MASTER

Feasible regularization of obstacle problems
convergence analysis and optimal control

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DEDICATION

This dissertation is dedicated to my wife Megabit, our two kids Natenael and Mahelat, my parents for their support and encouragement. A special dedication goes to my daughter, Mahelat, she was born during the beginning of the second semester of my first year study and I will see her when she is already one year old.

Abstract

In this thesis we investigate the analytical treatment of optimal control problems governed by a class of elliptic variational inequalities of the first kind with unilateral and bilateral constraints.

Since the control-to-state mapping $u \mapsto y$ is non-smooth and not Fréchet differentiable which makes difficult to get sharp optimality conditions and solve the problem numerically. To overcome this difficulty we used some smoothing (regularizing) technique to get smooth mapping $u \mapsto y_c$ where $c \to \infty$.

We considered a Moreau-Yosida approximation technique to reformulate the governing variational inequality of the first kind as an operator equation involving the max −function. Therefore, solving the variational inequality is equivalent to solving this regularized equation.

Regularized control problem is introduced and the convergence of the regularized optimal solutions towards a solution of the original control problem is verified. For each regularized problem an existence of necessary optimality conditions is derived and an optimality system for the original control problem is obtained as limit of the regularized ones. Thanks to the structure of the proposed regularization, complementarity relation between the variables involved are derived.

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Contents

Chapter 1

Introduction

1.1 Introductive sentences and overview

The theory of variational inequalities introduced by Stampacchia [6] in the early sixties has played a vital role in the study of a wide class of problems arising in pure and applied sciences including mechanics, optimization and optimal control, operations research, game theory, mathematical economics and engineering sciences.

In many physical processes "obstacles" appear in a natural way having strong influence on the character of the examined problem. A simple example of such a situation is the study of contrast between a vibrating membrane and a vibrating membrane set between obstacles. In the 1970's there was considerable interest in the analysis of obstacle problems. This was connected with the development of research on variational inequalities and has been studied by many authors. Although, in the words of J.L. Lions [17], this "simple, beautiful and deep" problem is naturally associated with partial differential equations of elliptic type, it arises in many other frameworks and in different kinds of free boundary problems (see [4] or [5], and their references) and it is related to variational inequalities. The variational formulations (also called weak formulations) of many non-linear boundary value problems result in variational inequalities rather than variational equations. Analogously to partial differential equations, variational inequalities can be of elliptic, parabolic, hyperbolic, etc. type.

The obstacle problem for elliptic partial differential equations appears classically in elasticity as the equation that models the shape of an elastic membrane that is constrained to remain above or below an obstacle (which pushes the membrane from below or respectively from above). Elliptic obstacle problems refer to find the equilibrium position of an elastic membrane whose boundary is held fixed, and which is constrained to lie above a given obstacle. It can be considered as a model problem for variational inequalities (see, e.g, [4]), and it has found applications in a number of different fields as elasticity and fluid dynamics. For example, applications include fluid filtration in porous media, optimal control, and financial mathematics. Numerous important optimization problems arising in continuum mechanics, economy, transportation networks etc. can be modeled as optimal control of variational inequalities or complementarity problems. Optimal control problems for elliptic variational inequalities have been much studied.

In this thesis we investigate the analytical treatment of optimal control problems governed by a class of elliptic variational inequalities of the first kind with unilateral and bilateral constraints. Moreover, we consider constraints on both the control and the state. These kind of problems have been extensively studied by many authors, as for example K. Ito and K. Kunisch [1], [13], or more recently Karl Kunisch and Daniel Wachsmuth [2],[3]).

In the optimal control problem of a variational inequality the main difficulty comes from the fact that the mapping between the control and the state (control-to-state operator) is not differentiable but only Lipschitz-continuous and so it is not easy to get first order optimality conditions. As a consequence of this, to get sharp optimality conditions and build numerical algorithms are difficult tasks. To overcome this difficulty different authors (see for example, K. Ito and K. Kunisch [1], [13], Karl Kunisch and Daniel Wachsmuth [2],[3] and the references therein) considered a Moreau-Yosida approximation technique to reformulate the governing variational inequality of the first kind as an operator equation involving the *max* function. After that, optimality conditions for the regularized problems are derived and an optimality system for the original control problem is obtained as limit of the regularized optimality systems. But the reformulation in terms of the *max* function in this case lead the authors to propose an semi-smooth Newton methods, or equivalently the iterative primal-dual active set strategy, for its numerical solution of the regularized control problem.

This thesis is primarily based on the results from the papers by Karl Kunisch and Daniel Wachsmuth [2], K. Ito and K. Kunisch [1].

The purpose of this thesis is to investigate the analytical background of an optimal control problem subject to elliptic variational inequalities of the first kind with unilateral and bilateral obstacle problems and develop a regularization method (i.e. to approximate the nondifferentiable ones depending on ($c \geq 0 \quad c \to \infty$)) for solving a nondifferentiable minimization problem. We also derive optimality conditions for the regularized problems and first order necessary optimality system for the original control problem is obtained as limit of the regularized optimality systems.

1.2 Outline of Thesis

Let us briefly outline the structure of the paper.
In chapter 2, we briefly mention some basic definitions and preliminaries of functional

analysis tools which we are going to use in the subsequent chapters. We recall some classical results and existence of solutions of Variational Inequalities of First Kind.

In chapter 3, we investigate optimal control problems governed by variational inequalities and involving constraints on both the control and the state. The formulation of the optimal control problem subject to unilateral obstacle problem is described, and we present regularity assumptions that we use throughout this paper. A regularized family of optimal control problems is introduced. The regularized problems are investigated and convergence of the regularized optimal solutions to the associated solutions of the obstacle problem is studied. Using a local smoothing of the max function, a first order optimality system for each regularized problem is derived. We also analyze properties of solutions of the regularized problem and their convergence as well as rate of convergence.

In chapter 4, we focus on the formulation of the optimal control problem subject to bilateral obstacle problem. A regularized family of optimal control problems is introduced. The regularized problems are investigated , and feasibility, regularity and convergence of the regularized optimal solutions is studied.

In chapter 5, we study an optimal control problem governed by semilinear elliptic regularized equation. Existence of a sequence of solutions of the regularized problem converging weakly and strongly to solutions of the original problem is studied. The regularity of the adjoint state and the state constraint multiplier is also studied. A sharp optimality system for the original control problem is obtained as limit of the regularized optimality systems.

In chapter 6, we formulate the obstacle problems for which the solution algorithms are developed. A regularized problem and iterative second-order algorithms for its solution are analyzed in infinite dimensional function spaces. Motivated from the dual formulation, a primal-dual active set strategy and a semismooth Newton method for a regularized problem are presented and their close relation is analyzed.

In chapter 7, we give a summary and concluding remarks.

In the Appendix, some general definitions for normed linear space are listed.
At the end of the thesis the list of considered literature is listed.

1.3 What is an optimal control problem?

An optimal control problem has the goal to find a control function for a given system such that a certain optimality criterion is achieved. The essential component of an optimal control problem is firstly the objective functional, a function of state and control variables, which is to be minimized (maximized). Moreover the problem includes in general partial differential equations of the state y and the control function u as constraints. In addition, further constraints on u and y may be given. The control function is to be chosen in that way such that the objective functional is minimal.

There are a lot of applications of optimal control problems for example in aeronautics, in robotic and in the control of chemical processes.

In this thesis, the constraint is not given by a partial differential equation, but by a variational inequality (VI). This significantly complicates the discussion of the associated optimal control problem, since VIs provide a certain non-smooth character. The finite dimensional counterpart of the optimal control problems with VIs are mathematical programs with equilibrium constraints (MPECs). As an example let us consider the following optimal control problem in which the state y is formulated as a solution of an elliptic variational inequality (cf. [7], page 2)

$$\begin{cases} \min J(y,u) \quad over \;\; y \in K, \quad u \in U, \\ s.t \;\; \langle Ay - g(u), v - y \rangle \geq 0 \;\; \forall v \in K \end{cases} \tag{1.1}$$

A is a second order linear elliptic partial differential operator and K denotes a closed convex cone in a Banach space. Moreover $g(\cdot)$ denotes a source term. If one introduces a slack variable ξ, the variational inequality can be formulated as a complementary problem.

$$\begin{aligned} &Ay - \xi = g(u), \quad y \geq 0 \;\text{ a.e. in }\; \Omega, \quad \langle \xi, v - y \rangle \geq 0 \;\; \forall v \in K \\ &\Leftrightarrow Ay - \xi = g(u), \quad y \geq 0, \quad \xi \geq 0, \quad \langle y, \xi \rangle = 0 \end{aligned}$$

This arising problem is called a mathematical program with complementarity constraints (MPCC) in the function space, which is a special form of a MPEC. For this form of an optimal control problem all classical constraint qualifications are violated. Thus one can not apply the standard optimality theory, which is the essential difficulty for this sort of optimization problems.

1.4 What is an obstacle problem?

In this master thesis we will concentrate on a special form of (1.1), where the VI is given by the obstacle problem. The obstacle problem is a classical instance for VIs and free boundary problems. It describes the problem to find the equilibrium position of an elastic membrane. In classical elasticity theory a membrane is a thin plate offering no resistance to bending, but acting only in tension. The boundary of the observed domain is held fixed and the membrane is constrained to lie above a given obstacle. The problem is related to the study of minimal surfaces and the capacity of a set in potential theory as well. To be more precise, we can formulate the obstacle problem as the problem to find the minimizer of the Dirichlet energy functional

$$J(u) = \int_D \frac{1}{2}|\nabla u|^2 dx, \tag{1.2}$$

where the function u represent the vertical displacement of the membrane. We denote by D a Lipschitz domain in $\mathbb{R}^n$. Let a smooth function $\varphi : \bar{D} \to \mathbb{R}$ be given, such that $\varphi|_{\partial D} \leq 0$. Moreover define the set $K = \{v \in H_0^1(D) : v|_{\partial D} = 0 \;\; and \;\; v \geq \varphi\}$, which is closed and convex. The membrane takes that form, which yields minimal potential energy. Then the solution of the obstacle problem is the function, which minimizes (1.2) overall functions v belonging to K. Since this is a convex optimization problem, we can equivalently reformulate it by its first-order optimality conditions: Seeking the energy minimizer in the set K it is equivalent to seek $u \in K$ such that

$$\int_D \nabla u \cdot \nabla(v - u) dx \geq 0 \quad \forall v \in K,$$

which is a VI of the same type as in (1.1).

Chapter 2

Notation, basic definitions and theorems

In this chapter we briefly mention some basic definitions and preliminaries of functional analysis tools which we are going to use in the subsequent chapters. We recall some classical results and existence of solutions of Variational Inequalities of First Kind. In the Appendix some basic facts about normed linear spaces are given.

2.1 Notation

We start with the introduction of some basic notation and with some assumptions on the quantities involved.
Throughout the thesis we will use the following notation: Let Ω be an open, bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega = \Gamma$. Throughout this thesis, unless specified, the $L^2(\Omega)$ inner product is defined by $(\cdot, \cdot)$. The duality pairing between $H_0^1(\Omega)$ and its dual $H_0^1(\Omega)^* = H^{-1}$ is often denoted by $\langle \cdot, \cdot \rangle$. It is well known that $H_o^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ with compact and dense injection.

2.2 Sobolev Spaces

Let $\Omega \in \mathbb{R}^N$ be a bounded Lipschitz domain. We denote by $|\Omega|$ its N-dimensional Lebesgue measure.

Definition 2.1. *(L^p-space)*
We consider by $L^p(\Omega), 1 \leq p < \infty$, the space of all (equivalence classes of) measurable functions $y : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega |y(x)|^p dx < \infty. \tag{2.1}$$

$L^p(\Omega)$ is equipped with the norm

$$\|y\|_{L^p(\Omega)} = (\int_\Omega |y(x)|^p dx)^{1/p}. \tag{2.2}$$

All functions, which are only different on sets of measure zero are seen to be equal. They belong to the same equivalence class.

Definition 2.2. *(L^∞ – space)*
By $L^\infty(\Omega)$ we denote the space of essentially bounded and measurable functions $y : \Omega \to \mathbb{R}$ equipped with the norm

$$\|y\|_{L^\infty(\Omega)} = ess \sup_{x\in\Omega} |y(x)|. \tag{2.3}$$

For all $p \in [1,\infty]$ the spaces $L^p(\Omega)$ are Banach spaces. $L^2(\Omega)$ is, when equipped, with the scalar product $(u,v)_{L^2(\Omega)} = \int_\Omega uvdx$ a Hilbert space.

Definition 2.3. *(Weak derivative)*
Let $y \in L^1_{loc}(\Omega)$ be given and $\alpha = (\alpha_1, \cdot\cdot\cdot, \alpha_n)$ be a multi-index. If there exists a function $w \in L^1_{loc}(\Omega)$ fulfilling

$$\int_\Omega y(x) D^\alpha v(x) dx = (-1)^{|\alpha|} \int_\Omega w(x) v(x) dx \qquad \forall v \in C_0^\infty(\Omega), \tag{2.4}$$

*then we call w the **weak derivative** of y and the derivative is denoted by $D^\alpha y$.*
The set $L^1_{loc}(\Omega)$ consists of all the functions $g : \Omega \to \mathbb{R}$, which for every compact subset $K \subset \Omega$ are Lebesgue-integrable, so being in $L^1(K)$.

