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Closed-loop reduction exploiting Hamiltonian structures

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Abstract

This paper presents a method for closed-loop order reduction of linear systems. A dynamic system with a Hamiltonian structure is obtained using optimal control techniques. The resulting linear time-invariant Hamiltonian system is then reduced in complexity applying standard reduction techniques. The method is implemented on illustrative examples.

1 Introduction

Simple controllers are preferred over complex ones as they are easier to implement and require less resources for their calculations. Simplicity is never a sole desire as a controller will have to meet certain performance specifications. This generally conflicts with the desire for simplicity so that trade-offs will have to be made. A common problem is how to arrive at a simple controller from a complex model for the plant. Three common model-based strategies in designing a controller of low complexity are graphically depicted in figure 1.

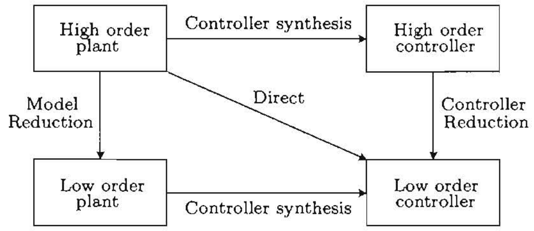

Figure 1: *Model-based strategies to obtain a low order controller*

Between these methods, direct and indirect methods can be distinguished. Between the indirect methods, better results are obtained when a high-order controller is designed for the high-order plant model. This is often computationally very demanding if not impossible for high-order plant models. First reducing the open-loop plant model and then designing a controller for it is computationally a lot more friendly, but there is no direct relation between steps in the design process and furthermore, potentially critical information is lost early on in the design process. A good approximation of the plant dynamics in open-loop is not likely to be but a decent approximation for the plant dynamics in closed-loop.

The benefits of direct methods are that there is a direct insight to the dynamics that are important in closed-loop, but only few of these methods exist today. More detailed information regarding the subject of model reduction for control can be found in [5].
In this paper, a new direct method is presented. The presented method exploits Hamiltonian structures as are obtained in optimal control. The Hamiltonian system describes closed-loop behavior of both plant and controller. Consequently, the Hamiltonian system describing the closed-loop is reduced using standard methods for linear time-invariant (LTI) models. The rest of this paper is organized as follows. In section 2 an introduction to Lagrangian theory for optimal control is be given for both the continuous time and the discrete time case. For both these cases, the optimality conditions can be written to the form of a dynamical system. Section 3 further discusses the dynamical system for the continuous time case. A linear time-invariant (LTI) system is constructed that generates optimal control. Consequently, this LTI system is reduced in complexity using standard model reduction techniques. Section 4 presents results via two illustrative examples, where the techniques as presented in this paper were implemented. Finally, sections 5 and 6 present conclusions and recommendations.

2 Lagrangian theory for optimal control

This section is concerned with Lagrangian theory for optimal control. In optimal control, an input vector is searched for that minimizes a specified cost function, equality constrained by the system's dynamics. Using duality, the optimization problem can be equivalently formulated in terms of an augmented cost function. In the following it will be shown that from the optimality conditions, a system of difference/differential equations results which can be rewritten as a dynamical system. Extensive information on optimal control can be found in [4] and information on constrained optimization in general can be found in [2].

2.1 Discrete time

Consider the case where we would like to drive the system's state from an initial state to the origin in an optimal way. Let the system's dynamics be governed by the following equation;

$$x_{k+1} = Ax_k + Bu_k \; ; \; x_k\Big|_{k=0} = x_0$$

where $x \in \mathbb{R}^n$ are the system's states, $u \in \mathbb{R}^m$ are the system's inputs and x_0 is the system's initial state. In linear quadratic regulation (LQR), a cost function of the following form is used to specify optimality;

$$\begin{aligned} J(x,u) &= \tfrac{1}{2} \sum_{k=0}^{N-1} \left[x_k^\top Q x_k + u_k^\top R u_k \right] + \tfrac{1}{2}\, x_N^\top E x_N \\ x &= (x_0, \ldots, x_N) \\ u &= (u_0, \ldots, u_N) \end{aligned}$$

where $N > 0$ is the control horizon, $E = E^{\top} > 0$ reflects the terminal cost, and $Q \geq 0$ and $R > 0$ are weighting matrices. Optimal control is defined as the input vector u that minimizes the function J given the system's dynamics and the initial state. The mathematical formulation of finding optimal control in primal form then becomes;

$$P_{\text{opt}} = \min_{x,u} \; J(x,u) = \Phi(x_k)\Big|_{k=N} + \sum_{k=0}^{N-1} F(x_k, u_k)$$

subject to;

$$x_{k+1} = Ax_k + Bu_k \;;\; x_k\Big|_{k=0} = x_0$$

where

$$F(x_k, u_k) = \tfrac{1}{2}\left[x_k^{\top} Q x_k + u_k^{\top} R u_k\right]$$
$$\Phi(x_k) = \tfrac{1}{2} x_k^{\top} E x_k$$

