

## BACHELOR

### Fractal phenomena in fracture

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# Fractal phenomena in fracture

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## **Abstract**

This paper concerns two aspects of the simulation of fracture. First analysis is done of the energy distribution within a lattice at different points during the fracture process by calculating several of its moments for different lattice sizes. The energy moments are hypothesized to exhibit multifractal scaling with the lattice size. Only for the point right before mechanical breakdown of the system, a deviation from monofractality is concluded.

Another aspect builds on the conclusion that random percolation can model strong-disorder fracture with respect to several properties that exhibit fractality. Analysis is done whether modifying random percolation into gradient percolation based on the damage profile from the fracture simulation can improve the similarity to strong-disorder fracture with respect to these properties. It is concluded that gradient percolation based on fracture's damage profile provides little improvement, but surprisingly, exaggerating this profile does improve the similarity.

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# Chapter 1

## Introduction

One area of physics that is not yet very well understood is fracture. When strenuous force is applied to a material, which eventually breaks, it is not clear what happens on a microscopic scale. While it is investigated in the lab, the process is analyzed theoretically as well. When the material is reduced to a 2D lattice (square or otherwise), predictions of the breaking process can be attempted using percolation theory. A lattice for this purpose could be a square lattice with resistors for bonds. The resistors have different resistivities which simulates the disorder of strength in materials. A voltage is applied on opposite sides of the lattice, and the currents are calculated in each resistor. If the current is above a certain threshold the resistor is removed from the lattice. This is the fuse model, and a lot is known about this model from percolation theory. However, it fails to incorporate an important aspect of fracture: in a material, forces are vectorial in nature, while currents are scalar. A modified lattice, where bonds are replaced by springs with different stiffnesses and thresholds, and nodes are not stationary but move according to the springs' will, does a better job. However, it is too complex to analyze by hand, which calls for the aid of computer simulation. In such a computer simulation, forces on each spring are calculated at each stage in the process, and springs are broken when the stress on them exceeds a certain threshold.

An alternative simulation is based on the same lattice, but instead of calculating whether any bond exceeds its strain threshold bonds are randomly broken. As it turns out, this random percolation resembles the fracture simulation in the case of high disorder. When fracture and random percolation are compared based on different properties, several, especially some fractal ones, coincide fairly well in the case of strong disorder.

In fracture however, the concentration of broken bonds is not constant along the axis parallel to the direction of the applied strain. A possible alternative to random percolation rises from this observation: break the bonds

randomly according to a non-uniform distribution. This is one question this paper answers: does this modified percolation simulate fracture better than uniform random percolation?

Another aspect of fracture analyzed in this paper is the fractality of energy moments during the process of fracture. During the applied displacement, forces and thus energies are imparted upon the bonds in the lattice. The distribution of these energies can be described by its moments. It is expected that the  $q$ th order moments of the energy distribution at a particular point in the fracture process scale with the lattice size  $L$  approximately as  $L$  to the power a constant times  $q$ . If this constant is not integer-valued, there is mention of fractality. The hypothesis in this paper, however, is that such a relation does not hold, and that there is in fact a non-linear dependence on  $q$  in the exponent of  $L$ . The other aim of this paper, therefore, is to show whether or not this “multifractality” appears in the fracture simulation, which may indicate the existence of multifractality of energy moments in fracture itself.

## 1.1 Fracture simulation

The simulation of fracture in a two-dimensional lattice in this paper takes a central-force spring lattice of unit length and  $L$  nodes on each edge. The nodes within the lattice are randomly distributed, and connected by the Delaunay tessellation criterion. Each bond is assigned a breaking threshold, distributed according to

$$P(t) = (1 - \alpha)\tau^{\alpha-1}t^{-\alpha}, \quad t \in [0, \tau], \quad (1.1)$$

where  $\alpha$  is the disorder parameter.  $\alpha$  is 0.7 for most of the research, since it compromises a high disorder (close to 1) with the fact that higher disorder is computationally very demanding. The top and bottom edges of the lattice are connected by a periodic boundary condition in order to eliminate their influence, and the left and right nodes are quasi-periodic in the sense that the difference in displacement of a left node and its corresponding right node is fixed.

In each step of the breaking process a prescribed displacement, or strain, is applied in the x-direction. A finite-element problem is solved to determine if any bonds experience a strain larger than their threshold. If so, the bond is removed from the lattice by setting the bond’s stiffness to (for calculation reasons) nearly 0. Then the stiffness matrix is modified, and a new set of linear equations is solved. This continues until the boundary conditions are satisfied or the system falls apart.

# Chapter 2

## Fractality of Energy Moments in fracture

### 2.1 Introduction

Scaling, or size dependence, of extensive properties systems can in many cases be written as a simple relation:

$$\frac{S(L_2)}{S(L_1)} = \left(\frac{L_2}{L_1}\right)^m,$$

where  $m$  is a scaling exponent. For common systems and properties such as area's dependence on length,  $m$  is integer valued. If you look at, for example, a snowflake and look at a window of linear size  $L$ , the mass in it will scale with  $L^{d_f}$ , with  $d_f$  some non-integer value. A geometry which has such a property is called fractal, and  $d_f$  its fractal dimension. In some cases, this exponent is a different fractal dimension for different  $L$ , in which case it is called multifractal. An example of multifractality is found in [Schmittbuhl]. Here, a material is fractured, and the height profile in one dimension is measured by a needle crossing over the surface. The  $q$ -th order moment of the crack roughness is then defined as  $\langle |\Delta h_l|^q \rangle$ , where  $\Delta h_l = h(x) - h(x + l)$ . In monofractal behavior, values of this property would be proportional to  $l^q$ . Instead, the exponent on  $l$  is a non-linear function of  $q$ , as seen in figure 2.1.

In this section, a property of the fracture process is analyzed that is hypothesized to show multifractality as well.

