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Convergence to Equilibrium of an Approximate Stochastic Model for Fiber Spinning

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Abstract

This paper introduces the mathematical modeling of fiber dynamics as it occurs in the practical application of melt-spinning processes of nonwoven materials and focus on the numerical simulation of this model. The model is based on a stochastic differential equation taking into account the motion of the fiber under the influence of turbulence. The model is used to compute the distribution of functionals of the process that might be helpful for the quality assessment of industrial fabrics.

Keywords: Fokker-Planck Equation, Ornstein-Unlenbeck Process, Monte-Carlo Method, Coarse Graining

Contents

1 Introduction

[1] Nonwoven materials/ fleece are webs of long flexible fibers that are used for composite materials (filters) as well as in the hygiene and textile industry. they are produced in melt-spinning processes. Here, hundreds of individual endless fibers are obtained by the continuous extrusion of a molten polymer granular through narrow nozzles. the viscous/ viscoelastic fibers are stretched and spun until they solidify due to cooling air streams. Highly turbulent air flows cause then entanglement and loop forming of the elastic fibers before they lay down on a moving conveyor belt to form a web. The quality of this web and the resulting nonwoven material- in terms of homogeneity and load capacity -depends essentially on they dynamics and the deposition of the fibers.

The software FIDYST developed on basis of the mathematical model of [7] at the Fraunhofer ITWM, Kaiserslautern enables the numerical simulation of the spinning and deposition regime in the nonwoven production processes, cf. Figure1.1. Because of the huge amount of physical details incorporated in FIDYST, the simulations of the fiber spinning and lay-down usually require extremely large computational effort and high memory storage which makes the optimization and control of the full process and particularly of the fleece quality difficult. Thus, a simplified stochastic model for the fiber lay-down process is presented in [2]. Under the assumption of a non-moving conveyor belt, this model describes the fiber position on the transport belt by a stochastic differential system containing characteristic process parameters, e.g. influence of the turbulence, that have to be identified from the full model and adapted in the reduced one. The associated Fokker-Planck equation and stationary solution are investigated. This provides and idea of the computation of the distribution of process functionals.

In this paper, we extend the stochastic model of [2] to a more realistic fiber lay-down model with a moving transport belt, Section 2. An derivation of the reduced Fokker-Planck equation, Section 3. An analytic analysis to the Fokker-Planck equation has been introduced, Section 4. The numerical results in Section 5 show already for moderate values of $\mathbf{A}$ surprisingly good agreements with the limit distributions.

2 Stochastic Model for Fiber Lay-down Process

[1] Consider a slender elastic inextensible endless fiber in a lay-down regime. Let the fiber be produced with the spinning speed v_{spin}, excited into motion by a surrounding highly turbulent air flow and lay down on the conveyor belt moving with the velocity v_{belt}. Due to its slenderness we prescribe the fiber on the two-dimensional transport belt as arc-length parameterized curve $\eta : \mathbb{R}_0^+ \to \mathbb{R}^2$. the fiber satisfies $\|d\eta/dt\| = 1$ because of its inextensibility. This leads to

$$d\eta = (\cos\alpha, \sin\alpha)^T dt.$$

Then, consider

$$\eta(t) = \xi(t) - \kappa t e_1$$

where $\kappa = v_{belt}/v_{spin} \in [0, 1]$ denotes the ratio between belt and spinning speed and $-e_1$ the direction of motion of the transport belt. Apart from the transport part, the actual fiber lay-down is modeled by the $\mathbb{R}^2$ -valued random process ξ. The full model is described by the following stochastic differential system

$$d\xi_1 = (\cos\alpha + \kappa)dt \tag{2.1a}$$

$$d\xi_2 = \sin\alpha dt \tag{2.1b}$$

$$d\alpha = c(\xi)(\xi_1 \sin\alpha - \xi_2 \cos\alpha)dt + A dW_t. \tag{2.1c}$$

Here, $\xi(t) = (\xi_1, \xi_2)(t)$ represents the contract point of the fiber point associated to the arc-length parameter t with the conveyor belt. The angle of the fiber relative to e_1 is given by α. Its change is characterized by the deterministic buckling/coiling $c(\xi)$ of the fiber that tends to turn back to its hitting point and the random fluctuations AdW_t of the fiber due to its interaction with the external turbulent air flow, where W denotes an one-dimensional Wiener process.

Remark 2.1. *The general deterministic coiling behavior of flexible fibers has been studied for example in [[8],[9]]. The function $c(\xi)$ in our model prescribes its amplitude that depends on the lay-down process, it is a scalar-valued function for isotropic processes, a matrix-valued one for anisotropic processes,[2]. For reasons that will become clear later on, cf. Eqn.(3.7), physically reasonable solutions can only be expected if $exp(-B(\xi) - \kappa\xi_1)$ is integrable for $\kappa \in \mathbb{R}$, where $\partial_{\xi_i} B(\xi) = c(\xi)\xi_i$. A typical example satisfying this condition is $c(\xi) = 1$ since then $B(\xi) = (\xi_1^2 + \xi_2^2)/2$.*

For the industrial application nonwoven materials with a homogeneous distribution of mass and fiber orientation are desired as they generally form for small κ and larger A. To get a deeper insight in the density distribution of the underlying ξ-process (2.1), we consider the associated Fokker-Planck equation

$$\partial_t p + (\cos\alpha + \kappa)\partial_{\xi_1} p + \sin\alpha \partial_{\xi_2} p - \partial_\alpha[c(\xi)(-\xi_1 \sin\alpha + \xi_2 \cos\alpha)p] = \frac{A^2}{2}\partial_\alpha^2 p \tag{2.2}$$

where $p = p(\xi_1, \xi_2, \alpha, t; A, \kappa)$ denotes the probatility density.

Remark 2.2. *In the case of a non-moving conveyor belt($\kappa = 0$), the processes η and ξ coincide. Then, it is advantageous to introduce polar coordinates $\xi_1 = r\cos\varphi$, $\xi_2 = r\sin\varphi$ and $\beta = \alpha - \varphi$ and to define $b(r) = \|\xi\| c(\|\xi\|)$ as done in [2]. The resulting system reduces then to two dimenstions and the associated Fokker-Planck equation for (r, β) reads*

$$\partial_t p + \cos\beta \partial_r p + (b(r) - \frac{1}{r})\partial_\beta (p \sin\beta) = \frac{A^2}{2}\partial_\beta^2 p \tag{2.3}$$

In the following we determine the evolution and the stationary solution of the Fokker-Planck equations(2.2) in the limit $A \to \infty$.

3 Derivation of the Reduced Fokker-Planck Equation

[1] In the case of a moving belt, the Fokker-Planck equation (2.2) reads as

$$\partial_t p + ((s + \kappa e_1) \cdot \nabla_\xi) p - \partial_\alpha [c(\xi)(n \cdot \xi) p] = \frac{1}{2\varepsilon} \partial_\alpha^2 p. \tag{3.1}$$

To simplify the notations, we have introduced $s = (\cos\alpha, \sin\alpha)$ and $n = \partial_\alpha s = (-\sin\alpha, \cos\alpha)$ as well as $\varepsilon = 1/A^2$. The distribution p satisfies the normalization condition

$$\int_{\mathbb{R}^2 \times [0,2\pi]} p(\xi, \alpha, t) d^2\xi d\alpha = 1.$$

Additionally we have the initial condition $p(\xi, \alpha, 0) = p_0(\xi, \alpha)$.
In the case of strong stochastic influence, i.e.$\varepsilon \ll 1$, we follow the main ideas of the previous case for $\kappa = 0$, i.e. the non-moving belt. Analogously we introduce the time scales: $\tau = t/\varepsilon$ and t.

Remark 3.1. *Note that in contrast to the case $\kappa = 0$, we do not introduce the slow scale $T = \varepsilon t$, but work with the unscaled time variable t. This is inspired by the fact that the deterministic drift term $\kappa\partial_{\xi_1} p$ due to the moving belt acts on the unscaled time variable. Therefore it is only natural to expect p to depend on t.*

For the distribution function $p = p(\xi, \alpha, t; \varepsilon, \kappa)$ we propose the following modified ansatz:

$$p = p^{(0)}(\xi, \alpha, \tau, t) + \varepsilon p^{(1)}(\xi, \alpha, \tau, t) + \varepsilon^2 p^{(2)}(\xi, \alpha, \tau, t) + \cdots$$

Pugging this into the Fokker-Planck equation(3.1) and collecting equal powers of ε, we obtain again a hierarchy of equations.
In leading order, the problem for $p^{(0)}$ reads as

$$Lp^{(0)} = 0 \tag{3.2a}$$

$$\int_{\mathbb{R}^2 \times [0,2\pi]} p^{(0)} = 1 \tag{3.2b}$$

$$p^{(0)}(\xi, \alpha, 0, 0) = p_0(\xi, \alpha). \tag{3.2c}$$

where $L = \partial_\tau - \partial_\alpha^2/2$ denotes the fast diffusion operator in the angular direction.
The transient solution of the problem (3.2) is given by

$$p^{(0)} = \mathcal{P}(\xi, t) + \sum_{j \in \mathbb{Z}\setminus\{0\}} e^{ij\alpha - j^2\tau/2} C_j(\xi) = \mathcal{P}(\xi, t) + e.s.t.$$

where e.s.t. abbreviates the exponentially small and decaying terms due to the initial condition and

$$C_j(\xi) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ij\alpha} p_0(\xi, \alpha) d\alpha$$

as the Fourier-coefficients of the initial condition. As in the previous case, the exponentially small terms due to the initial condition are of no importance and hence we neglect them.