Definition 2.4. ***(Sobolev Space)***
Let $1 \le p < \infty, k \in \mathbb{N}$. We define by $W^{k,p}(\Omega)$ the linear space of all $y \in L^p(\Omega)$, for which all weak derivatives $D^\alpha y$ with $|\alpha| \le k$ exists and belong to $L^p(\Omega)$, equipped with the norm

$$\|y\|_{W^{k,p}(\Omega)} = (\sum_{|\alpha|\le k} \int_\Omega |D^\alpha y(x)|^p dx)^{1/p}, \tag{2.5}$$

Moreover, we introduce $W^{k,\infty}(\Omega)$ for $p = \infty$ whose norm is given by

$$\|y\|_{W^{k,\infty}(\Omega)} = \max_{|\alpha|\le k} \|D^\alpha y\|_{L^\infty(\Omega)}. \tag{2.6}$$

*These spaces are called **Sobolev spaces**.*

All Sobolev spaces $W^{k,p}(\Omega)$ are Banach spaces. Of special interest is the case $p = 2$, where we put

$$H^k(\Omega) := W^{k,2}(\Omega). \tag{2.7}$$

In view of Definition (2.4) the space $H^1(\Omega)$, which we often need in our later discussion is defined

$$H^1(\Omega) := \{y \in L^2(\Omega) : D_i y \in L^2(\Omega) \quad i = 1, 2, \cdots, n\}, \tag{2.8}$$

and equipped with the norm

$$\|y\|_{H^1(\Omega)} = (\int_\Omega (y^2 + |\nabla y|^2)dx)^{1/2}. \tag{2.9}$$

By introducing the scaler product

$$(u, v)_{H^1(\Omega)} = \int_\Omega uvdx + \int_\Omega \nabla u \cdot \nabla vdx \tag{2.10}$$

$H^1(\Omega)$ becomes a Hilbert space.

Definition 2.5. *(The Sobolev space $W_0^{k,p}(\Omega)$)*
The closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ is called $W_0^{k,p}(\Omega)$. This space is equipped with same norm as $W^{k,p}(\Omega)$ and it is a closed subspace of $W^{k,p}(\Omega)$. In particular we define $H_0^k(\Omega) := W_0^{k,2}(\Omega)$.

For more detailed information about Sobolev spaces see for instance Fredi Tröltzsch [8]

The usefulness of Sobolev spaces is to a large extent determined by embedding results and trace theorems. We follow the standard text by Fredi Tröltzsch [8]

Theorem 2.6. ***(Sobolev Embedding)****.*
Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Moreover, let $1 < p < \infty$, and let m be a nonnegative integer. Then the following embeddings exist and are continuous:

- *for $mp < N$: $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ if $1 \leq q \leq \frac{Np}{N-mp}$*
- *for $mp = N$: $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ if $1 \leq q < \infty$*
- *for $mp > N$: $W^{m,p}(\Omega) \hookrightarrow C(\bar{\Omega})$.*

In particular, if $\Omega \subset \mathbb{R}^2$, then $H^1(\Omega) = W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leq q < \infty$, and if $\Omega \subset \mathbb{R}^3$, then $H^1(\Omega) \hookrightarrow L^6(\Omega)$. The smoothness properties of boundary values are described by the following result.

Theorem 2.7. *Let $m \in \mathbb{N}$ with $m > 0$, and let the boundary Γ be a class $C^{m-1,1}$. Then for $mp < N$ the trace operator τ is continuous from $W^{m,p}(\Omega)$ into $L^r(\Gamma)$, provided that $1 \leq r \leq \frac{(N-1)p}{N-mp}$. If $mp = N$, then τ is continuous for all $1 \leq r < \infty$.*

Theorem 2.8. *Suppose that Ω is a domain of class C^m, and let $1 < p < \infty$. Then the trace operator τ is continuous from $W^{m,p}(\Omega)$ onto $W^{m-1/p,p}(\Gamma)$.*

In particular, the continuity of the mapping $\tau : H^1(\Omega) \to H^{1/2}(\Gamma)$ follows; τ is even surjective.

Theorem 2.9. *(Rellich). Suppose that Ω is a bounded Lipschitz domain, and let $1 \leq p < \infty$ and $m \in \mathbb{N}$, with $m > 0$. Then every bounded set in $W^{m,p}(\Omega)$ is relatively compact in $W^{m-1,p}(\Omega)$.*

The above property is called a *compact embedding.* In particular, bounded subsets of $H^1(\Omega)$ are relatively compact in $L^2(\Omega)$.

Lemma 2.10. ***(Lax-Milgram)***
Let V be a real Hilbert space and $a : V \times V \to \mathbb{R}$ be a bilinear form with the following properties:
There exists positive constants ν_1 and ν_2 such that for all $y, v \in V$ the relations

$$\begin{aligned} |a(y,v)| &\leq C_b \|y\|_V \|v\|_V && \text{(Boundedness)} \\ a(y,y) &\geq C_c \|y\|_V^2 && \text{(V-Ellipticity or coercivity)} \end{aligned}$$

are fulfilled. Then the variational formulation

$$a(y,v) = \ell(v) \quad \text{for all} \ \ v \in V,$$

admits for every $\ell \in V^*$ exactly one solution $y \in V$ and there exists a constant μ independent of ℓ, such that the following inequality holds:

$$\|y\|_V \leq \mu \|\ell\|_{V^*}.$$

By the Riesz identification theorem we can uniquely identify $\ell \in V^*$ with a $f \in V$ so that $\ell(v) = (f, v)_V$ and $\|\ell\|_{V^*} = \|f\|_V$.

Theorem 2.11. ***(Hölder inequality)***
Let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$ and let $f \in L^p(\Omega), g \in L^q(\Omega)$, then it follows that

$$fg \in L^1(\Omega) \quad \text{and} \quad \|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

Lemma 2.12. ***(Young's inequality)***
Let $a, b \geq 0$, $p, q > 1$, $1/p + 1/q = 1$. Then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Lemma 2.13. ***(Modified Young's inequality)***
Let $a, b \geq 0$, $\varepsilon > 0$, $p, q > 1$, $1/p + 1/q = 1$. Then

$$ab \leq \frac{\varepsilon a^p}{p} + \frac{\varepsilon^{1-q} b^q}{q}.$$

Lemma 2.14. ***(Poincaré-Friedrichs inequality)***
Let Ω be a bounded Lipschitz domain. Then there exist a constant c_p, only depending on Ω, such that

$$\int_\Omega |y|^2 dx \leq c_p \int_\Omega |\nabla y|^2 dx$$

for all $y \in H_0^1(\Omega)$.

A **semilinear PDE** is a partial differential equation, whose main part of the differential operator is linear. An example for semilinear PDE is given by

$$\begin{aligned} -\Delta y + y^3 &= u \text{ in } \Omega \\ y = 0 &= \quad \text{on } \Gamma. \end{aligned}$$

2.3 Basic convergence concepts

Definition 2.15. ***(strong convergence)***
A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ converges to $x \in X$,i.e. $x_n \to x$, if

$$\lim_{n\to\infty} \|x_n - x\| = 0. \tag{2.11}$$

Definition 2.16. ***(Weak Convergence)***
Let X be a Banach space. We say that a sequence (x_n) converges weakly to $x \in X$, written

$$x_n \rightharpoonup x, \tag{2.12}$$

if

$$\langle x^*, x_n \rangle_{X^*,X} \to \langle x^*, x \rangle_{X^*,X} \quad as \quad n \to \infty \quad \forall x^* \in X^* \tag{2.13}$$

Definition 2.17. ***(Weakly continuous)***
Let X, Y be reflexive Banach spaces. A function $f : X \to Y$ is called weakly continuous if

$$x_n \rightharpoonup x \quad in \quad X \Longrightarrow f(x_n) \rightharpoonup f(x) \quad in \quad Y \tag{2.14}$$

Theorem 2.18. ***(Lower weakly Semicontinuity)***
Let X be a Banach space. Then any continuous, convex functional $f : X \to \mathbb{R}$ is weakly lower semicontinuous, i.e.

$$x_n \rightharpoonup x \Longrightarrow \liminf_{n\to\infty} f(x_n) \geq f(x). \tag{2.15}$$

Definition 2.19. ***(Radially unbounded)***
A function $f : X \to \mathbb{R}$ is called radially unbounded if for all sequences (x_n) in X it holds:

$$\|x_n\| \to +\infty \Longrightarrow f(x_n) \to +\infty \tag{2.16}$$

Theorem 2.20. *Let H be a Hilbert space. Then it holds:*

$$x_n \rightharpoonup x \text{ in } H \text{ and } \|x_n\|_H \to \|x\|_H \text{ for } n \to \infty \Longrightarrow x_n \to x \text{ in } H \text{ for } n \to \infty. \quad (2.17)$$

Definition 2.21. ***(Convexity of a Set)***
A set C of a normed linear space X is called convex, if

$$x, y \in C \Rightarrow \lambda x + (1-\lambda)y \in C \quad \forall \lambda \in [0,1].$$

Definition 2.22. ***(Convexity of a function)***
A functional $f : X \to \mathbb{R}$ in a normed linear space is called,

(i) *convex, if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in X$ and all $\lambda \in [0,1]$.

(ii) *strict convex, if the strict inequality holds for all $x \neq y$ and for all $\lambda \in (0,1)$.*

2.4 Differentiability in Banach Spaces

In the following let X and Y be Banach spaces, $U \subset X$ be an open set and $f : X \to Y$ be a given function.

Definition 2.23. ***(Directional Derivative)***
If for $x \in X$, $h \in X$ the limit

$$\delta f(x,h) = \lim_{t \downarrow 0} \frac{f(x+th) - f(x)}{t}$$

exists in Y, then f is directionally differentiable at x in direction h.

Definition 2.24. ***(Gâteaux Derivative)***
Let f be directionally differentiable at x in all directions $h \in X$.
If there exists an operator $A \in \mathcal{L}(X,Y)$ such that

$$\delta f(x,h) = Ah \quad \forall h,$$

then f is said to be Gâteaux differentiable at x.

Definition 2.25. ***(Fréchet Derivative)***
Let f be Gâteaux differentiable at x. If

$$\lim_{\|h\|_X \to 0} \frac{\|f(x+h) - f(x) - \delta f(x)h\|_Y}{\|h\|_X} = 0,$$

then f is called Fréchet-differentiable at x.

Definition 2.26. ***(Newton Derivative)***
If there is a mapping $\delta f : U \to Y$ *such that*

$$\lim_{\|h\|_X \to 0} \frac{\|f(x+h) - f(x) - \delta f(x+h)h\|_Y}{\|h\|_X} = 0,$$

then f *is called Newton-differentiable in* U*.*

Remark 2.27. *If* f *is Fréchet differentiable and* δf *is Lipschitz on an open set* $U \subset X$*, then* f *is Newton-differentiable on* U*.*

2.5 Existence of solutions of Variational Inequalities of First Kind

The obstacle problem is a classic motivating example in the mathematical study of variational inequalities and free boundary problems. The problem is to find the equilibrium position of an elastic membrane whose boundary is held fixed, and which is constrained to lie below a given obstacle.

Before we formulate the Model problem, it will be convenient to recall and discuss the following notations, definitions and theorems of the existence and uniqueness of variational inequalities of First Kind. Now let us begin with the state-equation which is a variational inequality. We denote the set of admissible controls by :

$$U_{ad} = \{u \in L^2(\Omega) : u_a \leq u \leq u_b\}$$

be a nonempty, closed, convex subset of $L^2(\Omega)$. For each $u \in U_{ad}$ we define $y = y(u)$ (the state function of the system) as the solution of the variational inequality of the first kind:
Find $y \in H_0^1(\Omega)$ such that y is a solution of the problem

$$a(y, v - y) \geq (u, v - y), \quad \forall v \in K, y \in K \tag{2.18}$$

where the set K is given by

$$K = \{v \in H_0^1(\Omega) : v \leq \psi\}$$

is a closed convex nonempty subset of $H_0^1(\Omega)$.
From the Riesz representation theorem for Hilbert spaces, there exists an operator $A \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$ such that

$$a(y, v) = \langle Ay, v \rangle, \quad \forall y, v \in H_0^1(\Omega).$$

The following results are taken from J.F.Rodrigues [4] and Roland Glowiniski [5]

Theorem 2.28. ***(Lions-Stampacchia)***
Let $K \subset H_0^1(\Omega)$ be a closed, nonempty, convex and $A : K \to H^{-1}(\Omega)$ a Lipschitz and coercive operator (not necessarily linear),that is,

$$\|Ay - Av\|_{H^{-1}} \leq C_b\|y - v\|_{H^1}, \quad \forall y, v \in K, \tag{2.19}$$

$$\langle Ay - Av, y - v\rangle \geq C_c\|y - v\|_{H^1}^2, \quad \forall y, v \in K, \tag{2.20}$$

for some constants C_b, $C_c > 0$. Then for each $u \in H^{-1}(\Omega)$, there exists a unique solution to the variational inequality

$$y \in K : \langle Ay - u, v - y\rangle \geq 0, \quad \forall v \in K. \tag{2.21}$$

*Moreover the (nonlinear) solution mapping is Lipschitz continuous, that is, if $u_j \in H^{-1}(\Omega)$
$(j = 1, 2)$ and y_j is the corresponding solution, then*

$$\|y_1 - y_2\|_{H^1} \leq \frac{1}{C_c}\|u_1 - u_2\|_{H^{-1}}. \tag{2.22}$$

Remark 2.29. *: In the case $K = H_0^1(\Omega)$ (or y is an interior point of K), (2.21) reduces to the equation $Ay - u = 0$, since then the $(v - y)$ ranges over a neighborhood of the origin in $H_0^1(\Omega)$.*