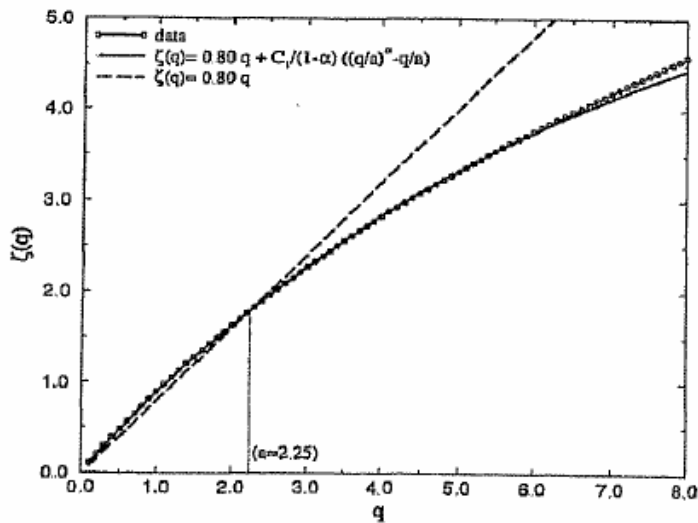


Figure 2.1: Exponents on  $l$  for the different moments  $q$  of the roughness profile. The deviation from linearity implies multifractality.

## 2.2 Theory

Energy moments within the lattice are defined as a function of the energies of the separate bonds as follows:

$$Z_q = \frac{1}{n_{int}} \sum_{i=1}^{n_{int}} E_i^q, \quad (2.1)$$

where *int* stands for “intact.” Scaling of these moments with lattice size  $L$  in the monofractal case is  $Z_q \propto L^{\zeta \cdot q}$ . However, in the case of multifractality, this exponent is not linear in  $q$ , but contains higher order terms that can be generalized as

$$Z_q \propto L^{\zeta \cdot q + K(q)} \equiv L^{\mu(q)}, \quad (2.2)$$

with  $K(q) = 0$  in the case of monofractality.

An estimate of what can be expected is derived as follows. Denoting the average energy of a single bond as  $\bar{E}$ ,  $Z_q$  can be estimated as

$$Z_q \approx \bar{E}^q \quad (2.3)$$

$\bar{E}$  can be related to  $L$  by the following reasoning. For each  $L$  at a specific point in the breaking process, perhaps the point of maximum stress, the displacement applied to the lattice is approximately the same, namely  $\Delta x$ . Subsequently each bond is stretched in the  $x$  direction so that the length of



each bond increases by a length proportional to  $\Delta x/L$ . The energy of a bond in this state is then approximately proportional to  $(\Delta x/L)^2$ . Thus we get:

$$Z_q \approx c_1 L^{-2q} \quad (2.4)$$

$$\log Z_q \approx c_2 - 2q \log L \quad (2.5)$$

$$\mu(q) \approx -2q \quad (2.6)$$

Of course there are several factors ignored in this approximation that will change the exponent or even introduce multifractality. For example, the bonds in the volume of the lattice may have different energies than on the edge, and the ratio of volume bonds to edge bonds is proportional to  $L^{-1}$ . Furthermore, when clusters form, energies will be rearranged in a manner which may or may not depend on  $L$ , but certainly might since the number of bonds affected scale with  $L$  in a similar manner as edge bonds. This goes without even mentioning the crudeness of the approximation in (2.3).

## 2.3 Methods

To calculate the energy moments in a lattice, the central-force lattice simulation calculated and output data for several moments of the energy distribution. This is done by comparing the length of each bond while strain is applied to its rest length (i.e. the length of the bond when no strain is applied; by definition there is no energy in the bonds in this state), and calculating  $E_i = 1/2 \cdot k_i(x_i - x_{i0})^2$ , where  $k_i$  is the stiffness of bond  $i$ . The energy moments are then defined as in (2.1). This is done for four points in the process, namely

**Random Percolation regime (RP)** The first point where the fractality is investigated is at a point in the regime where random percolation closely approximates fracture<sup>1</sup>. For practical purposes, it is observed that at the midpoint of the random percolation regime for  $L = 50$  the stress is 52% of the maximum stress, and it is assumed that this percentage is independent of  $L$ , i.e. the fraction of broken bonds  $p$  scales similarly with  $L$  for the midpoint of the random percolation regime and the point at which 52% of the maximum stress is attained.

**Maximum stress (MS)** This is the point at which the stress response to the applied strain is maximal.

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<sup>1</sup>Specifically, with respect to scaling properties of correlation length and the second moment of cluster size distribution.

**Maximum energy per bond (ME)** This is the point at which the average energy in the remaining bonds is maximal. This is in fact the point of the maximum of the first moment of the energy distribution. This point often coincides with the point of maximum stress.

**Stress drop (SD)** By prediction, multifractality is most likely to be observed beyond the maximum stress point, since the fragile state of the system may make energies exhibit grand-scale redistribution. For this reason the point right before mechanical breakdown is investigated for its fractality. In the calculation, the stress drop point is defined as the first point beyond the maximum stress point where the stress of the whole lattice drops below 5% of maximum stress. Since there is hardly any energy left here, the energy moments are calculated right before the bond breaks that mechanically breaks down the system.

The fracture simulation is run for system sizes  $L = [16, 24, 32, 48, 64, 96, 128]$ , for several repetitions each. A log-log plot of  $Z_q$  vs.  $L$  is then made for each value of  $q$  (= moment order). The slopes of these lines are equal to  $\mu(q)$  (see (2.5) and (2.6)). From this a plot can be made of  $\mu_q$  vs.  $q$ . If this plot is linear, no multifractality can be concluded.

## 2.4 Results

The log-log plot of the energy moments  $Z_q$  vs.  $L$  look like figure 2.2 in each case. The slopes of these lines are then plotted vs.  $q$  to obtain the graphs in figure 2.3. For all four of the curves, a linear fit and a linear fit through the origin are shown. Each of these curves can be accurately fit linearly. This strongly indicates monofractality. However, it is noteworthy that the lines do not pass straight through zero, which must happen by definition ( $Z_0 = 1 \propto L^0$ , see (2.1)). In the case of RP, MS, and ME, the y-intercepts are insignificant, with absolute values between 0.2 and 0.3 in each case. For SD however, the y-intercept is  $-0.60$ , which is significant, as can be confirmed by eye in the deviation of the fit from the fit through origin in figure 2.3.

