MASTER

LMI-approaches to the performance analysis of reset control systems

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Chapter 1

Introduction

Over the past decades control engineers have developed various methods in an attempt to control linear time-invariant (LTI) plants as 'good' as possible, or at least within design specifications. Various linear design methods, like classic control theory, state space control or $\mathcal{H}_\infty$ control, all have their own advantages, but each suffers from the same fundamental limitations (like Bode's Sensitivity integral) which are inevitable in the linear control of linear systems [22, 25]. For this reason several non-linear controllers for LTI plants have been suggested in literature [14]. This master's thesis intensively discusses one of those non-linear controllers, namely the reset controller.

A reset controller is basically a linear dynamic controller, whose states or subset of states are reset to zero whenever its input and/or output satisfy a certain algebraic relation. The idea of reset control can best be understood when considering an integrating controller in particular. An integrator 'sums up' the error over time in order to achieve a zero steady state error. However, when the error becomes zero for the first time, the integrator still has the 'summed' error stored in its states, which it has to get rid of. During this 'emptying' of the integrator it causes overshoot in the error.

A very obvious choice is then of course to reset the states of this integrator to zero as soon as its input (the error) becomes zero. This way the states, containing the 'summed' error, are cleared instantaneously, and hence the overshoot is avoided. This concept was indeed validated by simulations [3, 10], and the potential of reset control was furthermore shown in experiments [9, 12, 27]. In each case reset control seemed to be able to meet specifications which linear controllers could not meet.

Triggered by the apparent advantage of reset control over linear control, various people have tried to analyze the stability and the performance of reset control systems. Highlights in this respect are the H_β-condition for strictly proper controller in [5] and the $\mathcal{L}_2$ analysis for *first order reset elements* (FOREs) in [23, 26]. Multiple system descriptions and analysis methodologies have been tried in history, but the state space description and Lyapunov based stability results expressed in *linear matrix inequalities* (LMIs) of these papers can be called the most promising of them all.

Hence, inspired by these results, this master's thesis continues the work of [23, 26], formulating LMI-based performance measures for reset control systems. First, the results for $\mathcal{L}_2$ stability are extended in the paper in Chapter 2 such that the $\mathcal{H}_\infty$ norm can be calculated for general reset control systems. Hence, we can use any reset controller, instead of merely FOREs like in [23, 26]. Moreover, this extension offers a solution for tracking problems and suggests future research directions to further decrease the conservatism present in the analysis. In Chapter 3 a similar methodology is used in a paper where an LMI-based analysis to calculate the $\mathcal{H}_2$ norm of a reset control system is derived. Furthermore, the use of the $\mathcal{H}_2$ norm is illustrated with an example, where reset control is shown to outperform the 'optimal' linear controller in terms of a constrained $\mathcal{H}_2$ problem.

Extra background information about various related subjects is provided in several appendices at the end of this thesis. The very first known reset controller, called the Clegg integrator, is extensively described in Appendix A. The previously mentioned H_β-condition for stability is explained in Appendix B. Furthermore, Appendix C gives some necessary background information about dissipativity used in Chapters 2 and 3, whereas Appendices D and E give an overview of induced norm analysis LMIs for linear systems in continuous and discrete time, respectively. Finally, synthesis methods for linear systems are discussed in Appendices F (using a linearizing change of variables) and G (using the elimination lemma).

Chapter 2

$\mathcal{H}_\infty$ analysis of reset control systems

In this chapter a paper is presented in which the work of [23, 26] is generalized. This paper provides an $\mathcal{H}_\infty$ analysis tool, which is applicable to any reset control system which fits into the general $\mathcal{H}_\infty$ framework. It can hence be used for arbitrary linear plants and, more importantly, for arbitrary reset controllers (with linear flow dynamics). The $\mathcal{H}_\infty$ analysis itself is derived using dissipativity theory and piecewise quadratic Lyapunov functions, resulting in a computable set of LMIs.

Another improvement compared to [23, 26] is that the generalized system description remodels the resetting condition, such that e.g. tracking problems are successfully included in the $\mathcal{H}_\infty$ analysis. Possible conservatism which can be present when considering these tracking problems can be removed by including strictly proper input filters, thereby explicitly taking a priori knowledge about the input signals into account.

Other contributions of the following paper include various future research directions to reduce the conservatism even further, and a discussion on the usefulness of the $\mathcal{H}_\infty$ norm and other induced norms for reset control systems.

An LMI-based $\mathcal{H}_\infty$ performance analysis for reset control systems

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Abstract— **We present a general LMI-based analysis method to determine an upperbound on the $\mathcal{L}_2$ gain of a reset control system. These results are derived using piecewise quadratic Lyapunov functions. The computable sufficient conditions for $\mathcal{L}_2$ stability are suitable for all LTI plants and linear-based reset controllers, thereby generalizing the results available in literature. Our results furthermore extend literature by including tracking and measurement noise problems, while using proper input filters to reduce conservatism. We conclude by suggesting future research directions to reduce conservatism even further.**

Index Terms— **Switched systems, reset control, linear matrix inequality, Lyapunov stability, $\mathcal{L}_2$ gain, tracking.**

I. Introduction

In order to overcome the fundamental performance limitations which linear controllers are known to be subject to [1], [2], various nonlinear feedback controllers for linear time-invariant (LTI) plants were proposed in literature [3]. An example of such a nonlinear feedback is the reset controller, which is basically a linear controller whose states (or subset of states) are reset to zero whenever its input and output satisfy certain conditions.

The first resetting element was introduced in 1958 [4], when Clegg proposed an integrator which resets whenever the input is zero. The advantage of this Clegg integrator was illustrated using its describing function, which has the same magnitude plot as the linear integrator, but its performance limiting phase lag is 38,1° instead of 90°. However, the use and effect of this resetting integrator is not straightforward because of its nonlinear behavior, so its first use in a control design procedure [5] was not until 1974. Subsequently, a *first order reset element (FORE)* was introduced in [6], with which a controller design procedure was proposed based on frequency domain techniques. An overview of these results is given in [7].

At the end of the '90s there has been renewed interest in reset control systems, resulting in various stability analysis techniques. The first results were reported in [8], stating a stability criterion for zero-input closed loops with a second order plant and a Clegg integrator. However, the criterion involves explicit computation of reset times and closed loop solutions. A similar stability analysis for FOREs was published in [9], again restricted to second order plants, and hard to generalize for higher order systems. Additionally, in [10] a stability analysis using an integral quadratic constraint (IQC) was stated, which is however rather conservative due to the independency of reset times.

In following publications stability conditions were formulated using Lyapunov based conditions. This was first done in [11] and [12], considering only second order closed loops with constant inputs. The result has been extended in [13] to a sufficient criterion on BIBO (bounded input bounded output) stability, for which [14] has provided the proof. These results were generalized in [15], resulting in the so called H_β-condition. The same paper also addressed the tracking problem, based on the internal model principe. Furthermore, the possible advantages of reset controllers over linear ones have been shown both in simulations [9], [16] and experiments [7], [14], [17]. A clear overview of [11]–[17] is provided in [18], summarizing the H_β stability analysis for general reset systems.

The H_β-condition is a reformulation of Lyapunov based stability LMIs (linear matrix inequalities) using the Kalman-Yakubovich-Popov Lemma, in order to provide computable conditions to check the stability of zero-input reset control systems. The analysis consists of two stability LMIs, one corresponding to the *flowing* of the closed loop (i.e. smooth evolution of the state) and the other to the *reset* of the controller. These LMIs are coupled as a common quadratic Lyapunov function is searched for both. Therefore the H_β-condition is conservative, and is only necessary and sufficient for *quadratic* stability. Systems violating the H_β-condition can still be stable. Moreover, since the flowing LMI is solved for the complete state space, it requires the linear part of the closed loop dynamics to be stable. Since this is over-restrictive, it introduces additional conservatism.

This conservatism was largely being dealt with in more recent publications [19], [20]. To make the stability LMIs less conservative, the authors first suggested a slightly different resetting condition. Indeed, their idea to reset when controller in- and output have opposite sign instead of when the input is zero results in a much smaller flow region. This allows the flowing LMI to be solved over a smaller part of the state space, thereby releasing the stability constraint of the linear closed loop and thus sharpening the stability bounds of the reset system. Second, the authors allowed piecewise quadratic Lyapunov functions, thus approximating higher order Lyapunov functions to capture a broader class of stability problems. On top of this the analysis has been extended to $\mathcal{L}_2$ stability, such that the closed loop $\mathcal{H}_\infty$-gain from input to output of a reset control system can be approximated by an upperbound.

As such, the work in [19], [20] is the most general analysis framework for reset control systems currently available in literature. However, it is not generally applicable, since it treats only FOREs and Clegg integrators. Furthermore, it does not include a solution to the tracking problem, since its system description assumes a zero reference. Hence, our work will form an extension to [19], [20] in several directions. First, it generalizes the $\mathcal{H}_\infty$ analysis to general reset control systems fitting into the common $\mathcal{H}_\infty$ framework. Second, the tracking problem is successfully included in our results. A solution to reduce the possible conservatism that this might introduce, will be provided via input filtering. Third, additional remaining conservatism problems are discussed, and several possible solution directions suggested. Finally, in the discussion we will shortly reflect on synthesis possibilities, the usefulness of $\mathcal{H}_\infty$ for reset control systems and possible extensions to other induced norms.

This paper is organized as follows. First, Section II introduces the general $\mathcal{H}_\infty$ framework for reset control systems and describes the dynamics of the plant and the controller. In Section III our main results are derived, as well as a solution to deal with tracking problems. Furthermore, some future research directions are provided in Section IV. After the conclusions in Section V, we shortly discuss our results in Section VI.

Notation. The set of real numbers is denoted by $\mathbb{R}$. The identity matrix of dimension $n \times n$ is denoted by $I_n \in \mathbb{R}^{n\times n}$. Given two vectors x_1, x_2 we write (x_1, x_2) to denote $[x_1^T, x_2^T]^T$. A vector $x \in \mathbb{R}^n$ is nonnegative, denoted by $x \geq 0$, if its elements $x_i \geq 0$ for $i = 1, \dots, n$. A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite, denoted by $A \succ 0$ if $x^T A x > 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$. A sequence of scalars $(u^1, u^2, \dots, u^k)$ is called lexicographically nonnegative, written as $(u^1, u^2, \dots, u^k) \geq_\ell 0$, if $(u^1, u^2, \dots, u^k) = (0, 0, \dots, 0)$ or $u^j > 0$ where $j = \min\{p = 1, \dots, k : u^p \neq 0\}$. For a sequence of vectors $(x^1, x^2, \dots, x^k)$ with $x^j \in \mathbb{R}^n$, we write $(x^1, x^2, \dots, x^k) \geq_\ell 0$ when $(x_i^1, x_i^2, \dots, x_i^k) \geq_\ell 0$ for all $i = 1, \dots, n$.

II. General system description

In this section we present a mathematical description of the reset controller and the resulting closed loop. These descriptions are chosen to fit into the common multichannel $\mathcal{H}_\infty$ framework, as depicted in Figure 1. The augmented plant P, with state $x_p \in \mathbb{R}^{n_p}$, contains the system to be controlled, together with possible input- and output-weightings. The reset controller is denoted by K, whose states are denoted by $x_k \in \mathbb{R}^{n_k}$. The closed loop state is defined by $x \in \mathbb{R}^n$ with $x = (x_p, x_k)$. Moreover, $w \in \mathbb{R}^{n_w}$ and $z \in \mathbb{R}^{n_z}$ denote the exogenous inputs and outputs, and $y, u \in \mathbb{R}$ denote the controller input and output, respectively. Note that we consider SISO plants and controllers only, since reset control for MIMO systems is still an open issue.

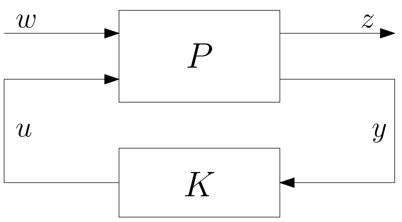


Fig. 1. General multichannel closed loop system

For the remainder of this paper we will consider LTI augmented plants P, whose dynamics are described by

$$\begin{aligned} \dot{x}_p &= Ax_p + Bu + B_w w \\ z &= C_z x_p + D_{zw} w + D_z u \\ y &= C x_p + D_w w. \end{aligned} \tag{1}$$

Note that there is no direct feedthrough from u to y, as is e.g. the case for general motion systems. Furthermore, for feedback control to make sense, we assume that (A, B) is at least stabilizable and (A, C) is at least detectable.

A. Reset controller

The controller K is described by a linear filter whose (subset of) states are reset whenever its input y and output u satisfy a certain condition, hence

$$\begin{aligned} \dot{x}_k &= A_K x_k + B_K y && \text{if } (y, u) \in \mathfrak{C}' \\ x_k^+ &= A_r x_k && \text{if } (y, u) \in \mathfrak{D}' \\ u &= C_K x_k + D_K y \end{aligned} \tag{2}$$

This reset controller can thus be seen as a hybrid system, whose states x_k flow linearly conform (A_K, B_K, C_K, D_K) in one part of the (y, u)-space and are reset instantaneously from x_k to x_k^+ in another part. These parts, or regions, are defined by a *flow set* $\mathfrak{C}'$ and a *reset set* $\mathfrak{D}'$ respectively, which are defined by the resetting condition. There are various resetting conditions possible, but here we will follow the lines of [19], [20], where resetting occurs whenever input and output have opposite sign (i.e. $yu \leq 0$). Compared to [18] this choice reduces the size of the flow set and thus allows a considerable relaxation of the Lyapunov conditions later on.

Furthermore, for robustness reasons the sets $\mathfrak{C}'$ and $\mathfrak{D}'$ should be closed and such that $\mathfrak{C}' \cup \mathfrak{D}' = \mathbb{R}^2$ [19]. To describe $\mathfrak{C}'$ and $\mathfrak{D}'$ we follow the method described in [21]. The controller states flow whenever $y \geq 0, u \geq 0$ or $y \leq 0, u \leq 0$, and reset otherwise, hence

$$\mathfrak{C}' := \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \mathbb{R}^2 : E_f \begin{bmatrix} y \\ u \end{bmatrix} \geq 0 \quad \text{or} \quad E_f \begin{bmatrix} y \\ u \end{bmatrix} \leq 0 \right\} \tag{3a}$$

$$\mathfrak{D}' := \left\{ \begin{bmatrix} y \\ u \end{bmatrix} \in \mathbb{R}^2 : E_R \begin{bmatrix} y \\ u \end{bmatrix} \geq 0 \quad \text{or} \quad E_R \begin{bmatrix} y \\ u \end{bmatrix} \leq 0 \right\} \tag{3b}$$

where

$$E_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad E_R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The flow and reset sets can also be expressed in terms of the closed loop state x. Therefore we introduce a transformation matrix $T = \left[\begin{array}{c|c} T_x & T_w \end{array}\right]$ defined as

$$\begin{bmatrix} y \\ u \end{bmatrix} = T \begin{bmatrix} x \\ w \end{bmatrix} = \left[\begin{array}{cc|c} C & 0 & D_w \\ D_K C & C_K & D_K D_w \end{array}\right] \begin{bmatrix} x_p \\ x_k \\ w \end{bmatrix},$$

such that

$$\begin{aligned}\mathfrak{C} \;:=\; \Bigg\{\begin{bmatrix}x\\w\end{bmatrix} \in \mathbb{R}^{n+n_w} :\;& E_f T\begin{bmatrix}x\\w\end{bmatrix} \geq 0\\ \text{or}\;\;& E_f T\begin{bmatrix}x\\w\end{bmatrix} \leq 0\Bigg\} \quad (4a)\\ \mathfrak{D} \;:=\; \Bigg\{\begin{bmatrix}x\\w\end{bmatrix} \in \mathbb{R}^{n+n_w} :\;& E_R T\begin{bmatrix}x\\w\end{bmatrix} \geq 0\\ \text{or}\;\;& E_R T\begin{bmatrix}x\\w\end{bmatrix} \leq 0\Bigg\}. \quad (4b)\end{aligned}$$

Note that $\mathfrak{C}$ and $\mathfrak{D}$ also depend on the input w when $D_w \neq 0$.

The instantaneous reset from x_k to x_k^+ itself is described by the discrete map $A_r \in \mathbb{R}^{n_k \times n_k}$, indicating to which values the controller states are reset to. Various choices for A_r are theoretically possible, but a reasonable and appropriate choice, commonly used in literature, is typically

$$A_r = \begin{bmatrix} I_{n_k - n_r} & 0\\ 0 & 0_{n_r}\end{bmatrix},$$

stating that the last n_r of the n_k controller states are reset to zero, while the others remain unchanged.

B. Closed loop dynamics

The dynamics of the augmented plant and the reset controller can be combined into one description for the closed loop dynamics, denoted by Σ:

$$\Sigma : \begin{cases} \dot{x} &= \mathcal{A}x + \mathcal{B}w \qquad \text{if } (x,w) \in \mathfrak{C}\\ x^+ &= A_R x \qquad\quad\;\; \text{if } (x,w) \in \mathfrak{D}\\ z &= \mathcal{C}x + \mathcal{D}w \end{cases} \quad (5)$$

where

$$\left[\begin{array}{c|c}\mathcal{A} & \mathcal{B}\\ \hline \mathcal{C} & \mathcal{D}\end{array}\right] = \left[\begin{array}{cc|c} A + BD_K C & BC_K & B_w + BD_K D_w\\ B_K C & A_K & B_K D_w\\ \hline C_z + D_z D_K C & D_z C_K & D_{zw} + D_z D_K D_w\end{array}\right]$$

$$A_R = \begin{bmatrix} I_{n_p} & 0\\ 0 & A_r\end{bmatrix}.$$

The linear closed loop system which results when resetting is omitted (i.e. when $A_R = I_n$, $\mathfrak{C} = \mathbb{R}^n$ and $\mathfrak{D} = \emptyset$), is called the *base linear system.*

At this point we emphasize that the reset controller should be such that multiple resets at one point in time are excluded, in order to guarantee local existence of solutions. For the remainder of this paper we thus assume that the closed loop states can flow after each reset for at least a small amount of time. For convenience we furthermore assume that the input w is analytic, denoted by $w \in \mathbb{A}^{n_w}$. We can then state our flow-assumption mathematically as follows:

Assumption 1 *The reset controller and the regions $\mathfrak{C}$ and $\mathfrak{D}$ must be such that*

$$\begin{bmatrix}x\\w\end{bmatrix} \in \mathfrak{D} \quad \Rightarrow \quad \begin{bmatrix}x^+\\w\end{bmatrix} \in \mathcal{F}_{\mathfrak{C}}, \quad (6)$$

where $\mathcal{F}_{\mathfrak{C}}$ is a subset of $\mathfrak{C}$ defined as

$$\begin{aligned}\mathcal{F}_{\mathfrak{C}} \;:=\; \Bigg\{\begin{bmatrix}x_0\\w\end{bmatrix} \in \mathbb{R}^n \times \mathbb{A}^{n_w} :&\\ \exists \epsilon > 0 \quad \forall \tau \in [0, \epsilon) \quad & \begin{bmatrix} x_{x_0,w}(\tau)\\ w(\tau)\end{bmatrix} \in \mathfrak{C}\Bigg\}. \quad (7)\end{aligned}$$

The set $\mathcal{F}_{\mathfrak{C}}$ thus defines all combinations (x_0, w) of initial conditions x_0 (or in this case x^+) and inputs w for which the closed loop solution trajectories $x_{x_0}(\tau)$ of (5) stay in $\mathfrak{C}$ for at least $\tau \geq \epsilon$.

This assumption implies that the state after a reset x^+ should either lie in the interior of $\mathfrak{C}$, or when it is on the boundary of $\mathfrak{C}$, should not be driven outside this region by either the dynamics or the external input. This can also be formulated with a lexicographic ordening:

Corollary 2 *The closed loop system (5) can flow after a reset $x^+ = A_R x$ at time t_r if at least one of the two following lexicographic ordenings hold whenever $(x, w) \in \mathfrak{D}$*

$$\begin{aligned}&(E_f T_x A_R x + E_f T_w w(t_r),\\ &\quad E_f T_x\left[\mathcal{A}A_R x + \mathcal{B}w(t_r)\right] + E_f T_w \dot{w}(t_r),\\ &\quad E_f T_x\left[\mathcal{A}^2 A_R x + \mathcal{A}\mathcal{B}w(t_r) + \mathcal{B}\dot{w}(t_r)\right]\\ &\qquad\qquad + E_f T_w \ddot{w}(t_r), \ldots) \geq_\ell 0 \quad (8a)\\ &(E_f T_x A_R x + E_f T_w w(t_r),\\ &\quad E_f T_x\left[\mathcal{A}A_R x + \mathcal{B}w(t_r)\right] + E_f T_w \dot{w}(t_r), \ldots) \leq_\ell 0. \quad (8b)\end{aligned}$$

Note again that $w \in \mathbb{A}^{n_w}$ is analytic, meaning that all its derivatives exist and are bounded.

We conclude our system description by making a small comparison to the work in [19], [20].

Remark 3 Consider a simple tracking problem, depicted in Figure 2, where H is a dynamical system with output $\bar{y}$, u is the output of the controller C, and r and e are the reference signal and the error, respectively. It is easy to see

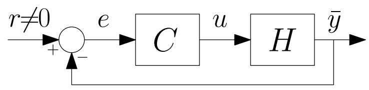


Fig. 2. Simple tracking problem

that in this case $w = r$ and thus, due to the direct feedthrough between r and e, $D_w = 1 \neq 0$. This means that the flow and reset regions $\mathfrak{C}$ and $\mathfrak{D}$ in (4) explicitly depend on w. Hence the input r clearly influences the reset moment, since resets should occur at signchanges of u or e. In the analysis of [20] however, the dependency on w was omitted, making it not applicable for tracking problems, but for disturbance rejection only. Indeed, the authors of [20] only include state information in the definition of $\mathfrak{C}$ and $\mathfrak{D}$, as the latter is defined as $\{x : x^T M x \leq 0\}$. This results in resets whenever u or $\bar{y}$ changes sign. It is easy to see that when $r \neq 0$ this resetting condition is not correct, since $e = r - \bar{y}$. □

Remark 4 In [19] Assumption 1 is simplified to

$$x \in \mathfrak{D} \quad \Rightarrow \quad x^+ \in \mathfrak{C}. \quad (9)$$

Besides the fact that the influence of w is neglected, this relation corresponds to only the first vector in the lexicographic ordening in (8). Hence, it is only necessary but not sufficient to guarantee a state flow after each reset. Still, one way to satisfy (9) for all $(x, w) \in \mathfrak{D}$ is to assume $C_K A_r = 0$. □

III. Main results

In this section we present our main results on $\mathcal{L}_2$ stability, applicable to any LTI plant (1) and any reset controller (2). These results use the following definition:

Definition 5 *The $\mathcal{L}_2$-gain $||\Gamma||_\infty^2$ of an arbitrary dynamical system Γ is defined as the worst case finite gain from a bounded input w to an output z, thus*

$$||\Gamma||_\infty^2 = \sup_{0<||w||_2<\infty} \frac{||z||_2^2}{||w||_2^2} \leq \gamma^2, \tag{10}$$

such that with initial condition $x(0) = 0$

$$\int_0^\infty z^T z dt \leq \gamma^2 \int_0^\infty w^T w dt. \tag{11}$$

Here $||.||_2^2$ denotes the square of the 2-norm of a signal.

Furthermore, we rely on the following lemma [22]:

Lemma 6 (Bounded Real Lemma) *An arbitrary dynamical system with state x has a finite $\mathcal{L}_2$ gain from input w to output z smaller than or equal to γ when it is strictly dissipative w.r.t. the supply function*

$$s(w, z) = \gamma^2 w^T w - z^T z. \tag{12}$$

This means that for this $s(w, z)$ and for all $t_1 \geq t_0$

$$V(x(t_1)) - V(x(t_0)) < \int_{t_0}^{t_1} s(w(t), z(t))dt, \tag{13}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is a storage function of the dissipative system. Furthermore, this system is stable if V is continuous and non-negative, and has a strong local minimum at the equilibrium point of the system. In that case the storage function V is a Lyapunov function.

The smallest possible value of γ for which Lemma 6 holds is known as the $\mathcal{H}_\infty$ norm of a dynamical system.

A. General $\mathcal{L}_2$ analysis

In order to find the $\mathcal{L}_2$ gain of a reset system we apply the Bounded Real Lemma to the closed loop system (5). As noted before, a reset $x^+ = A_R x$ is instantaneous, yielding $\int_{t_0}^{t_1} s(w, z)dt = 0$ since $t_0 = t_1$. With this and the differential form of (13) we can formulate a special case of Lemma 6:

Lemma 7 *The reset system (5) is stable and has a finite gain from w to z smaller than or equal to $\gamma > 0$ if there exists a non-negative Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that*

$$\frac{d}{dt}V(x) < \gamma^2 w^T w - z^T z \quad \text{if } (x, w) \in \mathfrak{C} \tag{14a}$$
$$V(x^+) - V(x) \leq 0 \quad \text{if } (x, w) \in \mathfrak{D} \tag{14b}$$

Finding the actual lowest possible value of γ can be very hard, since it depends on the used storage function V. Since we are dealing with a nonlinear closed loop system, the 'optimal' V will probably be a very complex function, possibly of high order. For the sake of computability we therefore choose to approximate such a complex function V by using *piecewise quadratic Lyapunov functions* [20]. This choice is motivated by the linear flow behavior in a large part of the state space, and results in computable LMIs. Nevertheless, it introduces some conservatism, which implies that the found minimal value of γ will only be an upperbound on the actual $\mathcal{H}_\infty$ norm.

The piecewise quadratic Lyapunov functions are obtained by partitioning the flow set $\mathfrak{C}'$ into smaller regions $\mathfrak{C}'_i$ and assigning a different quadratic Lyapunov function $V_i(x) = x^T P_i x$ to each of them [20], see Figure 3.

