BACHELOR

Categorizing models for water waves

Nijhuis, T.R.T.

Award date:
2011

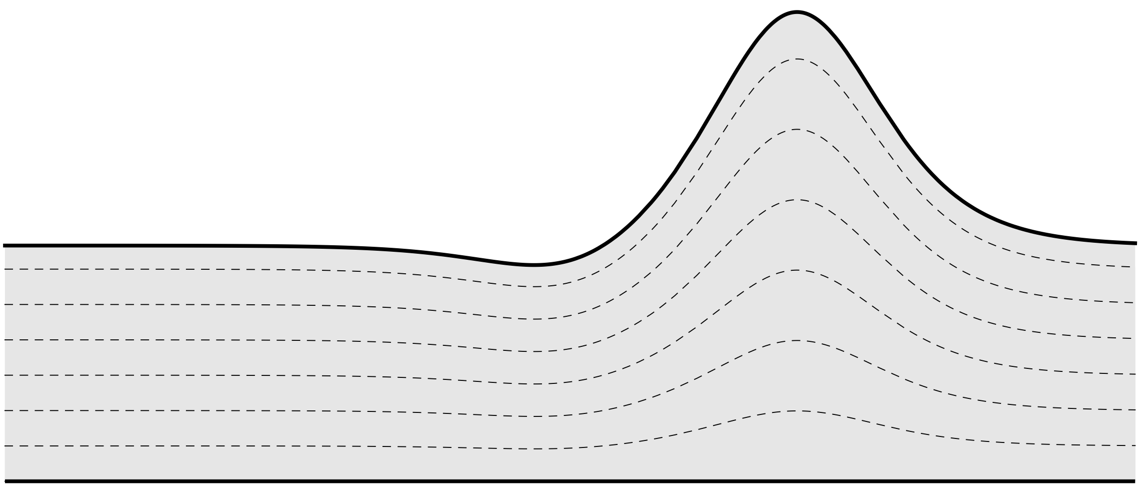
Eindhoven University of Technology
Department of Mathematics and Computer Science

Bachelor's thesis

Categorizing models for water waves

– T. Nijhuis –

20th July 2011



Supervised by:

dr. G. Prokert

Abstract

The study of waves on a water surface is involved with a great number of different models. In this report an overview is provided for the major ones of these equations. The most general equation will be derived by means of the Euler equation. The essential parameters are identified and these will be used to qualify the common assumptions of small waves and shallow water.

For small waves the non-dispersive wave equation and the linear dispersive equations will be derived by means of a Dirichlet-Neumann operator. This operator will be used to supply the exact solution of the linear system for small waves.

For shallow water the Green-Naghdi equations are derived and will be simplified further to the standard shallow water equations.

Lastly, some alternative methods and tools which do not fit in the framework of this report are discussed. These are unidirectionalization (leading to the Korteweg-de Vries equation) the tidal wave approximation and Boussinesq-type approximations. Also, an introduction to Hamiltonian theory on water waves is given. The report is concluded with an overview of the discussed models and with some remarks.

Cover ilustration:
The graph of a soliton traveling to right. This is a solution for the Boussinesq equations (Section 5.4)

Acknowledgements

This project has started in January 2009. It has been a work in process with a lengthy interruption since. This interruption, although motivated and not caused by poor discipline, must have been a nuisance to my coordinator, Georg Prokert. Nevertheless he managed to stay enthousiastic and was happy to see me return. Furthermore he called me on my flaws and even managed to repair them by conveying some of his *Deutsche Gründlichkeit* to me. As one of the most formal mathematicians I have had the pleasure of meeting, he was more than willing to get off the physics track and share some abstract results and elegant philosophical insights. The other side of the academic spectrum is held by my second examinator, Sjoerd Rienstra. Sjoerd is the model engineer who managed to show me what this project was actually about with his intuitive understanding and his pragmatic view on physical problems.
The interaction between Georg and Sjoerd has not only given me my best insights but was also a very entertaining view on the interface between a priori and a posteriori reasoning. I acknowledge both gentlemen not only for their help in my project but also for showing me the full span of meta-mathematics available.
The article of David Lannes and Phillipe Bonneton offered a great deal of the cohesion I was looking for. However it was written with a reader in mind who is more seasoned in the art of water waves than I used to be. David Lannes deserves my gratitude for kindly answering the most persistant of my questions.

Tom Nijhuis
July 2011

Contents

1 Introduction

1.1 Problem description and goal

We are concerned with water flow in a two-dimensional basin. The width of the basin stretches to infinity and the height can be any number. The lower bed of the basin is level and flat[1]. Furthermore, there are no external sources of flow. The restoring force is provided by gravity alone.

The upper boundary will be the wave profile itself. Thus we are dealing with a varying boundary. We are mostly concerned with the wave profile, the actual water flow will be eliminated from the problem wherever possible.

The goal of this report is to provide an overview of the major equations that can be used to model the wave profile under certain circumstances. It is not in our interest to actually solve or numerically implement the equations.

1.1.1 Mathematical description

The surface profile of the water can be modelled by a single function $\eta(x,t)$ to describe the difference between the average and the actual water height. Although the basin does stretch to infinity on both sides (as to not having to be concerned with boundary effects), we do assume periodicity on x. This is quite a natural assumption for waves.

The basin will be called $\Omega(t)$ and is the set of points occupied by water, so $\Omega(t) = \{(x,z)|0 < x < 2\pi L, -h_0 < z < \eta(x,t)\}$ with h_0 the average water depth and $2\pi L$ the spatial period. The water itself will be modelled by a vector field $\mathbf{v}(x,z,t)$ which describes the direction and speed of the particles at any point $(x,z) \in \Omega$. Figure 1 shows a typical description the situation.

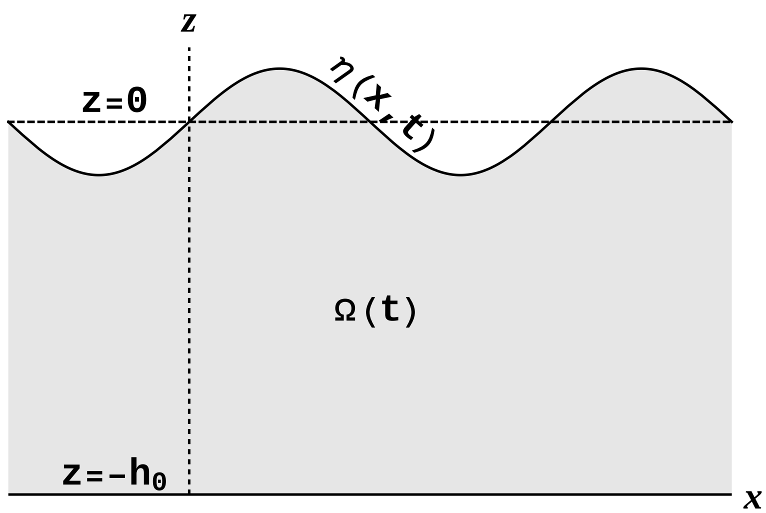


Figure 1: Graphical representation of the problem description.

[1]Most authors also investigate a three-dimensional basin with a bed which is not necessarily flat.

1.1.2 Goal

There are three main reasons to manipulate a system of equations. One, the original system may be too hard to implement numerically. For example because of an irregular domain or unknown scale effects. Two, a similar but easier model might provide some theoretical insight which is impossible to notice from the original system. Three, the original problem might be too much involved with information we do not care about. In this report, for example, we are interested in the surface profile. Therefore, we try to reduce additional information. For all reasons, a suitable model would be an evolution equation of the form:

$$\begin{pmatrix} \partial_t \mathbf{v} \\ \partial_t \eta \end{pmatrix} (x,t) = \mathscr{F} \begin{pmatrix} \mathbf{v} \\ \eta \end{pmatrix} (x,t) \tag{1}$$

In this description the heighth variable z would be eliminated and thus the only spatial variable would be x. Such models make no explicit use of the water flow in the interior of Ω and only require and yield information about the surface.

Mind that not only numerical calculations or simulations are favored, equations in some other form can also provide theoretical insights on the problem. An example of such an insight is dispersion (see Section 3.3).

Apart from showing how to derive some of the models, in this report we try to find unification between various models. The ultimate goal would be to provide some procedure of generating models that are valid in various parameter regions.

2 Deriving the equations of motion

The equations of motion for water in a domain with a free upper boundary are derived in almost all introductory literature [3, 9, 12]. In this report, we will derive them from the Euler equation.

2.1 Assumptions on flow

To derive the governing equations four main assumptions are used. The flow, described by a vector field $\mathbf{v}(x,z)$ is

- irrotational ($\nabla \times \mathbf{v} = 0$),
- incompressible ($\nabla \cdot \mathbf{v} = 0$) and
- inviscid.

The last one, no viscosity, means that the flow is not subject to shear stress. Lastly, we also assume that there is no surface tension. That means that no additional forces due to the irregularity of the boundary occur. These assumptions are reasonable since the viscous effects of water are very small and surface tension has no influence on the scale we are concerned with.
The first assumption, $\nabla \times \mathbf{v} = 0$, makes $\mathbf{v}$ a so called *potential flow*[2]. This means that the vector field is the gradient of a certain scalar function: $\mathbf{v} = \nabla\phi = (\partial_x\phi, \partial_z\phi)^{\mathsf{T}}$.
We can now always use ϕ instead of $\mathbf{v}$, then no more vector equations will be encountered. Also, the desired system (1) can be simplified to:

$$\begin{pmatrix}\partial_t\phi\\ \partial_t\eta\end{pmatrix}(x,t) = \mathscr{F}\begin{pmatrix}\phi\\ \eta\end{pmatrix}(x,t) \tag{2}$$

2.2 The interior PDE

Water flow is described by the incompressibility condition. In this, substitute the potential:

$$\begin{aligned}\operatorname{div}(\mathbf{v}) &= 0\\ \nabla\cdot\mathbf{v} &= 0\\ \nabla\cdot(\nabla\phi) &= 0\\ \nabla^2\phi &= 0 \end{aligned} \tag{3}$$

This is the well-known Laplace equation.

[2]In a simply connected domain, which is the case for Ω

2.3 Boundary conditions

2.3.1 Lower boundary

On the bottom $z = -h_0$ we have an impermeable boundary. This means that the flow in the normal direction $\mathbf{n}$ is zero

$$\mathbf{v} \cdot \mathbf{n} = 0$$

The bottom is level and flat, so $\mathbf{v} \cdot \mathbf{n} = \mathbf{v} \cdot (0, -1)^{\mathrm{T}} = -v_z$. Further, we use the potential:

$$\begin{aligned} v_z &= 0 \\ \partial_z \phi &= 0 \end{aligned} \tag{4}$$

2.3.2 Upper boundary

On top, we have two distinct boundary conditions. First, the kinematic condition which states that the normal particle velocity on the surface equals the normal velocity of the surface itself. This is really just a non-permeable boundary condition which includes the fact that the boundary is moving. Second, we have the pressure condition which states that the water pressure equals the atmospheric pressure on the surface.
For the kinematic condition, we start by finding the vector normal to the surface via the gradient of the contour which describes the surface (Figure 2)

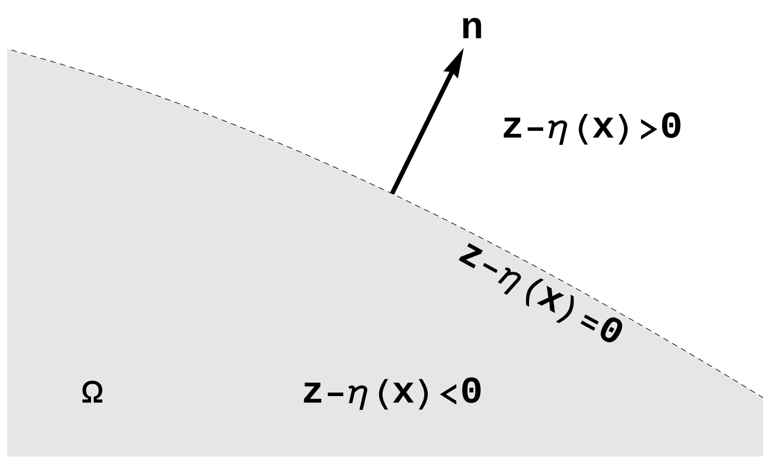


Figure 2: The normal of the boundary is found via the gradient of $z - \eta(z)$.

