthermore it is useful to look at a definition of dimension, which gives some understanding the way fractals are built up. In this paper it is also a point of interest to see whether something useful can be said about the measure of a fractal, since their dimension in general is not an integer.

## 2.2 Metric space $(\mathcal{H}(X), h)$

Before discussing the dimension and measure of fractals it is useful to define the space where fractals live in. This turns out to be the metric space $(\mathcal{H}(X), h)$.

**Definition 2.2.1** ([3, 4]).
Let $(X, d)$ be a complete metric space, the class of non-empty closed and bounded subsets of $X$ is called $\mathcal{H}(X)$.

**Definition 2.2.2** ([1, 2, 3, 4, 9]).
Let $A, B$ be sets (in $\mathcal{H}(X)$), the **Hausdorff distance** $h(A, B)$ is defined as follows,

$$h(A, B) = \sup\{d(a, B), d(b, A) : a \in A, b \in B\},$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$.

**Property 2.2.3.**
$h(A \cup B, C \cup D) \leq \max\{h(A, C), h(B, D)\}$.

*Proof.* $h(A \cup B, C \cup D) = \sup\{d(x, C \cup D), d(y, A \cup B) : x \in A \cup B, y \in C \cup D\}$
$\leq \max\{\sup\{d(a, C), d(c, A) : a \in A, c \in C\}, \sup\{d(b, D), d(d, B) : b \in B, d \in D\}\}$
$\leq \max\{h(A, C), h(B, D)\}$. □

As the name suggest the Hausdorff distance is a metric on $\mathcal{H}(X)$ recall that this means that the following properties hold.

**Property 2.2.4.**

**(i)** $\forall A \in \mathcal{H}(X),\ h(A, A) = 0.$

**(ii)** $\forall A, B \in \mathcal{H}(X), A \neq B,\ 0 < h(A, B) < \infty.$

**(iii)** $\forall A, B \in \mathcal{H}(X),\ h(A, B) = h(B, A).$

**(iv)** $\forall A, B, C \in \mathcal{H}(X),\ h(A, B) \leq h(A, C) + h(C, B).$

The following theorem is an important theorem that will be needed later, but that will not be proven.

**Theorem 2.2.5** ([4]).
*If $(X, d)$ is a complete metric space, then the metric space $(\mathcal{H}(X), h)$ is complete.*

## 2.3 Hausdorff measure and dimension

In the world of mathematics there exists many different definitions for the dimension of some object. The following definition of dimension is the one given by Hausdorff in 1919. It is known to be a rigorous definition of dimension [8].

### 2.3.1 The definition of Hausdorff measure and dimension

For fractals Hausdorff dimension is the one used most often for proofs of the dimension of a fractal. The definition however is not suitable for numerical estimation [3, 8, 9]. The Hausdorff measure and Hausdorff dimension are defined in the following way.

**Definition 2.3.1** ([1, 2, 3, 7, 8])**.**
Let $U$ be a non-empty subset of a metric space $(X, d)$, then $|U| := \operatorname{diam}(U) := \sup\{d(x, y) : x, y \in U\}$. Furthermore the family $U_i$ is a *$\delta$-cover* of $F$ if $F \subset \bigcup_{i=1}^{\infty} U_i$ with $0 \leq |U_i| \leq \delta$. Let $F$ be a subset of $\mathbb{R}^n$ and $s$ a non-negative number. For any $\delta > 0$ define

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} |U_i|^s : U_i \text{ is a } \delta\text{-cover of } F\right\}.$$

**Definition 2.3.2** ([1, 2, 3, 7, 8])**.**
The **($s$-dimensional) Hausdorff measure** exists for any element $F$ of metric space $(X, d)$ and is denoted by $\mathcal{H}^s(F)$, where

$$\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F).$$

*Remark* 2.3.3. Note that this is well defined, because $\mathcal{H}^s_\delta(F)$ is non-decreasing with respect to $\delta$. Which means that $\lim_{\delta\to 0} \mathcal{H}^s_\delta(F)$ either exists or not, but is not possible to have multiple limiting values.

**Definition 2.3.4** ([1, 2, 3, 7, 8])**.**
The **Hausdorff dimension** $\dim_H F$ is defined as follows,

$$\dim_H F = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

*Remark* 2.3.5. If $s = \dim_H F$, then $0 < \mathcal{H}^s(F) < \infty$ [3].

### 2.3.2 Properties of the Hausdorff measure

Since $\mathcal{H}^s(F)$ is a measure (as the name implies) the following properties hold.

**Property 2.3.6** ([3, 7])**.**

*(i)* $\mathcal{H}^s(\emptyset) = 0$ *however, strictly spoken the empty set is not an element of* $\mathcal{H}(X)$.

*(ii)* $\mathcal{H}^s(F_1) \leq \mathcal{H}^s(F_2)$ *if* $F_1 \subset F_2$.

*(iii)* $\mathcal{H}^s(\bigcup_{i=1}^{\infty} F_i) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(F_i)$

With these properties the following theorems can be proved. The theorems do not hold for metric spaces in general, neither is that required in case of fractals. For example when one wants to multiply a point with a number, like mentioned in the theorem below. So from now on it is assumed that $X \subset \mathbb{R}^n$.

**Theorem 2.3.7** ([1, 3])**.**
*If* $F \in \mathcal{H}(X)$ *with Hausdorff distance* $h$*,* $\lambda > 0$ *and* $\lambda F = \{\lambda x | x \in F\}$*, then*

$$\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F).$$

*Proof.* If $U_i$ is a $\delta$-cover of $F$ then $\lambda U_i$ is a $\lambda\delta$-cover of $\lambda F$. Hence

$$\mathcal{H}^s_{\lambda\delta}(\lambda F) \leq \sum |\lambda U_i|^s = \lambda^s \sum |U_i|^s \leq \lambda^s \mathcal{H}^s_\delta(F). \tag{2.1}$$

Since this holds for any $\delta$-cover, for $\delta \to 0$ gives that $\mathcal{H}^s(\lambda F) \leq \lambda^s \mathcal{H}^s(F)$. Replacing $\lambda$ with $1/\lambda$ and $F$ by $\lambda F$ gives the required opposite inequality [3]. □

**Theorem 2.3.8** ([3]).
*Let $F \in \mathcal{H}(X)$ with Hausdorff distance $h$ and suppose that $f : F \to X$ satisfies*

$$d(f(x), f(y)) \leq cd(x, y)^{\alpha},$$

*where $x, y \in F$and $\alpha > 0$, $c > 0$ are constants then,*

$$\forall s, \mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha}\mathcal{H}^{s}(F).$$

*Proof.* If $U_i$ is a $\delta$-cover of $F$ then, since $|f(F \cap U_i)| \leq c|U_i|^{\alpha}$, it follows that $f(F \cap U_i)$ is an $\epsilon$-cover of $f(F)$, where $\epsilon = c\delta^{\alpha}$. Thus $\sum_i |f(F \cap U_i)|^{s/\alpha} \leq c^{s/\alpha} \sum_i |U_i|^s$, so that $\mathcal{H}_{\epsilon}^{s/\alpha}(f(F)) \leq c^{s/\alpha}\mathcal{H}_{\delta}^{s}(F)$. As $\delta \to 0$, so $\epsilon \to 0$ giving $\mathcal{H}^{s/\alpha}(f(F)) \leq c^{s/\alpha}\mathcal{H}^{s}(F)$ [3]. □

### 2.3.3 Normalized Hausdorff measure

For the Hausdorff measure it would be convenient if for integer dimension $s$ (see Theorem 2.4.3) the Hausdorff measure would represent the $s$-dimensional volume. It turns out that this is not the case. Whenever $s$ is an integer the Hausdorff measure has to be multiplied by a certain constant to get the same value as the $s$-dimensional volume [4, 3, 2].

**Example 2.3.9.** Let $E$ be a unit square, meaning that all edges have length 1. The 2-dimensional volume of $E$ is the area which equals to 1. For the Hausdorff measure we need to find the smallest $\delta$ cover first. To make things easier it would be sufficient at first to find a cover for $\delta = 1$. Since a circle is the greatest set that can be created with some diameter it would be useful to create a cover with circles.

First put a circle with radius 1 in the middle. Then put 4 as large as possible circles with some radius at the corners and continue to do this. The infinite set of circles will indeed cover the square.

The area of a circle is $\pi r^2$ or $(\pi/4)d$, where $r$ is the radius and $d = 2r$. Since the square is covered with $U_i$ the set of circles it holds that $1 = \sum_{U_i} \pi/4|U_i|^2 = \pi/4 \sum_{U_i} |U_i|^2 = \pi/4\mathcal{H}_{\delta}^{s}(E)$. Which means that $\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}_{\delta}^{s}(E) = 4/\pi$ instead of 1.

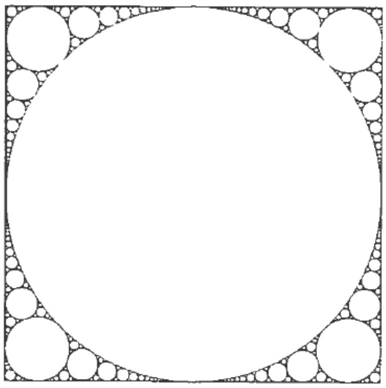


Figure 2.1: Cover of a unit square [2]

As suggested by the example it seems that the factor $\alpha_s$ needed to 'correct' the Hausdorff measure for integer dimension $s$ is equal to the volume of an $s$-dimensional ball with diameter 1. Which makes sense, because the figure with the largest possible area with given radius is a ball and not a square. The volume of an $s$-dimensional ball with radius 1/2 (meaning that the diameter is equal to 1) is equal to $2^{-n}\Gamma(1/2)^n/\Gamma(\frac{n}{2}+1)$, see Appendix A [4, 2, 3]. In this paper it is at some points better to discuss this 'corrected' Hausdorff measure, which shall be referred to as Normalized Hausdorff measure.

**Definition 2.3.10** ([2, 3, 4]).
Every cover used to find the $\delta$-cover has to be multiplied with that certain constant $\alpha_s$ which analog to Definition 2.3.1 gives us,

$${}_{\mathcal{N}}\mathcal{H}_{\delta}^{s}(F) = (\frac{2^{-s}\Gamma(1/2)^s}{\Gamma(\frac{s}{2}+1)}) \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : U_i \text{ is a } \delta\text{-cover of } F \right\}.$$

**Definition 2.3.11** ([2, 3, 4]).
The **normalized (s-dimensional) Hausdorff measure** exists for any $F$ element of $(\mathcal{H}(X), h)$ and is denoted by ${}_{\mathcal{N}}\mathcal{H}^s(F)$, where

$$ {}_{\mathcal{N}}\mathcal{H}^s(F) = \lim_{\delta \to 0} {}_{\mathcal{N}}\mathcal{H}^s_\delta(F). $$

## 2.4 Properties of the Hausdorff dimension

To find the Hausdorff dimension and prove correctness for fractals it might be nice to know something about the Hausdorff dimension of the union of subsets of that fractal. Proofs can be simplified by using the following theorem that is immediate from Property 2.3.6 (ii) and therefore needs no proof.