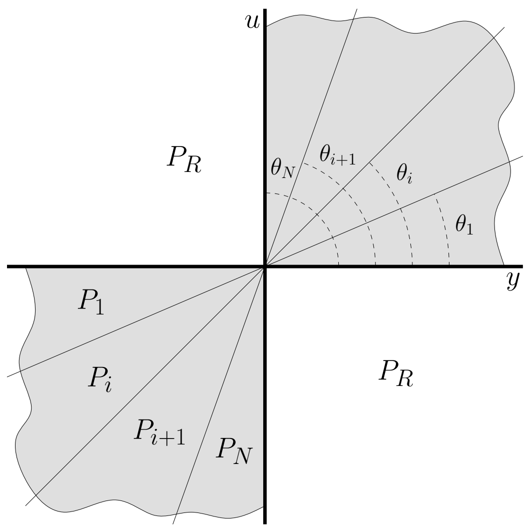


Fig. 3. Partitioning of the (y, u)-space

Each region $\mathfrak{C}'_i$ is bounded by two lines uniquely defined by the angles θ_i and θ_{i-1}. These angles should be chosen such that $0 < \theta_0 < \theta_1 < \ldots < \theta_N = \frac{\pi}{2}$, for example equidistantly distributed as $\theta_i = \frac{i}{N}\frac{\pi}{2}$, where $i = 0, \ldots, N$ and N is the number of desired subregions. Translated into the (x, w)-domain using the transformation matrix T, the regions $\mathfrak{C}_i$ and $\mathfrak{D}$ are then defined by

$$\mathfrak{C}_i := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathbb{R}^{n+n_w} : \; E_i T \begin{bmatrix} x \\ w \end{bmatrix} \geq 0 \text{ or } E_i T \begin{bmatrix} x \\ w \end{bmatrix} \leq 0 \right\} \tag{15a}$$

$$\mathfrak{D} := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathbb{R}^{n+n_w} : \; E_R T \begin{bmatrix} x \\ w \end{bmatrix} \geq 0 \text{ or } E_R T \begin{bmatrix} x \\ w \end{bmatrix} \leq 0 \right\} \tag{15b}$$

where

$$E_i = \begin{bmatrix} -\sin(\theta_{i-1}) & \cos(\theta_{i-1}) \\ \sin(\theta_i) & -\cos(\theta_i) \end{bmatrix}, \quad E_R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Moreover, using $T = \left[\, T_x \,|\, T_w \,\right]$, we introduce

$$E_i T = \left[\, E_i T_x \,|\, E_i T_w \,\right] = \left[\, E_{x,i} \,|\, E_{w,i} \,\right], \tag{16a}$$
$$E_R T = \left[\, E_R T_x \,|\, E_R T_w \,\right] = \left[\, E_{x,R} \,|\, E_{w,R} \,\right]. \tag{16b}$$

The region borders are defined by the equality

$$\begin{bmatrix} -\sin(\theta_i) & \cos(\theta_i) \end{bmatrix} T \begin{bmatrix} x \\ w \end{bmatrix} = \Phi_i \begin{bmatrix} x \\ w \end{bmatrix} = 0, \quad (17)$$

whose solutions are in the kernel of Φ_i. We can also use an image representation for these boundaries, yielding $\mathrm{im}(W_{\Phi i}) = \ker(\Phi_i)$, where $W_{\Phi i} \in \mathbb{R}^{(n+n_w)\times(n+n_w-1)}$ is a matrix with full columnrank, and $\mathrm{im}(W_{\Phi i})$ denotes its image.

Using these subregions and borders we can now formulate LMIs to calculate an upperbound on the $\mathcal{L}_2$ gain, as stated in the following theorem:

Theorem 8 *Consider the reset control system in (5) with a partitioning of the flow set given by (15). This system is stable and has an $\mathcal{L}_2$ gain from w to z smaller than or equal to $\gamma > 0$ if the following linear matrix inequalities and equalities in the variables $P_i, P_R, U_i, U_R, V_i, V_R$ are feasible:*

$$\begin{bmatrix} \mathcal{A}^T P_i + P_i\mathcal{A} + E_{x,i}^T U_i E_{x,i} & P_i\mathcal{B} + E_{x,i}^T U_i E_{w,i} & \mathcal{C}^T \\ \mathcal{B}^T P_i + E_{w,i}^T U_i E_{x,i} & -\gamma I + E_{w,i}^T U_i E_{w,i} & \mathcal{D}^T \\ \mathcal{C} & \mathcal{D} & -\gamma I \end{bmatrix} \prec 0, \quad i = 1,\dots,N \quad (18a)$$

$$\begin{bmatrix} A_R^T P_R A_R - P_R + E_{x,R}^T U_R E_{x,R} & E_{x,R}^T U_R E_{w,R} \\ E_{w,R}^T U_R E_{x,R} & E_{w,R}^T U_R E_{w,R} \end{bmatrix} \preceq 0 \quad (18b)$$

$$\begin{bmatrix} P_i - E_{x,i}^T V_i E_{x,i} & -E_{x,i}^T V_i E_{w,i} \\ -E_{w,i}^T V_i E_{x,i} & -E_{w,i}^T V_i E_{w,i} \end{bmatrix} \succ 0, \quad i = 1,\dots,N \quad (18c)$$

$$\begin{bmatrix} P_R - E_{x,R}^T V_R E_{x,R} & -E_{x,R}^T V_R E_{w,R} \\ -E_{w,R}^T V_R E_{x,R} & -E_{w,R}^T V_R E_{w,R} \end{bmatrix} \succ 0 \quad (18d)$$

$$W_{\Phi i}^T \begin{bmatrix} P_i - P_{i+1} & 0 \\ 0 & 0 \end{bmatrix} W_{\Phi i} = 0, \quad i = 1,\dots,N-1 \quad (18e)$$

$$W_{\Phi 0}^T \begin{bmatrix} P_R - P_1 & 0 \\ 0 & 0 \end{bmatrix} W_{\Phi 0} = 0 \quad (18f)$$

$$W_{\Phi N}^T \begin{bmatrix} P_N - P_R & 0 \\ 0 & 0 \end{bmatrix} W_{\Phi N} = 0 \quad (18g)$$

where $U_i, U_R, V_i, V_R \in \mathbb{R}^{2\times 2}$ are arbitrary symmetric matrices with non-negative elements. Minimizing (18) over γ returns an upperbound on the actual $\mathcal{H}_\infty$ norm of (5).

Proof: First, (18a) is the elaboration of the dissipativity equation (14a) using the quadratic supply function $s(w,z) = \gamma^2 w^T w - z^T z$ and piecewise quadratic Lyapunov functions $V_i(x) = x^T P_i x$:

$$\dot{x}^T P_i x + x^T P_i \dot{x} < \gamma^2 w^T w - z^T z$$

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} \mathcal{A}^T P_i + P_i\mathcal{A} & P_i\mathcal{B} \\ \mathcal{B}^T P_i & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < \begin{bmatrix} w \\ z \end{bmatrix}^T \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}$$

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} \mathcal{A}^T P_i + P_i\mathcal{A} & P_i\mathcal{B} \\ \mathcal{B}^T P_i & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} - \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} 0 & I \\ \mathcal{C} & \mathcal{D} \end{bmatrix}^T \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} 0 & I \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0$$

$$\Rightarrow \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} \mathcal{A}^T P_i + P_i\mathcal{A} + \mathcal{C}^T\mathcal{C} & P_i\mathcal{B} + \mathcal{C}^T\mathcal{D} \\ \mathcal{B}^T P_i + \mathcal{D}^T\mathcal{C} & \mathcal{D}^T\mathcal{D} - \gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} < 0$$

which should hold for at least all (x, w) belonging to $\mathfrak{C}_i$. As is explained in [21], this subset of (x, w) can be overapproximated with a quadratic form, hence

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} E_{x,i}^T \\ E_{w,i}^T \end{bmatrix} U_i \begin{bmatrix} E_{x,i} & E_{w,i} \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \geq 0, \quad (19)$$

where U_i is a square symmetric matrix with non-negative elements. Since any $(x, w) \in \mathfrak{C}_i$ also satisfies the quadratic constraint, we know that the subset $\{(x, w) : (19)\} \geq \mathfrak{C}_i$, as required. The S-procedure [22] can then be used to include (19) into the dissipativity inequality, resulting in an LMI. Finally, applying a Schur complement yields (18a). Moreover, (18b) results from (14b) in a similar way.

The Lyapunov functions $V_i(x) = x^T P_i x$ and $V_R(x) = x^T P_R x$ should furthermore be positive in their corresponding domain. Applying the S-procedure in a similar fashion results in the LMIs (18c) and (18d).

Finally, in Lemma 7 we have assumed that the Lyapunov function V is Lipschitz continuous. This also implies that V does not increase nor decrease whenever a boundary from $\mathfrak{C}_i$ to $\mathfrak{C}_{i+1}$ or vice versa in the (x, w)-plane is crossed, i.e. when switching between $x^T P_i x$ and $x^T P_{i+1} x$ takes place. Hence, we have to require continuity of the Lyapunov function, i.e. $x^T P_i x = x^T P_{i+1} x$ across the i-th border. These borders are spanned by $\ker(\Phi_i)$, or the matrix $W_{\Phi i}$ equivalently, so application of Finsler's Lemma [22] finally results in equalities (18e) to (18g). ■

Remark 9 As already mentioned earlier, the obtained minimal value of γ will always be an upperbound on the actual $\mathcal{H}_\infty$ norm. This upperbound can be lowered by increasing the number of subdivisions N, see [21]. □

Remark 10 Theorem 8 is a generalization of previous stability results in literature and is applicable to all augmented plants and reset controllers which fit into the common multi-channel $\mathcal{H}_\infty$ framework described in Section II. It therefore extends the work in [19], [20], which only considers FOREs (including the Clegg integrator) and plants in observability canonical form. Furthermore, as mentioned in Remark 3, our analysis includes tracking problems, while [19], [20] do not. Moreover, Theorem 8 is slightly less conservative, since P_i and P_R are not necessarily positive definite. Compared to [18], our result is much less conservative, includes $\mathcal{H}_\infty$ performance and is applicable to cases where $D_K \neq 0$. □

B. $\mathcal{L}_2$ for tracking problems

The analysis in Theorem 8 is capable of providing an upperbound on the actual $\mathcal{L}_2$ gain of any closed loop reset control system. However, in some situations the upperbound may still be too conservative, like in tracking problems as in Figure 2. The cause of this high upperbound, as is shown next, is the fact that $D_w \neq 0$ in those situations.

First note that $T \in \mathbb{R}^{2\times(n+n_w)}$ and $\Phi_i \in \mathbb{R}^{1\times(n+n_w)}$. Hence the kernel of Φ_i is spanned by $n + n_w - 1$ independent columns, so $W_{\Phi i} \in \mathbb{R}^{(n+n_w)\times(n+n_w-1)}$. Moreover, (18e) can be rewritten by removing its zeros, such that

$$W'^T_{\Phi i}(P_i - P_{i+1})W'_{\Phi i} = 0 \quad (20)$$

where $W'_{\Phi i} \in \mathbb{R}^{n\times(n+n_w-1)}$ are the first n rows of $W_{\Phi i}$. If $D_w = 0$ the last n_w entries of Φ_i are 0, which implies that the structure of $W_{\Phi i}$ is such (it contains columns $[0, \cdots, 0, I_{n_w}]^T$ or linear combinations of this with its other columns) that $W'_{\Phi i}$ consists of only $n - 1$ independent

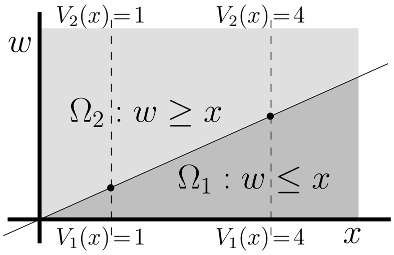


Fig. 4. Common Lyapunov function

columns. Hence its rank is $n-1$, which gives design freedom in $P_i - P_{i+1}$, since $P_i \in \mathbb{R}^n$. However, when $D_w \neq 0$ the rank of $W'_{\Phi i}$ is equal to n, and the only solution to (20) is $P_i - P_{i+1} = 0$ for all i. This means that all V_i are the same, yielding $V(x) = x^T Px$ for all subregions. Hence Theorem 8 is solved with a common quadratic Lyapunov function, which clearly introduces conservatism, and increases the upperbound on the $\mathcal{H}_\infty$ norm.

This conservatism also allows a more comprehensible interpretation. Each subregion $\mathfrak{C}_i$ has its own Lyapunov function $V_i(x)$, which solely depends on x, while (15) shows that the region itself is defined in terms of both x and w. Figure 4 illustrates this for the simple case where $x, w \in \mathbb{R}$. Continuity of $V(x)$ across the border between Ω_1 and Ω_2 requires that $V_1(x) = V_2(x)$, since $V(x)$ only depends on x (depicted by the dashed vertical lines).

The above problem of a common Lyapunov function arises in any situation where $D_w \neq 0$, including measurement noise and tracking problems. This drawback can be avoided however, by making some explicit assumptions about the structure of the augmented plant P.

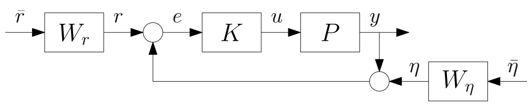


Fig. 5. Closed loop with filtered inputs

The solution we suggest is to force $D_w = 0$, which can be done by including strictly proper input filters for exogenous signals that enter the closed loop *before* the controller, see Figure 5. Since these strictly proper filters have no direct feedthrough of the input, there is also no direct feedthrough from w (containing $\bar{r}$ and $\bar{\eta}$) to y in the augmented plant in (1), so $D_w = 0$. By including input filters in the augmented plant we assume to have a priori knowledge of the inputs, which is often the case in practice. As such we are able to include this knowledge inside the state vector x_p. This way the Lyapunov function $V(x)$ also depends on this knowledge of the input, while $\mathfrak{C}_i$ no longer depends on w. Theorem 8 can now be simplified, since $T = T_x$, $E_{w,i} = E_{w,R} = 0$ and $\text{im}(\bar{W}_{\Phi i}) = \ker([-\sin(\theta_i), \cos(\theta_i)] \cdot T_x)$.

Theorem 11 *Consider the reset control system in (5), where as a special case $D_w = 0$. This system is stable and has an $\mathcal{L}_2$ gain from w to z smaller than or equal to $\gamma > 0$ if the following set of linear matrix inequalities and equalities in the variables $P_i, P_R, U_i, U_R, V_i, V_R$ are feasible:*

$$\begin{bmatrix} \mathcal{A}^T P_i + P_i \mathcal{A} + E_{x,i}^T U_i E_{x,i} & P_i \mathcal{B} & \mathcal{C}^T \\ \mathcal{B}^T P_i & -\gamma I & \mathcal{D}^T \\ \mathcal{C} & \mathcal{D} & -\gamma I \end{bmatrix} \prec 0, \quad i = 1, \ldots, N \quad \text{(21a)}$$

$$A_R^T P_R A_R - P_R + E_{x,R}^T U_R E_{x,R} \preceq 0 \quad \text{(21b)}$$

$$P_i - E_{x,i}^T V_i E_{x,i} \succ 0, \quad i = 1, \ldots, N \quad \text{(21c)}$$

$$P_R - E_{x,R}^T V_R E_{x,R} \succ 0 \quad \text{(21d)}$$

$$\bar{W}_{\Phi i}^T (P_i - P_{i+1}) \bar{W}_{\Phi i} = 0, \quad i = 1, \ldots, N-1 \quad \text{(21e)}$$

$$\bar{W}_{\Phi 0}^T (P_R - P_1) \bar{W}_{\Phi 0} = 0 \quad \text{(21f)}$$

$$\bar{W}_{\Phi N}^T (P_N - P_R) \bar{W}_{\Phi N} = 0 \quad \text{(21g)}$$

where $U_i, U_R, V_i, V_R \in \mathbb{R}^{2\times 2}$ are arbitrary symmetric matrices with non-negative elements. The smallest possible value of γ for which (21) holds, is an upperbound on the actual $\mathcal{H}_\infty$ norm of the considered system.

Remark 12 Theorem 11 is similar to the result in [20]. Note however that our result is applicable to all possible LTI plants and reset controllers which fit the $\mathcal{H}_\infty$ framework, as long as $D_w = 0$. In contrast to [20] it can cope with tracking problems and measurement noise, as long as these inputs are filtered with a strictly proper filter. □

Since the analysis in Theorem 11 encompasses a priori knowledge of the inputs, it can be performed for a broad class of desired input types (and corresponding input filters). Filter examples include

- unit step: $W_r(s) = \frac{1}{(s+\varepsilon)}$;
- unit ramp: $W_r(s) = \frac{1}{(s+\varepsilon)^2}$;
- sine wave with frequency ω: $W_r(s) = \frac{\omega}{(s+\varepsilon)^2+\omega^2}$;

where s is the Laplace variable and $\varepsilon > 0$ is a small number to force the eigenvalues of the filters to the LHP. This is common in $\mathcal{H}_\infty$ analysis and necessary to prevent closed loop poles at $s = 0$, which causes infeasibility of (21a).

IV. Future Research Directions

Remark 9 already stated that the above analysis tools only give an upperbound on the actual $\mathcal{H}_\infty$ norm of a reset control system, and are hence to some extend conservative. In this section we shortly address some future research possibilities to further reduce this conservatism.

A. Discontinuous Lyapunov function

The equalities (18e) to (18g) result from the assumption that the Lyapunov function is Lipschitz continuous. This is necessary when it is unclear whether solutions of (5) move from $\mathfrak{C}_i$ to $\mathfrak{C}_{i+1}$, or vice versa (or even both). However, in some situations we might be able to determine beforehand whether the closed loop solution follows a purely clockwise or counterclockwise path in the (y, u) plane. In these cases the Lipschitz continuity can be dropped, as we can allow decreases of the Lyapunov function across region boundaries. Since a pure (counter)clockwise path can be expressed in terms of y and u (e.g. by using $\frac{d}{dt}\frac{u}{y}$), we can formulate a priori tests. Hence if, depending on the plant and controller dynamics, either of the following two tests is valid, equalities (18e) to (18g) may be relaxed into inequalities as follows:

- Counterclockwise path in (y, u): $\quad y\dot{u} - u\dot{y} \geq 0$
  $\Rightarrow$ the equality signs in (18e) to (18g) can be replaced by 'greater or equal' signs;
- Clockwise path in (y, u): $\quad y\dot{u} - u\dot{y} \leq 0$
  $\Rightarrow$ the equality signs in (18e) to (18g) can be replaced by 'less or equal' signs;

These tests can be hard to use in practise however, so further research is necessary to formulate computable tests. Moreover, it still needs to be seen whether or not this relaxation is advantageous in specific situations.

B. Non-decreasing Lyapunov function

Since Theorem 8 is only a sufficient criterion, it can still be infeasible for systems where the actual closed loop is stable. Examples may include unstable linear systems stabilized through resets or unstable resets compensated by stable linear behavior. The Lyapunov functions as a function of time for such systems may evolve as depicted in Figure 6. Clearly, although the closed loop is stable, in either situation Theorem 8 is too restrictive to prove stability, since it requires a decreasing $V(x)$ over all time.

A possible strategy to prove stability for these cases, inspired by [23], is to find a Lyapunov function which decreases between flow-reset-cycles, marked by the dots in Figure 6. Consider the example of Figure 6(b) and for simplicity assume no input or output:

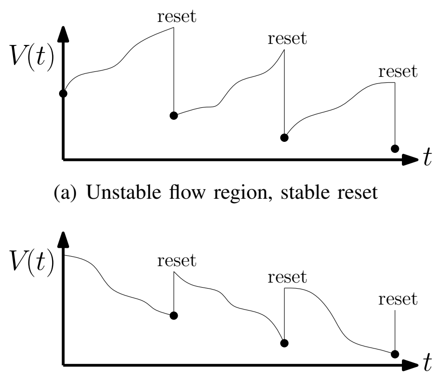
$$\Sigma : \left\{ \begin{array}{rcll} \dot{x} & = & \mathcal{A}x & \text{if } x \in \mathfrak{C} \\ x^+ & = & A_R x & \text{if } x \in \mathfrak{D} \end{array} \right. \tag{22}$$

where

$$\mathfrak{C} \; : \; \{x \in \mathbb{R}^n : \pm E_f T_x x \geq 0\} \tag{23a}$$
$$\mathfrak{D} \; : \; \{x \in \mathbb{R}^n : \pm E_R T_x x \geq 0\} \tag{23b}$$

Choose x_i to be the state right before a reset at time t_i, x_i^+ right after it, and x_{i+1} right before the next reset at t_{i+1}. Moreover, assume the flow region to be stable, hence

$$\dot{V}(x) + \alpha V(x) \leq 0, \qquad \text{with} \quad \alpha \geq 0, \tag{24}$$



(a) Unstable flow region, stable reset

(b) Stable flow region, unstable reset

Fig. 6. Possible Lyapunov functions $V(x)$ as a function of time t

If $V(x)$ may not increase between reset times, this yields

$$V(x_{i+1}) \leq \mathrm{e}^{-\alpha(t_{i+1}-t_i)} V(x_i^+) \leq V(x_i) \tag{25}$$

As an example, suppose we again use quadratic Lyapunov functions $V(x) = x^T P x$. Then (25) can be written as

$$x_i^T A_R^T \mathrm{e}^{\mathcal{A}^T(t_{i+1}-t_i)} P \mathrm{e}^{\mathcal{A}(t_{i+1}-t_i)} A_R x_i \leq x_i^T P x_i, \tag{26}$$

with must be valid for all $x_i \in \mathfrak{D}$. This then yields

$$\begin{array}{r} A_R^T \mathrm{e}^{\mathcal{A}^T(t_{i+1}-t_i)} P \mathrm{e}^{\mathcal{A}(t_{i+1}-t_i)} A_R - P \\ + E_{x,R}^T U_R E_{x,R} \;\; \preceq \;\; 0 \end{array} \tag{27}$$

Although this result might seem useful, it requires explicit knowledge of the reset times t_i and t_{i+1}, which are in practise hard to find and may be state dependent. Further research in this direction is therefore necessary, possibly in combination with non-quadratic $V(x)$.

Note that this approach does not include $\mathcal{L}_2$ gains yet. The needed introduction of w and z further complicates the analysis, while we believe that the stability proof by itself is already challenging enough.

V. Conclusions

In this paper we have derived a set of LMIs with which the $\mathcal{L}_2$ gain of any reset control system which fits into the $\mathcal{H}_\infty$ framework can be calculated, and is therefore a generalization of the work in [20]. Our analysis can also be applied to tracking and measurement noise problems; possible conservatism in these cases can be removed by including strictly proper input filters. Finally we have suggested possible future research directions to further improve stability and $\mathcal{L}_2$ analysis techniques for reset control systems.

VI. Discussion

As mentioned before, previous publications have shown in both simulations [9], [16] and experiments [7], [14], [17] that reset control can perform 'better' than linear control in some situations. This brings on the need for a performance measure to quantify this 'betterness'. Therefore the $\mathcal{H}_\infty$ analysis for reset control systems published in this paper and previously introduced in [19], [20] can be very useful.

Using $\mathcal{H}_\infty$ techniques it is now theoretically possible to define the best linear controller for an LTI system, e.g. by using the synthesis method in [24], and then search for a reset controller which outperforms it. The search for this reset controller might however be impossible.

First of all, it seems impossible to formulate $\mathcal{H}_\infty$ synthesis LMIs with the currently available knowledge, hence the $\mathcal{H}_\infty$ optimal reset controller is hard to find. This problem is to a large extend caused by the $E_{\cdot,\cdot}^T U . E_{\cdot,\cdot}$ terms in (18) and (21), which introduce nonlinear combinations of design variables. Unfortunately this nonlinearity cannot be eliminated using the results in [25] or linearized by the change of variables described in [24], so the synthesis does not yield an LMI.

Secondly, we question whether reset control can ever outperform the optimal $\mathcal{L}_2$-minimizing linear controller. As is shown in [26] and [27], for LTI plants there exists no nonlinear (possibly time-varying) controller which yields a lower

$\mathcal{L}_2$ gain than the optimal linear controller. Unfortunately, we expect reset control to be no exception to this. We therefore point out that Examples 2 and 3 in [19] do not provide a fair comparison between linear and reset controllers; in both examples it is easy to find linear controllers with much lower $\mathcal{H}_\infty$ norms.

We therefore suggest to use other norms than $\mathcal{H}_\infty$ to quantify reset control performance. Since the proof of Theorem 8 shows that (18a) is the result of using a specific quadratic supply function, we believe that it is possible to derive analysis LMIs for various other induced norms too, as long as they can be described using other specific quadratic supply functions. Moreover, the advantage of reset control was especially shown in the transient behavior of the step response, hence we believe the $\mathcal{H}_2$ analysis in particular to be very useful. Since one of the interpretations of $\mathcal{H}_2$ is the total output energy to specific initial values, we foresee reset opportunities in this area. Furthermore, it is recommended to examine whether reset control is advantageous in mixed problems, like in mixed $\mathcal{H}_2/\mathcal{H}_\infty$ analysis.

Finally, note that nonlinear controllers might be better than linear controllers in $\mathcal{H}_\infty$ sense, when we consider structured uncertainties, i.e. in the robust stabilization problem. Future research might indeed show that in these situations there are also possibilities for reset controllers.

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Chapter 3

$\mathcal{H}_2$ analysis of reset control systems

As already mentioned in Chapter 1, reset control is one of the possible non-linear control methods which might overcome some or all limitations which linear controllers are subject to. Indeed, both simulations [3, 10] and experiments [9, 12, 27] have shown the potential of reset control, especially in the transient behavior of the step response. Although there are some connections between $\mathcal{H}_\infty$ and step responses, the $\mathcal{H}_\infty$ performance analysis of Chapter 2 is mostly a steady state measure however. Hence, this brings on the need for a different performance measure, which has a clear transient time domain interpretation. A possible answer is the $\mathcal{H}_2$ norm, for which the paper in this chapter provides an analysis tool.