This is a standard equality constrained optimization problem which, using duality, can be reformulated as an equivalent optimization problem;

$$\begin{aligned}
\ell(\lambda) &= \min_{x,u} \; L(x,u,\lambda) \\
&= \min_{x,u} \sum_{k=0}^{N-1} \left[F(x_k,u_k) + \lambda_{k+1}^{\top}\left[Ax_k + Bu_k - x_{k+1}\right]\right] + \Phi(x_k)\Big|_{k=N} \\
&= \min_{x,u} \sum_{k=0}^{N-1} \left[H_k - \lambda_{k+1}^{\top} x_{k+1}\right] + \Phi(x_k)\Big|_{k=N}
\end{aligned}$$

$$H_k = F(x_k,u_k) + \lambda_{k+1}^{\top}\left[Ax_k + Bu_k\right]$$
$$\lambda = (\lambda_0, \ldots, \lambda_N)$$

where $L(x,u,\lambda)$ is called the Lagrangian and $\ell(\lambda)$ is called the dual cost. $H(x_k, u_k, \lambda_{k+1})$, or simply H_k, is called the Hamiltonian and it represents an energy function. The dual problem amounts to maximizing $\ell(\lambda)$ over all possible sequences λ;

$$D_{\text{opt}} = \max_{\lambda} \ell(\lambda)$$
$$\lambda = (\lambda_0, \ldots, \lambda_N)$$

By construction we have that $D_{\text{opt}} \leq P_{\text{opt}}$, where the difference $P_{\text{opt}} - D_{\text{opt}}$ is generally referred to as the duality gap.
The optimum of the dual problem exists and is, say ; (x^*, u^*, λ^*). Necessarily it satisfies $\nabla L(x^*, u^*, \lambda^*) = 0$. Moreover, if some assumptions hold, (x^*, u^*) will be a solution to the primal optimization problem. These assumptions include that A is invertible, the duality gap equals zero and that the Hessian of L is positive at (x^*, u^*, λ^*).

Differentiating L with respect to its independent variables leads to the following conditions that have to hold at the optimum;

$$\frac{\partial L}{\partial \lambda_{k+1}^*} = Ax_k^* + Bu_k^* - x_{k+1}^* = 0 \longrightarrow Ax_k^* + Bu_k^* = x_{k+1}^* \qquad \text{(state equation)}$$

$$\frac{\partial L}{\partial x_k^*} = \frac{\partial H_k}{\partial x_k^*} - \lambda_k^* = 0 \longrightarrow Qx_k^* + A^\top \lambda_{k+1}^* - \lambda_k^* = 0 \qquad \text{(costate equation)}$$

$$\frac{\partial L}{\partial u_k^*} = \frac{\partial H_k}{\partial x_k^*} = 0 \longrightarrow Ru_k^* + B^\top \lambda_{k+1}^* = 0 \qquad \text{(stationarity condition)}$$

these equations may be expressed as the following system;

$$\begin{pmatrix} x_{k+1}^* \\ \lambda_k^* \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^\top \\ Q & A^\top \end{pmatrix} \begin{pmatrix} x_k^* \\ \lambda_{k+1}^* \end{pmatrix}$$

which can be rearranged to equations (1) and (2);

$$\begin{pmatrix} x_k^* \\ \lambda_k^* \end{pmatrix} = \begin{pmatrix} A^{-1} & A^{-1}BR^{-1}B^\top \\ QA^{-1} & A^\top + QA^{-1}BR^{-1}B^\top \end{pmatrix} \begin{pmatrix} x_{k+1}^* \\ \lambda_{k+1}^* \end{pmatrix} \qquad (1)$$

$$\begin{pmatrix} x_{k+1}^* \\ \lambda_{k+1}^* \end{pmatrix} = \begin{pmatrix} A + BR^{-1}B^\top A^{-\top}Q & -BR^{-1}B^\top A^{-\top} \\ -A^{-\top}Q & A^{-\top} \end{pmatrix} \begin{pmatrix} x_k^* \\ \lambda_k^* \end{pmatrix} \qquad (2)$$

The resulting system is an autonomous system with two point boundary conditions, namely we know the state x at instant i to be x_0 and we know the costate λ at instant N to be Ex_N. Assume a new basis for the system as follows;

$$\begin{pmatrix} x_k^* \\ \lambda_k^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ P_k & I \end{pmatrix} \begin{pmatrix} x_k^* \\ \sigma_k^* \end{pmatrix} \iff \begin{pmatrix} x_k^* \\ \sigma_k^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ -P_k & I \end{pmatrix} \begin{pmatrix} x_k^* \\ \lambda_k^* \end{pmatrix}$$

and

$$\begin{pmatrix} x_{k+1} \\ \sigma_{k+1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ -P_{k+1} & I \end{pmatrix} \begin{pmatrix} x_{k+1} \\ \lambda_{k+1} \end{pmatrix} \iff \begin{pmatrix} x_{k+1} \\ \lambda_{k+1} \end{pmatrix} = \begin{pmatrix} I & 0 \\ P_{k+1} & I \end{pmatrix} \begin{pmatrix} x_{k+1} \\ \sigma_{k+1} \end{pmatrix}$$

then

$$\begin{pmatrix} x_{k+1}^* \\ \sigma_{k+1}^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ -P_{k+1} & I \end{pmatrix} \begin{pmatrix} A + BR^{-1}B^\top A^{-\top}Q & -BR^{-1}B^\top A^{-\top} \\ -A^{-\top}Q & A^{-\top} \end{pmatrix} \begin{pmatrix} I & 0 \\ P_k & I \end{pmatrix} \begin{pmatrix} x_k^* \\ \sigma_k^* \end{pmatrix}$$