**Theorem 2.4.1** ([1, 3]).
$\dim_H F_1 \leq \dim_H F_2$, *if* $F_1 \subset F_2$

The theorem for the Hausdorff dimension of the union is the following.

**Theorem 2.4.2** ([3]).
$\dim_H \bigcup_{i=1}^{\infty} F_i = \sup_{1 \leq i < \infty} \{\dim_H F_i\}$.

*Proof.* Define $F = \bigcup_{i=1}^{\infty} F_i$. To proof the theorem two statements will be proven.

$$ \dim_H F \geq \sup_i \{\dim_H F_i\} $$
$$ \dim_H F \leq \sup_i \{\dim_H F_i\} $$

Where the first statement is easy to proof since by Theorem 2.4.1 it is known that $\dim_H F \geq \dim_H F_i, \forall i$ so that $\dim_H F \geq \sup_i \{\dim_H F_i\}$.

Assume that for $F_i$ it holds that $s > \dim_H F_i$. then it is known that $\forall \epsilon_i > 0\ \mathcal{H}^s(F_i) < \epsilon_i$. Since the statement holds for any $\epsilon_i > 0$ it will also hold for $\epsilon_i = \frac{\epsilon}{2^i}$, $\epsilon > 0$. By Property 2.3.6 (iii) it is known that $\mathcal{H}^s(F) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(F_i)$. Which means that $\mathcal{H}^s(F) \leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon$. Since this holds for every $s > \dim_H F_i$ it follows that $\dim_H F \leq \sup_i \{\dim_H F_i\}$.

Since $\dim_H F \geq \sup_i \{\dim_H F_i\}$ and $\dim_H F \leq \sup_i \{\dim_H F_i\}$ this means that $\dim_H F = \sup_i \{\dim_H F_i\}$. ☐

For a definition of dimension one would expect that the dimension of a straight line is still 1 and that the dimension of a cube is still 3. This turns out to be true. More generally one can state the following theorem.

**Theorem 2.4.3** ([1, 2, 3]).
$\dim_H \mathbb{R}^n = n$.

In this paper no proof for the given theorem is presented.

**Theorem 2.4.4** ([3]).
*Let* $F \in \mathcal{H}(X)$ *with metric* $h$ *and suppose that* $f : F \to X$ *satisfies*

$$ d(f(x), f(y)) \leq c d(x, y)^\alpha $$

*where* $x, y \in F$*and* $\alpha > 0$, $c > 0$ *are constants* $\Longrightarrow \dim_H f(F) \leq \frac{1}{\alpha} \dim_H F$.

*Proof.* If $s > \dim_H F$ then by Theorem 2.3.8 it is known that $\mathcal{H}^{s/\alpha}(f(F)) \leq c^s/\alpha \mathcal{H}^s(F) = 0$. Which implies that $\dim_H f(F) \leq s/\alpha$ for all $s > \dim_H F$ [3]. ☐

## 2.5 Natural-measure for fractals

In classical geometry it is possible to say something about the length of the object if the dimension of that object is one. Whenever the dimension of an object is two, it is possible to say something about the area of a that object. However for fractals it is hard to say something useful about this. One might think that the length of a fractal is infinite anyway, so that there is no use to look at a measure anyway. However for the $\phi$-Koch curve (see example 3.1.7) the dimension changes with the choice of $\phi$. This on itself is not so interesting, but for $\phi = 0$ the fractal is a straight line with finite length and for $\phi = \pi/2$ the fractal is a triangle with finite area. This might imply that there is some way to measure fractals with respect to their dimension, which will be referred to as the Natural-measure.

For a Natural-measure $Nm$ the following properties are demanded in this paper.

- The Natural-measure may depend on some parameter $x$ of some fractal family. For example for every $\phi$ of the $\phi$-Koch curve there is a different Natural-measure.
- $0 < Nm(x) < \infty$, if bounded.
- when a figure has integer dimension $n$, $Nm(x)$ should give the expected $n$-dimensional volume.

This paper restricts the problem to self-similar fractals. Further information and details on this subject can be found in section 3.4

# Chapter 3

# Self-similar fractals

## 3.1 Introduction

In the previous chapter some things are discussed about fractals in general. It is possible to create some classes of fractals. Not all fractals fit in some class, but for this paper it does not matter. In this chapter only so called self-similar fractals are discussed. It will turn out that the property of being self-similar will lead to very nice behavior of the fractals, which will help to find the dimension and measure.

To discuss self-similarity and other properties of fractals a definition of invariance is needed.

**Definition 3.1.1** ([3, 4]).
Let $\mathcal{S} = \{S_1, S_2, ..., S_N\}$ be a finite set of contraction maps on $\mathbb{R}^n$. The compact set $K \subset \mathbb{R}^n$ is said to be **invariant** with respect to $\mathcal{S}$ if

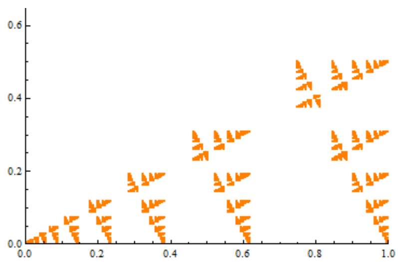
$$K = \bigcup_{i=1}^{N} S_i(K).$$

Conditions on the class of contraction maps will define what class of fractal the set $F$ may belong to.

**Definition 3.1.2** ([2, 9]).
If the set $K$ is invariant under a set of similarities, then $K$ is called **self-similar** (see also Example 3.1.3).

**Example 3.1.3.**

$$\begin{aligned} S_1(\underline{x}) &= 0.6\underline{x} \\ S_2(\underline{x}) &= 0.5\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\underline{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$



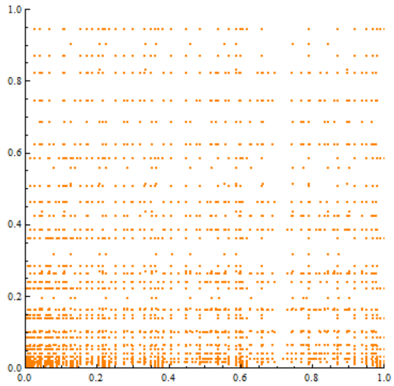

**Definition 3.1.4** ([2, 9]).
If the set $K$ is invariant under a set of affine transformations, then $K$ is called **self-affine**.

Note that by definition self-similar fractals are also self-affine, but self-affine fractals need not be self-similar (see also Example 3.1.5).

**Example 3.1.5.**

$$S_1(\underline{x}) = 0.618 * \underline{x}$$

$$S_2(\underline{x}) = 4 * \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \underline{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$




**Definition 3.1.6** ([9])**.**
If the set $K$ is invariant under a set of similarities with equal ratio, then $K$ is called **strictly self-similar** (see also Example 3.1.7).

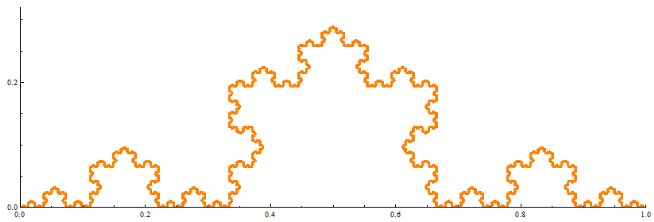
**Example 3.1.7.**

$$S_1(\underline{x}):\ 1/(2 + 2\cos\tfrac{\pi}{3})\underline{x}$$

$$S_2(\underline{x}):\ 1/(2 + 2\cos\tfrac{\pi}{3})\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}(\underline{x})\right)$$

$$S_3(\underline{x}):\ 1/(2 + 2\cos\tfrac{\pi}{3})\left(\begin{pmatrix} 1 + \cos\frac{\pi}{3} \\ \sin\frac{\pi}{3} \end{pmatrix} + \begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}(\underline{x})\right)$$

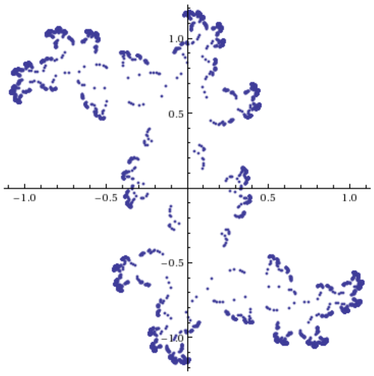
$$S_4(\underline{x}):\ 1/(2 + 2\cos\tfrac{\pi}{3})\begin{pmatrix} 1 + 2\cos\frac{\pi}{3} \\ 0 \end{pmatrix} + (\underline{x})$$




There are famous fractals not belonging to one of the classes above. A Julia set is for example a conformal transformation.

**Definition 3.1.8** ([2])**.**
If the set $K$ is invariant under a set of conformal transformations (so that derivatives $F'_i(x)$ are similarities for all $i$ and $x$), then $K$ is called **self-conformal** (see also Example 3.1.9).

**Example 3.1.9.**



## 3.2 Characterization of self-similar fractals

As stated before, you can view fractals as elements of the metric space $(\mathcal{H}(X), h)$. Note also that the definition of self-similarity suggests that there is a relation between contraction maps (a similarity is also a contraction map) and (strictly) self-similar fractals. A theorem that is of use to examine this possibility is the Banach fixed point theorem.

**Theorem 3.2.1.** ***(Banach fixed point theorem)***
*Let $(X, d)$ be a complete metric space and $f : X \to X$ be a contraction map. Then $f$ admits exactly one fixed point $x^* \in X$, which means $f(x^*) = x^*$. Furthermore this point is the limit of the sequence $x_n$, where $x_n = f(x_{n-1})$, starting with an arbitrary $x_0$.*

This theorem strengthens the intuition that there is a possibility to describe self-similar fractals with contraction maps. Even more so, this theorem states that every finite set of contraction maps might lead to a unique fractal. This turns out to be correct and the exact theorem is the following.

**Theorem 3.2.2** ([3, 4])**.**
*Let $S_1, ..., S_N$ be contraction maps on $D \subset \mathbb{R}^n$. Then there exists a unique non-empty compact set $F$ that is invariant for the $S_i$, i.e. which satisfies*

$$F = \bigcup_{i=1}^{N} S_i(F).$$

*Moreover, let a transformation $S$ be defined on $\mathcal{H}(X)$ by*

$$S(E) = \bigcup_{i=1}^{N} S_i(E).$$

*Let the $k$-th iteration be defined inductively by $S^0(E) = E$ and $S^k(E) = S(S^{k-1}(E))$ for $k \geq 1$, then*

$$F = \bigcap_{k=1}^{\infty} S^k(E)$$

*for any set $E$ in $\mathcal{H}(X)$ such that $S_i(E) \subset E$ for each $i$.*

*Proof.* Note that sets in $\mathcal{H}(X)$ are transformed by $S_i$ into other sets in $\mathcal{H}(X)$. Let $E$ be any set in $\mathcal{H}(X)$ such that $S_i(E) \subset E$ for all $i$. Then $S^k(E) \subset S^{k-1}(E)$ so $S^k(E)$ is a decreasing sequence of non-empty compact sets. It is known that $F = \bigcap_{k=1}^{\infty} S^k(E) \neq \emptyset$. Since $S^k(E)$ is a decreasing sequence it follows that $S(F) = F$.