Analogous to Chapter 2, this analysis is derived using dissipativity theory and piecewise quadratic Lyapunov functions. The result is again a computable set of LMIs, which returns an upperbound on the actual $\mathcal{H}_2$ norm for reset control systems. The usefulness of this LMI-based $\mathcal{H}_2$ norm calculation is shown in a simple though convincing example, where it is shown that reset control can outperform the 'optimal' linear controller for specific input-constrained $\mathcal{H}_2$ problems.

$\mathcal{H}_2$ performance analysis of reset control systems

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***Abstract*— To overcome fundamental limitations of linear controllers, reset controllers were proposed in literature. Since the closed loop system including such a reset controller is of a hybrid nature, it is difficult to determine its performance. The focus in this paper is the performance determination of a reset control system in $\mathcal{H}_2$ sense. The method is generally applicable in the sense that it is valid for any proper LTI plant and linear-based reset controller. We derive convex optimization problems in terms of LMIs to compute an upperbound on the total output energy, using dissipativity theory with piecewise quadratic Lyapunov functions. Finally, by means of a simple multiobjective tracking example, we show that reset control can outperform a linear controller obtained via a standard multiobjective control design method.**

***Index Terms*— Reset control, $\mathcal{H}_2$, stability, hybrid systems, linear matrix inequality, step response.**

I. Introduction

In order to overcome the fundamental performance limitations that linear controllers are known to be subject to [1]–[3], various nonlinear feedback controllers for linear time-invariant (LTI) plants were proposed in literature. An example of such a nonlinear feedback is the reset controller, which is basically a linear controller whose states (or subset of states) are reset to zero whenever its input and output satisfy certain conditions.

The concept of reset control was first introduced in 1958 by means of the resetting integrator of Clegg [4]. The describing function of the Clegg integrator has the same magnitude plot as a linear integrator, but its performance limiting phase lag is only 38,1° instead of the normal 90°. However, because of its nonlinear hybrid behavior, the use and effect of this resetting integrator is not straightforward, so it was not until 1974 that it was first used in a control design procedure [5]. Subsequently, in [6] a *first order reset element (FORE)* was introduced, which was used in a controller design procedure based on frequency domain techniques. An overview of these results is given in [7].

At the end of the '90s there has been renewed interest in reset control systems, resulting in various stability analysis techniques. The first results were reported in [8] and [9], stating stability criterions for zero-input closed loops with a second order plant and a Clegg integrator or a FORE, respectively. However, these criterions involve explicit computation of reset times and closed loop solutions, and are hard to generalize for higher order systems.

In following publications stability conditions were formulated using Lyapunov based conditions. This was first done in [10] and [11], in which only second order closed loops with constant inputs were considered. These results have been extended in [12] and [13] to a sufficient criterion for BIBO (bounded input bounded output) stability, and later to the so-called H_β-condition [14]. The latter paper also addressed the tracking problem, based on the internal model principe. The possible advantages of reset controllers over linear ones have been shown both in simulations [9], [15] and experiments [7], [13], [16]. A clear overview of this work is provided in [17], summarizing the H_β stability analysis for general reset systems.

The H_β-condition is a computable stability check for zero-input reset control systems, based on a reformulation of Lyapunov based stability linear matrix inequalities (LMIs) using the well known Kalman-Yakubovich-Popov Lemma. The H_β-condition is however conservative, since it considers only *quadratic* stability by using common quadratic Lyapunov functions, and it requires stability of the linear closed loop dynamics.

In more recent publications [18], [19] the authors were able to remove part of this conservatism by introducing two important adjustments. First, a slightly different resetting condition was suggested, i.e. resetting when the controller input and output have opposite sign instead of when the input is zero. Second, *piecewise quadratic* Lyapunov functions were used, to capture a broader class of stability problems than merely quadratic stability. On top of this the analysis has been extended to $\mathcal{L}_2$ stability, such that the closed loop $\mathcal{H}_\infty$-gain from input to output of a reset control system can be approximated by an upperbound. However, the results in [18] and [19] are not universally applicable, since it considers only Clegg integrators and FOREs.

Still, the proposed calculation of the $\mathcal{L}_2$ gain is very useful, since it expresses the performance of certain reset control systems (in this case FOREs) in a quantitative measure. Hence, it provides a measure to objectively compare the performance of reset controllers to that of linear ones. Although there are known to be some connections between $\mathcal{L}_2$ and closed loop step responses, the $\mathcal{L}_2$ gain is typically a steady state measure, whereas the advantage of reset control over linear control is especially apparent during the transient behavior [15]. In particular, it has been shown that reset controllers are able to reduce the overshoot of step responses, thereby decreasing the total energy of the error signal. This observation shows similarities with one of the interpretations of the $\mathcal{H}_2$ norm, which can be seen as the total output energy (of in this case the tracking error) of a closed loop system to either an impulse input or non-zero initial values. Because of this transient interpretation, we believe that the $\mathcal{H}_2$ norm might be a very helpful measure, even more than the $\mathcal{L}_2$ gain, to objectively show that reset controllers can outperform linear ones in specific situations.

For this reason this paper derives an LMI-based analysis method to calculate upperbounds on the $\mathcal{H}_2$ norm of a closed loop reset control system. The results can be used to approximate the energy content of the output resulting from specific input signals. We will use the same reset condition as in [19] and also adopt piecewise quadratic Lyapunov functions to reduce conservatism of the analysis. However, in contrast to [19] our results are not only useful for FOREs, but for any reset controller with linear flow dynamics. Moreover, they use $\mathcal{H}_2$ performance instead of the $\mathcal{L}_2$ gain. Finally, we provide a simple though convincing example to illustrate the accuracy of our proposed $\mathcal{H}_2$ norm calculation and show that, for an input-constrained $\mathcal{H}_2$ problem, reset control can indeed outperform a linear controller designed by a common multiobjective design method.

This paper is organized as follows. Section II provides some background on the $\mathcal{H}_2$ norm for linear systems. In Section III we introduce the closed loop layout under consideration, and give mathematical descriptions of the dynamics of the plant and the reset controller. Our main results on the $\mathcal{H}_2$ norm for reset control systems are derived in Section IV. Finally, the advantage of reset control in $\mathcal{H}_2$ sense is shown by an example in Section V.

Notation. The set of real numbers is denoted by $\mathbb{R}$, the set of positive real numbers is denoted by $\mathbb{R}_+$. The set of real symmetric matrices with non-negative elements is denoted by $\mathbb{S}_+$. The identity matrix of dimension $n\times n$ is denoted by $I_n\in\mathbb{R}^{n\times n}$. Given two vectors x_1, x_2 we write (x_1, x_2) to denote $[x_1^T, x_2^T]^T$. A vector $x \in \mathbb{R}^n$ is nonnegative, denoted by $x \geq 0$, if its elements $x_i \geq 0$ for $i = 1, \ldots, n$. A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is positive definite, denoted by $A \succ 0$ if $x^T Ax > 0$ for all $x \in \mathbb{R}^n\backslash\{0\}$. A sequence of scalars $(u^1, u^2, \ldots, u^k)$ is called lexicographically nonnegative, written as $(u^1, u^2, \ldots, u^k) \geq_\ell 0$, if $(u^1, u^2, \ldots, u^k) = (0, 0, \ldots, 0)$ or $u^j > 0$ where $j = \min\{p = 1, \ldots, k : u^p \neq 0\}$. For a sequence of vectors $(x^1, x^2, \ldots, x^k)$ with $x^j \in \mathbb{R}^n$, we write $(x^1, x^2, \ldots, x^k) \geq_\ell 0$ when $(x_i^1, x_i^2, \ldots, x_i^k) \geq_\ell 0$ for all $i = 1, \ldots, n$.

II. Linear $\mathcal{H}_2$ theory

Our main results on the $\mathcal{H}_2$ analysis of reset control systems uses some common $\mathcal{H}_2$ results for linear single input single output (SISO) systems

$$\Sigma : \begin{cases} \dot{x} &= \mathcal{A}x + \mathcal{B}w \\ z &= \mathcal{C}x. \end{cases} \tag{1}$$

where $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B} \in \mathbb{R}^{n\times 1}$, and $\mathcal{C} \in \mathbb{R}^{1\times n}$ are the system matrices, $x \in \mathbb{R}^n$ is the state, and $w \in \mathbb{R}$ and $z \in \mathbb{R}$ denote the input and output, respectively. We shortly summarize some of these results here (see [20], [21] for more details).

A. $\mathcal{H}_2$ norm for linear systems

It is well-known that one of the possible interpretations of the $\mathcal{H}_2$ norm is the total energy content of the output z due to an impulsive input w. The response of system (1) to such an impulsive input is equivalent to the response obtained when the system is subjected to the initial condition $x_0 = \mathcal{B}$. In this paper we focus on the latter and hence assume w.l.o.g. that $w = 0$.

Definition 1 *Consider the linear system (1), with initial state $x_0 \in \mathbb{R}^n$ and no input, so $w = 0$. The total output energy in z is then defined by:*

$$\int_0^\infty z^T z dt = \int_0^\infty x_0^T e^{\mathcal{A}^T t}\mathcal{C}^T\mathcal{C}e^{\mathcal{A}t}x_0 dt. \tag{2}$$

The square root of this integral is called the $\mathcal{H}_2$ norm for x_0 and is denoted by $||\Sigma||_{2,x_0}$.

To calculate (2) we introduce the observability gramian

$$M := \int_0^\infty e^{\mathcal{A}^T t}\mathcal{C}^T\mathcal{C}e^{\mathcal{A}t}dt, \tag{3}$$

so that the $\mathcal{H}_2$ norm is equal to

$$||\Sigma||_{2,x_0}^2 = \int_0^\infty z^T z dt = x_0^T M x_0. \tag{4}$$

It is well known that for Hurwitz matrices $\mathcal{A}$ the observability gramian M is the solution to the Lyapunov equality

$$\mathcal{A}^T M + M\mathcal{A} + \mathcal{C}^T\mathcal{C} = 0. \tag{5}$$

B. An LMI-approach to $\mathcal{H}_2$

The $\mathcal{H}_2$ norm of a linear system can also be formulated using the concept of dissipativity [20]:

Definition 2 *The linear system (1) with state $x \in \mathbb{R}^n$, input $w \in \mathbb{R}$ and output $z \in \mathbb{R}$ is stable and strictly dissipative w.r.t. a supply function $s : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ if there exists a storage function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that*

$$V(x(t_1)) - V(x(t_0)) < \int_{t_0}^{t_1} s(w(t), z(t))dt \tag{6}$$

for all $t_1 \geq t_0$ and $V(0) = 0$.

For the initial state $\mathcal{H}_2$ problem for (1) there is no input and we are only interested in the energy content of the output, hence we select $s(w, z) = -z^T z$. Using quadratic Lyapunov functions $V(x) = x^T Px$ with P positive definite, the differential form of (6) then yields

$$\begin{aligned} \tfrac{dV}{dx}\dot{x} &< s(w, z) & \forall x \neq 0 \\ \dot{x}^T Px + x^T P\dot{x} &< -z^T z & \forall x \neq 0 \\ x^T(\mathcal{A}^T P + P\mathcal{A})x + x^T\mathcal{C}^T\mathcal{C}x &< 0 & \forall x \neq 0 \\ \mathcal{A}^T P + P\mathcal{A} + \mathcal{C}^T\mathcal{C} &\prec 0, \end{aligned} \tag{7}$$

which is an LMI in the design variable $P \succ 0$. The actual $\mathcal{H}_2$ norm again follows from 6 and can be upperbounded by using $t_0 = 0$ and letting $t_1 \to \infty$, $V(x(t_1)) = 0$ and $s(w, z) = -z^T z$. We then obtain:

$$||\Sigma||_{2,x_0}^2 = \int_0^\infty z^T z dt < V(x(t_0)) = x_0^T Px_0. \tag{8}$$

We now approximate the $\mathcal{H}_2$ norm by infimizing γ^2 subject to the LMI constraints

$$\mathcal{A}^T P + P\mathcal{A} + \mathcal{C}^T\mathcal{C} \prec 0 \tag{9a}$$

$$x_0^T Px_0 < \gamma^2 \tag{9b}$$

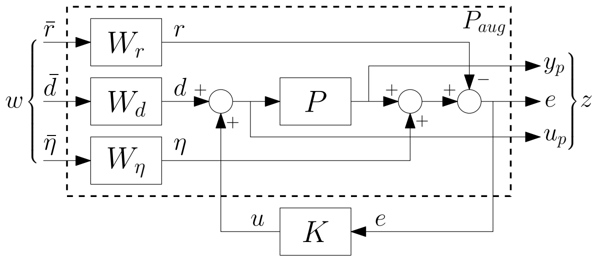

Fig. 1. Closed loop layout with input w and output z

which should be solved for $P \succ 0$ and γ^2. The infimum of this optimization problem retrieves $||\Sigma||^2_{2,x_0}$. These LMIs are, of course, closely related to (4) and (5).

III. System description

In this section we introduce the closed loop layout and reset control system for which the $\mathcal{H}_2$ norm will be calculated.

A. Closed loop layout

In recent papers (see e.g. [15]), reset control is shown to be advantageous when the transient response to specific input signals (like step functions) is considered. Therefore, we focus on the general reset control system layout depicted in Figure 1, consisting of various input filters, a linear plant P and a reset controller K. Our goal is to calculate the total output energy of the unfiltered output z, consisting of the signals e (tracking error), u_p (control) and y_p (plant output) or a subset of these signals, subject to certain specific exogenous inputs w, consisting of the signals r (reference), d (disturbance) and η (measurement noise) or a subset of these inputs. For ease of exposition, all these signals are assumed to take values in $\mathbb{R}$, i.e. we only consider SISO plants and controllers. The input w is assumed to be known a priori, and this knowledge is captured in input filters W_r, W_d and W_η. Possible filters include:

- unit step: $W(s) = \frac{1}{(s+\varepsilon)}$;
- unit ramp: $W(s) = \frac{1}{(s+\varepsilon)^2}$;
- sine wave with frequency ω: $W(s) = \frac{\omega}{(s+\varepsilon)^2+\omega^2}$;

where s is the Laplace variable and $\varepsilon > 0$ is a small offset to ensure stability of the filter. This offset is needed to guarantee closed loop stability, as is standard in $\mathcal{H}_2$ and $\mathcal{H}_\infty$ problems [22]. Note that when an impulse input is applied to these filters, the filter outputs indeed approximate a step, ramp or sine wave respectively, when ε tends to zero. Hence, the total energy in z as a result of these specific input signals gets arbitrarily close to the total output energy of the impulse response from w to z, which is one of the interpretations of the $\mathcal{H}_2$ norm. As discussed in Section II, to compute the $\mathcal{H}_2$ norm we can equivalently use the initial condition setting and assume $w = 0$. For reasons of generality, however, we will elaborate on the plant and controller dynamics without this assumption.

B. Plant dynamics

The plant P and all input filters are LTI systems. Together they form the augmented plant P_{aug}, depicted by the dashed box in Figure 1. Since this augmented plant is also LTI, we can describe the dynamics by:

$$\begin{array}{rcl} \dot{x}_p &=& Ax_p + B_w w + Bu \\ z &=& C_z x_p + D_{zw} w + D_z u \\ e &=& Cx_p + D_w w, \end{array} \tag{10}$$

where $A, B_w, B, C_z, D_{zw}, D_z, C, D_w$ are matrices of appropriate dimension, $x_p \in \mathbb{R}^{n_p}$ is the augmented plant state, and $w \in \mathbb{R}^{n_w}$ and $z \in \mathbb{R}^{n_z}$ are the exogenous input and output, respectively. The tracking or stabilization error is available for feedback and we assume that there is no direct feedthrough from u to e, as is e.g. the case for general motion systems.

C. Reset controller

The reset controller K is modeled as a linear controller which resets whenever its input e and output u satisfy a certain condition. This controller is thus described by

$$\begin{array}{rcll} \dot{x}_k &=& A_K x_k + B_K e & \text{if } (e,u) \in \mathfrak{C}' \\ x_k^+ &=& A_r x_k & \text{if } (e,u) \in \mathfrak{D}' \\ u &=& C_K x_k + D_K e, & \end{array} \tag{11}$$

where $x_k \in \mathbb{R}^{n_k}$ is the controller state. The closed loop state then becomes $x = (x_p, x_k)$ where $x \in \mathbb{R}^n$. The reset controller can be considered as a hybrid system with a *flow set* $\mathfrak{C}'$ and a *reset set* $\mathfrak{D}'$ [18]. Indeed, as long as $(e,u) \in \mathfrak{C}'$ the controller behaves linearly and its output flows conform (A_K, B_K, C_K, D_K). When $(e,u) \in \mathfrak{D}'$ the state is changed instantaneously from x_k to x_k^+ by the discrete map $A_r \in \mathbb{R}^{n_k \times n_k}$.

For analysis purposes the sets $\mathfrak{C}'$ and $\mathfrak{D}'$ should be closed and such that $\mathfrak{C}' \cup \mathfrak{D}' = \mathbb{R}^2$ [18]. The sets are defined by a resetting condition, for which many choices are possible, but here we will follow [18] and [19] where resetting occurs whenever input and output have opposite sign, i.e. $eu \leq 0$. Hence, the controller flows whenever $e \geq 0, u \geq 0$ or $e \leq 0, u \leq 0$, which yields, using the notation from [23],

$$\mathfrak{C}' := \left\{ \begin{bmatrix} e \\ u \end{bmatrix} \in \mathbb{R}^2 : E_f \begin{bmatrix} e \\ u \end{bmatrix} \geq 0 \;\text{ or }\; E_f \begin{bmatrix} e \\ u \end{bmatrix} \leq 0 \right\} \tag{12a}$$

$$\mathfrak{D}' := \left\{ \begin{bmatrix} e \\ u \end{bmatrix} \in \mathbb{R}^2 : E_R \begin{bmatrix} e \\ u \end{bmatrix} \geq 0 \;\text{ or }\; E_R \begin{bmatrix} e \\ u \end{bmatrix} \leq 0 \right\} \tag{12b}$$

where

$$E_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad E_R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The flow set (12a) and reset set (12b) can also be expressed in terms of x and w. Therefore we introduce a transformation matrix $T = \left[\begin{array}{c|c} T_x & T_w \end{array}\right]$:

$$\begin{bmatrix} e \\ u \end{bmatrix} = T \begin{bmatrix} x \\ w \end{bmatrix} = \left[\begin{array}{cc|c} C & 0 & D_w \\ D_K C & C_K & D_K D_w \end{array}\right] \begin{bmatrix} x_p \\ x_k \\ w \end{bmatrix},$$

such that

$$\tilde{\mathfrak{C}} := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathbb{R}^{n+n_w} : \; E_f T \begin{bmatrix} x \\ w \end{bmatrix} \geq 0 \;\; \text{or} \;\; E_f T \begin{bmatrix} x \\ w \end{bmatrix} \leq 0 \right\} \tag{13a}$$

$$\tilde{\mathfrak{D}} := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} \in \mathbb{R}^{n+n_w} : \ E_R T \begin{bmatrix} x \\ w \end{bmatrix} \geq 0 \right. \\ \left. \text{or} \quad E_R T \begin{bmatrix} x \\ w \end{bmatrix} \leq 0 \right\}. \quad (13b)$$

The flow and reset regions $\tilde{\mathfrak{C}}$ and $\tilde{\mathfrak{D}}$ depend in general on both the closed loop state x and the exogenous input w, due to the presence of D_w. However, it is known that the $\mathcal{H}_2$ norm of a linear system is bounded, only if the corresponding closed loop transfer matrix between w and z is strictly proper [21], [22]. As in our closed loop system the transfer between e.g. r and e is defined by the sensitivity function, which is biproper, we need a strictly proper W_r to obtain a strictly proper transfer matrix from $\bar{r}$ to e. A similar reasoning can be applied to the other input signals, which implies that the following assumption is needed to ensure a bounded $\mathcal{H}_2$ norm

Assumption 3 *The input filters W_r, W_d and W_η are all strictly proper.*

Note that strict properness of the input filters is quite natural in practice, as is also indicated by the three examples in Section III-A. The strict properness of the input filters implies that there is no direct feedthrough between the input w and the controller input signal e (i.e. $D_w = 0$) or between w and z (i.e. $D_{zw} = 0$). This simplifies the definitions of $\tilde{\mathfrak{C}}$ and $\tilde{\mathfrak{D}}$ to:

$$\mathfrak{C} := \{x \in \mathbb{R}^n : E_f T_x x \geq 0 \quad \text{or} \quad E_f T_x x \leq 0\} \quad (14a)$$
$$\mathfrak{D} := \{x \in \mathbb{R}^n : E_R T_x x \geq 0 \quad \text{or} \quad E_R T_x x \leq 0\} \quad (14b)$$

which now only depend on x. These flow and reset maps are used in the remainder of this paper.

The reset action itself is defined by the reset matrix A_r, indicating to which values the controller states are reset to. Various choices of A_r are possible, depending among others on the plant and controller dynamics. A common and appropriate choice in most cases, is to reset (a subset of) the states to zero, i.e.

$$A_r = \begin{bmatrix} I_{n_k - n_r} & 0 \\ 0 & 0_{n_r} \end{bmatrix},$$

stating that that only the last n_r of the n_k controller states are reset to zero, while the other states remain unchanged.

D. Closed loop dynamics

We can now combine the augmented plant and the reset controller into one closed loop system, described by Σ:

$$\Sigma : \begin{cases} \dot{x} &= \mathcal{A}x + \mathcal{B}w & \text{if } x \in \mathfrak{C} \\ x^+ &= A_R x & \text{if } x \in \mathfrak{D} \\ z &= \mathcal{C}x + \mathcal{D}w, \end{cases} \quad (15)$$

where, using $D_w = 0$ and $D_{zw} = 0$,

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] = \left[\begin{array}{cc|c} A + BD_K C & BC_K & B_w \\ B_K C & A_K & 0 \\ \hline C_z + D_z D_K C & D_z C_K & 0 \end{array}\right]$$
$$A_R = \begin{bmatrix} I_{n_p} & 0 \\ 0 & A_r \end{bmatrix}.$$

The linear closed loop system which results when resetting is omitted (i.e. when $A_R = I_n$, $\mathfrak{C} = \mathbb{R}^n$ and $\mathfrak{D} = \emptyset$), is called the *base linear system*. At this point we return to the assumption that $w = 0$ and consider non-zero initial values of the input filters. Hence, we assume $w = 0$ and $x_0 = \mathcal{B}_j$, where $\mathcal{B}_j$ denotes the j-th column of $\mathcal{B}$, corresponding to the j-th input (filter). Note that $x_0 = \mathcal{B}_i + \mathcal{B}_j$ where $i \neq j$ is also a valid initial condition.

To exclude multiple resets at one point in time, hence to guarantee that after each reset the system can flow on a non-trivial time interval (local existence of solutions), we adopt the following assumption:

Assumption 4 *The reset controller and the regions $\mathfrak{C}$ and $\mathfrak{D}$ are such that*

$$x \in \mathfrak{D} \quad \Rightarrow \quad x^+ \in \mathcal{F}_{\mathfrak{C}}. \quad (16)$$

where $\mathcal{F}_{\mathfrak{C}}$ is given by

$$\mathcal{F}_{\mathfrak{C}} := \{x_0 \in \mathfrak{C} : \exists \epsilon > 0 \quad \forall \tau \in [0, \epsilon) \quad x_{x_0}(\tau) \in \mathfrak{C}\} \quad (17)$$

defining all initial conditions x_0 (or in this case x^+) for which the closed loop solution $x_{x_0}(\tau)$ remains inside $\mathfrak{C}$ at least on the interval $\tau \in [0, \epsilon)$.

This assumption implies that the state after a reset x^+ should either lie in the interior of $\mathfrak{C}$, or the flow dynamics should not drive x outside $\mathfrak{C}$ when it is on the boundary of $\mathfrak{C}$. Hence, we can formulate a check for Assumption 4, based on a lexicographic ordering.

Corollary 5 *The closed loop system (15) can flow after a reset $x^+ = A_R x$ if at least one of the two following lexicographic orderings holds whenever $x \in \mathfrak{D}$*

$$\big(E_f T_x A_R x, E_f T_x \mathcal{A} A_R x, E_f T_x \mathcal{A}^2 A_R x, \\ \ldots, E_f T_x \mathcal{A}^{n-1} A_R x\big) \geq_\ell 0 \quad (18a)$$
$$\big(E_f T_x A_R x, E_f T_x \mathcal{A} A_R x, E_f T_x \mathcal{A}^2 A_R x, \\ \ldots, E_f T_x \mathcal{A}^{n-1} A_R x\big) \leq_\ell 0 \quad (18b)$$

Remark 6 As $\mathcal{F}_{\mathfrak{C}} \subseteq \mathfrak{C}$, we at least need

$$x \in \mathfrak{D} \quad \Rightarrow \quad x^+ \in \mathfrak{C}, \quad (19)$$

to satisfy (16), which corresponds to the first vector in the lexicographic ordering of (18). Hence, this relation as used in [18] is only a necessary condition that can be obtained from Corollary 5. One way to satisfy this condition is to assume that $C_K A_r = 0$ for all $x \in \mathfrak{D}$. □

The goal of this paper is to compute the $\mathcal{H}_2$ norm for reset control systems, which is defined as follows:

Definition 7 *The $\mathcal{H}_2$ norm of the reset control system (15), or equivalently the total energy in its output $z \in \mathbb{R}^{n_z}$ due to a non-zero initial value $x_0 \in \mathbb{R}^n$, is defined as*

$$||\Sigma||^2_{2,x_0} = \int_0^\infty z^T z dt \quad (20)$$

It turns out that the concept of dissipativity and the LMI formulation can be extended to calculate the $\mathcal{H}_2$ norm for reset control systems as will be done in the next section.

IV. MAIN RESULTS

We now present our main results on the LMI-based calculation of an upperbound on the $\mathcal{H}_2$ norm of reset control systems. For this initial state $\mathcal{H}_2$ analysis we extend the approach of Section II-B, i.e. we derive analysis LMIs based on dissipativity theory using a common quadratic Lyapunov function, after which we relax the Lyapunov conditions by introducing piecewise quadratic Lyapunov functions as in [24].