$$\begin{aligned} \mathbf{n} \cdot (\triangle x; \triangle z)^{\mathrm{T}} &= 0 \\ \mathbf{n} &= \frac{\nabla(z - \eta(z))}{|\nabla(z - \eta(z))|} \\ \mathbf{n} &= \frac{1}{\sqrt{1 + (\partial_x \eta)^2}} \begin{pmatrix} -\partial_x \eta \\ 1 \end{pmatrix} \end{aligned}$$

Now we use $\mathbf{n}$ to equal the normal change in the surface profile to the normal particle velocity. Observe that the velocity of the surface is given by $(0;\partial_t\eta)$

$$\begin{aligned}
(0;\partial_t\eta)^{\mathrm{T}}\cdot\mathbf{n} &= \mathbf{v}^{\mathrm{T}}\cdot\mathbf{n}\\
\left((0;\partial_t\eta)-\mathbf{v}^{\mathrm{T}}\right)\cdot\mathbf{n} &= 0\\
(-\partial_x\phi;\partial_t\eta-\partial_z\phi)\cdot(-\partial_x\eta;1)^{\mathrm{T}} &= 0\\
\partial_t\eta+\partial_x\phi\partial_x\eta-\partial_z\phi &= 0 \qquad (5)
\end{aligned}$$

For the pressure condition, we make use of the Euler equation. This is the main tool when modeling inviscid, incompressible flows. The vector equation as given below represents conservation of momentum in differential form [9]. The external force, i.e. gravity, is uniform and downwards: $\mathbf{F}=-\rho g\mathbf{e}_Z$

$$\begin{aligned}
\rho\left(\partial_t\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}\right) &= -\nabla p+\mathbf{F}\\
\rho\left(\partial_t\mathbf{v}+\tfrac{1}{2}\nabla(|\mathbf{v}|^2)+\mathbf{v}\times(\nabla\times\mathbf{v})\right) &= -\nabla p-\rho g\mathbf{e}_z\\
\partial_t\nabla\phi+\tfrac{1}{2}\nabla(|\nabla\phi|^2) &= -\tfrac{1}{\rho}\nabla p-g\mathbf{e}_z\\
\nabla\partial_t\phi+\tfrac{1}{2}\nabla\left[(\partial_x\phi)^2+(\partial_z\phi)^2\right]+g\mathbf{e}_z &= -\tfrac{1}{\rho}\nabla p\\
\nabla\left[\partial_t\phi+\tfrac{1}{2}\left[(\partial_x\phi)^2+(\partial_z\phi)^2\right]+gz\right] &= \nabla\left[-\tfrac{1}{\rho}p\right]
\end{aligned}$$

In the last equation we use that the density is constant, which is the case for incompressible flows. On the surface the fluid pressure equals the atmospheric pressure. Equality of the gradients implies that the functions differ only a constant C (which may depend on time).

$$\partial_t\phi+\tfrac{1}{2}(\partial_x\phi^2+\partial_z\phi^2)+g\eta=-\tfrac{1}{\rho}p_0+C(t)=0 \qquad (6)$$

In the last step, the constant of integration $C(t)$ is chosen to equal $\frac{1}{\rho}p_0$. This freedom is available since there are no further conditions on the pressure. Conceptually, this freedom implies that the behavior of the fluid is independent of both the atmospheric pressure p_0 and the density ρ.
Equation (6), known as the Bernoulli equation, gives the second upper boundary condition of the problem.

2.4 Combining the equations of motion

Combining the equations (3), (4), (5) and (6) gives:

$$\begin{aligned}
\partial_{xx}\phi+\partial_{zz}\phi &= 0 && \text{in } \Omega(t) && (7)\\
\partial_z\phi &= 0 && \text{on } z=-h_0 && (8)\\
\partial_t\eta+\partial_x\eta\partial_x\phi-\partial_z\phi &= 0 && \text{on } z=\eta(x,t) && (9)\\
\partial_t\phi+\tfrac{1}{2}\left[(\partial_x\phi)^2+(\partial_z\phi)^2\right]+g\eta &= 0 && \text{on } z=\eta(x,t) && (10)
\end{aligned}$$

This is a partial differential equation with non-linear boundary conditions on a free boundary. No solutions in closed form can be given for any general initial conditions. After defining the Dirichlet-Neumann operator in Section 3.2.1, we will try to recast the system as the desired evolution system (2) as much as possible.

2.5 Dimension analysis, scaling and essential parameters

2.5.1 Dimension analysis

Equations (7) to (10) involve quantities and variables with a certain dimension (lenght in m, time in s and potential speed in $m^2 s^{-1}$). We start with making the problem independent of these dimensions.
First, we substitute dimensionless variants of the variables and quantities. These are indicated by the hats:

$$\begin{aligned} x &= L\hat{x} & t &= T\hat{t} \\ z &= h_0\hat{z} & \phi &= \Lambda\hat{\phi} \\ \eta &= a\eta & \Omega &= h_0\hat{\Omega} \end{aligned}$$

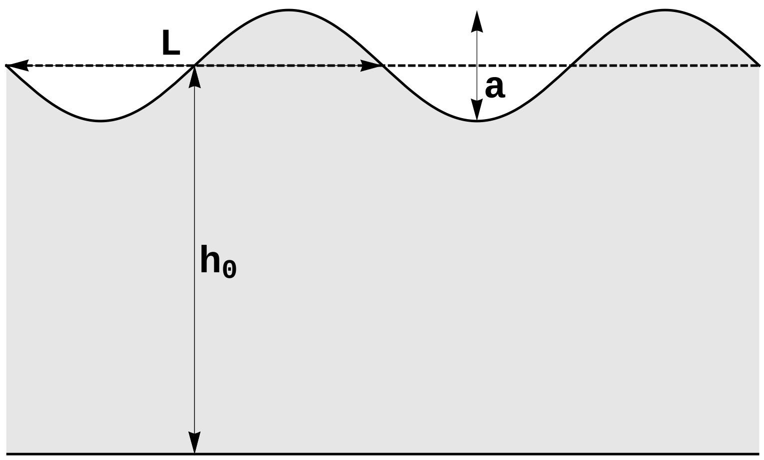


Figure 3: The three typical scales in the problem.

With h_0 the average water depth in meters, a the typical difference between the actual and the average water level and L is some typical horizontal length scale, given by the periodicity condition (see Figure 3). T is some time and Λ is some potential speed. These are to be specified later. Substituting these in (7) to (10).

$$\begin{aligned} \tfrac{\Lambda}{L^2}\partial_{\hat{x}\hat{x}}\hat{\phi} + \tfrac{\Lambda}{h_0{}^2}\partial_{\hat{z}\hat{z}}\hat{\phi} &= 0 && \text{in } \hat{\Omega} \\ \tfrac{\Lambda}{h_0}(\partial_{\hat{z}}\hat{\phi}) &= 0 && \text{on } h_0\hat{z} = -h_0 \\ \tfrac{a}{T}\partial_{\hat{t}}\hat{\eta} + \tfrac{a\Lambda}{L^2}\partial_{\hat{x}}\hat{\eta}\partial_{\hat{x}}\hat{\phi} - \tfrac{\Lambda}{h_0}\partial_{\hat{z}}\hat{\phi} &= 0 && \text{on } h_0\hat{z} = a\hat{\eta} \\ \tfrac{\Lambda}{T}\partial_{\hat{t}}\hat{\phi} + \tfrac{1}{2}\tfrac{\Lambda^2}{L^2}(\partial_{\hat{x}}\hat{\phi})^2 + \tfrac{1}{2}\tfrac{\Lambda^2}{h_0{}^2}(\partial_{\hat{z}}\hat{\phi})^2 + ga\hat{\eta} &= 0 && \text{on } h_0\hat{z} = a\hat{\eta} \end{aligned}$$

These equations are obtained with nothing but substituting. These can be cleaned up by dividing. Also, the hats are dropped which means that henceforth every variable and quantity is dimensionless:

$$\frac{h_0{}^2}{L^2}\partial_{xx}\phi + \partial_{zz}\phi = 0 \qquad \text{in } \Omega \tag{11}$$

$$\partial_z\phi = 0 \qquad \text{on } z = -1 \tag{12}$$

$$\frac{a}{T}\partial_t\eta + \frac{a\Lambda}{L^2}\partial_x\eta\partial_x\phi - \frac{\Lambda}{h_0}\partial_z\phi = 0 \qquad \text{on } z = \frac{a}{h_0}\eta \tag{13}$$

$$\frac{\Lambda}{T}\partial_t\phi + \frac{1}{2}\frac{\Lambda^2}{L^2}(\partial_x\phi)^2 + \frac{1}{2}\frac{\Lambda^2}{h_0{}^2}(\partial_z\phi)^2 + ga\eta = 0 \qquad \text{on } z = \frac{a}{h_0}\eta \tag{14}$$

2.5.2 Balancing and scaling

Scaling is used to simplify the order of magnitude of various quantities in a problem. It then provides a natural way of comparing, or balancing, independent quantities with one another. In the problem are various length scales available, they are pointed out in Figure 3. These will be used in a balancing argument to define Λ and T.
The balancing is started on (14). This is because it is an equation with all the scales present. Equation (14):

$$\frac{\Lambda}{T}\partial_t\phi + \frac{1}{2}\frac{\Lambda^2}{L^2}(\partial_x\phi)^2 + \frac{1}{2}\frac{\Lambda^2}{h_0{}^2}(\partial_z\phi)^2 + ga\eta = 0$$

There are four quantities balancing each other to zero. We can force one of them to be $\mathcal{O}(1)$ by dividing by its factor. Since the wave profile is going to be studied, this factor is chosen to be the one in front of η. So dividing by ga gives:

$$\frac{\Lambda}{Tga}\partial_t\phi + \frac{1}{2}\frac{\Lambda^2}{L^2ga}(\partial_x\phi)^2 + \frac{1}{2}\frac{\Lambda^2}{h_0{}^2ga}(\partial_z\phi)^2 + \eta = 0$$

Now, there is still some freedom in chosing Λ and T. We can use this to balance one of the other three terms with η. We want to consider timescales in which the solution changes not too slow nor too rapid, that is $\partial_t = \mathcal{O}(1)$. Therefore we choose the factor of $\partial_t\phi$ to be 1. This is done by setting $T = \frac{ga}{\Lambda}$:

$$\partial_t\phi + \frac{1}{2}\frac{\Lambda^2}{L^2ga}(\partial_x\phi)^2 + \frac{1}{2}\frac{\Lambda^2}{h_0{}^2ga}(\partial_z\phi)^2 + \eta = 0$$

Observe that the two terms in the center are actually coupled because they both originate from $|\nabla\phi|^2$. To not override the $\mathcal{O}(1)$-terms we just defined, force these terms to be small, say $\mathcal{O}(\gamma) \ll 1$.
Thus:

$$\gamma = \frac{\Lambda^2}{L^2ga}$$

And the full set looks like:

$$
\begin{aligned}
\frac{h_0{}^2}{L^2}\partial_{xx}\phi + \partial_{zz}\phi &= 0 \\
\partial_z\phi &= 0 \\
\partial_t\eta + \gamma\partial_x\eta\partial_x\phi - \gamma\frac{L^2}{h_0{}^2}\partial_z\phi &= 0 \\
\partial_t\phi + \frac{1}{2}\gamma(\partial_x\phi)^2 + \frac{1}{2}\gamma\frac{L^2}{h_0{}^2}(\partial_z\phi)^2 + \eta &= 0
\end{aligned}
$$

The parameter γ is yet to be defined. This is done in the next section.