Next thing to show is uniqueness of $F$ which follows from Property 2.2.3. Assume that sets $A, B \in \mathcal{H}(X)$ then,

$$h(S(A), S(B)) = h(\bigcup_{i=1}^{N} S_i(A), \bigcup_{i=1}^{N} S_i(B)) \leq \max_{i=1...m} \{h(S_i(A), S_i(B))\}.$$

This leads to the conclusion that,

$$h(S(A), S(B)) \leq \max_{i=1...m} \{c_i\} \, h(A, B),$$

where $c_i$ is the contraction factor of $S_i$. Furthermore if $S(A) = A$ and $S(B) = B$ then it follows that the equation only holds whenever $h(A, B) = 0$, which implies $A = B$ [3]. ∎

Note that from the theorem and the proof above it is clear that the following holds as well.

**Theorem 3.2.3** ([4]).
*Let $S_1, ..., S_N$ be contraction maps on $D \subset \mathbb{R}^n$ and $F$ be the unique non-empty compact set that is invariant for the $S_i$. Let a transformation $S$ be defined on $\mathcal{H}(X)$ on a set $E$ by $S(E) = \bigcup_{i=1}^{N} S_i(E)$. Let the $k$-th iteration be defined inductively by $S^0(E) = E$ and $S^k(E) = S(S^{k-1}(E))$ for $k \geq 1$, then for any non-empty bounded set $A \in \mathcal{H}(X)$ it holds that,*

$$S^k(A) \overset{k \to \infty}{\longrightarrow} F.$$

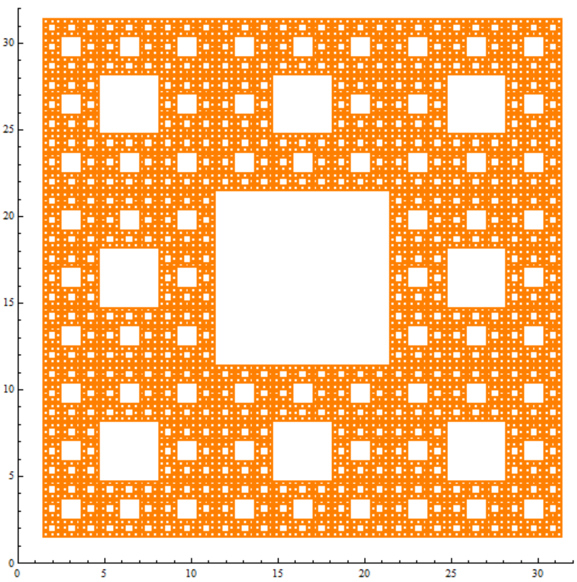
With the above theorem it is now possible to construct various strictly self-similar fractals. According to the first part of this theorem it would be sufficient to start with a point in $\mathbb{R}^n$ and apply the contraction maps iteratively on that point to create a fractal. To do this some examples are created with a program written in Mathematica.

### 3.2.1 Mathematica scripts

In this section some examples are shown on how a fractal can be drawn with use of the program Mathematica. The commands and the output are given for some well-known fractals. There are also some fractals included that were found while experimenting with transformation functions that looked nice enough to be shown here as well.

Listing 3.1: Sierpinski Carpet

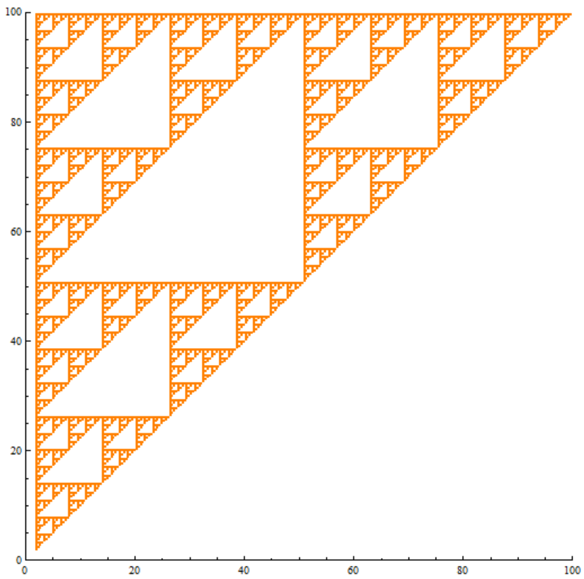
```
(* Sierpinski Carpet *)
f1 = Function[z, 1/3*z + 1 + I];
f2 = Function[z, 1/3*z + 1 + 11 I];
f3 = Function[z, 1/3*z + 1 + 21 I];
f4 = Function[z, 1/3*z + 11 + I];
f5 = Function[z, 1/3*z + 21 + I];
f6 = Function[z, 1/3*z + 21 + 11 I];
f7 = Function[z, 1/3*z + 11 + 21 I];
f8 = Function[z, 1/3*z + 21 + 21 I];
S = {f1[0], f2[0], f3[0], f4[0], f5[0], f6[0], f7[0], f8[0]};
S1 = S;
For[i = 1, i < 5, i++,
 S1 = Join[Map[f1, S1], Map[f2, S1], Map[f3, S1], Map[f4, S1],
   Map[f5, S1], Map[f6, S1], Map[f7, S1], Map[f8, S1]]
 ]
S1;
ListPlot[Table[{N[Re[p]], N[Im[p]]}, {p, S1}],
 PlotStyle -> Directive[PointSize[Small], Orange]]
```




Listing 3.2: Sierpinski Triangle

```
(*Sierpinski triangle *)
f1 = Function[z, 1/2*z + 1 + I];
f2 = Function[z, 1/2*z + 1 + 50 I];
```

```
f3 = Function[z, 1/2*z + 50 + 50 I];
S = {f1[0], f2[0], f3[0]};
S1 = S;
For[i = 1, i < 7,
 i++, (*strange I need one iteration less to get the same result as \
Maxim*)
 S1 = Join[Map[f1, S1], Map[f2, S1], Map[f3, S1]]
 ]
S1;
ListPlot[Table[{N[Re[p]], N[Im[p]]}, {p, S1}],
 PlotStyle -> Directive[PointSize[Small], Orange]]
```



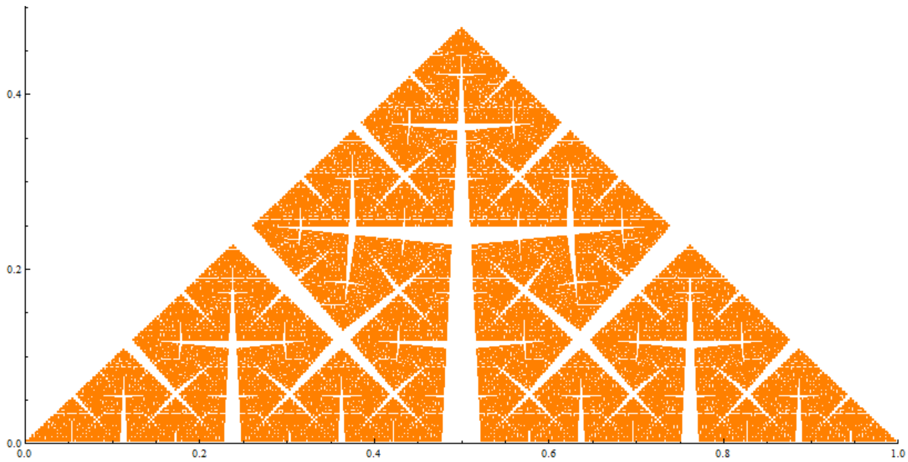




Figure 3.1: Koch curve, with $\cos\phi = 0.05$

Listing 3.3: $\phi$-Koch Curve

```
(*Koch Curve*)
x = 0.3 Pi;
radius[y_] = 1/(4*Sin[y]*(1 + Cos[y]));
cos[y_] := N[Cos[y]];
sin[y_] := N[Sin[y]];
contraction[y_] = (1/(2 + 2*Cos[y])); (*contraction factor*)

(*Description of Koch fractal*)
s1 = Function[z, 1/(2 + 2 cos[x])*z];
s2 = Function[z,
   1/(2 + 2 cos[x]) ({1,
```

```
        0} + {{cos[x], -sin[x]}, {sin[x], cos[x]}}.z)];
s3 = Function[z,
   1/(2 + 2 cos[x]) ({1 + cos[x],
        sin[x]} + {{cos[x], sin[x]}, {-sin[x], cos[x]}}.z)];
s4 = Function[z, 1/(2 + 2 cos[x]) ({1 + 2 cos[x], 0} + z)];
approx = {{1, 0}};
For[i = 1, i < 9, i++,
  newapprox =
   Join[Table[s1[p], {p, approx}], Table[s2[p], {p, approx}],
    Table[s3[p], {p, approx}], Table[s4[p], {p, approx}]];
  approx = newapprox;
  ];

KochCurve =
 ListPlot[Table[N[p], {p, approx}],
  PlotRange -> {{0, 1}, {0, sin[x]* 1/(2 + 2*Cos[x])}},
  AspectRatio -> Automatic,
  PlotStyle -> Directive[PointSize[Small], Orange]]
```

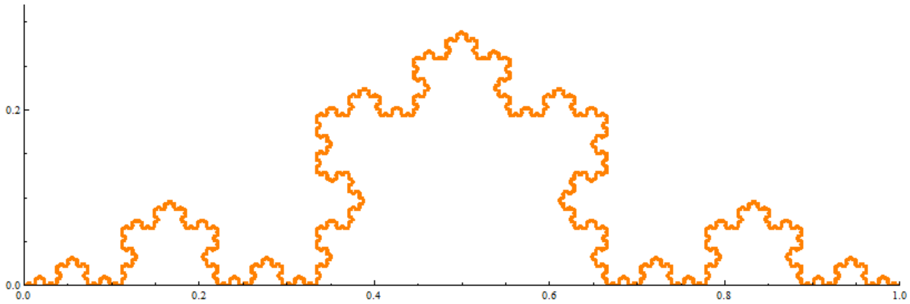
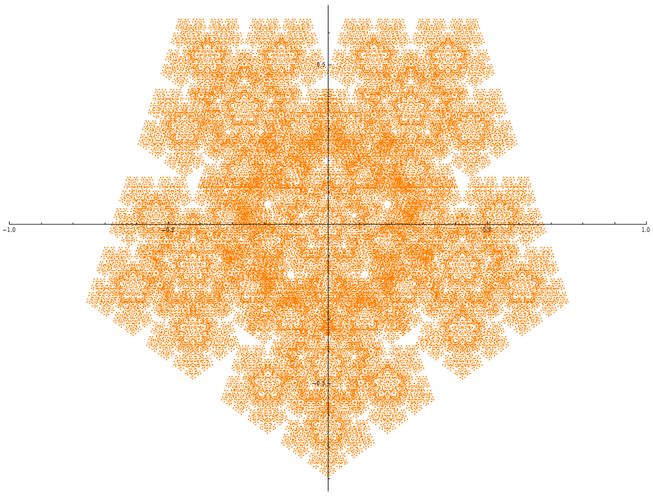