A closer look at SD is then appropriate. Zooming into the SD graph to the enclosed region in figure 2.3, figure 2.4 shows that there is a bend in the line. There seems to be significant deviation from a linear curve. However, in the region  $q = 2..8$  this deviation is gone, and a straight line can be fit for  $q > 2$ , albeit with a y-intercept unequal to zero.

A remark that should be made concerns the error bars. In the log-log graph of moments vs. lattice size they indicate 95% confidence intervals of the log of the means of the moments over all the samples at each lattice

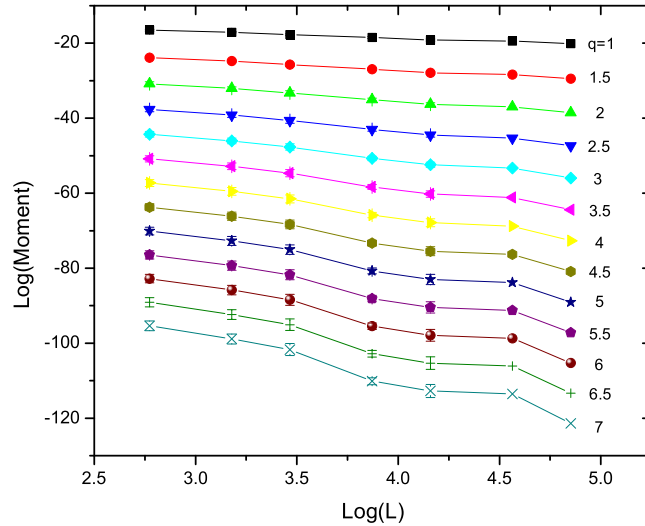


Figure 2.2: The calculated energy moments at the point of maximum stress for different lattice sizes  $L$ . Each line represents a different moment  $q$ .

size. These errors are taken into account when fitting the lines to obtain slopes that correspond to  $\mu(q)$ . While there is too much room implied by these bars to prove either linearity or non-linearity, the near-perfect lining-up of the points (and the points in the derivative of this graph) makes the error bars seem out of proportion. The explanation is that the points are correlated. All the moments were calculated from the same set of samples. The expectation is that redoing the experiment with an entirely new set of samples will produce a similar graph with a slope that is slightly different but fits inside the error bars in figure 2.4.

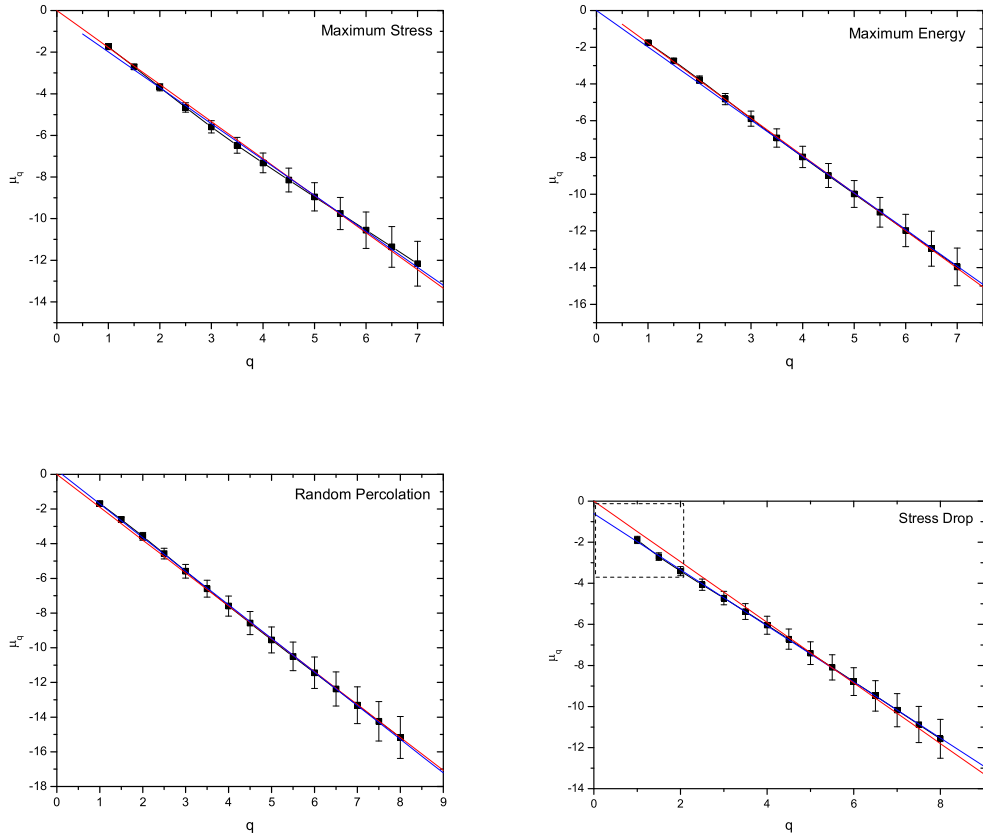


Figure 2.3: The plots of  $\mu_q$  vs.  $q$ . In each plot, a linear fit is shown in blue and a linear fit through the origin in red. The enclosed portion can be found in figure 2.4

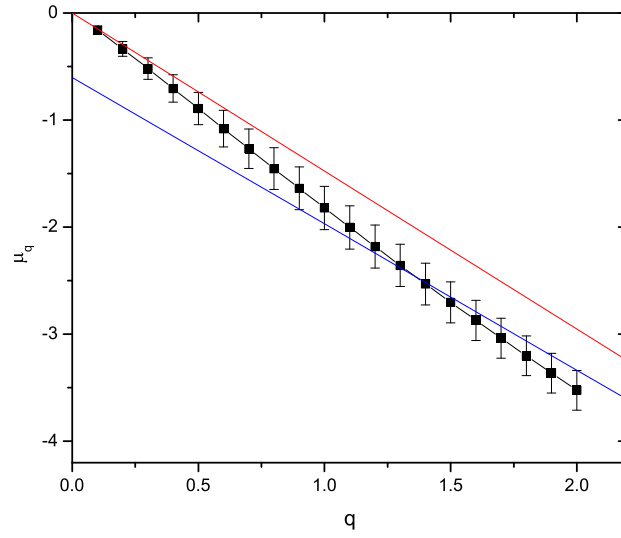


Figure 2.4: A closer look at the region enclosed in dotted lines in figure 2.3.