A. Common Lyapunov function

Although a reset control system behaves in a hybrid manner, the mathematical description of its dynamics (15) shows that both the flow and the reset part can be described in a linear fashion. This motivates our choice to use a common quadratic Lyapunov function $V(x) = x^T Px$.

Theorem 8 *Consider the reset control system (15) with $\mathfrak{C}$ and $\mathfrak{D}$ as defined in (14). The following statements are equivalent*

i) system (15) is asymptotically stable and $||\Sigma||_{2,x_0} < \gamma$
ii) there exists $P \succ 0$ and $U_f, U_R \in \mathbb{S}_+^{2\times 2}$ such that

$$\mathcal{A}^T P + P\mathcal{A} + \mathcal{C}^T\mathcal{C} + T_x^T E_f^T U_f E_f T_x \prec 0 \quad (21a)$$
$$A_R^T P A_R - P + T_x^T E_R^T U_R E_R T_x \preceq 0 \quad (21b)$$
$$\gamma^2 - x_0^T P x_0 > 0 \quad (21c)$$

Proof: The proof is based on showing that the hypothesis of the theorem imply, for $V(x) = x^T Px$, that

$$\frac{d}{dt}V(x) < s(w,z) \quad \text{when } x \in \mathfrak{C}\backslash\{0\} \quad (22a)$$
$$V(x^+) \leq V(x) \quad \text{when } x \in \mathfrak{D} \quad (22b)$$

Indeed, if (22) holds then

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} s(w(t), z(t))dt \quad (23)$$

which means that system (15) is dissipative w.r.t. the supply function $s(w,z)$. Furthermore, by letting $t_1 \to \infty$ and using $V(x(t_1)) = 0$, $s(w,z) = -z^T z$, this yields that

$$||\Sigma||_{2,x_0}^2 = \int_0^\infty z^T z dt = -\int_0^\infty s(w,z)dt \leq V(x(t_0)) < \gamma^2 \quad (24)$$

To show (22a) and (22b), note that

$$x \in \mathfrak{C} \quad \Rightarrow \quad x^T T_x^T E_f^T U_f E_f T_x x \geq 0 \quad (25a)$$
$$x \in \mathfrak{D} \quad \Rightarrow \quad x^T T_x^T E_R^T U_R E_R T_x x \geq 0. \quad (25b)$$

Hence, combining (25a) with (21a) yields that

$$x^T(\mathcal{A}^T P + P\mathcal{A} + \mathcal{C}^T\mathcal{C})x < 0 \quad \text{if } x \in \mathfrak{C}\backslash\{0\} \quad (26)$$

and combining (25a) with (21b) gives

$$x^T(A_R^T P A_R - P)x \leq 0 \quad \text{if } x \in \mathfrak{D}, \quad (27)$$

which are just reformulations of (22a) and (22b). Hence, the proof is complete. ∎

B. Piecewise quadratic Lyapunov functions

Although Theorem 8 provides an easy way to determine an upperbound on the $\mathcal{H}_2$ norm for a specific x_0, it can be conservative. Indeed, recalling the dissipativity inequalities

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} s(w(t), z(t))dt$$
$$\gamma^2 > V(x(t_0)),$$

it is easy to see that by considering quadratic Lyapunov functions we have restricted the storage function $V(x)$ corresponding to the computed $\mathcal{H}_2$ norm γ to be quadratic as well. However, since a reset control system (15) behaves in a hybrid manner, we expect the storage function corresponding to the actual $\mathcal{H}_2$ norm of the reset system, $V(x_0)_{||\Sigma||_{2,x_0}} = ||\Sigma||_{2,x_0}^2$, to be a complex function, possibly of high order.

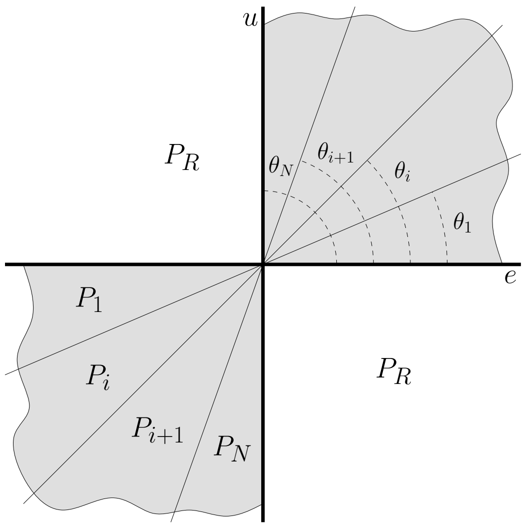

Fig. 2. Partitioning of the (e,u)-space

To reduce this conservatism, we will approximate this complex storage function by using *piecewise quadratic Lyapunov functions* [24], which also results in a set of LMIs. These piecewise Lyapunov functions are obtained by partitioning the flow set $\mathfrak{C}'$ into smaller regions $\mathfrak{C}'_i$ and assigning a different quadratic Lyapunov function $V_i(x) = x^T P_i x$ to each of them, see Figure 2. The angles θ_i and θ_{i-1} uniquely define two lines

$$u\cos(\theta_i) = e\sin(\theta_i)$$
$$u\cos(\theta_{i-1}) = e\sin(\theta_{i-1})$$

which bound each region $\mathfrak{C}'_i$. These angles should be chosen such that $0 < \theta_0 < \theta_1 < \ldots < \theta_N = \frac{\pi}{2}$. Here we choose to distribute θ_i equidistantly, so $\theta_i = \frac{i}{N}\frac{\pi}{2}$, where $i = 0, \ldots, N$ and N is the number of desired subregions. Using the coordinate transformation matrix T_x, we can now define regions $\mathfrak{C}_i$ and $\mathfrak{D}$ as

$$\mathfrak{C}_i := \{x \in \mathbb{R}^n : E_i T_x x \geq 0 \quad \text{or} \quad E_i T_x x \leq 0\} \quad (28a)$$
$$\mathfrak{D} := \{x \in \mathbb{R}^n : E_R T_x x \geq 0 \quad \text{or} \quad E_R T_x x \leq 0\} \quad (28b)$$

where

$$E_i = \begin{bmatrix} -\sin(\theta_{i-1}) & \cos(\theta_{i-1}) \\ \sin(\theta_i) & -\cos(\theta_i) \end{bmatrix}, \quad E_R = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Furthermore, the boundaries of the regions are defined by

$$\begin{bmatrix} -\sin(\theta_i) & \cos(\theta_i) \end{bmatrix} T_x x = \Phi_i x = 0, \tag{29}$$

whose solutions are in the kernel of Φ_i. We can also use an image representation for these boundaries, yielding $\text{im}(W_{\Phi i}) = \ker(\Phi_i)$, where $W_{\Phi i} \in \mathbb{R}^{n\times(n-1)}$ is a matrix with full column rank, and $\text{im}(W_{\Phi i})$ denotes its image.

Using this partitioning we can now formulate our main result on the calculation of an $\mathcal{H}_2$ upperbound, as stated in the following theorem:

Theorem 9 *The following statements are equivalent*

i) system (15) is asymptotically stable and $||\Sigma||_{2,x_0} < \gamma$

ii) there exists $P_i, P_R \succ 0$ *and* $U_i, U_R, V_i, V_R \in \mathbb{S}_+^{2\times 2}$ *such that*

$$\begin{aligned}
&\mathcal{A}^T P_i + P_i \mathcal{A} + \mathcal{C}^T \mathcal{C} + T_x^T E_i^T U_i E_i T_x \prec 0, && i = 1, \dots, N && \text{(30a)}\\
&A_R^T P_R A_R - P_R + T_x^T E_R^T U_R E_R T_x \preceq 0 && && \text{(30b)}\\
&P_i - T_x^T E_i^T V_i E_i T_x \succ 0 && i = 1, \dots, N && \text{(30c)}\\
&P_R - T_x^T E_R^T V_R E_R T_x \succ 0 && && \text{(30d)}\\
&W_{\Phi i}^T (P_i - P_{i+1}) W_{\Phi i} = 0, && i = 1, \dots, N-1 && \text{(30e)}\\
&W_{\Phi 0}^T (P_R - P_1) W_{\Phi 0} = 0 && && \text{(30f)}\\
&W_{\Phi N}^T (P_N - P_R) W_{\Phi N} = 0 && && \text{(30g)}\\
&\gamma^2 - x_0^T P_j x_0 > 0, && j \in I(x_0) && \text{(30h)}
\end{aligned}$$

where $I(x_0) := \{i : x_0 \in \mathfrak{C}_i\}$ *denotes the indices of the regions that contain* x_0.

Proof: We will show that V, defined as $V(x) = x^T P_i x$ when $x \in \mathfrak{C}_i$ and $V(x) = x^T P_R x$ when $x \in \mathfrak{D}$, is Lipschitz continuous and satisfies

$$\begin{aligned}
\frac{\partial V_i}{\partial x} \mathcal{A} x &< -z^T z && \text{if } x \in \mathfrak{C}_i,\ x \neq 0. && \text{(31a)}\\
V(x^+) - V(x) &\leq 0 && \text{if } x \in \mathfrak{D} && \text{(31b)}\\
\gamma^2 &> V(x_0), && && \text{(31c)}
\end{aligned}$$

Using the results in [25] it can be proven that (31a) and (31b) guarantee that

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} s(w(t), z(t)) dt \tag{32}$$

which yields similarly as in the proof of Theorem 8 that $||\Sigma||_{2,x_0}^2 \leq \gamma^2$. To show that V is Lipschitz continuous we will use (30e), (30f), and (30g). As $\mathfrak{C}_i \cap \mathfrak{C}_{i+1} \in \text{im}(W_{\Phi_i})$ we have that any $x \in \mathfrak{C}_i \cap \mathfrak{C}_{i+1}$ can be written as $x = W_{\Phi_i} \bar{x}$ for some $\bar{x}$ and thus

$$x^T (P_{i+1} - P_i) x = \bar{x} W_{\Phi_i}^T (P_{i+1} - P_i) W_{\Phi_i} \bar{x} \overset{(30e)}{=} 0. \tag{33}$$

The same applies for the intersections $\mathfrak{C}_1 \cap \mathfrak{D}$ and $\mathfrak{C}_N \cap \mathfrak{D}$. As $\mathfrak{D} \cup \cup_i \mathfrak{C}_i = \mathbb{R}^n$ this proves the Lipschitz continuity of V.

Applying a similar reasoning to (30a) and (30b) as in the proof of Theorem 8, yields (31a) and (31b), respectively.

Finally, to show that V is a positive definite function, note that for $x \in \mathfrak{C}_i \backslash \{0\}$ it holds that

$$V(x) = x^T P_i x \overset{(30c)}{>} x^T T_x^T E_i V_i E_i T_x x \geq 0 \tag{34}$$

as $x \in \mathfrak{C}_i \Rightarrow x^T T_x^T E_i V_i E_i T_x x \geq 0$ due to the fact that V_i has non-negative elements. The same applies for $x \in \mathfrak{D}$ using (30d). This establishes (31).

Applying now analogous arguments as in the proof of Theorem 8, completes this proof. ■

Remark 10 The results of Theorem 9 are related to [24], in which an $\mathcal{H}_2$ analysis is provided for piecewise affine systems. Analogously to [24] we thus have to conclude that the found minimal value of γ is only an upperbound on the actual $\mathcal{H}_2$ norm. However, as is explained in [24], this upperbound can be lowered by increasing the number of subregions N, thereby increasing the tightness of the approximation of the actual $\mathcal{H}_2$ norm. Hence, Theorem 9 is always less conservative than Theorem 8. □

Remark 11 Note that continuity of V is a very natural requirement, since we do not know beforehand whether or not closed loop solutions will go from $\mathfrak{C}_i$ to $\mathfrak{C}_{i+1}$ or vice versa. As such we use continuity to prevent the Lyapunov function to increase along border crossings. However, in some special cases we might be able to determine the path of the solution in the (e, u)-plane beforehand. If this movement is purely clockwise or counterclockwise, the Lipschitz continuity of V can be dropped, allowing decreases of the Lyapunov function across border crossings. This removes conservatism in the stability analysis even further. □

Remark 12 In situations where $D_w \neq 0$ and $w \neq 0$, e.g. in certain $\mathcal{H}_\infty$ problems, the flow and reset regions could still depend on the input w. In that case each piecewise quadratic Lyapunov function $V_i(x) = x^T P_i x$ depends only on x, but should be valid in a subregion $\tilde{\mathfrak{C}}_i$ which depends on both x and w, i.e. $V(x) = x^T P_i x$ when $(x, w) \in \tilde{\mathfrak{C}}_i$. Since for any x there is a w_i such that $(x, w_i) \in \tilde{\mathfrak{C}}_i$, it must hold that $x^T P_i x = V(x) = x^T P_j x$ and thus $P_i = P_j$. For example, with $(x, w) \in \mathbb{R}^{n+n_w}$:

$$\tilde{\mathfrak{C}}_i := \left\{ \begin{bmatrix} x \\ w \end{bmatrix} : E_i T \begin{bmatrix} x \\ w \end{bmatrix} \geq 0 \quad \text{or} \quad E_i T \begin{bmatrix} x \\ w \end{bmatrix} \leq 0 \right\}. \tag{35}$$

Since for any x we can always find a w such that (35) is valid, subregion $\tilde{\mathfrak{C}}_i$ covers all x, and hence V_i should hold for all x. Since this holds for all i, the piecewise Lyapunov approach reduces to a common quadratic one, which is a more conservative analysis, as discussed earlier. □

V. Example

As we mentioned before, performance improvement by using reset control is especially apparent in the transient closed loop behavior, which motivates our choice to consider the $\mathcal{H}_2$ norm of reset control systems. Indeed, in this section we show, by means of a multiobjective $\mathcal{H}_2/\mathcal{H}_\infty$ problem, that reset control can perform better than the optimal linear controller for this problem. We wish to minimize the energy content of the error e, subject to a maximum allowed control effort u. Moreover, the example illustrates the accuracy of our proposed LMI method in Theorem 9.

Example 1 (Tracking performance of an integrator plant) In this example we consider a closed loop system with an integrator plant, $G(s) = \frac{1}{s}$ where s is the Laplace variable, which should track a unit step reference $r(t) = 1(t)$. Our goal is to minimize the energy in the error e for this specific reference, subject to a maximum allowed control signal u to the plant, as is usually the case in practical situations. Hence, our design problem is defined by the multiobjective problem

$$\min_K \quad \sqrt{\int_0^\infty e^2 \, dt} \tag{36a}$$

$$\text{subject to} \quad |u(t)| < 1 \tag{36b}$$

The theoretical best non-linear controller for our integrator plant is described by the discontinuous feedback

$$u = \text{sign}(e). \tag{37}$$

This controller produces the maximum control signal as long as possible, and vanishes as soon as the plant output reaches the desired value. This way the plant reacts as fast as possible, without any overshoot, thus realizing a minimal amount of energy in e, i.e. $\sqrt{1/3} \approx 0.577$. The closed loop response resulting from this hybrid feedback is depicted in Figure 4 in grey.

Common multiobjective controller design methods rely on norm based optimization functionals and constraints [21]. Problem (36), however, is given in terms of time domain signals. As discussed in Section III, the step reference can be accurately approximated by including an input filter $W_r(s) = \frac{1}{s+\epsilon}$ and considering an impulsive input to this filter. A common attempt to capture the essence of time domain specifications such as (36b) is the reformulation into the frequency domain. In general such reformulations are either approximate or conservative. However, in [26] the authors propose to use a static output filter $W_u = \frac{m}{U_0}$ for a standard $\mathcal{H}_\infty$ optimization problem in order to obtain that $|u(t)| < U_0$ for the specific step reference $r(t) = m \cdot 1(t)$. These considerations result in the multiobjective problem as depicted in Figure 3, where, using $U_0 = m = 1$, $\epsilon = 10^{-4}$, the matrices defining the augmented plant P_{aug} are given by

$$A = \begin{bmatrix} 0 & 0 \\ 0 & -\varepsilon \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$
$$C_z = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 1 \end{bmatrix} \quad D_z = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This allows us to translate (36) into the mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem

$$\min_K \quad \gamma_2 = ||T_2||_2 \tag{38a}$$

$$\text{subject to} \quad ||T_\infty||_\infty < \gamma_\infty \tag{38b}$$

where T_2 denotes the transfer function from w_1 to z_1, T_∞ denotes the transfer between w_2 and z_2, and $\gamma_\infty = 1$. This

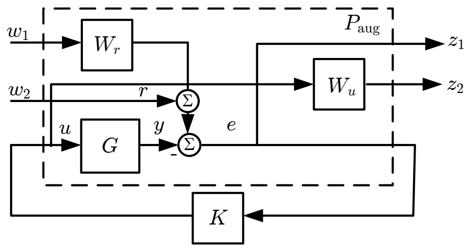

Fig. 3. Augmented plant description for the multiobjective problem

multiobjective problem is hard to solve in general since the corresponding LMIs involve products of the Lyapunov variables and controller parameters. The standard method to convexify (38) is by requiring a common Lyapunov function for both specifications, and is thus to some extend conservative [21]. It is therefore necessary to iteratively increase γ_∞ until the $\mathcal{H}_\infty$-norm of the control sensitivity function $R(s) = \frac{u(s)}{r(s)} = \frac{K}{1+KG}$ is just below 1, which in this case happens when $\gamma_\infty = 1.25$. Furthermore, the actual $\mathcal{H}_2$ norm with the obtained controller should be recalculated with a separate linear $\mathcal{H}_2$ analysis afterwards. Using this method, we obtain $\gamma_2 = 1$ and the static linear controller $K = 1$. The closed loop response using this controller is depicted in Figure 4 by the dashed line. The actual $\mathcal{H}_2$ norm from w_1 to z_1 is $\sqrt{1/2} \approx 0.707$ for this linear controller, which can be validated by either an LMI optimization or numerical integration of the output energy in e.

However, a better performance, i.e. a smaller minimum for (36a) while maintaining (36b), can easily be obtained by using the reset controller satisfying (11) where

$$A_K = \begin{bmatrix} -0.01 & 7.8125 \\ 0 & -62.5 \end{bmatrix}, \quad B_K = \begin{bmatrix} 0 \\ 8 \end{bmatrix},$$
$$C_K = \begin{bmatrix} 1.32 & 7.0313 \end{bmatrix}, \quad D_K = 0, \quad A_r = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The solid lines in Figure 4 show its closed loop response.

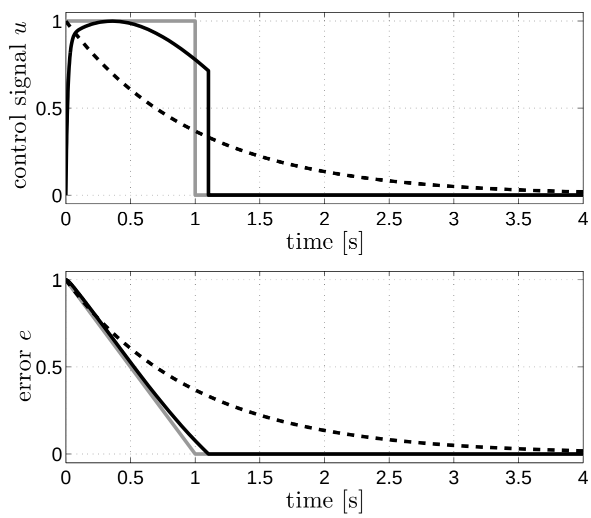

Fig. 4. Closed loop responses u and e using hybrid feedback (grey), the linear controller (dashed), and a reset controller (solid)

TABLE I

$\mathcal{H}_2$ NORMS FOR VARIOUS CONTROLLERS

	hybrid	linear	reset control N=2	N=5	N=10	N=50
Numerical int.	0.577	0.707	0.601	0.601	0.601	0.601
LMI approx.	0.577	0.707	0.721	0.618	0.604	0.601

Theorem 9 can now be applied to calculate the energy content in e for this controller by selecting the appropriate rows and columns of the augmented plant matrices. With $N = 5$ subregions to divide the state space, we obtain $\gamma_2 = 0.618$, which is much smaller than the linear $\mathcal{H}_2$ norm. It can be seen in Figure 4 that our reset controller approximates the discontinuous controller (37) fairly well, resulting in an almost as fast response and an $\mathcal{H}_2$ norm that is only slightly larger. The actual $\mathcal{H}_2$ norm from w_1 to z_1 obtained by the reset controller and calculated by numerical integration of the output energy in e equals $\sqrt{1/2} \approx 0.601$. To check the approximation power of Theorem 9, we divide the state space into more regions N. All results are summarized in Table I. Note that indeed our LMI-based $\mathcal{H}_2$ analysis converges to the correct value as N increases. □

VI. Conclusion

Motivated by recent publications on the potential advantages of reset control, we have developed a framework which can be used to calculate the $\mathcal{H}_2$ norm of a reset control system. The framework is based on LMIs obtained via Lyapunov and dissipativity theories. We removed much conservatism by introducing piecewise quadratic Lyapunov functions, which are much more flexible than quadratic ones. Furthermore, we have presented an example which shows that reset control can be close to the performance of the optimal (discontinuous) controller for a constrained $\mathcal{H}_2$ problem, while a common method to design a good linear controller provides a worse $\mathcal{H}_2$ performance. The example also shows the accuracy of our LMI-based calculation of the $\mathcal{H}_2$ norm in the sense that by increasing the number of regions in the piecewise quadratic Lyapunov function, we recover the actual $\mathcal{H}_2$ performance of the reset control system for this specific case.

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Chapter 4

Synthesis of reset controllers

Now Chapters 2 and 3 have provided analysis techniques to evaluate the performance of general reset control systems, a next step would be to synthesize reset controllers, based on the same $\mathcal{H}_\infty$ and $\mathcal{H}_2$ performance measures. It would be desirable to define a method, such as the ones for linear systems in Appendices F and G, which automatically returns the optimal reset controller. Unfortunately, with the current knowledge such an LMI-based synthesis is hard, if not impossible, to formulate for reset control systems. This chapter will shortly address why.

As described in Chapters 2 and 3, both the $\mathcal{H}_\infty$ and the $\mathcal{H}_2$ analysis for reset control systems contain the terms

$$\mathcal{A}^T P + P\mathcal{A} + T_x^T E_f^T U_f E_f T_x \tag{4.1}$$

in the flow region $\mathfrak{C}$, where

$$T_x = \begin{bmatrix} C & 0 \\ D_K C & C_K \end{bmatrix} \qquad \text{and} \qquad E_f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For synthesis problems, the terms $\mathcal{A}$, P, T_x and U_f in (4.1) are all design variables, hence (4.1) is a non-linear relation. However, for convenience and w.l.o.g. we assume that

$$U_f = \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

such that, using $D_K \in \mathbb{R}$,

$$T_x^T E_f^T U_f E_f T_x = T_x^T U_f T_x = \alpha \begin{bmatrix} 2C^T D_K C & C^T C_K \\ C_K^T C & 0 \end{bmatrix}. \tag{4.2}$$

If we furthermore fix α, this term becomes linear in the design variables C_K and D_K.

First consider the approach of the linearizing change of variables (LCV) described in Appendix F and [24]. This method removes the non-linearity in $\mathcal{A}^T P + P\mathcal{A}$ by pre- and post-multiplication with Π_1. However, this introduces non-linearities in $T_x^T U_f T_x$:

$$\begin{aligned} \Pi_1^T T_x^T U_f T_x \Pi_1 &= \begin{bmatrix} \boldsymbol{X} & \boldsymbol{M} \\ I & 0 \end{bmatrix} \begin{bmatrix} 2C^T D_K C & C^T C_K \\ C_K^T C & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{X} & I \\ \boldsymbol{M}^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2\boldsymbol{X} C^T D_K C \boldsymbol{X} + \boldsymbol{M} C_K^T C \boldsymbol{X} + \boldsymbol{X} C^T C_K \boldsymbol{M}^T & 2\boldsymbol{X} C^T D_K C + \boldsymbol{M} C_K^T C \\ 2C^T D_K C \boldsymbol{X} + C^T C_K \boldsymbol{M}^T & 2C^T D_K C \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{X} C^T \hat{\boldsymbol{C}} + \hat{\boldsymbol{C}}^T C \boldsymbol{X} & \hat{\boldsymbol{C}}^T C + \boldsymbol{X} C^T \hat{\boldsymbol{D}} C \\ C^T \hat{\boldsymbol{C}} + C^T \hat{\boldsymbol{D}} C \boldsymbol{X} & 2C^T \hat{\boldsymbol{D}} C \end{bmatrix}, \end{aligned} \tag{4.3}$$

which contains nonlinear combinations of the new design variables $\boldsymbol{X}$, $\hat{\boldsymbol{C}}$ and $\hat{\boldsymbol{D}}$. Hence (4.3) will turn the synthesis problem into a BMI, which is generally hard to solve. Unfortunately, the non-linear terms cannot be removed using e.g. a Schur complement like

$$\star + \Pi_1^T T_x^T U_f T_x \Pi_1 \prec 0 \qquad \rightarrow \qquad \begin{bmatrix} \star & \Pi_1^T T_x^T \\ T_x \Pi_1 & -U_f^{-1} \end{bmatrix} \prec 0,$$

since this requires $U_f \succ 0$ which is by definition not the case. In summary, the LCV of [24] linearizes the term $\mathcal{A}^T P + P\mathcal{A}$, at the cost of de-linearizing $T_x^T E_f^T U_f E_f T_x$. The synthesis problem can hence not be written as an LMI.

As an alternative one can try to use the elimination method described in Appendix G and [15]. Therefore, (4.1) should be written in the affine form $\Psi + K^T \Theta^T L + L^T \Theta K$. Again using (4.2), this yields

$$\mathcal{A}^T P + P\mathcal{A} + T_x^T E_f^T U_f E_f T_x = \left(A_0^T P + P A_0\right) + \bar{B}\Theta\bar{C} + \bar{C}^T \Theta^T \bar{B}^T \tag{4.4}$$

where A_0 is as defined in (G.3), and

$$\bar{C} = \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix} \qquad \text{and} \qquad \bar{B} = P \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix} + \begin{bmatrix} 0 & \alpha C^T \\ 0 & 0 \end{bmatrix}$$

The design variables in 4.4 are P and Θ, which appear non-linearly (through $\bar{B}$). Hence, P should be removed from $\bar{B}$, as is done by multiplication with P^{-1} in Appendix G, but this is impossible here. Removal of P from the first term in $\bar{B}$, introduces it in the second. As such, due to this dependency on P, it is impossible to find the kernel of $\bar{B}$, and thus to apply the elimination lemma.

