MASTER

# Identification of hydraulic conductivity in groundwater modeling

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# Identification of Hydraulic Conductivity in Groundwater Modeling

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*To my parents.*

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# Abstract

Saturated flow in groundwater systems is modeled by the following diffusion equation.

$$Q(x)\frac{\partial w}{\partial t} = \nabla \cdot (P(x)\nabla w) + R(x,t) \qquad \text{in } \Omega, \tag{1}$$

where $Q$ is the specific storage, $w$ is the piezometric head, $P$ is the hydraulic tensor and $R$ is the source/sink term for the flow. Here the domain $\Omega$ is such that $\Omega \subset \mathbf{R}^n$, $n = 2$ or 3.

In practical groundwater modeling one is interested in obtaining reliable values for $P, Q$ and $R$ from the knowledge of the values of $w$ inside $\Omega$ together with boundary values of $P$ and $w$. In this thesis we follow and explain the results obtained by the group of Professor Ian Knowles at the University of Alabama Birmingham on the recovery of $P, Q$ and $R$. We start by considering the existence and uniqueness of solutions to the forward Dirichlet boundary value problem obtained from the model problem (1) using a Laplace transform. Next, we state the inverse problem, give different uniqueness results and explain the related difficulties. Then we establish the Gateaux derivatives of the functionals used to detect the coefficients. We study in more details the case where $Q = 0$ and $R$ is given and we end by implementing the related conjugate gradient method and give some numerical results.

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# Chapter 1

# Introduction

One of the main objectives of groundwater modeling is to determine the properly working earth models in order to adequately explain the hydrogeological observations. From the mathematical point of view, such solutions can be found by optimization. Frequently, the inverse methods are used to determine the optimal parameter values of the groundwater models. Adjusting the estimates of the model parameters minimizes a special objective function, as a measure of the misfit or error, characterizing the deviation between the measured and calculated data.

## 1.1 Model problem

In this thesis we study the identification of hydraulic conductivities in groundwater modeling. We follow the work by Professor Ian Knowles, see [16, 15, 14, 13]. For simplicity we consider saturated flow in groundwater systems, modeled by the following equation,

$$Q(x)\frac{\partial w}{\partial t} = \nabla \cdot (P(x)\nabla w) + R(x,t) \tag{1.1.1}$$

over $x$ in a bounded region $\Omega \subset \mathbf{R}^n$, $n = 2$, or 3, and for $t > 0$. Here, $Q$ is the specific storage, $w$ is the piezometric head, $P$ is the hydraulic conductivity tensor and $R$ is the source/sink term for the flow.

A fundamentally important part of the practical modeling process is the full reconstruction problem, that is, the problem of obtaining reliable estimates for all of the various coefficient functions appearing in equations (1.1.1) from field measurements of the quantities $w$ (and some boundary data on $P$). The full parameter reconstruction problem for the groundwater model is a computationally complicated inverse problem, requiring the recovery of some 20 coefficient functions in the two-dimensional case alone. Methods that have been previously employed on the inverse groundwater problem typically focus only on the recovery of a scalar hydraulic conductivity, and range over educated guesswork (referred to as "trial and error calibration" in the hydrology literature) to various attempts at "automatic calibration". These calibration methods solve the groundwater model problem inversely by iteratively adjusting the unknown coefficient functions, for example, hydraulic conductivities until the solution matches the known piezometric head values, see [3]. Another approach is to reformulate the problem as an optimization problem which can be done in several ways. One can work directly to minimize the "equation error" as in [9]. Another optimization route makes use of a general idea of Tikhonov regularization, see [19]. All Tikhonov regularization methods make use of a regularization parameter whose critical value must be known quite accurately for the method to be effective. This general class methods is less effective because of lack of reliable methods for determining this critical value in practical situations, a problem which can even be more pronounced in the aquifer case due to the uncertainties in available data. A further point worthy of note is that in the current literature there are few universally applicable techniques for recovering the specific storage and even fewer viable methods available [3] for objectively assigning values to a time dependent recharge term. Once again rainfall is not readily measured as a local phenomenon, and the effect of supply and discharge from underground sources is even more difficult to measure directly. There are also essentially no viable methods for objectively obtaining the full hydraulic conductivity tensor. It is common in much of the literature to use only steady state flow data to compute groundwater parameters.

## 1.2 Organization of thesis

In chapter 2 we consider the forward Dirichlet boundary value problem (2.1.1). We reformulate this problem in distributional sense and prove existence and uniqueness results using Lax-Milgram and Riesz representation theorems. We also study regularity requirements of the coefficient functions and the boundary data in order to improve the regularity of the solution. In chapter 3 we consider some of the inverse problems for obtaining the coefficient function $p(x)$ for given $q(x)$ and $f(x)$ depending on the nature of given data, that is, if either boundary data is given or interior data for $u(x)$. For boundary data, we study results due to Calderón for identification of electrical conductivities using the so-called Dirichlet-to-Neumann map. We report some key papers where uniqueness results for identification of conductivities have been addressed. For interior data we study uniqueness results for three different cases, that is, uniqueness results for identification of one of the coefficient functions when given two of them, uniqueness results for identification of two of the coefficient functions when given one and the unique identification of a full set of the three coefficient functions. For the unique identification of the hydraulic conductivity $p$ in two dimension, results by G. Alessandrini play an important role. In chapter 4 we study in more details the case $q = 0$ and $f$ given. We define the functional for reconstruction of $p(x)$ from interior data $u(x)$ and also discuss the properties of the functional. We establish the Gateaux derivatives of this functional and explain the related conjugate gradient algorithm for its minimization. We include the numerical results in chapter 5 and introduce the elasticity problem in chapter 6. Finally in chapter 7 we give concluding remarks together with future work.

# Chapter 2

# The Forward Problem

In this chapter we consider the elliptic model problem(1.1.1), and study the existence and uniqueness of a solution to this problem.

## 2.1 Variational formulation

Assume for the moment that we have a heterogeneous isotropic flow, with the scalar function $P = P(x)$ representing the hydraulic conductivity. The basic idea is to Laplace transform data from solutions of (1.1.1) to solutions values $u(x, \lambda)$ of the elliptic equation

$$-\nabla \cdot (P(x)\nabla u) + \lambda Q(x)u = F(x, \lambda), \qquad x \in \Omega. \tag{2.1.1}$$

where $\lambda$ is the transform parameter, and $F$ depends on $R, Q$ and $\lambda$ in such a way that $R$ can be recovered if $Q$ and $F$ are known. We will show how the triple $(P, Q, F)$ can be found (under suitable conditions on the solutions $u(x, \lambda)$ and the form of $R$) as the unique global minimum of a convex functional which will be constructed in the forthcoming chapters.

Assume that $(P, Q, F) = (p, q, \tilde{f})$ are given, where $\tilde{f}$ is a function of $x$ only. That is, we assume that the source term $R(x,t)$ in equation (1.1.1) is constant in time. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with a $C^2$ boundary. Consider the differential equation

$$\begin{aligned} L_{p,q}\tilde{u} = -\nabla \cdot (p(x)\nabla\tilde{u}) + \lambda q(x)\tilde{u} &= \tilde{f}(x), \qquad x \in \Omega; \quad \lambda > 0. && (2.1.2) \\ \tilde{u} &= g(x), \qquad x \in \partial\Omega && (2.1.3) \end{aligned}$$

where $\partial\Omega$ denotes the boundary of the domain $\Omega$ and the coefficient functions are real-valued and have the following regularity conditions.

$$\tilde{f} \in \mathbf{C}^0(\bar{\Omega}), \quad q \in \mathbf{C}^0(\bar{\Omega}), \quad p \in \mathbf{C}^0(\bar{\Omega}). \tag{2.1.4}$$

Suppose $q(x) \geq 0$; $f \geq 0$; and $p$ satisfies

$$p(x) \geq \nu > 0, \qquad x \in \Omega \tag{2.1.5}$$

for some constant $\nu$.

Generally it is not possible to write an explicit formula for the classical solution $\tilde{u}$ to the problem (2.1.2). Hence we resort to the concept of weak solutions to elliptic problems. In the sequel we shall use the general Sobolev spaces $H^k(\Omega)$, where a function $\phi \in H^k(\Omega)$ if $\phi \in L^2(\Omega)$ and $D^\alpha\phi \in L^2(\Omega)$ for $|\alpha| \leq k$, $k \in \mathbf{R}$.

Let $g \in H^{\frac{1}{2}}(\partial\Omega)$, then using the inverse trace theorem, there exists $z \in H^1(\Omega)$ such that,

$$\mathcal{T}(z) = g, \qquad \text{on} \ \ \partial\Omega \tag{2.1.6}$$

where $\mathcal{T}$ denotes the trace operator.

If we make a change of variables, $\tilde{u} = u + z$ in (2.1.2), then the following Dirichlet problem arises,

$$\begin{cases} L_{p,q}u = f, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1.7}$$

where $f = (\tilde{f} - L_{p,q}z) \in H^{-1}(\Omega)$, the dual of $H^1(\Omega)$.

For the moment let us assume that $u$ is a smooth solution. We multiply the partial differential equation (2.1.7) by a smooth test function $v$, $v \in C_0^\infty(\Omega)$, and integrate over the domain $\Omega$. We find that;

$$\int_\Omega -\nabla \cdot (p(x)\nabla u)v + \lambda q(x)uvdx \quad = \quad \int_\Omega f(x)vdx \qquad \forall v \in C_0^\infty(\Omega).$$

Then using integration by parts we obtain,

$$\int_{\partial\Omega} -p(x)\nabla u \cdot nvdS + \int_\Omega p(x)\nabla u \cdot \nabla v + \lambda q(x)uvdx = \int_\Omega f(x)vdx,$$

where dS denotes a surface element on the boundary $\partial\Omega$ and n is the unit outward normal vector to $\partial\Omega$.

For all $v \in C_0^\infty$ we obtain,

$$\int_\Omega p(x)\nabla u \cdot \nabla v + \lambda q(x)uvdx = \int_\Omega f(x)vdx \qquad \forall v \in H_0^1(\Omega) = V, \tag{2.1.8}$$

since $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$.

Equation (2.1.8) can be written as,

$$a(u,v) = \langle f, v\rangle, \qquad \forall v \in V, \tag{2.1.9}$$

where,

$$a(u,v) \quad = \quad \int_\Omega p(x)\nabla u \cdot \nabla v + \lambda q(x)uvdx, \tag{2.1.10}$$

$$\langle f, v\rangle \quad = \quad \int_\Omega f(x)vdx \qquad \forall v \in H_0^1(\Omega) = V. \tag{2.1.11}$$

The expression $\langle ., .\rangle$ represents a duality pairing between elements in $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, the dual to $H_0^1(\Omega)$. Equation (2.1.9) is called a variational problem associated to the Dirichlet problem (2.1.7).

**Definition 2.1.1.** A function $\tilde{u}$ is said to be a weak or generalized solution of the Dirichlet problem (2.1.2), (2.1.3) if it satisfies $\tilde{u} = u + z$, where $u$ solves the variational problem (2.1.9) and $z$ is defined by (2.1.6).

## 2.2 Existence and uniqueness

In order to investigate the existence and uniqueness of a solution to the variational problem (2.1.9), we require the following definitions.

**Definition 2.2.1.** Let $V$ be a Hilbert space and $\mathbf{K}$ be a scalar field.

(i) A mapping $a(.,.) : V \times V \to \mathbf{K}$ is called a sesquilinear form if;

- a(.,.) is linear in the first argument. That is, for $\lambda, \mu \in \mathbf{K}$

$$a(\lambda u + \mu v, w) = \lambda a(u, w) + \mu(v, w) \qquad \forall u, v, w \in V$$

- a(.,.) is antilinear in the second argument. That is, for $\lambda, \mu \in \mathbf{K}$,

$$a(u, \lambda v + \mu w) = \bar{\lambda} a(u, w) + \bar{\mu}(v, w) \qquad \forall u, v, w \in V$$

In the rest of the definitions we assume that the scalar field $\mathbf{K} = \mathbf{R}$. Then in this case $a(.,.)$ is linear in both the first and second arguments and is called a bilinear form on $V$.

(ii) A bilinear form $a(.,.)$ is called bounded if and only if there is a constant $M > 0$ such that ,

$$|a(u, v)| \leq M \|u\|_V \|v\|_V \qquad \forall u, v \in V$$

where $\|.\|_V$ denotes the norm in the space $V$.

(iii) A bilinear form a(.,.) is called $V-$elliptic(or coercive) if and only if there is a constant $\alpha > 0$ such that;

$$a(u, v) \geq \alpha \|u\|_V^2 \qquad \forall u \in V$$

The following theorem gives the existence and uniqueness of the solution to the variational problem: find $u \in H_0^1(\Omega)$ such that equation (2.1.9) holds.

**Theorem 2.2.1.** *Let $\Omega \subset \mathbf{R}^n$ be a bounded $C^2$ domain and let $f \in L^2(\Omega)$. Then there exists a unique weak solution $u \in H_0^1(\Omega)$ fulfilling*

$$a(u, v) = \langle f, v \rangle, \qquad \forall v \in H_0^1(\Omega).$$

*This solution also satisfies the following estimate.*

$$\|u\|_{H^1(\Omega)} \leq c\|f\|_{L^2(\Omega)}, \tag{2.2.1}$$

*where the constant $c$ does not depend on $f$.*

*Proof.* The proof of Theorem 2.2.1 follows from the following Lemma.

**Lemma 2.2.2.** *(Lax-Milgram) Assume that $a(.,.)$ is a symmetric ,bounded, $V-$elliptic bilinear form on $V$. Then for any $f \in V^*$, $V^*$ the dual of $V$, there is precisely one $u \in V$ such that (2.1.9) holds. This solution satisfies an estimate*

$$\|u\|_V \leq c\|f\|_V*, \tag{2.2.2}$$

*where $c > 0$ is a constant.*

Before we provide a proof for Lemma 2.2.2, we need the following Lemma which will be important for the construction of the proof.

**Lemma 2.2.3.** *(Riesz) If $f \in V^*$ there exists a unique element $u_f \in V$ such that we have,*

$$\begin{aligned} \langle f, v \rangle &= (v, u_f)_V \qquad \forall v \in V. \\ \textit{with } \|f\|_{V^*} &= \|u_f\|_V, \end{aligned} \tag{2.2.3}$$

*where $(.,.)_V$ represents an inner product on the space $V$.*

*Proof.* This theorem follows in two parts,

(a) Uniqueness of $u_f$.

If $\langle f, v\rangle = (v, u_f^1)_V = (v, u_f^2)_V \qquad \forall v \in V$, we have,

$$(v, u_f^1 - u_v^2)_V \;=\; 0 \qquad\qquad \forall v \in V,$$

which implies that,

$$u_f^1 \;=\; u_f^2.$$

(b) Existence of $u_f$.

If $f = 0$ then $u_f = 0$ satisfies (2.2.3).

Assume $f \neq 0$ and let $N = Ker f = \{v; v \in V, \langle f, v\rangle = 0\}$ (the kernel of $f$)

$N$ is a closed subspace of $V$. If $N^\perp$ is the orthogonal subspace of $N$ in $V$, then $\dim N^\perp = 1$.

Let $u_0 \in N^\perp$ and put

$$u_f = \frac{\langle f, u_0\rangle}{\|u_0\|_v^2} u_0, \qquad\qquad u_f \in N^\perp$$

Then we have the following,

(1) if $v \in N$, then $(v, u_f)_V = 0 = \langle f, v\rangle$,

(2) if $v \in N^\perp$, then $v = \lambda u_0$ (since $\dim N^\perp = 1$)

This implies that $(v, u_f)_V = \lambda\langle f, u_0\rangle = \langle f, \lambda u_0\rangle = \langle f, v\rangle$, $\lambda \in \mathbf{R}$

(3) if $v \in V = N \oplus N^\perp$, $v = v_N + v_{N^\perp}$ and we have again (1) - (2) above. Then we have the result,

$$\langle f, v\rangle = (v, u_f)_V$$

In addition,

$$\|f\|_{V^*} = \sup_{\|v\|_V = 1} |\langle f, v\rangle| \;=\; \sup_{\|v\|_V = 1} |(v, u_f)_V| \leq \|u_f\|_V.$$

Also,

$$\|f\|_{V*} \geq |\langle f, \frac{u_f}{\|u_f\|_V}\rangle| \quad = \quad |(\frac{u_f}{\|u_f\|_V}, u_f)_V| = \|u_f\|_V.$$

Therefore

$$\|f\|_{V^*} \quad = \quad \|u_f\|_V.$$

□

Having established the Riesz representation Lemma, we now proceed to provide a proof for the Lax-Milgram theorem.

*Proof.* Lax-Milgram.

We reformulate (2.1.9) as an operator equation in $V$. For any fixed $v$, the mapping $u \rightarrow a(u,v)$ is a bounded linear functional on $V$, hence by Riesz representation theorem, there is a unique $w \in V$ such that $a(u,v) = (u,w) \quad \forall u \in V$.

We introduce the operator $A_V : V \rightarrow V$ defined by,

$$A_V u = w$$

Note that $A_V$ is bounded and this follows directly from the boundedness of $a(.,.)$. On the other hand, again form the Riesz representation theorem, there exists $b \in V$ such

$$\langle f, v\rangle = (v,b)_V \qquad\qquad \forall v \in V.$$

Setting for $u \in V$,

$$Tu = u - \rho(A_V u - b),$$

the problem reduces to finding the fixed points of T.

We have,

$$\begin{aligned}
\|Tu_2 - Tu_1\|_V^2 &= \|(u_2 - u_1) - \rho(A_V u_2 - A_V u_1)\|_V^2 \\
&= \|u_2 - u_1\|_V^2 - 2\rho a(u_2 - u_1, u_2 - u_1) + \rho^2 \|A_V(u_2 - u_1)\|_V^2 \\
&\leq (1 - 2\rho\alpha + \rho^2 M)\|u_2 - u_1\|_V^2.
\end{aligned}$$

Hence $T$ is a strict contraction in $V$ for $0 < 2\alpha/\rho M$, and as a result admits a unique fixed point $u$. The variational problem (2.1.9) therefore has a unique weak solution.