This results into

$$\begin{pmatrix} x_{k+1}^* \\ \sigma_{k+1}^* \end{pmatrix} = \begin{pmatrix} A - BR^{-1}B^\top A^{-\top}[P_k - Q] & -BR^{-1}B^\top A^{-\top} \\ \chi & A^{-\top} + P_{k+1}BR^{-1}B^\top A^{-\top} \end{pmatrix} \begin{pmatrix} x_k^* \\ \sigma_k^* \end{pmatrix} \qquad (3)$$

With boundary conditions;

$$\begin{pmatrix} x_i \\ \sigma_N \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$$

Equation (3) represents the same autonomous system with two point boundary conditions as does equation (2), but in a different coordinate system, where σ^* is independent of x^* only if;

$$\chi = Q + A^\top \Big[P_{k+1} - P_{k+1}B\left[R + B^\top P_{k+1}B\right]^{-1} B^\top P_{k+1}\Big] A - P_k = 0 \qquad (4)$$
$$P_N = E$$

Equation (4) is known as the Riccati control difference equation which in the infinite horizon case ($N \to \infty$) reduces to equation (5), the algebraic Riccati equation;

$$P = Q + A^\top \Big[P - PB\left[R + B^\top PB\right]^{-1} B^\top P\Big] A \qquad (5)$$

The coordinate change as used before already gives;

$$\lambda_k^* = P_k x_k^* + \sigma_k^*$$

hence

$$\begin{aligned} u_k^* &= -R^{-1}B^\top \lambda_{k+1}^* \\ &= -R^{-1}B^\top A^{-\top}\left[P_k x_k^* - Q x_k^*\right] \\ &= -R^{-1}B^\top P_{k+1} x_{k+1}^* \end{aligned} \qquad (6)$$

where P_k is obtained from the solution of (4) and σ_k^* equals 0 for all k. This is due to the fact that the σ-"system" is antistable and thus $\sigma_N = 0$ implies $\sigma_k = 0,\ \forall\ k$. Therefore equation (6) reduces to;

$$\begin{aligned} u_k^* &= -(B^\top P_{k+1} B + R)^{-1} B^\top P_{k+1} A x_k^* \\ &= -K_k x_k^* \end{aligned} \qquad (7)$$

Equation (7) implies that the optimal control may be implemented as a linear feedback on the states.
The Hamiltonian can now be restated as follows;

$$\begin{aligned} H(x_k^*, u_k^*, \lambda_{k+1}^*) &= \frac{1}{2}\left[x_k^{*\top} Q x_k^* + u_k^{*\top} R u_k^*\right] + \lambda_{k+1}^{*\top}\left[A x_k^* + B u_k^*\right] \\ &= \frac{1}{2} x_k^{*\top} Q x_k + \frac{1}{2}\lambda_{k+1}^{*\top} B R^{-1} B^\top \lambda_{k+1}^* + \lambda_{k+1}^{*\top}\left[A x_k^* - B R^{-1} B^\top \lambda_{k+1}^*\right] \\ &= \frac{1}{2} x_k^{*\top} Q x_k - \frac{1}{2}\lambda_{k+1}^{*\top} B R^{-1} B^\top \lambda_{k+1}^* + \lambda_{k+1}^{*\top} A x_k^* \\ &= \frac{1}{2} x_k^{*\top}\left[2P_k - K_k^\top R K_k - Q\right] x_k^* \end{aligned}$$

2.2 Continuous time

We may define an analogous problem for the continuous time case. Consider again a system's dynamics of the following form;

$$\dot{x}(t) = Ax(t) + Bu(t)\ ;\ x(0) = x_0$$

With system's states $x(t) \in \mathbb{R}^n$, inputs $u(t) \in \mathbb{R}^m$ and initial state x_0 respectively. Then define a continuous time cost function as follows;

$$
\begin{aligned}
J(x,u) &= \tfrac{1}{2}\int_{t_0}^{t_f} x^\top Q x + u^\top R u dt + \tfrac{1}{2}\, x^\top(t_f) E x(t_f) \\
x &= x(t),\ \ t \in [t_0, t_f] \\
u &= u(t),\ \ t \in [t_0, t_f]
\end{aligned}
$$

Minimization of J delivers optimal control trajectory u^* for the specified dynamical system. The mathematical formulation of the optimization problem in primal form then becomes;

$$
\begin{aligned}
&P_{\text{opt}} = \min_{x,u}\ J(x,u) = \int_{t_0}^{t_f} F(x,u)\,\mathrm{d}t + \Phi(x)\Big|_{t=t_f} \\
&\text{subject to;} \\
&\dot{x}(t) = Ax(t) + Bu(t)\ ;\ x(0) = x_0
\end{aligned}
$$

where

$$
\begin{aligned}
F(x,u) &= \tfrac{1}{2}\left[x^\top Q x + u^\top R u\right] \\
\Phi(x) &= \tfrac{1}{2}\, x^\top E x
\end{aligned}
$$