Figure 3.2: Koch curve, with $\cos\phi = 0.5$

Listing 3.4: Try out Result

```
c = 2 *Pi/5; d = 4/9;
(*nice parameter choices:
c=2 *Pi/5; d = 1/3; c=2 *Pi/5; d = 5/8; c=2 *Pi/5; d = 5/12; c=2 *Pi/5; d = 4/9;
c=2 *Pi/4; d = 5/12; c=2 *Pi/3; d = 5/12; c=2 *Pi/54; d = 1/2;
*)
f1 = Function[z, d*z];
f2 = Function[z, d*(z - I)];
f3 = Function[z, d*((Cos[c]*Re[z - I] - Sin[c]*Im[z - I]) +
        (Sin[c]*Re[z - I] + Cos[c]*Im[z - I])*I)];
f4 = Function[z, d*((Cos[2 c]*Re[z - I] - Sin[2 c]*Im[z - I]) +
        (Sin[2 c]*Re[z - I] + Cos[2 c]*Im[z - I])*I)];
f5 = Function[z, d*((Cos[3 c]*Re[z - I] - Sin[3 c]*Im[z - I]) +
        (Sin[3 c]*Re[z - I] + Cos[3 c]*Im[z - I])*I)];
f6 = Function[z, d*((Cos[4 c]*Re[z - I] - Sin[4 c]*Im[z - I]) +
        (Sin[4 c]*Re[z - I] + Cos[4 c]*Im[z - I])*I)];
S = {f1[0], f2[0], f3[0], f4[0], f5[0], f6[0]};
For[i = 1, i < 6, i++,
  S1 = Join[Table[f1[p], {p, S}], Table[f2[p], {p, S}],
    Table[f3[p], {p, S}], Table[f4[p], {p, S}], Table[f5[p], {p, S}],
    Table[f6[p], {p, S}]];
  S = S1;
  ];
S1;
ListPlot[Table[{N[Re[p]], N[Im[p]]}, {p, S1}],
 PlotStyle -> Directive[PointSize[Small], Orange]]
```



## 3.3 Dimension of strictly self-similar fractals

**Definition 3.3.1.** Let $S_1, ..., S_N$ be similarities on $\mathbb{R}^n$ under which $F$ is invariant, and $S_i$ has contraction factor $c_i$. Then the **Similarity dimension** $\dim_s F = s$, where $s$ is the solution of $\sum_{i=1}^{N} c_i^s = 1$.

*Remark* 3.3.2. Note that $\sum_{i=1}^{N} c_i^s = 1$ has a unique solution, because it is a monotone increasing function.

**Example 3.3.3.**
Let $F$ be the $\phi$-Koch curve as described in Listing 3.3
Starting with a straight line the $\phi$-Koch curve is created as follows,

- take the figure of the last iteration shrink the size with a factor $\frac{1}{2+2\cos\phi}$ and place it at the left most position.
- take the figure of the last iteration shrink the size with a factor $\frac{1}{2+2\cos\phi}$ angle it with $\phi$ radians at the righthand side of the already placed figure.
- take the figure of the last iteration shrink the size with a factor $\frac{1}{2+2\cos\phi}$ angle it with $\pi+\phi$ radians at the righthand side of the already placed figure.
- take the figure of the last iteration shrink the size with a factor $\frac{1}{2+2\cos\phi}$ and place it at the left most position.

In short this means four similarities were used each with ratio $\frac{1}{2+2\cos\phi}$. According to the above definition the following has to be solved,

$$\sum_{i=1}^{N} c_i^s = 1 \Longrightarrow 4\left(\frac{1}{2+2\cos\phi}\right)^s = 1 \Longrightarrow \dim_s F = \frac{\log 4}{\log(2+2\cos\phi)}.$$

Next it would be interesting to know when the similarity dimension is equal to the Hausdorff dimension. It would be convenient if these two dimensions are equal for strictly self-similar fractals. Unfortunately one extra condition is necessarily, namely the open set condition.

**Definition 3.3.4** ([1, 2, 3, 4])**.**
The family of contraction maps $S_i$ satisfies the **open set condition** if there exists an non-empty bounded set V such that $\bigcup_{i=1}^{N} S_i(V) \subset V$ and $S_i(V) \cap S_j(V) = \emptyset$ if $i \neq j$. )

With this extra condition it is possible to state when the Hausdorff dimension is equal to the Similarity dimension.

**Theorem 3.3.5** ([1, 2, 3, 4])**.**
*Suppose that the open set condition holds for similarities $S_i \subset \mathbb{R}^n$. If $F$ is the set invariant under $S_1, ..., S_N$, then $\dim_H F = \dim_s F = s$. Moreover, for this value $s$ it holds that $0 < \mathcal{H}^s(F) < \infty$.*

The proof for the above theorem can be found in [3, 4].

It seems that the open set condition is really necessarily to find the right dimension.

**Example 3.3.6.**
Let $S$ be the set of the following three similarities,

$S_1(x)$: $\frac{1}{2}x$

$S_2(x)$: $\frac{1}{2}x + 1$

$S_3(x)$: $\frac{1}{2}x + 2$.

The created figure $F$ is the interval $[0, 4)$. Let $V$ be the interval $[0, 4)$, then it holds that $V$ is the smallest bounded set such that $\bigcup_{i=1}^{3} S_i(V) \subset V$. And $S_1(V) = [0, 2)$, $S_2(V) = [1, 3)$ and $S_3(V) = [2, 4)$, so $S_i(V) \cap S_j(V) \neq \emptyset$ if $i \neq j$. Which means that the open set condition does not hold for $S$. Note however that the open set condition holds for $S_{13} = \{S_1, S_3\}$.

Note that $F$ is an interval and therefore Hausdorff dimension 1 is expected. The similarity dimension with respect to the set $S$ is equal to the solution of $s$ for $3(1/2)^s = 1$, so $s = \frac{\log 3}{\log 2} > 1$. The similarity dimension with respect to the set $S_{13}$ is equal to the solution of $s_{13}$ for $2(1/2)^{s_{13}} = 1$, so $s_{13} = \frac{\log 2}{\log 2} = 1$. Which shows that if the open set condition does not hold the similarity dimension may give wrong (too high) values.

## 3.4 Natural-measure for self-similar fractals

By Theorem 3.3.5 it is known that if $F$ is the set invariant under similarities $S_1, ..., S_N$, where the similarities meet the open set condition, that $0 < \mathcal{H}^s(F) < \infty$. Since this measure is also used to find the Hausdorff dimension, one might expect that the measure can be used as a Natural-measure. Note that this is particular interesting for self-similar fractals, since it is possible to find the Hausdorff dimension without use of the Hausdorff measure. Otherwise it might be that finding Hausdorff dimension and measure would end up in some sort of goose hunt.

Recall that in section 2.5 it was demanded that when the figure has integer dimension $n$, $Nm(x)$ should give the expected $n$-dimensional volume. As shown in section 2.3.3 that is not the case for the Hausdorff measure, but it is true for the Normalized Hausdorff measure. Since the only difference between the Hausdorff measure ($\mathcal{H}^s(F)$) and the Normalized Hausdorff measure (${}_{\mathcal{N}}\mathcal{H}^s(F)$) is a multiplication with a constant it still holds that if $F$ is the set invariant under similarities $S_1, ..., S_N$, where the similarities meet the open set condition, it is known that $0 < {}_{\mathcal{N}}\mathcal{H}^s(F) < \infty$.

The Hausdorff measure is defined as a limit over $\delta$. However as the name suggests a strictly self-similar fractal is self-similar after scaling with contraction factor $c$, which leads to the thought that $\mathcal{H}^s_\delta(F)$ is equivalent to $\mathcal{H}^s_{c^k\delta}(F)$ for all $k \in \mathbb{N}$. It turns out that the following theorem holds.

**Theorem 3.4.1** ([10])**.**
*Suppose $F$ is self-similar, $\delta > 0$ then $\mathcal{H}^s(F) = \mathcal{H}^s_\delta(F)$.*

*Proof.* Let $a = |U_i, i \geq 0|$ be a $\delta$-covering of $F$ with similarity dimension $s$. According to the definition it holds that $\mathcal{H}^s_\delta(F) \leq \sum_{i=0} \infty |U_i|^s$. Recall that a transformation $S$ is defined on $\mathcal{H}^s(F)$ by $S(E) = \bigcup_{i=1}^{N} S_i(E)$. Let $J_k$ denote the set of all sequences $(j_1...j_k)$. Let $j_l$ be a permutation of $(1, ..., N)$ and $S^k(E) = \bigcup S_{j_1}(S_{j_2}(...(S_{j_k}(E))...)$ for the $k$-th iteration. Where the $S_i$ have contraction factor $c_i$. Since $s$ is the similarity dimension it holds that, $\sum_{i=0}^{N} c_i^s = 1$.

Now $\mathcal{H}^s_\delta(F) \leq \sum_{(j_i...j_k)\in J_k} \sum_{i=0}^{\infty} c_{j_1}^s \cdots c_{j_k}^s |U_i|^s = \sum_{i=0}^{\infty} \sum_{(j_i...j_k)\in J_k} c_{j_1}^s \cdots c_{j_k}^s |U_i|^s = \sum_{i=0}^{\infty} (\sum_{j_1} c_{j_1}^s) \cdots (\sum_{j_k} c_{j_k}^s) |U_i|^s = \sum_{i=1}^{\infty} |U_i|^s$.

Now the following can be concluded. Suppose a $\delta$-cover is found with Hausdorff measure $\mathcal{H}^s_\delta(F)$. Let $\epsilon < \delta$, then there exists an $\epsilon$-cover such that $\mathcal{H}^s_\delta(F) = \mathcal{H}^s_\epsilon(F)$. However the Hausdorff measure is by definition the infimum, so $\mathcal{H}^s_\delta(F) \leq \mathcal{H}^s_\epsilon(F)$. Since $\epsilon < \delta$ it is by definition of the Hausdorff measure known that $\mathcal{H}^s_\delta(F) \geq \mathcal{H}^s_\epsilon(F)$. Which means that $\mathcal{H}^s_\delta(F) = \mathcal{H}^s_\epsilon(F)$. Since $\mathcal{H}^s(F) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(F)$ it is known that $\lim_{\epsilon\to 0} \mathcal{H}^s_\epsilon(F) = \mathcal{H}^s(F)$ is well defined. So $\lim_{\epsilon\to 0} \mathcal{H}^s_\delta(F) = \lim_{\epsilon\to 0} \mathcal{H}^s_\epsilon(F)$ gives $\mathcal{H}^s_\delta(F) = \mathcal{H}^s(F)$. □

This theorem is important, as it enables one to actually make estimates for the Hausdorff measure from concrete covers.