2.5.3 Essential parameters and dynamic scaling

In the previous section, note that the fraction $\frac{h_0}{L}$ pops up. This is a ratio between two of the inherent length scales and is therefore known as an *essential parameter* of the problem. We shall define: $\delta := \frac{h_0}{L}$. There are three inherent length scales (a, h_0 ,L), so we would at least need one more essential ratio to account for a. Define $\varepsilon := \frac{a}{h_0}$.
Choosing the ratios for the essential parameters is arbitrary from a strict mathematical viewpoint. However, the choice as given has some physical meaning: ε is the ratio between the surface variation and the water depth, δ is the ratio between the water depth and the wave length. Thus ε and δ will be small for small waves and shallow water respectively. These are the two main regions we will be considering (in Section 3 and Section 4)
Although it has no parameter assigned to it, the ratio $\frac{a}{L}$ has physical meaning as well. It is a measure for the steepness of the waves. This steepness will be adressed for the wave equation in Section 3.2.3 and for the Boussinesq approximation in Section 5.4.
The small parameter γ turns out to be a mere placeholder. We can pick it equal to either ε or δ for small waves and shallow water respectively. But since we want to consider as many models as possible from one general system, we do not want to rescale halfway through. This problem is solved by a method from [10] which we shall call *dynamic scaling*. Set:

$$
\begin{aligned}
z &= \nu\check{z} \\
\gamma &= \frac{\varepsilon}{\nu} \\
\text{with } \nu &= \frac{\tanh(\delta)}{\delta}
\end{aligned}
$$

Heuristically, the factor ν supplies a smooth transition between scaling z by L for small waves and by h_0 for shallow water.
Finally, the fundamental dimensionless, scaled problem looks like this:

$$\nu^2\delta^2\partial_{xx}\phi + \partial_{zz}\phi = 0 \qquad \text{in } \Omega \tag{15}$$

$$\partial_z\phi = 0 \qquad \text{on } z = -\frac{1}{\nu} \tag{16}$$

$$\partial_t\eta + \frac{\varepsilon}{\nu}\partial_x\eta\partial_x\phi - \frac{1}{\delta^2\nu^2}\partial_z\phi = 0 \qquad \text{on } z = \frac{\varepsilon}{\nu}\eta \tag{17}$$

$$\partial_t\phi + \frac{1}{2}\left[\frac{\varepsilon}{\nu}(\partial_x\phi)^2 + \frac{\varepsilon}{\delta^2\nu^3}(\partial_z\phi)^2\right] + \eta = 0 \qquad \text{on } z = \frac{\varepsilon}{\nu}\eta \tag{18}$$

This system will form the basis for all approximations.
With these choices, we can make the definitions of Λ and T explicit:

$$\Lambda = aL\sqrt{\frac{g}{h_0\nu}}$$
$$T = \frac{L}{\sqrt{gh_0\nu}}$$

Note:

3 Small waves

In this section the domain of small waves is analyzed. More accurately, the ratio between the typical elevation and the water depth is assumed to be small. In terms of the essential parameters, this means that $\varepsilon \ll 1$. An example of waves in this domain can be found in the middle of oceans. Here the wave height is in the order of meters and the water depth is in the order of kilometers (thus $\varepsilon \sim 10^{-3}$). For now δ is kept a constant of order one.
In the conclusion of this section, it will be clear that all regular models considering small waves are linear. Hence small waves also go by the name of *linear waves.*

3.1 Linearization

Fourier analysis is very suitable for linear problems. To make use of this it is convenient to work in a rectangular domain. This is obtained by a domain transform before the actual linearization starts. For using Fourier series, we will also need the periodicity on x. The endpoints $x = 0$ and $x = 2\pi$ are identified.

3.1.1 Domain transform

The domain Ω is a rectangle apart from the irregular upper boundary $z = \eta$. In this section it is transformed to a rectangle $\bar{\Omega}$. This is done by a suitable transformation on (x, z). Note that $z \in (-\frac{1}{\nu}, \frac{\varepsilon}{\nu}\eta)$. Choose $\bar{z} = -\frac{\varepsilon\eta - \nu z}{\varepsilon\eta+1}$ then $\bar{z} \in (-1, 0)$. No transformation is applied to x. Isolating z gives the following transformation:

$$\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\varepsilon\eta+1}{\nu} \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\varepsilon}{\nu}\eta \end{pmatrix}$$

And inverting the Jacobian gives: (in operator notation)

$$\begin{pmatrix} \partial_x \\ \partial_z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\varepsilon\frac{\nu\partial_{\bar{x}}\eta}{\varepsilon\eta+1} & \frac{\nu}{\varepsilon\eta+1} \end{pmatrix} \begin{pmatrix} \partial_{\bar{x}} \\ \partial_{\bar{z}} \end{pmatrix}$$

So $\partial_z = \frac{\nu}{\varepsilon\eta+1}\left(1 - \varepsilon\partial_{\bar{x}}\eta\right)\partial_{\bar{z}}$ and $\partial_{zz} = \left(\frac{\nu}{\varepsilon\eta+1}\right)^2\left(1 - \varepsilon\partial_{\bar{x}}\eta\right)^2\partial_{\bar{z}\bar{z}}$
The effect of the transformation applied to Ω is shown in Figure 4.
Substitute this in (15) to (18):

$$\begin{aligned} \nu^2\delta^2\partial_{\bar{x}\bar{x}}\phi + \tfrac{\nu^2}{(\varepsilon\eta+1)^2}\left(1 - \varepsilon\partial_x\eta\right)^2\partial_{\bar{z}\bar{z}}\phi &= 0 && \text{in } \bar{\Omega} \\ \tfrac{\nu}{\varepsilon\eta+1}\left(1 - \varepsilon\partial_x\eta\right)\partial_{\bar{z}}\phi &= 0 && \text{on } \bar{z} = -1 \\ \partial_t\eta + \tfrac{\varepsilon}{\nu}\partial_{\bar{x}}\eta\partial_{\bar{x}}\phi - \tfrac{\nu}{\varepsilon\eta+1}\left(1 - \varepsilon\partial_x\eta\right)\tfrac{1}{\delta^2\nu^2}\partial_{\bar{z}}\phi &= 0 && \text{on } \bar{z} = 0 \\ \partial_t\phi + \tfrac{1}{2}\left[\tfrac{\varepsilon}{\nu}(\partial_{\bar{x}}\phi)^2 + \tfrac{\varepsilon}{\delta^2\nu^3}\tfrac{\nu^2}{(\varepsilon\eta+1)^2}\left(1 - \varepsilon\partial_x\eta\right)^2(\partial_{\bar{z}}\phi)^2\right] + \eta &= 0 && \text{on } \bar{z} = 0 \end{aligned}$$

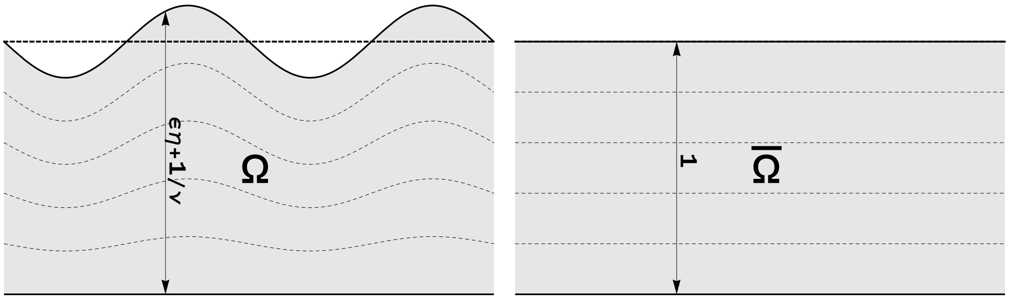


Figure 4: Ω and $\bar{\Omega}$ with some contours for $\bar{z}$

Rewrite and drop the bars:

$$(\varepsilon\eta+1)^2\delta^2\partial_{xx}\phi + \left(1-\varepsilon\partial_x\eta\right)^2\partial_{zz}\phi = 0 \qquad \text{in } \Omega \tag{19}$$

$$\partial_z\phi = 0 \qquad \text{on } z=-1 \tag{20}$$

$$(\varepsilon\eta+1)\partial_t\eta + \tfrac{\varepsilon}{\nu}(\varepsilon\eta+1)\partial_x\eta\partial_x\phi - \tfrac{1}{\delta^2\nu}\left(1-\varepsilon\partial_x\eta\right)\partial_z\phi = 0 \qquad \text{on } z=0 \tag{21}$$

$$(\varepsilon\eta+1)^2\partial_t\phi + \tfrac{1}{2}\left[\tfrac{\varepsilon}{\nu}(\varepsilon\eta+1)^2(\partial_x\phi)^2 + \tfrac{\varepsilon}{\delta^2\nu}\left(1-\varepsilon\partial_x\eta\right)^2(\partial_z\phi)^2\right] + (\varepsilon\eta+1)^2\eta = 0 \qquad \text{on } z=0 \tag{22}$$

Now the new domain Ω is rectangular and has no free boundaries. This simplification goes at the expense of additional non-linear terms. However, note that the nonlinearities are $\mathcal{O}(\varepsilon)$. It can be expected that these vanish for the small wave approximation $\varepsilon \ll 1$.

3.1.2 Expansion in ε

For the most fundamental equations (7) to (9) it can immediately be seen that $\phi_0 \equiv 0$, $\eta_0 \equiv 0$ is a solution. We will expand in a neighbourhood of this solution by using the small parameter ε:

$$\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \ldots = \varepsilon\phi_1 + \mathcal{O}(\varepsilon^2)$$
$$\eta = \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \ldots = \varepsilon\eta_1 + \mathcal{O}(\varepsilon^2)$$

After substituting this in (19) to (22), the terms of $\mathcal{O}(\varepsilon)$ are :

$$\delta^2\partial_{xx}\phi_1 + \partial_{zz}\phi_1$$
$$\partial_z\phi_1$$
$$\partial_t\eta_1 - \tfrac{1}{\delta^2\nu}\partial_z\phi_1$$
$$\partial_t\phi_1 + \eta_1$$

Ignore higher order terms and drop the indices. We obtain:

$$\delta^2\partial_{xx}\phi + \partial_{zz}\phi = 0 \qquad \text{in } \Omega \tag{23}$$

$$\partial_z\phi = 0 \qquad \text{on } z=-1 \tag{24}$$

$$\partial_t\eta - \tfrac{1}{\delta^2\nu}\partial_z\phi = 0 \qquad \text{on } z=0 \tag{25}$$

$$\partial_t\phi + \eta = 0 \qquad \text{on } z=0 \tag{26}$$

This system lies at the base of the first order small-wave approximations. It is linear and is defined on a rectangle. Therefore it can be solved by a Fourier technique.

3.2 The Dirichlet-Neumann operator

A useful tool when we are only interested in the behavior at the boundaries of PDE's is the so called Dirichlet-Neumann operator (DN operator). Before we consider its approximation via the linearized system, a definition is given.

3.2.1 General definition and properties

Given the stationary version of (15) to (18) with a Dirichlet boundary condition on the surface.

$$\begin{aligned} \nu^2\delta^2\partial_x\phi + \partial_z\phi &= 0 && \text{for } -\tfrac{1}{\nu} < z < \tfrac{\varepsilon}{\nu}\eta \\ \phi &= \psi && \text{for } z = \tfrac{\varepsilon}{\nu}\eta \\ \partial_z\phi &= 0 && \text{for } z = -\tfrac{1}{\nu} \end{aligned}$$

Then the **Dirichlet-Neumann operator** $\mathscr{A}$ is an operator defined on the surface such that:

$$\mathscr{A}\psi = \partial_{\mathbf{n}}\phi|_{z=\frac{\varepsilon}{\nu}\eta}$$

Where ϕ solves the given PDE.