This is a standard equality constrained optimization problem which, using duality, can be reformulated as an equivalent optimization problem;

$$
\begin{aligned}
\ell(\lambda) &= \min_{x,u}\ L(x,u,\lambda) = \min_{x,u} \int_{t_0}^{t_f} F(x,u) + \lambda^\top\Big[Ax + Bu - \dot{x}\Big]\mathrm{d}t + \Phi(x)\Big|_{t=t_f} \\
&= \min_{x,u} \Big\langle 1, F(x,u) + \lambda^\top\Big[Ax + Bu\Big]\Big\rangle - \Big\langle \lambda, \dot{x}\Big\rangle + \Phi(x)\Big|_{t=t_f} \\
&= \min_{x,u} \Big\langle 1, H(x,u,\lambda)\Big\rangle + \Big\langle \dot{\lambda}, x\Big\rangle + \Big\langle\!\!\Big\langle \lambda, x \Big\rangle\!\!\Big\rangle\Big|_{t=t_0} + \left\langle\!\!\left\langle \begin{pmatrix} 1 \\ -\lambda \end{pmatrix}, \begin{pmatrix} \Phi(x) \\ x \end{pmatrix} \right\rangle\!\!\right\rangle\Bigg|_{t=t_f} \\
H(x,u,\lambda) &= F(x,u) + \lambda^\top\Big[Ax + Bu\Big]
\end{aligned}
$$

Where $L(x,u,\lambda)$ is called the Lagrangian and $\ell(\lambda)$ the dual cost, $\langle \cdot, \cdot \rangle$ denotes an l_2 type inner product and $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes an Euclidean type inner product. $H(x,u,\lambda)$ is called the Hamiltonian and it represents an energy function. The dual problem is now defined as follows;

$$
D_{\text{opt}} = \max_{\lambda} \ell(\lambda) \leq P_{\text{opt}}
$$

As in the discrete time case, at the optimum all partial derivatives of the Lagrangian have to equal zero;

$$
\nabla L(x^*, u^*, \lambda^*) = 0 \tag{8}
$$

Furthermore, we assume that the duality gap equals zero and that the Hessian of L is positive at the optimum. Equation (8) leaves us to the following conditions;

$$\frac{\partial L}{\partial \lambda^*} = Ax^* + Bu^* - \dot{x}^* = 0 \longrightarrow Ax^* + Bu^* = \dot{x}^* \qquad \text{(state equation)}$$

$$\frac{\partial L}{\partial x^*} = \frac{\partial H}{\partial x^*} + \dot{\lambda}^* = 0 \longrightarrow Qx^* + \lambda^{*\top} A = -\dot{\lambda}^* \qquad \text{(costate equation)}$$

$$\frac{\partial L}{\partial u^*} = \frac{\partial H}{\partial u^*} = 0 \longrightarrow Ru^* + B^\top \lambda^* = 0 \qquad \text{(stationarity condition)}$$

which may be written down as the following system;

$$\begin{pmatrix} \dot{x}^* \\ \dot{\lambda}^* \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^\top \\ -Q & -A^\top \end{pmatrix} \begin{pmatrix} x^* \\ \lambda^* \end{pmatrix} \tag{9}$$

This is an autonomous system with two point boundary conditions, for which we may define a new basis as follows;

$$\begin{pmatrix} x^* \\ \lambda^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \begin{pmatrix} x^* \\ \sigma^* \end{pmatrix} \iff \begin{pmatrix} x^* \\ \sigma^* \end{pmatrix} = \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \begin{pmatrix} x^* \\ \lambda^* \end{pmatrix}$$

and

$$\tfrac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} x \\ \sigma \end{pmatrix} = \tfrac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} \iff \begin{pmatrix} \dot{x} \\ \dot{\sigma} \end{pmatrix} = \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\dot{P} & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix}$$

which gives;

$$\dot{\sigma} = -P\dot{x} + \dot{\lambda} - \dot{P}x$$

so

$$\begin{pmatrix} \dot{x}^* \\ \dot{\sigma}^* \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ -\dot{P} & 0 \end{pmatrix} + \begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \begin{pmatrix} A & -BR^{-1}B^\top \\ -Q & -A^\top \end{pmatrix} \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} \right] \begin{pmatrix} x^* \\ \sigma^* \end{pmatrix}$$

This results into;

$$\begin{pmatrix} \dot{x}^* \\ \dot{\sigma}^* \end{pmatrix} = \begin{pmatrix} A - BR^{-1}B^\top P & -BR^{-1}B^\top \\ \chi & -(A - BR^{-1}B^\top P)^\top \end{pmatrix} \begin{pmatrix} x^* \\ \sigma^* \end{pmatrix} \tag{10}$$

With boundary conditions;

$$\begin{pmatrix} x(0) \\ \sigma(t_f) \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}$$

Equation (10) represents the same autonomous system with two boundary point conditions as does equation (9), but in a different coordinate system, where σ^* is independent of x^* only if;

$$\chi = A^\top P + PA - PBR^{-1}B^\top P + Q + \dot{P} = 0 \tag{11}$$
$$P(t_f) = E$$