Besides this, note that the analysis LMI for the reset region $\mathfrak{D}$ contains a term similar to (4.1). Analogous to the drawback mentioned in the discussion at the end of Appendix G, the elimination method will check the feasibility of these LMIs separately. Hence this can only guarantee the existence of a Θ_{flow} and a Θ_{reset} separately, whilst not necessarily $\Theta = \Theta_{flow} = \Theta_{reset}$. The elimination method thus cannot guarantee the existence of a controller which makes both the flow and the reset LMI feasible.

In an attempt to make the last term in (4.1) independent on the design variables, one might choose to fix B_K, C_K and D_K and to synthesize only A_K. However, this cannot be solved using the linearizing change of variables, since this LCV demands full freedom in at least A_K, B_K and C_K. Fixing B_K, C_K and D_K results in an inequality linear in the design variables $\boldsymbol{X}$, $\boldsymbol{Y}$, $\boldsymbol{M}$ and $\boldsymbol{N}$. However, the matrices $\boldsymbol{M}$ and $\boldsymbol{N}$ should follow from $\boldsymbol{X}$ and $\boldsymbol{Y}$ afterwards, using the non-linear relation $\boldsymbol{M}\boldsymbol{N}^T = I - \boldsymbol{X}\boldsymbol{Y}$, hence the LMI-obtained $\boldsymbol{M}$ and $\boldsymbol{N}$ make no sense and cannot be used to construct P, let alone A_K. A similar problem arises when using the elimination lemma.

The above described problems hence make it impossible to construct computable synthesis LMIs. However, further research might yield that the resulting BMIs might still be solvable in specific situations. The described problems also arise in output feedback synthesis problems for general hybrid systems, which thus makes research in this direction very relevant.

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Appendix A

The Clegg integrator

In control, ordinary linear integrators are often needed to avoid steady-state errors. However, using integrators can also cause stability problems, since each linear integrator introduces a phase lag of 90 degrees for all frequencies:

$$\begin{array}{ll} I(s) = \frac{1}{s} \quad \Rightarrow & I(j\omega) = \frac{1}{j\omega} \\ \text{Magnitude:} & \frac{1}{|j\omega|} = \frac{1}{\sqrt{\omega^2}} = \frac{1}{\omega} \\ \text{Phase:} & \arctan(\frac{0}{1}) - \arctan(\frac{\omega}{0}) = 0 - \arctan(\infty) = -90^\circ \end{array} \tag{A.1}$$

To overcome this problem J.C. Clegg introduced a special non-linear integrator in 1958, which has much less phase loss. His integrator resets itself anytime the input of the integrator becomes 0. Figure A.1 shows how such an integrator can be modeled in Simulink® . The behavior of this Clegg integrator is illustrated in Figure A.2, where its response to a sinusoidal input $u(t) = \sin(\omega t)$ is depicted. Note that the output of a linear integrator (with initial condition 0) is given by

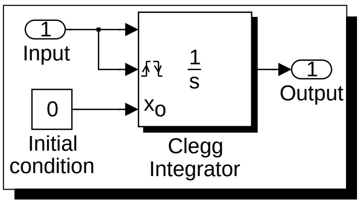

Figure A.1: Clegg integrator

$$\int \sin(\omega t) = -\frac{1}{\omega}\cos(\omega t) + \frac{1}{\omega}. \tag{A.2}$$

One can see in figure A.2 that the output of the Clegg integrator can be described as the sum of a square wave (in phase with the input) and a cosine (90° phase lag), both with

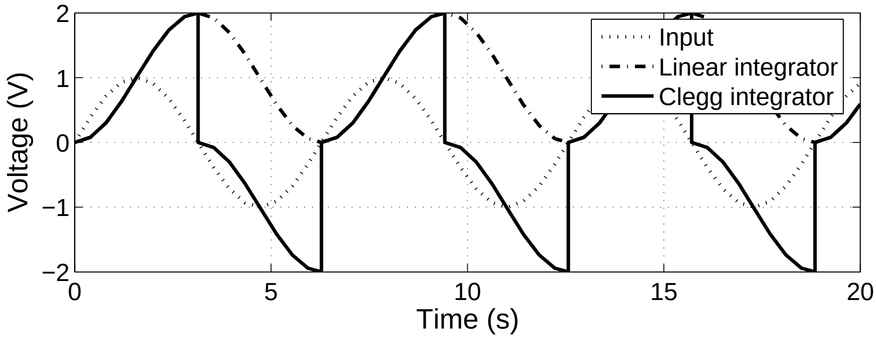

Figure A.2: Output of linear and Clegg integrator amplitude $\frac{1}{\omega}$. In order to estimate the effective amplitude and phase of the Clegg output, the describing function concept can be used. Therefore one calculates the first harmonic of its Fourier transform, which defines its fundamental component. The Fourier transform of a function $f(t)$ is given by:

$$\begin{aligned} f(t) &= \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt) \\ \text{where} \quad a_0 &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt \\ a_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)dt \\ b_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)dt. \end{aligned} \tag{A.3}$$

The cosine of course has only one fundamental component, so its magnitude and phase are $\frac{1}{\omega}$ and $-90°$. For a square wave with amplitude 1 and period 2π the integral over one period is zero, since the positive and the negative square cancel each other. Hence $a_0 = 0$, while the other components are:

$$\begin{aligned} a_1 &= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\cos(t)dt + \int_{0}^{\pi}\cos(t)dt\right) = \frac{1}{\pi}\left[-\sin(t)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[\sin(t)\right]_{0}^{\pi} = 0 \\ a_n &= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\cos(nt)dt + \int_{0}^{\pi}\cos(nt)dt\right) = 0 \end{aligned} \tag{A.4}$$

$$\begin{aligned} b_1 &= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin(t)dt + \int_{0}^{\pi}\sin(t)dt\right) = \frac{1}{\pi}\left[\cos(t)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\cos(t)\right]_{0}^{\pi} = \frac{1}{\pi}(2+2) \\ b_n &= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin(nt)dt + \int_{0}^{\pi}\sin(nt)dt\right) = \frac{1}{n\pi}\left(\left[\cos(nt)\right]_{-\pi}^{0} + \left[-\cos(nt)\right]_{0}^{\pi}\right) = \frac{4}{n\pi}. \end{aligned} \tag{A.5}$$

Hence a square wave $g(t)$ with amplitude $\frac{1}{\omega}$ and frequency ω can be approximated by:

$$g(t) = \frac{4}{\omega\pi}\sin(\omega t) + \frac{4}{3\omega\pi}\sin(3\omega t) + \ldots = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4}{n\omega\pi}\sin(n\omega t), \tag{A.6}$$

Its first harmonic is thus simply defined by $\frac{4}{\omega\pi}$. When combined with the cosine part of the Clegg output, defined by $\frac{1}{j\omega} = -\frac{j}{\omega}$, this then results in:

$$\begin{aligned} & y_{total} = y_{square} + y_{cos} && = \tfrac{1}{\omega}\left(\tfrac{4}{\pi} - j\right) \\ & \text{Amplitude:} && \tfrac{1}{\omega}\sqrt{\left(\tfrac{4}{\pi}\right)^2 + 1} \approx \tfrac{1.62}{\omega} \\ & \text{Phase:} && \arctan(-\tfrac{\pi}{4}) \approx -38.1° \end{aligned} \tag{A.7}$$

This indeed shows that the Clegg integrator has much less phase lag than a linear integrator, and on top of that, has a little bit more gain. Note however that the other harmonics are omitted in this. Since the Clegg integrator is a nonlinear element, it cannot simply be used as a linear filter, and care must be taken when implementing it in linear feedback systems.

Appendix B

Reset control and the H_β-condition

In this appendix the so called H_β-condition, or β positive real condition, is derived and explained, a theorem used to analyse the stability of reset control systems. Its most general form is given in [5], which will be used in this appendix.

Definition of a reset control system

Before continuing, the reset control system has to be defined explicitly. Here, the general description proposed in [5] and schematically shown in Figure B.1 is adopted. In this figure $P(s)$ with states $x_p \in \mathbb{R}^{n_p}$ represents the LTI plant and R represents the reset controller with states $x_r \in \mathbb{R}^{n_r}$. The latter is a system with linear flow dynamics, whose states or subset of states reset to zero when its input (in this case e) is zero. The plant $P(s)$ is described by:

$$\begin{aligned} \dot{x}_p(t) &= A_p x_p(t) + B_p u(t) \\ y_p(t) &= C_p x_p(t), \end{aligned} \tag{B.1}$$

where $A_p \in \mathbb{R}^{n_p \times n_p}$, $B_p \in \mathbb{R}^{n_p \times 1}$ and $C_p \in \mathbb{R}^{1 \times n_p}$. The reset controller is described by:

$$\begin{aligned} \dot{x}_r(t) &= A_r x_r(t) + B_r e(t) && \text{if } e(t) \neq 0 \\ x_r(t^+) &= A_\varrho x_r(t) && \text{if } e(t) = 0 \\ u_r(t) &= C_r x_r(t), \end{aligned} \tag{B.2}$$

where $A_r \in \mathbb{R}^{n_r \times n_r}$, $B_r \in \mathbb{R}^{n_r \times 1}$ and $C_r \in \mathbb{R}^{1 \times n_r}$. When $e(t) = 0$ the last n_ϱ of the n_r controller states are reset to zero, whilst the others remain unchanged. This is embedded in

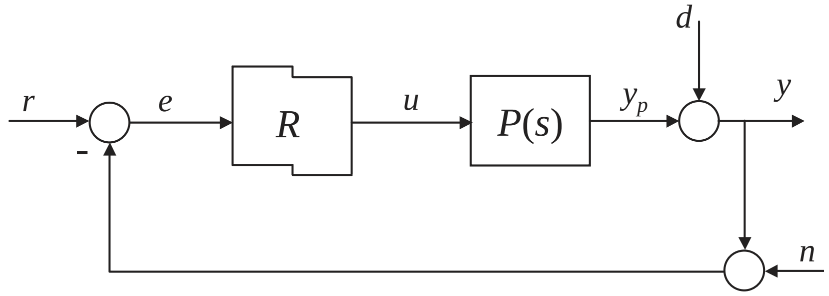

Figure B.1: Block diagram of a reset control system [5] the matrix $A_\varrho \in \mathbb{R}^{n_r \times n_r}$, which has the diagonal form

$$A_\varrho = \begin{bmatrix} I_{n_{\bar{\varrho}}} & 0 \\ 0 & 0_{n_\varrho} \end{bmatrix}.$$

The controller which arises when resetting is omitted (i.e. $A_\varrho = I_{n_r}$) is called the *base linear controller*, which is denoted by $R_{bl}(s) = C_r(sI - A_r)^{-1}B_r$.

Using $e(t) = r(t) - (y(t) + n(t)) = r(t) - (y_p(t) + d(t) + n(t))$ and $w(t) = r(t) - n(t) - d(t)$, one can write $e(t) = w(t) - y_p(t)$. With this the closed loop dynamics can be written as:

$$\begin{array}{rcll} \dot{x}(t) & = & A_{cl}x(t) + B_{cl}w(t) & \text{if } x(t) \notin \mathcal{M}(t) \\ x(t^+) & = & A_R x(t) & \text{if } x(t) \in \mathcal{M}(t) \\ y(t) & = & C_{cl}x(t) + d(t) & \end{array} \tag{B.3}$$

where

$$x(t) = \begin{bmatrix} x_p(t) \\ x_r(t) \end{bmatrix}, \quad A_{cl} = \begin{bmatrix} A_p & B_pC_r \\ -B_rC_p & A_r \end{bmatrix}, \quad B_{cl} = \begin{bmatrix} 0 \\ B_r \end{bmatrix}$$
$$C_{cl} = \begin{bmatrix} C_p & 0 \end{bmatrix}, \quad A_R = \begin{bmatrix} I_{n_p} & 0 \\ 0 & A_\varrho \end{bmatrix} = \begin{bmatrix} I_{n_p + n_{\bar{\varrho}}} & 0 \\ 0 & 0_{n_\varrho} \end{bmatrix},$$

so $A_{cl} \in \mathbb{R}^{n_{cl} \times n_{cl}}$, $A_R \in \mathbb{R}^{n_{cl} \times n_{cl}}$, $B_{cl} \in \mathbb{R}^{n_{cl} \times 1}$ and $C_{cl} \in \mathbb{R}^{1 \times n_{cl}}$, where $n_{cl} = (n_p + n_r)$. The reset surface $\mathcal{M}(t)$ defines the states where the controller (B.2) resets, so when $e(t) = 0$. These states are denoted by ξ. The trivial case where $\xi_r = 0$ is avoided, since a reset then again yields $\xi_r = 0$ and is thus useless. Hence $\mathcal{M}(t)$ is defined by:

$$\mathcal{M}(t) = \{\xi \in \mathbb{R}^{n_{cl}} : e(t) = 0, (I - A_R)\xi \neq 0\}. \tag{B.4}$$

This definition also states that when $x(t_*) \in \mathcal{M}(t_*)$, a single reset causes $x(t_*^+) \notin \mathcal{M}(t_*^+)$. Finally, note that when resets are omitted (hence $A_R = I_{n_{cl}}$) and $R_{bl}(s)$ is used, the resulting closed loop is LTI, which is called the *base linear system*:

$$C_{cl}(sI - A_{cl})^{-1}B_{cl}. \tag{B.5}$$

Derivation of H_β for stability

In the following stability analysis, based on Lyapunov theory, only zero-input stability is considered. Hence, (B.3) is simplified to the unforced case, i.e. $w = 0$, resulting in

$$\begin{array}{rcll} \dot{x}(t) & = & A_{cl}x(t) & \text{if } x(t) \notin \mathcal{M} \\ x(t^+) & = & A_R x(t) & \text{if } x(t) \in \mathcal{M} \\ y(t) & = & C_{cl}x(t), & \end{array} \tag{B.6}$$

with initial condition $x(0) = x_0$ and where $\mathcal{M} = \{\xi \in \mathbb{R}^{n_{cl}} : C_{cl}\xi = 0, (I - A_R)\xi \neq 0\}$. The equilibrium point $x = 0$ of this system is globally asymptotically stable if there exists a continuously-differentiable positive definite function $V(x) : \mathbb{R}^{n_{cl}} \to \mathbb{R}$, such that

$$\dot{V}(x) = \left[\frac{\partial V}{\partial x}\right] A_{cl}x < 0 \qquad \text{if } x \neq 0 \tag{B.7}$$
$$\Delta V(x) = V(A_R x) - V(x) \leq 0 \qquad \text{if } x \in \mathcal{M}. \tag{B.8}$$

Furthermore, (B.6) is quadratically stable if $V(x) = x^T P x$, with P a symmetric positive-definite matrix. For convenience this matrix P is divided into a number of submatrices:

$$P = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{P_1} & \begin{array}{c} P_2^T \\ P_{\bar{\varrho}}^T \end{array} \\ \hline P_2 & P_{\bar{\varrho}} & P_\varrho \end{array}\right], \quad \text{with:} \quad \begin{array}{l} P_1 \in \mathbb{R}^{(n_p+n_{\bar{\varrho}})\times(n_p+n_{\bar{\varrho}})}, P_2 \in \mathbb{R}^{n_\varrho \times n_p} \\ P_{\bar{\varrho}} \in \mathbb{R}^{n_\varrho \times n_{\bar{\varrho}}}, P_\varrho \in \mathbb{R}^{n_\varrho \times n_\varrho} \end{array}$$

Based on (B.7) and (B.8), a reset control system is thus quadratically stable if

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T P x + x^T P \dot{x} = x^T A_{cl}^T P x + x^T P A_{cl} x \\ &= x^T (A_{cl}^T P + P A_{cl}) x < 0 \qquad \text{if } x \neq 0 \qquad \text{(B.9)} \\ \Delta V(x) &= x^T A_R^T P A_R x - x^T P x = x^T (A_R^T P A_R - P) x \leq 0 \qquad \text{if } x \in \mathcal{M} \qquad \text{(B.10)} \end{aligned}$$

Since $\mathcal{M} = \{\xi \in \mathbb{R}^{n_{cl}} : C_{cl}\xi = 0, (I - A_R)\xi \neq 0\}$ (B.10) is valid for all (non-zero) states $\xi \in \mathbb{R}^{n_{cl}}$ for which $C_{cl}\xi = 0$. These states ξ are all in the kernel of C_{cl}. Hence one can define $\text{im}(\Theta) = \ker(C_{cl})$, where $\text{im}(\Theta)$ denotes the image of Θ, such that $\xi = \Theta\eta$ for any η. Hence

$$\left.\begin{array}{l} C_{cl}\xi = 0 \\ \Theta\eta = \xi, \quad \forall \eta \in \mathbb{R}^{n_{cl}-1} \end{array}\right\} \Rightarrow \quad C_{cl}\Theta\eta = 0, \quad \forall \eta \in \mathbb{R}^{n_{cl}-1} \quad \Rightarrow \quad C_{cl}\Theta = 0 \qquad \text{(B.11)}$$

where $C_{cl} \in \mathbb{R}^{1\times n_{cl}}$ and hence $\Theta \in \mathbb{R}^{n_{cl}\times(n_{cl}-1)}$. Now $\mathcal{M}$ can be rewritten as

$$\mathcal{M} = \{\Theta\eta : \eta \in \mathbb{R}^{n_{cl}-1}\}, \qquad \text{(B.12)}$$

such that (B.10), using Finsler's Lemma, now becomes:

$$\Theta^T (A_R^T P A_R - P)\Theta \leq 0. \qquad \text{(B.13)}$$

Since the first $(n_p+n_{\bar{\varrho}})\times(n_p+n_{\bar{\varrho}})$ block of A_R is identity and the other blocks zero, and P is symmetric, $A_R^T P A_R - P$ returns a symmetric $n_{cl} \times n_{cl}$ matrix, which is equal to $-P$ with zeros on the first $(n_p+n_{\bar{\varrho}})\times(n_p+n_{\bar{\varrho}})$ elements. Schematically:

$$A_R^T P A_R - P = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{0} & \begin{array}{c} -P_2^T \\ -P_{\bar{\varrho}}^T \end{array} \\ \hline -P_2 & -P_{\bar{\varrho}} & -P_\varrho \end{array}\right].$$

Furthermore, the kernel of C_{cl}, spanned by $\Theta \in \mathbb{R}^{n_{cl}\times(n_{cl}-1)}$, can be related to the null space of C_p, spanned by $\Theta_p \in \mathbb{R}^{n_p\times(n_p-1)}$, since $C_{cl} = [C_p, 0]$:

$$\Theta = \left[\begin{array}{cc} \Theta_p & 0_{n_p\times n_r} \\ 0_{n_r\times(n_p-1)} & I_{n_r\times n_r} \end{array}\right]. \qquad \text{(B.14)}$$

The left hand side of (B.13) thus returns a symmetric $(n_{cl}-1)\times(n_{cl}-1)$ matrix, with zeros on the first $(n_p+n_{\bar{\varrho}}-1)\times(n_p+n_{\bar{\varrho}}-1)$ elements. Schematically:

$$\Theta^T (A_R^T P A_R - P)\Theta = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{0} & \begin{array}{c} -\Sigma^T \\ -P_{\bar{\varrho}}^T \end{array} \\ \hline -\Sigma & -P_{\bar{\varrho}} & -P_\varrho \end{array}\right] \leq 0$$

This matrix should thus be negative semi-definite. Because of its shape, this is only possible if $\Sigma = P_{\bar{\varrho}} = 0$ and if P_ϱ is positive definite. The latter is always true, since P is positive definite, but the former specifies restrictions on P. Because of its symmetry, it suffices to look only to the last n_ϱ rows of (B.13), which then yields

$$\begin{bmatrix} -P_2 & -P_{\bar\varrho} & -P_\varrho \end{bmatrix} \Theta = \begin{bmatrix} -\Sigma & -P_{\bar\varrho} & -P_\varrho \end{bmatrix} = \begin{bmatrix} 0 & 0 & -P_\varrho \end{bmatrix}.$$

Because of the structure of Θ, see (B.14), this gives:

$$P_{\bar\varrho} = 0 \qquad \text{and} \qquad P_2\Theta_p = 0$$

Since Θ_p is defined by $C_p\Theta_p = 0$ (similar to (B.11)), this latter statement can only be true when P_2 consists of n_ϱ linear combinations of C_p: $P_2 = \beta C_p$ with $\beta \in \mathbb{R}^{n_\varrho \times 1}$. In summary, the last n_ϱ rows of the symmetric positive definite matrix P should be equal to $[\beta C_p, 0, P_\varrho]$:

$$\begin{bmatrix} 0 & 0 & I_{n_\varrho} \end{bmatrix} P = \begin{bmatrix} \beta C_p & 0_{n_\varrho \times n_{\bar\varrho}} & P_\varrho \end{bmatrix} \qquad \text{or} \qquad B^\sharp P = C^\sharp \tag{B.15}$$

Hence, system (B.6) is quadratically stable if there exists a positive definite P which satisfies both (B.9) and (B.15). These relations correspond to the Kalman-Yakubovich-Popov Lemma, which states that this P exists *if and only if* the system $\dot{x} = A_{cl}x + B^\sharp u$, $y = C^\sharp x$, or equivalently

$$H_\beta(s) = \begin{bmatrix} \beta C_p & 0_{n_\varrho \times n_{\bar\varrho}} & P_\varrho \end{bmatrix} (sI - A_{cl})^{-1} \begin{bmatrix} 0_{n_p \times n_\varrho} \\ 0_{n_{\bar\varrho} \times n_\varrho} \\ I_{n_\varrho} \end{bmatrix} \tag{B.16}$$

is strictly positive real (SPR), for some $\beta \in \mathbb{R}^{n_\varrho \times 1}$. So if H_β is SPR for some value of β and some positive definite P_ϱ, the unforced reset control system is quadratically stable. Note that this H_β-condition is both necessary and sufficient for quadratic stability, and only sufficient to prove stability. Moreover, note that (B.9) indicates that the H_β-condition at least requires stability of the base linear system, which is quite restrictive.

Example

Consider the plant $P(s) = 1/s$ and reset controller $R_{bl}(s) = 1/(s+1)$. This means that

$$A_p = 0, \quad B_p = 1, \quad C_p = 1, \quad A_r = -1, \quad B_r = 1, \quad C_r = 1, \quad A_\varrho = 0,$$
$$A_{cl} = \begin{bmatrix} A_p & B_pC_r \\ -B_rC_p & A_r \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}.$$

Without loss of generality one can take $P_\varrho = 1$, such that the H_β-condition becomes

$$\begin{aligned} H_\beta &= \begin{bmatrix} \beta & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \beta & 1 \end{bmatrix} \frac{1}{s^2+s+1} \begin{bmatrix} s+1 & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\beta + s}{s^2+s+1} \end{aligned}$$

which is a minimum phase stable system for $\beta > 0$. Its SPR-ness is tested with $H_\beta(j\omega)$:

$$H_\beta(j\omega) = \frac{j\omega + \beta}{1 - \omega^2 + j\omega} \cdot \frac{1 - \omega^2 - j\omega}{1 - \omega^2 - j\omega} = \frac{j\omega(1 - \omega^2 - \beta) + \omega^2(1-\beta) + \beta}{[1-\omega^2]^2 + \omega^2}$$

$$\Rightarrow \mathrm{Re}[H_\beta(j\omega)] = \frac{\omega^2(1-\beta) + \beta}{[1-\omega^2]^2 + \omega^2} > 0 \quad \forall \omega > 0, \quad \text{if} \quad 0 \le \beta \le 1.$$

So indeed there exists a β (here $0 \le \beta \le 1$) for which H_β is SPR, so the reset control system is quadratically stable.

Other results

As a consequence of the quadratic stability result defined above, [5] also states bounded-input bounded-state (BIBS) stability and asymptotic tracking results. Without further detailed explanation this appendix will shortly summarize these results.

First note that the H_β analysis only holds for unforced systems, so $w(t) = 0$. However, in [5] is was shown that systems satisfying the H_β condition are also BIBS-stable, meaning that $x(t)$ is bounded for bounded $w(t)$, if the controller system matrix A_r has the shape

$$A_r = \begin{bmatrix} A_{r11} & A_{r12} \\ 0 & A_{r22} \end{bmatrix}, \tag{B.17}$$

which means that the derivatives of the resetting controller states not explicitly depend on the non-resetting states.

BIBS stability does not guarantee asymptotic tracking of a reference $r(t)$. However, if an *internal model* of $r(t)$ is present in the plant $P(s)$ or in the non-resetted part of the controller R and the closed loop reset control system satisfies the H_β-condition, asymptotic tracking is achieved [5]. Hence, when $r(t)$ is an initial condition response of a linear system $M(s)$, then $P(s)$ (or the non-resetted part of R) should contain the same system $M(s)$. For example, for step references $r(t) = 1(t)$ it is known that $M(s) = 1/s$, so $P(s)$ should also contain an integrator to guarantee asymptotic tracking.

It is important to note that the internal model should not be present in the resetted part of the reset controller R. This is illustrated by the following example. Consider a plant $P(s) = 1/(s + 1)$ controlled by a Clegg integrator. The H_β-condition yields

$$\begin{aligned} H_\beta &= \begin{bmatrix} \beta & 1 \end{bmatrix} \begin{bmatrix} s+1 & -1 \\ 1 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{s+\beta+1}{s^2+s+1} \\ \Rightarrow \quad \mathrm{Re}[H_\beta(j\omega)] &= \frac{\beta(1-\omega^2)+1}{[1-\omega^2]^2+\omega^2} > 0 \quad \forall \omega > 0, \quad \text{if} \quad -1 \le \beta \le 0. \end{aligned}$$

Hence, this reset control system is (zero-input) quadratically stable and thus $x \to 0$ for $t \to \infty$. Furthermore, since all states of the reset controller are reset, as required in (B.17), the system is also BIBS stable, so any bounded input results in a bounded output. However, when a step reference ($1/s$) is applied, there is no guaranteed asymptotic tracking, since $P(s)$ lacks an internal model of the reference. The plant $P(s)$ needs a persistent input u to keep its output at the desired value. The Clegg integrator however reset this u to zero as soon as $y = r$. The three described situations are shown in figure B.2.