For this solution, using coercivity of the bilinear form $a(.,.)$, we have

$$\|u\|_V^2 \leq \frac{1}{\alpha}a(u,u) = \frac{1}{\alpha}\langle f, u\rangle.$$

Then by using Cauchy Schwartz inequality we have,

$$\|u\|_V^2 \leq \frac{1}{\alpha}\|u\|_V\|f\|_{V^*}.$$

Hence,

$$\|u\|_V \leq \frac{1}{\alpha}\|f\|_{V^*}.$$

□

Next we consider $a(.,.)$ defined by equation (2.1.10) to check whether the assumptions of Lax-Milgram Lemma are satisfied on the space $V = H_0^1(\Omega)$.

(i) Symmetry of $a(.,.)$.

$$\begin{aligned} a(u,v) &= \int_\Omega p(x)\nabla u\nabla v + \lambda q(x)uvdx \\ &= \int_\Omega p(x)\nabla v\nabla u + \lambda q(x)vudx \\ &= a(v,u). \end{aligned}$$

Hence $a(.,.)$ is symmetric

(ii) $a(.,.)$ is bilinear. Let $\alpha, \beta \in \mathbf{R}$, and $u, v, w \in V$. Then,

$$\begin{aligned} a(\alpha u + \beta v, w) &= \int_\Omega p(x)\nabla(\alpha u + \beta v)\nabla w + \lambda q(x)(\alpha u + \beta v)wdx \\ &= \alpha\int_\Omega p(x)\nabla u\nabla w + \lambda q(x)uwdx \\ &\quad +\beta\int_\Omega p(x)\nabla v\nabla w + \lambda q(x)vwdx \\ &= \alpha a(u,v) + \beta a(v,w). \end{aligned}$$

For $\gamma, \mu \in \mathbf{R}$ we have,

$$\begin{aligned}
a(u, \gamma v + \mu w) &= \int_\Omega p(x)\nabla u(\gamma v + \mu w) + \lambda q(x) u(\gamma v + \mu w) dx \\
&= \gamma \int_\Omega p(x)\nabla u \nabla v + \lambda q(x) uv dx \\
&\quad + \mu \int_\Omega p(x)\nabla u \nabla w + \lambda q(x) uw dx \\
&= \gamma a(u, v) + \mu a(v, w).
\end{aligned}$$

The above two results justify the claim that $a(.,.)$ is a bilinear form on $V$.

(iii) Boundedness of $a(.,.)$

$$|a(u, v)| = |\int_\Omega p(x)\nabla u \nabla v + \lambda q(x) uv dx|$$

Using the triangle inequality we obtain,

$$|a(u, v)| \leq \int_\Omega |p(x)||\nabla u||\nabla v| + |\lambda||q(x)||u||v| dx$$

Then using Cauchy Schwartz inequality gives,

$$\begin{aligned}
|a(u, v)| &\leq \sup_{x\in\Omega} |p(x)| \; \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + |\lambda| \sup_{x\in\Omega} |q(x)| \; \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\
&\leq \left( \sup_{x\in\Omega} |p(x)| + |\lambda| \sup_{x\in\Omega} |q(x)| \right) \|u\|_V \|v\|_V \\
&= M \|u\|_V \|v\|_V.
\end{aligned}$$

Hence the bilinear form $a(.,.)$ is bounded.

(iv) $a(.,.)$ is $V-$elliptic.

$$a(u, u) = \int_\Omega p(x)\nabla u \nabla u + \lambda q(x) uu$$

Since $\lambda > 0$ and $q > 0$, we have

$$\begin{aligned}
a(u, u) &\geq \int_\Omega p(x)|\nabla u|^2 \\
&= \frac{1}{2}\int_\Omega p(x)|\nabla u|^2 + \frac{1}{2}\int_\Omega p(x)|\nabla u|^2,
\end{aligned}$$

Using Poincaré inequality we obtain,

$$\begin{aligned} a(u,u) &\geq \frac{1}{2}\inf_{x\in\Omega} p(x)\left(\int_\Omega |\nabla u|^2 + C\int_\Omega u^2\right) \\ &\geq \alpha\|u\|_V^2, \end{aligned}$$

where $\alpha = \frac{1}{2}\inf_{x\in\Omega} p(x) \times \min\{1, C\}$.

Hence $a(.,.)$ is a symmetric, bounded and $V-$elliptic bilinear form on $V = H_0^1(\Omega)$.

(v) The function $F$ defined by $F(v) = \langle f, v\rangle$ is linear and bounded.

Let $\alpha, \beta \in \mathbf{R}$ and $v, w \in V$. Then,

$$\begin{aligned} F(\alpha v + \beta w) &= \int_\Omega f(x)(\alpha v + \beta w), \\ &= \alpha\int_\Omega f(x)v + \beta\int_\Omega f(x)w, \\ &= \alpha F(v) + \beta F(w), \end{aligned}$$

which shows that $F$ is linear.

$$\begin{aligned} |F(v)| &= |\int_\Omega f(x)v|, \\ &= \int_\Omega |f(x)|\ |v|, \\ &= \leq \|f\|\|v\|. \end{aligned}$$

Hence $F$ is bounded.

By Lax-Milgram Lemma, the variational problem (2.1.9) has a unique weak or generalized solution $u \in H_0^1(\Omega)$. The estimate (2.2.1) follows from (2.2.2). □

## 2.3 Regularity

We note that we can control the regularity of solutions $u$ to (2.1.7) depending on the regularity of the coefficient functions $p, q, f$ and the boundary $\partial\Omega$. This regularity theory can be found in [18].

**Theorem 2.3.1.** *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with $\mathbf{C}^\infty$ boundary $\partial\Omega$. Let $m \in \mathbf{R}$ and $k \in \mathbf{N}$ satisfy $m > \frac{1}{2}n + k$. Then*

$$H^m(\Omega) \hookrightarrow C^k(\bar{\Omega})$$

Theorem (2.3.1) tells us that the Sobolev space $H^m(\Omega)$ is continuously embedded in $C^k(\bar{\Omega})$, where,

$$C^k(\bar{\Omega}) = \{v : D^\alpha v \in \mathbf{C}(\bar{\Omega}), \forall\alpha, \text{ with } |\alpha| \leq k\}.$$

For $(x_1, x_2, \ldots, x_n) \in \mathbf{R}^n$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $D^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \ldots \partial_{x_n}^{\alpha_n}$.
For the proof of Theorem 2.3.1, see [8].
In two dimension, if the right hand side function f, is such that $f \in H^{1+s}(\Omega)$, then from standard theory of elliptic partial differential equations, the solution $u$ is such that $u \in H^{3+s}(\Omega)$ for $s > 0$. Using theorem (2.3.1), $u \in \mathbf{C}^2(\bar{\Omega})$ and $f \in \mathbf{C}^0(\bar{\Omega})$. If the solution is such that $u \in \mathcal{C}^2(\bar{\Omega})$, then it satisfies the elliptic problem (2.1.2) in a classical sense. In order to see this we consider equation (2.1.8) and proceed with an integration by parts as follows.

$$\begin{aligned} \int_\Omega p(x)\nabla u \cdot \nabla v + \lambda q(x)uvdx &= \int_\Omega f(x)vdx \qquad \forall v \in H_0^1(\Omega) \\ \int_{\partial\Omega} p(x)\nabla u \cdot nvdx + \int_\Omega -\nabla \cdot (p(x)\nabla u)v + \lambda q(x)u\ v &= \int_\Omega f(x)vdx. \end{aligned}$$

This implies that,

$$\int_\Omega \left(-\nabla \cdot (p(x)\nabla u) + \lambda q(x)u - f(x)\right) vdx = 0 \qquad \forall v \in H_0^1(\Omega) \qquad (2.3.1)$$

If we make the substitution $\phi = -\nabla \cdot (p(x)\nabla u) + \lambda q(x)u - f(x)$, then we can see from equation (2.3.1) that $\phi$ is orthogonal to all elements $v \in H_0^1(\Omega)$, that is, $\phi \in \overline{H_0^1(\Omega)}^\perp$. Since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$, then $\overline{H_0^1(\Omega)}^\perp = L^2(\Omega)^\perp$. Therefore $\phi \in L^2(\Omega)^\perp$ which implies that $\phi = 0$ in $L^2(\Omega)$ and hence $u$ satisfies,

$$\begin{aligned} -\nabla \cdot (p(x)\nabla u) + \lambda q(x)u &= f(x) \qquad a.e\ \ x \in \Omega, \qquad (2.3.2) \\ u &= 0 \qquad x \in \partial\Omega. \end{aligned}$$

Since $q(x), p(x), f(x)$ are all continuous, the equality in equation (2.3.2) holds everywhere in the domain $\Omega$. Hence $u$ satisfies the Dirichlet problem (2.1.7) in a classical sense.

The solution of the variational problem (2.1.9) is also the solution of the optimization problem, find $u \in V$ satisfying,

$$J(u) = \inf_{v \in V} J(v) \tag{2.3.3}$$

where $J : V \to \mathbf{R}$ is the quadratic function defined by,

$$J(v) = \frac{1}{2} a(v, v) - \langle f, v \rangle \tag{2.3.4}$$

Indeed, the gradient $J'(u)$ at $u$,

$$\begin{aligned} J'(u)[v] &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [J(u + \varepsilon v) - J(u)] \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \frac{1}{2} a(u + \varepsilon v, u + \varepsilon v) - \langle f, u + \varepsilon v \rangle - \frac{1}{2} a(u, u) + \langle f, u \rangle \right] \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ \varepsilon a(u, v) - \varepsilon \langle f, v \rangle + \frac{\varepsilon^2}{2} a(v, v) \right] \\ &= a(u, v) - \langle f, v \rangle \end{aligned}$$

The optimality condition $J'(u) = 0$ for the solution of problem (2.3.3) is thus,

$$a(u, v) = \langle f, v \rangle \qquad \forall v \in V, \tag{2.3.5}$$

which is nothing but the problem (2.1.9).

The functional (2.3.4) represents the potential energy of the system described by (2.1.2). This motivates the functional that we will use in section (4.1) for the inverse problem.

# Chapter 3

# The inverse problem

In this section we consider the recovery of the coefficient functions $f, p, q$ in equation (2.1.2) for given values of $\lambda$ and either few or many boundary data $g(x)$ or the values of few solutions $u(x)$ in the interior of the region of interest.

## 3.1 Recovery from boundary measurements.

The task is to reconstruct the coefficient function $p(x)$ from boundary measurements using the so-called Dirichlet to Neumann map which we will describe later. We consider the steady-state aquifer with zero recharge term, that is , $q(x) = 0$ and $f(x) = 0$. Then the problem (2.1.2) reduces to,

$$\begin{cases} \nabla \cdot (p(x)\nabla u(x)) = 0, & x \in \Omega; \\ u(x) = g(x), & x \in \partial\Omega. \end{cases} \tag{3.1.1}$$

The inverse problem for the model (3.1.1) was first addressed by A.P Calderón in 1980. The problem he proposed is whether it is possible to determine the conductivity of a body by making current and voltage measurements at the boundary, that is, the Dirichlet to Neumann map. This problem arises for instance in geophysical prospection. More recently this non-invasive inverse method, also refered to as Electrical Impedance Tomography, has been proposed as a possible diagnostic tool in medical imaging, see [27, 11]. One concrete clinical application, which seems to be very promising, is in the monitoring of pulmonary edema, see [12]. We now describe more precisely the mathematical problem and Calderón's approach to determine the coefficient function $p(x)$.

Let $\Omega \subseteq \mathbf{R}$ be a bounded domain with smooth boundary. The conductivity of $\Omega$ is represented by $p(x)$ such that $p(x) \geq \nu > 0, \quad x \in \Omega$, that is , $p(x)$ is strictly positive.

Given $g(x) \in H^{1/2}(\partial\Omega)$ on the boundary, the induced $u \in H^1(\Omega)$ solves the Dirichlet problem (3.1.1). We associate to $u$ the trace of its derivative $\frac{\partial u}{\partial n} \in H^{-1/2}(\partial\Omega)$, where $n$ denotes the unit outer normal to $\partial\Omega$. The Dirichlet to Neumann map $\Lambda_p : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is given by;

$$\Lambda_p(g(x)) = p(x)\frac{\partial u(x)}{\partial n} \qquad\qquad x \in \partial\Omega,$$

The inverse problem is to determine $p(x)$ from the knowledge of $\Lambda_p$. It is difficult to find a systematic way of prescribing measurements of $g(x)$ at the boundary to be able to find the conductivity $p(x)$. Calderón instead took a different route.

Consider problem (3.1.1), multiply both sides by $u(x)$ and integrate over $\Omega$. This gives,

$$\int_\Omega \nabla \cdot (p(x)\nabla u)u \; dx \;\; = \;\; 0,$$

which we write after integration by parts as,

$$\int_{\partial\Omega} p(x)\nabla u \cdot nu \; dS - \int_\Omega p(x)|\nabla u|^2 \; dx \;\; = \;\; 0. \tag{3.1.2}$$

From (3.1.2) Calderón considers the quadratic form defined by,

$$Q_p(g) = \int_\Omega p(x)|\nabla u|^2 \; dx = \int_{\partial\Omega} \Lambda_p(g)g \; dS \tag{3.1.3}$$

where $dS$ denotes the surface measure and $u$ is the solution of (3.1.1). $Q_p(g)$ is the quadratic form associated to the linear map $\Lambda_p$. That is, to know $\Lambda_p(g)$ or $Q_p(g)$ for all $g \in H^{1/2}(\partial\Omega)$ is equivalent. If one looks at $Q_p$, the problem is changed to find enough solutions $u \in H^1(\Omega)$ of the equation (3.1.1) in order to find $p(x)$ in the interior.

Next, we give Calderón's procedure for the linearization of the map

$$p \to Q_p$$

and the approximation formula to reconstruct a conductivity $p(x)$ which is apriori close to a constant conductivity. Calderón proved in [6] that the map $Q$ is analytic and the Fréchet derivative of $Q$ at $p = p_0$ in the direction $h$ is given by,

$$dQ|_{p=p_0}(h)(g,w) = \int_\Omega h\nabla u \cdot \nabla v \, dx \tag{3.1.4}$$

where $u, v \in H^1(\Omega)$ solve

$$\begin{cases} \nabla \cdot (p_0 \nabla u) = \nabla \cdot (p_0 \nabla v) = 0, & \text{in } \Omega; \\ u = g \in H^{1/2}(\partial\Omega), \quad v = w \in H^{1/2}(\partial\Omega), & \text{on } \partial\Omega. \end{cases} \tag{3.1.5}$$

Calderón proved that the linearized map is injective in the case $p_0 =$ constant, which is assumed for simplicity to be a constant function 1. The original idea of Calderón is to reduce this problem to the denseness in $L^2(\Omega)$ of the products of gradients of harmonic functions. To explain this, consider the following harmonic functions

$$u = e^{x\cdot\rho}, \qquad v = e^{-x\cdot\bar\rho} \tag{3.1.6}$$

where $\rho \in \mathbf{C}^n$ with

$$\rho \cdot \rho = 0. \tag{3.1.7}$$

We remark that the condition (3.1.7) is equivalent to the following,

$$\begin{aligned} \rho &= \frac{\eta + ik}{2}, & \eta, k \in \mathbb{R}^n \\ |\eta| &= |k|, & \eta \cdot k = 0. \end{aligned} \tag{3.1.8}$$

Substituting the solutions (3.1.6) into equation (3.1.4) for $dQ|_{p_0=1}(h) = 0$, we obtain

$$\begin{aligned} \int_\Omega h\nabla e^{x\cdot\rho}\cdot\nabla e^{-x\cdot\bar{\rho}}\;\; dx &= 0 \\ \int_\Omega -h\rho e^{x\cdot\rho}\cdot\bar{\rho}e^{-x\cdot\bar{\rho}}dx &= 0 \\ \int_\Omega -h\rho\cdot\bar{\rho}e^{x(\rho-\bar{\rho})}dx &= 0 \end{aligned}$$

Using (3.1.8), we obtain,

$$\int_\Omega -h\frac{|k|^2}{2}e^{ikx}dx = 0,$$

which can also be written as,

$$|k|^2(\chi_\Omega h)^\wedge(k) = 0 \qquad \forall k \in \mathbb{R}^n, \tag{3.1.9}$$

where $\chi_\Omega$ denotes the characteristic function of $\Omega$ and $f^\wedge$ denotes the Fourier transform of $f$. Then by the injectivity of the Fourier transform we conclude that $h = 0$. However one can not apply the implicit function theorem to conclude that the map $p \to Q_p$ is invertible near a constant since conditions on the range of $Q$ that would allow the use of the implicit function theorem are either false or not known.

Calderón observed that using the solutions (3.1.6) one can find an approximation for the conductivity $p$ near a constant.