Equation (11) is known as the Riccati control differential equation which in the infinite horizon case ($t_f \to \infty$) reduces to equation (12), the algebraic Riccati equation;

$$\chi = A^\top P + PA - PBR^{-1}B^\top P + Q = 0 \tag{12}$$

The coordinate change as used before already gave;

$$\lambda^* = Px^* + \sigma^*$$

hence

$$\begin{aligned} u^* &= -R^{-1}B^\top \lambda^* \\ &= -R^{-1}B^\top \left[Px^* + \sigma^*\right] \end{aligned}$$

with P the solution to the Ricatti equation as stated in (11) and where σ^* equals 0 due to the fact that the σ-"system" is antistable and thus $\sigma(t_f) = 0$ implies $\sigma = 0 \ \forall \ t$. Therefore the above reduces to;

$$u^* = -R^{-1}B^\top Px^* = -Kx^*$$

which implies that the optimal control may be implemented as a linear feedback on the states.
The Hamiltonian can now be restated as follows;

$$\begin{aligned} H(x^*, u^*, \lambda^*) &= \frac{1}{2}\left[x^{*\top}Qx^* + u^{*\top}Ru^*\right] + \lambda^{*\top}\left[Ax^* + Bu^*\right] \\ &= \frac{1}{2}x^{*\top}Qx^\top + \frac{1}{2}\lambda^{*\top}BR^{-1}B^\top\lambda^* + \lambda^{*\top}\left[Ax^* - BR^{-1}B^\top\lambda^*\right] \\ &= \frac{1}{2}x^{*\top}Qx^\top - \frac{1}{2}\lambda^{*\top}BR^{-1}B^\top\lambda^* + \lambda^{*\top}Ax^* \\ &= \frac{1}{2}x^{*\top}\left[Q + 2PA - K^\top RK\right]x^* \end{aligned}$$

3 Model reduction on the Hamiltonian system

In this section, the order of the Hamiltonian system as constructed in the previous section will be reduced using standard model reduction techniques. As these techniques are very well known they will be discussed quite briefly. More detailed information can be found in e.g. [1].

3.1 Unconstrained Hamiltonian system

In this section, we consider the continuous time formulation of the Hamiltonian system. As was presented in section 2, a system of differential equations were obtained from the optimality conditions. This system describes the evolution of state $x \in \mathbb{R}^n$ and costate $\lambda \in \mathbb{R}^n$ in time, both in closed-loop. Provided that P is time-invariant, we may regard the system to be an autonomous LTI system that incorporates the closed-loop dynamics. A time-invariant P can be obtained by solving equation (12) (equation (5) for discrete time). n stable and n antistable poles are present, or better: if ω is an eigenvalue of the state evolution matrix in equation (9) than so is $-\omega$ (ω^{-1} for discrete time). In order for the closed-loop behavior to be bounded over an infinite horizon, it was shown that state and costate have to be linearly dependent via $\lambda(t) = Px(t)$. Furthermore, it was shown that the optimal control $u^*(t)$ was a linear projection of the costate. Therefore, we shall introduce the following LTI system (13) with an output $u^*(t)$, and an input $v(t)$ in order to properly initialize the state and costate. The latter is taken care of if we choose $v(t) = x_0\delta(t)$, under the assumption that Σ_H contains no energy initially.

$$\begin{gathered}\begin{pmatrix} \dot{x}^*(t) \\ \dot{\lambda}^*(t) \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^\top \\ -Q & -A^\top \end{pmatrix} \begin{pmatrix} x^*(t) \\ \lambda^*(t) \end{pmatrix} + \begin{pmatrix} I \\ P \end{pmatrix} v(t) \\ u^*(t) = \begin{pmatrix} 0 & -R^{-1}B^\top \end{pmatrix} \begin{pmatrix} x^*(t) \\ \lambda^*(t) \end{pmatrix}\end{gathered} \tag{13}$$

which may also be denoted in compact form by;

$$\Sigma_H = \left(\begin{array}{cc|c} A & -BR^{-1}B^\top & I \\ -Q & -A^\top & P \\ \hline 0 & -R^{-1}B^\top & 0 \end{array}\right)$$

The resulting Hamiltonian system is a linear time-invariant (LTI) system. Therefore, standard reduction techniques for these kind of systems could be investigated. Because Σ_H is non-minimal due to the chosen mapping from input to state, we shall first discuss its minimization.

In section 2.2, a similarity transformation based upon the solution of the Riccati equation was presented. If we perform this transformation on Σ_H we obtain;

$$\begin{gathered}\begin{pmatrix} \dot{x}^*(t) \\ \dot{\sigma}^*(t) \end{pmatrix} = \begin{pmatrix} A - BR^{-1}B^\top P & -BR^{-1}B^\top \\ 0 & -\left(A - BR^{-1}B^\top P\right)^\top \end{pmatrix} \begin{pmatrix} x^*(t) \\ \sigma^*(t) \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} v(t) \\ u^*(t) = \begin{pmatrix} -R^{-1}B^\top P & -R^{-1}B^\top \end{pmatrix} \begin{pmatrix} x^*(t) \\ \sigma^*(t) \end{pmatrix}\end{gathered}$$

From this it is clear that the σ states are uncontrollable, which we desired because of their anti-stability. A minimal realization of Σ_H may now be obtained by truncating the uncontrollable σ states, which yields the following stable system;

$$\Sigma_H^- = \left(\begin{array}{c|c} A - BR^{-1}B^\top P & I \\ \hline -R^{-1}B^\top P & 0 \end{array}\right) \tag{14}$$

This system will serve as a departure point for model reduction in sections 3.1.1 and 3.1.2.