$\mathscr{A}$ is a global operator which means that it cannot be calculated in the restriction of any arbitrary interval $(x_0, x_0 + d)$.
To stress that $\mathscr{A}$ acts on the boundary we will keep using the symbol $\psi(x) = \phi(x,z)|_{z=\frac{\varepsilon}{\nu}\eta}$ to indicate the surface potential.
Note: *Heuristically the operator $\mathscr{A}$ transforms Dirichlet conditions to Neumann conditions.*

3.2.2 The DN operator on small waves

In this linear case we solve the system for any ψ:

$$\delta^2\partial_{xx}\phi + \partial_{zz}\phi = 0 \qquad \text{in } \Omega \tag{27}$$

$$\phi = \psi \qquad \text{in } z = 0 \tag{28}$$

$$\partial_z\phi = 0 \qquad \text{in } z = -1 \tag{29}$$

As noted, Fourier techniques are very suitable for this problem. We substitute the common ansatz. (Summations are over $k \in \mathbb{Z}$):

$$\begin{aligned}\phi(x,z) &= \sum \hat{\phi}_k(z) e^{ikx} \\ &\Rightarrow \begin{cases} \partial_{xx}\phi = -\sum k^2 \hat{\phi}_k e^{ikx} \\ \partial_{zz}\phi = \sum \hat{\phi}_k'' e^{ikx} \end{cases} \\ \psi(x) &= \sum \hat{\psi}_k e^{ikx}\end{aligned}$$

Then, for $\hat{\phi}_k$ holds:

$$\begin{aligned}-k^2\delta^2\hat{\phi}_k + \hat{\phi}_k'' &= 0 \\ \hat{\phi}_k(0) &= \hat{\psi}_k \\ \hat{\phi}_k'(-1) &= 0\end{aligned}$$

Note that $\hat{\phi}_k$ are functions of z and $\hat{\psi}_k$ are just numbers. The first equation is a second order ODE which is solved by.

$$\begin{aligned}k = 0 : \hat{\phi}_0(z) &= C_0^1 z + C_0^2 \\ k \neq 0 : \hat{\phi}_k(z) &= C_k^1 e^{k\delta z} + C_k^2 e^{-k\delta z}\end{aligned}$$

Including the boundary conditions:

$$\begin{aligned}\hat{\phi}_0(0) &= C_0^2 = \hat{\psi}_0 \\ \hat{\phi}_0'(-1) &= C_0^1 = 0 \\ \hat{\phi}_k(0) &= (C_k^1 + C_k^2) = \hat{\psi}_k \\ \hat{\phi}_k'(-1) &= k\delta(C_k^1 e^{-k\delta} - C_k^2 e^{k\delta}) = 0\end{aligned}$$

Which gives:

$$C_0^1 = 0 \text{ and } C_0^2 = \hat{\psi}_0$$
$$C_k^1 = \frac{e^{k\delta}}{e^{k\delta} + e^{-k\delta}}\hat{\psi}_k \text{ and } C_k^2 = \frac{e^{-k\delta}}{e^{k\delta} + e^{-k\delta}}\hat{\psi}_k$$

So:

$$\begin{aligned}\hat{\phi}_0(z) &\equiv \hat{\psi}_0 \\ \hat{\phi}_k(z) &= \frac{e^{k\delta(z+1)} + e^{-k\delta(z+1)}}{e^{k\delta} + e^{-k\delta}}\hat{\psi}_k = \frac{\cosh(k\delta(z+1))}{\cosh(k\delta)}\hat{\psi}_k\end{aligned}$$

Because of the rectangular domain, the surface normal derivative is found to be only ∂_z which makes the decoupling of z and x very convenient. Thus the Fourier symbol $\hat{\mathscr{A}}_k$ of the DN operator is found to be.

$$
\begin{aligned}
\mathscr{A}\psi &= \sum \hat{\mathscr{A}}_k \hat{\psi}_k e^{ikx} \\
&= \partial_{\mathbf{n}}\phi|_{z=0} = \partial_z \phi|_{z=0} \\
\sum \hat{\mathscr{A}}_k \hat{\psi}_k e^{ikx} &= \left[\partial_z \sum \hat{\phi}_k e^{ikx}\right]_{z=0} \\
&= \left[\partial_z \left(\sum \frac{\cosh(k\delta(z+1))}{\cosh(k\delta)} \hat{\psi}_k e^{ikx} + \hat{\psi}_0\right)\right]_{z=0} \\
&= \sum \frac{k\delta \sinh(k\delta)}{\cosh(k\delta)} \hat{\psi}_k e^{ikx} \qquad (30)
\end{aligned}
$$

Note: *The constant term $\hat{\psi}_0$ plays no role. That means that $\mathscr{A}(\psi + c) = \mathscr{A}\psi$ for any constant c. This is not surprising due to the fact that ψ is actually a potential.*
The Fourier symbol of the DN operator follows from (30).

$$
\hat{\mathscr{A}}_k = k\delta \tanh(k\delta) \qquad (31)
$$

Operator $\mathscr{A}$ now contains all information from the interior of Ω we need. Furthermore, it takes functions on the boundary as an argument and it yields functions on the boundary. This is a huge advantage since we can replace every occurence of $\partial_z\phi$ on $z = \frac{\varepsilon}{\nu}\eta$ by $\mathscr{A}\psi$. For the first term we would still need z whereas in the second this variable is eliminated. This reduction goes at the expense of explicit information about the flow in the interior of Ω, but since we are studying the surface we are not concerned with this.
Making use of this for the linearized system (23) to (26)

$$
\begin{aligned}
\partial_t \eta &= \tfrac{1}{\delta^2 \nu}\mathscr{A}\psi && \text{for } x \in (0, 2\pi) \qquad (32) \\
\partial_t \psi &= -\eta && \text{for } x \in (0, 2\pi) \qquad (33)
\end{aligned}
$$

As announced, the dependence on z has completely vanished. Including any more of the four original equations would be redundant since any information they carry is included in $\mathscr{A}$. Finally, we have arrived at the desired evolution equation (2).
Since we are especially interested in the wave profile η, it would make sense to eliminate ψ. This is done by differentiating (32) with respect to t and substituting (33):

$$
\partial_{tt}\eta = -\tfrac{1}{\delta^2 \nu}\mathscr{A}\eta \qquad (34)
$$

3.2.3 The wave equation

A natural first approach for modelling water waves would be to consider the wave equation. This model actually forms the base of equation (34) and can be obtained by expanding the Fourier coefficients of $\mathscr{A}$ in a Taylor series. Although δ was initially kept $\mathscr{O}(1)$, we will now investigate for $\delta \ll 1$. This means that the wave length is much larger than the water depth, thus $\delta k \ll 1$. Using the Taylor expansion for the symbol.

$$\hat{\mathscr{A}}_k = k\delta \tanh(k\delta) = (\delta k)^2 + \tfrac{1}{3}(\delta k)^4 + \tfrac{2}{15}(\delta k)^6 + \dots$$

and thus:

$$\mathscr{A} = -\delta^2\partial_{xx} - \tfrac{1}{3}\delta^4\partial_x^4 - \tfrac{2}{15}\delta^6\partial_x^6 + \dots$$

Applying this to (34):

$$\partial_{tt}\eta = \tfrac{1}{v}\partial_{xx}\eta + \tfrac{1}{3v}\delta^2\partial_x^4 - \tfrac{2}{15v}\delta^4\partial_x^6 + \dots \tag{35}$$

By ignoring terms from $\mathcal{O}(\delta^2)$ onwards, we arrive at the wave equation.
We already assumed $\varepsilon \ll 1$. When simultaneously considering $\delta \ll 1$ it is necessary to regard the order relation between the both of them. Recall $\varepsilon = \frac{a}{h_0}$ and $\delta = \frac{h_0}{L}$. If both parameters are small, this implies $L \gg h_0 \gg a$.
As stated in Section 2.5.3, the relation between a and L, given by $\varepsilon\delta = \frac{a}{L}$, is a measure for $\partial_x\eta$, the steepness of the wave. Therefore the wave equation holds when $\varepsilon\delta \ll 1$. This kind of waves is known as very long and flat .

3.2.4 Alternative approximations of $\mathscr{A}$

The approximating $\mathscr{A}$ by its first term shows to be poor because we do not expect water to behave the way solutions of the wave equation do. One could argue that including more terms of the Taylor series would improve the solution, but really better options are, among others:

$$\hat{A}_k \approx (\delta k)^2\left(1 + \tfrac{1}{6}(\delta k)^2\right)^2$$
$$\hat{A}_k \approx (\delta k)^2\left(1 - \tfrac{1}{3}(\delta k)^2\right)^{-1}$$

These choices seem somewhat arbitrary, but they have the advantage that they yield equations with a squared simple operator and an inverted simple operator respectively:

$$\partial_{tt}\eta \approx -\tfrac{\delta^2}{v}\mathbf{G}^2\partial_{xx}\eta$$
$$\partial_{tt}\eta \approx -\tfrac{\delta^2}{v}\mathbf{H}^{-1}\partial_{xx}\eta \Rightarrow \partial_{tt}\mathbf{H}\eta \approx -\tfrac{\delta^2}{v}\partial_{xx}\eta$$

with:

$$\mathbf{G} = 1 - \tfrac{1}{6}\delta^2\partial_{xx}$$
$$\mathbf{H} = 1 + \tfrac{1}{3}\delta^2\partial_{xx}$$

This kind of operators allows for some numerical convenience whereas the truncation of the Taylor series does not necessarily. However, numerics are not investigated in this report. A more in-depth analysis involving the operator kernels of **G** and **H** can be found in [1].

3.3 Dispersion

Dispersion is the effect that the frequency of the wave is dependent of its wave number. In this case, a wave profile composed of several frequencies will be dispersed along the surface as the wave propagates. To find this relation we substitute $\eta(x,t) = e^{ik(x-ct)} = e^{i(kx-\omega t)}$ (a single simple wave with speed c, wave number k and frequency ω) in equation (34). Then only the k^{th} term of the Fourier series will be extracted

$$\begin{aligned} \partial_{tt}\eta &= -\tfrac{1}{\delta^2 \nu}\mathscr{A}\eta \\ -k^2 c^2 e^{ik(x-ct)} &= -\tfrac{1}{\delta^2 \nu} k\delta \tanh(k\delta) e^{ik(x-ct)} \end{aligned}$$

A relation between the speed and the wave number is then given by (recall $\nu = \frac{\tanh(\delta)}{\delta}$):

$$c^2 = \frac{1}{\nu}\frac{\tanh(k\delta)}{k\delta} = \frac{\tanh(k\delta)}{k\tanh(\delta)}$$

Thus, the wave speed and the frequency would be given:

$$\begin{aligned} c &= \pm\sqrt{\frac{\tanh(k\delta)}{k\tanh(\delta)}} \\ \omega = kc &= \sqrt{\frac{k\tanh(k\delta)}{\tanh(\delta)}} \end{aligned}$$

To get a feeling of the effects of dispersion, review the limiting cases for deep water and shallow water:

$$\begin{aligned} \lim_{\delta\to\infty} c &= \pm\tfrac{1}{\sqrt{k}} \\ \lim_{\delta\to 0} c &= \pm 1 \end{aligned}$$

From which we conclude that dispersive effects increase as the depth increases.
The speed c is known as the phase speed. This is the propagation speed of a simple wave. For waves composed of several frequencies, a group speed is defined as well. This is the speed of the enveloping profile and is given by $\frac{\mathrm{d}\omega}{\mathrm{d}k}$. For small waves, the group speed is:

$$\begin{aligned} v_g = \tfrac{\mathrm{d}\omega}{\mathrm{d}k} &= \frac{\sqrt{\tanh(\delta k)}}{2\sqrt{k\tanh(\delta)}} + \frac{\delta\sqrt{k}}{2\sqrt{\tanh(\delta k)\tanh(\delta)}} \\ &= \tfrac{1}{2}c\left(1 + \frac{\delta k}{\sinh(\delta k)\cosh(\delta k)}\right) \end{aligned}$$