Suppose $p = 1 + h$, with $h$ small enough in $L^\infty$ norm. We are given;

$$G_p := Q_p(e^{x.\rho}|_{\partial\Omega}, e^{-x.\bar{\rho}}|_{\partial\Omega}),$$

where $\rho \in \mathbf{C}^n$. Writing the quadratic form(3.1.3) as a bilinear form, we have

$$G_p = \int_\Omega(1+h)\nabla u\cdot\nabla v + \int_\Omega(1+h)(\nabla\delta u\cdot\nabla v+\nabla u\cdot\nabla\delta v) + \int_\Omega(1+h)\nabla\delta u\cdot\nabla\delta v, \tag{3.1.10}$$

where $u, v$ solve equation (3.1.5) and

$$\begin{aligned} \nabla\cdot(p\nabla(u+\delta u)) &= \nabla\cdot(p\nabla(v+\delta v)) = 0 \qquad \text{in } \Omega \\ \delta u|_{\partial\Omega} &= \delta v|_{\partial\Omega} = 0 \end{aligned} \tag{3.1.11}$$

We note after integration by parts that

$$\begin{aligned}\int_{\Omega}(\nabla\delta u\cdot\nabla v+\nabla u\cdot\nabla\delta v) &= \int_{\partial\Omega}(\nabla v\cdot n\delta u+\nabla u\cdot n\delta v)dS\\ &\quad -\int_{\Omega}(\delta u\Delta v+\delta v\Delta u)dx\\ &= 0,\end{aligned}$$

where $dS$ denotes the surface element and $n$ is the unit outer normal vector to the boundary of the domain $\Omega$. Equation(3.1.10) then becomes,

$$G_p=\int_{\Omega}(1+h)\nabla u\cdot\nabla v+\int_{\Omega}h(\nabla\delta u\cdot\nabla v+\nabla u\cdot\nabla\delta v)+\int_{\Omega}(1+h)\nabla\delta u\cdot\nabla\delta v. \quad (3.1.12)$$

We also note that for the solution $u$, equation(3.1.11) can be written as

$$\begin{cases}\nabla\cdot(p\nabla\delta u)=\nabla\cdot(h\nabla u), & \text{in } \Omega;\\ \delta u=0, & \text{on } \partial\Omega.\end{cases} \quad (3.1.13)$$

The right-hand side of the first equation in (3.1.13) is such that: $\nabla\cdot(p\nabla u)\in H^{-1}(\Omega)$. Thus from standard elliptic estimates we have

$$\|\nabla\delta u\|_{L^2(\Omega)}\leq C\|h(\nabla u)\|_{L^2(\Omega)}$$

for $C>0$. Taking the smallest ball or radius $r$ around $\Omega$ and integrating over this ball for solutions $u$ of the form (3.1.6) we have

$$\|h(\nabla u)\|_{L^2(\Omega)}=\leq C\|h\|_{L^\infty(\Omega)}|k|e^{1/2r|k|}.$$

Thus we have the following estimates for $\delta u$ and $\delta v$.

$$\|\nabla\delta u\|_{L^2(\Omega)},\quad \|\nabla\delta v\|_{L^2(\Omega)}\leq C\|h\|_{L^\infty(\Omega)}|k|e^{1/2r|k|}. \quad (3.1.14)$$

Substituting for $u,v$ into (3.1.10) and using equation(3.1.9) we obtain

$$\widehat{\chi_\Omega p}(k)=-\frac{2G_\gamma}{|k|^2}+R(k)=\hat{F}(k)+R(k),$$

where $F$ is determined by $G_p$ and is therefore known. Using estimate (3.1.14), we have

$$|R(k)| \leq C\|h\|_{L^\infty(\Omega)} e^{r|k|}.$$

In other words we know $\widehat{\chi_\Omega p}(k)$ up to a term that is small for k small enough. More precisely, let $1 < \alpha < 2$. Then for $|k| \leq \frac{2-\alpha}{r} \log \frac{1}{\|h\|_{L^\infty(\Omega)}} =: \alpha$, we have

$$|R(k)| \leq c\|h\|^{\alpha}_{L^\infty(\Omega)}.$$

We take $\hat{\eta}$ a $C^\infty$ cut-off so that $\hat{\eta}(0) = 1$, supp $\hat{\eta}(k) \subset \{k \in \mathbf{R}^n, |k| \leq 1\}$ and $\eta_\alpha(\chi) = \sigma^n \eta(\sigma x)$. Then we obtain

$$\widehat{\chi_\Omega p}(k)\hat{\eta}(k/\sigma) = \frac{-2G_p p}{|k|^2}\hat{\eta}(k/\sigma) + R(k)\hat{\eta}(k/\sigma).$$

Using this result, we get the following estimate

$$|\rho(x)| \leq C\|h\|_{L^\infty(\Omega)}[\log \frac{1}{\|h\|_{L^\infty(\Omega)}}]^n, \tag{3.1.15}$$

where $\rho(x) = (\chi_\Omega p * \eta_\sigma)(x) - (F * \eta_\sigma)(x)$. Inequality (3.1.15) gives an approximation to the smoothed out conductivity, $\chi_\Omega p * \eta_\sigma$ for $h$ sufficiently small and uses the harmonic exponentials for low frequencies.

Does the knowledge of $\Lambda_p$ uniquely determine p? This issue has been addressed by many authors producing hundreds of papers on the subject. We mention just few key papers. After the fundamental work by Calderón, Kohn and Vogelius in [17] answered this uniqueness question in the case where $p$ is piecewise analytic. Their idea is to use highly oscillating solutions near the boundary to prove uniqueness of all the derivatives of the coefficient $p$. Then they use the analyticity of $p$ to conclude the uniqueness inside.

Sylevester and Uhlmann in [27, 26] constructed complex geometrical optics solutions for the Schrödinger equation associated to a bounded potential. These solutions behave like Calderón's complex exponential solutions for large complex frequencies.

For dimension $n \geq 3$, Sylevester and Uhlmann used geometrical optics solutions to prove global uniqueness result in [26]. In the dimension $n \geq 3$, the uniqueness is still open in the large space $L^\infty(\Omega)$. In two dimension A. Nachman proved in [20] that one can uniquely determine conductivities in $W^{2,q}(\Omega)$, for some $q > 1$, from $\Lambda_p$. Recently, Brown and Uhlmann in [5] have improved uniqueness for a less regularity requirement than A. Nachman. Finally, more recently Astala and Paivarinta solved completely this uniqueness question in $L^\infty(\Omega)$, see [4]. Hence the Calderón problem is completely solved in the two dimensional case.

## 3.2 Recovery from interior measurements.

In groundwater modeling, one is usually given the piezometric head data $u(x)$ in the interior of the region of interest together with boundary values of the conductivity $p$. This data is measured practically using piezometric head methods which are available to geologists. In such cases one is interested in constructing functionals that can be used to recover the coefficient functions $f, p, q$ for given values of $\lambda$ using the available interior information on $u(x)$.

In the remaining part of this thesis, we consider the inverse problem with the assumption that we know the solution $u(x)$ in the interior of the region of interest.

## 3.3 Uniqueness for the inverse problem

In this section we answer the question of whether the knowledge of the interior values of the solution $u = u_{p,q,f}$ to the elliptic problem (2.1.2) guarantees the unique determination of any one of these coefficient functions.

It is worth noting that under conditions (2.1.4) and (2.1.5), and assuming the condition that the Dirichlet data $g(x) > 0$, the solutions $u_{p,q,f,\lambda}$ are positive everywhere in $\Omega$. This result follows from the strong maximum principle for elliptic equations. Three cases are considered for the discussion on uniqueness, that is, one-coefficient case, two-coefficient case and three-coefficient case.

1. **The one-coefficient case**.

   Let $P, Q, F$ be the coefficient functions for which the inverse problem is considered.

   If any two of these functions are known , then the remaining coefficient can be determined from the knowledge of the solution $u_{P,Q,F}$ for a given value of $\lambda$. Three cases arise in this situation.

   a. Given $P$ and $Q$.

      Assume that the functions $W = u_{P,Q,F,\lambda}$ and $w = u_{P,Q,f,\lambda}$ satisfy;

$$
\begin{aligned}
-\nabla \cdot (P(x)\nabla W) + \lambda Q(x)W &= F(x), \qquad x \in \Omega; \\
-\nabla \cdot (P(x)\nabla w) + \lambda Q(x)w &= f(x), \qquad x \in \Omega;
\end{aligned}
$$

      and that $W = w$ on $\Omega$. Then trivially, we have;

$$
f(x) = F(x) \qquad\qquad x \in \Omega.
$$

      This ensures the unique determination of the function $F$ from the knowledge of the solution to the elliptic problem (2.1.2).

   b. Given values of $P$ and $F$.

      Suppose that $W = u_{P,Q,F,\lambda}$ and $w = u_{P,q,F,\lambda}$ satisfy;

$$
\begin{aligned}
-\nabla \cdot (P(x)\nabla W) + \lambda Q(x)W &= F(x), \qquad x \in \Omega; \\
-\nabla \cdot (P(x)\nabla w) + \lambda q(x)w &= F(x), \qquad x \in \Omega;
\end{aligned}
$$

and that $W = w$ on $\Omega$. Subtracting the two equations above we have,

$$\lambda(Q(x) - q(x))W = 0. \tag{3.3.1}$$

The unique determination of $Q(x)$ is possible under conditions that $\lambda > 0$ and that the solution $W > 0$ on $\Omega$. With the assumption that the Dirichlet data $g(x) > 0$, and the conditions (2.1.4) and (2.1.5), the maximum principle then ensures that the solution $W(x) > 0$ on $\Omega$. Thus the coefficient function $Q(x)$ can be uniquely determined from equation (3.3.1).

c. Given $Q$ and $F$.

Suppose that $W = u_{P,Q,F,\lambda}$ and $w = u_{p,Q,F,\lambda}$ satisfy;

$$\begin{aligned}
-\nabla \cdot (P(x)\nabla W) + \lambda Q(x)W &= F(x), \qquad x \in \Omega; \\
-\nabla \cdot (p(x)\nabla w) + \lambda Q(x)w &= F(x), \qquad x \in \Omega;
\end{aligned}$$

and that $W = w$ on $\Omega$. This case is more involved than the two situations which we have already considered. The question to be answered here is "Under what conditions on $W$ is it true that $P = p$?".

Subtracting the two equations above gives;

$$-\nabla \cdot ((P - p)(x)\nabla W) = 0,$$

which implies that

$$\nabla(P - p)(x) \cdot \nabla W(x) + (P - p)(x)\Delta W(x) = 0. \tag{3.3.2}$$

Equation (3.3.2) is a first order differential equation in $(P - p)(x)$, the solution to which depends on the coefficients $\nabla W(x)$ and $\Delta W(x)$.

**Theorem 3.3.1.** *If*

$$\inf_{x\in\Omega}[\max\{|\Delta W(x)|, |\nabla W(x)|\}] > 0, \tag{3.3.3}$$

*then $P = p$ in $\Omega$. If $n = 2$, the condition (3.3.3) may be weakened to the requirement that at points $x = (\xi, \eta) \in \Omega$ for which $\nabla W(x) = 0$, not all of $W_{\xi\xi}(x), W_{\xi\eta}(x), W_{\eta\eta}(x)$ vanish.*

The proof of Theorem 3.3.3 can be obtained from [13].

**Remark**. In two dimension, the zeros of $\nabla W(x)$ also called the critical points of $W$ can be controlled on the domain $\Omega$. In [1], G. Alessandrini states the following important results in a two dimensional domain $\Omega$ for $f(x) = 0$.

(i) If the set of points of relative maximum of the function $g(x)$ on the boundary $\partial\Omega$ is made of $N$ connected components, then the interior critical points of $W(x)$ are finite in number, and denoting by $m_1, m_2, \ldots, m_k$ the respective multiplicities, the following estimate holds:

$$\sum_{i=1}^{k} m_i \leq N - 1 \tag{3.3.4}$$

Here the number $N$ is used to refer to the number of maxima(minima) of $g(x)$ on $\partial\Omega$.

(ii) If $\partial\Omega$ and $g(x)$ are sufficiently smooth, we obtain a lower bound on $|\nabla W(x)|$. Namely, for every $\Omega' \subset\subset \Omega$ there exists a positive constant $C$ depending only on $\Omega$, $\Omega'$ , $g(x)$ and the coefficients $\{p(x), q(x)\}$ such that for every $x \in \Omega'$,

$$|\nabla W(x)| \geq C \prod_{i=1}^{k} |x - x_i|^{m_i} \tag{3.3.5}$$

where $x_1, x_2, \ldots, x_k$ are the interior critical points of $W(x)$ and $m_1, m_2, \ldots, m_k$ are the respective multiplicities.

From the above two results we are then sure that at all non critical points where $\Delta W = 0$, the gradient $|\nabla W|$ is bounded below away from zero using

(3.3.5). Inequality (3.3.4) shows that if the boundary function $g(x)$ has, say one maximum point, then we are sure that the solution $W(x)$ has no critical points in the interior of the domain. Thus $\nabla W \neq 0$ everywhere in the domain $\Omega$.

2. **The two-coefficient case**.

   In this section it is assumed that one of the coefficient functions $P, Q, F$ is known. The other two coefficients are to be determined from the knowledge of the solution $u_{P,Q,F,\lambda}$ for two values of $\lambda$ with additional conditions as will be explained later.

   Three different cases are discussed below.

   a. First consider the case where $P$ is known and one has two solutions, $u_{P,Q,F,\lambda_1}$ and $u_{P,Q,F,\lambda_2}$ where $\lambda_1 \neq \lambda_2$.

      Let $W = u_{P,Q,F,\lambda_1}$, $w = u_{P,q,f,\lambda_1}$ and $V = u_{P,Q,F,\lambda_2}$, $v = u_{P,q,f,\lambda_2}$, where $W = w$ and $V = v$ on $\Omega$. Substitution into equation (2.1.2) yields the four equations below.

$$\begin{aligned} -\nabla \cdot (P(x)\nabla W) + \lambda_1 Q(x) W &= F, \qquad x \in \Omega \\ -\nabla \cdot (P(x)\nabla w) + \lambda_1 q(x) w &= f, \qquad x \in \Omega \\ -\nabla \cdot (P(x)\nabla V) + \lambda_2 Q(x) V &= F, \qquad x \in \Omega \\ -\nabla \cdot (P(x)\nabla v) + \lambda_2 q(x) v &= f, \qquad x \in \Omega \end{aligned}$$

      Subtraction then yields the homogeneous system;

$$\begin{aligned} \lambda_1 (Q(x) - q(x)) W - (F - f) &= 0 \\ \lambda_2 (Q(x) - q(x)) V - (F - f) &= 0. \end{aligned}$$

      A trivial solution to the above system occurs if and only if

$$\begin{aligned} \lambda_1 W &\neq \lambda_2 V \\ \text{or } \lambda_1 u_{P,Q,F,\lambda_1} &\neq \lambda_2 u_{P,Q,F,\lambda_2} \end{aligned} \tag{3.3.6}$$

Condition (3.3.6) provides for the unique determination of $Q$ and $F$.

b. Next assume that the function $Q$ is known and that one is given the functions $W = u_{P,Q,F,\lambda_1}$, $w = u_{p,Q,f,\lambda_1}$ and $V = u_{P,Q,F,\lambda_2}$, $v = u_{p,Q,f,\lambda_2}$ satisfying,

$$\begin{aligned} -\nabla\cdot(P(x)\nabla W) + \lambda_1 Q(x)W &= F, \qquad x\in\Omega \\ -\nabla\cdot(p(x)\nabla w) + \lambda_1 Q(x)w &= f, \qquad x\in\Omega \\ -\nabla\cdot(P(x)\nabla V) + \lambda_2 Q(x)V &= F, \qquad x\in\Omega \\ -\nabla\cdot(p(x)\nabla v) + \lambda_2 Q(x)v &= f, \qquad x\in\Omega \end{aligned}$$

If $W = w$ and $V = v$ on $\Omega$,then after subtraction we obtain,

$$-\nabla\cdot((P-p)(x)\nabla W) = F - f, \tag{3.3.7}$$

$$-\nabla\cdot((P-p)(x)\nabla V) = F - f \tag{3.3.8}$$

Hence

$$-\nabla\cdot((P-p)(x)\nabla(W-V)) = 0 \tag{3.3.9}$$

Then one seeks conditions on $W, V$ to ensure that $P = p$ and $F = f$. One answer is given by the following theorem.

**Theorem 3.3.2.** *If $W - V$ satisfies the conditions satisfied by $W$ in Theorem 3.3.1,then $P = p$ and $F = f$ on $\Omega$.*

*Proof.* The equality $P = p$ follows from Theorem 3.3.1 and the equality $F = f$ follows from equation (3.3.7). □

c. Finally assume that $F$ is known and one seeks uniqueness for $P$ and $Q$. Given functions $W = u_{P,Q,F,\lambda_1}$, $w = u_{p,q,F,\lambda_1}$ and $V = u_{P,Q,F,\lambda_2}$, $v =$

$u_{p,q,F,\lambda_2}$ for $\lambda_1 \neq \lambda_2$, then,

$$\begin{aligned}
-\nabla \cdot (P(x)\nabla W) + \lambda_1 Q(x) W &= F, \qquad x \in \Omega \\
-\nabla \cdot (p(x)\nabla w) + \lambda_1 q(x) w &= F, \qquad x \in \Omega \\
-\nabla \cdot (P(x)\nabla V) + \lambda_2 Q(x) V &= F, \qquad x \in \Omega \\
-\nabla \cdot (p(x)\nabla v) + \lambda_2 q(x) v &= F, \qquad x \in \Omega
\end{aligned}$$

with $W = w$ and $V = v$ on $\Omega$.

Subtraction then yields,

$$-\nabla \cdot ((P-p)(x)\nabla W) + \lambda_1 (Q-q)(x) W = 0, \qquad (3.3.10)$$

$$-\nabla \cdot ((P-p)(x)\nabla V) + \lambda_2 (Q-q)(x) V = 0 \qquad (3.3.11)$$

Multiplying equation (3.3.10) by $\lambda_2 V(x)$, equation (3.3.11) by $\lambda_1 W(x)$ and subtracting the results, we obtain,

$$\lambda_1 W(x) \nabla \cdot ((P-p)(x)\nabla V(x)) - \lambda_2 V(x) \nabla \cdot ((P-p)(x)\nabla W(x)) = 0,$$

which we rewrite as:

$$\nabla (P-p)(x) \cdot M(x) + (P-p)(x) N(x) = 0.$$

where,

$M(x) = \lambda_1 W(x) \nabla V(x) - \lambda_2 V(x) \nabla W(x)$ and

$N(x) = \lambda_1 W(x) \Delta V(x) - \lambda_2 V(x) \Delta W(x)$

The following theorem gives conditions which guarantee uniqueness for the values of $P$ and $Q$.

**Theorem 3.3.3.** *If the flow generated by the vector field $M(x)$ on $\Omega$ has the property that every point exits at the boundary of $\Omega$ (i.e, lies on a flow line starting at the boundary), then $P = p$ and $Q = q$ in $\Omega$.*

For the proof of this theorem, see [13].

3. **The three-coefficient case**.

In this section, assume that $\lambda_1,\ \lambda_2$ and $\lambda_3$ are given, together with functions $U = u_{P,Q,F,\lambda_1},\ \ u = u_{p,q,f,\lambda_1}, V = u_{P,Q,F,\lambda_2},\ \ v = u_{p,q,f,\lambda_2}$ and $W = u_{P,Q,F,\lambda_3},\ \ w = u_{p,q,f,\lambda_3}$ satisfying;

$$\begin{aligned} -\nabla\cdot(P(x)\nabla U) + \lambda_1 Q(x)U &= F, \qquad x\in\Omega \\ -\nabla\cdot(p(x)\nabla u) + \lambda_1 q(x)u &= f, \qquad x\in\Omega \\ -\nabla\cdot(P(x)\nabla V) + \lambda_2 Q(x)V &= F, \qquad x\in\Omega \\ -\nabla\cdot(p(x)\nabla v) + \lambda_2 q(x)v &= f, \qquad x\in\Omega \\ -\nabla\cdot(P(x)\nabla W) + \lambda_3 Q(x)W &= F, \qquad x\in\Omega \\ -\nabla\cdot(p(x)\nabla w) + \lambda_3 q(x)w &= f, \qquad x\in\Omega \end{aligned} \tag{3.3.12}$$

and $U = u,\ V = v$ and $W = w$ on $\Omega$. The following theorem provides for the unique determination of the three coefficient functions $P, Q, F$.