3.1.1 Reduction via modal truncation

In this section, the stable minimal Hamiltonian system Σ_H^- as presented in equation (14) is reduced by implementing modal truncation methodology. Prior to reduction, a similarity transformation will be performed on Σ_H^- where transformation Ψ is such that;

$$\Psi\left(A - BR^{-1}B^\top P\right)\Psi^{-1} = \Omega = \begin{pmatrix} \Omega(\omega_1) & 0 & \cdots & 0 \\ 0 & \Omega(\omega_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Omega(\omega_n) \end{pmatrix}$$

Where Ω is a diagonal matrix containing the modes or closed-loop poles of Σ_H^- in Jordan blocks. The modes are ordered from slow to fast; $0<\mathrm{Re}(\omega_1)\leq\mathrm{Re}(\omega_2)\leq \ldots \leq\mathrm{Re}(\omega_n)$. After transformation, the equivalent representation of Σ_H^- in the new coordinate system is as follows;

$$\Sigma_H^- = \left(\begin{array}{c|c} \Omega & \Psi^{-1} \\ \hline -R^{-1}B^\top P\Psi & 0 \end{array}\right) = \left(\begin{array}{c|c} \Omega & \Gamma \\ \hline \Upsilon & 0 \end{array}\right)$$

which may be partitioned as follows, where $\Omega_{11} \in \mathbb{R}^{k\times k}$;

$$\Sigma_H^- = \left(\begin{array}{cc|c} \Omega_{11} & \Omega_{12} & \Gamma_1 \\ \Omega_{21} & \Omega_{22} & \Gamma_2 \\ \hline \Upsilon_1 & \Upsilon_2 & 0 \end{array}\right)$$

As system dynamics are dominated by slow modes rather than fast modes the system, Σ_H^- can be simplified by keeping only the first $k \leq n$ modes and truncating the other $n - k \geq 0$. This results in the following reduced order system of order k;

$$\hat{\Sigma}_H^k = \left(\begin{array}{c|c} \Omega_{11} & \Gamma_1 \\ \hline \Upsilon_1 & 0 \end{array}\right) \tag{15}$$

The system as defined in equation (15) is both stable and minimal, and the output it is able to generate is a subset of the output that (14) is able to generate.

3.1.2 Reduction via balanced truncation

Another reduction method considered for reducing the Hamiltonian system is balanced truncation, which is based upon the Hankel singular values of Σ_H^-. Using information from both the observability and reachability gramians, the state space is parameterized such that states are ordered according to the Hankel singular values of the system. Hankel singular values provide direct insight to both the observability and reachability of corresponding states. A k^{th} order approximation may consequently be obtained by preserving the first $k < n$

states and truncating the remaining $n - k \geq 0$ states of the system in balanced form. This reduced model preserves the k best reachable as well as observable states.

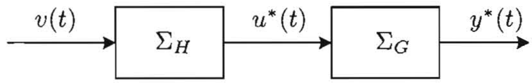

Figure 2: *Frequency weighting of* $u^*(t)$

A more advanced balancing-based reduction technique is a frequency weighted balanced truncation. Adding frequency weighting is an interesting option here since we actually prefer a good approximation of $y^*(t)$ over one of $u^*(t)$ itself (figure 2). In figure 2, G denotes a model describing the dynamics of the plant to be controlled. By adding a model for the plant dynamics as an output frequency weighting, the internal balancing of the controller is tailored to the plant to be controlled. Detailed information on frequency weighted balancing can be found in [3]. Once a description of Σ_H^- in frequency weighted balanced form is obtained, it may be truncated to a lower order. This yields an approximated version of our controller which takes into account the behavior of Σ_H^- which is most efficient and desirable for the control of G. An example may be found in section 4.

4 Illustrative examples

The methodology as presented in section 3.1 is illustrated on two examples. In both these examples, a Dirac-pulse $\delta(t)$ multiplied by x_0 was used as an input signal for the Hamiltonian systems. Furthermore ,the value of the weighting matrix Q in the cost functions was taken to be $C^\top C$, while the value of $R \succeq 0$ was varied. As the value of weighting matrix R decreases for a fixed value of Q, the closed-loop poles shift further away from the imaginary axis in the complex plane. However, for the following examples illustrative values of R were determined. In the following examples, reduced order Hamiltonian systems were used to generate reduced order inputs for the full order plant.

4.1 A 2^{nd} order system

For our first example, we consider a single-input single-output (SISO) system with the following transfer function;

$$G(s) = \frac{s+1}{(s+10)(s+0.1)}$$

Control in feed-forward was performed on this system, meaning the only input of the controller is the initial state of the system. In figure 3, results are plotted where G is driven to the origin of its state space from a given initial condition for a value of $R = 0.001$. Results for a similar test with a value of $R = 0.1$ are depicted in figure 4. Both figures depict both the truly optimal $y^*(t)$ and

$u^*(t)$ alongside results obtained with controllers reduced to order 1. In both figures, reduction method 1 refers to modal truncation as described in section 3.1.1. For method 1, the added 'slow' and 'fast' indicate which eigenmode was kept in place. Method 2 refers to balanced truncation as described in section 3.1.2 and 'FW' denotes plant frequency weighted balancing whereas no addition refers to plain balancing.
As becomes clear from especially figure 3, only the true optimal control and the frequency weighted reduced version thereof are able to drive the system to the origin in a reasonable amount of time. This originates from the fact that the other controllers are unaware that there are multiple states which also interact. As the feed-forward controllers do not address the energy contained in the other state, the energy has to be dissipated in a natural way via the dynamics of G. By introducing frequency weighting in the reduction, the reduced controller has information about the open-loop plant behavior and is able to act accordingly. Furthermore, by introducing frequency weighting, we obtained a better fit on $y(t)$ rather than on $u(t)$, which is more desirable in practical applications. With no frequency weighting applied, the reduced controller may use additional freedom in trying to fit dynamics of $u(t)$ which have little influence on $y(t)$. This effect can also be observed in figure 3.