# Chapter 3

## Gradient Percolation

### 3.1 Introduction

This section deals with the ability of percolation in simulated lattices to model fracture. In [Malakhovsky] a comparison is made between fracture and random percolation (RP). For the latter, bonds in the lattice are sequentially broken, chosen entirely at random. As is explained more in detail below, some properties of RP matched those of fracture, while others did not. The aim of this part of the project is to observe whether using random percolation where not all bonds have an equal probability of being broken can improve the similarity between fracture and percolation. “Gradient percolation” (GP) is the term used for percolation where the probability of a bond breaking depends on its location in the lattice. We assess how well GP matches fracture on account of several properties.

### 3.2 Damage profile/distribution

The choice of the distribution that is used to assign breaking probabilities to each bond in the lattice is based on the damage profile obtained from the fracture simulation in [Malakhovsky]. To generate a damage profile, vertical lines are dropped at 200 x-coordinates in the lattice, and the fraction of bonds crossed by this line that are broken is recorded. This is done at different points in the breaking process, but the focus lies at the point of maximum stress. A typical (normalized and averaged over all samples) damage profile looks like figure 3.1, for  $L = 50$ ,  $\alpha = 0.7$ , and fraction of broken bonds  $p = 0.181$ , which is the point of maximum stress. Since the shape of the damage profile is that of a bell curve, an obvious candidate to emulate this profile is a Gaussian distribution. It is elevated in the sense that the pdf is

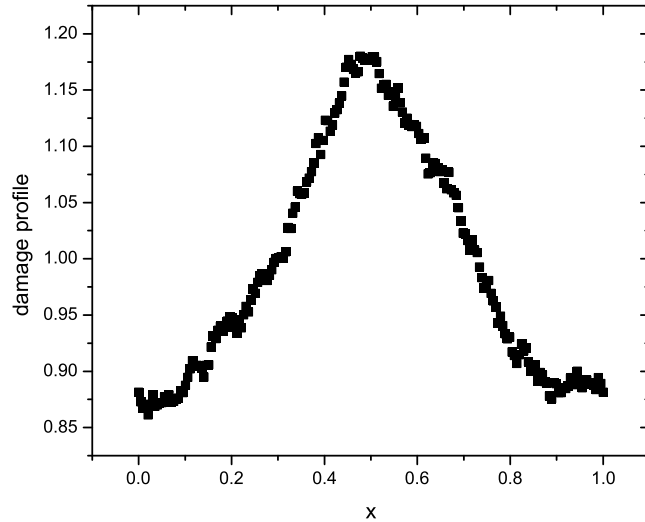


Figure 3.1: A typical damage profile from a fracture simulation for  $L = 50$ ,  $\alpha = 0.7$ , and  $p = 0.181$ .

raised by some constant (and the pdf is accordingly normalized) and cropped in the sense that only the part of the distribution between  $x = 0$  and  $x = 1$  is considered. The pdf therefore becomes

$$f(x) = \begin{cases} \frac{1-c}{Z} e^{-\frac{(x-1/2)^2}{2\sigma^2}} + c & 0 \leq x \leq 1 \\ 0 & elsewhere \end{cases} \quad (3.1)$$

where  $Z$  normalizes the cropped Gaussian distribution ( $c = 0$ ). There are two main drawbacks to using this function to generate breaking probabilities.

1. There are two parameters ( $\sigma$  and  $c$ ). While they can be estimated from the fracture damage profiles, both fluctuate as the lattice size  $L$  and the disorder parameter  $\alpha$  change, in an unapparent fashion.
2. Any analytical predictions of gradient percolation are unlikely since the function is so complex.

A simpler function that matches the density profile sufficiently well is the elevated tent. This is an upside down absolute value function, centered at  $1/2$  and elevated by a constant  $c$ , while maintaining normalization. Its pdf

is

$$f(x) = \begin{cases} 4(1-c)[-|x - \frac{1}{2}| + \frac{1}{2}] + c & 0 \leq x \leq 1 \\ 0 & \textit{elsewhere} \end{cases} \quad (3.2)$$

where  $c = f(0) = f(1)$ .

The value of  $c$  is estimated from the damage profile for each  $L$  and  $\alpha$  by minimizing the Kolmogorov-Smirnov statistic. This function is preferable to the elevated cropped Gaussian distribution for the following reasons:

1. There is only one parameter.
2. There already is some knowledge of gradient percolation from a linear profile, which is similar to each of the two halves of the tent function. It is more likely that some properties can be derived if the tent profile is assumed than if more complex elevated cropped normal distribution is considered.

While it is possible to use the corresponding estimate of  $c$  for each lattice size  $L$  for the disorder  $\alpha = 0.7$ ,  $c$  has been fixed at  $c = \frac{6}{7}$ ; since the results that are sought after are qualitative, varying  $c$  over its range of about 0.15 ( $17 \leq L \leq 200, \alpha = 0.7$ ) will not alter the results in a manner that outweighs the luxury of eliminating one factor from the experiment.  $\frac{6}{7}$  has been chosen rather arbitrarily around the middle of the aforementioned range. It is slightly lower than the estimated value at  $L = 50, \alpha = 0.7$ , which are the parameters for the simulation on which most of the analysis is based. The experiment is also carried out for  $c = \frac{3}{5}$ , which gives a tent function that exaggerates the damage profile found in fracture. It is conceivable that applying a gradient that is larger than the gradient found in fracture will increase the effect of matching percolation to fracture.

### 3.3 Comparison of GP and fracture

In gradient percolation, lattice bonds are assigned probabilities based on the x-coordinate of their midpoints, according to the tent distribution. At each stage, a bond is broken according to these probabilities, after which a new set of probabilities is calculated.