**Theorem 3.3.4.** *If the flow generated by the vector field*

$$\begin{aligned} N(x) &= (\lambda_2 V(x) - \lambda_3 W(x))\nabla U(x) + (\lambda_3 W(x) - \lambda_1 U(x))\nabla V(x) \\ &\quad + (\lambda_1 U(x) - \lambda_2 V(x))\nabla W(x) \end{aligned} \tag{3.3.13}$$

*on $\Omega$ has the property that every point exits at the boundary of $\Omega$, that is, lies on a flow line starting at the boundary and at least one of*

$$\lambda_2 V(x) - \lambda_3 W(x), \quad \lambda_3 W(x) - \lambda_1 U(x), \quad \lambda_1 U(x) - \lambda_2 V(x)$$

*in not zero at every $x \in \Omega$, then $P = p$, $Q = q$ and $F = f$ in $\Omega$.*

*Proof.* From the set of equations (3.3.12), one obtains;

$$\begin{aligned}
-\nabla\cdot((P-p)(x)\nabla U)+\lambda_1(Q-q)(x)U &= F-f,\\
-\nabla\cdot((P-p)(x)\nabla V)+\lambda_2(Q-q)(x)V &= F-f, \qquad (3.3.14)\\
-\nabla\cdot((P-p)(x)\nabla W)+\lambda_3(Q-q)(x)W &= F-f,
\end{aligned}$$

it follows by subtraction that,

$$\begin{aligned}
-\nabla\cdot((P-p)(x)\nabla(U-V))+(\lambda_1U-\lambda_2V)(Q-q)(x)=0, \qquad (3.3.15)\\
-\nabla.((P-p)(x)\nabla(U-W))+(\lambda_1U-\lambda_3W)(Q-q)(x)=0.
\end{aligned}$$

Elimination of the terms $Q-q$ then gives,

$$\nabla(P-p)(x)\cdot N(x)+(P-p)(x)T(x)=0$$

where,

$$\begin{aligned}
N(x) &= [(\lambda_2V(x)-\lambda_3W(x))\nabla U(x)+(\lambda_3W(x)-\lambda_1U(x))\nabla V(x)\\
&\quad +(\lambda_1U(x)-\lambda_2V(x))\nabla W(x)]\\
T(x) &= [(\lambda_2V(x)-\lambda_3W(x))\Delta U(x)+(\lambda_3W(x)-\lambda_1U(x))\Delta V(x)\\
&\quad +(\lambda_1U(x)-\lambda_2V(x))\Delta W(x)]
\end{aligned}$$

From the given property on the vector field $N(x)$, the above equation will have the unique solution $P-p=0$ and that $Q=q$, $F=f$ follows from equations (3.3.15) and (3.3.14) respectively. □

# Chapter 4

# Methods for recovering parameters from interior data

In this section we consider the functional for the reconstruction of the hydraulic conductivity $P$ from interior data of the solutions $u(x)$ to the problem (2.1.2). This functional gives twice the difference between the energy of the systems whose solutions are $u = u_{P,Q,F}$ and $u_{p,q,f}$ and it is motivated by the equivalence of the minimization problem (2.3.3) to the variational problem (2.1.9). We give the properties of this functional in form of a theorem and some stability and convergence results for the conjugate gradient algorithm which we use for minimization.

## 4.1 Reconstruction functional

Suppose the solution $u := u_{P,Q,F}$ to equation (2.1.2) is given for which $P, Q, F$ are the coefficients representing the parameters $p, q, f$ of the problem.
Define
$\mathcal{D}_G = \{(p, q, f, \lambda) : p, q, f \;\text{ satisfy }\; (2.1.4), (2.1.5), \lambda > 0 \;\text{ and }\; p|\partial\Omega = P|\partial\Omega\},$
It is natural to consider a functional that gives the energy of the system described by

the elliptic equation (2.1.2). For $(p, q, f, \lambda)$ in $\mathcal{D}_G$, define

$$\begin{aligned} G(p,q,f,\lambda) = & \int_\Omega p(x)(|\nabla u|^2 - |\nabla u_{p,q,f,\lambda}|^2) + \lambda q(x)(u^2 - u^2_{p,q,f,\lambda}) \\ - & 2f(x)(u - u_{p,q,f,\lambda}). \end{aligned} \tag{4.1.1}$$

The following theorem summarizes some of the properties of the functional $G(p, q, f, \lambda)$. In the theorem it will be convenient to set $c = (p, q, f, \lambda)$.

**Theorem 4.1.1.** *(a) For any $c$ in $\mathcal{D}_G$,*

$$G(c) = \int_\Omega p(x)|\nabla(u - u_c)|^2 + \lambda q(x)(u - u_c)^2 dx = (L_{p,q}(u - u_c), (u - u_c))$$

*(b) $G(c) \geq 0$ for all $c$ in $\mathcal{D}_G$, and $G(c) = 0$ if and only if $u = u_c$.*

*(c) For $c_1 = (p_1, q_1, f_1, \lambda)$ and $c_2 = (p_2, q_2, f_2, \lambda)$ in $\mathcal{D}_G$, we have*

$$\begin{aligned} G(c_1) - G(c_2) \quad = \quad & \int_\Omega (p_1 - p_2)(|\nabla u|^2 - \nabla u_{c_1}.\nabla u_{c_2}) + \lambda(q_1 - q_2)(u^2 - u_{c_1}.u_{c_2}) \\ & -2(f_1 - f_2)(u - \frac{u_{c_1} + u_{c_2}}{2}) \end{aligned}$$

*(d) The first Gâteaux differential for $G$ is given by*

$$G'(c)[h_1, h_2, h_3] = \int_\Omega (|\nabla u|^2 - |\nabla u_c|^2)h_1 + \lambda q(x)(u^2 - u_c^2)h_2 - 2f(x)(u - u_c)h_3$$

*for $h_1, h_2 \in \mathcal{L}^\infty(\Omega)$; $\;h_3 \in \mathcal{L}^2(\Omega)$ and $G'(c) = 0$ if and only if $u = u_c$.*

*(e) The second Gâteaux differential for $G$ is given by*

$$G''(c)[h, k] = 2(L^{-1}_{p,q}(e(h)), e(k)),$$

*where $h = (h_1, h_2, h_3)$, $\;k = (k_1, k_2, k_3)$ and the functions $h_1, h_2, k_1, k_2 \in \mathcal{L}^\infty(\Omega)$, $h_3, k_3 \in \mathcal{L}^2(\Omega)$, with $h_1|_{\partial\Omega} = k_1|_{\partial\Omega} = 0$ and $e(h) = -\nabla \cdot (h_1 \nabla u_c) + h_2 u_c - h_3$. $(.,.)$ denotes the usual inner product in $\mathcal{L}^2(\Omega)$.*

*Proof.* If $u$ is the generalized solution of the Dirichlet problem (2.1.2) with boundary conditions (2.1.3) ,we have.

$$(L_{p,q}u,\phi) = \int_\Omega p(x)\nabla u \cdot \nabla\phi + \lambda q(x)u\ \phi \qquad \forall\phi \in H_0^1(\Omega), \tag{4.1.2}$$

since

$$\int_\Omega \phi\nabla \cdot (p(x)\nabla u) = -\int_\Omega p(x)\nabla u \cdot \nabla\phi, \tag{4.1.3}$$

for any function $\phi \in H_0^1(\Omega)$. The latter formula is integration by parts and will be used more often in the rest of this proof.

Note that for arbitrary values $a$ and $b$

$$a^2 - b^2 \;=\; (a-b)^2 + 2ab - 2b^2 \tag{4.1.4}$$

and

$$ab \;\leq\; \frac{(a^2+b^2)}{2}. \tag{4.1.5}$$

(a) Using equation (4.1.4) we obtain,

$$\begin{aligned} G(c) &= \int_\Omega p|\nabla u - \nabla u_c|^2 + 2p\nabla u \cdot \nabla u_c - 2p|\nabla u_c|^2 + \lambda q(u^2 - u_c^2) - 2f(u-u_c) \\ &= \int_\Omega p|\nabla(u-u_c)|^2 + 2p\nabla u_c \cdot \nabla(u-u_c) + \lambda q(u^2-u_c^2) - 2f(u-u_c). \end{aligned}$$

By using integration by parts, that is, formula (4.1.3) with $\phi = (u - u_c) \in H_0^1(\Omega)$, we have

$$G(c) = \int_\Omega p|\nabla(u-u_c)|^2 - 2(u-u_c)\nabla\cdot(p\nabla u_c) + \lambda q(u^2-u_c^2) - 2f(u-u_c).$$

From equation (2.1.2), we then have

$$\begin{aligned} G(c) &= \int_\Omega p|\nabla(u-u_c)|^2 - 2(u-u_c)(\lambda q u_c - f) + \lambda q(u^2-u_c^2) - 2f(u-u_c) \\ &= \int_\Omega p|\nabla(u-u_c)|^2 - 2\lambda q u_c(u-u_c) + \lambda q(u^2-u_c^2) \\ &= \int_\Omega p|\nabla(u-u_c)|^2 + \lambda q(u-u_c)^2 \\ &= (L_{p,q}(u-u_c),(u-u_c)). \end{aligned}$$

(b) Since $q(x) \geq 0, \quad p(x) \geq \nu > 0$ and $\lambda > 0 \quad \forall x \in \Omega$, it follows from $(a)$ above that $G(c) \geq 0$.

Suppose $u = u_c$. Then $G(c) = 0$.

On the other hand, if $G(c) = 0$ , then from $(a)$ we have,

$$\int_\Omega p|\nabla(u - u_c)|^2 + \lambda q(u - u_c)^2 = 0 \tag{4.1.6}$$

This implies that $\nabla(u - u_c) = 0$, showing that $u - u_c =$ constant on $\Omega$. Since $u - u_c = 0$ on the boundary of $\Omega$, then $u = u_c$ everywhere. Hence $G(c) = 0$ if and only if $u = u_c$.

(c) From the definition of the functional $G(c)$, we have;

$$\begin{aligned} G(c_1) - G(c_2) &= \int_\Omega p_1(|\nabla u|^2 - |\nabla u_{c_1}|^2) - p_2(|\nabla u|^2 - |\nabla u_{c_2}|^2) + \lambda q_1(u^2 - u_{c_1}^2) \\ &\quad -\lambda q_2(u^2 - u_{c_2}^2) - 2f_1(u - u_{c_1}) + 2f_2(u - u_{c_2}) \\ &= \int_\Omega p_1(|\nabla u|^2 - |\nabla u_{c_1}|^2) - p_1(|\nabla u|^2 - |\nabla u_{c_2}|^2) \\ &\quad +p_1(|\nabla u|^2 - |\nabla u_{c_2}|^2) - p_2(|\nabla u|^2 - |\nabla u_{c_2}|^2) \\ &\quad +\lambda q_1(u^2 - u_{c_1}^2) - \lambda q_1(u^2 - u_{c_2}^2) + \lambda q_1(u^2 - u_{c_2}^2) \\ &\quad -\lambda q_2(u^2 - u_{c_2}^2) - 2f_1(u - u_{c_1}) + 2f_2(u - u_{c_2}) \\ &= \int_\Omega p_1(|\nabla u_{c_2}|^2 - |\nabla u_{c_1}|^2) + \lambda q_1(u_{c_2}^2 - u_{c_1}^2) \\ &\quad +(p_1 - p_2)(|\nabla u|^2 - |\nabla u_{c_2}|^2) \\ &\quad +\lambda(q_1 - q_2)(u^2 - u_{c_2}^2) - 2f_1(u - u_{c_1}) + 2f_2(u - u_{c_2}) \\ &= \int_\Omega p_1\nabla(u_{c_2} + u_{c_1}).\nabla(u_{c_2} - u_{c_1}) + \lambda q_1(u_{c_2}^2 - u_{c_1}^2) \\ &\quad +(p_1 - p_2)(|\nabla u|^2 - |\nabla u_{c_2}|^2) + \lambda(q_1 - q_2)(u^2 - u_{c_2}^2) \\ &\quad -2f_1(u - u_{c_1}) + 2f_2(u - u_{c_2}). \end{aligned}$$

Using integration by parts, that is, formula (4.1.3) with $\phi = (u_{c_1} - u_{c_2}) \in H_0^1(\Omega)$, we have

$$
\begin{aligned}
& G(c_1) - G(c_2) \\
= \; & \int_\Omega (u_{c_1} - u_{c_2})(\nabla \cdot (p_1 \nabla u_{c_1}) + \nabla \cdot (p_1 \nabla u_{c_2})) + \lambda q_1 (u_{c_2}^2 - u_{c_1}^2) \\
& + (p_1 - p_2)(|\nabla u|^2 - |\nabla u_{c_2}|^2) + \lambda (q_1 - q_2)(u^2 - u_{c_2}^2) \\
& - 2 f_1 (u - u_{c_1}) + 2 f_2 (u - u_{c_2}) \\
= \; & \int_\Omega (u_{c_1} - u_{c_2})\{\nabla \cdot (p_1 \nabla u_{c_1}) + \nabla \cdot (p_2 \nabla u_{c_2}) + \nabla \cdot ((p_1 - p_2) \nabla u_{c_2})\} \\
& + \lambda q_1 (u_{c_2}^2 - u_{c_1}^2) + (p_1 - p_2)(|\nabla u|^2 - |\nabla u_{c_2}|^2) \\
& + \lambda (q_1 - q_2)(u^2 - u_{c_2}^2) - 2 f_1 (u - u_{c_1}) + 2 f_2 (u - u_{c_2}) \\
= \; & \int_\Omega (u_{c_1} - u_{c_2})(\lambda q_1 u_{c_1} + \lambda q_2 u_{c_2} - f_1 - f_2) \\
& - (p_1 - p_2) \nabla u_{c_2} \cdot \nabla (u_{c_1} - u_{c_2}) \\
& + (p_1 - p_2)(|\nabla u|^2 - |\nabla u_{c_2}|^2) + \lambda q_1 (u_{c_2}^2 - u_{c_1}^2) \\
& + \lambda (q_1 - q_2)(u^2 - u_{c_2}^2) - 2 f_1 (u - u_{c_1}) + 2 f_2 (u - u_{c_2}).
\end{aligned}
$$

After an arrangement, we then obtain

$$
\begin{aligned}
G(c_1) - G(c_2) \; = \; & \int_\Omega (p_1 - p_2)(|\nabla u|^2 - \nabla u_{c_1} . \nabla u_{c_2}) + \lambda (q_1 - q_2)(u^2 - u_{c_1} . u_{c_2}) \\
& - 2 (f_1 - f_2)(u - \frac{u_{c_1} + u_{c_2}}{2})
\end{aligned}
$$

The proof of $(d)$ and $(e)$ requires the following important results.

(i) **Poincare inequality.** Let $\Omega$ be a bounded Lipschitz domain. There exists a constant $c_p > 0$ such that;

$$
\|v\|_{L^2(\Omega)} \leq c_p \left( \int_\Omega |\nabla v|^2 dx \right)^{\frac{1}{2}} \qquad \forall \;\; v \in H_0^1(\Omega) \tag{4.1.7}
$$

(ii) For any fixed $c = (p, q, f, \lambda)$ and $h = (h_1, h_2, h_3)$ we have,

$$
\lim_{\varepsilon \to 0} u_{c + \varepsilon h} = u_c \qquad in \quad H^1(\Omega) \tag{4.1.8}
$$

This result follows from subtracting the following two equations.

$$-\nabla\cdot(p\nabla u_c)+\lambda q u_c = f, \tag{4.1.9}$$

$$-\nabla\cdot((p+\varepsilon h_1)\nabla u_{c+\varepsilon h})+\lambda(q+\varepsilon h_2)u_{c+\varepsilon h} = f+\varepsilon h_3$$

to obtain,

$$L_{p,q}(u_{c+\varepsilon h}-u_c)=\varepsilon(\nabla\cdot(h_1\nabla u_{c+\varepsilon h})-\lambda h_2 u_{c+\varepsilon h}+h_3) \tag{4.1.10}$$

Multiplying equation (4.1.10) by $u_{c+\varepsilon h}-u_c$ and integrating, we obtain,

$$\begin{aligned}
&(L_{p,q}(u_{c+\varepsilon h}-u_c),(u_{c+\varepsilon h}-u_c))\\
=\;& \int_\Omega (u_{c+\varepsilon h}-u_c)\varepsilon(\nabla\cdot(h_1\nabla u_{c+\varepsilon h})-\lambda h_2 u_{c+\varepsilon h}+h_3)\\
=\;& \varepsilon\int_\Omega h_1\nabla u_{c+\varepsilon h}\cdot\nabla(u_{c+\varepsilon h}-u_c)-\lambda h_2 u_{c+\varepsilon h}(u_{c+\varepsilon h}-u_c)\\
& +h_3(u_{c+\varepsilon h}-u_c)\\
=\;& \varepsilon\int_\Omega -(h_1|\nabla(u_{c+\varepsilon h}-u_c)|^2+h_1\nabla u_c\cdot\nabla(u_{c+\varepsilon h}-u_c))\\
& -\lambda h_2(u_c(u_{c+\varepsilon h}-u_c)+(u_{c+\varepsilon h}-u_c)^2)+h_3(u_{c+\varepsilon h}-u_c).
\end{aligned}$$

After repeated use of inequality (4.1.5), we obtain

$$\begin{aligned}
&(L_{p,q}(u_{c+\varepsilon h}-u_c),(u_{c+\varepsilon h}-u_c))\\
\le\;& \varepsilon\int_\Omega |h_1||\nabla(u_{c+\varepsilon h}-u_c)|^2+|\frac{h_1}{2}|(|\nabla u_c|^2+|\nabla(u_{c+\varepsilon h}-u_c)|^2)\\
& +|\lambda||\frac{h_2}{2}|(u_c^2+(u_{c+\varepsilon h}-u_c)^2)+|\lambda||h_2|(u_{c+\varepsilon h}-u_c)^2\\
& +\frac{1}{2}(|h_3|^2+(u_{c+\varepsilon h}-u_c)^2).
\end{aligned} \tag{4.1.11}$$

The left hand side of the above inequality is given by,

$$\begin{aligned}
& \int_\Omega p|\nabla(u_{c+\varepsilon h}-u_c)|^2+\lambda q(u_{c+\varepsilon h}-u_c)^2\\
\ge\;& \int_\Omega p|\nabla(u_{c+\varepsilon h}-u_c)|^2\\
=\;& \frac{1}{2}\int_\Omega p|\nabla(u_{c+\varepsilon h}-u_c)|^2+\frac{1}{2}\int_\Omega p|\nabla(u_{c+\varepsilon h}-u_c)|^2
\end{aligned}$$

Using Poincaré inequality (4.1.7), with $v = u_{c+\varepsilon h} - u_c \in H_0^1(\Omega)$ for the second integral on the right hand side, we have

$$\begin{aligned}
& \int_\Omega p|\nabla(u_{c+\varepsilon h} - u_c)|^2 + \lambda q(u_{c+\varepsilon h} - u_c)^2 \\
\geq \;& \frac{p}{2}\left(\int_\Omega |\nabla(u_{c+\varepsilon h} - u_c)|^2 + \frac{1}{c_p}(u_{c+\varepsilon h} - u_c)^2\right) \\
\geq \;& C\|u_{c+\varepsilon h} - u_c\|_{H^1(\Omega)}
\end{aligned}$$

where $C = \min\left(\frac{p}{2}, \frac{p}{c_p}\right)$.