Once again, we consider the limiting cases for deep water and shallow water:

$$\begin{aligned} \lim_{\delta\to\infty} v_g &= \pm\tfrac{1}{2\sqrt{k}} && (= \tfrac{1}{2}c) \\ \lim_{\delta\to 0} v_g &= \pm 1 && (= c) \end{aligned}$$

From these limits we conclude that the group speed is equal to the phase speed in shallow water and equal to half the phase speed in deep water.
Some graphs of the results from this section are provided in Figure 5.
Note: *If we would not have used the dynamical scaling with ν in Section 2.5.3, some of the limits would be indeterminate.*

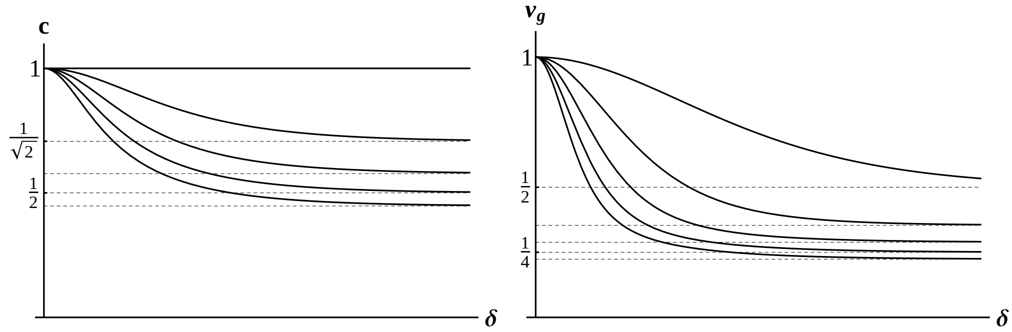


Figure 5: Plots of the phase speed c and the group speed v_g as function of δ for values $k = 1,\dots,5$

3.4 The exact solution

In this linear case we can, unlike in the other discussed models, actually provide an exact solution in terms of Fourier series. This is not a goal of this report and, if nothing else, the derivation is just a showcase for the DN operator.

We will solve the following evolution system with initial conditions:

$$\begin{aligned}
\partial_{tt}\eta &= -\tfrac{1}{\delta^2\nu}\mathscr{A}\eta && \text{for } x \in (0,2\pi), t > 0\\
\eta(x,0) &= \mu(x) && \text{for } x \in (0,2\pi), t = 0\\
\partial_t\eta(x,0) &= \sigma(x) && \text{for } x \in (0,2\pi), t = 0
\end{aligned}$$

With $\hat{\eta}_k(t)$, $\hat{\mu}_k$ and $\hat{\sigma}_k$ the Fourier coefficients of the various functions, we can write:

$$\sum \hat{\eta}_k'' e^{ikx} = \tfrac{1}{\delta^2\nu}\sum \hat{\eta}_k k\delta \tanh(k\delta) e^{ikx}$$
$$\Rightarrow \hat{\eta}_k'' = -\frac{k\tanh(k\delta)}{\tanh\delta}\hat{\eta}_k$$

This is a second order ODE which is solved by:

$$\hat{\eta}_k(t) = C_k^1 e^{i\omega_k t} + C_k^2 e^{-i\omega_k t}$$

With $\omega_k = \sqrt{\frac{k\tanh(k\delta)}{\tanh\delta}}$, as found in the previous section.
The initial conditions:

$$\begin{aligned}
\hat{\eta}_k(0) &= C_k^1 + C_k^2 = \hat{\mu}_k \\
\hat{\eta}'_k(0) &= i\omega_k(C_k^1 - C_k^2) = \hat{\sigma}_k \\
&\Rightarrow \\
C_k^1 &= \tfrac{1}{2}(\hat{\mu}_k - \tfrac{\hat{\sigma}_k}{\omega_k} i) \\
C_k^2 &= \tfrac{1}{2}(\hat{\mu}_k + \tfrac{\hat{\sigma}_k}{\omega_k} i)
\end{aligned}$$

Which gives the solution in its fullest:

$$\begin{aligned}
\eta &= \sum \left[\tfrac{1}{2}(\hat{\mu}_k - \tfrac{\hat{\sigma}_k}{\omega_k} i) e^{i\omega_k t} + \tfrac{1}{2}(\hat{\mu}_k + \tfrac{\hat{\sigma}_k}{\omega_k} i) e^{-i\omega_k t} \right] e^{ikx} \\
&= \sum \left[\hat{\mu}_k \cos(\omega_k t) + \frac{\hat{\sigma}_k}{\omega_k} \sin(\omega_k t) \right] e^{ikx}
\end{aligned}$$

4 Shallow water

In this section we shall review shallow water. That is, $\delta \ll 1$. Now we will keep ε at $\mathscr{O}(1)$. The orignal scaled system (15) to (18) had nonlinearities of $\mathscr{O}(\varepsilon)$, therefore the small wave approximations were linear. In this case we do not expect to get linear models.

4.1 The general Dirichlet-Neumann operator

Before any approximations occur. We will define a new DN operator. It will, however, not work out quite as nice as before because of the lack of linearity. We cannot expect to find the operator (or its Fourier symbol) explicitly. To stress that the operator we will be using now is no longer a linear approximation as in Section 3.2.2, the symbol $\mathscr{G}$ is assigned to it. This is the actual DN operator for the original problem (15) to (18).
Again, $\mathscr{G}$ is defined by means of the normal derivative:

$$\mathscr{G}\psi := \left[-\nu\delta^2\varepsilon\partial_x\eta\partial_x\phi + \partial_z\phi\right]_{z=\frac{\varepsilon}{\nu}\eta} \tag{36}$$

With ϕ solving the PDE related to (15) to (18):

$$\nu^2\delta^2\partial_x\phi + \partial_z\phi = 0 \qquad \text{for } -\tfrac{1}{\nu} < z < \tfrac{\varepsilon}{\nu}\eta \tag{37}$$

$$\phi = \psi \qquad \text{for } z = \tfrac{\varepsilon}{\nu}\eta \tag{38}$$

$$\partial_z\phi = 0 \qquad \text{for } z = -\tfrac{1}{\nu} \tag{39}$$

Now once again, we use the equality (36) to eliminate all occurrences of $\partial_z\phi$ in (15) to (18)

$$\partial_t\eta - \tfrac{1}{\delta^2\nu^2}\mathscr{G}\,\psi = 0 \tag{40}$$

$$\partial_t\psi + \tfrac{\varepsilon}{2\nu}(\partial_x\psi)^2 + \tfrac{\varepsilon}{\delta^2\nu^2}\left(\mathscr{G}\psi + \nu\delta^2\varepsilon\partial_x\eta\partial_x\psi\right)^2 + \eta = 0 \tag{41}$$

And we arrive at an evolution system as desired in (2). It is only defined on the boundary and thus does not depend on z.

4.2 Depth-averaged horizontal velocity

First, we set $\nu = 1$. For shallow water, this gives the desired scaling.
The *depth-averaged horizontal velocity* $\bar{v}$ is defined as the average horizontal velocity over some vertical column of water.

$$\bar{v}(x) := \frac{1}{h(x)}\int_{-1}^{\varepsilon\eta(x)} \partial_x\phi(x,z)\,\mathrm{d}z$$

Now, we consider $\partial_x(h\bar{v})$. We have, by making use of both the continuity equation (15) and the non-permeable lower boundary (16).

$$\begin{aligned}
\partial_x(h\bar{v}) =& \partial_x\left[\int_{-1}^{\varepsilon\eta(x)} \partial_x\phi(x,z)\,\mathrm{d}z\right] \\
=& \int_{-1}^{\varepsilon\eta} \partial_{xx}\phi\,\mathrm{d}z + \left[\partial_x(\varepsilon\eta)\partial_x\phi\right]_{z=\varepsilon\eta} \\
\overset{(15)}{=}& \int_{-1}^{\varepsilon\eta} -\tfrac{1}{\delta^2}\partial_{zz}\phi\,\mathrm{d}z + \varepsilon\left[\partial_x\eta\partial_x\phi\right]_{z=\varepsilon\eta} \\
=& \left[-\tfrac{1}{\delta^2}\partial_z\phi\right]_{z=\varepsilon\eta} - \left[-\tfrac{1}{\delta^2}\partial_z\phi\right]_{z=-1} + \varepsilon\left[\partial_x\eta\partial_x\phi\right]_{z=\varepsilon\eta} \\
\overset{(16)}{=}& \varepsilon\partial_x\eta\partial_x\psi - \tfrac{1}{\delta^2}\partial_z\psi
\end{aligned}$$

Now note the similarity with definition (36). Evidently we have:

$$\mathscr{G}\psi = -\delta^2\partial_x(h\bar{v}) \tag{42}$$

This is an unexpected result and one might reject the earlier statement that we cannot find $\mathscr{G}$ explicitly. Result (42) sure seems explicit enough. This is not true, though. Note that $\bar{v}$ requires information from the full range of z and thus does not qualify for our desirable evolution system (2).
It would make sense if we had a certain depth, say z^*, at which the horizontal velocity equals $\bar{v}$. But in general, this depth is hard to point out. For shallow water, however, we would be always close to this depth in the sense that $|\frac{\varepsilon}{\nu}\eta - z^*| < \delta \ll 1$. So $|\bar{v} - \partial_x\psi| \le \delta\partial_z\phi(x,z^*)|\psi(x) - \phi(x,z^*)|$. Therefore it will be useful to examine $\bar{v}$ after all.

4.3 Expansion in δ: Shallow water and Green-Naghdi

Now we get to the actual approximation of $\delta \ll 1$. As before, we will expand the quantities with respect to the small parameter. Different from the former case is that we will not try to find the solution ϕ immediately. This is because the domain is not rectangular, and the equations are not linear.
Therefore, the operator $\mathscr{G}$ will be approximated, and then approximating equations for $\bar{v}$ are found by the coupling of (42).
Start with the perturbation:

$$\phi = \phi_0 + \delta\phi_1 + \delta^2\phi_2 + \ldots$$

Plug it in the Dirichlet problem (37) to (39):

$$\begin{aligned}
\delta^2\partial_{xx}\phi + \partial_{zz}\phi &= 0 && \text{for } z \in (-1,\eta) \\
\phi &= \psi && \text{for } z = \eta \\
\partial_z\phi &= 0 && \text{for } z = -1
\end{aligned}$$

This gives a recurrent relationship linking the different orders of δ:

$$\begin{aligned} \partial_{xx}\phi_{i-2} + \partial_{zz}\phi_i &= 0 && \text{for } z \in (-1, \varepsilon\eta) \\ \phi_0 &= \psi && \text{for } z = \varepsilon\eta \\ \phi_i &= 0 && \text{for } z = \varepsilon\eta, i \neq 0 \\ \partial_z\phi_i &= 0 && \text{for } z = -1 \end{aligned}$$

The even and odd terms of the sequence are decoupled. Also, the odd terms are fully homogenous. Therefore we can pick the trivial solution $\phi_i \equiv 0$ for odd values of i. By convention, we also set $\phi_{-2} \equiv \phi_{-1} \equiv 0$. For even values, the system is solvable up to any order. Starting with ϕ_0:

$$\begin{aligned} \partial_{zz}\phi_0 &= 0 \\ &\Rightarrow \phi_0 = C_0^1(x)z + C_0^2(x) \end{aligned}$$

The boundaries for ϕ_0

$$\begin{aligned} \phi_0(\varepsilon\eta, z) &= C_0^1\varepsilon\eta + C_0^2 = \psi \\ \partial_z\phi_0(-1, z) &= C_0^1 = 0 \\ &\Rightarrow \phi_0 = \psi \end{aligned}$$

Using this solution for the next term, ϕ_2:

$$\begin{aligned} \partial_{zz}\phi_2 &= -\partial_{xx}\phi_0 = -\partial_{xx}\psi \\ &\Rightarrow \phi_2 = -\tfrac{1}{2}\partial_{xx}\psi z^2 + C_2^1 z + C_2^2 \end{aligned}$$

And its boundaries

$$\begin{aligned} \phi_2(\varepsilon\eta, z) &= -\tfrac{1}{2}\partial_{xx}\psi z^2 + C_2^1 z + C_2^2 = 0 \\ \partial_z\phi_2(-1, z) &= \partial_{xx}\psi + C_2^1 = 0 \\ &\Rightarrow \phi_2 = \left[(\tfrac{1}{2}\varepsilon^2\eta^2 + \varepsilon\eta) - (\tfrac{1}{2}z^2 + z)\right]\partial_{xx}\psi \end{aligned}$$

Combining ϕ_0 and ϕ_2 gives the approximation up to $\mathcal{O}(\delta^4)$:

$$\begin{aligned} \phi &= \phi_0 + \delta^2\phi_2 + \mathcal{O}(\delta^4) \\ &= \psi + \delta^2\left[(\tfrac{1}{2}\varepsilon^2\eta^2 + \varepsilon\eta) - (\tfrac{1}{2}z^2 + z)\right]\partial_{xx}\psi + \mathcal{O}(\delta^4) \end{aligned}$$

It would be possible to repeat this procedure as often as desired for higher orders but in this report we will restrict to a maximal accuracy of $\mathcal{O}(\delta^4)$.