The key to find the Hausdorff measure will be to find a clever $\delta$-cover. This turns out to be really hard so for this paper the research is restricted to the $\phi$-Koch curve.

## 3.5 Natural-measure for the $\phi$-Koch curve

First thing to do is to check whether the open set condition holds for the $\phi$-Koch curve. The $\phi$-Koch curve is given by the following set of functions, where input $x$ is a 2 dimensional vector.

$S_1(\underline{x})$**:** $1/(2+2\cos\phi)\underline{x}$

$S_2(\underline{x})$**:** $1/(2+2\cos\phi)\left(\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}(\underline{x})\right)$

$S_3(\underline{x})$**:** $1/(2+2\cos\phi)\left(\begin{pmatrix}1+\cos\phi\\ \sin\phi\end{pmatrix}+\begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}(\underline{x})\right)$

$S_4(\underline{x})$**:** $1/(2+2\cos\phi)\begin{pmatrix}1+2\cos\phi\\ 0\end{pmatrix}+(\underline{x})$

**Theorem 3.5.1.** *The open set condition (see also Definition 3.3.4) holds for the $\phi$-Koch curve.*

*Proof.* For the proof it is first needed to choose some set $T$ such that $\bigcup_{i=1}^{4} S_i(T)\subset T$. A nice choice for $T$ would be a set that is symmetric around $x=\frac{1}{2}$. Then due to the same symmetry of the fractal it is enough to show that $S_1(T)\subset T$ and $S_2(T)\subset T$. Suppose $T$ is the triangle with where boundary of $T$ is given by,

$T_1(x)$**:** $0$ if $x\in[0,1]$

$T_2(x)$**:** $2\sin\phi * \frac{1}{2+2\cos\phi}x$ if $x\in[0,1/2]$

$T_3(x)$**:** $2\sin\phi * \frac{1}{2+2\cos\phi}(1-x)$ if $x\in[1/2,1]$

Since the functions $S_1$, $S_2$, $S_3$ and $S4$ only consists of rotating and resizing, the triangle properties of $T$ remain after applying the functions. Which means that to show that $\bigcup_{i=1}^{2} S_i(T)\subset T$ it is sufficient to proof this only for the corner points of $T$. If the corner points of $T$ remain in $T$ after applying one of the $S_i$ than it is known that $T$ remains in $T$ after applying one of the $S_i$. The corner points of T are $\{0,0\}$, $\left\{1/2, \sin\phi * \frac{1}{2+2\cos\phi}\right\}$, $\{1,0\}$.

$S_1(T)$

- $S_1(\{0,0\})=\{0,0\}\in T$.
- $S_1(\left\{1/2,\sin\phi\cdot\frac{1}{2+2\cos\phi}\right\})=\left\{1/2\cdot\frac{1}{2+2\cos\phi},\sin\phi\cdot\left(\frac{1}{2+2\cos\phi}\right)^2\right\}$. Note that $T_2(1/2\cdot\frac{1}{2+2\cos\phi})=\sin\phi\cdot\left(\frac{1}{2+2\cos\phi}\right)^2$, so $\left\{1/2\cdot\frac{1}{2+2\cos\phi},\sin\phi\cdot\left(\frac{1}{2+2\cos\phi}\right)^2\right\}\in T$.
- $S_1(\{1,0\})=\left\{\frac{1}{2+2\cos\phi},0\right\}\in T$.

$S_2(T)$

- $S_2(\{0,0\})=\left\{\frac{1}{2+2\cos\phi},0\right\}\in T$.
- $S_2(\left\{1/2,\sin\phi\cdot\frac{1}{2+2\cos\phi}\right\})=$ $\frac{1}{2+2\cos\phi}\cdot\left\{1+1/2\cos\phi-\sin\phi^2\cdot\frac{1}{2+2\cos\phi},1/2\sin\phi+\cos\phi\cdot\sin\phi\cdot\left(\frac{1}{2+2\cos\phi}\right)\right\}$. $T_2(\frac{1}{2+2\cos\phi}*\Big(1+1/2\cos\phi-\sin$ $2\sin\phi\left(\frac{1}{2+2\cos\phi}\right)^2\left(1+1/2\sin\phi-\sin\phi^2\cdot\left(\frac{1}{2+2\cos\phi}\right)\right)=$ $1/2\sin\phi\frac{1}{2+2\cos\phi}\left(\frac{4}{2+2\cos\phi}-4\sin\phi^2\left(\frac{1}{2+2\cos\phi}\right)\right)+\cos\phi\sin\phi\left(\frac{1}{2+2\cos\phi}\right)^2>$ $1/2\sin\phi+\cos\phi\cdot\sin\phi\cdot\left(\frac{1}{2+2\cos\phi}\right)$, for this to be true it is needed that $\frac{4}{2+2\cos\phi}-4\sin\phi^2\frac{1}{2+2\cos\phi}>1$ so it must hold that $1-\frac{\sin\phi^2}{2+2\cos\phi}>\frac{2+2\cos\phi}{4}$. Which indeeds holds, because $\frac{2+2\cos\phi}{4}+\frac{\sin\phi^2}{2+2\cos\phi}=\frac{(2+2\cos\phi)^2+4\sin\phi^2}{4(2+2\cos\phi)}<\frac{4+8\cos\phi+2\cos\phi^2+4\sin\phi^2+(2\cos\phi^2-2\cos\phi^2)}{8+8\cos\phi}=\frac{8+8\cos\phi-2\cos\phi}{8+8\cos\phi}<1$ Now it is proven that $S_2(\left\{1/2,\sin\phi\cdot\frac{1}{2+2\cos\phi}\right\})\in T$.
- $S_2(\{1,0\})=\left\{1/2,\sin\phi\cdot\frac{1}{2+2\cos\phi}\right\}\in T$

Next to proof is the property $S_i(T) \cap S_j(T) = \emptyset$ if $i \neq j$. This however is not the case since only the corner points of the triangle coincide. This can easily solved as follows suppose $V$ is the triangle with where boundary of $V$ is given by,

$V_1$**:** $y = 0$ if $x \in [0, 1)$

$V_2$**:** $y = 2\sin\phi \cdot \frac{1}{2+2\cos\phi} \cdot x$ if $x \in [0, 1/2)$

$V_3$**:** $y = 2\sin\phi \cdot \frac{1}{2+2\cos\phi} \cdot (1 - x)$ if $x \in [1/2, 1)$

Now it is proven that the open-set condition holds for the $\phi$-Koch curve. □

If the Normalized Hausdorff measure turns out to be the Natural-measure for the $\phi$-Koch curve, the Normalized Hausdorff measure should be 1 if $\phi \to 0$ (dimension = 1) and 1/4 if $\phi = \pi/2$ (dimension = 2).

*Remark* 3.5.2. For describing sets in a cover with shapes as filled triangles and filled circles they will be referred to as triangles and circles.

The $\phi$-Koch curve is described in Listing 3.3 which is based on a set of points. However to find the $\delta$ cover it is more convenient to start with a triangle. Then it is clear that at first step that the set can best be covered with a set with diameter 1 and the Hausdorff measure (not Normalized Hausdorff measure) will be 1. Next step the triangle is replaced by triangles reduced with ratio $\frac{\log 4}{\log(2+2\cos\phi)}$, so the next cover consist of 4 triangles with diameter $\frac{\log 4}{\log(2+2\cos\phi)}$. However the measure will be $4(\frac{1}{\log(2+2\cos\phi)})^{\frac{\log 4}{\log(2+2\cos\phi)}} = 4\left(\log(2+2\cos\phi)\right)^{\frac{\log 4}{\log(2+2\cos\phi)}^{-1}} = 4{\cdot}4^{-1} = 1$. That would mean that there is no improvement, which means the triangles have to be scaled 'smarter' to find a smaller measure.

Note that for calculating the Hausdorff measure only the diameter is relevant. Since a circle is the figure with greatest covered area for a given diameter the search for a good cover can be restricted to finding smart circles instead of triangles.

### 3.5.1 Problem Model

As noted before, finding the (Normalized) Hausdorff measure is hard. To simplify the problem one can formulate restrictions on the cover. As noted above there is no use in placing 4 circles with diameter $\frac{\log 4}{\log(2+2\cos\phi)}$. Which leads to the solution to combine two pieces in one circle. If $F$ is the entire fractal then the most obvious choice would be to cover at least $S_2(F) \cup S_3(F)$ with one circle. If one such circle is found it induces a possible cover for the whole fractal. If for example there is a circle which contains only $S_2(F) \cup S_3(F)$, then half the fractal is covered. Which means that the Hausdorff measure of the whole fractal is twice the Hausdorff measure of that circle.

For finding coverings with circle $C$ for $S_2(F) \cup S_3(F)$ the following properties are used.

**Property 3.5.3.**
*If $\phi = 0$, then it should hold that ${}_{\mathcal{N}}\mathcal{H}(F) = 1$. If $\phi = \frac{\pi}{2}$, then it should hold that ${}_{\mathcal{N}}\mathcal{H}(F) = \frac{1}{4}$.*

*Proof.* Note that for the $\phi$-Koch Curve the similarity dimension is equal to the Hausdorff dimension (Theorem 3.3.5)of which Theorem 2.4.3 states that $\dim_H \mathbb{R}^n = n$. As shown in Example 3.3.3 $\dim_s F = \dim_H F = \frac{\log 4}{\log(2+2\cos\phi)}$. If $\phi = 0$ then $\dim_H F = 1$ and the Koch curve is a straight line. The length of the line is 1, so ${}_{\mathcal{N}}\mathcal{H}(F) = 1$. If $\phi = \frac{\pi}{2}$ then $\dim_H F = 2$ and the Koch curve is a filled triangle. The area of this triangle is $\frac{1}{4}$, so ${}_{\mathcal{N}}\mathcal{H}(F) = \frac{1}{4}$. □

**Property 3.5.4** (Sine rule)**.**
*Let $\alpha, \beta, \gamma, a, b, c$ and $r$ as shown in picture 3.3. The sine rule states $\frac{a}{\sin[\alpha]} = \frac{b}{\sin[\beta]} = \frac{c}{\sin[\gamma]} = 2r$. So the radius of the circumscribed circle is $\frac{a}{2\sin[\alpha]} = \frac{b}{2\sin[\beta]} = \frac{c}{2\sin[\gamma]}$.*

**Property 3.5.5.**
*If $\left\{(\frac{1}{2+2\cos\phi}, 0), (1/2, \frac{\sin\phi}{2+2\cos\phi}), (1 - \frac{1}{2+2\cos\phi}, 0)\right\} \in C$ then $S_2(F) \cup S_3(F) \in C$ for all $\phi \in (0, \frac{\pi}{2})$.*