When the terms in $u_{c+\varepsilon h} - u_c$ from the right hand side of equation (4.1.11) are transferred to the left hand side, the resulting expression is bounded below by $C_2\|u_{c+\varepsilon h} - u_c\|_{H^1(\Omega)}$ which is less than $C\|u_{c+\varepsilon h} - u_c\|_{H^1(\Omega)}$. We note that in this case, $\varepsilon$ should be taken small enough in order to have a positive term on the left hand side after the difference. The resulting equation then becomes,

$$C_2\|u_{c+\varepsilon h} - u_c\|_{H^1(\Omega)} \;\; \leq \;\; O(\varepsilon)$$

which implies that,

$$\lim_{\varepsilon\to 0} u_{c+\varepsilon h} \;\; = \;\; u_c \qquad\qquad in \quad H^1(\Omega)$$

(iii) For any function $\eta \in L^\infty(\Omega)$, we have

$$\|\nabla.(\eta\nabla u_{c+\varepsilon h})\|_{H^{-1}(\Omega)} \leq K \tag{4.1.12}$$

where the constant $K$ is independent of $\varepsilon$. In order to obtain this result, we consider a functional $F$ defined by

$$\begin{aligned}
F(\phi) \;\; &= \;\; \int_\Omega \eta\nabla u_{c+\varepsilon h}.\nabla\phi \qquad\qquad \text{on} \quad H_0^1(\Omega) \\
&= \;\; \int_\Omega -\nabla\cdot(\eta\nabla u_{c+\varepsilon h})\phi
\end{aligned}$$

This functional is linear and bounded, that is, it satisfies, $|F(\phi)| \leq K\|\phi\|_{H_0^1(\Omega)}$. Consequently, $F \in (H_0^1(\Omega))^*$, the dual of $H_0^1(\Omega)$. Thus $\nabla.(\eta\nabla u_{c+\varepsilon h}) \in H^{-1}(\Omega)$

and $\|F\|_{H^{-1}(\Omega)} = \|\nabla \cdot (\eta \nabla u_{c+\varepsilon h})\|_{H^{-1}(\Omega)}$

The estimate (4.1.12) then follows from the boundedness of $F$.

(d) The first Gâteaux derivative of the functional $G$ is given by,

$$G'(c)[h] := \lim_{\varepsilon \to 0} \frac{G(c+\varepsilon h) - G(c)}{\varepsilon}$$

If we use property (c) of the theorem we obtain,

$$\begin{aligned}
\frac{G(c+\varepsilon h) - G(c)}{\varepsilon} &= \varepsilon^{-1} \int_\Omega (p + \varepsilon h_1 - p)(|\nabla u|^2 - \nabla u_{c+\varepsilon h} \cdot \nabla u_c) \\
&\quad + \lambda(q + \varepsilon h_2 - q)(u^2 - u_{c+\varepsilon h} u_c) \\
&\quad - 2(f + \varepsilon h_3 - f)(u - \frac{u_{c+\varepsilon h} + u_c}{2}) \\
&= \int_\Omega h_1(|\nabla u|^2 - \nabla u_{c+\varepsilon h} \cdot \nabla u_c) + \lambda h_2(u^2 - u_{c+\varepsilon h} u_c) \\
&\quad - 2h_3(u - \frac{u_{c+\varepsilon h} + u_c}{2})
\end{aligned}$$

Using equation (4.1.8), we obtain

$$G'(c)[h_1, h_2, h_3] = \int_\Omega (|\nabla u|^2 - |\nabla u_c|^2) h_1 + \lambda(u^2 - u_c^2) h_2 - 2(u - u_c) h_3 dx.$$

(e) The second Gâteaux differential of the functional $G(c)$ is given by,

$$G''(c)[h, k] := \lim_{\varepsilon \to 0} \frac{G'(c+\varepsilon h)[k] - G'(c)[k]}{\varepsilon} \tag{4.1.13}$$

Using part $(d)$ of theorem (4.1.1) and some algebra, the right hand side of (4.1.13) is given by,

$$\begin{aligned}
&= \varepsilon^{-1} \int_\Omega (|\nabla u_c|^2 - |\nabla u_{c+\varepsilon h}|^2) k_1 + \lambda(u_c^2 - u_{c+\varepsilon h}^2) k_2 - 2(u_c - u_{c+\varepsilon h}) k_3 \\
&= \varepsilon^{-1} \int_\Omega k_1 \nabla(u_c - u_{c+\varepsilon h}) \cdot \nabla(u_c + u_{c+\varepsilon h}) + \lambda(u_c^2 - u_{c+\varepsilon h}^2) k_2 - 2(u_c - u_{c+\varepsilon h}) k_3.
\end{aligned}$$

Using integration by parts we obtain,

$$\begin{aligned} G''(c)[h,k] &= \varepsilon^{-1}\int_\Omega (u_{c+\varepsilon h}-u_c) \\ &\quad \times\{\nabla\cdot(k_1\nabla(u_c+u_{c+\varepsilon h}))-\lambda(u_c+u_{c+\varepsilon h})k_2+2k_3\} \end{aligned}$$

Then applying equation (4.1.10) we have,

$$\begin{aligned} & G''(c)[h,k] \\ = & \int_\Omega L_{p,q}^{-1}(-\nabla\cdot(h_1\nabla u_c)+\lambda h_2 u_c - h_3) \\ & \times\{-\nabla\cdot(k_1\nabla(u_c+u_{c+\varepsilon h}))+\lambda(u_c+u_{c+\varepsilon h})k_2-2k_3\} \\ = & \ 2\int_\Omega L_{p,q}^{-1}(-\nabla\cdot(h_1\nabla u_c)+\lambda h_2 u_c - h_3)\{-\nabla\cdot(k_1\nabla u_c)+\lambda u_c k_2 - 2k_3\} \\ & + \int_\Omega L_{p,q}^{-1}(-\nabla\cdot(h_1\nabla(u_{c+\varepsilon h}-u_c)+\lambda h_2(u_{c+\varepsilon h}-u_c)) \\ & \times\{-\nabla\cdot(k_1\nabla(u_c+u_{c+\varepsilon h}))+\lambda(u_c+u_{c+\varepsilon h})k_2-2k_3\} \\ & + \int_\Omega L_{p,q}^{-1}(-\nabla\cdot(h_1\nabla u_c+\lambda h_2 u_c - h_3) \\ & \times\{-\nabla\cdot(k_1\nabla(u_{c+\varepsilon h}-u_c))+\lambda(u_{c+\varepsilon h}-u_c)k_2\} \end{aligned} \tag{4.1.14}$$

It remains to show that the second and third integrals in equation (4.1.14) tend to zero as $\varepsilon\to 0$.

As the operator $L_{p,q}^{-1}$ is self-adjoint, if we set

$$w_\varepsilon = -\nabla\cdot(k_1\nabla(u_c+u_{c+\varepsilon h}))+\lambda(u_c+u_{c+\varepsilon h})k_2-k_3,$$

the second integral may be rewritten as;

$$\begin{aligned} & \int_\Omega (-\nabla\cdot(h_1\nabla(u_{c+\varepsilon h}-u_c)+\lambda h_2(u_{c+\varepsilon h}-u_c))L_{p,q}^{-1}w_\varepsilon \\ = & \int_\Omega h_1\nabla(u_{c+\varepsilon h}-u_c)\cdot\nabla(L_{p,q}^{-1}w_\varepsilon)+\lambda h_2(u_{c+\varepsilon h}-u_c)L_{p,q}^{-1}w_\varepsilon \end{aligned}$$

Now, from (4.1.12), $w_\varepsilon$ is uniformly bounded in $H^{-1}(\Omega)$ and as $L_{p,q}^{-1}$ may be extended uniquely as a bounded linear operator from $L^2(\Omega)$ to $H^{-1}(\Omega)$, $L_{p,q}^{-1}w_\varepsilon$

is bounded independently of $\varepsilon$ in $H^1(\Omega)$. From the boundedness of $\nabla$ on $H^1(\Omega)$ to $L^2(\Omega) \times L^2(\Omega)$ it follows that $|\nabla(L_{p,q}^{-1} w_\varepsilon)|$ is bounded independently of $\varepsilon$ in $L^2(\Omega)$.

Using (4.1.8) it now follows that the second integral in (4.1.14) tends to zero with $\varepsilon$. Finally, note that $L_{p,q}^{-1}(-\nabla \cdot (h_1 \nabla u_c) + \lambda h_2 u_c - h_3)$ lies in $H^1(\Omega)$. After an integration by parts and applying (4.1.8), the third integral of (4.1.14) also vanishes as $\varepsilon \to 0$. This completes the proof of theorem (4.1.1).

$\square$

In the remaining part of this thesis, we consider the elliptic problem (2.1.2) for the case $q(x) = 0$. Then property $(b)$ in theorem (4.1.1) becomes,

$$G'(p)h = \int_\Omega h(|\nabla u|^2 - |\nabla u_p|^2) \tag{4.1.15}$$

**Remark**

From equation (4.1.15) it follows that $G'(p)$ is in the dual space of $L^\infty(\Omega)$. Then this equation can be written as a duality pairing between elements of $L^\infty(\Omega)$ and $L^\infty(\Omega)^*$, where $()^*$ denotes the dual space. Then we have,

$$\begin{aligned} \langle G'(p), h \rangle_{L^{\infty *}, L^\infty} &= \langle |\nabla u|^2 - |\nabla u_p|^2, h \rangle_{L^{\infty *}, L^\infty} \\ &= \langle |\nabla u|^2 - |\nabla u_p|^2, h \rangle_{L^{1**}, L^\infty} \end{aligned}$$

Since $L^1(\Omega)$ is isometrically embedded in $L^1(\Omega)^{**}$, we can make the identification,

$$G'(p) = \nabla G = |\nabla u|^2 - |\nabla u_p|^2,$$

since $|\nabla u|^2 - |\nabla u_p|^2 \in L^1(\Omega)$.

## 4.2 A conjugate gradient algorithm

In this section we discuss an algorithm for minimization of the functional $G(c)$ based upon the use of certain conjugate gradient descent directions. In the remaining part of this thesis we consider an elliptic problem for the case $q(x) = 0$. Equation(2.1.2) then becomes,

$$\begin{aligned} -\nabla \cdot (p(x)\nabla u) &= f, & x \in \Omega \\ u(x) &= g(x), & x \in \partial\Omega \end{aligned} \tag{4.2.1}$$

In this case we note that the functional $G(p)$ is strictly convex, see Theorem(4.1.1), since $G''(p) > 0$. Thus the minimization should be computationally effective since from the properties of $G(p)$ discussed in theorem (4.1.1), $P$ is not only the unique global minimum for $G$, but also the unique zero for the gradient $\nabla G$. Thus provided that the descent is stable, it cannot get "stuck" at a function other than P. We also note that in general one cannot use the $L^1$-gradient $\nabla G$, because there are numerical problems associated with this gradient in the descent procedure, stemming from the fact that the $L^1$-gradient need not be zero on the boundary of the domain $\Omega$. In this case the updated $P$ need not preserve the given boundary data. Instead we shall use the Neuberger gradient, $\nabla_N G$, chosen so that

$$G'(p)h = (\nabla_N G(p), h)_1, \qquad h \in H_0^1(\Omega) \cap L^\infty(\Omega) \tag{4.2.2}$$

where $(.,.)_1$ denotes the usual inner product in $H^1(\Omega)$.
If we compute the function $g$ from

$$\begin{aligned} -\Delta g + g &= \nabla G(p), \\ g|_{\partial\Omega} &= 0, \end{aligned} \tag{4.2.3}$$

it follows that, for $h \in H_0^1(\Omega) \cap L^\infty(\Omega)$,

$$(g, h)_1 = \int_\Omega \nabla g \cdot \nabla h + gh.$$

After an integration by parts we obtain,

$$\begin{aligned}(g,h)_1 &= \int_\Omega (-\Delta g + g)h, \\ &= \int_\Omega h\nabla G(p).\end{aligned}$$

Using equation (4.1.15) we have,

$$\begin{aligned}(g,h)_1 &= \int_\Omega h(|\nabla u|^2 - |\nabla u_p|^2) \\ &= G'(p)h\end{aligned}$$

Thus we can set $g = \nabla_N G(p)$.

**Definition 4.2.1.** A vector $h \in \mathbf{R}^n$ is a descent direction for a functional $G$ at a point $p$ if

$$\frac{dG(p+\alpha h)}{d\alpha}|_{\alpha=0} = \nabla G(p)^T h < 0.$$

If we choose $h = -g$ as the descent direction, we obtain

$$G(p - \alpha g) - G(p) \approx -\alpha G'(p) g.$$

Then using equation (4.2.2) we have,

$$G(p - \alpha g) - G(p) \approx -\alpha \|g\|_1^2 < 0.$$

Therefore $-g$ is a descent update direction for $p$ that preserves the values of $p$ at the boundary of $\Omega$. We note from equation (4.2.3) that

$$\begin{aligned}(-\Delta + I)\nabla_N G(p) &= \nabla G(p) \\ \text{or}\ \ \nabla_N G(p) &= (-\Delta + I)^{-1}\nabla G(p).\end{aligned}$$

Therefore the Neuberger gradient is a preconditioned version of the $L^1-$gradient. A detailed discussion of this approach can be found in [21].

Since we will use the Neuberger gradient $\nabla_N G$ to update the iterates $p \in \mathcal{D}_{\mathcal{G}}$, we need to be sure that $\nabla_N G \in L^\infty(\Omega)$. This is a consequence of an elliptic regularity estimate of De Giorgi [10]. If $u$ is a solution of the elliptic problem (4.2.1), with $f = 0$, then $u \in C^{0,\alpha}(\Omega)$ for some constant $\alpha > 0$ and if the concentric balls $B_\rho$ lie in $\Omega$ for $\rho < R_0$, then for some constant $c$ and any $\rho < R < R_0$,

$$\int_{B_\rho} |\nabla u|^2 \leq c(\rho/R)^{M+2\alpha} \int_{B_R} |u - u_R| \tag{4.2.4}$$

where $u_R$ denotes the average value of $u$ over the ball $B_R$.

We note that, if A is the homogeneous Dirichlet operator $-\Delta + I$ on $\Omega$ then the solution $g$ to (4.2.3) can be written as

$$g = \mathcal{G} * \nabla G$$

where $\mathcal{G}$ is the Green's function associated with the operator A.

Let $\tilde{\Omega}$ be such that $\Omega \subset\subset \tilde{\Omega}$. Then the Green's function $\mathcal{G}$ satisfies

$$\begin{aligned} A\mathcal{G} &= -\delta(x,z) && \text{in } \tilde{\Omega} \\ \mathcal{G} &= 0 && \text{on } \partial\tilde{\Omega} \end{aligned}$$

where $\delta(.,.)$ is the dirac delta function. We recall that, $\Phi(x,z) := \frac{C_M}{|x-z|^{M-2}}$ is the fundamental solution to $-\Delta\Phi = \delta(x,z)$ in $\mathbf{R}^M$, where $c_M$ is a constant depending only on $M$.