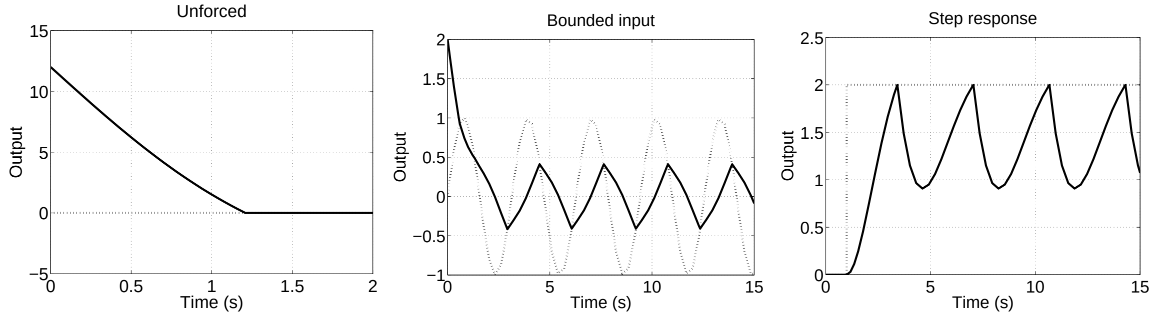

Figure B.2: Typical responses of a reset control system: $P(s) = 1/(s + 1)$ with Clegg

Appendix C

Dissipativity and stability

Dissipativity and stability of dynamical systems are often linked to each other in literature. In particular, this can be used to derive LMIs to determine closed loop stability and performance (e.g. the $\mathcal{L}_2$ gain). This appendix will shortly explain dissipative behavior of dynamical systems and how this can be linked to stability. This knowledge will then be used to derive stability LMIs, after which some examples will be given.

Dissipativity

For a dynamical system Σ described by

$$\Sigma : \begin{cases} \dot{x} &= f(x, w) \\ z &= g(x, w) \end{cases} \tag{C.1}$$

dissipativity can be defined as follows:

Definition 1 *The system Σ is dissipative with respect to a certain supply function s if there exists a storage function $V : X \to \mathbb{R}$ such that*

$$V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} s(w(t), z(t))dt \tag{C.2}$$

for all $t_1 \geq t_0$ and all signals satisfying (C.1).

Whether or not a certain system is labeled 'dissipative' hence depends on s. A system Σ could be dissipative with respect to a certain s_1 but not with respect to s_2. This supply function (or supply rate) should thus be chosen with care, depending on each specific situation.
Basically the dissipativity definition just states that 'energy' is lost, or dissipated, inside the system. It states that the difference between the final and the initial energy, given by the storage functions $V(x(t_1)) - V(x(t_0))$, can never exceed the energy supplied to the system, given by the 'sum' of the supply rate, $\int_{t_0}^{t_1} s(w(t), z(t))dt$.

Note that in (C.2) the storage function V is not necessarily non-negative to show dissipative behavior of (C.1). Such constraints on V are only needed to link dissipativity to stability:

Definition 2 *When the system in (C.1) has an equilibrium point x^* and its storage function V satisfying (C.2) for some s is in fact continuous and non-negative and attains a strong local minimum at x^*, then the point x^* is in fact a stable point of the dissipative system Σ. Furthermore, the storage function V is then a Lyapunov function in the neighborhood of x^*.*

This definition thus states that $V(x(t)) - V(x^*) \geq 0$, $\forall t$, so if the initial condition of (C.1) is at the equilibrium $x(t_0) = x^*$, the supply function for a stable system must satisfy

$$\int_{t_0}^{t_1} s(w(t), z(t))dt \geq V(x(t_1)) - V(x^*) \geq 0, \qquad \forall t_1 \geq 0. \tag{C.3}$$

Linear systems and quadratic supply and storage functions

This section limits itself to linear systems, for which it will derive some general dissipativity and stability results. First consider a simple linear system with a non-zero initial condition

$$\dot{x} = Ax, \qquad x(0) = x_0 \tag{C.4}$$

and no in- or output, so $s = 0$. Following Definitions 1 and 2 its equilibrium point $x^* = 0$ is stable if and only if there exists a positive definite Lyapunov function V, with a strong local minimum at x^*, e.g. $V(x^*) = 0$, satisfying

$$V(x(t_0)) - V(x(t_1)) \geq 0, \qquad \forall t_1 \geq t_0,$$

which is thus monotonically decreasing over time, i.e. $\frac{d}{dt}V < 0$. For linear systems one can even define asymptotic stability by taking a quadratic Lyapunov function $V(x) = x^T Px$, whose derivative $\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = x^T(A^T P + PA)x$ should be negative. Hence (C.4) is asymptotically stable if and only if the following inequalities hold

$$\begin{aligned} & x^T Px > 0, \quad \text{and} \quad x^T(A^T P + PA)x < 0, \qquad \forall x \neq 0 & \text{(C.5)} \\ \Rightarrow \quad & P \succ 0, \quad \text{and} \qquad A^T P + PA \prec 0. & \text{(C.6)} \end{aligned}$$

Now expanded system (C.4) with an input w and an output z, so that

$$T : \left\{ \begin{array}{rcl} \dot{x} & = & Ax + Bw \\ z & = & Cx + Dw. \end{array} \right. \tag{C.7}$$

For such linear systems we can assume quadratic supply and storage functions without loss of generality. In other words, in (C.2) one can assume

$$s(w,z) = \begin{bmatrix} w \\ z \end{bmatrix}^T S \begin{bmatrix} w \\ z \end{bmatrix} \qquad \text{and} \qquad V(x) = x^T Px,$$

so that the derivative of the positive definite Lyapunov function $V(x)$ then satisfies

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T Px + x^T P\dot{x} \\ &= (x^T A^T + w^T B^T)Px + x^T P(Ax + Bw) \\ &= x^T(A^T P + PA)x + w^T B^T Px + x^T PBw \\ &= \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \end{aligned} \tag{C.8}$$

Using this and $\begin{bmatrix} w \\ z \end{bmatrix} = \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}$ in the differential form of (C.2), one then obtains:

$$\frac{d}{dt}V(x) = \frac{d}{dt}x(t)^T Px(t) \quad \leq \quad s(w(t), z(t)) \tag{C.9}$$

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \quad \leq \quad \begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix}, \tag{C.10}$$

or in LMI formulation:

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \prec \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \tag{C.11}$$

According to Definition 1, if this LMI is feasible for some arbitrary P, the original system (C.7) is dissipative *with respect to the chosen quadratic supply function* s. Then (C.7) is said to satisfy a certain quadratic performance measure, defined by the matrix S. Furthermore, following Definition 2, if the Lyapunov function is restricted to be non-negative with a local minimum $V(x^*) = 0$, hence if (C.11) is feasible for some $P \succ 0$, system (C.7) is also stable.

The only remaining problem is now how to choose the supply function s. For a system with input w and output z a reasonable choice could be

$$s = \text{'input energy'} - \text{'output energy'} = ||w||_2^2 - ||z||_2^2.$$

In this equation $||w||_2^2$ can be seen as the total amount of energy *put into* the system and $||z||_2^2$ the amount of energy that *comes out* of the system. The difference is the net amount of *supplied energy* and is thus an appropriate choice for the supply function. In this interpretation the internal energy storage is then given by the Lyapunov function $V(x) = x^T Px$.

The previous supply function can be written in matrix form:

$$s = w^T w - z^T z = \begin{bmatrix} w \\ z \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} \tag{C.12}$$

So in order to check the stability of (C.7) and its dissipativity w.r.t the supply function (C.12), one has to test the feasibility of the following LMIs:

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \prec 0, \quad P \succ 0. \tag{C.13}$$

Hence, if the LMIs (C.13) are feasible, it means that the system is asymptotically stable and dissipative with respect to $||w||_2^2 - ||z||_2^2$. The latter means that the *performance* of (C.7) is such that the output energy is always less than the supplied input energy.

Stability and $\mathcal{H}_\infty$ performance

The above dissipativity theory can hence be used to derive quadratic performance measures in terms of LMIs. A well-known example is the $\mathcal{H}_\infty$ norm, also known as the $\mathcal{L}_2$ induced norm, which is the worst case energy gain from w to z. For system (C.7) this norm is defined as

$$||T||_\infty^2 = \frac{||z||_2^2}{||w||_2^2} \leq \gamma^2,$$

Whether or not $||T||_\infty$ is smaller than γ can be tested with appropriate LMIs. Therefore define the supply function s as:

$$s(w,z) = \gamma^2 ||w||_2^2 - ||z||_2^2 = \gamma^2 w^T w - z^T z = \begin{bmatrix} w \\ z \end{bmatrix}^T \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} \tag{C.14}$$

Hence inserting $S = \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix}$ in (C.11), then yields the following LMI:

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} - \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T S \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \prec 0 \tag{C.15a}$$

$$\begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} + \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \prec 0 \tag{C.15b}$$

$$\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \gamma^2 I \end{bmatrix} \prec 0 \tag{C.15c}$$

This LMI can be reformulated by multiplying it with $\gamma^{-1} > 0$, substituting $\mathcal{P} = \gamma^{-1} P$, and applying a Schur complement:

$$\begin{bmatrix} A^T \mathcal{P} + \mathcal{P} A + \gamma^{-1} C^T C & \mathcal{P} B + \gamma^{-1} C^T D \\ B^T \mathcal{P} + \gamma^{-1} D^T C & \gamma^{-1} D^T D - \gamma I \end{bmatrix} \prec 0$$

$$\begin{bmatrix} A^T \mathcal{P} + \mathcal{P} A & \mathcal{P} B \\ B^T \mathcal{P} & -\gamma I \end{bmatrix} + \begin{bmatrix} \gamma^{-1} C^T C & \gamma^{-1} C^T D \\ \gamma^{-1} D^T C & \gamma^{-1} D^T D \end{bmatrix} \prec 0$$

$$\begin{bmatrix} A^T \mathcal{P} + \mathcal{P} A & \mathcal{P} B \\ B^T \mathcal{P} & -\gamma I \end{bmatrix} - \begin{bmatrix} C^T \\ D^T \end{bmatrix} (-\gamma^{-1} I) \begin{bmatrix} C & D \end{bmatrix} \prec 0$$

$$\xrightarrow{Schur} \left[\begin{array}{cc:c} A^T \mathcal{P} + \mathcal{P} A & \mathcal{P} B & C^T \\ B^T \mathcal{P} & -\gamma I & D^T \\ \hdashline C & D & -\gamma I \end{array}\right] \prec 0 \tag{C.16}$$

Note again that if this LMI holds for some $\mathcal{P}$, that $||T||_\infty < \gamma$. If (C.16) is feasible for some $\mathcal{P} \succ 0$, then (C.7) is also stable. The latter can easily be verified by noting that element (1,1) of (C.16), i.e. $A^T \mathcal{P} + \mathcal{P} A \prec 0$, together with $\mathcal{P} \succ 0$ again form the stability LMI in (C.6). The result in (C.16) is also known as the *Bounded Real Lemma*.

Example

As an example, consider the following system:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w \\ z &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{aligned}$$

Suppose a supply rate as in (C.12) is chosen. Then (C.13) is feasible, with e.g.

$$P = \begin{bmatrix} 2.3672 & 0.5352 \\ 0.5352 & 0.5791 \end{bmatrix}.$$

Note however, that if one chooses the opposite supply function $s = ||z||_2^2 - ||w||_2^2$, the resulting LMI is infeasible. This shows that for this system $||z||_2^2 \leq ||w||_2^2$, and not $||z||_2^2 \geq ||w||_2^2$. Hence the input energy always exceeds the output energy.

One can also perform an $\mathcal{H}_\infty$ analysis on this system by solving the LMI in (C.16) for some γ. When the objective to minimize γ is added to this LMI, one finds the following optimal solution:

$$P = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 0.4358 \end{bmatrix} \qquad \text{for} \qquad \gamma = 0.5$$

Appendix D

Overview of system analysis LMIs

This appendix shortly summarizes some of the most important continuous time LMIs, used to analyse linear time-invariant systems in terms of induced norms. Moreover, some common interpretations of these norms are provided. The following system description will be used:

$$T : \left\{ \begin{array}{lcl} \dot{x} & = & \mathcal{A}x + \mathcal{B}w \\ z & = & \mathcal{C}x + \mathcal{D}w, \end{array} \right. \tag{D.1}$$

where x is the system state, w is the input and z is the output.

$\mathcal{H}_\infty$ analysis

The definition of the $\mathcal{H}_\infty$-norm, or $\mathcal{L}_2$ induced norm, of system (D.1) is the following:

$$||T||_\infty = \sup_{0<||w||_2<\infty} \frac{||z||_2}{||w||_2} = \sup_{0<||w||_2<\infty} \frac{||z||_{pow}}{||w||_{pow}} = \sup_{\omega \in \mathbb{R}} \sigma_{max}(T(j\omega)) \tag{D.2}$$

It thus denotes the maximum possible gain from the input energy of w to the output energy of z. It is the largest possible gain when the system T is fed by harmonic input signals, and thus it corresponds to the maximum singular value (denoted by σ_{max}) of T over all ω.

The system (D.1) is asymptotically stable and $||T||_\infty < \gamma$ for some value of $\gamma > 0$, if and only if there exists a $\mathcal{P} \succ 0$ such that

$$\begin{bmatrix} \mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B}_j & \mathcal{C}_j^T \\ \mathcal{B}_j^T\mathcal{P} & -\gamma I & \mathcal{D}_j^T \\ \mathcal{C}_j & \mathcal{D}_j & -\gamma I \end{bmatrix} \prec 0. \tag{D.3}$$

$\mathcal{H}_2$ analysis

The (ordinary) $\mathcal{H}_2$-norm of a system T given in (D.1) is defined by:

$$||T||_2^2 = \frac{1}{2\pi} \operatorname{trace} \int_{-\infty}^{\infty} T(j\omega)T(j\omega)^* d\omega \tag{D.4}$$

and is thus equal to the surface beneath the magnitude of the Bode plot of $T(j\omega)$. The $\mathcal{H}_2$-norm coincides with the total output energy of the impulse responses of T. A more practical interpretation is that the $\mathcal{H}_2$-norm corresponds to the asymptotic variance of the output when it is excited by white noise input signals.

The $\mathcal{H}_2$ analysis can be done with the following definition of the $\mathcal{H}_2$-norm:

$$||T||_2^2 = \text{trace}(\mathcal{C}_j W \mathcal{C}_j^T) = \text{trace}(\mathcal{B}_j^T M \mathcal{B}_j) < \gamma^2 \tag{D.5}$$

where W and M are the controllability and the observability gramians defined by:

$$\mathcal{A}W + W\mathcal{A}^T + \mathcal{B}_j\mathcal{B}_j^T = 0 \tag{D.6a}$$

$$\mathcal{A}^T M + M\mathcal{A} + \mathcal{C}_j^T\mathcal{C}_j = 0 \tag{D.6b}$$

A typical computation of the $\mathcal{H}_2$-norm is equivalent to: there exists a $X \succ 0$, such that

$$\mathcal{A}X + X\mathcal{A}^T + \mathcal{B}_j\mathcal{B}_j^T \prec 0, \qquad \text{trace}(\mathcal{C}_j X \mathcal{C}_j^T) < \gamma^2$$

Pre- and post-multiplying this with $\mathcal{P} = X^{-1}$ and introducing an auxiliary variable Q, this means there exists a $\mathcal{P} \succ 0$ such that

$$\mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} + \mathcal{P}\mathcal{B}_j\mathcal{B}_j^T\mathcal{P} \prec 0, \qquad \mathcal{C}_j\mathcal{P}^{-1}\mathcal{C}_j^T \prec Q, \qquad \text{trace}(Q) < \gamma^2$$

Using two Schur complements this finally leads to the result: system (D.1) is asymptotically stable and $||T||_2 < \gamma$ for some value $\gamma > 0$, if and only if there exists a $\mathcal{P} \succ 0$ and Q such that

$$\begin{bmatrix} \mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B}_j \\ \mathcal{B}_j^T\mathcal{P} & -I \end{bmatrix} \prec 0 \tag{D.7a}$$

$$\begin{bmatrix} \mathcal{P} & \mathcal{C}_j^T \\ \mathcal{C}_j & Q \end{bmatrix} \succ 0 \tag{D.7b}$$

$$\text{trace}(Q) < \gamma^2. \tag{D.7c}$$

Note that the $\mathcal{H}_2$-norm is bounded, thus $||T||_2 < \infty$, if and only if $\mathcal{D}$=0.

Generalized $\mathcal{H}_2$ analysis

The generalized $\mathcal{H}_2$-norm, or $\mathcal{L}_2$-$\mathcal{L}_\infty$ induced norm, differs slightly from the ordinary $\mathcal{H}_2$-norm, since the former is defined by:

$$||T||_{2g} = ||T||_{2,\infty} = \sup_{0<||w||_2<\infty} \frac{||z||_\infty}{||w||_2}. \tag{D.8}$$

This norm is also called the 'energy to peak' norm, since it measures the peak amplitude (in time) of the output signal z over all unity energy inputs w.
An alternative definition of the generalized $\mathcal{H}_2$-norm, which resembles the ordinary $\mathcal{H}_2$-norm to a large extend, is given by

$$||T||_{2g}^2 = \mathcal{C}_j W \mathcal{C}_j^T < \gamma^2 I, \qquad \text{with} \qquad \mathcal{A}W + W\mathcal{A}^T + \mathcal{B}_j\mathcal{B}_j^T = 0 \tag{D.9}$$

In LMI-form this thus becomes

$$\mathcal{A}X + X\mathcal{A}^T + \mathcal{B}_j\mathcal{B}_j^T \prec 0, \qquad \mathcal{C}_j X \mathcal{C}_j^T \prec \gamma^2 I$$

The rest of the derivation is completely analogous to the ordinary $\mathcal{H}_2$-norm. And thus system (D.1) is asymptotically stable and $||T||_{2g} < \gamma$ for some $\gamma > 0$, if and only if there exists a $\mathcal{P} \succ 0$ such that

$$\begin{bmatrix} \mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B}_j \\ \mathcal{B}_j^T\mathcal{P} & -I \end{bmatrix} \prec 0 \tag{D.10a}$$

$$\begin{bmatrix} \mathcal{P} & \mathcal{C}_j^T \\ \mathcal{C}_j & \gamma^2 I \end{bmatrix} \succ 0. \tag{D.10b}$$

Peak to peak analysis

The peak to peak gain, or $\mathcal{L}_\infty$ induced norm, is simply defined by

$$||T||_{peak} = \sup_{0<||w||_\infty<\infty} \frac{||z||_\infty}{||w||_\infty} \tag{D.11}$$

which defines a mapping from the (bounded) maximum amplitude of the input w to the (bounded) maximum amplitude of the output z. Note that $||T||_\infty \leq ||T||_{peak}$.
If, for some value $\gamma > 0$, there exists a $\mathcal{P} \succ 0$, $\lambda > 0$ and μ such that

$$\begin{bmatrix} \mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} + \lambda\mathcal{P} & \mathcal{P}\mathcal{B}_j \\ \mathcal{B}_j^T\mathcal{P} & -\mu I \end{bmatrix} \prec 0 \tag{D.12a}$$

$$\begin{bmatrix} \lambda\mathcal{P} & 0 & \mathcal{C}_j^T \\ 0 & (\gamma-\mu)I & \mathcal{D}_j^T \\ \mathcal{C}_j & \mathcal{D}_j & \gamma I \end{bmatrix} \succ 0. \tag{D.12b}$$

is feasible, then (D.1) is asymptotically stable and $||T||_{peak} < \gamma$. Note that this condition is sufficient but not necessary.

Appendix E

Discrete time analysis LMIs

In an answer to the previous appendix, some analysis LMIs for discrete time systems will now be derived. The results in this appendix cover stability LMIs as well as $\mathcal{H}_\infty$, $\mathcal{H}_2$ and generalized $\mathcal{H}_2$ results. Starting point is the following linear discrete time system:

$$T : \begin{cases} x_{k+1} &= Ax_k + Bw_k \\ z_k &= Cx_k + Dw_k \end{cases} \tag{E.1}$$

where x_k is the state, w_k the input and z_k the output at timestep k. Furthermore, quadratic Lyapunov (or storage) functions will be used, such that $V(x_k)$ at timestep k is defined by

$$V(x_k) = x_k^T P x_k > 0 \quad \text{for} \quad x_k \neq 0, \qquad \text{hence} \quad P \succ 0. \tag{E.2}$$

Discrete time stability LMI

Internal asymptotic stability of the state x_k is determined by the system matrix A, since for internal stability $w_k = 0$ and hence it follows that $x_{k+1} = Ax_k$. Internal asymptotic stability for such linear systems can be proven if the quadratic Lyapunov function (E.2) is monotonically decreasing in time, hence the *change* in the Lyapunov function $\Delta V = V(x_{k+1}) - V(x_k)$ should be negative, in other words

$$\begin{aligned} \Delta V = x_{k+1}^T P x_{k+1} - x_k^T P x_k &< 0 \\ x_k^T A^T P A x_k - x_k^T P x_k &< 0 \\ x_k^T (A^T P A - P) x_k &< 0 \end{aligned} \tag{E.3}$$

Hence, system (E.1) is internally asymptotically stable if and only if the inequalities

$$A^T P A - P \prec 0, \qquad P \succ 0, \tag{E.4}$$

are feasible. Using a Schur complement (E.4) can be transformed into a single LMI :

$$\begin{aligned} P - A^T P A \succ 0, &\qquad P \succ 0 \\ P - (A^T P) P^{-1} (PA) \succ 0, &\qquad P \succ 0 \\ \xrightarrow{Schur} \quad \begin{bmatrix} P & A^T P \\ PA & P \end{bmatrix} &\succ 0 \end{aligned} \tag{E.5}$$

Discrete time $\mathcal{H}_\infty$ LMI

The derivation of the LMI for the $\mathcal{H}_\infty$ norm is done by using the concept of dissipativity, see Definition 1 in Appendix C. For the discrete case the inequality (C.2) can be stated as

$$\Delta V = V(x_{k+1}) - V(x_k) \leq s(w_k, z_k), \tag{E.6}$$

where $s(w_k, z_k)$ is a certain supply function. For the discrete system (E.1) ΔV becomes

$$\begin{aligned}
\Delta V &= x_{k+1}^T P x_{k+1} - x_k^T P x_k \\
&= (x_k^T A^T + w_k^T B^T) P (A x_k + B w_k) - x_k^T P x_k \\
&= x_k^T (A^T P A - P) x_k + w_k^T B^T P A x_k + x_k^T A^T P B w_k + w_k^T B^T P B w_k \\
&= \begin{bmatrix} x_k \\ w_k \end{bmatrix}^T \begin{bmatrix} A^T P A - P & A^T P B \\ B^T P A & B^T P B \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix}
\end{aligned}$$

Since here the $\mathcal{H}_\infty$-norm of (E.1) is considered, i.e.

$$||T||_\infty = \sup_{0 < ||w||_2 < \infty} \frac{||z||_2}{||w||_2} < \gamma$$

the supply function $s(w_k, z_k)$ in E.6 should be chosen as

$$\begin{aligned}
s(w_k, z_k) &= \gamma^2 w_k^T w_k - z_k^T z_k \\
&= \begin{bmatrix} w_k \\ z_k \end{bmatrix}^T \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} w_k \\ z_k \end{bmatrix} \\
&= \begin{bmatrix} x_k \\ w_k \end{bmatrix}^T \begin{bmatrix} 0 & I \\ C & D \end{bmatrix}^T \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} 0 & I \\ C & D \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \\
&= - \begin{bmatrix} x_k \\ w_k \end{bmatrix}^T \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D - \gamma^2 I \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix}
\end{aligned}$$

By using this ΔV and $s(w_k, z_k)$ in (E.6), it can then be concluded that system (E.1) is asymptotically stable and $||T||_\infty < \gamma$ if and only if the following LMI in $P \succ 0$

$$\begin{aligned}
-\begin{bmatrix} A^T P A - P & A^T P B \\ B^T P A & B^T P B \end{bmatrix} - \begin{bmatrix} C^T C & C^T D \\ D^T C & D^T D - \gamma^2 I \end{bmatrix} &= \\
\begin{bmatrix} P - A^T P A - C^T C & -A^T P B - C^T D \\ -B^T P A - D^T C & -B^T P B - D^T D + \gamma^2 I \end{bmatrix} &\succ 0,
\end{aligned} \tag{E.7}$$

is feasible. Using a Schur complement this LMI can be rewritten:

$$\begin{aligned}
\begin{bmatrix} P - C^T C & -C^T D \\ -D^T C & -D^T D + \gamma^2 I \end{bmatrix} - \begin{bmatrix} A^T P A & A^T P B \\ B^T P A & B^T P B \end{bmatrix} &= \\
\begin{bmatrix} P - C^T C & -C^T D \\ -D^T C & -D^T D + \gamma^2 I \end{bmatrix} - \begin{bmatrix} (A^T P) P^{-1} (P A) & (A^T P) P^{-1} (P B) \\ (B^T P) P^{-1} (P A) & (B^T P) P^{-1} (P B) \end{bmatrix} &= \\
\begin{bmatrix} P - C^T C & -C^T D \\ -D^T C & -D^T D + \gamma^2 I \end{bmatrix} - \begin{bmatrix} A^T P \\ B^T P \end{bmatrix} P^{-1} \begin{bmatrix} P A & P B \end{bmatrix} &\succ 0 \\
\xrightarrow{Schur} \quad \left[\begin{array}{c|cc} P & PA & PB \\ \hline A^T P & P - C^T C & -C^T D \\ B^T P & -D^T C & -D^T D + \gamma^2 I \end{array}\right] &\succ 0
\end{aligned}$$

Now multiply with $\gamma^{-1} > 0$, define $\mathcal{P} = \gamma^{-1}P$, and apply another Schur complement:

$$\begin{bmatrix} \mathcal{P} & \mathcal{P}A & \mathcal{P}B \\ A^T\mathcal{P} & \mathcal{P} - \gamma^{-1}C^TC & -\gamma^{-1}C^TD \\ B^T\mathcal{P} & -\gamma^{-1}D^TC & -\gamma^{-1}D^TD + \gamma I \end{bmatrix} =$$

$$\begin{bmatrix} \mathcal{P} & \mathcal{P}A & \mathcal{P}B \\ A^T\mathcal{P} & \mathcal{P} & 0 \\ B^T\mathcal{P} & 0 & \gamma I \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & \gamma^{-1}C^TC & \gamma^{-1}C^TD \\ 0 & \gamma^{-1}D^TC & \gamma^{-1}D^TD \end{bmatrix} =$$

$$\begin{bmatrix} \mathcal{P} & \mathcal{P}A & \mathcal{P}B \\ A^T\mathcal{P} & \mathcal{P} & 0 \\ B^T\mathcal{P} & 0 & \gamma I \end{bmatrix} - \begin{bmatrix} 0 \\ C^T \\ D^T \end{bmatrix} \gamma^{-1} \begin{bmatrix} 0 & C & D \end{bmatrix} \succ 0$$

$$\xrightarrow{Schur} \left[\begin{array}{ccc:c} \mathcal{P} & \mathcal{P}A & \mathcal{P}B & 0 \\ A^T\mathcal{P} & \mathcal{P} & 0 & C^T \\ B^T\mathcal{P} & 0 & \gamma I & D^T \\ \hdashline 0 & C & D & \gamma I \end{array}\right] \succ 0 \tag{E.8}$$

Of course, Schur complements are not unique; it is allowed to pre- and post-multiply the obtained matrix to interchange rows and columns, e.g.:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} \mathcal{P} & \mathcal{P}A & \mathcal{P}B & 0 \\ A^T\mathcal{P} & \mathcal{P} & 0 & C^T \\ B^T\mathcal{P} & 0 & \gamma I & D^T \\ 0 & C & D & \gamma I \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathcal{P} & 0 & A^T\mathcal{P} & C^T \\ 0 & \gamma I & B^T\mathcal{P} & D^T \\ \mathcal{P}A & \mathcal{P}B & \mathcal{P} & 0 \\ C & D & 0 & \gamma I \end{bmatrix} \succ 0 \tag{E.9}$$

One thus needs just a single LMI (E.8) or (E.9), if desired with the objective to minimize γ, to solve the $\mathcal{H}_\infty$ problem. Note that the additional constraint $\mathcal{P} \succ 0$ is no longer necessary, since it is already contained inside LMIs.