#### 3.3.1 Properties for comparison GP and fracture

In [Malakhovskiy] a comparison was made between fracture and random percolation on account of the comparison of several properties. Some properties, such as the relationship between cluster weight and gyration radius, the



joining probability, and cluster-size distribution all showed similarity for fracture and RP and matched the predictions for RP. However, some properties matched in certain regimes, or showed no similarity at all. For some of these properties, it is investigated whether GP data more closely matches fracture data, or over a broader regime. The investigated properties are:

**Young's modulus** This property is one with explicit physical meaning. When the material obeys Hooke's law, Young's modulus measures the ratio of response force of the lattice as a whole (stress) to displacement (strain). Denoting  $\sigma$  the stress and  $\epsilon$  the strain,

$$Y = \frac{\sigma}{\epsilon}. \quad (3.3)$$

**Scaling of  $\xi$  and  $M_2$**  These properties say something about the formation of clusters during the different processes. At several values of  $p$  (the fraction of broken bonds) the various clusters of broken bonds are identified, and several properties are calculated. Calling  $n_s$  the number of clusters of size  $s$  divided by the total number of sites in the lattice, the moments of their distribution are defined by

$$M_i = \sum_s n_s s^i, \quad (3.4)$$

where the second moment, denoted by  $M_2$ , is of interest here. The correlation length is a measure of the radius of clusters which give the main contribution to  $M_2$ , defined by

$$\xi^2 = \frac{2 \sum_s R_s^2 s^2 n_s}{\sum_s s^2 n_s}, \quad (3.5)$$

where the gyration radius  $R_s$  is defined by

$$R_s^2 = \left\langle \sum_{i=1}^s \frac{|\mathbf{r}_i - \mathbf{r}_{cm}|^2}{s} \right\rangle_{(all\ s\ clusters)}. \quad (3.6)$$

For RP in an infinite 2D lattice,  $M_2$  and  $\xi$  exhibit universal scaling laws

$$M_2 \propto |p - p_c|^{-\gamma}, \quad \gamma = \frac{43}{18} \quad (3.7)$$

$$\xi \propto |p - p_c|^{-\nu}, \quad \nu = \frac{4}{3} \quad (3.8)$$

which are universal in the sense that the exponents do not depend on the lattice details, although the proportionality factors and  $p_c$ , the percolation threshold<sup>1</sup>, do.

**Survival probability** For percolation, survival is defined as not containing a spanning cluster. The survival probability is the fraction of lattice samples that survive with fraction of broken bonds  $p$ . Specifically, we look at  $p_{50}$ , the fraction of broken bonds at which 50% of lattices survive, as a function of  $L$ . With an infinite lattice,  $p_{50} = p_c$ , and for finite lattices  $p_{50}$  is a good estimate of  $p_c$ . In RP, a spanning cluster coincides with  $\xi$  approaching  $L$ , so (3.8) can be used to derive a relation for the finite-size scaling of  $p_{50}$ :

$$|p_{50} - p_c| \propto L^{-\frac{1}{\nu}}, \quad \nu = 4/3. \quad (3.9)$$

The prediction matches the RP data well, but does not match the fracture data at all.

**Anisotropy** Anisotropy measures the tendency of clusters to form in one direction more than another. In the case of fracture, the displacement is applied in the x-direction, causing different behavior in crack propagation along the x- and y-directions. In RP, there is no difference between behavior in the x- and y-directions, so no anisotropy is expected, and indeed, only anisotropy within the bounds of white noise is found. In GP, varying probabilities are assigned to bonds based on their x-coordinates but not their y-coordinates, possibly causing anisotropy.

Anisotropy is defined as

$$\Phi = \frac{\xi_{\perp} - \xi_{\parallel}}{\xi_{\perp} + \xi_{\parallel}} \quad (3.10)$$

with  $\xi_{\parallel}$  and  $\xi_{\perp}$  the x- and y-components of  $\xi$ :

$$\begin{aligned} \xi_{\parallel}^2 &= \frac{2 \sum_s X_s^2 s^2 n_s}{\sum_s s^2 n_s}, & X_s^2 &= \left\langle \sum_{i=1}^s \frac{|x_i - x_{cm}|^2}{s} \right\rangle_{(all\ s\ clusters)} \\ \xi_{\perp}^2 &= \frac{2 \sum_s Y_s^2 s^2 n_s}{\sum_s s^2 n_s}, & Y_s^2 &= \left\langle \sum_{i=1}^s \frac{|y_i - y_{cm}|^2}{s} \right\rangle_{(all\ s\ clusters)} \end{aligned} \quad (3.11)$$

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<sup>1</sup>The percolation threshold is defined as the lowest fraction of broken bonds which cause an infinite spanning cluster to form in an infinite lattice.

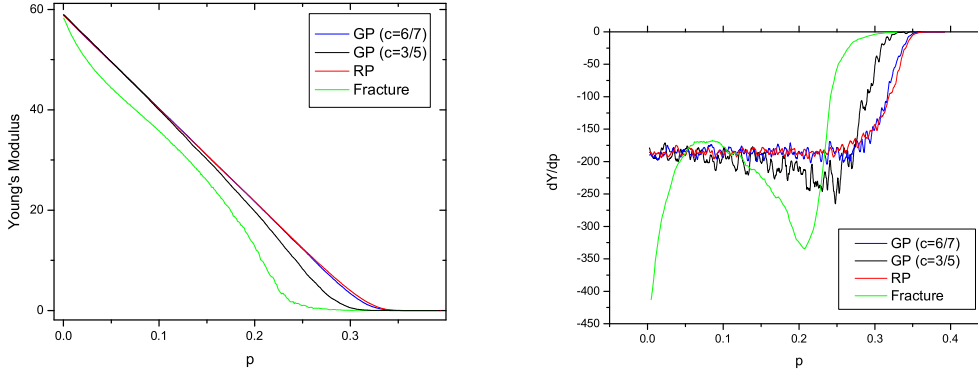


Figure 3.2: Young's Modulus as a function of fraction of broken bonds  $p$  for the different processes, and its derivative.