If we write,

$$\mathcal{G} = \Phi + \tilde{\Phi},$$

then,

$$\begin{aligned} (-\Delta + I)\tilde{\Phi}(x,z) &= -\Phi(x,z) && \text{in } \tilde{\Omega} \\ \tilde{\Phi}(x,z) &= -\Phi(x,z) && \text{on } \partial\tilde{\Omega}. \end{aligned} \tag{4.2.5}$$

Multiplying both sides of equation(4.2.5) by $\mathcal{G}(x,t)$ and integrating over $\tilde{\Omega}$ we obtain,

$$\int_{\tilde{\Omega}}(-\Delta+I)\tilde{\Phi}(t,z)\mathcal{G}(x,t)dt \quad = \quad \int_{\tilde{\Omega}}-\Phi(t,z)\mathcal{G}(x,t)dt.$$

Integration by parts then gives,

$$\begin{aligned}&\underbrace{-\int_{\partial\tilde{\Omega}}\nabla\tilde{\Phi}(t,z).n\mathcal{G}(x,t)\,dS}_{=0}+\int_{\tilde{\Omega}}\nabla\tilde{\Phi}(t,z).\nabla\mathcal{G}(x,t)+\tilde{\Phi}(t,z).\mathcal{G}(x,t)dt\\ = \quad &\int_{\tilde{\Omega}}-\Phi(t,z)\mathcal{G}(x,t)dt.\end{aligned}$$

Further integration by parts leads to,

$$\int_{\partial\tilde{\Omega}}\nabla\mathcal{G}(x,t).n\tilde{\Phi}(t,z)dS+\int_{\tilde{\Omega}}(-\Delta+I)\mathcal{G}(x,t).\tilde{\Phi}(t,z)dt \quad = \quad \int_{\tilde{\Omega}}-\Phi(t,z)\mathcal{G}(x,t)dt,$$

which implies that,

$$\begin{aligned}\int_{\tilde{\Omega}}\delta(x,t)\tilde{\Phi}(x,z)dt &= \int_{\tilde{\Omega}}-\Phi(t,z)\mathcal{G}(x,t)dt-\int_{\partial\tilde{\Omega}}\nabla\mathcal{G}(x,t).n\tilde{\Phi}(t,z)dS\\ \tilde{\Phi}(x,z) &= \int_{\tilde{\Omega}}-\Phi(t,z)\mathcal{G}(x,t)dt-\int_{\partial\tilde{\Omega}}\nabla\mathcal{G}(x,t).n\tilde{\Phi}(t,z)dS\end{aligned}$$

From the estimates on Green's functions, [23, 24],

$$|\tilde{\Phi}(x,z)|\leq C\left\{\int_{\tilde{\Omega}}\frac{1}{|x-t|^{M-2}}\frac{1}{|t-z|^{M-2}}+\int_{\partial\tilde{\Omega}}\frac{1}{|x-t|^{M-1}}\frac{1}{|t-z|^{M-1}}\right\}$$

For $x\in\Omega$ we have,

$$\int_{\partial\tilde{\Omega}}\frac{1}{|x-t|^{M-1}}\frac{1}{|t-z|^{M-1}}dS(t)<\infty,\quad \text{since}\ \ \Omega\subset\subset\tilde{\Omega}, \tag{4.2.6}$$

we also have,

$$\int_{\tilde{\Omega}}\frac{1}{|x-t|^{M-2}}.\frac{1}{|t-z|^{M-2}}dt\leq c\begin{cases}c/|x-z|^{M-4}, & \text{if } M>4;\\ c\ln|x-z|, & \text{if } M=4;\\ c|x-z|^{4-M}, & \text{if } M<4.\end{cases} \tag{4.2.7}$$

It follows from (4.2.6) and (4.2.7) that

$$|\tilde{\Phi}(x,z)| \leq c \begin{cases} c/|x-z|^{M-4}, & \text{if } M > 4; \\ c \ln |x-z|, & \text{if } M = 4; \\ c|x-z|^{4-M}, & \text{if } M < 4. \end{cases} \tag{4.2.8}$$

Since $\Phi(x,z)$ and $\tilde{\Phi}(x,z)$ are smooth if $x \neq z$, then using (4.2.8), we have

$$|\tilde{\Phi}(x,z)| \leq C|\Phi(x,z)|, \qquad \text{for every } x \neq z \text{ in } \Omega. \tag{4.2.9}$$

Note that for $x \in \Omega$ and $M > 2$ (the proof for M=2 is similar),

$$\begin{aligned} \nabla_N G(x) &= \int_\Omega \mathcal{G}(x,y)(|\nabla u|^2(y) - |\nabla u_p|^2(y))dy \\ &= C \int_\Omega |x-y|^{2-M}(|\nabla u|^2(y) - |\nabla u_p|^2(y))dy + S(x), \end{aligned} \tag{4.2.10}$$

where $\mathcal{G}$ denotes the Green function for the homogeneous Dirichlet operator $-\Delta + I$ on $\Omega$, $C$ is a constant which depends only on $M$. The quantity $S$ is dominated by the first integral expression by using the result in (4.2.9). Therefore we have $\nabla_N G \in L^\infty(\Omega)$ if we can prove that the first integral expression is bounded.

If $B_\varepsilon$ and $B_{2\varepsilon}$ are balls centered at a fixed $x$ and $\varepsilon$ is chosen so that $B_\varepsilon \subset B_{2\varepsilon} \subset \Omega$, we have,

$$\int_\Omega |x-y|^{2-M}|\nabla u|^2(y)dy = \int_{B_\varepsilon} |x-y|^{2-M}|\nabla u|^2(y)dy + \int_{\Omega - B_\varepsilon} |x-y|^{2-M}|\nabla u|^2(y)dy$$

Since $x$ is fixed we can write $r = x - y$, $m = 2 - M$ and rewrite the first integral on the right-hand side of the above equation in polar coordinates. This leads to,

$$
\begin{aligned}
\int_{B_\varepsilon} |x-y|^{2-M}|\nabla u|^2(y)dy &= \int_{S^{M-1}}\int_0^\varepsilon r^{2-M}|\nabla u(x-r)|^2 r^{M-1} dr d\theta \\
&= \int_{S^{M-1}}\int_0^\varepsilon m r^{1-M}\left(\int_0^r |\nabla u|^2 t^{M-1} dt\right) dr d\theta \\
&+ \lim_{r\to 0}\int_{S^{M-1}}\int_0^\varepsilon m r^{2-M}\int_0^r |\nabla u|^2 t^{M-1} dt d\theta \\
&- \varepsilon^{2-M}\int_{S^{M-1}}\int_0^\varepsilon |\nabla u|^2 t^{M-1} dr d\theta \\
&= (M-2)\int_0^\varepsilon r^{1-M}\left(\int_{B_r}|\nabla u|^2 dr\right) \\
&- \lim_{r\to 0} r^{2-M}\int_{B_r}|\nabla u|^2 + \varepsilon^{2-M}\int_{B_\varepsilon}|\nabla u|^2
\end{aligned}
$$

Applying the De Giorgi estimate (4.2.4) to the above equation, we obtain that all the terms on the right-hand side are uniformly bounded. We can estimate the remaining part of equation (4.2.10) in the same way.

Next we describe briefly a steepest descent algorithm for the minimization of the functional $G(p)$.

### 4.2.1 Steepest descent algorithm

Step 0: Start with an initial guess $p_0 \in \mathbf{R}^n$.

Step 1: Solve for $g_0 = -\nabla_N G(p_0)$ using equation (4.2.3). If $g_0 = 0$ stop, else set $n = 0$ and go to step 2.

Step 2: Compute $\alpha_n \geq 0$ such that

$$G(p_n + \alpha_n g_n) = \min_{\alpha \geq 0} G(p_n + \alpha g_n)$$

We note here that for the implementation of step 2, we use the Golden section method as will be explained later.

Step 3: Set $P_{n+1} = p_n + \alpha_n g_n$, set $n = n+1$ and go to step 2.

**The Golden section method**, see [22].

In the implementation of the line search algorithm in step 2, we use the Golden section method. In this method, we consider minimization of the function

$$f(\alpha) = G(p + \alpha g)$$

where $f$ is continuously differentiable. It is assumed that the derivative of the function $f$ at $\alpha = 0$ is given by

$$f'(0) = \nabla G(x)' g < 0,$$

that is, $g$ is a descent direction at $p$. It is also assumed that $f(\alpha)$ is strictly unimodal in the interval $[0, s]$, where $s > 0$ is a scalar.

**Definition 4.2.2.** A strictly unimodal function $f$ over an interval $[0, s]$ is defined as a function that has a unique global minimum $\alpha^* \in [0, s]$ and if $\alpha_1, \alpha_2$ are two points in $[0, s]$ such that $\alpha_1 < \alpha_2 < \alpha^*$ or $\alpha^* < \alpha_1 < \alpha_2$, then $f(\alpha_1) > f(\alpha_2) > f(\alpha^*)$ or $f(\alpha^*) < f(\alpha_2) < f(\alpha_1)$ respectively. For example a strictly convex function over $[0, s]$ is unimodal.

The Golden section method minimizes $f$ over $[0, s]$ by determining at the $k^{th}$ iteration an interval $[\alpha_k, \bar{\alpha}_k]$ containing $\alpha^*$, the minimum of $f(\alpha)$. These intervals are obtained using the number $\tau = \frac{3-\sqrt{5}}{2}$ which satisfies $\tau = (1-\tau)^2$ and is related to the Fibonacci number sequence.

**Golden section algorithm**

Step 0: Take $[\alpha_0, \bar{\alpha}_0] = [0, s]$, set $k = 0$.

Step 1: Determine $[\alpha_{k+1}, \bar{\alpha}_{k+1}]$ so that $\alpha^* \in [\alpha_{k+1}, \bar{\alpha}_{k+1}]$ as follows.

Calculate

$$\begin{aligned} b_k &= \alpha_k + \tau(\bar{\alpha}_k - \alpha_k) \\ \bar{b}_k &= \bar{\alpha}_k - \tau(\bar{\alpha}_k - \alpha_k) \;\text{ and }\; f(b_k), f(\bar{b}_k). \end{aligned}$$

Step 2: If $f(b_k) < f(\bar{b}_k)$, set

$$\begin{aligned} \alpha_{k+1} &= \alpha_k, \quad \bar{\alpha}_{k+1} = b_k \;\text{ if }\; f(\alpha_k) \leq f(b_k) \\ \alpha_{k+1} &= \alpha_k, \quad \bar{\alpha}_{k+1} = \bar{b}_k \;\text{ if }\; f(\alpha_k) > f(b_k) \end{aligned}$$

elseif $f(b_k) > f(\bar{b}_k)$, set

$$\begin{aligned} \alpha_{k+1} &= \bar{b}_k, \quad \bar{\alpha}_{k+1} = \bar{\alpha}_k \;\text{ if }\; f(\bar{b}_k) \geq f(\bar{\alpha}_k) \\ \alpha_{k+1} &= b_k, \quad \bar{\alpha}_{k+1} = \bar{\alpha}_k \;\text{ if }\; f(\bar{b}_k) < f(\bar{\alpha}_k) \end{aligned}$$

else $f(b_k) = f(\bar{b}_k)$, set

$$\alpha_{k+1} = b_k, \quad \bar{\alpha}_{k+1} = \bar{b}_k$$

Based on the definition of a strictly unimodal function it can be shown (see figure (4.1)) that the intervals $[\alpha_k, \bar{\alpha}_k]$ contain $\alpha^*$ and their lengths converge to zero. In practice, the computation is terminated once $(\bar{\alpha}_k - \alpha_k)$ becomes smaller than a prescribed tolerance.

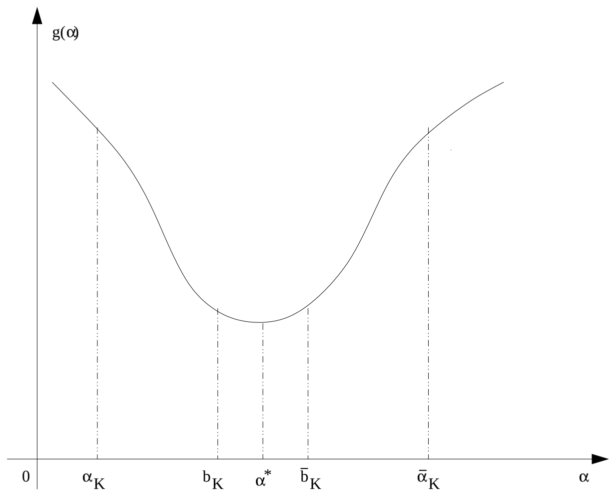



Figure 4.1: Golden Section search

### 4.2.2 Convergence results

Next we present some convergence results for the steepest descent method. For simplicity we assume throughout this section that $f = 0$, $M = 2$, $\partial\Omega$ is $C^2$ and the coefficients $p = p_1$ and $p = p_2$ in (4.2.1) are $C^1$, with solutions $u_{p_1},\ u_{p_2}$ having boundary data with at most $N$ maxima and minima in $\partial\Omega$. It is known in G. Alessandrini's paper [2] that for every $d, \theta,\quad d > 0$ and $0 < \theta < \frac{1}{2}$, and some constant C

$$\|p_1 - p_2\|_{L^\infty(\Omega_d)} \leq C\{\|u_{p_1} - u_{p_2}\|_2 |\Omega|^{-\frac{1}{2}}\}^{(1/2-\theta)/(2N+1)}$$

where $\Omega_d = \{x \in \Omega : d(x, \partial\Omega) > d\}$. This says roughly, that the problem becomes well-posed if one can provide $u$ data sufficient that the second derivatives of $u$ can be accurately approximated. When one only knows that the coefficients are bounded, (and therefore that in general the solutions have no better smoothness than $C^{0,\alpha}(\Omega), \alpha < 1$), one must expect a considerably weaker kind of convergence.
Before we give convergence results, we need the following Lemma.

**Lemma 4.2.1.** *With $p, u, \nu$ defined such that*

$$\begin{aligned} -\nabla \cdot (p\nabla u) &= 0 && \textit{in } \Omega \\ u &= g && \textit{on } \partial\Omega \\ p \geq \nu &> 0 && p \in \mathcal{D}_G, \end{aligned}$$

a) $\|u - u_p\|_{H^1(\Omega)} \leq \frac{c_p^2+1}{\nu} \|u\|_{H^1(\Omega)} \|P - p\|_\infty$

b) $\frac{\nu}{c_p^2+1} \|u - u_p\|^2_{H^1(\Omega)} \leq G(p) \leq \frac{c_p^2+1}{\nu} \|u\|^2_{H^1(\Omega)} \|P - p\|^2_\infty$

*Proof.* First we consider the left-hand side inequality in (b).

$$\frac{\nu}{c_p^2 + 1} \|u - u_p\|^2_{H^1(\Omega)} = \int_\Omega \frac{\nu}{c_p^2 + 1} (|u - u_p|^2 + |\nabla(u - u_p)|^2)$$

Using the condition $p \geq \nu > 0$, we have,

$$\frac{\nu}{c_p^2+1}\|u-u_p\|^2_{H^1(\Omega)} \;\leq\; \int_\Omega \frac{p}{c_p^2+1}(|u-u_p|^2+|\nabla(u-u_p)|^2),$$

which, after using Poincaré inequality, leads to,

$$\begin{aligned}\frac{\nu}{c_p^2+1}\|u-u_p\|^2_{H^1(\Omega)} &\leq \int_\Omega p(|\nabla(u-u_p)|^2)\\ &= G(p).\end{aligned}$$

Secondly, using part (c) of theorem (4.1.1), we have

$$G(p)-G(P) \;=\; \int_\Omega (p-P)(|\nabla u|^2-\nabla u_p.\nabla u_P).$$

Since $G(P)=0$ and $u=u_P$, we have

$$G(p) \;=\; \int_\Omega (p-P)\nabla u(\nabla u-\nabla u_p).$$

Cauchy Schwartz inequality then leads to,

$$\begin{aligned}G(p) &\leq \|P-p\|_\infty\|\nabla u\|_{L^2(\Omega)}\|\nabla(u-u_p)\|_{L^2(\Omega)},\\ &\leq \|u\|_{H^1(\Omega)}\|P-p\|_\infty\|u-u_p\|_{H^1(\Omega)}.\end{aligned}$$

It follows from these results that

$$\frac{\nu}{c_p^2+1}\|u-u_p\|^2_{H^1(\Omega)} \;\leq\; G(p)\leq \|u\|_{H^1(\Omega)}\|P-p\|_\infty\|u-u_p\|_{H^1(\Omega)}, \quad (4.2.11)$$

which implies that,

$$\begin{aligned}\frac{\nu}{c_p^2+1}\|u-u_p\|^2_{H^1(\Omega)} &\leq \|u\|_{H^1(\Omega)}\|P-p\|_\infty\|u-u_p\|_{H^1(\Omega)}\\ \|u-u_p\|_{H^1(\Omega)} &\leq \frac{c_p^2+1}{\nu}\|u\|_{H^1(\Omega)}\|P-p\|_\infty\end{aligned}$$

This proves part (a) of Lemma (4.2.1).

Using part (a) and inequality (4.2.11) we obtain the following result.

$$\frac{\nu}{c_p^2+1}\|u-u_p\|^2_{H^1(\Omega)} \;\leq\; G(p)\leq \|u\|_{H^1(\Omega)}\|P-p\|_\infty\frac{c_p^2+1}{\nu}\|u\|_{H^1(\Omega)}\|P-p\|_\infty.$$

This implies that,

$$\frac{\nu}{c_p^2+1}\|u-u_p\|^2_{H^1(\Omega)} \;\; \le \;\; G(p) \le \frac{c_p^2+1}{\nu}\|u\|^2_{H^1(\Omega)}\|P-p\|^2_\infty,$$

which justifies part (b) of Lemma (4.2.1). □

**Theorem 4.2.2.** *Let $\{p_n \subset \mathcal{D}_G\}$ be uniformly bounded below and such that $G(p_n) \to 0$. Then $u_{p_n} \to u$ in $H^1(\Omega)$ and $\nabla\cdot((P-p_n)\nabla u_{p_n})$ converges weakly to zero in $H^1_0(\Omega)$.*

*Proof.* It follows from part (b) of Lemma (4.2.1) that $u_{p_n} \to u$ in $H^1(\Omega)$. Also, for $h \in H^1_0(\Omega)$,

$$\begin{aligned} |\int_\Omega h\nabla\cdot((P-p_n)\nabla u_{p_n})| &= |\int_\Omega h\nabla\cdot(P\nabla u_{p_n})|, \\ &= |\int_\Omega h\nabla\cdot(P\nabla(u_{p_n}-u))|. \end{aligned}$$

Using integration by parts, we have

$$|\int_\Omega h\nabla\cdot((P-p_n)\nabla u_{p_n})| \;\; = \;\; |\int_\Omega P\nabla(u_{p_n}-u)\cdot\nabla h|,$$

which, after using Cauchy Schwartz inequality, gives

$$|\int_\Omega h\nabla\cdot((P-p_n)\nabla u_{p_n})| \;\; \le \;\; \|P\|_\infty\|u_{p_n}-u\|_{H^1(\Omega)}\|h\|_{H^1(\Omega)} \to 0.$$

□

**Lemma 4.2.3.** *Let $p_n \to p^*$ in the space $L^\infty(\Omega)$ and let $g_n = \nabla_N G(p_n)$ and also $\tilde{g}_n = \nabla G(p_n)$. Then $\tilde{g}_n$ tends to zero in $L^1(\Omega)$ and $g_n$ tends to zero in $H^1(\Omega)$.*

*Proof.* We begin by showing that $u_{p_n} \to u_{p^*}$ in $H^1(\Omega)$. If $L_{p^*}$ denotes the Dirichlet operator in $L^2(\Omega)$ formed from (4.2.1) with $p = p^*$, we have,

$$\begin{aligned} L_{p^*}(u_{p_n}-u_{p^*}) &= -\nabla\cdot(p^*\nabla(u_{p_n}-u_{p^*})) \\ &= -\nabla\cdot(p^*\nabla u_{p_n}) + \underbrace{\nabla\cdot(p^*\nabla u_{p^*})}_{=0}, \end{aligned}$$

since $\nabla \cdot (p_n \nabla u_{p_n}) = 0$, we have

$$-\nabla \cdot (p^* \nabla(u_{p_n} - u_{p^*})) \;=\; -\nabla \cdot ((p^* - p_n)\nabla u_{p_n}).$$