4.2 A binary distillation column

The second system used for illustrating the methodology is a more complex plant model. We used a linearized time-invariant model of a stabilized distillation column with 41 stages. A detailed description of this originally non-linear model can be found in [6]. A schematic representation of the distillation column with nomenclature is depicted in figure 5. Flow units are in kmol/min, holdups in kmol, and compositions in mole fraction. The model contains two proportional controllers in order to stabilize the levels using the product flows. In this study, we consider only V_B and L_T to exert control over only the product compositions X_B and X_D. The resulting plant model is a stable LTI model with 2 inputs, 2 outputs and 82 states. The value of R in the cost function was taken to be $0.001 \cdot I_2$, with I_2 being the 2×2 identity matrix.
For a reduced order controller of the distillation column, only reduction method 2 was considered because of the complexity of the plant model and the difficulty in deciding which modes to truncate. In figure 6, feed-forward results are depicted where the Hamiltonian system was reduced to an order of 2. As can be seen, there is some difference in the control signals but there is virtually no difference between the induced plant outputs.

5 Conclusions

In this paper, a new direct method for closed-loop controller reduction was presented. A Hamiltonian system was obtained using Lagrangian theory for optimal control. This Hamiltonian system can be represented as a normal LTI system describing the complete closed-loop behavior of both plant and controller. Modal truncation and balanced truncation were used to obtain low-order controllers. The methods were implemented in two different examples in which it was observed that between the alternatives presented, frequency weighted balanced truncation delivers the most desirable low-order controller. Using frequency weighted balanced truncation on the Hamiltonian system of a distillation column, a controller of order 2 was obtained that induced plants outputs which were nearly indistinguishable from those induced by the optimal controller of order 82. The methods as presented in this paper therefore seem to be a new and viable alternative to existing techniques. In general, reduction methods that allow frequency weighting are expected to produce low-order controllers that perform better than ones obtained with non-frequency weighted reduction techniques. This originates from the desire to have a low-order controller that produces plant outputs resembling plant outputs induced by the optimal controller rather than one that produces control signals resembling optimal control signals.

6 Recommendations

Results obtained in this paper might further improve by using a frequency weighted optimal Hankel norm approximation to reduce the Hamiltonian system. Optimal Hankel norm approximation is an excellent candidate to reduce the Hamiltonian system as the Hankel norm operator maps past inputs to future outputs. This fits perfectly with the character of the Hamiltonian system as this system generates optimal control from an input signal that is only nonzero at $t = t_0$. Frequency weighted Hankel norm approximation is expected to have similar benefits as are described above for frequency weighted balanced truncation.

The reduced order controllers in this paper were provided feed-forward control to the plant. Placing a reduced order controller in feed-back with the plant will improve robustness and performance of control.

Currently only the optimization problem was only constrained by the plant's dynamics. Inequality constraints could possibly be added to the problem to provide low-order (sub-optimal) constrained control.

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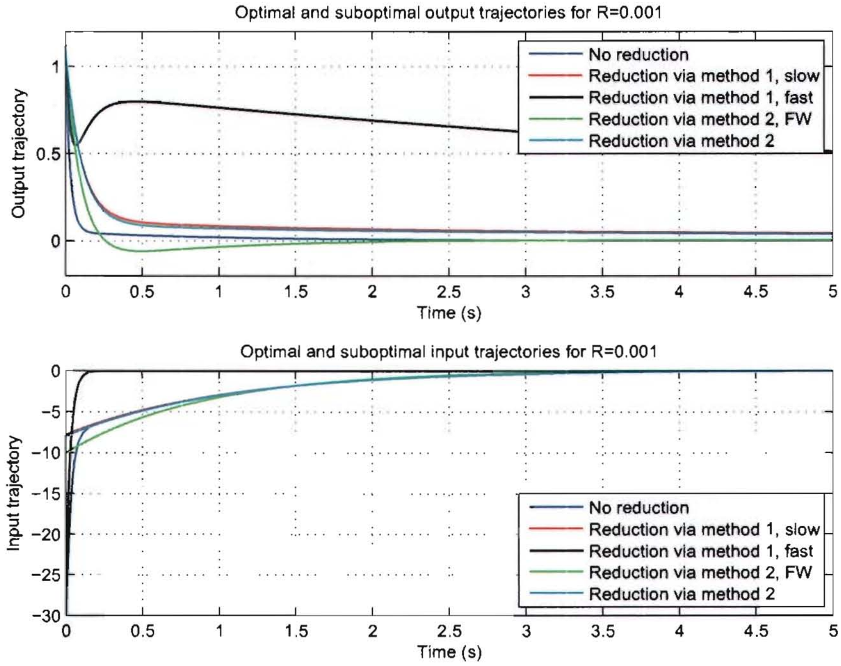