## 3.4 Results

### 3.4.1 Young's Modulus

Young's Modulus is a property that will show whether GP improves the similarity of percolation to fracture in a physical sense. While of course in a percolation simulation there is no displacement and no force, a strain can be applied to the lattice and its stress response can be measured. Data for Young's modulus is shown in figure 3.2. It is apparent that for GP with the tent function that matches the fracture damage profile (fracture profile GP, or  $GP_{FP}$ ,  $c = 6/7$ ) there is no improvement over RP regarding matching fracture's Young's modulus progression. However, exaggerated profile GP ( $GP_{EP}$ ,  $c = 3/5$ ) clearly has a Young's modulus curve deviating from the RP curve in the direction of fracture. To make sure this effect is based on more than the fact that  $GP_{EP}$  has a lower percolation threshold, i.e. that the system breaks apart sooner than RP, the derivative is analyzed, and it turns out that while  $GP_{FP}$  follows RP in having a linearly decreasing Young's modulus up until the breaking point,  $GP_{EP}$  shows behavior similar to fracture when its Young's modulus declines more steeply near the end of the process before the breakdown.

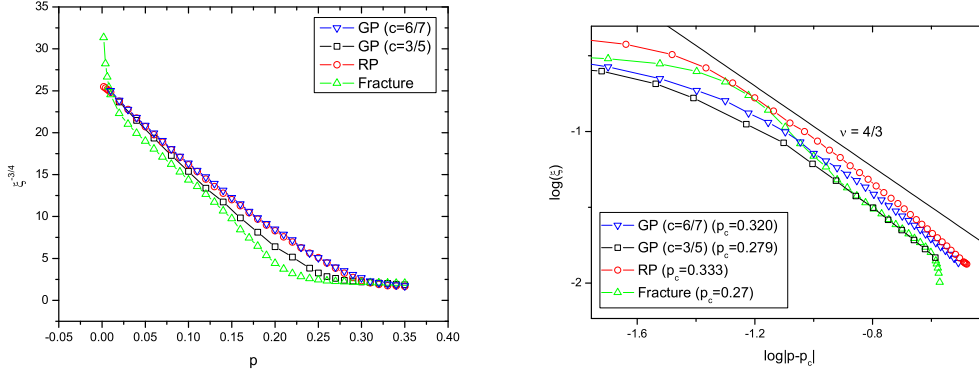


Figure 3.3: Scaling of correlation length as a function of fraction of broken bonds  $p$  (left) and distance from the percolation threshold (right), at lattice size  $L = 50$ .

### 3.4.2 Correlation length and second moment of cluster-size distribution

Figure 3.3 shows the evolution of the correlation length along the processes of fracture, RP, and GP for each selected value of  $c$ . The results of  $GP_{FP}$  lie right on top of the results of RP, indicating no improvement of GP over RP. However,  $GP_{EP}$  results clearly choose a middle road between the RP (and  $GP_{FP}$ ) and fracture.

The same pattern can be found in the cluster-size distribution data. Figure 3.4 shows that  $GP_{FP}$  results are hardly different than RP results, and clearly suggest no improvement over RP. However, the  $GP_{EP}$  results deviate from this in the direction of the fracture data.

While it can be conjectured that the decrease in percolation threshold solely accounts for this improvement, it is not the case. This can be seen in the right graphs in figures 3.3 and 3.4, where the behavior near the percolation threshold is analyzed independent of the particular thresholds. Here it can be seen that  $GP_{EP}$  matches fracture data better than RP and  $GP_{FP}$  in both cases.

### 3.4.3 Survival Probability

The survival threshold  $p_{50}$ , which characterizes the survival probability, is shown for each of the processes in figure 3.5. In fracture, different conditions for not surviving are shown: having reached maximum stress (MS), having

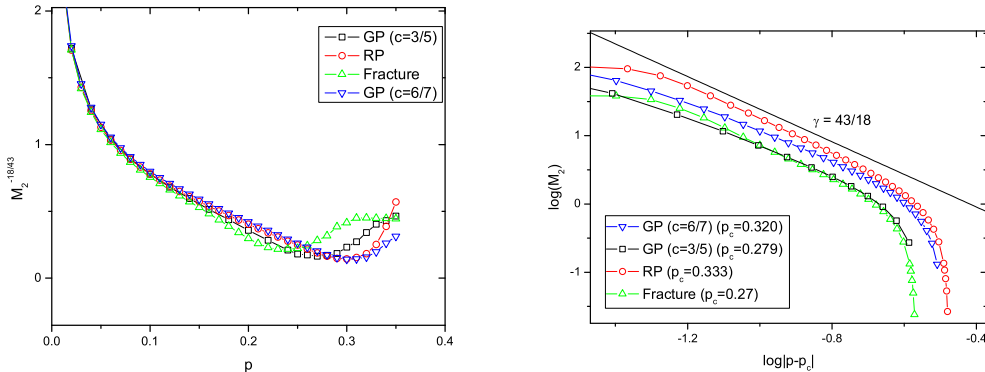


Figure 3.4: Scaling of second moment of cluster-size distribution as a function of fraction of broken bonds  $p$  (left) and distance from the percolation threshold (right) for lattice size  $L = 50$ .

reached stress drop (SD), and containing a spanning cluster (SC). For physical systems the MS and SD criteria are important, while the SC criterion is interesting to include in the comparison between the different processes.

On the one hand, the survival thresholds of  $GP_{FP}$  and  $GP_{EP}$  as a function of lattice size  $L$  do not exhibit a shape that matches fracture data any better than RP. Like RP and unlike fracture,  $p_{50}$  is a linear function of  $L^{-3/4}$ , whereas this property for all fracture criteria is certainly not. Moreover, the trend in GP as well as RP is that the threshold increases with increasing lattice size, while fracture features the opposite trend. This is of course devastating evidence that RP and GP do not simulate fracture with reference to lattice size scaling.

On the other hand,  $GP_{FP}$  and  $GP_{EP}$  do have percolation thresholds more like fracture than RP. Like fracture, GP concentrates its breaking, causing a spanning cluster or crack earlier.