*Proof.* Let $\phi \in (0, \frac{\pi}{2})$ and let $C$ be a circle such that $\left\{(\frac{1}{2+2\cos\phi}, 0), (1/2, \frac{\sin\phi}{2+2\cos\phi}), (1 - \frac{1}{2+2\cos\phi}, 0)\right\} \in C$. Let $C2$ be the circle such that $\left\{(0,0), (1/2, \frac{\sin\phi}{2+2\cos\phi}), (1, 0)\right\} \in C2$ as described in paragraph 3.5.2.1. By definition $C2$ covers the whole fractal. To show that $S_2(F) \cup S_3(F) \in C$ for all $\phi \in (0, \frac{\pi}{2})$ it is sufficient to show that $a = b$ in Figure 3.5.1. Which is indeed the case, because $S_2(F)$ and $S_3(F)$ are both similar to $F$. □

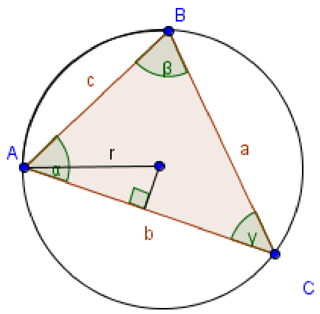
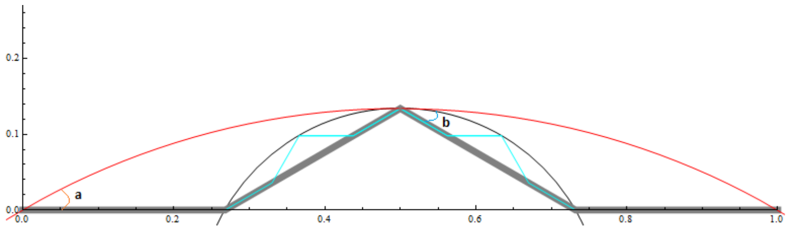



Figure 3.3: Sine rule




### 3.5.2 Possible covers

A few circles are found that cover at least $S_2(F) \cup S_3(F)$. With the program Mathematica it was easy to calculate the Normalized Hausdorff measure of each of them and to find the minimum of all. The circles will be described with names corresponding to the given Mathematica script and output.

#### 3.5.2.1 Circles $C1$ and $C2$

A simple first covering consisting of a circle that covers $S_2(F) \cup S_3(F)$ is the circle covering the whole fractal. There exist many circles that cover the whole fractal. In this paper two circles are discussed. The first circle (C1) has midpoint (1/2,0) with radius $r$ for all $\phi$. The other circle (C2) is the circle around of which the points $(0,0)$, $(1/2, \frac{\sin\phi}{2+2\cos\phi})$ and $(1,0)$ lay on the edge. To determine the radius one can use property 3.5.4. Here the $\alpha$ is known to be $2*\phi$ and $a = 1$, so the radius of the circle will be $\frac{1}{2\sin\phi}$. So C2 has midpoint $(1/2, \frac{\sin\phi}{2+2\cos\phi} - \frac{1}{2\sin\phi})$ with radius $\frac{1}{2\sin\phi}$.

#### 3.5.2.2 Circles $C3$ and $C4$

A simple guess for a circle covering $S_2(F) \cup S_3(F)$ would be circle $C3$ with midpoint $1/2, 0$ and radius such that $(1/2, \frac{\sin\phi}{2+2\cos\phi}) \in S_2(F) \cup S_3(F)$ is on the edge. That means the radius of $C3$ equals $\sin\phi/(2+2\cos\phi)$. However it turns out that for $\phi < \pi/4$ the $C3$ does not cover $S_2(F) \cup S_3(F)$.

**Property 3.5.6.**
*$S_2(F) \cup S_3(F) \subset C3$, for $\phi \geq Pi/4$.*

*Proof.* By construction of $C3$ one has $(1/2, \frac{\sin\phi}{2+2\cos\phi}) \in C3$. Due to Property 3.5.5 the only thing left to do is to proof that $(\frac{1}{2+2\cos\phi}, 0) \in C3$ and $(1 - \frac{1}{2+2\cos\phi}) \in C3$ for $\phi \leq \frac{\pi}{4}$. Due to the symmetry it is sufficient to show that $(\frac{1}{2+2\cos\phi}, 0) \in C3$ for $\phi \leq \frac{\pi}{4}$. For $\phi = \frac{\pi}{2}$ it holds that $\frac{1}{2} - \frac{\sin[\frac{\pi}{2}]}{2+2\cos[\frac{\pi}{2}]} = 0 < \frac{1}{2+2\cos[\frac{\pi}{2}]} = \frac{1}{2}$. Next to show is that the greatest $\phi < \frac{\pi}{2}$ such that $\frac{1}{2} - \frac{\sin\phi}{2+2\cos\phi} = \frac{1}{2+2\cos\phi}$ is $\phi = \frac{\pi}{4}$, this can be done with Mathematica.

```
sol = Solve[ 1/2 - Sin[y]/(2 + 2*Cos[y]) == 1/(2 + 2*Cos[y]), y, InverseFunctions -> True];
Select[ y /. sol, # > 0 &]
```

Which gives the output $\{\frac{\pi}{4}\}$.

$\square$

One could also define circle $C4$ with midpoint $1/2, 0$ and radius such that the two points $(\frac{1}{2+2\cos\phi}, 0) \in S_2(F) \cup S_3(F)$ and $(1 - \frac{1}{2+2\cos\phi}) \in S_2(F) \cup S_3(F)$ are on the edge. That means the radius of $C4$ equals $1/2 - \frac{1}{2+2\cos\phi}$. However it turns out that for $\phi > \pi/4$ the $C4$ does not cover $S_2(F) \cup S_3(F)$.

**Property 3.5.7.**
*$S_2(F) \cup S_3(F) \subset C4$, for $\phi \leq Pi/4$.*

*Proof.* By construction of $C4$ one has $(\frac{1}{2+2\cos\phi}, 0) \in C4$ and $(1 - \frac{1}{2+2\cos\phi}) \in C4$. Due to Property 3.5.5 the only thing left to do is to proof that $(1/2, \frac{\sin\phi}{2+2\cos\phi}) \in C4$ for $\phi \leq \frac{\pi}{4}$. If $\phi = 0$ then $\frac{1}{2} - \frac{1}{2+2\cos[0]} = \frac{1}{4} > 0 = \frac{\sin[0]}{2+2\cos[0]}$. Next to show is that the smallest $\phi > 0$ such that $\frac{1}{2} - \frac{1}{2+2\cos\phi} = \frac{\sin\phi}{2+2\cos\phi}$ is $\phi = \frac{\pi}{4}$, this can be done with Mathematica.

```
sol = Solve[1/2 - 1/(2 + 2*Cos[y]) == Sin[y]*1/(2 + 2*Cos[y]), y, InverseFunctions -> True];
Select[ y /. sol, # > 0 &]
```

Which gives the output $\{\frac{\pi}{4}\}$.

□

#### 3.5.2.3 Circle C5

Combining the properties of $C3$ and $C4$ once can construct a circle around the triangle formed by the points, $(\frac{1}{2+2\cos\phi}, 0)$, $(1/2, \frac{\sin\phi}{2+2\cos\phi})$ and $(1 - \frac{1}{2+2\cos\phi})$. According to property 3.5.4 the radius of $C5$ equals $\frac{1}{2\sin\phi(2+2\cos\phi)}$. Which means that $C5$ has midpoint $(1/2, \frac{\sin\phi}{2+2\cos\phi} - \frac{1}{2\sin\phi(2+2\cos\phi)})$.

**Theorem 3.5.8.** *$S_2(F) \cup S_3(F) \subset C5$ for $0 \leq \phi \leq \pi/2$*

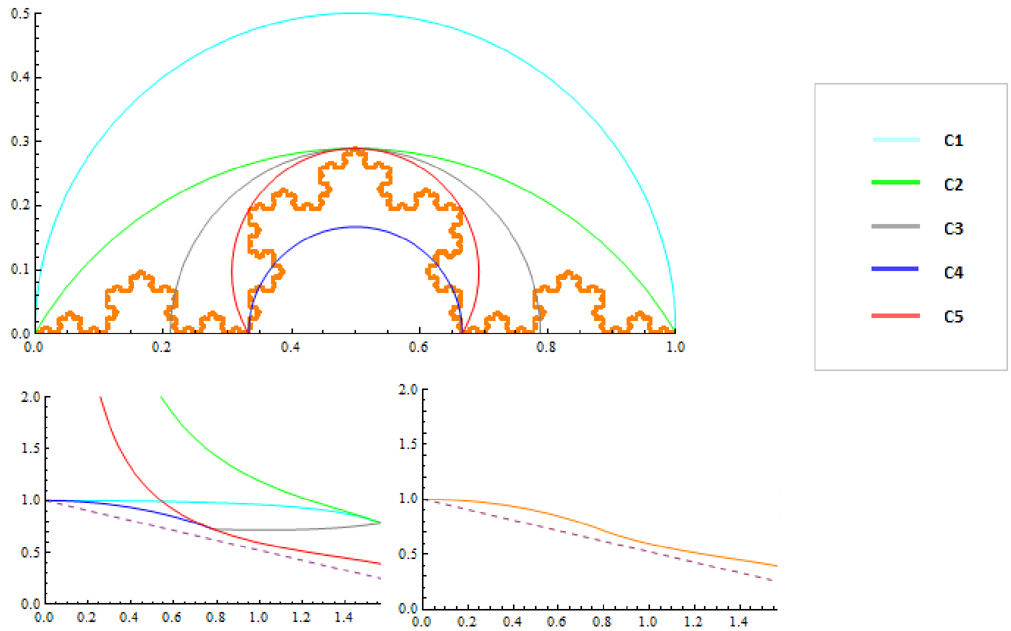
*Proof.* In property 3.5.5 this has already been proven. □

### 3.5.3 Estimating the Normalized Hausdorff measure

By definition the Normalized Hausdorff measure is the limit ($\delta \to 0$) of an infimum of $\delta$-covers, see also Definition 2.3.10 and Definition 2.3.11. By Theorem 3.4.1 it is known that for the $\phi$-Koch curve it is sufficient to find the infimum for some $\delta$. Which means that there is no need to take the limit of the covers found so far, due to self-similarity.

For the covers described above it is now possible to calculate the Normalized Hausdorff measure (with respect to these covers) as a function of $\phi$ and compare them to each other. For $C1$ and $C2$ it is easy since the cover consists of one circle with given radius. Using the definition it once can find the Normalized Hausdorff Measure. For $C4$ and $C5$ it is also easy since the circles cover half a fractal. However for $C3$ a problem occurs since it covers more than half a fractal. It is hard to determine what part of the fractal is covered and what not. To get an idea one can approximate it with the linear function $\frac{1}{\phi} \cdot \frac{\pi}{2}$. Because half the fractal is covered for $\phi = \frac{\pi}{4}$ and the whole fractal is covered for $\phi = \frac{\pi}{2}$, which indeed holds.