4.3.1 Up to $\mathcal{O}(\delta^4)$: Green-Naghdi

For the Green-Naghdi equations the derivation from [10] follows. These result in a system on $\bar{v}$ and η which is not the original formulation from [5].
As announced, the surface potential is used to approximate $\bar{v}$ (recall: $h = \varepsilon\eta + 1$):

$$\begin{aligned}
\bar{v}h &= \int_{-1}^{\varepsilon\eta(x)} \partial_x \phi(x,z)\,dz + \mathcal{O}(\delta^4) \\
&= \int_{-1}^{\varepsilon\eta(x)} \partial_x \left[1 + \delta^2(\tfrac{1}{2}\varepsilon^2\eta^2 + \varepsilon\eta) - \delta^2(\tfrac{1}{2}z^2 + z)\partial_{xx}\right]\psi\,dz + \mathcal{O}(\delta^4) \\
&= h\partial_x\psi + \left[\int_{-1}^{\varepsilon\eta(x)} \delta^2(\tfrac{1}{2}\varepsilon^2\eta^2 + \varepsilon\eta)\,dz - \int_{-1}^{\varepsilon\eta(x)} \delta^2(\tfrac{1}{2}z^2 + z)\,dz\right]\partial_x^3\psi + \mathcal{O}(\delta^4) \\
&= h\partial_x\psi + \left[h\delta^2(\tfrac{1}{2}\varepsilon^2\eta^2 + \varepsilon\eta) - \delta^2(\tfrac{1}{6}\varepsilon^3\eta^3 + \tfrac{1}{2}\varepsilon^2\eta^2 - \tfrac{1}{3})\right]\partial_x^3\psi + \mathcal{O}(\delta^4) \\
&\Rightarrow \\
\bar{v} &= \partial_x\psi + \tfrac{\delta^2}{h}\left[h(\tfrac{1}{2}\varepsilon^2\eta^2 + \varepsilon\eta) - (\tfrac{1}{6}\varepsilon^3\eta^3 + \tfrac{1}{2}\varepsilon^2\eta^2 - \tfrac{1}{3})\right]\partial_x^3\psi + \mathcal{O}(\delta^4) \\
&= \partial_x\psi + \delta^2\left[(\tfrac{1}{2}h^2 - \tfrac{1}{2}) - (\tfrac{1}{6}h^2 - \tfrac{1}{2})\right]\partial_x^3\psi + \mathcal{O}(\delta^4) \\
&= \partial_x\psi + \tfrac{1}{3}\delta^2 h^2\partial_x^3\psi + \mathcal{O}(\delta^4)
\end{aligned} \tag{43}$$

To simplify the way this last equations will be inverted, define the operator $\mathbf{T} := \frac{1}{3}h^2\partial_{xx}$. Then (43) reads:

$$\begin{aligned}
\bar{v} &= (1 + \delta^2\mathbf{T})\partial_x\psi + \mathcal{O}(\delta^4) \\
&\Rightarrow \\
(1 + \delta^2\mathbf{T})^{-1}\,\bar{v} &= \partial_x\psi + \mathcal{O}(\delta^4) \\
(1 - \delta^2\mathbf{T})\,\bar{v} &= \partial_x\psi + \mathcal{O}(\delta^4) \\
\Rightarrow \partial_x\psi &= \left[1 - \tfrac{1}{3}\delta^2 h^2\partial_{xx}\right]\bar{v} + \mathcal{O}(\delta^4)
\end{aligned} \tag{44}$$

In the last step the operator is inverted by means of a Neumann series. This introduces an extra error of $\mathcal{O}(\delta^4)$. Therefore the choice to restrict to this order was a convenient one.
To arrive at th Green-Naghdi model, we manipulate the system of (40) and (41). First, use (42) to eliminate $\mathcal{G}$ from (40). Then it reads:

$$\partial_t\eta + \partial_x(h\bar{v}) = 0 \tag{45}$$

This is known as the classical formulation of the kinematic condition.
Now in (41) first replace $\mathcal{G}$ by means of (42), then take the derivative to x and finally replace

$\partial_x\psi$ by means of (44). Throughout, terms of $\mathcal{O}(\delta^4)$ will be collected on the right hand side.

$$\partial_t\psi + \tfrac{1}{2}\varepsilon(\partial_x\psi)^2 + \tfrac{\varepsilon}{\delta^2}\left(\mathcal{G}\psi + \delta^2\varepsilon\partial_x\eta\partial_x\psi\right)^2 + \eta = \mathcal{O}(\delta^4)$$
$$\partial_t\psi + \tfrac{\varepsilon}{2}(\partial_x\psi)^2 + \tfrac{\varepsilon}{\delta^2}\left(-\delta^2\partial_x(h\bar{v}) + \delta^2\varepsilon\partial_x\eta\partial_x\psi\right)^2 + \eta = \mathcal{O}(\delta^4)$$
$$\partial_t\partial_x\psi + \varepsilon\partial_x\psi\partial_{xx}\psi + \varepsilon\delta^2\partial_x\left(-\partial_x(h\bar{v}) + \varepsilon\partial_x\eta\partial_x\psi\right)^2 + \partial_x\eta = \mathcal{O}(\delta^4)$$

$$\partial_t\left[(1-\delta^2T)\bar{v}\right] + \varepsilon\left[(1-\delta^2T)\bar{v}\right]\partial_x\left[(1-\delta^2T)\bar{v}\right]$$
$$+\varepsilon\delta^2\partial_x\left(-\partial_x(h\bar{v}) + \varepsilon\partial_x\eta\left[(1-\delta^2T)\bar{v}\right]\right)^2 + \partial_x\eta = \mathcal{O}(\delta^4)$$

$$(1-\delta^2T)\partial_t\bar{v} + \varepsilon\bar{v}\partial_x\bar{v} - \varepsilon\delta^2\left[\bar{v}T\partial_x\bar{v} + \partial_x\bar{v}T\bar{v}\right]$$
$$+\varepsilon\delta^2\partial_x\left[(\partial_x(h\bar{v}))^2 + \varepsilon^2(\bar{v}\partial_x\eta)^2 - 2\varepsilon\partial_x(h\bar{v})\partial_x\eta\bar{v} - 2\varepsilon\delta^2\partial_x\eta\bar{v}T\bar{v}\right] + \partial_x\eta = \mathcal{O}(\delta^4)$$

$$(1-\delta^2T)\partial_t\bar{v} + \varepsilon\bar{v}\partial_x\bar{v} + \varepsilon\delta^2\left[\tfrac{1}{3}h^2\partial_x(\bar{v}\partial_{xx}\bar{v})\right] - \varepsilon\delta^2\left[\partial_x(h\partial_x\bar{v})^2\right] + \partial_x\eta = \mathcal{O}(\delta^4)$$

$$(1-\tfrac{1}{3}h^2\delta^2\partial_{xx})\partial_t\bar{v} + \varepsilon\bar{v}\partial_x\bar{v} - \tfrac{\varepsilon\delta^2}{3h}\partial_x\left[h^3\bar{v}\partial_{xx}\bar{v} - (\partial_x\bar{v})^2\right] + \partial_x\eta = \mathcal{O}(\delta^4) \tag{46}$$

Now the Green-Naghdi equations are those for $\bar{v}$ and η with ignoring $\mathcal{O}(\delta^4)$:

$$\partial_t\eta + \partial_x(h\bar{v}) = 0 \tag{47}$$
$$(1-\tfrac{1}{3}h^2\delta^2\partial_{xx})\partial_t\bar{v} + \varepsilon\bar{v}\partial_x\bar{v} - \tfrac{\varepsilon\delta^2}{3h}\partial_x\left[h^3\bar{v}\partial_{xx}\bar{v} - (\partial_x\bar{v})^2\right] + \partial_x\eta = 0 \tag{48}$$

These are very similar to the desired system (2). Differences are the fact that they use $\bar{v}$ instead of ϕ and the system has an operator $(1-\frac{1}{3}h^2\delta^2\partial_{xx})$ in front of $\partial_t\bar{v}$. In fact, this operator is desirable for it appears to give some numerical regularisation. (Remark 3.3 in [10])

4.3.2 Up to $\mathcal{O}(\delta^2)$: Standard shallow water

The standard shallow water equations are not yet discussed. This is because the derivations are easy after those of the Green-Naghdi equations. Shallow water follows the same derivation, but then up to $\mathcal{O}(\delta^2)$ instead of $\mathcal{O}(\delta^4)$. The model can be obtained from (47) and (48) by ignoring $\mathcal{O}(\delta^2)$ terms:

$$\partial_t\eta + \partial_x(h\bar{v}) = 0 \tag{49}$$
$$\partial_t\bar{v} + \varepsilon\bar{v}\partial_x\bar{v} + \partial_x\eta = 0 \tag{50}$$

Note: *By ignoring $\mathcal{O}(\varepsilon)$ effects and eliminating $\bar{v}$, the wave equation is obtained.*

5 Other methods

This section analyzes some other methods used for finding various models. About half of the authors use a *Hamiltonian framework* to approach water waves. We will give an introduction to this method.
Apart from this completely alternative approach, there are also three ad hoc methods that are applicable in specific cases. The first one being *uni-directionalisation* which is the main tool towards the Korteweg-de Vries equation. The second method is the *tidal wave* approximation in which vertical fluid motion is ignored. And lastly, we have the *Boussinesq* approximations.

5.1 Hamiltonian description

The theory of Hamiltonians is applicable in a great range of physical applications. In the case of water waves, [6] and [1] provide an excellent view of the possibilties. In this report, the general idea is given.
Note: *Hamiltonian considerations are concise and formal. They do, however, not provide any new results unobtainable from the methods in the rest of this report. A very important reason for using them is the fact that the Hamiltonian structure can be used to yield discretisation schemes that are guaranteed to be numerically stable.*
The framework is defined by the following:
Define a Hamiltonian scalar $H(\phi,\eta)$ as the total energy in the basin. That is, the sum of potential and kinetic energy.

$$H(\phi,\eta) = P(\eta) + K(\phi,\eta)$$

These energies can be found (observe that g and ρ have been scaled out of the problem):

$$\begin{aligned} P &= \int\int_\Omega z \,\mathrm{d}z\,\mathrm{d}x = \int \tfrac{1}{2}h^2\,\mathrm{d}x = \int \tfrac{1}{2\nu}(\varepsilon\eta+1)^2\,\mathrm{d}x \\ K &= \int\int_\Omega \tfrac{1}{2}|\nabla\phi|^2\,\mathrm{d}z\,\mathrm{d}x \end{aligned}$$

With ϕ solving (37) to (39).
Now a Hamiltonian system is described by (proof in [6]):

$$\begin{pmatrix}\partial_t\phi\\ \partial_t\eta\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\delta_\phi H\\ \delta_\eta H\end{pmatrix}$$

With δ_ϕ and δ_η indicating functional derivatives and now using conjugate variables ϕ and η.
Of the two functionals, the kinetic energy K is the most difficult since it depends implicitly (via the Laplace-system) on η. Approximating K is done via approximating ϕ in a similar fashion as in this report.