### 3.4.4 Anisotropy

From figure 3.6 it can be seen that fracture and RP obey the predictions made in section 3.3: for RP, the anisotropy stays at 0 plus some white noise. For fracture, we see that the beginning is similar to RP in that there is no significant preference for cluster formation in one direction more than the other, which is because cluster formation has not yet started. As the process progresses, localization takes place, and clusters are formed. Clearly the

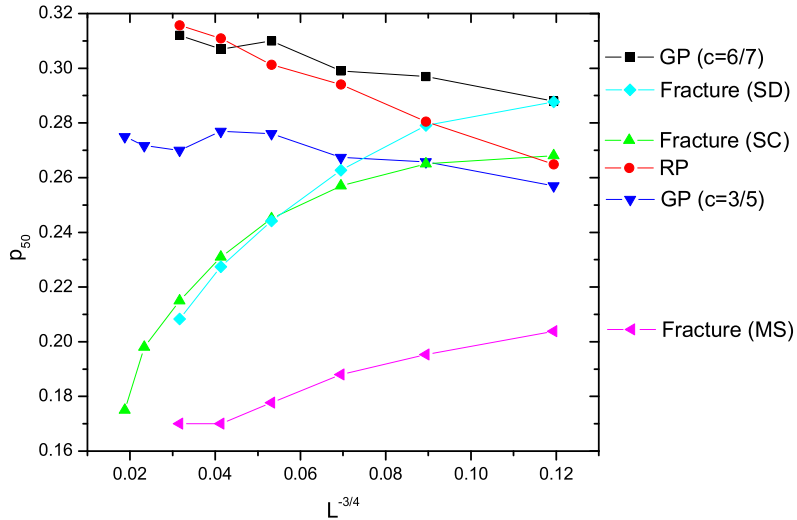


Figure 3.5: Survival threshold. This shows at which fraction of broken bonds 50% of samples of lattice size  $L$  survive, i.e. do not form spanning clusters. For fracture, three different criteria are used for not surviving: having reached maximum stress (MS), having reached stress drop (SD), and containing a spanning cluster (SC).

clusters tend to grow into the  $y$ -direction.

The GP results are similar to those in section 3.4.2. Up until the fraction of broken bonds where fracture endures a stress drop,  $GP_{FP}$  shows similar behavior to RP, i.e. no anisotropy. While the process is forced into concentrating its broken bonds in the center, there is no rule to encourage breaking onto a cluster of broken bonds, so cluster formation only really starts to happen after a certain fraction of bonds is broken, which is indicated by the increase after  $p = 0.25$ . When the finite size of the cluster saturates the growth in the  $y$ -direction, the further growth in the  $x$ -direction evens out the anisotropy, causing  $\Phi$  to decrease.

In  $GP_{EP}$  the anisotropy clearly matches the fracture data better than RP and  $GP_{FP}$  do. Because the breakings are forced towards the horizontal center even more than in  $GP_{FP}$ , clusters are more quickly given a chance to grow, creating a crack along the horizontal middle, which has positive anisotropy according to the definition (3.10).

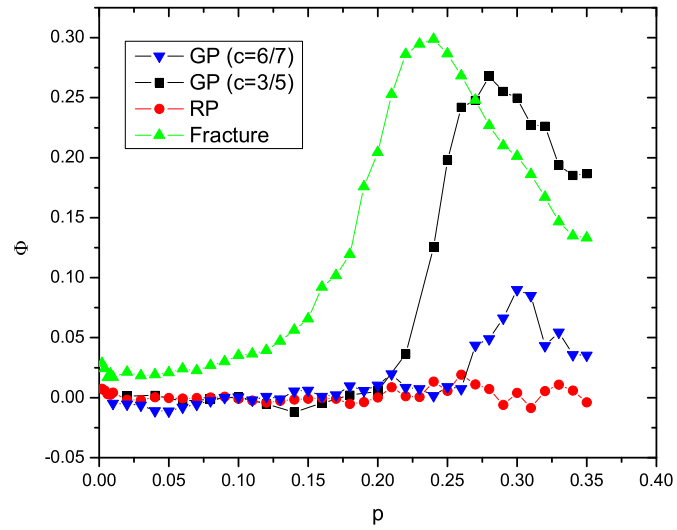


Figure 3.6: Anisotropy  $\Phi$  of the various processes as it develops over the range of broken bond fractions  $p$ . The lattice size is  $L = 50$ .

# Chapter 4

## Discussion and Conclusions

### 4.1 Fractality of Energy Moments in fracture

At the maximum stress, maximum energy, and random percolation points, no conclusion can be drawn about the nature of the fractality of energy moments. While the dependence of  $\mu$  on  $q$  can be well fitted linearly, the uncertainties in the points and the possibility that the y-intercept is significantly unequal to 0 after all (given the significant deviation in the y-intercept in the SD graph) discourage concluding monofractality. In any case, there is no multifractality of polynomial order greater than 1, as in [Schmittbuhl] (figure 2.1).

At the stress drop point, monofractality can be rejected. For low values of  $q$ , there is deviation from the line that characterizes the fractality for  $q > 2$ , and a certain curvature. What this curve is and why it appears will be left for further research. Again, however, multifractality of the form as in [Schmittbuhl] is not observed.

The results are consistent with those found in [Herrmann]. There energy moment analysis was done of the fuse model. While this model exhibits shortcomings with respect to the forces involved as mentioned before, the energy moments display the same multifractality as in this experiment and likewise, this multifractality occurs only near the mechanical breakdown point. Of course the calculation of energy ignores the vectorial nature of the forces in the lattice, but it is still remarkable that the energy moments display similar multifractal behavior in the two models.