To include also the effect of the slow scale $T = \varepsilon t$ in the expansion, we write

$$\mathcal{P}(\xi, t) = \mathcal{P}(\xi, t, \varepsilon t) \quad \text{and} \quad \partial_t \mathcal{P} = F^{(0)} + \varepsilon F^{(1)}. \tag{3.3}$$

Remark 3.2. *The ansatz (3.3) is a variant of a Chapman-Enskog expansion.*

In the $\mathcal{O}(\varepsilon)$-order, we have to solve

$$\begin{aligned} Lp^{(1)} &= -(s \cdot \nabla_\xi)\mathcal{P} - \kappa \partial_{\xi_1} \mathcal{P} - \partial_\alpha (c(\xi)(s \cdot \xi)\mathcal{P} - F^{(0)}, \\ \int_{\mathbb{R}^2 \times [0,2\pi]} p^{(1)} &= 0, \\ p^{(1)}(\xi, \alpha, 0, 0) &= 0. \end{aligned}$$

Canceling the secular terms in this equation leads to the condition

$$0 = \kappa \partial_{\xi_1} \mathcal{P} + F^{(0)} \tag{3.4}$$

This condition reflects the transport of $\mathcal{P}$ with the belt velocity κ in the ξ_1-direction happening on the original time scale t.Furthermore, we get

$$p^{(1)} = -2[s \cdot (\nabla_\xi + c(\xi)\xi)\mathcal{P}]$$

To determine the reduced Fokker-Planck equation , we have to take the second order into account

$$\begin{aligned} Lp^{(2)} &= -(s \cdot \nabla_\xi)p^{(1)} - \kappa \partial_{\xi_1} p^{(1)} - \partial_\alpha (c(\xi)(s \cdot \xi)p^{(1)} - F^{(1)}, \\ \int_{\mathbb{R}^2 \times [0,2\pi]} p^{(2)} &= 0, \\ p^{(2)}(\xi, \alpha, 0, 0) &= 0. \end{aligned}$$

As in the case of the non-moving belt, we identify the secular terms and eliminate them

$$0 = \nabla_\xi \cdot (\nabla_\xi + c(\xi)\xi)\mathcal{P} - F^{(1)} \tag{3.5}$$

Inserting the conditions (3.4)and (3.5) in (3.3) yields the reduced equation

$$\partial_t \mathcal{P} = \nabla_\xi \cdot (\varepsilon \nabla_\xi + \varepsilon c(\xi)\xi - \kappa e_1)\mathcal{P} \tag{3.6}$$

The stationary solution $\mathcal{P}_s(\xi)$ is characterized by

$$\nabla \cdot (\varepsilon \nabla + \varepsilon c(\xi)\xi - \kappa e_1)\mathcal{P}_s = 0$$

together with the normalization condition

$$\int_{\mathbb{R}^2} \mathcal{P}_s d^2\xi = 1$$

The solution of this linear PDE is given by

$$\mathcal{P}_s(\xi;\kappa) = \kappa e^{-B(\xi)-\kappa\xi_1/\varepsilon} \tag{3.7}$$

where $\nabla B(\xi) = c(\xi)\xi$ and κ is the normalization constant.

The Fokker-Planck equation describes the evolution of the probability density corresponding to the diffusion law at time t, see [3]
Let $(\mathrm{X}(t), t \geq 0)$ be the unique solution of the SDE

$$dX(t) = \mathbf{b}(X(t))dt + \sigma(X(t))d\mathbf{B}(t), \qquad X(0) = X$$

with $\mathbf{b}$ and σ Lipschitz functions and X square integrable random variable that is $\mathcal{F}_0$-measurable and has the probability density $p_0(\mathrm{x})$. The infinitesimal generator $\mathbf{A}$ of this diffusion is given by

$$(Af)(\mathrm{x}) = \frac{1}{2}\sum_{i,j=1}^{n} a_{i,j}(\mathrm{x})\partial_{i,j}f(\mathrm{x}) + \sum_{j=1}^{n} b_j(\mathrm{x})\partial_j f(\mathrm{x})$$

with $a(\mathrm{x}) = \sigma(\mathrm{x})\sigma^T(\mathrm{x})$. Moreover, let μ_t denote the probability law of $\mathrm{X}(t)$,

$$\mu_t(f) = \mathbb{E}[f(\mathrm{X}(t))], \qquad f \qquad \text{continuous, bounded}$$

Theorem 3.1. *Let f be a bounded, twice continuously differentiable function with bounded first and second order partial derivatives. Then*

$$\mu_t(f) = \mu_0(f) + \int_0^t \mu_s(Af)ds$$

Based on Theorem (3.1) we establish the Fokker-Planck equation. Therefore, we introduce the adjoint operator $A^\star$ of the infinitesimal generator $\mathbf{A}$ by

$$(A^\star f)(\mathrm{x}) = \frac{1}{2}\sum_{i,j=1}^{n} \partial_{ij}(a_{ij}(\mathrm{x})f(\mathrm{x})) - \sum_{j=1}^{n} \partial_j(b_j(\mathrm{x})f(\mathrm{x})).$$

Then, the following relation holds

$$\int_{\mathbb{R}^n} (Af)(\mathrm{x})g(\mathrm{x}) = \int_{\mathbb{R}^n} f(x)(A^\star g)(\mathrm{x})dx$$

for twice continuously differentiable functions f and g of which at least one has compact support.

Theorem 3.2. *Assume that the law of the random variable* $\mathbf{X}(t)$ *has a probability density* $p \in \mathcal{C}^{1,2}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R})$. *Then this density satisfies the Fokker-Planck equation*

$$\partial_t p(t, \mathrm{x}) = (A^\star p)(t, \mathrm{x}) \qquad (t, \mathrm{x}) \in \mathbb{R}^+ \times \mathbb{R}^n$$

$$p(0, \mathrm{x}) = p_0(\mathrm{x}) \qquad \textit{almost surely in } \mathrm{x}$$

According to the Theorem(3.2), we got the associated SDE in our problem which is

$$d\xi = -\varepsilon c(\xi)\xi dt + \kappa e_1 dt + \sqrt{2\varepsilon} dW_t$$

Remark 3.3. *Obviously, we obtain a stationary distribution independent of* ε *only if* κ *is proportional to* $\varepsilon = 1/A^2$. *This means, we deal with the case of large* A *and small* κ.

Remark 3.4. *As in the case of the non-moving belt, we consider the special case* $c(\xi) = 1$, *i.e.* $b(r) = r$. *Then* $B(\xi) = \xi_1^2/2 + \xi_2^2/2$ *and we obtain the Ornstein-Uhlenbeck type process prescribed by*

$$d\xi = -\varepsilon \xi dt + \kappa e_1 dt + \sqrt{2\varepsilon} dW_t$$

or respectively

$$\partial_t \mathcal{P} = \nabla \cdot (\varepsilon \nabla + \varepsilon \xi - \kappa e_1)\mathcal{P}$$

4 Analytical Analysis of the Fokker-Planck Equation

This section deals with analytical analysis of the Fokker-Planck equation. From Section 3, we have already obtained the Ornstein-Unlenbeck type process prescribed by

$$d\xi = -\varepsilon c(\xi)\xi dt + \kappa e_1 dt + \sqrt{2\varepsilon} dW_t$$

or respectively

$$\partial_t \mathcal{P} = \nabla \cdot (\varepsilon \nabla + \varepsilon c\xi - \kappa e_1)\mathcal{P}.$$

Its stationary density distribution is Gaussian, centered at $(\xi_1 - \kappa/\varepsilon, \xi_2)$ with variance $\sigma = 1$

$$\mathcal{P}_s(\xi;\kappa) = \frac{1}{2\pi} e^{-(\xi_1-\kappa/\varepsilon)^2/2 - \xi_2^2/2} \tag{4.1}$$