Corollary 2.30. **(i)** ***(Stampacchia theorem)*** *[5]*
Let $K \subset H_0^1(\Omega)$ be a closed, nonempty, convex set, $u \in H^{-1}(\Omega)$ and $a(\cdot, \cdot)$ a continuous and coercive bilinear form. Then there exists a unique solution to the variational inequality

$$y \in K : a(y, v - y) \geq \langle u, v - y\rangle, \quad \forall v \in K. \tag{2.23}$$

(ii) ***(Lax-Milgram theorem)***
In the case $K = H_0^1(\Omega)$, (i) reduces to Lax-Milgram, one has unique solvability of

$$y \in H_0^1(\Omega) : \quad a(y, v) = \langle u, v\rangle, \quad \forall v \in H_0^1(\Omega). \tag{2.24}$$

Proposition 2.31. *In $H_0^1(\Omega)$, the variational inequality*

$$y \leq \psi : \quad \langle Ay - u, v - y\rangle \geq 0, \quad \forall v \leq \psi \tag{2.25}$$

for any $\psi \in H^1(\Omega)$, with $\psi|_{\partial\Omega} \geq 0$ is equivalent to the nonlinear complementarity problem

$$y \leq \psi, \quad Ay - u \geq 0 \quad \text{and} \quad \langle Ay - u, y - \psi\rangle = 0. \tag{2.26}$$

Remark 2.32. *1. Let $\phi : \mathbb{R} \to \mathbb{R}$, $\phi \in C^0(\mathbb{R})$, nondecreasing with $\phi(0) = 0$ and $u \in H^{-1}(\Omega)$.*

The nonlinear elliptic equation defined by:
Find $y \in H_0^1(\Omega)$ *such that*

$$\begin{cases} a(y,v) + \langle \phi(y), v \rangle = \langle u, v \rangle, \quad \forall v \in H_0^1(\Omega) \\ \phi(y) \in L^1(\Omega) \cap H^{-1}(\Omega) \end{cases} \tag{2.27}$$

is equivalent to

$$Ay + \phi(y) = u, \quad y \in H_0^1(\Omega), \quad \phi(y) \in L^1(\Omega) \cap H^{-1}(\Omega). \tag{2.28}$$

2. *(2.27) has a unique solution.*

Chapter 3

Optimal control with unilateral constraints

In this chapter we investigate the analytical treatment of optimal control problems governed by a class of elliptic variational inequalities of the first kind with unilateral constraints. Moreover, we consider constraints on the control.

In the optimal control problem of a variational inequality the main difficulty comes from the fact that the mapping between the control and the state (control-to-state operator) is not Gâteaux differentiable (the reason for this fact is that its derivative is also a solution of a variational inequality in (P) and therefore it is not linear with respect to the direction) but only Lipschitz-continuous and so it is not easy to get first order optimality conditions. As a consequence of this, to get sharp optimality conditions and build numerical algorithms are difficult tasks. To overcome this difficulty different authors (see for example, K. Ito and K. Kunisch [1], [13], Karl Kunisch and Daniel Wachsmuth [2],[3] and the references therein) consider a Moreau-Yosida approximation technique to reformulate the governing variational inequality of the first kind as an operator equation involving the *max* function.

Problems in robotics and biomechanics such as trajectory planning or resolution of redundancy can be effectively solved using optimal control. Such systems are often subject to unilateral constraints. Examples include tasks involving contacts (e.g., walking, running, multifingered or multiarm manipulation), and other tasks that may not involve contacts but in which the system state or the inputs must satisfy inequality conditions (e.g., limits on actuator forces), to read more see [20].

3.1 Formulation of the optimal control problem in the variational form

We start with the introduction of some basic notation and with some assumptions on the quantities involved that we use in the forthcoming sections. In this section we will discuss the regularization method by introducing a Lagrange multiplier for the non-differentiable term (or we have approximate the variational inequality by a complementarity constraint formulation) to overcome the difficulty associated with the nondifferentiability of the functional J in (P). The idea of the regularization method is to approximate the non-differentiable term by a sequence of differentiable ones. Approximation of the unilateral obstacle problem to a certain semilinear elliptic equation is shown.

Suppose that Ω is an open, and bounded subset of $\mathbb{R}^N (N \leq 3)$ with Lipschitz-continuous boundary $\Gamma = \partial\Omega$.
For the definition of Sobolev spaces we refer the reader section 2.2.
We define the bilinear form $a(\cdot,\cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ by

$$a(y,v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij} \frac{\partial y}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \sum_{i=1}^{N} \int_\Omega b_i \frac{\partial y}{\partial x_i} v dx + \int_\Omega c_0 y v dx \tag{3.1}$$

where a_{ij}, b_i ,c_0 belong to $L^\infty(\Omega)$. Moreover, we suppose that $a_{ij} \in C^{0,1}(\bar{\Omega})$ (the space of Lipschitz continuous function in Ω, where $\bar{\Omega}$ is the closure of Ω) and $c_0 \geq 0$, to ensure a “good” regularity of the solution and satisfying the conditions $a_{ij} = a_{ji}$ and

$$\sum_{i,j=1}^{N} a_{ij}(x) \xi_i \xi_j \geq \delta_0 \|\xi\|^2 \text{ a.e. on } \Omega \quad \forall \xi \in \mathbb{R}^N. \tag{3.2}$$

By construction the bilinear form $a(\cdot,\cdot)$ is continuous on $H_0^1(\Omega) \times H_0^1(\Omega)$:

$$\exists C_b > 0, \quad \forall (y,v) \in H_0^1(\Omega) \times H_0^1(\Omega) \qquad a(y,v) \leq C_b \|y\|_{H_0^1} \|v\|_{H_0^1} \tag{3.3}$$

and is coercive (H^1-ellipticity):

$$\exists C_c > 0, \quad \forall y \in H_0^1(\Omega) \qquad a(y,y) \geq C_c \|y\|^2_{H_0^1(\Omega)} \tag{3.4}$$

We call $A : H_0^1 \to H^{-1}$ the linear (elliptic) operator associated to $a(\cdot,\cdot)$ such that

$$\langle Ay, v \rangle := a(y,v) \qquad \forall\, y,v \in H_0^1(\Omega). \tag{3.5}$$

The operator A is an elliptic differential operator defined by

$$(Ay)(x) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}(a_{ij}(x) \frac{\partial}{\partial x_i} y(x)) + \sum_{i=1}^{N} b_i \frac{\partial}{\partial x_i} y(x) + c_0(x) y(x) \tag{3.6}$$

We note that the coercivity assumption (3.4) on a implies that

$$\forall y \in H_0^1(\Omega) \quad \langle Ay, y\rangle \geq C_c \|y\|^2_{H_0^1(\Omega)} \tag{3.7}$$

Consider the following variational inequality

$$a(y, v-y) \geq (u, v-y), \quad \forall v \in K, \quad u \in U_{ad} \tag{3.8}$$

where

$$K = \{v \in H_0^1(\Omega) : v \leq \psi \ \text{ a.e. on } \ \Omega\}$$

and the set of admissible controls

$$U_{ad} = \{u \in L^2(\Omega) : u_a \leq u \leq u_b\}$$

are nonempty, closed, and convex subset of $H_0^1(\Omega)$ and $L^2(\Omega)$ respectively.

The optimal control problems subject to variational inequality (obstacle problem) to be studied in this chapter can be set in the following general form:

$$(P) \quad \begin{cases} \min J(y,u) = g(y) + j(u) \\ over \ \ y \in K, \quad u \in U_{ad} \\ s.t \ \ a(y, v-y) \geq (u, v-y) \ \ \forall v \in K \end{cases}$$

Here we call variable y is the state and u is the control. The function ψ (obstacle) denotes the bound constraint on the state, i.e., $y \leq \psi$ has to hold pointwise almost everywhere (a.e.) in Ω. Note that the mapping control-to-state is not differentiable (and even not continuous if we define it on the whole space $H_0^1(\Omega)$).

It is well known that under the conditions that has been specified above on the coefficients of the bilinear form a and introducing a multiplier λ, the obstacle problem (3.8) can be equivalently written as complementary condition as follows:

$$\begin{cases} Ay + \lambda = u, \\ y \leq \psi, \\ \lambda \geq 0, \\ (\lambda, y-\psi) = 0 \end{cases} \tag{3.9}$$

where $\lambda \in H^{-1}(\Omega)$ is the associated Lagrange multiplier to the solution (P) and $\psi \in H^1(\Omega)$ with $\psi|_\Gamma \geq 0$.

In this way the optimal control of variational inequality (P) is interpreted as optimization with complementarity constraints. If λ has extra regularity in sense that $\lambda \in L^2(\Omega)$, the optimality system (3.9) can equivalently be expressed as

$$\begin{cases} Ay + \lambda = u \ \ in \ \ L^2(\Omega) \\ \\ \lambda = \max(0, \lambda + c(y-\psi)) \end{cases} \tag{3.10}$$

for any $c > 0$ and where max denotes the pointwise a.e. maximum operation. Since this max −function is non-smooth, the variational inequality in (P) makes the optimal control problem (P) non-smooth.
The optimal control problem (P) therefore can be equivalently expressed as minimizing $J(y, u)$ subject to (3.10).
Since $x \to \max(0, x)$ is not C^1 regular (not Gâteaux differentiable), to regularize the max −function in (3.10) we are tempted to use the well known smoothing (C^1−approximation)

$$\max_c(0, x) = \begin{cases} x, & \text{for } x \geq \frac{1}{2c} \\ \frac{c}{2}(x + \frac{1}{2c})^2, & \text{for } |x| \leq \frac{1}{2c} \\ 0, & \text{for } x \leq -\frac{1}{2c}. \end{cases} \tag{3.11}$$

where $c > 0$. Then $\max_c(0, x) = \int_{-\infty}^{x} \operatorname{sgn}_c(s)ds$, where $\operatorname{sgn}_c(x)$ is defined by

$$\operatorname{sgn}_c(x) = \begin{cases} 1, & \text{for } x \geq \frac{1}{2c} \\ c(x + \frac{1}{2c}), & \text{for } |x| \leq \frac{1}{2c} \\ 0, & \text{for } x \leq -\frac{1}{2c}. \end{cases} \tag{3.12}$$

Equation (3.10), can be approximated by the following smooth semilinear equation

$$Ay + \max_c(c^s\bar{\lambda} + c(y - \psi)) = u, \tag{3.13}$$

where $0 < s < 1/2$ and $\inf \bar{\lambda} > 0$, $\bar{\lambda} \in L^\infty(\Omega)$, the max −operation was replaced by a generalized Moreau-Yosida type regularization. Regularization refers to the fact that the inequality involving the operator A is replaced by an equality by means of an appropriate Lagrangian variable.
As a consequence the regularized control problems that we are interested are given by

$$(P_c) \quad \begin{cases} \min J(y, u) = g(y) + j(u) \\ over \;\; u \in U_{ad}, \quad \text{subject to} \\ Ay + \max_c(c^s\bar{\lambda} + c(y - \psi)) = u, \;\; y \in H_0^1(\Omega) \end{cases}$$

where $0 < s < 1/2$ and $\inf \bar{\lambda} > 0$, $\bar{\lambda} \in L^\infty(\Omega)$, is fixed during the regularization process given, and $\max_c$ is a C^1−approximation of $x \to \max(0, x)$. If g and j are C^1−regular, then the first order optimality system for (P_c) is given by

$$\begin{cases} Ay_c + \max_c(c^s\bar{\lambda} + c(y_c - \psi)) = u_c, \\ A^*p_c + c\operatorname{sgn}_c(c^s\bar{\lambda} + c(y_c - \psi))p_c + g'(y_c) = 0, \\ (j'(u_c) - p_c, u - u_c) \geq 0, \;\; \forall u \in U_{ad} \end{cases} \tag{3.14}$$

where sgn_c in (3.12) and expressions

$$\lambda_c = \max_c(c^s\bar{\lambda} + c(y_c - \psi)) \text{ and } \mu_c = c\text{sgn}_c(c^s\bar{\lambda} + c(y_c - \psi))p_c$$

in (3.14) tend to measure Lagrange multipliers as $c \to \infty$, here c is the regularization and smoothing parameter.
We are now in position to state the following regularity assumptions that we use throughout this thesis paper: (cf. [2])

Standing assumptions

Assumption 1. (i) The domain $\Omega \subset \mathbb{R}^N$, $N \in \{2, 3\}$ is bounded, and its boundary is of class $C^{1,1}$.

(ii) The operator A is an elliptic differential operator defined by

$$(Ay)(x) = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial}{\partial x_i}y(x)\right) + \sum_{i=1}^{N} b_i \frac{\partial}{\partial x_i}y(x) + c_0(x)y(x)$$

with functions $a_{ij} \in C^{0,1}(\bar{\Omega})$, b_j, $\frac{\partial}{\partial x_j}b_j$, $c_0 \in L^\infty(\Omega)$ satisfying the conditions $a_{ij}(x) = a_{ji}(x)$ and

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \delta_0\|\xi\|^2 \text{ a.e. on } \Omega \ \ \forall \xi \in \mathbb{R}^N.$$

with some $\delta_0 > 0$. Additionally, we require $c_0(x) \geq \delta_1 \geq 0$ with δ_1 sufficiently large such that the bilinear form $a(\cdot, \cdot)$ induced by A fulfills the coercivity condition (3.4).