Figure 3: *Results with the 2^{nd} order system*

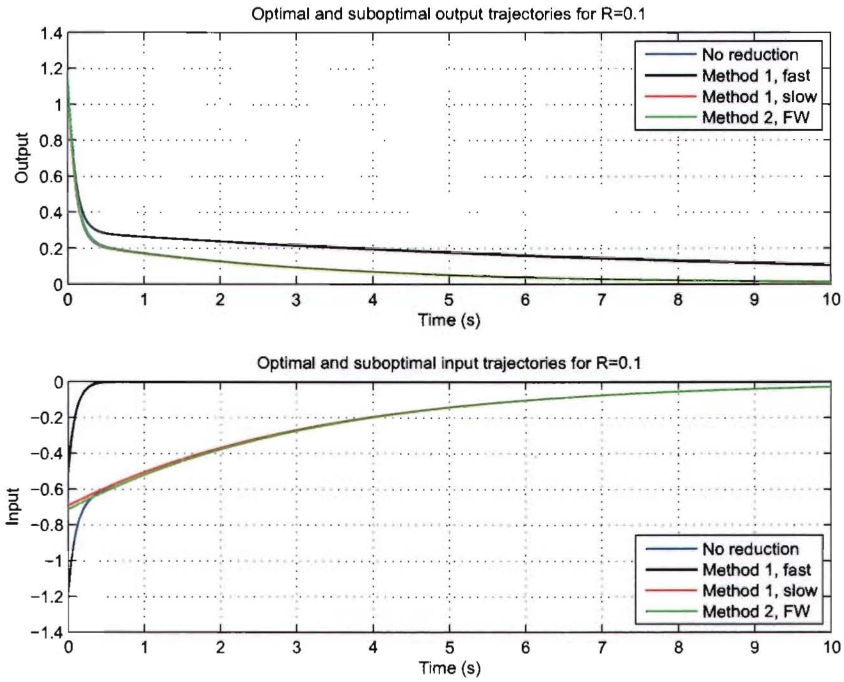
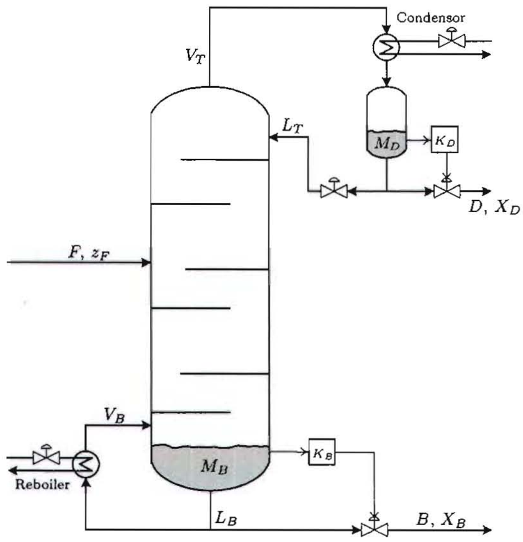


Figure 4: *Results with the 2^{nd} order system*



X_B	Bottom composition
X_D	Distillate composition
V_B	Boilup vapor flow
L_T	Reflux flow
L_B	Bottom liquid flow
V_T	Top vapor flow
M_B	Reboiler holdup
M_D	Condensor holdup
B	Bottom product flow
D	Distillate product flow
K_B	Stabilizing P-controller
K_D	Stabilizing P-controller

Figure 5: *Distillation column*

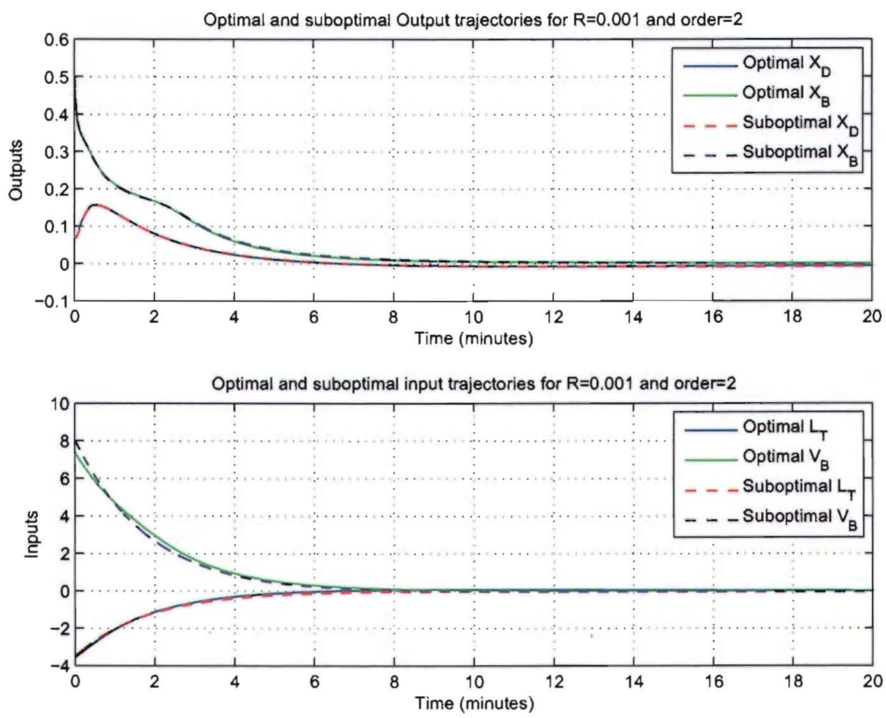


Figure 6: *Results with the distillation column*