Multiplying both sides of the above equation by $u_{p_n} - u_{p^*}$ and using integration by parts leads to,

$$\begin{aligned}
& \int_\Omega p^* |\nabla(u_{p_n} - u_{p^*})|^2 \\
= & \int_\Omega (p^* - p_n)\nabla u_{p_n} \cdot \nabla(u_{p_n} - u_{p^*}) \\
= & \int_\Omega (p^* - p_n)\{\nabla(u_{p_n} - u_{p^*})(\nabla(u_{p_n} - u_{p^*}) + \nabla u_{p^*} \cdot \nabla(u_{p_n} - u_{p^*}))\} \\
= & \int_\Omega (p^* - p_n)\{|\nabla(u_{p_n} - u_{p^*})|^2 + \nabla u_{p^*} \cdot \nabla(u_{p_n} - u_{p^*})\}
\end{aligned}$$

Using inequality (4.1.5), we have

$$\begin{aligned}
\int_\Omega p^* |\nabla(u_{p_n} - u_{p^*})|^2 &\leq \int_\Omega (p^* - p_n)\{\frac{3}{2}|\nabla(u_{p_n} - u_{p^*})|^2 + \frac{1}{2}|\nabla u_{p^*}|^2\} \\
\int_\Omega (p^* - \frac{3}{2}(p^* - p_n))|\nabla(u_{p_n} - u_{p^*})|^2 &\leq \frac{1}{2}\|p_n - p^*\|_\infty \int_\Omega |\nabla u_{p^*}|^2.
\end{aligned}$$

For $n$ large enough, and the fact that $p* \geq \nu > 0$, we obtain that

$$u_{p_n} \to u_{p^*} \quad \text{in} \quad H^1(\Omega)$$

From the above result we deduce that;

$$\begin{aligned}
\|\tilde{g}_n - \tilde{g}_{p^*}\|_{L^1(\Omega)} &= \|\nabla G(p_n) - \nabla G(p^*)\|_{L^1(\Omega)} \\
&= \|(|\nabla u|^2 - |\nabla u_{p_n}|^2) - (|\nabla u|^2 - |\nabla u_{p^*}|^2)\|_{L^1(\Omega)} \\
&= \||\nabla u_{p^*}|^2 - |\nabla u_{p_n}|^2\|_{L^1(\Omega)} \to 0.
\end{aligned}$$

Therefore $\tilde{g}_n - \tilde{g}_{p^*}$ converges to zero in $L^1(\Omega)$. Next we have to show that $g_n - g_{p^*}$ converges to zero in $H^1(\Omega)$.

$$(g_n - g_{p^*})(x) \;=\; \int_\Omega \mathcal{G}(x,y)(\tilde{g}_n - \tilde{g}_{p^*})(y)dy$$

where $\mathcal{G}$ denotes the Green's function for the Dirichlet operator $A = -\Delta + I$ which is used in defining the Neuberger gradients. We have,

$$\begin{aligned}
A(g_n - g_{p^*}) &= \tilde{g}_n - \tilde{g}_{p^*} \\
\int_\Omega A(g_n - g_{p^*})(g_n - g_{p^*})(x)dx &= \int_\Omega (\tilde{g}_n - \tilde{g}_{p^*})(g_n - g_{p^*})(x)dx \\
\|g_n - g_{p^*}\|_1^2 &= \int_\Omega (\tilde{g}_n - \tilde{g}_{p^*})(g_n - g_{p^*})(x)dx \\
&= \int_\Omega (\tilde{g}_n - \tilde{g}_{p^*})(x) \int_\Omega \mathcal{G}(x,y)(\tilde{g}_n - \tilde{g}_{p^*})(y)dydx.
\end{aligned}$$

One can now extract the singular part of $\mathcal{G}$ and (within a ball of radius $\varepsilon > 0$, for $\varepsilon$ small enough), change to polar coordinates, integrate by parts and use the estimate of De Giorgi as was done earlier to show that the resulting integrands are $O(\varepsilon)$. The convergence to zero of $\|g_n - g_{p^*}\|_{H^1(\Omega)}$ then follows from the convergence of $\tilde{g}_n - \tilde{g}_{p^*}$ to zero in $L^1(\Omega)$.

Since $g_n$ are steepest descent gradients, and thus these directions are conjugate, that is $(g_n, g_{n+1}) = 0$, for all $n$, it follows that $g_n \to 0$ in $H^1(\Omega)$. This idea is originally due to H.B. Curry [7]. Therefore $g_{p^*} = 0$. We also have $\tilde{g}_{p^*} = 0$ since,

$$\tilde{g}_{p^*} = (-\Delta + I)g_{p^*}$$

This then implies that $\tilde{g}_n \to 0$ in $L^1(\Omega)$. $\square$

Finally we describe some conditions under which a steepest descent implementation would converge in this manner.

**Theorem 4.2.4.** *Let $\{p_n\} \subset \mathcal{D}_G$ be the sequence of steepest descent iterates obtained via the Neuberger gradient, that is, $p_{n+1} = p_n + \alpha_n g_n$. Assume also that $\{p_n\}$ is uniformly bounded both above and below, and*

$$G(p_n) - G(p_{n+1}) \geq \alpha_n \|\tilde{g}_n\|_{L^1(\Omega)} \|g_n\|_\infty, \tag{4.2.12}$$

*where $\tilde{g}_n = \nabla G(p_n)$. Then $G(p_n) \to 0$ as $n \to \infty$.*

*Proof.* First we claim that there is a subsequence $\{\tilde{g}_{\phi(n)}\}$ converging strongly to zero in $L^1$. To see this, assume by way of contradiction that there is a number $\delta > 0$ such that

$$\|\tilde{g}_n\|_{L^1(\Omega)} \geq \delta \tag{4.2.13}$$

for all n.

Then from (4.2.12) and (4.2.13),

$$G(p_n) - G(p_{n+1}) \geq \|\tilde{g}_n\|_{L^1(\Omega)}\|p_{n+1} - p_n\|_\infty \geq \delta\|p_{n+1} - p_n\|_\infty$$

for all n. Hence for $n > r$,

$$\begin{aligned} G(p_n) - G(p_{n+1}) &\geq \delta\|p_{n+1} - p_n\|_\infty \\ G(p_{n-1}) - G(p_n) &\geq \delta\|p_n - p_{n-1}\|_\infty \\ &\vdots \\ G(p_{r+1}) - G(p_{r+2}) &\geq \delta\|p_{r+2} - p_{r+1}\|_\infty \end{aligned}$$

The triangle inequality then implies,

$$G(p_{r+1}) - G(p_{n+1}) \geq \delta \sum_{i=r+1}^{n} \|p_{i+1} - p_i\|_\infty \geq \delta\|p_{r+1} - p_{n+1}\|_\infty$$

Now $\{G(p_n)\}$ is convergent as it is monotonically decreasing and bounded below. Hence $\{p_n\}$ is a Cauchy sequence in $L^\infty$ and must converge strongly to some function $p^*$ in $L^\infty$ . So by Lemma (4.2.3) the sequence $\{\tilde{g}_n\}$ converges to zero strongly in $L^1$ and this contradicts (4.2.13). It follows that we can find a subsequence $\{\tilde{g}_{\phi(n)}\}$ converging to zero strongly in $L^1$. In addition, as the sequence $\{p_n\}$ is bounded in $L^\infty$, it follows that $G(p_{\phi(n)}) \to 0$ as $n \to \infty$. As the sequence $G(p_n)$ is decreasing, it must also converge to zero. □

# Chapter 5

# Numerical Results

In this chapter we give numerical results for some examples of $p$. The test data is obtained by solving the forward problem with known values of $p$ and then using the conjugate gradient methods to minimize the related functional $G(p)$. All boundary value problems are solved using finite elements method with the help of the matlab PDE tool box. Line minimization is done using the Golden Section method as explained in chapter 4.

The algorithm is tasted on synthetic data obtained by using the following values of the coefficient function $p(x)$ on a square domain $\Omega = [-1, 1] \times [-1, 1]$.

$$p_1(x, y) = e^{-(\frac{x+y}{4})} + 0.5$$

$$p_2(x, y) = \begin{cases} 0.5, & \text{if } x < 0 \text{ ;} \\ 1.0, & \text{elswhere.} \end{cases}$$

$$p_3(x, y) = \begin{cases} 1.5, & \text{if } -0.25 < x < 0 \text{ and } 0 < y < 0.25; \\ 1.0, & \text{if } |x| < 0.5 \text{ and } |y| < 0.5; \\ 0.5, & \text{elsewhere.} \end{cases}$$

$$p_4(x, y) = \begin{cases} 1.0, & \text{if } |x| < 0.5 \text{ and } |y| < 0.5; \\ 0.5, & \text{elswhere.} \end{cases}$$

$$p_5(x,y) \;=\; \begin{cases} 2.0, & \text{if } |x| < 0.5 \text{ and } |y| < 0.5; \\ 0.5, & \text{elswhere.} \end{cases}$$

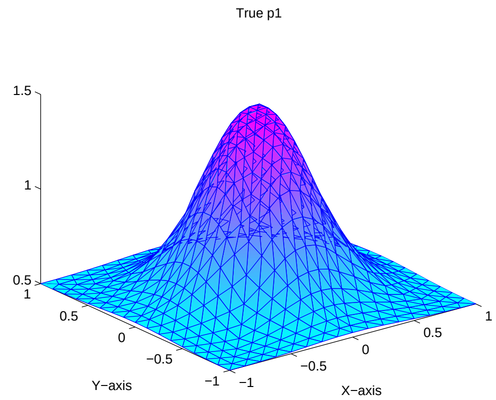
In all the different cases for $p_i, i = 1, 2, \ldots, 5$, the boundary function $g(x,y) = x+y+4$ is used together with the right-hand side function $f(x,y) = x + y + 4$.
First we consider $p_1$ which is continuous and analyze the results from the algorithm using different initial guesses and different values of the step size $\alpha$. We also compare these results with results obtained using line minimization at each descent step.
Secondly we analyze convergence results for the discontinuous cases $p_i,\ i = 2, 3, 4, 5$.
We conclude this chapter by considering the results obtained when noise is added to the solution data $u$.
We note that during the implementation for noisy data, we add noise according to the following formula.

$$\frac{\|u_{noise} - u\|_{L^1(\Omega)}}{\|u\|_{L^1(\Omega)}} = \delta,$$

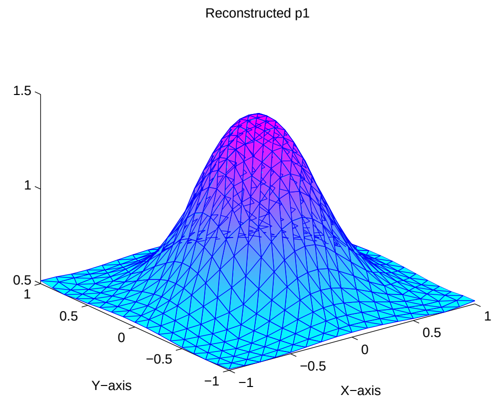
where $u_{noise}$ is the noisy data and $\delta$ is the percentage noise level.

## 5.1 Continuous case

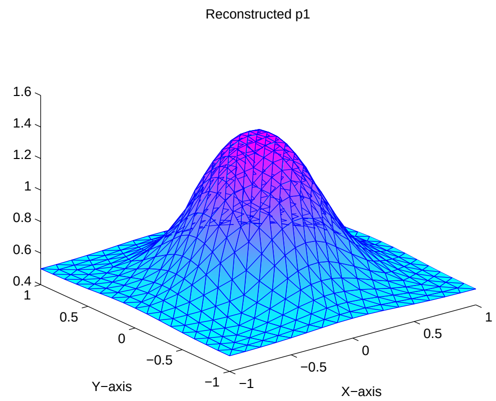
Here we consider the case where the hydraulic conductivity is assumed to be continuous. As taste data, we use the coefficient function $p_1$ which is a gaussian function. This function is used in the forward problem to obtain synthetic data $u$ which is then used for the inverse problem. The following results are obtained for this case.



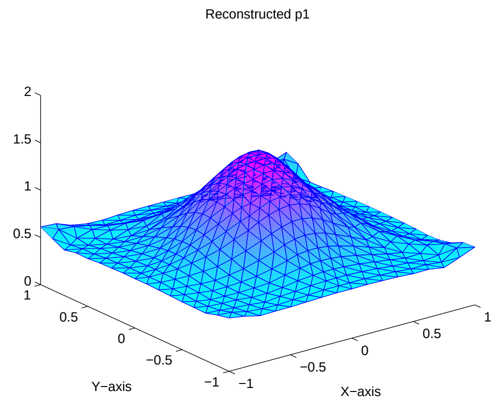

(a) True $p_1(x, y)$




(b) Reconstructed $p_1$, $p_0 = 0.75$




(c) Reconstructed $p_1$, $p_0 = 0.5$




(d) Reconstructed $p_1$, $p_0 = 2.0$

Figure 5.1: Reconstruction of $p_1$

Figure 5.1(a) shows the true plot of $p_1(x, y)$. Figure 5.1(b) was obtained using the steepest descent method with an initial guess $p_0 = 0.75$ and step size $\alpha = 0.1$. For this result there is a 0.03 $L_1$ error between the true $p_1$ and the reconstructed one after 3124 descent steps. However, when the step size $\alpha$ was set to 0.35, the same reconstruction was obtained after 891 descent steps. This shows that there exists a value of $\alpha$ for which the reconstruction is faster.

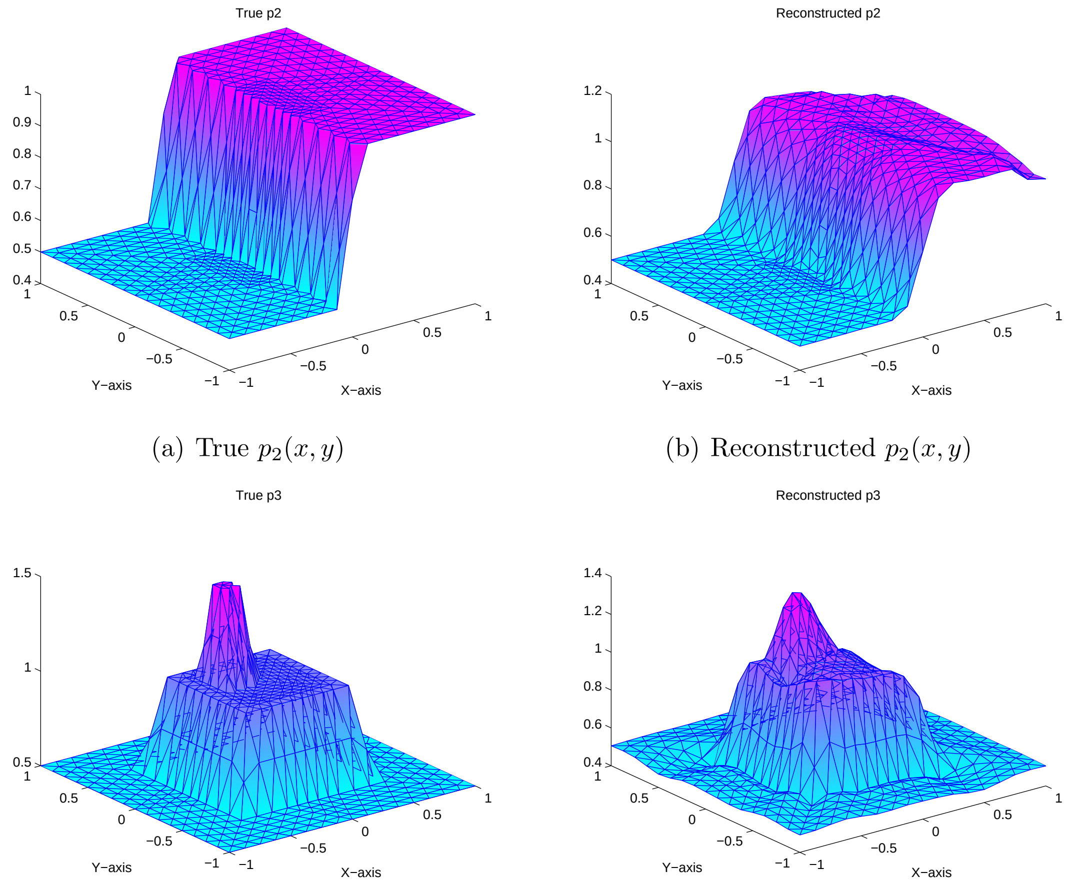
In order to make an improvement on the above results, we carried out a line search minimization in the descent direction at each step as described earlier on in section (4.2). An implementation of the Golden section line search algorithm gives rise to the same $L_1$ error after 446 iterations.

We also note that the initial guess plays an important role in the nature of results obtained during the reconstruction process. Figure 5.1(c) is obtained with an initial guess $p_0 = 0.5$. The same $L_1$ error was obtained after 372 iterations using the line minimization algorithm.

However, the initial guess $p_0 = 2.0$ gives rise to a 0.067 $L_1$ error after 800 iterations of the steepest descent method. This shows that when one chooses an initial guess which is further away from the true value, it takes more descent steps before the true value. This result is represented in figure 5.1(d). It is also worthy noting that even higher values of $p_0$ were tasted and there is convergence of the algorithm. This is due to the fact that the functional is strictly convex. From the above results, we suggest that it is always advisable to choose an initial guess which is close to the given boundary value of the coefficient function $p$.

## 5.2 Discontinuous case

Next we considered the case where the hydraulic conductivity $p$ is assumed to be discontinuous. As taste data we use the values $p_i, i = 2, \dots, 5$. For all the results obtained below, we used an initial guess $p_0 = 0.5$ with a line search at each iterate using the Golden section method.