(iii) The obstacle ψ fulfills $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$ with $A\psi \in L^\infty(\Omega)$ and $\psi|_\Gamma \geq 0$.

(iv) $g : L^2(\Omega) \to \mathbb{R}$ is continuously Fréchet-differentiable and bounded from below, moreover the restriction $g : H_0^1(\Omega) \to \mathbb{R}$ is twice continuously Fréchet-differentiable.

(v) $j : L^2(\Omega) \to \mathbb{R}$ is twice continuously Fréchet-differentiable and weakly lower semicontinuous. Moreover, we assume j to be radially unbounded,i.e., $j(u_n) \to +\infty$ whenever $\|u_n\|_{L^2(\Omega)} \to \infty$ with $u_n \in L^2(\Omega)$.

Some results can be obtained under weaker requirements on g.
Let us introduce the adjoint operator A^* to A by

$$(A^*p)(x) = -\sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{N} a_{ij}(x)\frac{\partial}{\partial x_i}p(x) + b_j(x)p(x)\right) + c_0(x)p(x).$$

3.2 Necessary Optimality Conditions for (P)

In this section we investigate first-order necessary optimality conditions for optimal control problem (P) of variational inequality. The derivation of necessary optimality conditions is challenging due to the lack of Fréchet differentiability of the associated control-to-state map. To overcome this difficulty many authors have used either approximation of the variational inequality by penalization, or the differentiability almost everywhere for Lipschitz continuous mappings or the generalized gradient.
Let us briefly summarize known results about unique solvability of the underlying variational inequality (3.9).

Lemma 3.1. **(i)** *For each $u \in H^{-1}(\Omega)$ the variational inequality (3.8) admits a unique solution $y^* \in H_0^1(\Omega)$,*

(ii) *if $u \in H^{-1}(\Omega)$, then the mapping $u \mapsto y$ is Lipschitz continuous from $u \in H^{-1}(\Omega)$ to $H_0^1(\Omega)$.*

(iii) *if $u \in L^2(\Omega)$, then the mapping $u \mapsto y$ is Lipschitz continuous from $u \in L^2(\Omega)$ to $L^\infty(\Omega)$.*

Proof. (i) Suppose the contrary to our claim, that there exist two solutions y^* and $\tilde{y}$ of the variational inequality in (P), i.e.,

$$a(y^*, v - y^*) \geq \langle u, v - y^* \rangle \quad \forall v \in K, \quad y^* \in K \tag{3.15}$$

$$a(\tilde{y}, v - \tilde{y}) \geq \langle u, v - \tilde{y} \rangle \quad \forall v \in K, \quad \tilde{y} \in K \tag{3.16}$$

Now let us insert $v = \tilde{y}$ in (3.15) and $v = y^*$ in (3.16) and add the arising inequalities (3.15) and (3.16), giving in turn

$$\begin{aligned} a(y^* - \tilde{y}, \tilde{y} - y^*) &\geq \langle u - u, \tilde{y} - y^* \rangle = 0 \\ \Leftrightarrow a(y^* - \tilde{y}, y^* - \tilde{y}) &\leq 0. \end{aligned}$$

Since one knows that $a(y^* - \tilde{y}, y^* - \tilde{y}) \geq 0$, by the coercivity of a and the positivity of norms, it follows that $a(y^* - \tilde{y}, y^* - \tilde{y}) = 0$. From this we conclude

$$\|y^* - \tilde{y}\|_{H^1(\Omega)} = 0 \Leftrightarrow \tilde{y} = y^* \quad \text{a.e. in } \Omega$$

Hence $y^* \in K$ is the unique solution of the variational inequality in the original optimal control problem (P).
(ii) Our proof starts with the observation that

$$y_m = y(u_m) \Leftrightarrow a(y(u_m), v - y(u_m)) \geq \langle u_m, v - y(u_m) \rangle \quad \forall v \in K, y(u_m) \in K \tag{3.17}$$

$$y_n = y(u_n) \Leftrightarrow a(y(u_n), v - y(u_n)) \geq \langle u_n, v - y(u_n) \rangle \quad \forall v \in K, y(u_n) \in K \tag{3.18}$$

where $u_m, u_n \in H^{-1}(\Omega)$ are arbitrary. Inserting $v = y(u_n)$ in (3.17) and $v = y(u_m)$ in (3.18) and adding the arising inequalities yield

$$\begin{aligned} a(y(u_m) - y(u_n), y(u_n) - y(u_m)) &\geq \langle u_m - u_n, y(u_n) - y(u_m)\rangle \\ \Leftrightarrow a(y(u_m) - y(u_n), y(u_m) - y(u_n)) &\leq \langle u_m - u_n, y(u_m) - y(u_n)\rangle. \end{aligned}$$

In the following step we use the coercivity of the bilinear form, therefore one receives

$$\begin{aligned} C_c\|y(u_m) - y(u_n)\|^2_{H_0^1(\Omega)} &\leq a(y(u_m) - y(u_n), y(u_m) - y(u_n)) \leq \langle u_m - u_n, y(u_m) - y(u_n)\rangle \\ &\leq |\langle u_m - u_n, y(u_m) - y(u_n)\rangle| \leq \|u_m - u_n\|_{H^{-1}(\Omega)}\|y(u_m) - y(u_n)\|_{H_0^1(\Omega)} \end{aligned}$$

The last inequality follows from the definition of the operator norm $\|u_m - u_n\|_{H^{-1}(\Omega)}$. Thus we obtain

$$\|y(u_m) - y(u_n)\|_{H_0^1(\Omega)} \leq L\|u_m - u_n\|_{H^{-1}(\Omega)},$$

where $L = \frac{1}{C_c}$. For the proof of (iii) we refer the reader to see Lemma 2.2 in [2] on page 6. This completes the proof. □

Under the strong regularity assumptions (1) above we get the following result (cf. the result can be found from Brezis and Stampacchia [19]).

Lemma 3.2. *For $u \in L^2(\Omega)$ the unique solution (y, λ) of (3.9) belongs to $(H_0^1(\Omega) \cap H^2(\Omega)) \times L^2(\Omega)$. If in addition $u \in L^p(\Omega)$ and $\max(0, A\psi - u) \in L^p(\Omega)$, for $p \in [2, \infty)$, then $(y, \lambda) \in W^{2,p}(\Omega) \times L^p(\Omega)$.*

Now we prove that conditions (iv) and (v) of assumption 1 together with Lemma 3.1 imply the existence of at least one solution (y^*, u^*) with $y^* = y(u^*)$ to (P).

Proposition 3.3. *Let $j : L^2(\Omega) \to \mathbb{R}$ be weakly lower semi-continuous. There exists a solution $(y^*, u^*) \in H_0^1(\Omega) \times L^2(\Omega)$ to (P).*

Proof. Since j is radially unbound and g is bounded below, every minimizing sequence $\{(y(u_n), u_n)\}$ to (P) has a weakly convergent subsequence, denoted by the same symbol, with weak limit $u^* \in L^2(\Omega)$ and $(y^*, \psi^*) \in H_0^1 \times U_{ad}$ such that $u_n \to u^*$ weakly in L^2 and $y_n \to y^*$ weakly in H_0^1. Moreover, it follows that $y_n \to y^*$ strongly in H_0^1 and that y^* is the solution to (3.8) with $u = u^*$.
Due to weak lower semi-continuity of j and continuity of $g : H_0^1(\Omega) \to \mathbb{R}$

$$J(y(u^*), u^*) \leq \lim_{n\to\infty} \inf J(y(u_n), u_n), \tag{3.19}$$

and consequently (y^*, u^*) is a solution to (P). □

In [25], Anton Schiela and Daniel Wachsmuth had obtained the following optimality condition system

Theorem 3.4. *Let (y^*, u^*) be a locally optimal pair for the optimal control problem (P) with associated multiplier $\lambda^* \in L^2(\Omega)$. Then there exist adjoint states $p^* \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and $\mu^* \in H^{-1}(\Omega) \cap (L^\infty(\Omega))^*$ such that*

$$A^*p^* + \mu^* + g'(y^*) = 0 \quad \text{and} \quad p^* \geq 0 \quad \text{where} \quad y^* = \psi, \tag{3.20}$$

$$\lambda^* p^* = 0 \quad \text{a.e. on } \Omega, \quad \text{and} \quad \langle \mu^*, p^* \rangle \geq 0 \tag{3.21}$$

$$\langle \mu^*, y^* - \psi \rangle = 0, \tag{3.22}$$

$$\langle \mu^*, \phi \rangle \geq 0 \quad \text{for all} \quad \phi \in H_0^1(\Omega) \quad \text{with} \quad \langle \lambda^*, \phi \rangle = 0 \quad \text{and} \quad \phi \geq 0 \quad \text{on} \quad \{y^* = \psi\}, \tag{3.23}$$

$$(j'(u^*) - p^*, u - u^*) \geq 0 \quad \forall u \in U_{ad} \tag{3.24}$$

Moreover, we have the following sign condition for μ^* on the biactive set $B = \{\lambda = 0, y = \psi\}$:

$$\langle \mu^*, \phi \rangle \geq 0 \quad \text{for all} \quad \phi \in H_0^1(\Omega), \quad \phi \geq 0 \quad \text{on} \quad B, \quad \phi = 0 \quad \text{on} \quad \Omega \backslash B. \tag{3.25}$$

This last condition (3.25) is true only if the admissible set U_{ad} is the whole space $L^2(\Omega)$.

Note that conditions (3.20) to (3.24) is called C-stationary and the C-stationary together with the last condition is called strong stationary.

It is well known that in the case of nonlinear equations the first order conditions are in general not sufficient for optimality. A second order sufficient optimality conditions for a class of elliptic boundary control problems is derived in [2].

3.3 A semilinear elliptic regularized problem

The elliptic equation occurring in problem (3.13) is semilinear. In this and the next two sections we will discuss existence, regularity and feasibility of the solution of the semilinear elliptic regularized problem of (3.13) with homogenous Dirichlet boundary value problem.

$$\left.\begin{aligned} Ay + \max_c(c^s\bar{\lambda} + c(y - \psi)) = u \ \text{ in } \Omega \\ y = 0 \ \text{ on } \Gamma \end{aligned}\right\} \tag{3.26}$$

The elliptic differential operator A is assumed to take the form (3.6), and the function u will play the role of the controls. This class of elliptic problems exhibits the essential difficulties associated with nonlinear equations.
We recall that problems in which the control occurs as a source term on the right-hand side of the partial differential equation are termed distributed control problems.

Now we give a review of some classical existence results for weak solutions to Dirichlet problems concerning nonlinear elliptic operators (3.26). First of all, we refer to some classical results involving the so-called monotone operators and then we show how these results can be applied to Dirichlet problems for nonlinear elliptic operators.

Definition 3.5. *A mapping $B : K \to H^{-1}(\Omega)$ is called*

- *monotone if*
$$\langle By - Bv, y - v\rangle \geq 0, \quad \forall y, v \in H_0^1(\Omega), \tag{3.27}$$
- *strictly monotone if it is monotone and*
$$\langle By - Bv, y - v\rangle > 0, \quad \forall y, v \in H_0^1(\Omega), y \neq v, \tag{3.28}$$
- *hemicontinuous if*
$$\langle B(y_1 + ty_2), v\rangle \to \langle By_1, v\rangle \quad \text{as} \quad t \to 0^+ \quad, \forall y_1, y_2, v \in H_0^1(\Omega). \tag{3.29}$$

Obviously if A is a continuous operator, then A is also hemicontinuous, but the contrary is not true in general. Nevertheless hemicontinuity plus monotonicity and boundedness of an operator yields the continuity.
A bounded, hemicontinuous and monotone operator is not enough to get an existence theorem. This result may be proved by assuming that the operator is coercive.

- *coercive if*
$$\frac{\langle By, y\rangle}{\|y\|_{H_0^1}} \to +\infty \quad \text{as} \quad \|y\|_{H_0^1} \to +\infty \quad \text{for any} \quad y \in K \tag{3.30}$$
- *strongly monotone if*
$$\langle By - Bv, y - v\rangle \geq \beta_0 \|y - v\|_{H_0^1}^2, \quad \forall y, v \in H_0^1(\Omega), \tag{3.31}$$

To prove the existence of solutions of the regularized problem (3.26) we use the following theorem which is due to Browder and Minty (cf. Fredi Tröltzsch [8]).

Theorem 3.6. *(Main Theorem on Monotone Operators)*
Let V be a separable Hilbert space, and let a mapping $B : V \to V^*$ be monotone, coercive, and hemicontinuous. Then the equation $By = f$ has for every $f \in V^*(\Omega)$ a solution $y \in V(\Omega)$. The set of all solutions is bounded, closed and convex. If B is strictly monotone, then y is uniquely determined. If B is moreover strongly monotone, then the inverse $B^{-1} : V^* \to V$ is Lipschitz continuous mapping.

Proof. Its proof can be found in, e.g., Eberhard Zeidler, Léo F. Boron [21]. □

We apply this theorem to problem (3.26) in the space $V = H_0^1(\Omega)$ and $f = u$. To do this, we first have to define the notation of a weak solution to the nonlinear elliptic boundary value problem (3.26). The idea is simple: we bring the nonlinear term $\max_c(c^s\bar{\lambda} + c(y - \psi))$ in (3.26) to the right-hand side of the equation, thus obtaining a boundary value problem with the right-hand side $\tilde{u} = u - \max_c(c^s\bar{\lambda} + c(y - \psi))$ and linear differential operators on the left-hand side. For this purpose , we use the variational formulation for linear boundary value problem.