5.2 Uni-directionalisation and the Korteweg-de Vries

5.2.1 Uni-directionalisation

The technique of uni-directionalisation is derived from an order argument. First, assume similarity between ε and δ^2. Then in leading order, the solution of the Green-Naghdi equations ((47) and (48)) is zero, that is $\bar{v} - \eta = \mathcal{O}(\varepsilon, \delta^2)$. In the next order, that of $\varepsilon \sim \delta^2$, the solution is given by the wave equation. The wave equation is solved by travelling waves. If we pick out one direction, say increasing x, then we have a solution F, such that $\partial_x F = -\partial_t F$. The higher orders of δ^4, ε^2 and $\varepsilon\delta^2$ will be collected in one term. The uni-directional ansatz sums up these order arguments.
The trick is to consider the travelling waves of $\mathcal{O}(\varepsilon)$ and those of $\mathcal{O}(\delta^2)$ seperately.

$$\bar{v} = \eta + \varepsilon F + \delta^2 G + \mathcal{O}(\varepsilon^2, \delta^4, \varepsilon\delta^2) \tag{51}$$

With $\bar{v}$ and η satisfying the wave equation and with F and G travelling waves in increasing direction.
Equation (51) is called the uni-directional ansatz. This is a useful form because the first order wave equation allows to easily equate derivatives to t with derivatives to x.

5.2.2 Korteweg-de Vries equation

We apply the uni-directional ansatz (51) to the Green-Naghdi equations (47), (48). Starting with (47):

$$\begin{aligned}
\partial_t \eta + \partial_x (h\bar{v}) &= 0 \\
\partial_t \eta + \partial_x \left((1+\varepsilon\eta)(\eta + \varepsilon F + \delta^2 G) \right) &= \mathcal{O}(\varepsilon^2, \delta^4, \varepsilon\delta^2) \\
\partial_t \eta + \partial_x \left(\eta + \varepsilon\eta^2 + \varepsilon F + \delta^2 G \right) &= \mathcal{O}(\varepsilon^2, \delta^4, \varepsilon\delta^2) \\
\partial_t \eta + \partial_x \eta + \varepsilon \left(2\eta\partial_x\eta + \partial_x F \right) + \delta^2 \partial_x G &= \mathcal{O}(\varepsilon^2, \delta^4, \varepsilon\delta^2)
\end{aligned} \tag{52}$$

And then in (48):

$$\begin{aligned}
(1 - \tfrac{1}{3} h^2 \delta^2 \partial_{xx}) \partial_t \bar{v} + \varepsilon \bar{v} \partial_x \bar{v} - \tfrac{\varepsilon\delta^2}{3h} \partial_x \left[h^3 \bar{v} \partial_{xx} \bar{v} - (\partial_x \bar{v})^2 \right] + \partial_x \eta &= 0 \\
(1 - \tfrac{1}{3}(\varepsilon\eta+1)^2 \delta^2 \partial_{xx}) \partial_t \left(\eta + \varepsilon F + \delta^2 G \right) + \varepsilon \left(\eta + \varepsilon F + \delta^2 G \right) \partial_x \left(\eta + \varepsilon F + \delta^2 G \right) + \partial_x \eta &= \mathcal{O}(\varepsilon^2, \delta^4, \varepsilon\delta^2) \\
\partial_t \eta + \varepsilon \partial_t F + \delta^2 \partial_t G - \tfrac{1}{3} \delta^2 \partial_{xxt} \eta + \varepsilon \eta \partial_x \eta + \partial_x \eta &= \mathcal{O}(\varepsilon^2, \delta^4, \varepsilon\delta^2) \\
\partial_t \eta + \partial_x \eta + \varepsilon \left(\partial_t F + \eta \partial_x \eta \right) + \delta^2 \left(\partial_t G - \tfrac{1}{3} \partial_{xxt} \eta \right) &= \mathcal{O}(\varepsilon^2, \delta^4, \varepsilon\delta^2) \\
\partial_t \eta + \partial_x \eta + \varepsilon \left(\partial_t F + \eta \partial_x \eta \right) + \delta^2 \left(\partial_t G + \tfrac{1}{3} \partial_{xxx} \eta \right) &= \mathcal{O}(\varepsilon^2, \delta^4, \varepsilon\delta^2)
\end{aligned} \tag{53}$$

In the very last step we have used that that $\partial_t \eta = -\partial_x \eta$. This is a consequence of the fact that η satisfies the wave equation and is a travelling wave in negative direction.
Subtracting (53) from (52) and dropping the higher order terms gives:

$$\varepsilon \left(\partial_t F - \partial_x F - \eta \partial_x \eta \right) + \delta^2 \left(\partial_t G - \partial_x G + \tfrac{1}{3} \partial_{xxx} \eta \right) = 0$$

This should hold for al ε and δ^2 in the region, so we can decouple this equation in $\mathcal{O}(\varepsilon)$ en $\mathcal{O}(\delta^2)$.

$$\begin{aligned}\partial_x F - \partial_t F + \eta\partial_x\eta &= 0\\ \partial_x G - \partial_t G - \tfrac{1}{3}\partial_{xxx}\eta &= 0\end{aligned}$$

Now use the first order wave equation for F and G:

$$\begin{aligned}2\partial_x F &= -\tfrac{1}{2}\partial_x(\eta^2)\\ 2\partial_x G &= +\tfrac{1}{3}\partial_{xxx}\eta\\ &\Rightarrow\\ F &= -\tfrac{1}{4}(\eta)^2 + C_F(t)\\ G &= +\tfrac{1}{6}\partial_{xx}\eta + C_G(t)\end{aligned}$$

To not violate the wave equation, $C_F(t)$ and $C_G(t)$ should be constant in t. They can be chosen zero because F and G are defined up to a constant.
Replacing F and G in the uni-directional ansatz gives:

$$\bar{v} = \eta - \tfrac{1}{4}\varepsilon\eta^2 + \tfrac{1}{6}\delta^2\partial_{xx}\eta + \mathcal{O}(\varepsilon^2, \delta^2, \varepsilon\delta^2)$$

Now, this equation can be used to eliminate $\bar{v}$ by substituting it in (47) and collecting higher order terms.

$$\begin{aligned}\partial_t\eta + \partial_x(h\bar{v}) &= 0\\ \partial_t\eta + \partial_x\eta\big((1+\varepsilon\eta)(\eta - \tfrac{1}{4}\varepsilon\eta^2 + \tfrac{1}{6}\delta^2\partial_{xx}\eta)\big) &= \mathcal{O}(\varepsilon^2, \delta^2, \varepsilon\delta^2)\\ \partial_t\eta + \partial_x\eta + \tfrac{3}{2}\eta\partial_x\eta + \tfrac{1}{6}\delta^2\partial_{xxx}\eta &= \mathcal{O}(\varepsilon^2, \delta^2, \varepsilon\delta^2) \qquad (54)\end{aligned}$$

Equation (54), without the higher order terms, is essentially the Korteweg-de Vries equation. But its common form is different. This is a result of translating and scaling. First we translate: $\eta = \tilde{\eta} - \frac{2}{3\varepsilon}$

$$\begin{aligned}\partial_t\tilde{\eta} + \partial_x\tilde{\eta} + \tfrac{3}{2}(\tilde{\eta} - \tfrac{2}{3\varepsilon})\partial_x\tilde{\eta} + \tfrac{1}{6}\delta^2\partial_{xxx}\tilde{\eta} &= 0\\ \partial_t\tilde{\eta} + \partial_x\tilde{\eta} + \tfrac{3}{2}\varepsilon\tilde{\eta}\partial_x\tilde{\eta} + \tfrac{1}{6}\delta^2\tilde{\eta} &= 0\end{aligned}$$

Next, we rescale $\tilde{\eta}$ and t:

$$\begin{aligned}\bar{\eta} = \tfrac{\delta}{\varepsilon}\tilde{\eta} &\text{ and } \bar{t} = \tfrac{1}{\delta}t\\ \Rightarrow \partial_{\bar{t}}\bar{\eta} + \tfrac{3}{2}\bar{\eta}\partial_{\bar{x}}\bar{\eta} &+ \tfrac{1}{6}\partial_{\bar{x}\bar{x}\bar{x}}\bar{\eta} = 0\end{aligned}$$

Dropping the bars gives the final form of the Korteweg-de Vries equation:

$$\partial_t\eta + \tfrac{3}{2}\eta\partial_x\eta + \tfrac{1}{6}\partial_{xxx}\eta = 0 \qquad (55)$$

5.3 Tidal wave approximation

The tidal wave approximation is the result of the assumption that $\partial_z\phi = 0$. That is, vertical fluid motion is ignored [6]. Thus, $\phi(x,z,t) = \psi(x,t)$ for all z. Substituting this in the surface conditions (16) and (17) gives:

$$\begin{aligned}\partial_t\eta + \tfrac{\varepsilon}{\nu}\partial_x\eta\partial_x\psi &= 0\\ \partial_t\psi + \tfrac{1}{2}\tfrac{\varepsilon}{\eta}(\partial_x\psi)^2 + \eta &= 0\end{aligned}$$

It is unclear in which regions the assumption of negligible vertical fluid motion is justified.

5.4 Boussinesq

The parametric region of the Boussnesq approximation is $\varepsilon \sim \delta^2$ small. This domain lies somewhere between the small waves and shallow water.
Starting with the upper boundary conditions (17), (18). We set $\delta^2 = \varepsilon$:

$$\begin{aligned}\partial_t\eta + \tfrac{\varepsilon}{\nu}\partial_x\eta\partial_x\phi - \tfrac{1}{\varepsilon\nu^2}\partial_z\phi &= 0\\ \partial_t\phi + \tfrac{1}{2}\left[\tfrac{\varepsilon}{\nu}(\partial_x\phi)^2 + \tfrac{1}{\nu^3}(\partial_z\phi)^2\right] + \eta &= 0\end{aligned}$$

In these equations, still a term $\partial_z\phi$ is present. This term is defined in the interior of Ω. Boussinesq found a way to approximate $\partial_z\phi$ in terms of the horizontal velocity at the bottom by means of a Taylor expansion [4].
Expand $\phi(x,z,t)$ around the bed $z = -\frac{1}{\nu}$.

$$\begin{aligned}\phi(x,z,t) = \phi(x,-\tfrac{1}{\nu},t) &+ (z+\tfrac{1}{\nu})\partial_z\phi(x,-\tfrac{1}{\nu},t) + \tfrac{1}{2}(z+\tfrac{1}{\nu})^2\partial_z^2\phi(x,-\tfrac{1}{\nu},t)\\ &+ \tfrac{1}{6}(z+\tfrac{1}{\nu})^3\partial_z^3\phi(x,-\tfrac{1}{\nu},t) + \tfrac{1}{24}(z+\tfrac{1}{\nu})^4\partial_z^4\phi(x,-\tfrac{1}{\nu},t) + \ldots\end{aligned}$$

Now, with (15) the derivatives can be rewritten. This gives a distinction between the even and odd terms. For orders one to four:

$$\begin{aligned}\partial_z\phi &= 0\\ \partial_{zz}\phi &= \nu^2\delta^2\partial_{xx}\phi\\ \partial_{zzz}\phi &= \nu^2\delta^2\partial_{zxx}\phi = \nu^2\delta^2\partial_{xx}\partial_z\phi = 0\\ \partial_z^4\phi &= \nu^2\delta^2\partial_{zzxx}\phi = \nu^2\delta^2\partial_{xxzz}\phi = \nu^4\delta^4\partial_z^4\phi\end{aligned}$$

Repeating the argument gives $\partial_z^{2n}\phi = (\nu\delta)^{2n}\partial_x^{2n}\phi$ and $\partial_z^{2n+1}\phi = 0$
Now substitute the surface $z = \frac{\varepsilon}{\nu}\eta$ This gives (with height $h = \frac{1}{\nu}(\varepsilon\eta+1)$ and a bottom potential: $\beta(x,t) := \phi(x,-\frac{1}{\nu},t)$):

$$\phi(x,\tfrac{\varepsilon}{\nu}\eta,t) = \beta - \tfrac{1}{2}\nu^2\varepsilon h^2\partial_x^2\beta + \tfrac{1}{24}\nu^4\varepsilon^2h^4\partial_x^4\beta + \ldots \tag{56}$$

We managed once again to eliminate z from the problem.
Substituting (56) in (17), (18) and ignoring $\mathcal{O}(\varepsilon^2)$ gives:

$$\partial_t\eta + \frac{\varepsilon}{\nu}\partial_x\eta\partial_x\beta = 0$$
$$\partial_t\beta + \frac{1}{2}\varepsilon\partial_{xxt}\beta + \frac{1}{2}\frac{\varepsilon}{\nu}(\partial_x\beta)^2 + \eta = 0$$

These are the original Boussinesq equations. The last approximation would be between β and ϕ. Since we have $\delta^2 \sim \varepsilon \ll 1$, the water depth is small and thus the difference between the water speed at the bottom and at top would not introduce larger errors.
Approximating $\partial_z\phi$ in terms of some other velocity below the surface are known as Boussinesq-type approximations. An overview of the options is provided in [11]
The graph of a singleton solution, travelling to the right, for the Boussinesq equations is given on the cover of this report.