[1] To investigate the relaxation to the stationary solution in more detail, we focus on the case $c(\xi) = 1$. To compute the density of the process explicitely, we assume, that the initial distribution is a Dirac delta centered at some point $\mu_0 \in \mathbb{R}$. We make the following ansatz for the transient distribution

$$\mathcal{P}(\xi,t) = \frac{f(t)}{2\pi} e^{-(\xi-\mu(t)/\varepsilon)^2/(2\sigma(t))}, \tag{4.2}$$

i.e. a Gaussian with moving center $\mu(t)$, variance $\sigma(t)$ and normalization constant $f(t)$. Plugging this ansatz into the reduced Fokker-Planck equation(3.6) and equating for all ξ_1, ξ_2 yields after some calculations

$$\begin{aligned} \frac{d\sigma}{dt} &= 2\varepsilon(1-\sigma) \\ \frac{d\mu}{dt} &= \varepsilon(\kappa e_1 - \mu) \\ \frac{df}{dt}\sigma + f\frac{d\sigma}{dt} &= 0 \end{aligned}$$

Together with the initial conditions $\sigma(0) = 0$,$\mu(0) = \mu_0$ and $f(0) = 1$, we obtain $f = 1/\sigma$ and the following motions of the center and variance

$$\begin{aligned} \sigma(t) &= 1 - e^{-2\varepsilon t} \\ \mu(t) &= \kappa e_1(1 - e^{-\varepsilon t}) + \mu_0 e^{-\varepsilon t} \end{aligned}$$

Compare this result with the explicit solution formulas for linear stochastic differential equations, see[4].

Remark 4.1. *Note that the relaxation to the stationary solution, i.e. $\sigma = 1$ and $\mu = \kappa$, happens on the slow time scale $T = \varepsilon t$. Furthermore the decay rate for the variance is twice the decay rate of the center.*

We have derived the reduced Fokker-Planck equation (3.6)

$$\partial_t \mathcal{P} = \nabla \cdot (\varepsilon \nabla \mathcal{P} + (\varepsilon c\xi - \kappa e_1)\mathcal{P})$$

in the case of dominating stochastic forcing $A^2 = 1/\varepsilon \gg 1$. The "relative velocity" κ of the lay-down process as well as the function $c = c(\xi)$ governing the deterministic fiber bending are still arbitrary. The stationary distribution $\mathcal{P}_s$(3.7) is of Gaussian type

$$\mathcal{P}_s = \kappa e^{-B(\xi)-\kappa\xi_1/\varepsilon}$$

with $\nabla B(\xi) = c(\xi)\xi$.
The convergence against this stationary solution can be proven by classical arguments for a recent discussion. Let us introduce the Kullback relative entropy

$$\mu = \int \mathcal{P} \ln \frac{\mathcal{P}}{\mathcal{P}_s} \tag{4.3}$$

The rate of dissipation of the entropy is given by

$$\partial_t \mu = \int \partial_t \mathcal{P} \ln \frac{\mathcal{P}}{\mathcal{P}_s} = \int \ln \frac{\mathcal{P}}{\mathcal{P}_s} \nabla \cdot [\varepsilon \nabla \mathcal{P} + (\varepsilon c\xi - \kappa e_1)\mathcal{P}]$$

and after integration by parts

$$\partial_t \mu = -\int [\nabla \ln \frac{\mathcal{P}}{\mathcal{P}_s}] \cdot [\varepsilon \nabla \mathcal{P} + (\varepsilon c\xi - \kappa e_1)\mathcal{P}]$$

Using the fact, that $\varepsilon \nabla \mathcal{P}_s = -(\varepsilon c\xi - \kappa e_1)\mathcal{P}_s$, we get

$$\partial_t \mu = -\varepsilon \int \mathcal{P} (\nabla \ln \frac{\mathcal{P}}{\mathcal{P}_s})^2 \geq 0$$

Hence, the entropy is monotonically decaying in time and $\mu = 0$ if and only if $\mathcal{P} = \mathcal{P}_s$. Applying the logarithmic Sobolev inequality [4], we obtain

$$\partial_t \mu \geq -2\varepsilon \mu \tag{4.4}$$

and hence a decay rate of $e^{-2\varepsilon t}$ for the entropy μ. Using the Csiszar-Kullback inequality one obtains a decay rate of $e^{-\varepsilon t}$ for the L_1-distance of $\mathcal{P}$ and $\mathcal{P}_s$.

5 Numerical Investigation of the F-P Equation

5.1 Approximation of $\mathcal{P}$ by Monte-Carlo Method

This section deals with numerical analysis of the Fokker-Planck equation, which using Monte-Carlo methods to approximate $\mathcal{P}$. From Section 3, we have already obtained the Ornstein-Unlenbeck type process prescribed by

$$d\xi = -\varepsilon c(\xi)\xi dt + \kappa e_1 dt + \sqrt{2\varepsilon} dW_t$$

or respectively

$$\partial_t \mathcal{P} = \nabla \cdot (\varepsilon \nabla + \varepsilon c\xi - \kappa e_1)\mathcal{P}.$$

The Feynman-Kac formula enables the probabilistic interpretaton of parabolic problems,then the solutions of parabolic PDEs can be probabilistically represented as an expectation of a functional on a diffusion process, i.e. $\mathbb{E}[\psi(X(s))]$, where $(X(s), s \geq 0)$ is the solution of a stochastic differential equation,see [3].

Consider a $\mathbb{R}^n$-valued diffusion process that solves the SDE

$$d\mathrm{X}(t) = \mathbf{b}(\mathrm{X}(t))dt + \sigma(\mathrm{X}(t))d\mathrm{B}(t)$$

Since the coefficients $\mathbf{b}$ and σ do not explicitely depend on time t such a process is said to be homogeneous. Denot the unique solution of

$$\mathrm{X}(s) = \mathrm{x} + \int_t^s \mathbf{b}(\mathrm{X}(u))du + \int_t^s \sigma(\mathrm{X}(u))d\mathrm{B}(u), \qquad s \geq t$$

by($\mathrm{X}^{t,\mathrm{x}}(s)$,$s \geq t$). This process is the solution of the stated SDE starting from x at time t, i.e. $\mathrm{X}^{t,\mathrm{x}}(s) = \mathrm{x}$. Abbreviate $\mathrm{X}^{\mathrm{x}} := \mathrm{X}^{0,\mathrm{x}}$. Moreover let $f : \mathbb{R}^n \to \mathbb{R}$ and c be a bounded continuous function.

Theorem 5.1. *(Generalized Feynman-Kac Formula). Let f and $g : \mathbb{R}^n \to \mathbb{R}$ be continuous and $c : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ be bounded from below. Let $u \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R}^n, \mathbb{R})$ with bounded derivatives in* x *that satisfies the PDE*

$$\begin{aligned} (\partial_t u + Au - cu)(t,\mathrm{x}) &= f(\mathrm{x}), \qquad (t,\mathrm{x}) \in [0,T] \times \mathbb{R}^n \\ u(T,\mathrm{x}) &= g(\mathrm{x}), \qquad \mathrm{x} \in \mathbb{R}^n \end{aligned}$$

Denote $\beta_{t,s} = e^{-\int_t^s c(\mathrm{X}^{t,\mathrm{x}}(\eta))d\eta}$, then for all $(t,\mathrm{x}) \in [0,T] \times \mathbb{R}^n$, u has the following probabilistic representation

$$u(t,\mathrm{x}) = \mathbb{E}[\beta_{t,T} g(\mathrm{X}^{t,\mathrm{x}}(T)) - \int_t^T \beta_{t,\eta} f(\mathrm{X}^{t,\mathrm{x}}(\eta))d\eta]$$

Remark 5.1. *Note that the linear operator A of the PDE is the infinitesimal generator of the diffusion process $(\mathrm{X}^{t,\mathrm{x}}(s), s \geq t)$. Since the diffusion process is homogeneous, the probability law of $(\mathrm{X}^{t,\mathrm{x}}(s), s \geq t)$ equals the one of $(\mathrm{X}^{\mathrm{x}}(s), s \geq 0)$. Consequently, the result of Theorem 5.1 can be rewritten as*

$$u(t,\mathrm{x}) = \mathbb{E}[\beta_{0,T-t} g(\mathrm{X}^{\mathrm{x}}(T-t)) - \int_0^{T-t} \beta_{0,\eta-t} f(\mathrm{X}^{\mathrm{x}}(\eta)) d\eta].$$

Remark 5.2. *Setting $c = 0$ and $f = 0$ in the Feynman-Kac formula, the regular solution u of the PDE*

$$\begin{aligned} (\partial_t u + Au)(t,\mathrm{x}) &= 0, \qquad (t,\mathrm{x}) \in [0,T] \times \mathbb{R}^n \\ u(T,\mathrm{x}) &= g(\mathrm{x}), \qquad \mathrm{x} \in \mathbb{R}^n \end{aligned}$$

has the probabilistic representation

$$u(t,\mathrm{x}) = \mathbb{E}[g(\mathrm{X}^{t,\mathrm{x}}(T))] = \mathbb{E}[g(\mathrm{X}^{\mathrm{x}}(T-t))].$$

This equation is traditionally called a retrograde Kolmogorov equation.