Definition 3.7. *A function $y \in H_0^1(\Omega)$ is called a weak solution to problem (3.26) if we have, for every $v \in H_0^1(\Omega)$,*

$$a(y, v) + \int_\Omega max_c(c^s\bar{\lambda} + c(y - \psi))vdx = \int_\Omega uvdx. \tag{3.32}$$

Assumption 2. Let $\Omega \subset \mathbb{R}^N, \ \ N \leq 3$, is a bounded Lipschitz domain with boundary Γ, and A is an elliptic differential operator of the form (3.6) with bounded and measurable coefficient functions a_{ij} that satisfy the symmetry condition and the condition (3.2) of uniform ellipticity.

Note that $\max_c(c^s\bar{\lambda} + c(y - \psi))$ is bounded, since

$$|\max_c(c^s\bar{\lambda} + c(y - \psi))| \leq |c^s\bar{\lambda} + c(y - \psi)| + \frac{1}{2c},$$

and it is monotone increasing with respect to y almost everywhere in Ω.

Theorem 3.8. *Suppose that Assumption 2 hold. Then for each $c > 0$ and for every right-hand side $u \in L^2(\Omega)$ there exists a unique solution $y_c \in H_0^1(\Omega)$ of the semilinear elliptic equation (3.26). Moreover, there is some constant $\gamma > 0$ such that*

$$\|y_c\|_{H^1(\Omega)} \leq \gamma\|u\|_{L^2(\Omega)}. \tag{3.33}$$

Proof. We apply the main theorem on monotone operators in $V = H_0^1(\Omega)$.

(i) <u>*Definition of a monotone operator $B : H_0^1 \longrightarrow H^{-1}$*</u>

It follows from section 3.1 that the bilinear form (3.1) generates a continuous linear operator $A : H_0^1 \longrightarrow H^{-1}$ through the relation

$$\langle Ay_c, v\rangle = a(y_c, v).$$

This is the linear part of nonlinear operator B. The nonlinear part of B is formally defined by the identity $(A_1y_c)(x) := \max_c(c^s\bar{\lambda} + c(y_c(x) - \psi))$. The sum of the two operators yield the the operator B, i.e., $B = A + A_1$.

(ii) *Monotonicity*

We show that the operators A and A_1 are monotone so that this property then also holds for B. First, A is monotone, since $a(y_c, y_c) \geq 0$ for all $y_c \in H_0^1(\Omega)$. Next, we consider A_1. Owing the monotonicity of $\max_c$ in y_c, we have

$$(\max_c(c^s\bar{\lambda} + c(y_{c,1}(x) - \psi)) - \max_c(c^s\bar{\lambda} + c(y_{c,2}(x) - \psi)))(y_{c,1}(x) - y_{c,2}(x)) \geq 0$$

for all $y_{c,1}, y_{c,2} \in H_0^1(\Omega)$ and all x. Therefore, for all $y_{c,1}, y_{c,2} \in H_0^1(\Omega)$

$$\langle A_1(y_{c,1}) - A_1(y_{c,2}), y_{c,1} - y_{c,2}\rangle$$

$$= \int_\Omega (\max_c(c^s\bar{\lambda}+c(y_{c,1}(x)-\psi))-\max_c(c^s\bar{\lambda}+c(y_{c,2}(x)-\psi)))(y_{c,1}(x)-y_{c,2}(x))dx \geq 0$$

Note that the boundedness condition for $\max_c$ guarantees that the function

$$x \mapsto \max_c(c^s\bar{\lambda} + c(y_{c,1}(x) - \psi)) - \max_c(c^s\bar{\lambda} + c(y_{c,2}(x) - \psi))$$

is square integrable for $y_{c,1}, y_{c,2} \in L^2(\Omega)$, so that the above integral exists. In conclusion, A_1 is monotone.

(iii) *Coercivity of B*

A is coercive follows from (3.4) and (3.5),i.e.,

$$\langle Ay_c, y_c\rangle = a(y_c, y_c) \geq C_c\|y_c\|_{H_0^1}^2 \quad \forall y_c \in H_0^1(\Omega). \tag{3.34}$$

For all $y_c \in H_0^1(\Omega)$

$$\begin{aligned}\langle A_1 y_c, y_c\rangle &= \langle \max_c(c^s\bar{\lambda} + c(y_c - \psi)), y_c\rangle \\ &= \int_\Omega [\max_c(c^s\bar{\lambda} + c(y_c - \psi)) - \max_c(c^s\bar{\lambda})](y_c - \psi) \\ &+ \int_\Omega \max_c(c^s\bar{\lambda} + c(y_c - \psi))\psi + \int_\Omega \max_c(c^s\bar{\lambda})(y_c - \psi)\end{aligned}$$

By the monotonicity of $\max_c$,

$$\int_\Omega [\max_c(c^s\bar{\lambda} + c(y_c - \psi)) - \max_c(c^s\bar{\lambda})](y_c - \psi) \geq 0 \tag{3.35}$$

and the last two integrals above can be estimated as

$$\begin{aligned}|\int_\Omega \max_c(c^s\bar{\lambda} + c(y_c - \psi))\psi| &\leq \int_\Omega (c^s\bar{\lambda} + c|y_c - \psi|)|\psi| \\ &\leq (c^s\|\bar{\lambda}\|_{L^\infty} + c(\|y_c\|_{H^1} + \|\psi\|_{L^2}))\|\psi\|_{L^2}\end{aligned} \tag{3.36}$$

$$\begin{aligned}|\int_\Omega \max_c(c^s\bar{\lambda})(y_c - \psi)| &\leq \int_\Omega c^s\bar{\lambda}|y_c - \psi| \\ &\leq c^s\|\bar{\lambda}\|_{L^\infty}(\|y_c\|_{H_0^1} + \|\psi\|_{L^2})\end{aligned} \tag{3.37}$$

Using equations (3.34), (3.35), (3.36) and (3.37) one can show

$$\frac{\langle By_c, y_c\rangle}{\|y_c\|_{H_0^1}} = \frac{\langle Ay_c + A_1 y_c, y_c\rangle}{\|y_c\|_{H_0^1}} \longrightarrow +\infty \text{ as } \|y_c\|_{H_0^1} \longrightarrow +\infty \text{ for any } y_c \in K \quad (3.38)$$

Then this proves the claim that $B = A + A_1$ is coercive.

(iv) *Hemicontinuity of B*

We show that the operators A and A_1 are continuous so that this property then also holds for B and follows the claim. Observe that for any fixed $y \in H_0^1(\Omega)$ the linear mapping $a_y : H_0^1(\Omega) \longrightarrow \mathbb{R}, v \mapsto a(y,v)$ is continuous on $H_0^1(\Omega)$. For the linear operator $A : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega), y \mapsto a_y$ we have

$$\begin{aligned} \|Ay\|_{H^{-1}} &= \sup_{\|v\|_{H_0^1}=1} |a_y(v)| = \sup_{\|v\|_{H_0^1}=1} |a(y,v)| \\ &\le \sup_{\|v\|_{H_0^1}=1} C_b \|y\|_{H_0^1}\|v\|_{H_0^1} = C_b\|y\|_{H_0^1}. \end{aligned}$$

Clearly, this implies that $\|A\| \le C_b$. Hence, A is bounded and implies it is continuous.

Referring section 4.3.3 in [8], one conclude that our Nemytskii operator $A_1 : L^2 \longrightarrow L^2$ is continuous. Applying the embedding property $H_0^1 \hookrightarrow L^2 \hookrightarrow H^{-1}$ we get

$$\begin{aligned} y_{c,n} \longrightarrow y_c \text{ in } H_0^1(\Omega) &\Rightarrow y_{c,n} \longrightarrow y_c \text{ in } L^2(\Omega) \\ &\Rightarrow A_1(y_{c,n}) \longrightarrow A_1(y_c) \text{ in } L^2(\Omega) \\ &\Rightarrow A_1(y_{c,n}) \longrightarrow A_1(y_c) \text{ in } H^{-1}(\Omega) \end{aligned}$$

This proves that $A_1 : H_0^1 \longrightarrow H^{-1}$ is continuous.

(v) *Well-posedness of the solution*

Existence and uniqueness of a weak solution $y_c \in H_0^1(\Omega)$ now follow directly from the main theorem on monotone operators. Since B is obviously strongly monotone, the asserted estimate also holds. Now to prove the estimate we take y_c itself as the test function to obtain

$$a(y_c, y_c) + \langle \max_c(c^s\bar{\lambda} + c(y_c - \psi)), y_c\rangle = \int_\Omega u y_c dx.$$

$$\begin{aligned} a(y_c, y_c) &= \int_\Omega u y_c dx - \int_\Omega \max_c(c^s\bar{\lambda} + c(y_c - \psi)) y_c dx \\ &\le \int_\Omega |u y_c| dx + \int_\Omega |c^s\bar{\lambda} + c(y_c - \psi)||y_c| dx \\ \nu\|y_c\|_{H^1}^2 &\le \|u\|_{L^2}\|y_c\|_{L^2} + [c^s\|\bar{\lambda}\|_{L^\infty} + c(\|y_c\|_{L^2} + \|\psi\|_{L^2})]\|y_c\|_{L^2} \\ &\le \|u\|_{L^2}\|y_c\|_{H^1} + [c^s\|\bar{\lambda}\|_{L^\infty} + c(\|y_c\|_{H^1} + \|\psi\|_{L^2})]\|y_c\|_{H^1} \\ \nu\|y_c\|_{H^1} &\le \|u\|_{L^2} + c^s\|\bar{\lambda}\|_{L^\infty} + c\|y_c\|_{H^1} + c\|\psi\|_{L^2} \end{aligned}$$

absorbing the third term in the right hand side by the left hand side term then the asserted estimate follows. This concludes the proof of the theorem. □

3.4 Elliptic regularity and continuity of solutions of regularized problems

In this section, we will prove the results concerning essential boundedness of the solution to the semilinear elliptic boundary value problem (3.26) in section 3.3.

Theorem 3.9. *Suppose that Assumption 2 hold, and let $r > N/2$. Then for any pair $u \in L^r(\Omega)$, we have $y_c \in L^\infty(\Omega)$.*

To prove this theorem we apply a method of Stampacchia. It makes use of the following auxiliary result (cf. David Kinderlehrer, Guido Stampacchia [18]).

Lemma 3.10. *Let $k_0 \in \mathbb{R}$, and suppose that φ is nonnegative and nonincreasing function defined in $[k_0, \infty)$ and having the following property: for every $h > k \geq k_0$,*

$$\varphi(h) \leq \frac{C}{(h-k)^a}\varphi(k)^b$$

with constants $C > 0, a > 0,$ and $b > 1$. Then $\varphi(k_0 + \delta) = 0$, where

$$\delta^a = C\varphi(k_0)^{b-1}2^{\frac{ab}{b-1}}.$$

Proof. of Theorem 3.9
The idea of proofs to the general theorems stated in this section were obtained from Fredi Tröltzsch [8] for Neumann boundary conditions to deal with the present case.

(i) *Preliminaries*

To this end, we will test the solution y_c to (3.26) in the variational formulation with the part of y_c that is larger than $k > 0$ in absolute value, and then show that this part vanishes for sufficiently large k. Integrability property of u was postulated in the statement of the theorem. Here, we denote the order of integrability by $\tilde{r}$. We thus have $u \in L^{\tilde{r}}(\Omega)$, where $\tilde{r} > N/2$.
We first assume $N \geq 3$ and explain at the end of the proof which modifications have to made for the case of $N = 2$. We fix some $\lambda \in (1, \frac{N-1}{N-2})$ sufficiently close to unity such that

$$\tilde{r} > r := \frac{N}{N - \lambda(N-2)}.$$

Since $N \geq 3$, and owing to the choice of λ, we obviously have $r > 1$. If we succeed in proving the result for r, then it will be valid for all $\tilde{r} > r$. The conjugate exponent r' for r is given by

$$\frac{1}{r'} = 1 - \frac{1}{r} = \lambda\frac{N-2}{N}, \tag{3.39}$$

Below, we will use the embedding estimate

$$\|v\|_{L^p(\Omega)} \leq \beta \|v\|_{H^1(\Omega)} \quad \text{for} \quad \frac{1}{p} = \frac{1}{2} - \frac{1}{N} = \frac{N-2}{2N} = \frac{1}{2\lambda r'}, \tag{3.40}$$

Since $2r' \leq p$, this implies that

$$\|v\|_{L^{2r'}(\Omega)} \leq \beta \|v\|_{H^1(\Omega)} \tag{3.41}$$

Next, we define for each $k > 0$ a function $v_k \in H_0^1(\Omega)$, such that

$$v_k(x) = \begin{cases} y_c(x) - k, & \text{if } \; y_c(x) \geq k \\ 0, & \text{if } \; |y_c(x)| < k \\ y_c(x) + k, & \text{if } \; y_c(x) \leq -k. \end{cases} \tag{3.42}$$

We aim to show that v_k vanishes almost everywhere for sufficiently large k, which then implies the boundedness of y_c. For the sake of brevity, we suppress the subscript k, writing v_k simply as v. We introduce the set

$$\Omega(k) = \{x \in \Omega : |y_c(x)| \geq k\}$$

(ii) *Convergence of monotonicity*

We claim that

$$\int_\Omega \max\nolimits_c(c^s\bar{\lambda} + c(y_c - \psi))v dx \geq 0 \tag{3.43}$$