Discrete time $\mathcal{H}_2$ LMI

The definition for the $\mathcal{H}_2$-norm is the discrete case differs slightly from the continuous case. For discrete systems the $\mathcal{H}_2$-norm is defined by

$$||T||_2^2 = \text{trace}(DD^T + CWC^T) < \gamma^2, \tag{E.10}$$

where W is the controllability gramian, defined by the solution of

$$AWA^T - W + BB^T = 0. \tag{E.11}$$

These definitions are equivalent to the following inequalities, where $X \succ 0$:

$$AXA^T - X + BB^T \prec 0, \qquad DD^T + CXC^T \prec Z, \qquad \text{trace}(Z) < \gamma^2 \tag{E.12}$$

One can again apply various Schur complements, multiplications and substitutions to simplify the above LMIs. The first Schur complement results in

$$X - BB^T - AXX^{-1}XA^T \succ 0, \qquad X \succ 0$$

$$\xrightarrow{Schur} \begin{bmatrix} X & XA^T \\ AX & X - BB^T \end{bmatrix} \succ 0$$

Pre- and post-multiplication by X^{-1}, substitution of $P = X^{-1}$ and a Schur complement yields:

$$\begin{bmatrix} X^{-1} & 0 \\ 0 & X^{-1} \end{bmatrix}^T \begin{bmatrix} X & XA^T \\ AX & X - BB^T \end{bmatrix} \begin{bmatrix} X^{-1} & 0 \\ 0 & X^{-1} \end{bmatrix} =$$

$$\begin{bmatrix} X^{-1} & A^T X^{-1} \\ X^{-1}A & X^{-1} - X^{-1}BB^T X^{-1} \end{bmatrix} = \begin{bmatrix} P & A^T P \\ PA & P - PBB^T P \end{bmatrix} =$$

$$\begin{bmatrix} P & A^T P \\ PA & P \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & PBB^T P \end{bmatrix} =$$

$$\begin{bmatrix} P & A^T P \\ PA & P \end{bmatrix} - \begin{bmatrix} 0 \\ PB \end{bmatrix} I \begin{bmatrix} 0 & B^T P \end{bmatrix} \succ 0$$

$$\xrightarrow{Schur} \left[\begin{array}{cc|c} P & A^T P & 0 \\ PA & P & PB \\ \hline 0 & B^T P & I \end{array}\right] \succ 0 \tag{E.13}$$

Again, one might wish to switch rows and columns to obtain a more common LMI:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T \begin{bmatrix} P & A^T P & 0 \\ PA & P & PB \\ 0 & B^T P & I \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} P & PA & PB \\ A^T P & P & 0 \\ B^T P & 0 & I \end{bmatrix} \succ 0 \tag{E.14}$$

Note that the constraint $X = P^{-1} \succ 0$ does not have to be formulated explicitly anymore, since $P \succ 0$ is already explicitly included in (E.14).
The second inequality in (E.12) can also be rewritten, again using $P = X^{-1}$:

$$Z - CXC^T - DD^T = Z - CP^{-1}C^T - DD^T \succ 0, \qquad P \succ 0$$

$$\xrightarrow{Schur} \left[\begin{array}{c|c} I & D^T \\ \hline D & Z - CP^{-1}C^T \end{array}\right] \succ 0, \qquad P \succ 0$$

$$\begin{bmatrix} I & D^T \\ D & Z \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & CP^{-1}C^T \end{bmatrix} =$$

$$\begin{bmatrix} I & D^T \\ D & Z \end{bmatrix} - \begin{bmatrix} 0 \\ C \end{bmatrix} P^{-1} \begin{bmatrix} 0 & C^T \end{bmatrix} \succ 0, \qquad P \succ 0$$

$$\xrightarrow{Schur} \left[\begin{array}{c|cc} P & 0 & C^T \\ \hline 0 & I & D^T \\ C & D & Z \end{array}\right] \succ 0 \tag{E.15}$$

In summary, the discrete time system given in (E.1) is asymptotically stable and its performance such that its $\mathcal{H}_2$ norm $||T||_2 < \gamma$, if and only if the following LMIs in the variables P and Z are feasible:

$$\begin{bmatrix} P & PA & PB \\ A^T P & P & 0 \\ B^T P & 0 & I \end{bmatrix} \succ 0, \qquad \begin{bmatrix} P & 0 & C^T \\ 0 & I & D^T \\ C & D & Z \end{bmatrix} \succ 0, \qquad \text{trace}(Z) < \gamma^2 \tag{E.16}$$

If desired, the objective to minimize γ^2 can be added, in order to find an upperbound on the actual $\mathcal{H}_2$-norm of the system.

Discrete time generalized $\mathcal{H}_2$ LMI

As in the continuous case, the generalized $\mathcal{H}_2$-norm resembles the ordinary $\mathcal{H}_2$-norm to a large extend. For a discrete system (E.1) its definition is given by

$$||T||_{2g}^2 = DD^T + CWC^T < \gamma^2 I, \tag{E.17}$$

where W is the controllability gramian, defined by the solution of

$$AWA^T - W + BB^T = 0. \tag{E.18}$$

Introducing $X \succ 0$, these equations are equivalent to the inequalities

$$AXA^T - X + BB^T \prec 0, \qquad DD^T + CXC^T \prec \gamma^2 I,$$

which, completely analogous to the ordinary $\mathcal{H}_2$-norm, finally results in the LMIs:

$$\begin{bmatrix} P & PA & PB \\ A^T P & P & 0 \\ B^T P & 0 & I \end{bmatrix} \succ 0, \qquad \begin{bmatrix} P & 0 & C^T \\ 0 & I & D^T \\ C & D & \gamma^2 I \end{bmatrix} \succ 0 \tag{E.19}$$

Hence, if and only if the above LMIs in the variable P are feasible, then the discrete time system (E.1) is asymptotically stable and its performance such that $||T||_{2g} < \gamma I$.

Appendix F

Controller synthesis using LMIs and a change of variables

The analysis methods mentioned in Appendices D and E can also be used to synthesize controllers. However, these methods often result in non-linear sets of equations, which are numerically hard to solve. Fortunately, for these situations a smart linearizing change of variables (LCV) can be found [24], which results in computable sets of LMIs. These LCVs are discussed in this appendix.

State feedback: stability

The simplest illustration of the necessity of a change of variables is the stabilizing state feedback control situation. Consider the following system

$$T : \begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{cases} \tag{F.1}$$

which should be stabilized using a state feedback controller, $u = Kx$, where the controller gain K has to be designed. The stability of the closed loop

$$\dot{x} = Ax + BKx = (A + BK)x$$

is determined by $(A + BK)$. The stability analysis and controller synthesis follow from the same optimization problem, using a quadratic Lyapunov function $V(x) = x^T Px > 0$, $\forall x \neq 0$:

$$\begin{aligned} \dot{V}(x) &= \dot{x}^T Px + x^T P\dot{x} \\ &= x^T (A + BK)^T Px + x^T P(A + BK) \\ &= x^T (A^T P + PA + K^T B^T P + PBK)x < 0 \end{aligned}$$

resulting in the following inequality in the variables $P \succ 0$ and K:

$$A^T P + PA + K^T B^T P + PBK \prec 0, \tag{F.2}$$

which is non-linear in its decision variables, since it contains products of K and P. However, pre- and post-multiplying with P^{-1} and introducing $Q = P^{-1}$ and $Y = KQ = KP^{-1}$ yields:

$$\begin{aligned} P^{-1}(A^T P + PA + K^T B^T P + PBK)P^{-1} &\prec 0, \qquad P \succ 0 \\ P^{-1}A^T + AP^{-1} + P^{-T}K^T B^T + BKP^{-1} &\prec 0, \qquad P \succ 0 \\ QA^T + AQ + Y^T B^T + BY &\prec 0, \qquad Q \succ 0 \end{aligned} \tag{F.3}$$

These inequalities are now linear in its decision variables Q and Y, and are thus LMIs which can easily be solved. The actual controller can be retrieved using $P = Q^{-1}$ and $K = YP$.

State feedback: pole placement

LMIs can also be used to place the poles of a closed loop system. In that case a so called LMI region should be defined, inside which the poles of the closed loop system matrix must lie. In the state feedback case the LMI region thus defines a set of allowed eigenvalues of the closed loop system matrix $(A + BK)$.

An LMI region $\mathcal{R}$ is a subset of the complex plane $\mathbb{C}$, usually defined as

$$\mathcal{R} := \left\{ s \in \mathbb{C} : L + sM + \bar{s}M^T + \bar{s}Ns < 0 \right\}, \tag{F.4}$$

where $\bar{s}$ is the complex conjugate of s, or alternatively by

$$\mathcal{R}_T := \left\{ s \in \mathbb{C} : \begin{pmatrix} I \\ sI \end{pmatrix}^* T \begin{pmatrix} I \\ sI \end{pmatrix} \prec 0 \right\} \qquad \text{with} \qquad T = \begin{bmatrix} L & M \\ M^T & N \end{bmatrix}. \tag{F.5}$$

In practice, often $N = 0$, so an LMI region is the set of all s satisfying $L + sM + \bar{s}M^T < 0$. For example, the region $\mathrm{Re}(s) < -\alpha$ can be written as an LMI region by choosing

$$T = \begin{bmatrix} L & M \\ M^T & N \end{bmatrix} = \begin{bmatrix} 2\alpha & 1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad 2\alpha + s + \bar{s} = 2\alpha + 2\mathrm{Re}(s) < 0$$

These LMI regions can be used to check the location of the eigenvalues of a matrix $\mathcal{A}$. Assuming $N = 0$, all these eigenvalues are inside the LMI region $\mathcal{R}$ defined by (F.4) if and only if there exists a matrix $P \succ 0$ satisfying

$$\left[l_{ij}P + m_{ij}\mathcal{A}^T P + m_{ji}P\mathcal{A} \right]_{i,j} \prec 0 \tag{F.6a}$$

$$\begin{bmatrix} l_{11}P + m_{11}\mathcal{A}^T P + m_{11}P\mathcal{A} & \cdots & l_{1n}P + m_{1n}\mathcal{A}^T P + m_{n1}P\mathcal{A} \\ l_{21}P + m_{21}\mathcal{A}^T P + m_{12}P\mathcal{A} & \cdots & l_{2n}P + m_{2n}\mathcal{A}^T P + m_{n2}P\mathcal{A} \\ \vdots & \ddots & \vdots \\ l_{n1}P + m_{n1}\mathcal{A}^T P + m_{1n}P\mathcal{A} & \cdots & l_{nn}P + m_{nn}\mathcal{A}^T P + m_{nn}P\mathcal{A} \end{bmatrix} \prec 0 \tag{F.6b}$$

where l_{ij} and m_{ij} are the i, j-th elements of the $n \times n$ matrices L and M respectively. Note that when state feedback is considered $\mathcal{A} = (A + BK)$. To synthesize the controller K in that case, (F.2) is replaced by the following inequality in the variables $P \succ 0$ and K:

$$\left[l_{ij}P + m_{ij}(A^T P + K^T B^T P) + m_{ji}(PA + PBK) \right]_{i,j} \prec 0 \tag{F.7}$$

Pre- and post-multiplying by P^{-1} and using the same change of variables as before ($Q = P^{-1}$ and $Y = KQ = KP^{-1}$), this results in the LMI:

$$\left[l_{ij}Q + m_{ij}(QA^T + Y^T B^T) + m_{ji}(AQ + BY) \right]_{i,j} \prec 0, \qquad Q \succ 0 \tag{F.8}$$

As an example, consider the case where the closed loop poles should be placed inside a certain vertical band in the left half plane (to achieve a certain speed) and within a cone to assure some amount of damping. These regions can be defined separately and then stacked together. The actual LMI region is then the intersection of these regions.

First define the vertical band: $-\alpha_2 < \mathrm{Re}(s) < -\alpha_1$. Hence we specify L and M in (F.4) as

$$L+sM+\bar{s}M^T = \begin{bmatrix} 2\alpha_1 & 0 \\ 0 & 2\alpha_2 \end{bmatrix} + s\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \bar{s}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

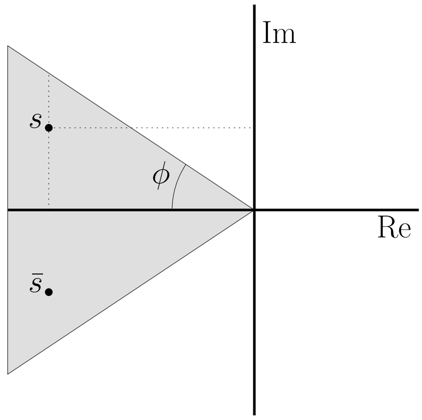

Figure F.1: Conic stability region

such that (F.8), with $Q \succ 0$, becomes

$$\begin{bmatrix} 2\alpha_1 Q + QA^T + Y^TB^T + AQ + BY & 0 \\ 0 & -2\alpha_2 Q - QA^T - Y^TB^T - AQ - BY \end{bmatrix} \prec 0.$$

The idea of the cone is illustrated in figure F.1. The conic region is defined by the inequality $\mathrm{Re}(s)\tan(\phi) < -|\mathrm{Im}(s)|$, and thus the matrices L and M should be specified as

$$L + sM + \bar{s}M^T = s\begin{bmatrix} \sin(\phi) & \cos(\phi) \\ -\cos(\phi) & \sin(\phi) \end{bmatrix} + \bar{s}\begin{bmatrix} \sin(\phi) & -\cos(\phi) \\ \cos(\phi) & \sin(\phi) \end{bmatrix}$$

The closed loop poles are inside this region if and only if there exists a $Q \succ 0$ such that

$$\begin{bmatrix} (QA^T + Y^TB^T + AQ + BY)\sin(\phi) & (QA^T + Y^TB^T - AQ - BY)\cos(\phi) \\ (QA^T + Y^TB^T - AQ - BY)^T\cos(\phi) & (QA^T + Y^TB^T + AQ + BY)\sin(\phi) \end{bmatrix} \prec 0.$$

If the system is controllable, solving both LMIs simultaneously and applying $P = Q^{-1}$ and $K = YP$ will finally result in a control matrix K placing the poles in the predefined region.

Output feedback: $\mathcal{H}_\infty$ controller synthesis

This section will discuss the $\mathcal{H}_\infty$ controller synthesis for output feedback systems. The considered general system layout is given in figure F.2, where P is the augmented plant with states $x \in \mathbb{R}^n$ and K is the controller with states $x_k \in \mathbb{R}^{n_K}$, which uses y to calculate the control signal u. Notice there are n_w exogenous inputs w and n_z exogenous outputs z. Moreover, with w_j and z_j specific in/outputs or combinations of in/outputs will be denoted.

The augmented plant P in figure F.2 is defined by

$$P : \begin{cases} \dot{x} &= Ax + B_w w + Bu \\ z &= C_z x + D_{zw} w + D_z u \\ y &= Cx + D_w w \end{cases} \tag{F.9}$$

Notice that $D = 0$, so there is no direct coupling between u and y, hence P is strictly proper. The controller K can be described by

$$K : \begin{cases} \dot{x}_k &= A_K x_k + B_K y \\ u &= C_K x_k + D_K y \end{cases} \tag{F.10}$$

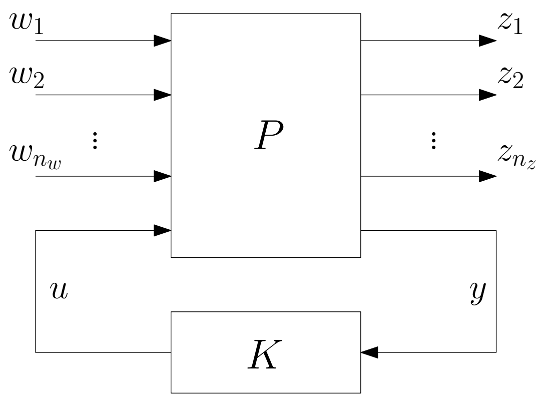

Figure F.2: General multichannel closed loop system

With these descriptions the closed loop dynamics can be obtained:

$$\begin{aligned}
\dot{x} &= Ax + B_w w + B(C_K x_k + D_K(Cx + D_w w)) \\
&= (A + BD_K C)x + BC_K x_k + (B_w + BD_K D_w)w \\
\dot{x}_k &= A_K x_k + B_K(Cx + D_w w) = B_K Cx + A_K x_k + B_K D_w w \\
z &= C_z x + D_{zw} w + D_z(C_K x_k + D_K(Cx + D_w w)) \\
&= (C_z + D_z D_K C)x + D_z C_K x_k + (D_{zw} + D_z D_K D_w)w
\end{aligned}$$

With $x_{cl}^T = [x^T, x_k^T]$ this closed loop T can be written as

$$T : \left\{ \begin{array}{rcl} \dot{x}_{cl} &=& \mathcal{A}x_{cl} + \mathcal{B}w \\ z &=& \mathcal{C}x_{cl} + \mathcal{D}w \end{array} \right. \quad \Rightarrow \quad \left[\begin{array}{c} \dot{x} \\ \dot{x}_k \\ \hline z \end{array} \right] = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array} \right] \left[\begin{array}{c} x \\ x_k \\ \hline w \end{array} \right] \tag{F.11}$$

where

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array} \right] = \left[\begin{array}{cc|c} A + BD_K C & BC_K & B_w + BD_K D_w \\ B_K C & A_K & B_K D_w \\ \hline C_z + D_z D_K C & D_z C_K & D_{zw} + D_z D_K D_w \end{array} \right].$$

That the dimensions of this matrix are indeed correct is shown in figure F.3. The closed loop in (F.11) represents the transfer from all inputs w to all outputs z. If only specific in- and

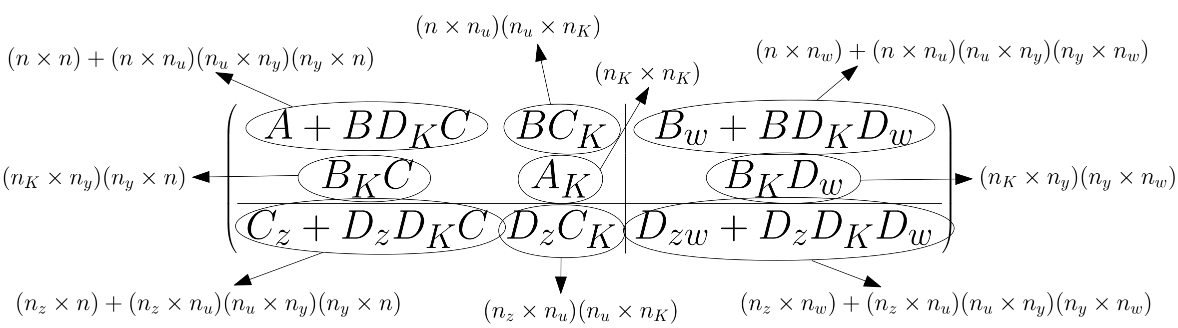

Figure F.3: Size check of closed loop matrix. Total size is $(n+n_K+n_z) \times (n+n_K+n_w)$.

outputs w_j and z_j should be considered (w_j and z_j can still be multidimensional!), one can apply a matrix multiplication to select the right channels and/or combination of channels:

$$T_j = L_j T R_j \qquad \Rightarrow \qquad w = R_j w_j \quad \text{and} \quad z_j = L_j z, \tag{F.12}$$

where $L_j \in \mathbb{R}^{n_{zj} \times n_z}$ and $R_j \in \mathbb{R}^{n_w \times n_{wj}}$. Here $n_{wj} \leq n_w$ and $n_{zj} \leq n_z$ are the number of considered inputs and outputs in the $\mathcal{H}_\infty$ synthesis. With this transformation any input and any output (or combinations of these) can now be selected. Hence, the j-th realization of the closed loop transfer $T_j = L_j T R_j$ from w_j to z_j is now completely described by

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B}_j \\ \hline \mathcal{C}_j & \mathcal{D}_j \end{array}\right] = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B}R_j \\ \hline L_j\mathcal{C} & L_j\mathcal{D}R_j \end{array}\right] = \left[\begin{array}{cc|c} A + BD_KC & BC_K & B_j + BD_KF_j \\ B_KC & A_K & B_KF_j \\ \hline C_j + E_jD_KC & E_jC_K & D_j + E_jD_KF_j \end{array}\right] \tag{F.13}$$

where

$$\begin{array}{lll} B_j = B_w R_j, & C_j = L_j C_z, & D_j = L_j D_{zw} R_j, \\ E_j = L_j D_z, & F_j = D_w R_j. & \end{array}$$

The $\mathcal{H}_\infty$ controller synthesis starts with the analysis LMI (C.16), derived in Appendix C. It states that the closed loop system T_j is asymptotically stable and $||T_j||_\infty < \gamma$ if and only if there exists a $\mathcal{P} \succ 0$ and $\gamma > 0$ such that

$$\begin{bmatrix} \mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B}_j & \mathcal{C}_j^T \\ \mathcal{B}_j^T\mathcal{P} & -\gamma I & \mathcal{D}_j^T \\ \mathcal{C}_j & \mathcal{D}_j & -\gamma I \end{bmatrix} \prec 0. \tag{F.14}$$

Furthermore, minimization of γ in (F.14) returns the actual $\mathcal{H}_\infty$ norm of T_j.