Remark 5.3. *For evolution problems with time-independent operator A and initial condition Theorem 5.1 can be re-expressed in the following way. Let u be a (regular) function satisfying*

$$\begin{aligned} \partial_t u(t,\mathrm{x}) &= Au(t,\mathrm{x}) - cu(t,\mathrm{x}) - f(\mathrm{x}), \qquad (t,\mathrm{x}) \in [0,T] \times \mathbb{R}^n \\ u(0,\mathrm{x}) &= g(\mathrm{x}), \qquad \mathrm{x} \in \mathbb{R}^n \end{aligned}$$

then its probabilistic representation reads

$$u(t,\mathrm{x}) = \mathbb{E}[\beta_{0,t} g(\mathrm{X}^{\mathrm{x}}(t)) - \int_0^t \beta_{0,\eta} f(\mathrm{X}^{\mathrm{x}(\eta)}) d\eta].$$

Result 5.1. *In our case, the solution $\mathcal{P}$ of the PDE*

$$\begin{aligned} \partial_t \mathcal{P} &= \nabla \cdot (\varepsilon \nabla + \varepsilon c \xi - \kappa e_1)\mathcal{P}, \qquad (t,\xi) \in [0,T] \times \mathbb{R}^2 \\ \mathcal{P}(0,\xi) &= P_0(\xi), \qquad \xi \in \mathbb{R}^2 \end{aligned}$$

has the probabilistic representation

$$\mathcal{P}(t,\xi) = \mathbb{E}[P_0(\xi^{0,\xi}(t))] = \mathbb{E}[P_0(\xi^{\xi}(t))]. \tag{5.1}$$

Now, we need to simulate the above result(6.1) by Monte-Carlo Method(MCM). The implementation of a MCM contains three steps:

Step1. Simulation of a trajectory of the diffusion process at a finite number of time points $0 \leq t_1 \leq t_2 < \cdots < t_n$. Here, the solution process of the SDE is approximated by help of the Euler scheme since it is not known explicitely.

Step2. Approximation of the functional $\psi(X(s))$ by a quantity of the type ψ_n being a unique function of $X(t_1),\cdots, X(t_n)$.

Step3. Repeat Step1 and Step2 M times and then compute the mean value $\frac{1}{M}\sum\psi_M$ for every path we get one function value.

Algorithm 5.1. *(Euler-Maruyama)*
Assume that we are interested in the solution of a parabolic problem that is represented via a $\mathbb{R}^n$-valued diffusion process given by

$$d\mathrm{X}(t) = \boldsymbol{b}(\mathrm{X}(t))dt + \sigma(\mathrm{X}(t))d\mathbf{B}(t), \qquad \mathrm{X}(0) = \mathrm{x},$$

with $\boldsymbol{b} : \mathbb{R}^n \to \mathbb{R}^n$,$\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ and $(\mathbf{B}(t), t \geq 0)$ d- dimensional Brownian motion. In the absence of an exact solution form, we approximate the solution of the SDE by help of Euler schemes:
Let the time interval, $t \in [0, T]$, be discretized by $t_i = ih, i = 0, 1, \ldots, N$ with time step $h > 0$. Initialize the numerical approximation $\bar{X}$ of the diffusion process X *by $\bar{X}(0) = \mathrm{x}$. Then, the Euler scheme reads*

$$\bar{X}((i+1)h) = \bar{X}(ih) + \boldsymbol{b}(\bar{X}(ih))h + \sigma(\bar{X}(ih))(\mathbf{B}((i+1)h) - \mathbf{B}(ih)), i = 1, \ldots, N$$

The sampling of the random variables $(\mathbf{B}((i+1)h)-\mathbf{B}(ih)) \in \mathcal{N}(0, hI)$ with identity matrix $\mathrm{I} \in \mathbb{R}^{d\times d}$ is carried out as followings :
Brownian motion $(B(t), 0 \leq t \leq T)$ is the basis for the simulation of more complex diffusion . The probability law of $B(T)$ is Gaussian centered with variance T. As a Brownian motion has independent increments, the simulation of a path at $t_1 < t_2 < \ldots < t_n$ reduces to

$$\begin{aligned} B(t_1) &= \sqrt{t_1}g_1 \\ B(t_k) - B(t_{k-1}) &= \sqrt{t_k - t_{k-1}}g_k, \qquad k = 2, \ldots, n \end{aligned}$$

where $g_k \in \mathcal{N}(0, 1)$ independent, $k = 1, \ldots, n$. Hence, the law of the n-tupel $(B(t_1), \ldots, B(t_n))$ can be simulated exactly.

Using Algorithm (5.1), we can simulate the Stochastic Differential Equation to get the trajectories, see figure(1):

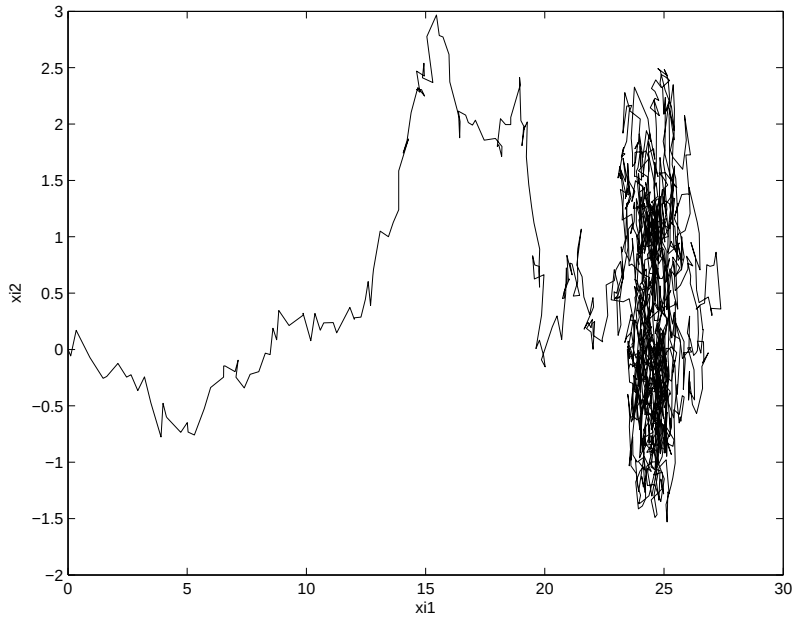

Figure 1: Trajectories from simulation of SDE

5.2 Computation of μ by Coarse Graining Method

To investigate the convergence of the reduced Fokker-Planck equation (3.6), we introduce the Kullback relative entropy

$$\mu = \int \mathcal{P} \ln \frac{\mathcal{P}}{\mathcal{P}_s}$$

Here, we use a Particle Methods—**Coarse Graining**[5] to compute the nonlinear functionals above.

Let $f : \mathbb{R} \to \mathbb{R}^+$ be a density function, i.e. f is positive and $\mathcal{L}^1(\mathbb{R})$-integrable with $\int_{\mathbb{R}} f(v)dv = 1$. We write $f \in \mathcal{L}^{1,1}_+(\mathbb{R})$. The particle ensembles $\{(v_1, \ldots, v_N)\}_{N\in\mathbb{N}}$ with $v_j \in \mathbb{R}$, $j = 1, \ldots, N$, approximate f, if the measure $\omega_N := (1/N)\sum_{i=1}^N \delta(v_i - v)$, $N \in \mathbb{N}$ generated by the above point sets converges weakly in the measure theoretic sense to $f(v)dv$. Here $\delta(v_i - v)$ denotes the Dirac measure at the point v_i. We write $\omega_N \to f$ weakly as $N \to \infty$. It is equivalent to requiring that the discrepancy $D(\omega_N, f)$ tends to 0 as N tends to infinity with

$$D(\omega_N, f) := sup_{a<b} \mid \frac{1}{N}\sum_{j=1}^{N} \chi_{[a,b)}(v_j) - \int_{[a,b)} f(v)dv \mid .$$

Here we denoted by $\chi_{[a,b)}$ the characteristic function of the interval $[a, b) \subset \mathbb{R}$.