For the Normalized Hausdorff measure of the $\phi$-Koch curve it was demanded that the measure should be 1 if $\phi = 0$ and $\frac{1}{4}$ if $\phi = \frac{\pi}{2}$ (see also Property 3.5.3). In Mathematica one can easily plot the Normalized Hausdorff measure belonging to the covers described above. In Mathematica one could plot the minimum of all these functions for each $\phi$ to get an estimation for the Normalized Hausdorff measure (see Section B.3 for Mathematica input). The results are shown in in Figure 3.4. In this figure the colours of the plots correspond with the colour of the cover. Meaning that the red line corresponds to the Hausdorff Measure of the red coloured $C5$.




| $\frac{\pi}{2}$ | $\frac{\pi}{3}$ | $\frac{\pi}{4}$ | $\frac{\pi}{5}$ | $\frac{\pi}{6}$ | $\frac{\pi}{7}$ | $\frac{\pi}{8}$ | $\frac{\pi}{9}$ | $\frac{\pi}{10}$ |
|---|---|---|---|---|---|---|---|---|
| 0.392699 | 0.573634 | 0.724903 | 0.83033 | 0.884651 | 0.916359 | 0.936512 | 0.950134 | 0.959781 |

Figure 3.4: Normalized Hausdorff Measure

# Chapter 4

# Discussion and Conclusion

## 4.1 Discussion

Most theorems in this paper are known results summed up in this paper. The research on the Normalized Hausdorff measure for the $\phi$-Koch curve is new, since so far it looks like the $\phi$-Koch curve has only been studied for $\phi = \frac{\pi}{3}$. The results of the research given above looks nice since and it seems to lay rather close to the linear line $l\phi = (1 - \frac{3}{4}\frac{\pi}{2})\phi$. However the result only gives an upper bound, since it is unknown what the true Normalized Hausdorff measure is. It is even unknown whether the linear line is a good approximation or not. The only thing that is known is that the Normalized Hausdorff measure should be equal to 1 for $\phi = 0$ and equal to $\frac{1}{4}$ for $\phi = \frac{\pi}{2}$ (see also Property 3.5.3). The results found show indeed that the found measure is indeed 1 if $\phi = 0$, but for $\phi = \frac{\pi}{2}$ the found measure is higher than $\frac{1}{4}$. This indicates that it is possible to find covers that are better than the ones found so far. This paper suggests that a circle that is found induces a possible cover for the whole fractal, so there is a possibility to combine the different circles to create a cover for the fractal.

For $\phi = \frac{\pi}{3}$ an upper bound is ${}_{\mathcal{N}}\mathcal{H}^s(F) < \frac{1}{4}2^s$, where $s$ is the dimension which is $\frac{\log 4}{\log 3}$. In that case it holds that ${}_{\mathcal{N}}\mathcal{H}^s(F) < 0.5995$ [10]. Even more, it is known that $0.52878539 < {}_{\mathcal{N}}\mathcal{H}^s(F) < 0.58905161$ [5]. The found Natural measure fits between the bounds which suggests that the found results are nor far from the real results, at least in the case that $\phi = \frac{\pi}{3}$. Interesting is that $l[\frac{\pi}{3}] = (1 - \frac{3}{4}\frac{\pi}{2}) * \frac{\pi}{3} = \frac{1}{2}$, which is lower than the lowerbound. Meaning that the Normalized Hausdorff measure is not linear dependent on $\phi$.

## 4.2 self-similar fractals

For a self-similar fractal it is rather simple to plot them in a program like Mathematica. One only needs a beginning point and a set of contraction maps. This is sufficient since a set of contraction maps uniquely defines a fractal. Another point of interest is the dimension and the Natural-measure for fractals. The similarity dimension is easy to compute, while the Hausdorff dimension is hard to find. For self-similar fractals for which the open set condition holds it turns out that the similarity dimension is equal to the Hausdorff dimension. A famous fractal for which this holds is the $\phi$-Koch curve.

Furthermore it turns out that the Normalized Hausdorff measure meets most likely all desired properties of the Natural-measure for self-similar fractals. For fractals that are not self-similar this is unknown since there is no guarantee that the Normalized Hausdorff measure will be bounded or not.

## 4.3 $\phi$-Koch curve

It is known that the open set condition holds for the $\phi$-Koch curve. Which means that the Hausdorff dimension equals the Similarity dimension which is $\frac{\log 4}{\log(2+2\cos\phi)}$, with $\phi \in [0, \frac{\pi}{2}]$. It turns out to be hard to find the Normalized Hausdorff measure. For $\phi = \frac{\pi}{3}$ it is known that the Hausdorff measure is bounded between $(0.52878539, 0.58905161)$. In this paper some further research has been done on finding the Normalized Hausdorff measure.

For finding the Normalized Hausdorff measure one needs to find smart covers for the fractal. The fractal is described by the following functions.

**$S_1(x)$:** $1/(2 + 2\cos\phi)x$

$S_2(x)$**:** $1/(2+2\cos\phi)\left(\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}\cos\phi & -\sin\phi\\ \sin\phi & \cos\phi\end{pmatrix}(x)\right)$

$S_3(x)$**:** $1/(2+2\cos\phi)\left(\begin{pmatrix}1+\cos\phi\\ sin\phi\end{pmatrix}+\begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}(x)\right)$

$S_4(x)$**:** $1/(2+2\cos\phi)\begin{pmatrix}1+2\cos\phi\\ 0\end{pmatrix}+(x)$

It turns out to be useless to find a smart cover for each $S_i(x)$ instead of one cover for the whole fractal. Which means it is necessarily to find a circle that covers more than one part. To simplify the problem this paper restricted itself to finding smart covers for $S_2(F) \cup S_3(F)$. For such cover there were five possibilities represented.

- A circle $C1$ with radius a half that simply covered the whole fractal.
- A circle $C2$ that was a circle around the three outer points of the fractal.
- A circle $C3$ that contained the top of the fractal. Only legal when $\phi > \frac{\pi}{4}$.
- A circle $C4$ that contained the widest two points of the two inner parts. Only legal when $\phi < \frac{\pi}{4}$.
- A circle $C5$ that is a circle around the three outer points of the two inner parts.

The Hausdorff dimension is known which makes it possible to find the Normalized Hausdorff measure with respect to of these covers. Then the minimum of all these measures is taken to find the best approximation for the Normalized Hausdorff measure. In the table below some values can be found.

| $\frac{\pi}{2}$ | $\frac{\pi}{3}$ | $\frac{\pi}{4}$ | $\frac{\pi}{5}$ | $\frac{\pi}{6}$ | $\frac{\pi}{7}$ | $\frac{\pi}{8}$ | $\frac{\pi}{9}$ | $\frac{\pi}{10}$ | |
|---|---|---|---|---|---|---|---|---|---|
| 0.392699 | 0.573634 | 0.724903 | 0.83033 | 0.884651 | 0.916359 | 0.936512 | 0.950134 | 0.959781 | |

The only thing that is known is that for $\phi = \frac{\pi}{3}$ the found measure fits within the known upper and lower bound. Furthermore it is certain that the measure is not good enough since for $\phi = \frac{\pi}{2}$ the found measure is higher than $\frac{1}{4}$.

# Appendix A

# Volume of n-dimensional ball

## Volume of n-dimensional unit ball

Let $V^{(n)}[1] = V^{(n)}$ be the volume of an $n$-dimensional unit ball (meaning that the radius is 1). The following equations holds,

$$V^{(n)} = V^{(n-1)} \int_{-1}^{1} (1-x^2)^{n/2} dx = V^{(n-1)} 2 \int_{0}^{1} (1-x^2)^{n/2} dx.$$

With substitution of $u = 1 - x^2$ it follows that, $x = \sqrt{1-u}$ and $dx = \frac{-du}{2\sqrt{1-u}}$. Which leads to the equation,

$$V^{(n)} = V^{(n-1)} \int_{-1}^{1} (1-x^2)^{n/2} dx = V^{(n-1)} 2 \int_{0}^{1} u^{n/2} (1-u)^{-1/2} du.$$

This may seem as a difficult integral but there is a predefined Beta function which helps to solve the equation.

**Definition A.0.1.** Let $Re(x), Re(y) > 0$, then

$$B(x,y) = \int_{0}^{1} t^{x-1} (1-t)^{y-1} dt.$$

Furthermore it is known that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

That means that,

$$V^{(n)} = V^{(n-1)} 2 \int_{0}^{1} u^{n/2} (1-u)^{-1/2} du = V^{(n-1)} B(n/2+1, 1/2) = V^{(n-1)} \frac{\Gamma(n/2+1)\Gamma(1/2)}{\Gamma(n/2+3/2)}.$$

Knowing that $V^{(1)} = 1$, the following holds,

$$V^{(n)} = V^{(1)} \left( \frac{\Gamma(n/2+1)\Gamma(1/2)}{\Gamma((n/2+3/2)} \right) \left( \frac{\Gamma((n-1)/2+1)\Gamma(1/2)}{\Gamma((n-1)/2+3/2)} \right) \cdots \left( \frac{\Gamma(2/2+1)\Gamma(1/2)}{\Gamma(2/2+3/2)} \right).$$

$$V^{(n)} = \Gamma(1/2)^n \frac{\Gamma(2)}{\Gamma((n/2+3/2)} = \Gamma(1/2)^n / \Gamma((n/2+1).$$

## Volume of $n$-dimensional ball

With the following theorem it is easy so calculate the volume of an $n$-dimensional ball with some radius $r$.

**Theorem A.0.2.** *Let $V^{(n)}[r]$ be a n-dimensional ball with radius r, then $V^{(n)}[r] = r^n V^{(n)}$.*

*Proof.* The theorem shall be proved by induction. It is known that $V^{(1)}[r] = rV^{(1)} = r$. Assume that

$$V^{(n)}[r] = r^n V^{(n)}.$$

Then

$$V^{(n+1)}[r] = V^{(n)} \int_{-r}^{r} V^{(n)}[\sqrt{r - x^2}]dx.$$

$$V^{(n+1)}[r] = V^{(n)} r \int_{-1}^{1} V^{(n)}[\sqrt{r - (rx)^2}]dx.$$

$$V^{(n+1)}[r] = V^{(n)} \int_{-1}^{1} V^{(n)}[r\sqrt{1 - x^2}]dx.$$

$$V^{(n+1)}[r] = V^{(n)} \int_{-1}^{1} r^n V^{(n)}[\sqrt{1 - x^2}]dx. = r^{n+1} V^{(n+1)}$$