To see this, let

$$\Omega_+(k) := \{x \in \Omega : y_c(x) > k\}.$$

On the set $\Omega_+(k)$ one can immediately observe that

$$\int_{\Omega_+(k)} \max\nolimits_c(c^s\bar{\lambda} + c(y_c - \psi))v dx = \int_{\Omega_+(k)} \max\nolimits_c(c^s\bar{\lambda} + c(y_c - \psi))(y_c - k) dx \geq 0 \tag{3.44}$$

On the set $\Omega_-(k) := \{x \in \Omega : y_c(x) < -k\}$; since $\max_c(c^s\bar{\lambda} + c(y_c - \psi))$ is monotonic increasing with respect to y_c, we have

$$\begin{aligned} \max\nolimits_c(c^s\bar{\lambda} + c(y_c - \psi)) &\leq \max\nolimits_c(c^s\bar{\lambda} + c(-k - \psi)) \\ &\leq 0 \quad \text{ if k is large enough.} \end{aligned}$$

This implies that

$$\int_{\Omega_-(k)} \max\nolimits_c(c^s\bar{\lambda} + c(y_c - \psi))v dx = \int_{\Omega_-(k)} \max\nolimits_c(c^s\bar{\lambda} + c(y_c - \psi))(y_c - k) dx \geq 0 \tag{3.45}$$

Then (3.44) and (3.45) prove (3.43).
From the variational formulation for y_c, we infer that, with the bilinear form $a(y_c, v)$ defined in (3.1),

$$a(y_c, v) + \int_\Omega \max\nolimits_c(c^s\bar{\lambda} + c(y_c - \psi))v dx = \int_\Omega uv dx$$

hence from (3.43), we have

$$a(y_c, v) \leq \int_\Omega uvdx \tag{3.46}$$

(iii) <u>*Estimation of* $\|v\|_{H^1(\Omega)}$</u>

We claim that

$$a(v, v) \leq a(y_c, v) \tag{3.47}$$

Indeed, we obviously have $\frac{\partial y_c}{\partial x_i} = \frac{\partial v}{\partial x_i}$ in $\Omega(k)$ and $v = 0$ in $\Omega\backslash\Omega(k)$, hence it follows that

$$\int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial y_c}{\partial x_i} \frac{\partial v}{\partial x_j} dx = \int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} dx$$

$$\int_\Omega \sum_{i=1}^N b_i \frac{\partial y_c}{\partial x_i} vdx = \int_\Omega \sum_{i=1}^N b_i \frac{\partial v}{\partial x_i} vdx$$

Moreover, since $y_c - k > 0$ in $\Omega_+(k)$, $\;y_c + k < 0$ in $\Omega_-(k)$, and $v = 0$ in $\Omega\backslash\Omega(k)$,

$$\begin{aligned}
\int_\Omega c_0 y_c v dx &= \int_{\Omega_+(k)} c_0 y_c (y_c - k) dx + \int_{\Omega_-(k)} c_0 y_c (y_c + k) dx \\
&= \int_{\Omega_+(k)} c_0 [(y_c - k)^2 + (y_c - k)k] dx + \int_{\Omega_-(k)} c_0 [(y_c + k)^2 - (y_c + k)k] dx \\
&\geq \int_\Omega c_0 v^2 dx.
\end{aligned}$$

From (3.46) and (3.47) and the coercivity property of the elliptic boundary value problem, we conclude that, with some $\theta > 0$,

$$\theta \|v\|^2_{H^1(\Omega)} \leq \int_\Omega uvdx. \tag{3.48}$$

(iv) <u>*Estimation of both sides of* (3.48)</u>

Now recall the embedding inequality in (3.40) and Young's inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$ for all $a, b \in \mathbb{R}$ and $\varepsilon > 0$. With some generic constant $\beta > 0$, and using Hölder's inequality, we can estimate the right-hand side as follows:

$$\begin{aligned}
|\int_\Omega uvdx| &\leq \|u\|_{L^r(\Omega)} \|v\|_{L^{r'}(\Omega)} \\
&\leq \|u\|_{L^r(\Omega)} [(\int_{\Omega(k)} |v|^{2r'} dx)^{\frac{1}{2}} (\int_{\Omega(k)} 1 dx)^{\frac{1}{2}}]^{\frac{1}{r'}} \\
&= \|u\|_{L^r(\Omega)} \|v\|_{L^{2r'}(\Omega)} |\Omega(k)|^{\frac{1}{2r'}} \\
&\leq \beta \|u\|_{L^r(\Omega)} \|v\|_{H^1(\Omega)} |\Omega(k)|^{\frac{1}{2r'}} \\
&\leq \beta \|u\|^2_{L^r(\Omega)} |\Omega(k)|^{\frac{1}{r'}} + \varepsilon \|v\|^2_{H^1(\Omega)} \\
&= \beta \|u\|^2_{L^r(\Omega)} |\Omega(k)|^{\lambda \frac{2}{p}} + \varepsilon \|v\|^2_{H^1(\Omega)}
\end{aligned}$$

The number $\varepsilon > 0$ is yet to be determined. Now, by choosing $\varepsilon := \theta/2$, we may absorb into the left-hand side of (3.48) the term $\varepsilon\|v\|^2_{H^1(\Omega)}$ occurring in the last inequality.
Invoking (3.48) and the estimate given in (3.40), we find that

$$(\int_{\Omega(k)} |v|^p dx)^{\frac{2}{p}} \leq \beta\|v\|^2_{H^1(\Omega)},$$

hence, by the definition of v,

$$(\int_{\Omega(k)} (|y_c| - k)^p dx)^{\frac{2}{p}} \leq \beta\|v\|^2_{H^1(\Omega)}, \tag{3.49}$$

(v) *Application of Lemma* 3.10
Suppose that $h > k$. Then $\Omega(h) \subset \Omega(k)$, and thus $|\Omega(h)| \leq |\Omega(k)|$. Therefore,

$$\begin{aligned}(\int_{\Omega(k)} (|y_c| - k)^p dx)^{\frac{2}{p}} &\geq (\int_{\Omega(h)} (|y_c| - k)^p dx)^{\frac{2}{p}} \\ &\geq (\int_{\Omega(h)} (h - k)^p dx)^{\frac{2}{p}} \\ &= (h-k)^2|\Omega(h)|^{\frac{2}{p}}.\end{aligned}$$

Finally, we infer from (3.49) and (3.48) that

$$\begin{aligned}(h-k)^2|\Omega(h)|^{\frac{2}{p}} &\leq \beta\|u\|^2_{L^r(\Omega)}|\Omega(k)|^{\lambda\frac{2}{p}} \\ &= \beta\|u\|^2_{L^r(\Omega)}(|\Omega(k)|^{\frac{2}{p}})^{\lambda}\end{aligned}$$

Putting $\varphi(h) := |\Omega(h)|^{\frac{2}{p}}$, we therefore obtain the inequality

$$(h-k)^2\varphi(h) \leq \beta\|u\|^2_{L^r(\Omega)}\varphi(k)^{\lambda},$$

for all $h > k \geq 0$. We now apply Lemma 3.10 with the specifications

$$a = 2, \quad b = \lambda > 1, \quad k_0 = 0, \quad C = \beta\|u\|^2_{L^r(\Omega)}.$$

We obtain $\delta^2 = \tilde{\beta}\|u\|^2_{L^r(\Omega)}$, hence the assertion follows: in fact $\varphi(\delta) = 0$ means that $|y_c(x)| \leq \delta$ for almost every $x \in \Omega$.

(vi) *Modification for the case N=2*
Let $r > \frac{N}{2} = 1$ be the order of integrability of u assumed in the theorem. In the case of $N = 2$, the embedding inequality (3.40) is valid for all $p < \infty$. We therefore define p with $\lambda > 1$ by

$$\frac{1}{p} = \frac{1}{2\lambda r'}.$$

With this specification, all conclusions subsequent to (3.40) in the $N \geq 3$ case carry over to $N = 2$, resulting the validity of the assertion.

This concludes the proof. □

Remark 3.11. *: In particular, if we consider the case where $r = 2$, then we have $r = 2 > N/2$ which implies $N \leq 3$. In the above theorem, we have already proved for the case where $N = 2$ and $N = 3$. Therefore we conclude that if $u \in L^2(\Omega)$, then the weak solution $y_c \in H_0^1(\Omega)$ is actually essentially bounded(regular), i.e., $y_c \in L^\infty(\Omega)$.*

3.5 Feasibility of solutions of regularized problem for large c

For fixed $\bar{\lambda} \geq \inf \bar{\lambda} > 0$, $0 < s < 1/2$ and for large c sufficiently large the optimal states y_c of (P_c) are feasible, i.e., $y_c \leq \psi$.
We modify a proof given in Lemma 3.10 of [2] to deal with our case. Let us first prove an estimate on the norm of the violation of the constraint $y \leq \psi$. Let us define for solutions y_c of the regularized equation.

$$\phi_c := \max(0, y_c - \psi).$$

Then from the definition of the max we have $\phi_c \geq 0$ and $\phi_c \in H_0^1(\Omega)$. Furthermore, we have the following Lemma.

Lemma 3.12. *Let y_c be a solution of the regularized equation (3.13) for a given right-hand side $u \in L^2(\Omega)$. Then the violation of $y_c \leq \psi$ can be estimated by*

$$\|\phi_c\|_{H^1}^2 + c\|\phi_c\|_{L^2}^2 \leq Cc^{-1}\| \max(u - A\psi - c^s\bar{\lambda}, 0)\|_{L^2}^2$$

with a constant C independent of $c, \bar{\lambda}, u, y_c$.
Furthermore, if c is chosen large enough in the above estimate, then the solutions y_c are feasible,i.e., $y_c \leq \psi$.

Proof. Testing the regularized equation

$$Ay_c + \max_c(c^s\bar{\lambda} + c(y_c - \psi)) = u$$

with $\phi_c = \max(0, y_c - \psi) \geq 0$ and using the fact that $\max_c(x) \geq \max(0, x)$ we obtain

$$\langle Ay_c - A\psi, \phi_c\rangle + (\max(0, c^s\bar{\lambda} + c(y_c - \psi)), \phi_c) \quad \leq \quad \langle u - A\psi, \phi_c\rangle$$

Since from the definition $\max(0, x) \geq x$ and $(y_c - \psi, \phi_c) = \|\phi_c\|_{L^2}^2$, it follows

$$\begin{aligned}(\max(0, c^s\bar{\lambda} + c(y_c - \psi)), \phi_c) &\geq (c^s\bar{\lambda} + c(y_c - \psi), \phi_c) \\ &= (c^s\bar{\lambda}, \phi_c) + c\|\phi_c\|_{L^2}^2\end{aligned}$$

This implies

$$\langle Ay_c - A\psi, \phi_c\rangle + (c^s\bar{\lambda}, \phi_c) + c\|\phi_c\|_{L^2}^2 \leq (u - A\psi, \phi_c)$$

and the claim follows with

$$\begin{aligned}\langle A\phi_c, \phi_c\rangle + c\|\phi_c\|_{L^2}^2 &\le (u - A\psi - c^s\bar\lambda, \phi_c) \\ &\le (\max(u - A\psi - c^s\bar\lambda, 0), \phi_c) \\ &\le \|\max(u - A\psi - c^s\bar\lambda, 0)\|_{L^2}\|\phi_c\|_{L^2} \\ &= \left(\frac{1}{\sqrt{c}}\|\max(u - A\psi - c^s\bar\lambda, 0)\|_{L^2}\right)\cdot\left(\sqrt{c}\|\phi_c\|_{L^2}\right) \\ &\le \frac{1}{2c}\|\max(u - A\psi - c^s\bar\lambda, 0)\|_{L^2}^2 + \frac{c}{2}\|\phi_c\|_{L^2}^2\end{aligned} \tag{3.50}$$

by the assumption on the elliptic operator A. □

Proposition 3.13. *Let $\rho \ge 0$ and let $y_c \in H_0^1(\Omega)$ denote the solution to $Ay_c + \max_c(c^s\bar\lambda + c(y_c - \psi)) = u$ with $u \in B_\rho = \{u : \|u\|_{L_\infty} \le \rho\}$. If $c^s\bar\lambda \ge \max(0, \rho - A\psi)$ for any $c > 0$, then y_c is feasible,i.e., $y_c \le \psi$ for each $c > 0$.*

Proof. Let $u \in B_\rho$. Due to the assumption on $\bar\lambda$ we have since $\phi_c \ge 0$

$$(u - A\psi - c^s\bar\lambda, \phi_c) \le (\rho - A\psi - \max(0, \rho - A\psi), \phi_c) \le 0$$

Then the equation (3.50) implies $\phi_c = 0$ and hence $y_c \le \psi$. □

3.6 Convergence Analysis of regularized Equations

In this section we study on the convergence properties of the solutions of the semilinear elliptic regularized equation (3.13) as the regularization parameter $c \to \infty$:

$$Ay + \max_c(c^s\bar\lambda + c(y - \psi)) = u,$$

and define

$$\lambda_c = \max_c(c^s\bar\lambda + c(y - \psi)).$$

We will consider a sequence of solutions of the regularized problem converging weakly/strongly to solutions of the original problem.
In the following section, we require that $\bar\lambda \in L^\infty(\Omega)$ and $\bar\lambda \ge \inf\bar\lambda > 0$, which is fixed during the following process $c \to \infty$. The following convergence result is the modification of the Lemma in K. Ito and K. Kunisch [13] to our case.