As a first step we approximate the density function f in a strong sense. Let $\{\Delta_i^{(n)}\}_{i\in\mathbb{Z}}$

be an equidistant partition of $\mathbb{R}$ into intervals $\Delta_i^{(n)} = [a_i^{(n)}, b_i^{(n)})$ of size $\mid \Delta_i^{(n)} \mid = 1/n$, $\Delta_i^{(n)} \cap \Delta_j^{(n)} = \emptyset$, $i \neq j$, $\cup_{i\in\mathbb{Z}} \Delta_i^{(n)} = \mathbb{R}$. Then we define for $v \in \mathbb{R}$ and $\Delta_i^{(n)}(v)$ the interval of the above partition containing v

$$P_n\omega_N(v) := \frac{1}{\mid \Delta_i^{(n)}(v) \mid}\omega_N(\Delta_i^{(n)}(v)) = \frac{n}{N}\sum_{j=1}^{N}\chi_{\Delta_i^{(n)}(v)}(v_j).$$

Analogously,

$$P_n f(v) := n\int_{\Delta_i^{(n)}(v)} f(v')dv'.$$

In the next lemma we give an error estimate for the approximation of f by $P_n\omega_N$ as n and N tend to infinity.

Lemma 5.1. *Let $f \in \mathcal{L}_+^{1,1}(\mathbb{R})$ be Lipschitz continuous with Lipschitz constant L and ω_N point measures approximating f, i.e. $\omega_N \to f$ weakly, then we have $\forall v \in \mathbb{R}$:*

$$\mid P_n\omega_N(v) - f(v) \mid \leq \frac{L}{n} + nD(\omega_N, f)$$

In particular $P_n\omega_N \to f$ pointwise in $\mathbb{R}$, if n, $N \to \infty$ and $D(\omega_N, f)$ tends faster to 0 than $1/n$.

Remark 5.4. *Pointsets $\{(v_1, \ldots, v_N)\}_{N\in\mathbb{N}}$ can be easily constructed for any $f \in \mathcal{L}_+^{1,1}(\mathbb{R})$ such that $D(\omega_N, f) \sim 1/N$. This rate is optimal. If $\{(v_1, \ldots, v_N)\}_{N\in\mathbb{N}}$ is generated by a sequence of points $v_1, \ldots, v_N, \ldots$, then the optimal conmvergence rate is $D(\omega_N, f) \sim (\ln N/N)$,see Kuipers et al. This means that for n tending to infinity like N^k, $0 < k < 1$ we get $P_n\omega_N \to f$. In contrast if $n \sim N^k$,$k \geq 1$ it is easy to construct a counterexample even for points with an optimal convergence rate.*

We use this approximation to compute nonlinear functionals of f. Let f be Lipschitz continuous and in $\mathcal{L}_+^{1,1}(\mathbb{R})$ as before. Moreover we assume f to be 0 outside of $B_R := \{v \in \mathbb{R} \mid\mid v \mid \leq R\}$,$R > 0$ and the approximating points $\{(v_1, \ldots, v_N)\}_{N\in\mathbb{N}}$, $\omega_N \to f$ weakly to be in B_R. According to the above lemma we assume also that $nD(\omega_N, f)$ tends to 0. The assumptions on f are not really necessary: we could have also assumed f to decay fast enough as v tends to infinity.

We are interested in functionals of the form

$$\int \phi(f(v))dv,$$

where $\phi : \mathbb{R}^+ \to \mathbb{R}$ is continuous and $\phi(0) = 0$. Define for $h > 0$ and $C > 0$ the modulus of continuity

$$M_\phi^C(h) := sup\{\mid \phi(x) - \phi(y) \mid\mid x, y \in [0, C], \mid x - y \mid \leq h\}.$$

Proposition 5.1.

$$| \int \phi(f(v))dv - \sum_{i\in\mathbb{Z}} \frac{1}{n}\phi(\frac{n}{N}\sum_{j=1}^{N} \chi_{\Delta_i^{(n)}}(v_j)) | \leq C_f M_\phi^{C_f}(\frac{L}{n} + nD(\omega_N, f))$$

where C_f is a constant depending on f.

Remark 5.5. *We remark that the conditions on ϕ are fullfilled for example if $\phi(x) = x^p$,$p \in \mathbb{N}$ or for the entropy functional, i.e.$\phi(x) = -x\ln x$, if $x > 0$ and $\phi(x) = 0$, if $x = 0$. Moreover we have for these choices of ϕ by an easy computation $M_\phi^C(h) \leq Ah$ and $M_\phi^C(h) \leq Ah \mid \ln h \mid$, respectively, for h small and A a constant depending on C.*

This means that M_ϕ^C tends to 0 for h tending to 0 with a certain rate. Choosing e.g. an optimal sequence of points $v_1, \ldots, v_N, \ldots$ s.t. $D(\omega_N, f) \sim (\ln N/N)$ and $n = (N/\ln N)^{1/2}$, we get for the entropy functional convergence rates for the approximation in Proposition 5.1 proportional to $((\ln N)^{3/2}/N^{1/2})$.

Result 5.2. *In our case, we could substitute $\phi(f(v))$ by $\phi(\mathcal{P}(\xi))$, and set*

$$\phi(\mathcal{P}(\xi)) = \mathcal{P}(\xi)\ln\frac{\mathcal{P}(\xi)}{\mathcal{P}_s(\xi)},$$

and then we could approximate μ by

$$\mu = \int \phi(\mathcal{P}(\xi))d\xi \to \sum_{i\in\mathbb{Z}} \frac{1}{n}\phi(\frac{n}{N}\sum_{j=1}^{N} \chi_{\Delta_i^{(n)}}(\xi_j)). \tag{5.2}$$

To simulate the result in (5.2),we do the following two steps:

Step1. Generation of the uniform grid within $[\min(\xi_1), \max(\xi_1)] \times [\min(\xi_2), \max(\xi_2)]$, see figure (2).

Step2. Using a counter to calculate the number of points in each cell which generated by step 1, see figure (3).

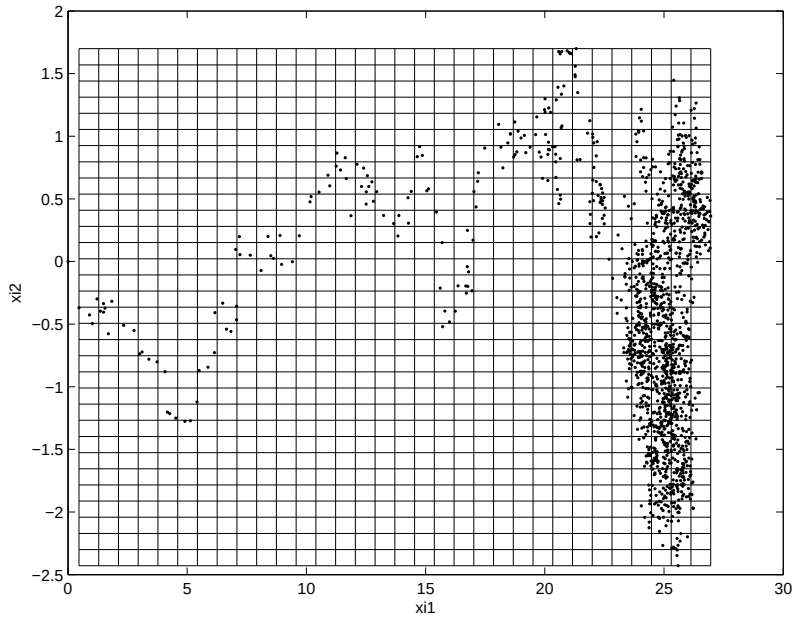

Figure 2: Uniform grid within $[\min(\xi_1), \max(\xi_1)] \times [\min(\xi_2), \max(\xi_2)]$

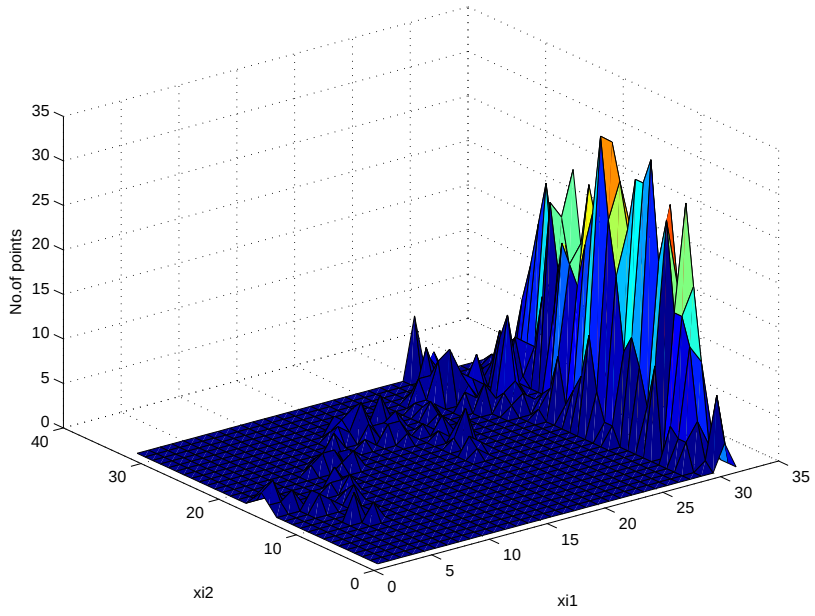

Figure 3: Number of points in each cell

6 Numerical Results of the Fokker-Planck Equation

6.1 Distribution of Stationary Solution $\mathcal{P}_s$

The stationary solution reads

$$\mathcal{P}_s(\xi;\kappa) = \frac{1}{2\pi} e^{-(\xi_1-\kappa/\epsilon)^2/2-\xi_2^2/2}$$

being independent of ϵ for $\kappa A^2 = k$, $k \in \mathbb{R}$. To test the approximation equality of this l limit distribution we compare it with the results for various A. Figure (4-8) shows the trajectories of the SDE and the stationary density distributions of the components ξ_1 for $k = 0.5$ respectively. They are computed from 1500 Monte-Carlo simulations of the ξ-process (2.1). Whereas the distribution functions for $A < 1$, largely differ, they already approach qualitatively for $A = 1$ and show good agreement for $A \geq 2$.