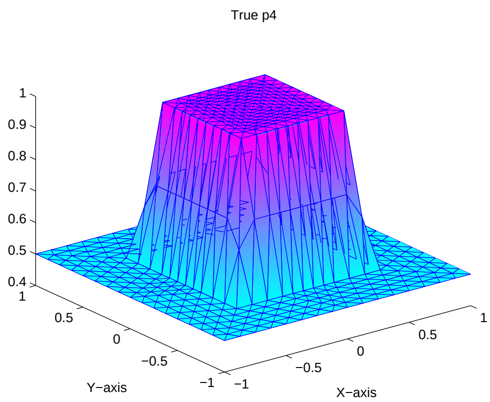



(a) True $p_2(x, y)$ (b) Reconstructed $p_2(x, y)$

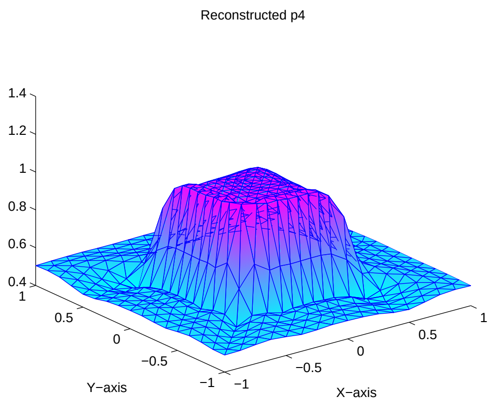
(c) True $p_3(x, y)$ (d) Reconstructed $p_3(x, y)$

Figure 5.2: Reconstruction of $p_2$ and $p_3$

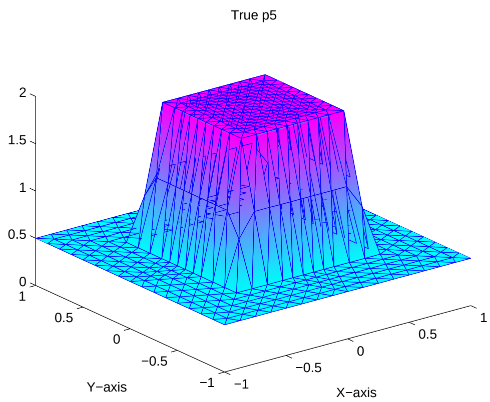
Figure 5.2(b) shows a reconstruction of $p_2(x, y)$ (figure 5.2(a). In this case, the $L_1$ error between the true value of $p_2$ and the reconstructed value is equal to 0.0941 after 600 descent steps. The reconstruction of $p_3$ (figure 5.2(c) is shown in figure 5.2(d). In this case a 0.1856 $L_1$ error was obtained between the true value of $p_3$ and the reconstructed one after 600 iterations. We note that the algorithm works better for continuous values of $p$ than for discontinuous values. This is evident from the $L_1$ errors obtained for $p_1$ results and the results for $p_2$ and $p_3$. However, the discontinuity is well localized in all cases, that is, the position of the discontinuity is accurately determined by the algorithm.



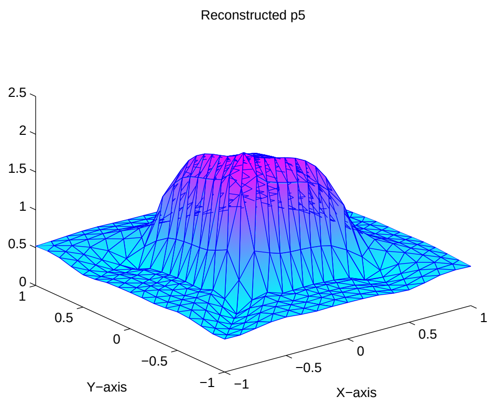

(a) True $p_4(x, y)$




(b) Reconstructed $p_4(x, y)$




(c) True $p_5(x, y)$

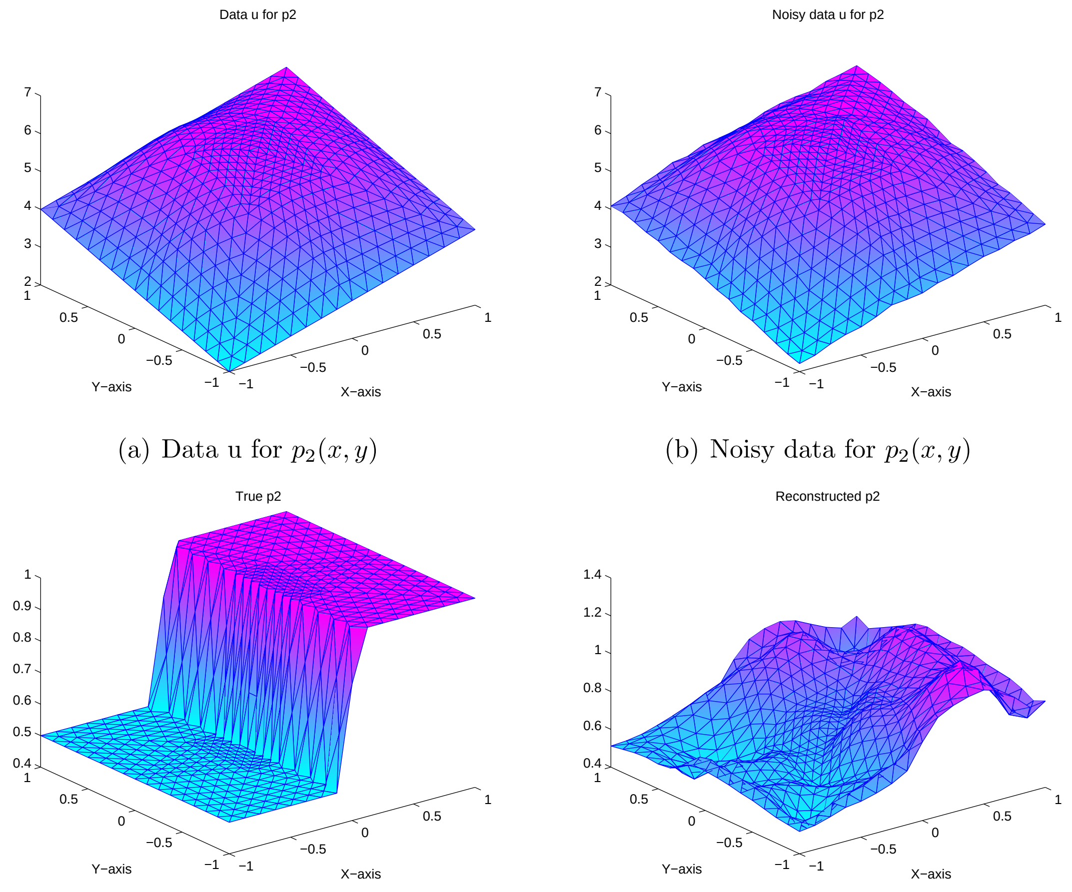



(d) Reconstructed $p_5(x, y)$

Figure 5.3: Comparison of contrast

In figures 5.3(a),(b), (c) and (d) we compare the results obtained for different contrast values between $p$, that is, the difference between the minimum and maximum values of any given $p$. For the reconstruction of $p_4$ (low contrast) we obtained a 0.1636 $L_1$ error while the reconstruction of $p_5$ (high contrast) gave rise to a 0.6703 $L_1$ error,

both results after 600 iterations. Thus it is more difficult to reconstruct discontinuous conductivities with high contrast than ones with low contrast.
Next we analyze the nature of results obtained by adding noise to the synthetic data $u$.




(a) Data u for $p_2(x, y)$

(b) Noisy data for $p_2(x, y)$

(c) True $p_2(x, y)$

(d) Reconstructed $p_2(x, y)$

Figure 5.4: Noisy data

Figure 5.4(a) shows the data $u(x, y)$ for $p_2(x, y)$ while figure 5.4(b) shows the same data after an addition of 2% noisy of the original data. After 14 iterations of the algorithm figure 5.4(d) was obtained as a reconstruction of the true value 5.4(c) of $p_2$.

The $L_1$ error between the true $p_2$ and the reconstructed one is equal to 0.414. We note that the algorithm is sensitive to noise due to the ill posedness of differentiation of data during calculation of the $L_1$ gradient $\nabla G(p)$ and although the noise percentage is quite small, we don't have convergence. It is also important to note that the solution data is obtained by solving the forward problem with a known value of the coefficient function $p$. During this process there are errors between the would be exact solution and the one obtained after finite element discretization. Such errors act as noise to the data and contribute considerably to the ill-posedness of the inverse problem.

# Chapter 6

# The elasticity Problem

In this section, we discuss briefly the elasticity problem. We give the fundamental equations of linear elasticity and state the necessary boundary conditions for the forward problem. We include some possible inverse problems that arise in elasticity theory. Our focus is the identification of coefficient functions that appear in the dynamic elasticity equation when we write the solution as a time-harmonic motion.

## 6.1 Introduction

Elasticity theory describes the reversible deformation of solid bodies subjected to excitations of various physical nature, for example, mechanical, thermal, electromagnetic etc. Such excitations applied as distributions over the body (for example, gravitation, Lorentz forces, thermal expansion) or over the boundary (pressure, contact forces), generate strains and stresses in the material. Elasticity is a mechanical constitutive property of the material whereby there is a one-one relationship between instantaneous strains and stresses on the current deformed configuration and the material reverts to its initial state if the excitation history is reversed. Almost all natural or manufactured solid materials have a deformation range within which their mechanical behavior can be modeled by elasticity theory. For sufficiently small strains, the elastic behavior is considered as linear, that is, strains and stresses are assumed to be proportional to each other.

The theory of linearized elasticity has developed into one of the now classical areas of mathematical physics and is also of great importance in medical section. In the medical section doctors carry out analysis of their patients' health owing to information about the elasticity of tissue in the body. Information on the elastic behavior of tissue can be obtained using magnetic resonance elastography (MRE) which is an innovative method for visualizing strain waves in an object. For a detailed description of MRE, see [25].

Equilibrium problems are governed by elliptic partial differential equations, similar to those of electrostatics but more complex in that physical quantities of interest are described by tensor fields rather than vector fields. Dynamical conditions give rise to hyperbolic partial differential equations.

## 6.2 Review of governing equations

### 6.2.1 Fundamental equations for three dimensional elasticity

The deformation of an elastic body, occupying in its undeformed state the region $\Omega \subset \mathbf{R}^3$ bounded by the surface $S$, is usually described in terms of a vector displacement field $u(x,t); x \in \Omega$ which is such that the deformation process moves a small material element lying at $x$ to its new position $x + u(x,t)$. The linearized elasticity theory is established on the assumption of small strains, namely $|\nabla u(x,t)| \ll 1$. In that case, the changes in metric induced by the deformation are described by the linearized symmetric strain tensor $\varepsilon(x,t)$, defined as a differential operator on $u$ by,

$$\varepsilon[u](x,t) = (\nabla u(x,t) + \nabla u^T(x,t))/2. \tag{6.2.1}$$

This equation is often referred to as the compatibility equation for small deformations. The material is characterized by two constitutive parameters, that is, its mass density distribution $\rho(x)$, associated with the kinetic energy $T(u) = \frac{1}{2}\int_\Omega \rho|u_t|^2 dV$, where $u_t$ denotes time differentiation and the fourth order tensor of elastic moduli $C(x)$, hereafter referred to as the elasticity tensor, associated with the elastic strain energy $E(u) = \frac{1}{2}\int_\Omega \varepsilon[u] : C : \varepsilon[u] dV$.

The stress tensor $\sigma$ describes internal forces and the traction vector $\tau_n$ is such that

$$\tau_n(x,t) = \sigma(x,t).n(x), \tag{6.2.2}$$

where $n(x)$ is the unit normal vector located $x \in \Omega$.

The fundamental balance equation of the dynamics of deformable bodies is given by,

$$\mathrm{div}\sigma(x,t) + f(x,t) = \rho(x) u_{tt}(x,t), \tag{6.2.3}$$

where $f(x,t)$ is a given distribution of body forces. The constitutive assumption of linearized elasticity postulates that the stress tensor $\sigma(x,t)$ depends linearly on the linearized strain tensor, that is,

$$\sigma(x,t) = C(x) : \varepsilon[u](x,t). \tag{6.2.4}$$

Combining the three field equation (6.2.1), (6.2.3) and (6.2.4) and eliminating $\varepsilon$ and $\sigma$, the displacement field is found to be governed by the partial differential equation

$$\mathrm{div}(C(x) : \varepsilon[u](x,t)) + f(x,t) = \rho(x) u_{tt}(x,t). \tag{6.2.5}$$

Equation (6.2.5) is the analogue for linear elasticity of the hyperbolic linear wave equation.

### 6.2.2 Direct problems

For a well-posed problem, elastodynamic equation (6.2.5) should be solved together with initial conditions

$$u(x,0) = u_0(x) \qquad u_t(x,0) = v_0(x), \qquad x \in \Omega \tag{6.2.6}$$

and boundary conditions on the boundary $S$ of $\Omega$, for instance, displacements $\xi$ and traction $\phi$ prescribed on complementary portions $S_u$ and $S_p = S \setminus S_u$ of S,

$$\begin{aligned} u(x,t) &= \xi(x,t) & x \in S_u, \quad t \in [0,T], & \qquad (6.2.7)\\ \tau_n[u](x,t) &= \phi(x,t) & x \in S_p, \quad t \in [0,T] & \qquad (6.2.8) \end{aligned}$$

The geometry $(\Omega)$, the physical characteristics $(C, \rho)$ of the elastic body, the structure of boundary conditions, the prescribed values $(\xi, \phi)$ on the boundary, and the initial data $u_0, v_0$ are assumed to be known.
The direct elastic equilibrium problem is given by the following equation,

$$\operatorname{div}(C(x) : \varepsilon[u](x)) + f(x) = 0, \tag{6.2.9}$$

together with boundary conditions

$$\begin{aligned} u(x) &= \xi(x) & x \in S_u, & \qquad (6.2.10)\\ \tau_n[u](x) &= \phi(x) & x \in S_p. & \qquad (6.2.11) \end{aligned}$$

### 6.2.3 Inverse problem

The inverse problem in elasticity can be a parameter identification problem where one obtains the elasticity moduli distribution. It can also be a problem of identification of inclusions in an object such as cracks or buried objects.
Our concern here is an inverse problem of identification of elasticity moduli $C(x)$.

For the case of isotropic elasticity, $C$ can be expressed in terms of the Lamé coefficients $(\lambda, \mu)$ in the form $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il})$. In this case equation (6.2.5) becomes,

$$\nabla(\lambda\nabla \cdot \overrightarrow{u}) + \nabla \cdot (\mu(\nabla\overrightarrow{u} + \nabla\overrightarrow{u}^T)) + f(x,t) = \rho(x)\overrightarrow{u}_{tt}(x,t), \tag{6.2.12}$$

where $\overrightarrow{u}$ is the vector elastic displacement in an isotropic medium.
When we consider time-harmonic motions, that is, when $u(x,t)$ has the form

$$u(x,t) = \mathrm{Re}[u(x)e^{i\omega t}], \tag{6.2.13}$$

where $\omega$ is the frequency, then the complex-valued unknown field $u(x)$ solves the following equation

$$\nabla(\lambda\nabla \cdot \overrightarrow{u}(x)) + \nabla \cdot (\mu(\nabla\overrightarrow{u}(x) + \nabla\overrightarrow{u}^T(x))) + \rho\omega^2 u(x) + f(x) = 0 \tag{6.2.14}$$

together with boundary conditions such as (6.2.7) and (6.2.8), where the prescribed data $\xi, \phi$ are also complex valued and obey the time-harmonic convention (6.2.13).
We raise a question of whether it is possible to obtain information of tissue stiffness characteristics, that is, $\lambda, \mu$ from the knowledge of the time-independent displacement $u(x)$ using equation (6.2.14). Such a problem is still an open question.

# Chapter 7

# Conclusion and Future work

It is worth noting that although we have mainly followed the work by Professor Ian Knowles, in this thesis we have spent some time in analyzing the solvability of the forward problem and also considered the possibility of solving the inverse problem using the so-called Dirichlet to Neumann map. Our numerical simulations have also been done using finite elements method with the help of matlab PDE tool box instead of finite difference methods.

We have considered the general elliptic forward problem that arises from groundwater modeling and analyzed the existence and uniqueness of solution to the problem. We have also considered ways how to improve the regularity of the solution using the theory of Sobolev spaces. For a more regular solution we require that the coefficient functions as well as the boundary of the domain should satisfy some smoothness assumptions. However in general, the boundedness of these coefficient functions is enough to guarantee existence and uniqueness of solutions to elliptic problems. In chapter 3 we have discussed briefly how to solve the inverse problem by use of Calderón's idea which uses data from boundary measurements. We also presented uniqueness results for the inverse problem. In chapter 4 we have presented a functional which uses the potential energy of a system described by elliptic partial differential equations to reconstruct the hydraulic conductivities in groundwater modeling from interior measurements. This functional is differentiable and convex. This convexity property allows for a stable minimization using the conjugate gradient method.

We have implemented the steepest descent method including a line search method for updating the step size $\alpha$.
As evident from the nature of results, that is, from the number of iterations, we can improve these results by implementing faster algorithms such as the Polak-Ribière conjugate gradient method. It is also worthy to note that the nature of results obtained from such implementations in inverse problems is strongly influenced by the way how the gradient $\nabla u$ of the solution is implemented. This is due to the fact that differentiation is an ill posed inverse problem.
Secondly, in the numerical results we have used synthetic data obtained by solving the forward boundary value problem for known values of $p$. An important step forward is to validate the algorithm using real data obtained from piezometric head methods. It will also be interesting if we can address the problem for the recovery of the full hydraulic conductivity tensor $P$.
In chapter 6 we stated the elasticity problem where one is interested in identification of the Lamé parameters $\lambda$ and $\mu$ from the knowledge of time-independent data in the interior of the domain of consideration. It would be useful to investigate whether we can use functionals such as the one discussed in chapter 4 to identify tissue properties from interior measurements and also find out if such results can be meaningful.

All in all, the field of groundwater modeling remains a rich area for scientific research. No viable methods have been developed to fully recover the hydraulic conductivity tensor.

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# Eidesstattliche Erklärung

Ich, Vincent Ssemaganda, erkläre an Eides statt, dass ich die vorliegende Masterarbeit selbstständig und ohne fremde Hilfe verfasst, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt bzw. die wörtlich oder sinngemäß entnommenen Stellen als solche kenntlich gemacht habe.

Author: ______________________

Vincent SSemaganda.

# Curriculum Vitae

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| Surname: | Ssemaganda |
| Given Names: | Vincent |
| Place of Birth: | Kawempe, Uganda |
| Date of Birth: | 11th, March, 1981 |

## Education

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|---|---|
| Johannes Kepler University Linz (Austria) | 2006-2007 |
| Techinische Universiteit Eindhoven (The Netherlands) | 2005-2006 |
| Makerere University kampala (Uganda) | 2001-2004 |
| City High School(Uganda) | 1999-2000 |
| Najjanankumbi Young Christian School(Uganda) | 1995-1998 |
| ST Kizito primary school(Uganda) | 1987-1994 |

## Special Activities

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|---|---|
| Participant 20th ECMI modeling week Lyngby-Denmark | 2006 |
| Participant modeling week Mathematics for Industry-Eindhoven Netherlands | 2007 |