### 4.2 Gradient Percolation

In almost all properties measured, similar results are found: gradient percolation assuming the tent function based on the damage profile of fracture



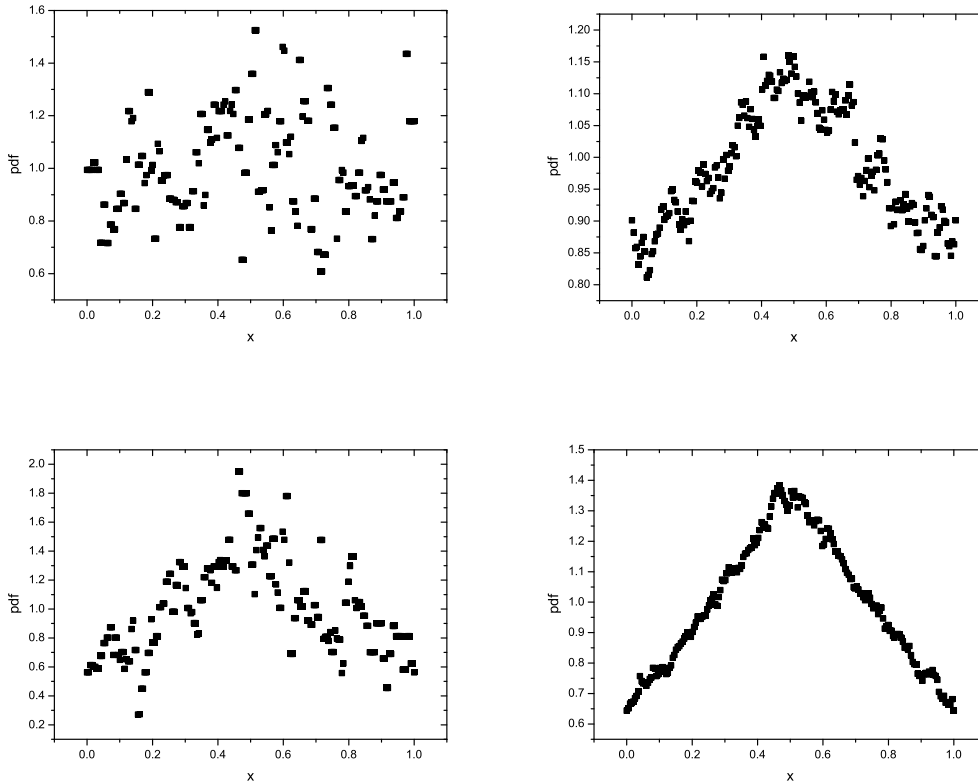


Figure 4.1: The damage profile of GP using  $c = 6/7$  for one sample (top left) and averaged over 50 samples (top right), and of GP using  $c = 3/5$  for one sample (bottom left) and averaged over 100 samples (bottom right).

does not improve the similarity to fracture with respect to random percolation. However, when the tent function that is used in gradient percolation is exaggerated, i.e. preferring the horizontal center more than fracture does and breaking less bonds at the horizontal edges than fracture, there is significant improvement of the similarity.

A main contributor to this surprising conclusion is the fact that a single  $\text{GP}_{\text{FP}}$  simulation does not show enough dissimilarity to a single random percolation simulation, but a single  $\text{GP}_{\text{EP}}$  simulation does. The tent function is too close in distribution to a uniform distribution to overrule the white noise, as seen in figure 4.1. However, averaged over many samples, the tent function is clearly visible. Using the exaggerated tent function, however, the profile is visible in a single sample (see figure 4.1). Still, the fact that all the measured properties tend towards the fracture results and away from random percola-

tion results could not be predicted beforehand but gives rise to possibilities to model fracture by percolation. It raises the question whether exaggerating the profile even more will make percolation approach the fracture process even more or distort the improvement seen thus far. Of course, one obvious property that will not be improved this way is the damage profile.

On the other hand, choosing the distribution based on the fracture process's own distribution, and then finding that the processes are dissimilar on account of several properties supports the idea that fracture processes, and physical processes in general, are not random. The anisotropy encountered in fracture is not inherent, but caused by the process, evident by observing that applied anisotropy does not produce the same results.

### 4.3 Recommendations

**Fractality of energy moments in percolation** Now that the fractality of energy moments is analyzed in fracture, and it is concluded that gradient percolation in some form or another can match fracture, the question that is raised is what sort of fractality energy moments show in the various percolation processes. The distribution of energy moments is a physical property which says a lot about a process; therefore, it will certainly help support or reject the notion of substituting percolation for fracture. In any case, it is interesting to see whether percolation will show clear multifractality or clear monofractality.

**Analyze distribution of energies** Now that it is determined that near the mechanical breakdown of the system the energy moments as a function of length start to display deviant behavior, it is easy to wonder what the energy distribution looks like. The histogram of the energies of the bonds may change over the course of the process, especially near the stress drop point, a critical point.

**Determine the shape of  $\mu$  vs.  $q$  graph for  $q = 0..2$**  Especially in the SD graph there is apparent nonlinearity at least for small moments ( $q < 2$ ). To discover the nature of this nonlinearity will be interesting.

**More exaggerated tent profile** As mentioned in section 4.2, a hypothesis that results from the conclusion drawn about  $GP_{FP}$  and  $GP_{EP}$  is that a further exaggerated tent profile (or perhaps Gaussian, cf. 3.1) will tend its properties towards those of fracture or start to distort them, denying further improvement. A possible choice is selecting a value for

$c$  for which the percolation threshold is the same as in fracture, for each different lattice size.

**Analyze avalanche-like behavior around and beyond maximum stress**

It is concluded that the final crack localization and mechanical breakdown cannot be modelled by a random process, so to find out what the driving force is behind the formation of the crack is an important step towards understanding the process of fracture.

# Bibliography

- [Herrmann] H.J. Herrmann, *Fractures*. A. Bundle and S. Havlin (eds.), *Fractals and disordered systems*. Berlin: Springer-Verlag, 1991.
- [Malakhovsky] I. Malakhovsky and M.A.J. Michels, *Scaling and localization in fracture of disordered central-force spring lattices: comparison with random damage percolation*. Physical Review B, **74**, 014206 (2006).
- [Schmittbuhl] J. Schmittbuhl, F. Schmitt, and C. Scholz, *Scaling invariance of crack surfaces*. Journal of Geophysical Research, **100**, 5953-5973 (1995).
- [Stauffer] D. Stauffer and A. Aharony, *Introduction to Percolation Theory*. London, Washington, DC: Taylor & Francis, 1992.