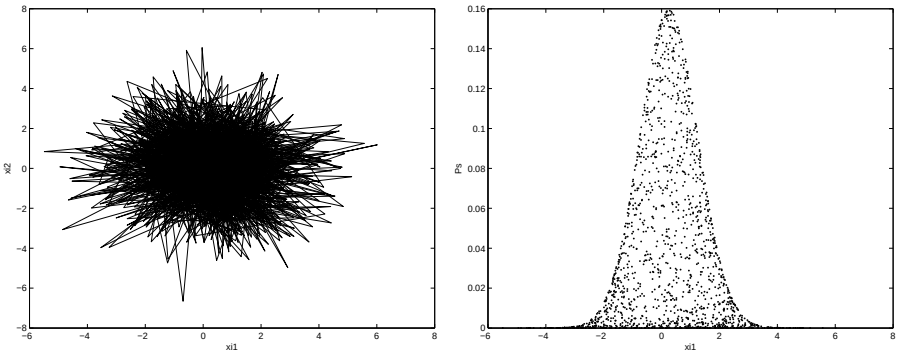

Figure 4: $c(\xi) = 1$,$A = 0.70711$,$k = 0.5$

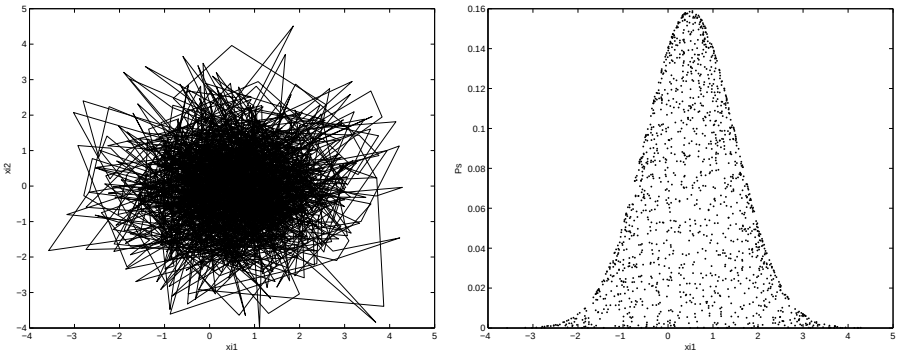

Figure 5: $c(\xi) = 1$,$A = 1$,$k = 0.5$

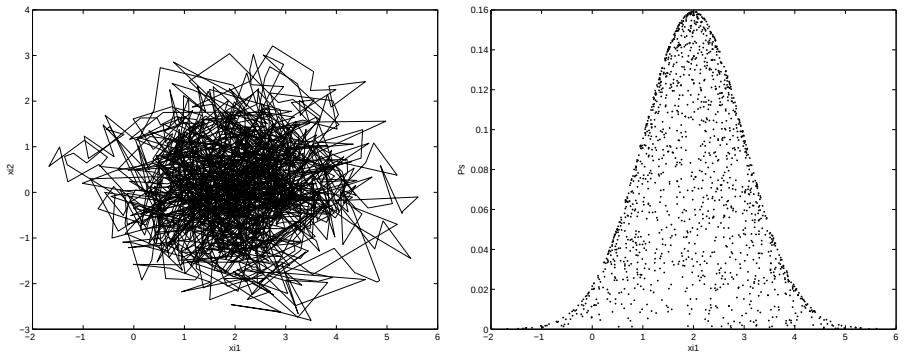

Figure 6: $c(\xi) = 1$,$A = 2$,$k = 0.5$

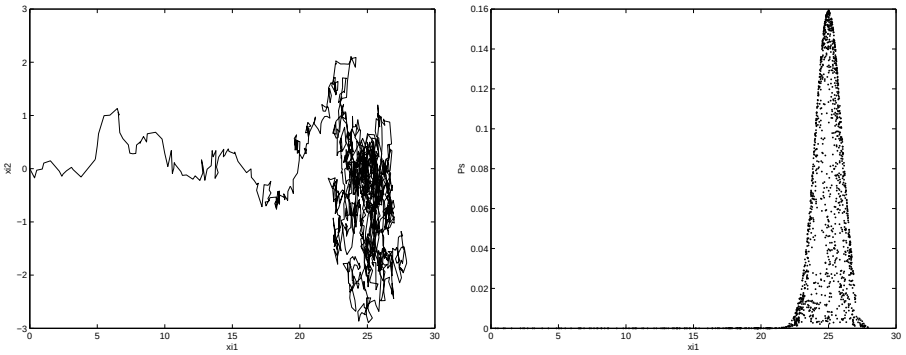

Figure 7: $c(\xi) = 1$,$A = 7.0711$,$k = 0.5$

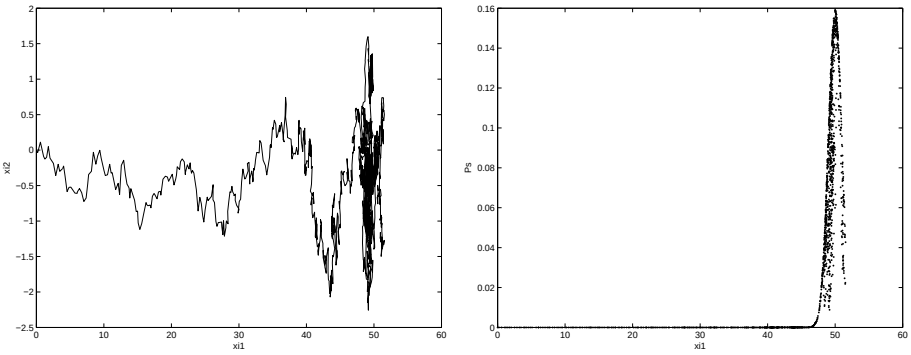

Figure 8: $c(\xi) = 1$,$A = 10$,$k = 0.5$

6.2 Transient Distribution of $\mathcal{P}$

From Result 5.1, we have the solution $\mathcal{P}$ of the PDE

$$\begin{aligned} \partial_t \mathcal{P} &= \nabla \cdot (\varepsilon \nabla + \varepsilon c \xi - \kappa e_1)\mathcal{P}, \qquad (t, \xi) \in [0, T] \times \mathbb{R}^2 \\ \mathcal{P}(0, \xi) &= P_s(\xi), \qquad \xi \in \mathbb{R}^2 \end{aligned}$$

has the probabilistic representation

$$\mathcal{P}(t, \xi) = \mathbb{E}[P_s(\xi^{0,\xi}(t))] = \mathbb{E}[P_s(\xi^{\xi}(t))].$$

and then doing the MCM implementation. Figure (9-12) shows the trajectories of the SDE and the solution distributions $\mathcal{P}$ for different t respectively. Figure 11 and Figure 12 also give the stationary distribution, which shows that when $t \to \infty$, $\mathcal{P}$ convergence to $\mathcal{P}_s$.