Lemma 3.14. *Let $\bar\lambda \in L^\infty(\Omega)$ be given. For $u \in L^2(\Omega)$ let $(y_c, \lambda_c) \in H_0^1(\Omega) \times L^2(\Omega)$ be the solution of the regularized problem (3.13). Then the solution (y_c, λ_c) converges to the solution (y, λ) of (3.9) in the sense that $y_c \to y = y(u)$ strongly in $H_0^1(\Omega)$ and $\lambda_c \to \lambda$ strongly in $H^{-1}(\Omega)$ as $c \to \infty$.*

Proof. For every $c > 0$ from (3.9) and (3.13), $y_c \in H_0^1(\Omega)$ satisfy

$$\left.\begin{aligned} a(y_c, y_c - y) + (\lambda_c, y_c - y) = (u, y_c - y) \\ \lambda_c = \max_c(c^s\bar\lambda + c(y_c - \psi)) \end{aligned}\right\} \tag{3.51}$$

Since $\lambda_c \geq 0$ and $y \in K$, i.e., $\psi - y \geq 0$ we have

$$\begin{aligned}(\lambda_c, y_c - y) &= (\lambda_c, \frac{\bar{\lambda}}{c^{1-s}} + y_c - \psi + \psi - y - \frac{\bar{\lambda}}{c^{1-s}}) \\ &\geq \frac{1}{c}(\lambda_c, c^s\bar{\lambda} + c(y_c - \psi)) - \frac{1}{c^{1-s}}(\lambda_c, \bar{\lambda})\end{aligned}$$

Note that for $|x| \leq \frac{1}{2c}$ where $x := c^s\bar{\lambda} + c(y_c - \psi)$ and $\lambda_c := \max_c(x)$, we have

$$\begin{aligned}x &\geq \max_c(x) - \frac{1}{2c} \\ \Rightarrow \max_c(x).x &\geq \max_c(x)^2 - \frac{1}{2c}.\max_c(x) \\ &\geq \max_c(x)^2 - \frac{1}{4c^2} \qquad (\text{since } \max_c(x) \leq \frac{1}{2c})\end{aligned}$$

and hence

$$(\lambda_c, y_c - y) \geq \frac{1}{c}\|\lambda_c\|^2_{L^2} - \frac{1}{4c^3} - \frac{1}{c^{1-s}}(\lambda_c, \bar{\lambda}) \tag{3.52}$$

Using this inequality in equation (3.51) and employing Young's inequality we get

$$\begin{aligned}a(y_c, y_c) + \frac{1}{c}\|\lambda_c\|^2_{L^2} &\leq a(y_c, y) + (u, y_c - y) + \frac{1}{4c^3} + \frac{1}{c^{1-s}}(\lambda_c, \bar{\lambda}) \\ &\leq \frac{\nu}{4}\|y_c\|^2_{H^1} + \frac{1}{\nu}\|y\|^2_{H^1} - (u, y) + \frac{\nu}{4}\|y_c\|^2_{H^1} + \frac{1}{\nu}\|u\|^2_{L^2} \\ &+ \frac{1}{4c^3} + \frac{1}{2c}\|\lambda_c\|^2_{L^2} + \frac{1}{2c^{1-2s}}\|\bar{\lambda}\|^2_{L^2} \\ &\leq \frac{\nu}{2}\|y_c\|^2_{H^1} + \frac{1}{\nu}\|y\|^2_{H^1} + \|u\|_{L^2}\|y\|_{H^1} + \frac{1}{\nu}\|u\|^2_{L^2} + \frac{1}{4c^3} \\ &+ \frac{1}{2c}\|\lambda_c\|^2_{L^2} + \frac{1}{2c^{1-2s}}\|\bar{\lambda}\|^2_{L^2}\end{aligned}$$

The terms $\frac{\nu}{2}\|y_c\|^2_{H^1}$ and $\frac{1}{2c}\|\lambda_c\|^2_{L^2}$ are absorbed by the left hand side implies the right hand side is independent of c. Since a is coercive, this implies that

$$\nu\|y_c\|^2_{H^1} + \frac{1}{c}\|\lambda_c\|^2_{L^2}$$

is uniformly bounded with respect to $c \geq 1$ and hence by (3.13) the family $\{\lambda_c\}_{c\geq 1}$ is bounded in $H^{-1}(\Omega)$.
Consequently there exist $(y^*, \lambda^*) \in H^1_0(\Omega) \times H^{-1}(\Omega)$ and a sequence $\{(y_{c_n}, \lambda_{c_n})\}$ with $\lim c_n = \infty$ such that

$$(y_{c_n}, \lambda_{c_n}) \rightharpoonup (y^*, \lambda^*) \text{ in } H^1_0(\Omega) \times H^{-1}(\Omega).$$

Henceforth we drop the subscript n with c_n. Implies that y_c converges to y^* a.e. in Ω and therefore $y^* \leq \psi$ since y_c is feasible, i.e., $y_c \leq \psi$.
From (3.9) and (3.13) we also have

$$a(y_c - y, y_c - y) + \langle \lambda_c - \lambda, y_c - y \rangle = 0,$$

By the Young's inequality we have

$$\begin{aligned}\frac{1}{c^{1-s}}(\lambda_c,\bar{\lambda}) &= \left(\frac{1}{\sqrt{c}}\lambda_c, \frac{1}{c^{1/2-s}}\bar{\lambda}\right)\\ &\le \frac{1}{c}\|\lambda_c\|_{L^2}^2 + \frac{1}{4c^{1-2s}}\|\bar{\lambda}\|_{L^2}^2\end{aligned}$$

and by (3.52)

$$(\lambda_c, y_c - y) \ge \frac{1}{c}\|\lambda_c\|_{L^2}^2 - \frac{1}{4c^3} - \frac{1}{c}\|\lambda_c\|_{L^2}^2 - \frac{1}{4c^{1-2s}}\|\bar{\lambda}\|_{L^2}^2$$

implies

$$(\lambda_c, y_c - y) \ge -\frac{1}{4c^3} - \frac{1}{4c^{1-2s}}\|\bar{\lambda}\|_{L^2}^2.$$

Hence

$$0 \le \overline{\lim_{c\to\infty}}\,\nu\|y_c - y\|_{H_0^1}^2 \le \lim_{c\to\infty}\langle\lambda, y_c - y\rangle = \langle\lambda, y^* - \psi\rangle \le 0,$$

where we used the complementarity condition $\langle\lambda, y - \psi\rangle = 0$ and $y^* \le \psi$. It follows that $\lim_{c\to\infty} y_c = y$ in $H_0^1(\Omega)$ and hence $y^* = y$. Taking the limit in

$$a(y_c, v) + (\lambda_c, v) = (u, v) \quad \forall v \in H_0^1(\Omega),$$

we find

$$a(y, v) + \langle\lambda^*, v\rangle = (u, v) \quad \forall v \in H_0^1(\Omega),$$

This equation is also satisfied with λ^* replaced by λ and consequently $\lambda = \lambda^*$. Since (y, λ) is the unique solution to (3.10) the whole family $\{(y_c, \lambda_c)\}$ converges in the sense given in the statement of the theorem. □

3.7 Convergence Rate of regularized Equations

In this section we derive the rate of convergence of the family $\{y_c\}_{c>0}$ to y in $L^\infty(\Omega)$. The idea of proofs to the general theorems stated in this section were obtained from [2].

Proposition 3.15. *Let $\bar{\lambda} \in L^\infty(\Omega)$ be given. Let y and y_c denote the corresponding solutions of the variational inequality (3.8) and the regularized equation (3.13), respectively. Then*

$$\|y_c - y\|_{L^\infty(\Omega)} \le \frac{\|\bar{\lambda}\|_{L^\infty}}{c^{1-s}} + \frac{1}{2c^2}.$$

for c sufficiently large.

Proof. Let the test function be defined as

$$\phi_k := \begin{cases} y - y_c - k, & \text{if } \; y - y_c > k \\ y - y_c + k, & \text{if } \; y - y_c < -k \\ 0, & \text{else} \end{cases}$$

where $k \in \mathbb{R}^+$. Subtracting (3.13) from the first equation in (3.9) and testing with ϕ_k gives

$$a(y - y_c, \phi_k) + \langle \lambda - \max_c(c^s\bar{\lambda} + c(y_c - \psi)), \phi_k \rangle = 0 \tag{3.53}$$

Since $\max(\phi_k, 0)$ and $\min(\phi_k, 0)$ belong to $H_0^1(\Omega)$, we can split $\phi_k = \max(\phi_k, 0) + \min(\phi_k, 0)$. Since $\lambda \geq 0$ we get $\langle \lambda, \max(\phi_k, 0) \rangle \geq 0$. By feasibility of y_c we find

$$\begin{aligned} 0 \geq \min(\phi_k, 0) &= \min(y - y_c + k, 0) \geq \min(y - \psi + k, 0) \\ &\geq \min(y - \psi, 0) = y - \psi. \end{aligned}$$

By the complementarity condition this implies

$$0 \geq \langle \lambda, \min(\phi_k, 0) \rangle \geq \langle \lambda, y - \psi \rangle = 0.$$

On the set $\{\phi_k > 0\}$ since $y - y_c > k \Rightarrow \psi - y_c > k$, we get

$$\begin{aligned} \max_c(c^s\bar{\lambda} + c(y_c - \psi)) &\leq \max_c(c^s\bar{\lambda} + c(y_c - y)) \\ &\leq \max_c(c^s\bar{\lambda} - ck) \leq \max_c(-\frac{1}{2c}) = 0, \quad \text{for} \quad k \geq \frac{\|\bar{\lambda}\|_{L^\infty}}{c^{1-s}} + \frac{1}{2c^2}, \end{aligned}$$

and hence

$$(\max_c(c^s\bar{\lambda} + c(y_c - \psi)), \max(\phi_k, 0)) = 0.$$

Since from the definition $\max_c(c^s\bar{\lambda} + c(y_c - \psi)) \geq 0$, we have

$$-(\max_c(c^s\bar{\lambda} + c(y_c - \psi)), \min(\phi_k, 0)) \geq 0.$$

Thus from (3.53), we get $a(y - y_c, \phi_k) \leq 0$ for $k \geq \frac{\|\bar{\lambda}\|_{L^\infty}}{c^{1-s}} + \frac{1}{2c^2}$.
Due to properties of the bilinear form a and the definition of ϕ_k, we have $a(y - y_c, \phi_k) \geq a(\phi_k, \phi_k)$, which implies $\phi_k = 0$. □

Up to now, we studied convergence of solutions for fixed right-hand side u in (3.13). Let us now turn to the case, where the right-hand side is a (possibly weakly) convergent sequence. Due to the monotonicity of the $\max_c$-function we obtain the following Lipschitz continuity result for the solutions of the regularized equation.

Lemma 3.16. *Let $\bar{\lambda} \in L^\infty(\Omega)$ be given. Let $u_m, u_n \in H^{-1}(\Omega)$ be given. Then there exists a constant $L > 0$ independent of c such that*

$$\|y_c(u_m) - y_c(u_n)\|_{H_0^1} \leq L\|u_m - u_n\|_{H^{-1}}.$$

Proof. The proof is similar to the one of Lemma 3.1. The main idea again is to take

$$a(y_c, v) + (\max_c(c^s\bar{\lambda} + c(y_c - \psi)), v) = (u, v)$$

with $u = u_n$ and $u = u_m$. Then we insert $v = y_c(u_m) - y_c(u_n)$ and $v = y_c(u_n) - y_c(u_m)$ in the arising equalities. The summation of both equations yields

$$\begin{aligned} 0 &= a(y_c(u_n), y_c(u_m) - y_c(u_n)) - a(y_c(u_m), y_c(u_m) - y_c(u_n)) \\ &+ (\max_c(c^s\bar{\lambda} + c(y_c(u_n) - \psi)) - \max_c(c^s\bar{\lambda} + c(y_c(u_m) - \psi)), y_c(u_m) - y_c(u_n)) \\ &+ \langle u_m - u_n, y_c(u_m) - y_c(u_n) \rangle \end{aligned}$$

$$\begin{aligned}\Leftrightarrow 0 &= a(y_c(u_m), y_c(u_m) - y_c(u_n)) - a(y_c(u_n), y_c(u_m) - y_c(u_n)) \\ &+ (\max_c(c^s\bar{\lambda} + c(y_c(u_m) - \psi)) - \max_c(c^s\bar{\lambda} + c(y_c(u_n) - \psi)), y_c(u_m) - y_c(u_n)) \\ &- \langle u_m - u_n, y_c(u_m) - y_c(u_n)\rangle\end{aligned} \tag{3.54}$$

Because of the monotonicity of the max function it is obvious that

$$(\max_c(c^s\bar{\lambda} + c(y_c(u_m) - \psi)) - \max_c(c^s\bar{\lambda} + c(y_c(u_n) - \psi)), y_c(u_m) - y_c(u_n)) \geq 0$$

According to the above equality (3.54), and with the coercivity of the bilinear form, we have

$$\begin{aligned}0 &\geq a(y_c(u_m), y_c(u_m) - y_c(u_n)) - a(y_c(u_n), y_c(u_m) - y_c(u_n)) - \langle u_m - u_n, y_c(u_m) - y_c(u_n)\rangle \\ &\geq a(y_c(u_m) - y_c(u_n), y_c(u_m) - y_c(u_n)) - \|u_m - u_n\|_{H^{-1}(\Omega)}\|y_c(u_m) - y_c(u_n)\|_{H_0^1(\Omega)} \\ &\geq C_c\|y_c(u_m) - y_c(u_n)\|^2_{H_0^1(\Omega)} - \|u_m - u_n\|_{H^{-1}(\Omega)}\|y_c(u_m) - y_c(u_n)\|_{H_0^1(\Omega)}\end{aligned}$$

$$\Rightarrow \quad \|y_c(u_m) - y_c(u_n)\|_{H_0^1(\Omega)} \leq \frac{1}{C_c}\|u_m - u_n\|_{H^{-1}(\Omega)} \tag{3.55}$$

which establishes the case with $L := \frac{1}{C_c}$. □

Lemma 3.17. *Let $\bar{\lambda} \in L^\infty(\Omega)$ be given. Let $u_c \rightharpoonup u$ in $L^2(\Omega)$. Then $y_c \to y$ in $H_0^1(\Omega)$ strongly.*

Proof. It holds

$$\|y_c(u_c) - y(u)\|_{H^1} \leq \|y_c(u_c) - y_c(u)\|_{H^1} + \|y_c(u) - y(u)\|_{H^1}.$$

The first addend can be majorized by $L\|u_c - u\|_{H^{-1}}$ due to Lemma 3.16. By compact embeddings this term tends to zero for $c \to \infty$. The second addend tends to zero according to Lemma 3.14. □

Chapter 4

Optimal control with bilateral constraints

In this chapter we investigate the analytical treatment of optimal control problems governed by a class of elliptic variational inequalities of the first kind with bilateral constraints. Moreover, we consider constraints on the control.