6 Conclusion

To obtain most of the models for water waves available, we have used two main entrance points: small waves and shallow water. These models are entirely different from one-another. The differences start already with scaling: the problem of having to scale z different for deep water and shallow water is solved by introducing the dynamic scaling. Throughout the report, this eased comparing the orders of δ and ε.
The Dirichlet-Neumann operator has shown great value for problems like these in which we are only interested in the boundary.
The desired goal of trying to find a unifying theory to generate wave models has only partially been reached. A lot of models are obtained by specific methods, some of which described in Section 5, and by different similarity choices between ε and δ. Including these in this report would not contribute to the main goal of finding a general method of generating these models.

6.1 Small waves

The case of small waves is easily adressed because the $\mathcal{O}(\varepsilon)$ approximations are linear. This even allows, by means of the Dirichlet-Neumann operator for a theoretical exact solution (Section 3.4). The different approximations in this region are all involved with numerically smart choices of approximating this operator.
Exact small wave equation:

$$\partial_{tt}\eta = -\frac{1}{\delta^2 \nu}\mathscr{A}\eta$$

Possible approximations:

$$\begin{aligned} \partial_{tt}\eta &= -\left[1-\tfrac{1}{6}\delta^2\partial_{xx}\right]^2 \tfrac{1}{\nu}\partial_{xx}\eta \\ \left[1-\tfrac{1}{3}\delta^2\partial_{xx}\right]\partial_{tt}\eta &= -\partial_{xx}\eta \end{aligned}$$

The DN operator $\mathscr{A}$ is easily found and provides much more insight than the original system. It is used to find the dispersive relation.

6.2 Shallow water

The equations from the assumption of shallow water are more interesting and challenging than the former. The non-linear DN operator $\mathscr{G}$ is, as if by some marvellous coincidence, accessable through the depth-averaged horizontal velocity $\bar{v}$. This provides a way to recast the system into an $\mathcal{O}(\delta^4)$ evolution system of η and $\bar{v}$ called the Green-Naghdi equation. Further simplification to $\mathcal{O}(\delta^2)$ yield the standard shallow water equation.
Green-Naghdi:

$$\begin{aligned} \partial_t\eta + \partial_x(h\bar{v}) &= 0 \\ (1-\tfrac{1}{3}h^2\delta^2\partial_{xx})\partial_t\bar{v} + \varepsilon\bar{v}\partial_x\bar{v} - \tfrac{\varepsilon\delta^2}{3h}\partial_x\left[h^3\bar{v}\partial_{xx}\bar{v} - (\partial_x\bar{v})^2\right] + \partial_x\eta &= 0 \end{aligned}$$

Shallow water:

$$\partial_t \eta + \partial_x (h\bar{v}) = 0 \tag{57}$$

$$\partial_t \bar{v} + \varepsilon \bar{v} \partial_x \bar{v} + \partial_x \eta = 0 \tag{58}$$

6.3 Other methods

We showed a completely aternative approach to water waves, namely the Hamiltonian structure. Further, there are a lot of ad hoc methods to arrive at various models. We considered three of them: uni-directionalisation, the tidal wave approximation and the Boussinesq approximations.

Hamiltonian framework

It is possible to rewrite the complete theory in a Hamiltonian structure like this:

$$\begin{pmatrix} \partial_t \phi \\ \partial_t \eta \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta_\phi H \\ \delta_\eta H \end{pmatrix}$$

With ϕ and η as conjugate variables and using the energy as Hamiltonian:

$$H(\phi, \eta) = P(\eta) + K(\phi, \eta) = \int \tfrac{1}{2\nu} (\varepsilon \eta + 1)^2 \, \mathrm{d}x + \int \int_\Omega \tfrac{1}{2} |\nabla \phi|^2 \, \mathrm{d}z \, \mathrm{d}x$$

No extra results can be obtained from this formulation, but discretisation schemes that make use of the Hamiltonian structure are guaranteed to be numerically stable.

Uni-directionalisation and Korteweg-de Vries

The method of uni-directionalisation restrict solutions to travelling waves in one direction by starting wih the ansatz (51):

$$\bar{v} = \eta + \varepsilon F + \delta^2 G + \mathcal{O}(\varepsilon^2, \delta^4, \varepsilon \delta^2)$$

Korteweg-de Vries is obtained by using this ansatz for Green-Naghdi and order arguments. After eliminating $\bar{v}$ and translating and rescaling on t and η, this results in (55):

$$\partial_t \eta + \tfrac{3}{2} \eta \partial_x \eta + \tfrac{1}{6} \partial_{xxx} \eta = 0$$

Boussinesq approximation

Boussinesq is applicable in the specific domain of weakly non-linear dispersive waves ($\delta^2 \sim \varepsilon \ll 1$). The Boussinesq approximation is used to eliminate z from the problem and by ignoring $\mathcal{O}(\varepsilon^2)$ we have:

$$\partial_t \eta + \tfrac{\varepsilon}{\nu} \partial_x \eta \partial_x \beta = 0$$

$$\partial_t \beta + \tfrac{1}{2} \varepsilon \partial_{xxt} \beta + \tfrac{1}{2} \tfrac{\varepsilon}{\nu} (\partial_x \beta)^2 + \eta = 0$$

6.4 Unresolved questions and remarks

Major issues

- Some important models have not been adressed in this report. Most notably Camassa-Holm and Benjamin-Bona-Mahony. In [10] it is sugested that these should follow fairly easily from Green-Naghdi, but these derivations seem too much involved compared to the straightforward ones.

- Does there really exist a unifying theory to acces every model from our main entrance points of small waves and shallow water? Can we unify the ad hoc methods from Section 5 in some large scheme?

- Dispersion, as reviewed in Section 3.3, is not only applicable to linear waves. For all wave equations some relation between c and k can be found. This is usually not done by one general method. Does such a general method for finding dispersion relations exist?

- Numerics are mentioned a couple of times without any in depth analyses. Since wave models are very applicable, it is important to get a better feeling of the problems arising in numerics.

Minor issues

- The parameter δ only appears in squared form in the equations (15) to (18). Virtually every author therefore identifies $\frac{{h_0}^2}{L^2}$ as a parameter.

- A factor $\delta^2\partial_{xx}$ can be extracted from $\mathscr{A}$. Many authors favor this convention [1, 2, 6, 10]

- Would it be possible to use Fourier integrals as opposed to Fourier sums in Section 3.1 for non-periodic domains? If so, there is no longer a maximum wavelength, what is then the meaning of L?

- Korteweg-de Vries rescales time with $\frac{1}{\delta}$. It thus supplies solutions on a timescale different fromt that of Green-Naghdi. What is the physical significance?

- The tidal wave approximation is almost exclusively mentioned by E.M. Groesen without statements about its feasibility. Is it a plausible approximation?

- The jargon is inconsistent and confusing. Does ‘fairly long waves’ mean the same as ‘rather long waves’? Is the ‘non-linear effect’ just ε and the ‘dispersive effect’ just δ^2?

References

[1] *Article:* L.J.F. Broer - *Approximate equations for long water waves*
(Applied Scientific Research vol. 31, 1975)

[2] *PhD Thesis:* E. Cahyono - *Analytical wave codes for predicting surface waves in a laboratory basin*
(University of Twente, 2002)

[3] *Book:* G.D. Crapper - *Introduction to water waves*
(Ellis Horwood Limited, 1984)

[4] *Book:* M.W. Dingemans - *Water wave propagation over uneven bottoms*
(World Scientific Publishing Co. Pte. Ltd., 1997)

[5] *Article:* A.E. Green and P.M. Naghdi - *A derivation of equations for wave propagation in water of variable depth*
(Journal of fluid mechanics vol. 78, 1976)

[6] *Book:* E van Groesen, E.M. de Jager - *Mathematical structures in continous dynamical systems.*
(North-Holland, 1994)

[7] E.M. de Jager - *On the Origin of the Korteweg-de Vries Equation*
(2006)

[8] *Article:* R.S. Johnson - *Camassa-Holm, Korteweg-de Vries and related models for wter waves.*
(Journal of fluid mechanics vol. 455, 2002)

[9] *Book:* R.S. Johnson - *Modern Introduction to Mathematical Theory of Water Waves*
(Cambridge University Press, 2001)

[10] *Article:* D. Lannes and P. Bonneton - *Derivation of asymptotic two-dimensional time-dependant equations for surface water wave propagation*
(Physics of fluids vol. 21, 2009)

[11] *Article:* P.A. Madsen and H.A. Schäffer - *A review of Boussinesq-type equations for surface gravity waves*
(Advances in Coastal and Ocean Engineering vol. 5, 1999)

[12] *Book:* J.J. Stoker - *Water waves, Volume IV*
(Interscience Publishers Inc., 1957)

A Glossary

Variables and sets:

- x the horizontal location
- z the vertical location
- t time
- Ω the set of points occupied by water

Inherent lengths and parameters:

- h_0 the average water depth
- a the typical difference between the actual and the average water height
- L the typical (maximum) wave length
- ε the essential parameter $\frac{a}{h_0}$
- δ the essential parameter $\frac{h_0}{L}$
- ν the dynamic scaling factor $\frac{\tanh(\delta)}{\delta}$

Quantities and graphs:

- η the graph describing the surface profile
- $\mathbf{v}$ the velocity field denoting the local particle speed
- ϕ the potential of $\mathbf{v}$
- ψ the restriction of ϕ to the surface
- β the restriction of ϕ to the bottom
- h the water height at point x, $h = \frac{1}{\nu}(\varepsilon\eta + 1)$
- p the scalar field describing the pressure
- g the acceleration due to gravity
- ρ the scalar field describing the density

Operators:

- ∂_s differentiation with respect to a variable s
- $\partial_{\mathbf{n}}$ directional differentiation along a unit normal vector $\mathbf{n}$

∇	an operator-vector: (∂_x, ∂_z)
∇^2	the Lapace operator: $\partial_{xx} + \partial_{zz}$
$\mathscr{G}$	the general Dirichlet-Neuman operator, acting on the surface
$\mathscr{A}$	the DN operator for linear waves, acting on the surface

Miscellaneous:

$\hat{f}_k$	the k$^{\text{th}}$ Fourier coefficient of f
$\sim$	an infix operator to indicate order similarity, $f \sim g \Leftrightarrow f = \mathscr{O}(g) \wedge g = \mathscr{O}(f)$
$\mathscr{O}(f,g)$	simultaneous order: $\mathscr{O}(f,g) = \mathscr{O}(f) + \mathscr{O}(g)$
C	an integration constant