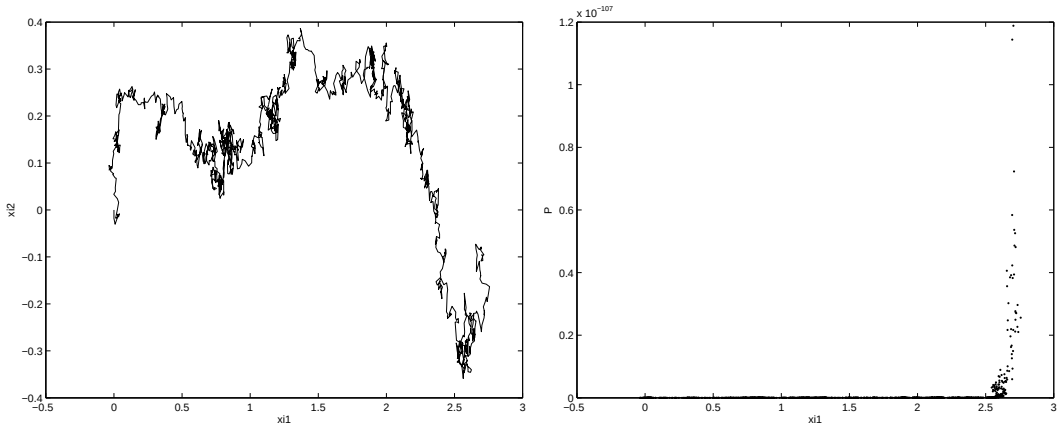

Figure 9: $c(\xi) = 1$,$A = 7.0711$,$k = 0.5$,$t = 5$

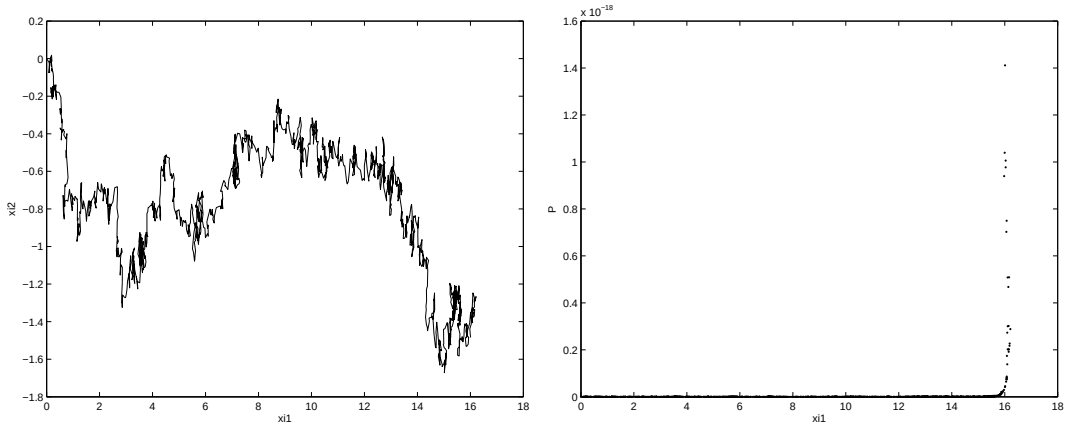

Figure 10: $c(\xi) = 1$,$A = 7.0711$,$k = 0.5$,$t = 50$

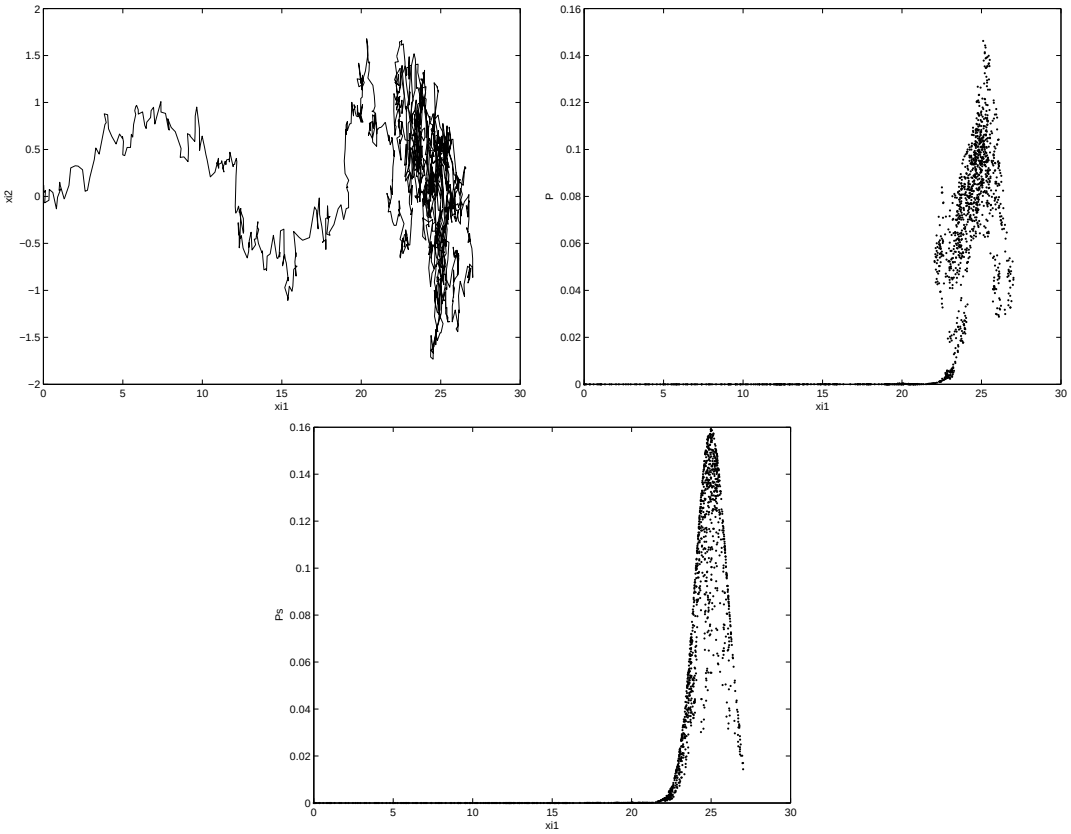

Figure 11: $c(\xi) = 1$,$A = 7.0711$,$k = 0.5$,$t = 500$

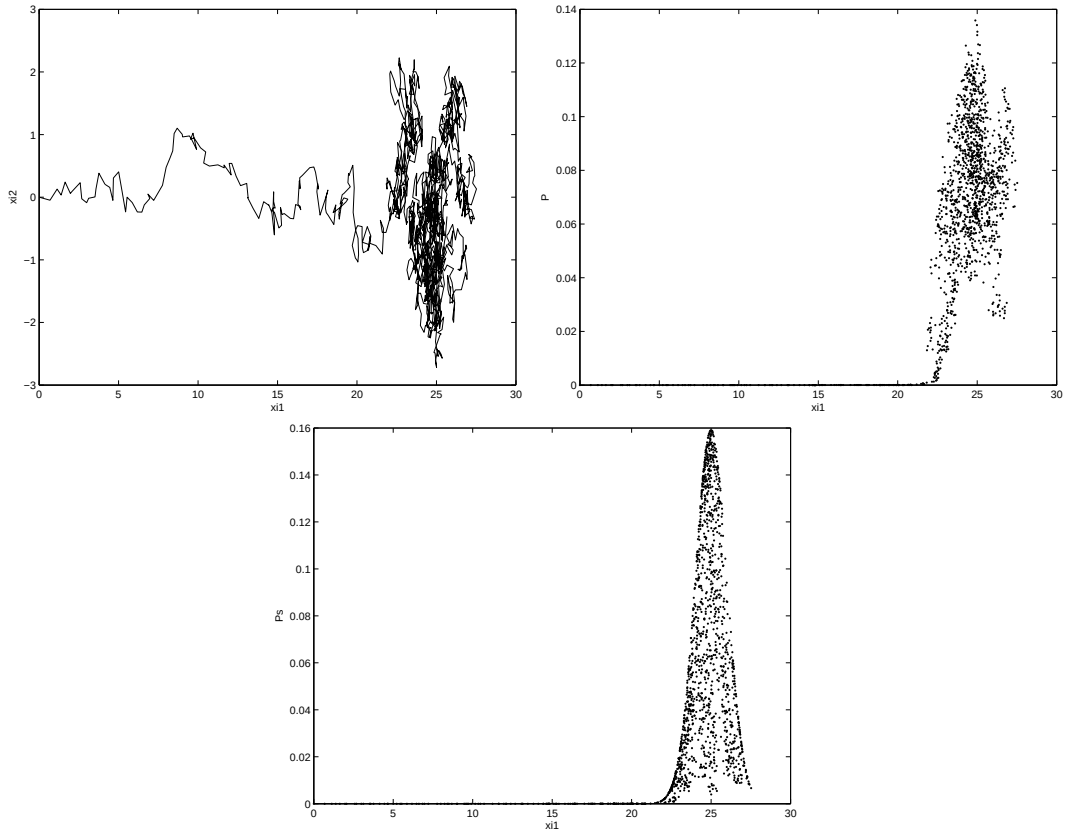

Figure 12: $c(\xi) = 1$,$A = 7.0711$,$k = 0.5$,$t = 1000$

6.3 Convergence of the entropy μ

For $A \to \infty$ we have proved the exponential convergence against the stationary Gaussian-type distribution. Choosing $c(\xi) = 1$. We note that for $N = 1500$ Monte-Carlo simulations, the distances between the stationary solutions for $A > 2$ are within the range of the approximation error. Consequently, the limit distribution is a good approximation of the true distributions-already for moderate values of A, see Figure (13-16).