□

# Appendix B

# Mathematica input

## B.1 Drawing of Property 3.5.5

Listing B.1: Drawing of Property 3.5.5

```
x = Pi/6;
cos = N[Cos[x]]; sin = N[Sin[x]];
cos2 = N[Cos[2*x]]; sin2 = N[Sin[2*x]];
c = 1/(2 + 2*cos); (*contraction factor*)
r = 1/(4*sin*(1 + cos));

Cirkle =
  Graphics[{Circle[{1/2, sin*c - r}, r]}, Axes -> True,
   PlotRange -> {{0, 1}, {0, 1/2}}];

Circle2 =
  Graphics[{Red, Circle[{1/2, sin*c - 1/(2*sin)}, 1/(2*sin)]},
   Axes -> True, PlotRange -> {{0, 1}, {0, 1/2}}];

Innerpoints =
  Graphics[{PointSize[Large], Orange,
    Point[{{c, 0}, {1/2, sin*c}, {1 - c, 0}}]}];
PossibleInnerP =
  Graphics[{PointSize[Large], Green,
    Point[{{c + cos*c^2 + cos2*c^2,
       sin (c - c^2)}, {1 - c - (cos + cos2)*c^2, sin (c - c^2)}}]}];

Triangle1 =
  Graphics[{Thickness[0.009], Gray,
    Line[{{0, 0}, {c, 0}, {1/2, sin*c}, {1 - c, 0}, {1, 0}}]} ,
   Axes -> True, PlotRange -> {{0, 1}, {0,
      1/2}}];(*Length of each line-segment = c *)

FractalShape1 =
  Graphics[{Cyan,
    Line[{{c, 0}, {c + cos*c^2, sin*c^2}, {c + cos*c^2 + cos2*c^2,
       sin (c - c^2)} , {c + cos*(c - c^2), sin (c - c^2)}, {1/2, sin*c}} ]} ,
       Axes -> True, PlotRange -> {{0, 1}, {0, 1/2}}];
(*Length of each line-segment = c^2 *)

FractalShape2 =
  Graphics[{Cyan,
    Line[{{1 - c, 0}, {1 - c - cos*c^2,
       sin*c^2}, {1 - c - (cos + cos2)*c^2,
       sin (c - c^2)} , {1 - c - cos*(c - c^2), sin (c - c^2)},
      {1/2, sin*c}}]} , Axes -> True, PlotRange -> {{0, 1}, {0, 1/2}}];
(*Length of each line-segment = c^2*)

FractalShape3 =
  Graphics[{Cyan,
    Line[{{c + (cos + cos2)*c^2, sin (c - c^2)} , {c + cos*(c - c^2),
       sin (c - c^2)},
      {1/2, sin*c}, {1 - c - cos (c - c^2),
       sin (c - c^2)}, {1 - c - (cos + cos2)*c^2, sin (c - c^2)} }
      ]} , Axes -> True, PlotRange -> {{0, 1}, {0, 1/2}}];
(*Length of each line-segment = c^2*)

Show[Cirkle , Triangle1 , FractalShape3 , FractalShape2 , FractalShape1 , \
Circle2]
```

## B.2 Covers Koch Curve

Listing B.2: Plot of C1,..., C5

```
verdeeltwee[{a_, b_}] :=
  Block[{}, {a, (1 - \[Lambda])*a + \[Lambda]*b, (a + b)/2 + \[Mu]*
      Cross[b - a], \[Lambda]*a + (1 - \[Lambda])*b, b}];

knoop[l_] := Flatten[Append[Most /@ Most[l], Last[l]], 1];

KC[l_, n_Integer] :=
  If[n == 5, l, KC[knoop[verdeeltwee /@ Partition[l, 2, 1]], n + 1]];

Manipulate[\[Lambda] = 1/(2*(1 + Cos[\[Alpha]])); \[Mu] =
  Sin[\[Alpha]]/2;
 Graphics[{Orange, Line[KC[{{0, 0}, {1, 0}}, 0 ]]},
  PlotRange -> {{-0.1, 1.1}, {-0.

(* Define fractal *)
 Clear[dimKphi, bolvolume, hdmaat, c]
 phi = 1/3 Pi; (* phi in [0,Pi/2] *)
 dimKphi[phi_] :=
   Log[4]/Log[2 + 2*N[Cos[phi], 10]]; (* Hausdorffdimension K(phi) *)

 bolvolume[d_] :=
  Gamma[1/2]^d/Gamma[d/2 + 1]; (* volume d-dimensional bal *)
 hdmaat[dim_, radius_] :=
  bolvolume[dim]*
   radius^dim(* Normalized Hausdorffmeasure of a set in dimension \
dim. *)
 c = N[Cos[phi]];
 s = Sqrt[1 - c^2]; (* For more speed use  numerical results *)
 s1 = Function[z, 1/(2 + 2 c)*z];
 s2 = Function[z, 1/(2 + 2 c) ({1, 0} + {{c, -s}, {s, c}}.z)];
 s3 = Function[z, 1/(2 + 2 c) ({1 + c, s} + {{c, s}, {-s, c}}.z)];
 s4 = Function[z, 1/(2 + 2 c) ({1 + 2 c, 0} + z)];
 approxorder = 9;
 approx = {{1, 0}};
 For[i = 1, i < approxorder, i++,
   newapprox =
    Join[Table[s1[p], {p, approx}], Table[s2[p], {p, approx}],
     Table[s3[p], {p, approx}], Table[s4[p], {p, approx}]];
   approx = newapprox;
   ];

 (*circles*)
 r = 1/(4*s*(1 + c));
 C1 = Circle[{1/2, 0}, 1/2];
 C2 = Circle[{1/2, s/(2 + 2*c) - 1/(2*s)}, 1/(2*s)];
C3 = Circle[{1/2, 0},    s/(2 + 2*c)];
C4 = Circle[{1/2, 0}, 1/2 - 1/(2 + 2*c)];
C5 = Circle[{1/2, s/(2 + 2*c) - r}, r];


(*Draw fractal with circles*)
Show[
 ListPlot[Table[N[p], {p, approx}], PlotRange -> {{0, 1}, {0, 1}},
  AspectRatio -> Automatic,
  PlotStyle -> Directive[PointSize[Small], Orange]],
 Graphics[{Cyan, C1}],
 Graphics[{Green, C2}],
 Graphics[{Gray, C3}],
 Graphics[{Blue, C4}],
 Graphics[{Red, C5}],
 PlotRange -> {{0, 1}, {0, 1/2}}
 ]
```

## B.3 Hausdorff Measure

Listing B.3: Hausdorff Measure

```
clear[r, c, s, x, C1, C2, C3, C4, C5];
x = 1/3 * Pi;
s = Sin[x]; c = Cos[x];
r = 1/(4*s*(1 + c));

s1 = Function[z, 1/(2 + 2 cos[x])*z];
s2 = Function[z,
```

```
   1/(2 + 2 cos[x]) ({1,
       0} + {{cos[x], -sin[x]}, {sin[x], cos[x]}}.z)];
s3 = Function[z,
   1/(2 + 2 cos[x]) ({1 + cos[x],
       sin[x]} + {{cos[x], sin[x]}, {-sin[x], cos[x]}}.z)];
s4 = Function[z, 1/(2 + 2 cos[x]) ({1 + 2 cos[x], 0} + z)];
approx = {{1, 0}};
For[i = 1, i < 9, i++,
  newapprox =
   Join[Table[s1[p], {p, approx}], Table[s2[p], {p, approx}],
    Table[s3[p], {p, approx}], Table[s4[p], {p, approx}]];
  approx = newapprox;
  ];

C1 = Circle[{1/2, 0}, 1/2];
C2 = Circle[{1/2, s/(2 + 2*c) - 1/(2*s)}, 1/(2*s)];
C3 = Circle[{1/2, 0}, s/(2 + 2*c)];
C4 = Circle[{1/2, 0}, 1/2 - 1/(2 + 2*c)];
C5 = Circle[{1/2, s/(2 + 2*c) - r}, r];

(* Draw fractal with covers *)
Show[
 ListPlot[Table[N[p], {p, approx}], PlotRange -> {{0, 1}, {0, 1}},
  AspectRatio -> Automatic,
  PlotStyle -> Directive[PointSize[Small], Orange]],
 Graphics[{Cyan, C1}], Graphics[{Green, C2}], Graphics[{Gray, C3}],
 Graphics[{Blue, C4}], Graphics[{Red, C5}],
 PlotRange -> {{0, 1}, {0, 1/2}}
 ]


(* Drawing plots for Normalized Hausdorff measure estimation *)
Clear[phi, cos, sin];
Clear[dimKphi, bolvolume, hdmaat, c];
dimKphi[phi_] :=
  Log[4]/Log[2 + 2*N[Cos[phi], 10]]; (* Hausdorffdimension *)


bolvolume[d_] :=
  Gamma[1/2]^d/
   Gamma[d/2 + 1]; (* Volume unitball of dimension d *)
hdmaat[dim_, radius_] :=
 bolvolume[dim]*
  radius^dim
cos[phi_] := N[Cos[phi]];
sin[phi_] := N[Sin[phi]];

(*C3 and C5 don't cover requested part for certain phi*)
sol = Solve[ 1/2 - Sin[y]/(2 + 2*Cos[y]) == 1/(2 + 2*Cos[y]), y,
   InverseFunctions -> True];
opt3 = Select[ y /. sol, # > 0 &];
sol2 = Solve[1/2 - 1/(2 + 2*Cos[y]) == Sin[y]*1/(2 + 2*Cos[y]), y,
   InverseFunctions -> True];
opt4 = Select[ y /. sol2, # > 0 &];

(*Determine Hausdorff measure*)
maat1[phi_] := hdmaat[dimKphi[phi], 1/2]; (*covers whole fractal*)
maat2[phi_] :=
  hdmaat[dimKphi[phi], 1/(2*sin[phi])]; (*covers whole fractal*)
maat3[phi_] :=
  1/( 2/Pi* phi)*
   Piecewise[{{2,
      phi < opt3[[1]]}, {hdmaat[dimKphi[phi],
       sin[phi]/(2 + 2*cos[phi])],
      phi >= opt3[[1]]}}];(*covers ?? a fractal, => linear estimation *)


maat4[phi_] :=
  2* Piecewise[{ {hdmaat[dimKphi[phi], 1/2 - 1/(2 + 2*cos[phi])],
      phi <= opt4[[1]]}, {2,
      phi > opt4[[1]]} }]; (*covers half a fractal*)
maat5[phi_] :=
  2* hdmaat[dimKphi[phi],
    1/(4*sin[phi]*(1 + cos[phi]))]; (*covers half a fractal*)


(*Plot results*)
{Plot[{maat1[phi], maat2[phi], maat3[phi], maat4[phi],  maat5[phi],
   1 - (3/4)/(Pi/2)*phi}, {phi, 0, Pi/2},
  PlotStyle -> {Cyan, Green , Gray, Blue, Red, Dashed},
  PlotRange -> {{0, Pi/2}, {0, 2}}],

 (* plot minimum *)
 Plot[{{Min[maat1[phi], maat2[phi], maat3[phi], maat4[phi],
     maat5[phi]]}, {1 - (3/4)/(Pi/2)*phi}}, {phi, 0, Pi/2},
```

```
  PlotStyle -> {Orange, Dashed}, PlotRange -> {{0, Pi/2}, {0, 2}}]}

Grid[Transpose[
  Table[{Pi/2*1/n,
    Min[maat1[Pi/2*1/n], maat2[Pi/2*1/n], maat3[Pi/2*1/n],
     maat4[Pi/2*1/n],  maat5[Pi/2*1/n]]}, {1, 2, 10, 1}]],
 Frame -> All, Alignment -> Left, Spacings -> {2, 1}]
```

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