When a controller needs to be designed, the matrices A_K, B_K, C_K and D_K are unknown and should be found using the above inequality. However, $\mathcal{P}$ is also unknown, resulting in non-linear terms like $\mathcal{P}\mathcal{A}$ and $\mathcal{P}\mathcal{B}_j$. Thus a linearizing change of variables is needed. This LCV starts with the partitioning of the positive definite matrix $\mathcal{P}$ and its inverse $\mathcal{P}^{-1}$:

$$\mathcal{P} = \begin{bmatrix} \boldsymbol{Y} & N \\ N^T & \star \end{bmatrix} \quad \text{and} \quad \mathcal{P}^{-1} = \begin{bmatrix} \boldsymbol{X} & M \\ M^T & \star \end{bmatrix} \tag{F.15}$$

where $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n \times n}$ and symmetric. Since $\mathcal{P}\mathcal{P}^{-1} = I$ it holds that $\mathcal{P}\left[\begin{smallmatrix} \boldsymbol{X} \\ M^T \end{smallmatrix}\right] = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, and thus

$$\mathcal{P}\Pi_1 = \Pi_2 \qquad \text{with} \quad \Pi_1 = \begin{bmatrix} \boldsymbol{X} & I \\ M^T & 0 \end{bmatrix} \quad \text{and} \quad \Pi_2 = \begin{bmatrix} I & \boldsymbol{Y} \\ 0 & N^T \end{bmatrix}.$$

With this, by pre- and post-multiplying with Π_1, the constraint $\mathcal{P} \succ 0$ can be converted into

$$\Pi_1^T\mathcal{P}\Pi_1 = \Pi_1^T\Pi_2 = \begin{bmatrix} \boldsymbol{X} & I \\ I & \boldsymbol{Y} \end{bmatrix} \succ 0. \tag{F.16}$$

The $\mathcal{H}_\infty$ LMI (F.14) should also be pre- and post-multiplied with the scaling Π_1:

$$\begin{bmatrix} \Pi_1^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathcal{A}^T\mathcal{P} + \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{B}_j & \mathcal{C}_j^T \\ \mathcal{B}_j^T\mathcal{P} & -\gamma I & \mathcal{D}_j^T \\ \mathcal{C}_j & \mathcal{D}_j & -\gamma I \end{bmatrix} \begin{bmatrix} \Pi_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \prec 0$$

$$\begin{bmatrix} \Pi_1^T\mathcal{A}^T\mathcal{P}\Pi_1 + \Pi_1^T\mathcal{P}\mathcal{A}\Pi_1 & \Pi_1^T\mathcal{P}\mathcal{B}_j & \Pi_1^T\mathcal{C}_j^T \\ \mathcal{B}_j^T\mathcal{P}\Pi_1 & -\gamma I & \mathcal{D}_j^T \\ \mathcal{C}_j\Pi_1 & \mathcal{D}_j & -\gamma I \end{bmatrix} \prec 0 \tag{F.17}$$

Now calculate all the terms containing Π_1 and define new variables $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{B}}$, $\hat{\boldsymbol{C}}$ and $\hat{\boldsymbol{D}}$:

$$\begin{aligned}
\mathcal{D}_j &= D_j + E_j D_K F_j = D_j + E_j \hat{\boldsymbol{D}} F_j \\
\Rightarrow \hat{\boldsymbol{D}} &= D_K \qquad \text{(F.18a)} \\
\mathcal{C}_j \Pi_1 &= \begin{bmatrix} C_j + E_j D_K C & E_j C_K \end{bmatrix} \begin{bmatrix} \boldsymbol{X} & I \\ M^T & 0 \end{bmatrix} \\
&= \begin{bmatrix} C_j \boldsymbol{X} + E_j (D_K C \boldsymbol{X} + C_K M^T) & C_j + E_j D_K C \end{bmatrix} \\
&= \begin{bmatrix} C_j \boldsymbol{X} + E_j \hat{\boldsymbol{C}} & C_j + E_j \hat{\boldsymbol{D}} C \end{bmatrix} \\
\Rightarrow \hat{\boldsymbol{C}} &= D_K C \boldsymbol{X} + C_K M^T \qquad \text{(F.18b)} \\
\Pi_1^T \mathcal{P} \mathcal{B}_j &= \Pi_2^T \mathcal{B}_j = \begin{bmatrix} I & 0 \\ \boldsymbol{Y} & N \end{bmatrix} \begin{bmatrix} B_j + B D_K F_j \\ B_K F_j \end{bmatrix} \\
&= \begin{bmatrix} B_j + B D_K F_j \\ \boldsymbol{Y} B_j + (\boldsymbol{Y} B D_K + N B_K) F_j \end{bmatrix} = \begin{bmatrix} B_j + B \hat{\boldsymbol{D}} F_j \\ \boldsymbol{Y} B_j + \hat{\boldsymbol{B}} F_j \end{bmatrix} \\
\Rightarrow \hat{\boldsymbol{B}} &= \boldsymbol{Y} B D_K + N B_K \qquad \text{(F.18c)} \\
\Pi_1^T \mathcal{P} \mathcal{A} \Pi_1 &= \Pi_2^T \mathcal{A} \Pi_1 = \begin{bmatrix} I & 0 \\ \boldsymbol{Y} & N \end{bmatrix} \begin{bmatrix} A + B D_K C & B C_K \\ B_K C & A_K \end{bmatrix} \begin{bmatrix} \boldsymbol{X} & I \\ M^T & 0 \end{bmatrix} \\
&= \begin{bmatrix} A\boldsymbol{X} + B(D_K C \boldsymbol{X} + C_K M^T) & A + B D_K C \\ \star\text{big term}\star & \boldsymbol{Y} A + (\boldsymbol{Y} B D_K + N B_K) C \end{bmatrix} \\
&= \begin{bmatrix} A\boldsymbol{X} + B\hat{\boldsymbol{C}} & A + B D_K C \\ \hat{\boldsymbol{A}} & \boldsymbol{Y} A + \hat{\boldsymbol{B}} C \end{bmatrix} \\
\Rightarrow \hat{\boldsymbol{A}} &= \boldsymbol{Y} A \boldsymbol{X} + \boldsymbol{Y} B D_K C \boldsymbol{X} + N B_K C \boldsymbol{X} + \boldsymbol{Y} B C_K M^T + N A_K M^T \qquad \text{(F.18d)}
\end{aligned}$$

Using this change of variables, (F.17) now becomes linear in the decision variables $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{B}}$, $\hat{\boldsymbol{C}}$, $\hat{\boldsymbol{D}}$, $\boldsymbol{X}$ and $\boldsymbol{Y}$, thereby creating a computable set of LMIs:

$$\begin{bmatrix} \boldsymbol{X} & I \\ I & \boldsymbol{Y} \end{bmatrix} \succ 0 \qquad \text{(F.19a)}$$

$$\begin{bmatrix} A\boldsymbol{X} + \boldsymbol{X}A^T + B\hat{\boldsymbol{C}} + \hat{\boldsymbol{C}}^T B^T & \hat{\boldsymbol{A}}^T + (A + B\hat{\boldsymbol{D}}C) & \star & \star \\ \hat{\boldsymbol{A}} + (A + B\hat{\boldsymbol{D}}C)^T & \boldsymbol{Y}A + A^T\boldsymbol{Y} + \hat{\boldsymbol{B}}C + C^T\hat{\boldsymbol{B}}^T & \star & \star \\ (B_j + B\hat{\boldsymbol{D}}F_j)^T & (\boldsymbol{Y}B_j + \hat{\boldsymbol{B}}F_j)^T & -\gamma I & \star \\ C_j\boldsymbol{X} + E_j\hat{\boldsymbol{C}} & C_j + E_j\hat{\boldsymbol{D}}C & D_j + E_j\hat{\boldsymbol{D}}F_j & -\gamma I \end{bmatrix} \prec 0 \qquad \text{(F.19b)}$$

When all inputs w and outputs z are considered (so $L_j = R_j = I$), LMI (F.19b) becomes:

$$\begin{bmatrix} A\boldsymbol{X} + \boldsymbol{X}A^T + B\hat{\boldsymbol{C}} + \hat{\boldsymbol{C}}^T B^T & \hat{\boldsymbol{A}}^T + (A + B\hat{\boldsymbol{D}}C) & \star & \star \\ \hat{\boldsymbol{A}} + (A + B\hat{\boldsymbol{D}}C)^T & \boldsymbol{Y}A + A^T\boldsymbol{Y} + \hat{\boldsymbol{B}}C + C^T\hat{\boldsymbol{B}}^T & \star & \star \\ (B_w + B\hat{\boldsymbol{D}}D_w)^T & (\boldsymbol{Y}B_w + \hat{\boldsymbol{B}}D_w)^T & -\gamma I & \star \\ C_z\boldsymbol{X} + D_z\hat{\boldsymbol{C}} & C_z + D_z\hat{\boldsymbol{D}}C & D_{zw} + D_z\hat{\boldsymbol{D}}D_w & -\gamma I \end{bmatrix} \prec 0 \qquad \text{(F.20)}$$

The new variables can then be used to derive the controller matrices and the Lyapunov matrix $\mathcal{P}$. First note that $\mathcal{P}^{-1}\mathcal{P} = I$ yields $MN^T = I - \boldsymbol{XY}$. Since $\left[\begin{smallmatrix} \boldsymbol{X} & I \\ I & \boldsymbol{Y} \end{smallmatrix}\right] \succ 0$ and thus $I - \boldsymbol{XY} \prec 0$, the matrix MN^T is nonsingular. Thus one can always find square and nonsingular matrices M and N satisfying $MN^T = I - \boldsymbol{XY}$, for example by making an SVD-decomposition of $I - \boldsymbol{XY}$:

$$I - \boldsymbol{XY} = MN^T = U\Sigma V^T = U\sqrt{\Sigma}\sqrt{\Sigma}V^T$$

$$M = U\sqrt{\Sigma} \quad \Rightarrow \quad M^{-1} = U^T \frac{1}{\sqrt{\Sigma}} \quad \Rightarrow \quad M^{-T} = U\frac{1}{\sqrt{\Sigma}}$$
$$N = \sqrt{\Sigma}V \quad \Rightarrow \quad N^{-1} = \frac{1}{\sqrt{\Sigma}}V^T$$

Once M^{-T} and N^{-1} are found, the controller matrices can easily be computed:

$$D_K = \hat{\boldsymbol{D}} \tag{F.21a}$$
$$C_K = (\hat{\boldsymbol{C}} - D_K C \boldsymbol{X})M^{-T} \tag{F.21b}$$
$$B_K = N^{-1}(\hat{\boldsymbol{B}} - \boldsymbol{Y} B D_K) \tag{F.21c}$$
$$\begin{aligned} A_K = {} & N^{-1}\hat{\boldsymbol{A}}M^{-T} - B_K C \boldsymbol{X} M^{-T} \\ & -N^{-1}\boldsymbol{Y} B C_K - N^{-1}\boldsymbol{Y}(A + BD_K C)\boldsymbol{X} M^{-T} \end{aligned} \tag{F.21d}$$

It can easily be shown that the dimensions of the new linearizing variables are:

$$\begin{array}{lll} \boldsymbol{X} : n \times n & \hat{\boldsymbol{D}} : n_u \times n_y & \hat{\boldsymbol{B}} : n \times n_y \\ \boldsymbol{Y} : n \times n & \hat{\boldsymbol{C}} : n_u \times n & \hat{\boldsymbol{A}} : n \times n \end{array}$$

With this, and the fact that the matrices M and N must be invertible and thus nonsingular, it can be proven that both M and N should be $n \times n$, yielding that $n_K = n$. In other words, the order of the controller is equal to the order of the plant.

Appendix G

Alternative $\mathcal{H}_\infty$ controller synthesis

This appendix will discuss an alternative way to formulate the $\mathcal{H}_\infty$ controller synthesis, as is described in [15]. This synthesis does not use a change of variables, as in Appendix F, but instead pulls the controller parameters out of the inequalities and *eliminates* them to find an appropriate Lyapunov function. After that, the controller is subtracted using a similar LMI.

Problem description

The method is illustrated by means of an ordinary linear system. Therefore, the same closed loop configuration as in figure F.2 is used, and furthermore the dynamics of the augmented plant and the controller are identical to the ones in (F.9) and (F.10). The closed loop dynamics, with state $x_{cl} = [x^T, x_k^T]^T$, is thus given by

$$\begin{array}{rcl} \dot{x}_{cl} & = & \mathcal{A}x_{cl} + \mathcal{B}w \\ z & = & \mathcal{C}x_{cl} + \mathcal{D}w \end{array} \quad \Rightarrow \quad \left[\begin{array}{c} \dot{x} \\ \dot{x}_k \\ \hline z \end{array}\right] = \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] \left[\begin{array}{c} x \\ x_k \\ \hline w \end{array}\right] \tag{G.1}$$

where

$$\left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & \mathcal{D} \end{array}\right] = \left[\begin{array}{cc|c} A + BD_K C & BC_K & B_w + BD_K D_w \\ B_K C & A_K & B_K D_w \\ \hline C_z + D_z D_K C & D_z C_K & D_{zw} + D_z D_K D_w \end{array}\right].$$

The controller parameters A_K, B_K, C_K and D_K can be pulled out of these matrices, and written in one matrix Θ:

$$\Theta := \left[\begin{array}{cc} A_K & B_K \\ C_K & D_K \end{array}\right] \tag{G.2}$$

so that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ can be written as

$$\begin{array}{rcl} \mathcal{A} & = & \left[\begin{array}{cc} A & 0 \\ 0 & 0 \end{array}\right] + \left[\begin{array}{cc} BD_K C & BC_K \\ B_K C & A_K \end{array}\right] \\ & = & \left[\begin{array}{cc} A & 0 \\ 0 & 0 \end{array}\right] + \left[\begin{array}{cc} 0 & B \\ I & 0 \end{array}\right] \Theta \left[\begin{array}{cc} 0 & I \\ C & 0 \end{array}\right] = A_0 + B_1 \Theta C_1 \end{array} \tag{G.3a}$$

$$\begin{aligned}
\mathcal{B} &= \begin{bmatrix} B_w \\ 0 \end{bmatrix} + \begin{bmatrix} BD_KD_w \\ B_KD_w \end{bmatrix} \\
&= \begin{bmatrix} B_w \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & B \\ I & 0 \end{bmatrix} \Theta \begin{bmatrix} 0 \\ D_w \end{bmatrix} = B_0 + B_1\Theta D_2 \qquad \text{(G.3b)} \\
\mathcal{C} &= \begin{bmatrix} C_z & 0 \end{bmatrix} + \begin{bmatrix} D_zD_KC & D_zC_K \end{bmatrix} \\
&= \begin{bmatrix} C_z & 0 \end{bmatrix} + \begin{bmatrix} 0 & D_z \end{bmatrix} \Theta \begin{bmatrix} 0 & I \\ C & 0 \end{bmatrix} = C_0 + D_1\Theta C_1 \qquad \text{(G.3c)} \\
\mathcal{D} &= D_{zw} + D_zD_KD_w = D_{zw} + \begin{bmatrix} 0 & D_z \end{bmatrix} \Theta \begin{bmatrix} 0 \\ D_w \end{bmatrix} = D_{zw} + D_1\Theta D_2 \qquad \text{(G.3d)}
\end{aligned}$$

Since A_0, B_0, C_0, D_{zw}, B_1, C_1, D_1 and D_2 are all parameters of the augmented plant and thus known beforehand, one can conclude that the closed loop matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ depend *affinely* on the controller parameters Θ, a property that will be used in the next section.

Moreover, the remainder of this appendix will use the following elimination lemma.

Lemma 1 *Given a symmetric matrix Ψ and two matrices K and L, consider the problem of finding a matrix Θ which satisfies the LMI*

$$\Psi + K^T\Theta^TL + L^T\Theta K \prec 0 \qquad \text{(G.4)}$$

If W_K and W_L denote the null spaces of K and L respectively, then this LMI is feasible if and only if the LMIs

$$\begin{cases} W_K^T\Psi W_K \prec 0 \\ W_L^T\Psi W_L \prec 0 \end{cases} \qquad \text{(G.5)}$$

are feasible. Note that the actual solution Θ is eliminated in the latter LMIs.

Feasibility LMIs

Recall from (C.16) that the $\mathcal{H}_\infty$-norm of a system can be found with the LMI

$$\begin{bmatrix} \mathcal{A}^TP + P\mathcal{A} & P\mathcal{B} & \mathcal{C}^T \\ \mathcal{B}^TP & -\gamma I & \mathcal{D}^T \\ \mathcal{C} & \mathcal{D} & -\gamma I \end{bmatrix} \prec 0. \qquad \text{(G.6)}$$

If this LMI is feasible for a certain $P \succ 0$ and $\gamma > 0$, then the closed loop system (G.1) is asymptotically stable and the $\mathcal{L}_2$-gain $\frac{||z||_2}{||w||_2} < \gamma$. When the definitions of (G.3) are substituted into (G.6), the following result is obtained:

$$\begin{aligned}
&\begin{bmatrix} (A_0 + B_1\Theta C_1)^TP + P(A_0 + B_1\Theta C_1) & P(B_0 + B_1\Theta D_2) & (C_0 + D_1\Theta C_1)^T \\ (B_0 + B_1\Theta D_2)^TP & -\gamma I & (D_{zw} + D_1\Theta D_2)^T \\ C_0 + D_1\Theta C_1 & D_{zw} + D_1\Theta D_2 & -\gamma I \end{bmatrix} \\
&= \begin{bmatrix} A_0^TP + PA_0 + C_1^T\Theta^TB_1^TP + PB_1\Theta C_1 & PB_0 + PB_1\Theta D_2 & C_0^T + C_1^T\Theta^TD_1^T \\ B_0^TP + D_2^T\Theta^TB_1^TP & -\gamma I & D_{zw}^T + D_2^T\Theta^TD_1^T \\ C_0 + D_1\Theta C_1 & D_{zw} + D_1\Theta D_2 & -\gamma I \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} A_0^TP + PA_0 & PB_0 & C_0^T \\ B_0^TP & -\gamma I & D_{zw}^T \\ C_0 & D_{zw} & -\gamma I \end{bmatrix} + \begin{bmatrix} C_1^T\Theta^TB_1^TP & 0 & C_1^T\Theta^TD_1^T \\ D_2^T\Theta^TB_1^TP & 0 & D_2^T\Theta^TD_1^T \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} PB_1\Theta C_1 & PB_1\Theta D_2 & 0 \\ 0 & 0 & 0 \\ D_1\Theta C_1 & D_1\Theta D_2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} A_0^TP + PA_0 & PB_0 & C_0^T \\ B_0^TP & -\gamma I & D_{zw}^T \\ C_0 & D_{zw} & -\gamma I \end{bmatrix} + \begin{bmatrix} C_1^T \\ D_2^T \\ 0 \end{bmatrix} \Theta^T \begin{bmatrix} B_1^TP & 0 & D_1^T \end{bmatrix} + \begin{bmatrix} PB_1 \\ 0 \\ D_1 \end{bmatrix} \Theta \begin{bmatrix} C_1 & D_2 & 0 \end{bmatrix}$$

$$= \Psi + K^T\Theta^TL_P + L_P^T\Theta K \prec 0$$

According to Lemma 1, there is a parameter set Θ for which this inequality is feasible if and only if the LMIs

$$\begin{cases} W_K^T\Psi W_K \prec 0 \\ W_{L_P}^T\Psi W_{L_P} \prec 0 \end{cases}$$

are feasible for some $P \succ 0$ and $\gamma > 0$. Here W_K and W_{L_P} are matrices whose columns span the kernels of K and L_P respectively. These should be known beforehand, but W_{L_P} contains P which is yet unknown. However, for any $\text{im}(W_\Omega) = \ker(\Omega)$ it holds that

$$\Omega W_\Omega = 0 \quad \Rightarrow \quad \Omega Q Q^{-1} W_\Omega = 0 \quad \Rightarrow \quad \text{im}(Q^{-1}W_\Omega) = \ker(\Omega Q).$$

So for $L = \begin{bmatrix} B_1^T & 0 & D_1^T \end{bmatrix}$ and $L_P = L \begin{bmatrix} P & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$, then $W_{L_P} = \begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} W_L$.

This thus means that $W_{L_P}^T\Psi W_{L_P} \prec 0$ can be rewritten using W_L:

$$W_L^T \begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_0^TP + PA_0 & PB_0 & C_0^T \\ B_0^TP & -\gamma I & D_{zw}^T \\ C_0 & D_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} P^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} W_L$$

$$= W_L^T \begin{bmatrix} A_0P^{-1} + P^{-1}A_0^T & B_0 & P^{-1}C_0^T \\ B_0^T & -\gamma I & D_{zw}^T \\ C_0P^{-1} & D_{zw} & -\gamma I \end{bmatrix} W_L \quad = \quad W_L^T\Phi W_L \prec 0$$

Thus there exists a controller which yields a stable closed loop with an $\mathcal{L}_2$-gain smaller than $\gamma > 0$ if and only if the following LMIs are feasible for some $P \succ 0$:

$$\begin{cases} W_K^T\Psi W_K \prec 0 \\ W_L^T\Phi W_L \prec 0 \end{cases} \tag{G.7}$$

with $\text{im}(W_K) = \ker(\begin{bmatrix} C_1 & D_2 & 0 \end{bmatrix})$, $\text{im}(W_L) = \ker(\begin{bmatrix} B_1^T & 0 & D_1^T \end{bmatrix})$ and

$$\Psi = \begin{bmatrix} A_0^TP + PA_0 & PB_0 & C_0^T \\ B_0^TP & -\gamma I & D_{zw}^T \\ C_0 & D_{zw} & -\gamma I \end{bmatrix} \quad \text{and} \quad \Phi = \begin{bmatrix} A_0P^{-1} + P^{-1}A_0^T & B_0 & P^{-1}C_0^T \\ B_0^T & -\gamma I & D_{zw}^T \\ C_0P^{-1} & D_{zw} & -\gamma I \end{bmatrix}$$

However, since these inequalities should be solved for P, they are not linear (due to P^{-1}). This can be solved by assuming the following structure of P:

$$P := \begin{bmatrix} Y & N \\ N^T & \star \end{bmatrix} \succ 0 \quad \text{and} \quad P^{-1} := \begin{bmatrix} X & M \\ M^T & \star \end{bmatrix} \succ 0 \tag{G.8}$$

By introducing a proper multiplication matrix (see Appendix F), the constraint $P \succ 0$ can be written as

$$\begin{bmatrix} \boldsymbol{X} & I \\ M^T & 0 \end{bmatrix}^T \begin{bmatrix} Y & N \\ N^T & \star \end{bmatrix} \begin{bmatrix} \boldsymbol{X} & I \\ M^T & 0 \end{bmatrix} = \begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succ 0$$

The structure of (G.8) and the definitions of (G.3) can be substituted into the matrices Ψ and Φ, which then become

$$\Psi = \left[\begin{array}{cc:c:c} A^TY + YA & A^TN & YB_w & C_z^T \\ N^TA & 0 & N^TB_w & 0 \\ \hdashline B_w^TY & B_w^TN & -\gamma I & D_{zw}^T \\ \hdashline C_z & 0 & D_{zw} & -\gamma I \end{array}\right]$$

$$\Phi = \left[\begin{array}{cc:c:c} AX + XA^T & AM & B_w & XC_z^T \\ M^TA^T & 0 & 0 & M^TC_z^T \\ \hdashline B_w^T & 0 & -\gamma I & D_{zw}^T \\ \hdashline C_zX & C_zM & D_{zw} & -\gamma I \end{array}\right]$$

Now since $L = \begin{bmatrix} 0 & I & 0 & 0 \\ B^T & 0 & 0 & D_z^T \end{bmatrix}$ and $K = \begin{bmatrix} 0 & I & 0 & 0 \\ C & 0 & D_w & 0 \end{bmatrix}$ it holds that the second row of both W_L and W_K contains solely zeros:

$$W_L = \begin{bmatrix} W_{l1} & 0 \\ 0 & 0 \\ 0 & 1 \\ W_{l2} & 0 \end{bmatrix} \quad \text{and} \quad W_K = \begin{bmatrix} W_{k1} & 0 \\ 0 & 0 \\ W_{k2} & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore the second rows and columns in Ψ and Φ do not influence $W_K^T\Psi W_K \prec 0$ and $W_L^T\Phi W_L \prec 0$ and can thus be canceled. Furthermore, if one defines $\mathcal{N}_X = \begin{bmatrix} W_{l1} \\ W_{l2} \end{bmatrix}$ and $\mathcal{N}_Y = \begin{bmatrix} W_{k1} \\ W_{k2} \end{bmatrix}$, where the columns of $\mathcal{N}_X$ span $\ker(\begin{bmatrix} B^T & D_z^T \end{bmatrix})$ and the columns of $\mathcal{N}_Y$ span $\ker(\begin{bmatrix} C & D_w^T \end{bmatrix})$, this finally results in the following set of feasibility LMIs:

$$\begin{bmatrix} \mathcal{N}_X & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} AX + XA^T & B_w & XC_z^T \\ C_zX & D_{zw} & -\gamma I \\ B_w^T & -\gamma I & D_{zw}^T \end{bmatrix} \begin{bmatrix} \mathcal{N}_X & 0 \\ 0 & I \end{bmatrix} \prec 0 \tag{G.9a}$$

$$\begin{bmatrix} \mathcal{N}_Y & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} A^TY + YA & YB_w & C_z^T \\ B_w^TY & -\gamma I & D_{zw}^T \\ C_z & D_{zw} & -\gamma I \end{bmatrix} \begin{bmatrix} \mathcal{N}_Y & 0 \\ 0 & I \end{bmatrix} \prec 0 \tag{G.9b}$$

$$\begin{bmatrix} X & I \\ I & Y \end{bmatrix} \succ 0, \tag{G.9c}$$

It is clear to see that these inequalities are linear in their decision variables X and Y, and can thus easily be solved. If these LMIs are feasible for a certain $\gamma > 0$, then there exists a controller which can achieve this γ. By minimizing over γ one can find optimal values of X and Y, which guarantees the existence of a controller at this optimal value of γ.

Controller synthesis

The obtained X and Y can then be used to derive P according to

$$MN^T = I - XY, \tag{G.10}$$

which is solvable by using e.g. the SVD-decomposition of $I - XY$. In was shown in the previous section that this P guarantees the existence of a controller, which can again be found using (C.16):

$$\begin{bmatrix} \mathcal{A}^T P + P\mathcal{A} & P\mathcal{B} & \mathcal{C}^T \\ \mathcal{B}^T P & -\gamma I & \mathcal{D}^T \\ \mathcal{C} & \mathcal{D} & -\gamma I \end{bmatrix} = \Psi + K^T \Theta^T L_P + L_P^T \Theta K \prec 0 \tag{G.11}$$

Now all terms on the left hand side of this inequality are known, except for the controller Θ. Hence (G.11) is linear in its design variable Θ. So using the obtained P and γ, this LMI will result in a controller Θ with an $\mathcal{L}_2$-gain smaller or equal to γ.

Discussion

Compared to the method of the linearizing change of variables described in Appendix F, the above described elimination method has a major drawback, which is the reason why the former is more commonly used. Namely, the elimination method will not work for multichannel or mixed problems. Take for example a mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem, where the elimination lemma will result in two sets of feasibility LMIs. Feasibility of the first set only guarantees the *existence* of an $\mathcal{H}_\infty$-controller Θ_∞ and the second only guarantees the *existence* of an $\mathcal{H}_2$-controller Θ_2, however, these two are not necessarily equal! Even if these LMIs are solved for a common P, there is no guarantee that the controller synthesis LMI can find one $\Theta = \Theta_\infty = \Theta_2$, since this equality is not taken into account in the first feasibility LMI. In the LCV method however, all controller variables are explicitly present in the same LMI which checks the feasibility, hence the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ LMIs can be solved simultaneously using a common controller (and necessarily a common P). This controller is then guaranteed to satisfy both the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ constraints, which is impossible with the elimination method.