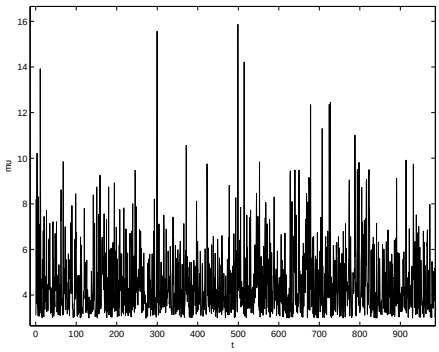

Figure 13: $c(\xi) = 1$,$A = 1$,$k = 0.5$,$t = 1000$

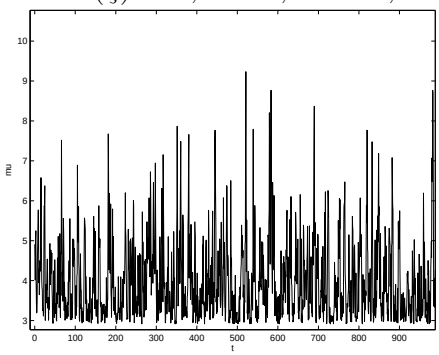

Figure 14: $c(\xi) = 1$,$A = 2$,$k = 0.5$,$t = 1000$

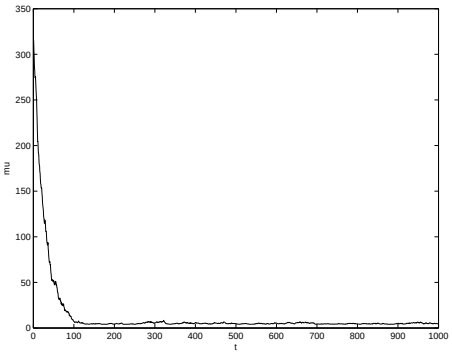

Figure 15: $c(\xi) = 1$,$A = 7.0711$,$k = 0.5$,$t = 1000$

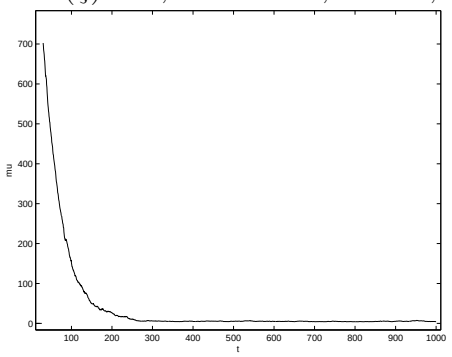

Figure 16: $c(\xi) = 1$,$A = 10$,$k = 0.5$,$t = 1000$

7 Conclusion

In this work we have presented an extended stochastic model for the fiber lay-down regime in a nonwoven production process that contains a moving conveyor belt. From the associated Fokker-Planck equation we have explicitely determined the stationary and transient density distribution for the hydrodynamic limit, $A \to \infty$. Using a multiple scale ansatz and a Chapman-Enskog expansion we have generally found a Gaussian-type stationary solution in leading order to which the transient distribution converges with exponential rate of decay. For the special choice $c = 1$ the center of its reduced Gaussian distribution depends on the relation of 'relative process velocity' and turbulent noise, κA^2. Already for moderate values of A, i.e. $A \geq 2$, this limit distribution turns out to be a very good approximation according to our numerical simulations.

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A Appendix

A.1 Monte-Carlo for approximation of $\mathcal{P}$

```
% Monte-Carlo for approximation of P
clear;
  value=[0.70711 1 2 7.0711 10];
  A=value(4);
  k=0.5;
  epsilon=1/(A^2);
  p = k*A^2
  N = 1500;             % number of steps to take
  T = 1000;                % maximum time
  M = 5;
  h = T/(N-1);              % time step
  t = 0:h:T;          % t is the vector [0 1h 2h 3h ... Nh]
  xi1 = zeros(size(t));   % prepare place to store locations
  xi2 = zeros(size(t));

  s = sqrt(2*epsilon);              % \sigma
  xi1(1) = 0;             % initial height of si1
  xi2(1) = 0;             % initial height of si2
  Ps(1) = exp(-(k/epsilon)^2/2)/(2*pi);     % stationary density distribution
  P_T = zeros([1,N]);
  P = zeros([1,N]);

for j=1:M

  for i = 1:N-1  % start taking steps
    xi1(i+1) = xi1(i)-epsilon*1*xi1(i)*h+k*h+s*sqrt(h)*randn;
    xi2(i+1) = xi2(i)-epsilon*1*xi2(i)*h+s*sqrt(h)*randn;
    Ps(i+1) = exp(-(xi1(i+1)-k/epsilon)^2/2-(xi2(i+1)^2)/2)/(2*pi);
  end;

  P_T = P_T + Ps;

end;
  P = P_T./M;

  figure(1)
  plot(xi1,xi2,'-') % plot more permanently
  xlabel('xi1')
  ylabel('xi2')

  figure(2)
  plot(xi1,Ps,'.')
  xlabel('xi1')
  ylabel('Ps')
```

```
  figure(3)
  plot(xi1,P,'.')
  xlabel('xi1')
  ylabel('P')
```

A.2 Coarse Graining for computation of μ

```
%Coarse Graining for computation of mu
 clear; close all;
 value=[0.70711 1 2 7.0711 10];
  A=value(4);
  k=0.5;
  epsilon=1/(A^2);
  p = k*A^2
  N=1500;              % number of steps to take
  T=1000;                 % maximum time
  h=T/(N-1);               % time step
  t= 0:h:T;           % t is the vector [0 1h 2h 3h ... Nh]
  xi1 = zeros(size(t));   % prepare place to store locations
  xi2 = zeros(size(t));
  s = sqrt(2*epsilon);              % \sigma

  xi1(1)=0;             % initial height of si1
  xi2(1)=0;             % initial height of si2
  Ps(1) = exp(-(k/epsilon)^2/2)/(2*pi);     % density distribution

% Simulation of Ornstein-Uhlenbeck process
  for i = 1:N-1           % start taking steps
    xi1(i+1) = xi1(i)-epsilon*1*xi1(i)*h+k*h+s*sqrt(h)*randn;
    xi2(i+1) = xi2(i)-epsilon*1*xi2(i)*h+s*sqrt(h)*randn;
    Ps(i+1) = exp(-(xi1(i+1)-k/epsilon)^2/2-(xi2(i+1)^2)/2)/(2*pi);
  end;

% Discrete x,y-coordinates
figure(4)
points = 33;
 Nx = points;
 Ny = points;  % Nx, Ny
 minxi1 = min(xi1(2:end));
 maxxi1 = max(xi1(2:end));
 minxi2 = min(xi2(2:end));
 maxxi2 = max(xi2(2:end));
 n = (points-1) * (points-1);

 dltx = (maxxi1-minxi1)/(Nx-1);
 dlty = (maxxi2-minxi2)/(Ny-1);

 x = minxi1:dltx:maxxi1;
 y = minxi2:dlty:maxxi2; % Matlab's meshgrid is used to create 2D grid from specified divisons above
```

```
 [X,Y] = meshgrid(x,y);
 for j = 1:points
     plot(X(j,:),Y(j,:));
     hold on;
 end;
 for k = 1:points
     plot(X(:,k),Y(:,k));
 end;
plot(xi1,xi2,'.') % plot more permanently
xlabel('xi1')
ylabel('xi2')

% Range Query & Counting Points
k = 0;
A = zeros([points-1,points-1]);
A_xi = zeros([1,N]);
P = zeros([1,N]);
for r = 1:points-1
    for s = 1:points-1
        k=0;
        for i = 1:N-1
          if (xi2(i) >= y(r) && xi2(i) < y(r+1) && xi1(i) >= x(s) && xi1(i) < x(s+1))
           k = k+1;
         end;
        A(r,s) = k;
        end;
    end;
 end;
figure(5)
surf(A)
xlabel('xi1')
ylabel('xi2')
zlabel('No.of points')

% Coarse Graining
mu = zeros([1,N]);
for i = 1:N
    if Ps(i) > 0
        for r = 1:points-1
            for s = 1:points-1
                if A(r,s) > 0
                    mu(i) = mu(i) + (A(r,s)*log((n*A(r,s))/(Ps(i)*N)))/N;
                end;
            end;
        end;
    end;
end;
figure(6)
plot(t,mu)
xlabel('t')
ylabel('mu')
```