Approximation of the bilateral obstacle problem to a certain semilinear elliptic equation is shown. The treatment of bilateral constraints gives rise to some additional difficulties. Existence results are given and an optimality system is derived.

4.1 Formulation of the optimal control problem in the variational form

In this section we formulate the optimal control problem governed by bilateral state constraints. We consider approximation of the variational inequality by an equation where the maximal monotone operator (which is in this case the subdifferential of a Lipschitz function) is approached by a differentiable single-value mapping, with Moreau-Yosida approximations techniques.

Let Ω be an open, bounded subset of $\mathbb{R}^N$ $(N \leq 3)$ with a smooth boundary $\Gamma = \partial\Omega$. The optimal control problems subject to bilateral obstacle problem to be studied in this chapter can be set in the following general form:

$$(P') \quad \begin{cases} \min J(y,u) = g(y) + j(u) \\ over \ \ y \in K', \ \ u \in U_{ad} \\ s.t \ \ a(y, v-y) \geq (u, v-y) \ \ \forall v \in K' \end{cases}$$

where ψ_a, $\psi_b \in H^1(\Omega)$, $\psi_a \leq \psi_b$ are given functions.

$$K' = \{v \in H_0^1(\Omega) : \psi_a \leq v \leq \psi_b \ \ \text{a.e. on} \ \ \Omega\}$$

and

$$U_{ad} = \{u \in L^2(\Omega) : u_a \leq u \leq u_b\}$$

are nonempty, closed, and convex subset of $H_0^1(\Omega)$ and $L^2(\Omega)$ respectively.
All the properties of the bilinear form $a(\cdot,\cdot)$ and its Riesz representative A are the same as we defined in chapter 3.
Consider the following variational inequality in (P'):

$$a(y, v-y) \geq (u, v-y), \quad \forall v \in K', \tag{4.1}$$

where u belongs to U_{ad} as a source term. Introducing multipliers λ^b and λ^a, the variational inequality in (P') can equivalently be expressed as complementary condition:

$$\begin{cases} Ay + \lambda^b - \lambda^a = u, \\ \psi_a \leq y \leq \psi_b, \\ \lambda^b \geq 0 \ \text{ and } \ \lambda^a \geq 0, \\ (\lambda^b, y-\psi_b) = (\lambda^a, y-\psi_a) = 0 \end{cases} \tag{4.2}$$

where $\lambda^b, \lambda^a \in H^{-1}(\Omega)$ are the associated Lagrange multipliers to the solution (P') and $\psi_a, \psi_b \in H^1(\Omega)$ and $\psi_a|_\Gamma \leq 0 \leq \psi_b|_\Gamma$.
In this way the optimal control of variational inequality (P') is interpreted as optimization with complementarity constraints. If λ^b, λ^a has extra regularity in sense that $\lambda^b, \lambda^a \in L^2(\Omega)$, the optimality system (4.2) can equivalently be expressed as

$$\begin{cases} Ay + \lambda^b - \lambda^a = u \ \ in \ \ L^2(\Omega) \\ \\ \lambda^b - \lambda^a = \max(0, \lambda + c(y-\psi_b)) - \max(0, \lambda + c(\psi_a - y)) \end{cases} \tag{4.3}$$

for any $c > 0$ and where max denotes the pointwise a.e. maximum operation. The optimal control problem (P') therefore can be equivalently expressed as minimizing $J(y,u)$ subject to (4.3). This is the reason that the constraint (4.1) makes the optimal control problem (P') non-smooth.
Since $x \rightarrow \max(0,x)$ is not C^1 regular(not Gâteaux differentiable), to regularize the $\max -$function in (4.3) we use the well known smoothing (C^1-approximation), $\max_c$. Then the complementarity system (4.3) will be approximated by means of the regularized state equation

$$Ay + \max_c(c^s\bar{\lambda} + c(y-\psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y)) = u, \tag{4.4}$$

where $0 < s < 1/2$ and $\inf \bar{\lambda} > 0$, the $\max -$operation was replaced by a generalized Moreau-Yosida type regularization.
As a consequence the optimal control problem subject to regularized equation is given by

$$(P_c') \begin{cases} \min J(y,u) = g(y) + j(u) \\ over \ \ u \in U_{ad}, \ \ \text{subject to} \\ Ay + \max_c(c^s\bar{\lambda} + c(y-\psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y)) = u, \ \ y \in H_0^1(\Omega) \end{cases} \tag{4.5}$$

where $\inf \bar{\lambda} > 0$, $\bar{\lambda} \in L^\infty(\Omega)$, is fixed during the regularization process given, and $\max_c$ is a C^2-approximation of $x \rightarrow \max(0,x)$. If g and j are C^1-regular, then the first order optimality system for (P'_c) is given by

$$\begin{cases} Ay_c + \max_c(c^s\bar{\lambda} + c(y_c - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y_c)) = u_c, \\ A^*p_c + c(\operatorname{sgn}_c(c^s\bar{\lambda} + c(y_c - \psi_b)) + \operatorname{sgn}_c(c^s\bar{\lambda} + c(\psi_a - y_c)))p_c + g'(y_c) = 0, \\ (j'(u_c) - p_c, u - u_c) \geq 0, \quad \forall u \in U_{ad} \end{cases} \tag{4.6}$$

where expressions

$$\lambda_c^b - \lambda_c^a = \max_c(c^s\bar{\lambda} + c(y_c - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y_c))$$

and

$$\mu_c^b + \mu_c^a = c(\operatorname{sgn}_c(c^s\bar{\lambda} + c(y_c - \psi_b)) + \operatorname{sgn}_c(c^s\bar{\lambda} + c(\psi_a - y_c)))p_c$$

in (4.6) tend to measure Lagrange multipliers as $c \rightarrow \infty$.

4.2 A semilinear elliptic regularized problem

The elliptic equation occurring in problem (4.4) is semilinear. In this and the next two sections we will discuss existence, regularity and feasibility of the solution of the regularized problem (4.4) with homogenous Dirichlet elliptic boundary value problem.

$$\left.\begin{aligned} Ay + \max_c(c^s\bar{\lambda} + c(y - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y)) = u \;\text{ in }\; \Omega \\ y = 0 \;\text{ on }\; \Gamma \end{aligned}\right\} \tag{4.7}$$

The elliptic differential operator A is assumed to take the form (3.6), and the function u will play the role of the controls.

To prove the existence and uniqueness of solutions of the regularized problem (4.7), we apply Main Theorem on Monotone Operators 3.6. We follow the same procedure as in the proof of Theorem 3.8. To do this, we first have to define the notation of a weak solution to the nonlinear elliptic boundary value problem (4.7). The idea is simple: we bring the nonlinear term $\max_c(c^s\bar{\lambda} + c(y - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y))$ in (4.7) to the right-hand side of the equation, thus obtaining a boundary value problem with the right-hand side $\tilde{u} = u - \max_c(c^s\bar{\lambda} + c(y - \psi_b)) + \max_c(c^s\bar{\lambda} + c(\psi_a - y))$ and linear differential operators on the left-hand side. For this purpose , we use the variational formulation for linear boundary value problem.

Definition 4.1. *A function $y \in H_0^1(\Omega)$ is called a weak solution to problem (4.7) if we have, for every $v \in H_0^1(\Omega)$,*

$$a(y,v) + \int_\Omega (max_c(c^s\bar{\lambda} + c(y - \psi_b)) - max_c(c^s\bar{\lambda} + c(\psi_a - y)))vdx = \int_\Omega uvdx. \tag{4.8}$$

Note that $\max_c(c^s\bar{\lambda} + c(y - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y))$ is bounded as,

$$|\max_c(c^s\bar{\lambda}+c(y-\psi_b))-\max_c(c^s\bar{\lambda}+c(\psi_a-y))| \leq |c^s\bar{\lambda}+c(y-\psi_b)|+|c^s\bar{\lambda}+c(\psi_a-y)|+\frac{1}{2c},$$

and monotone increasing almost everywhere in Ω.

Theorem 4.2. *Suppose that Assumption 2 hold. Then the for each $c > 0$ and for every right-hand side $u \in L^2(\Omega)$ there exists a unique solution $y_c \in H_0^1(\Omega)$ of the semilinear elliptic equation (4.7). Moreover, there is some constant $\gamma > 0$ such that*

$$\|y_c\|_{H^1(\Omega)} \leq \gamma \|u\|_{L^2(\Omega)}. \tag{4.9}$$

Proof. We apply the main theorem on monotone operators in $V = H_0^1(\Omega)$.

(i) *Definition of a monotone operator $B : H_0^1 \longrightarrow H^{-1}$*
It follows from section (3.1) that the bilinear form (3.1) generates a continuous linear operator $A : H_0^1 \longrightarrow H^{-1}$ through the relation

$$\langle Ay_c, v\rangle = a(y_c, v).$$

This is the linear part of nonlinear operator B. The nonlinear part of B is formally defined by the identity $(A_2 y)(x) := \max_c(c^s\bar{\lambda}+c(y_c(x)-\psi_b))-\max_c(c^s\bar{\lambda}+c(\psi_a - y_c(x)))$. The sum of the two operators yield the the operator B, i.e., $B = A + A_2$.

(ii) *Monotonicity*
We show that the operators A and A_2 are monotone so that this property then also holds for B. First, A is monotone, since $a(y_c, y_c) \geq 0$ for all $y_c \in H_0^1(\Omega)$. Next, we consider A_2. Owing the monotonicity of $\max_c$ in y_c, we have

$$[(\max_c(c^s\bar{\lambda} + c(y_{c,1}(x) - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y_{c,1}(x))))$$
$$-(\max_c(c^s\bar{\lambda}+c(y_{c,2}(x)-\psi_b))-\max_c(c^s\bar{\lambda}+c(\psi_a-y_{c,2}(x))))](y_{c,1}(x)-y_{c,2}(x)) \geq 0$$

for all $y_{c,1}, y_{c,2} \in H_0^1(\Omega)$ and all x. Therefore, for all $y_{c,1}, y_{c,2} \in H_0^1(\Omega)$

$$\langle A_2(y_{c,1}) - A_2(y_{c,2}), y_{c,1} - y_{c,2}\rangle$$
$$= \int_\Omega [(\max_c(c^s\bar{\lambda} + c(y_{c,1}(x) - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y_{c,1}(x))))$$
$$-(\max_c(c^s\bar{\lambda}+c(y_{c,2}(x)-\psi_b))-\max_c(c^s\bar{\lambda}+c(\psi_a-y_{c,2}(x))))](y_{c,1}(x)-y_{c,2}(x))dx \geq 0$$

Note that the boundedness condition for $\max_c$ guarantees that the function

$$\begin{aligned} x \mapsto [(\max_c(c^s\bar{\lambda} + c(y_{c,1}(x) - \psi_b)) &- \max_c(c^s\bar{\lambda} + c(\psi_a - y_{c,1}(x)))) \\ -(\max_c(c^s\bar{\lambda} + c(y_{c,2}(x) - \psi_b)) &- \max_c(c^s\bar{\lambda} + c(\psi_a - y_{c,2}(x))))] \end{aligned}$$

is square integrable, so that the above integral exists. In conclusion, A_2 is monotone.

(iii) *Coercivity of B*

A is coercive follows from (3.4) and (3.5),i.e.,

$$\langle Ay_c, y_c\rangle = a(y_c, y_c) \geq C_c\|y_c\|^2_{H_0^1} \quad \forall y_c \in H_0^1(\Omega). \tag{4.10}$$

For all $y_c \in H_0^1(\Omega)$

$$\begin{aligned}
\langle A_2 y_c, y_c\rangle &= \langle \max_c(c^s\bar{\lambda} + c(y_c - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y_c)), y_c\rangle \\
&= \int_\Omega [\max_c(c^s\bar{\lambda} + c(y_c - \psi_b)) - \max_c(c^s\bar{\lambda})](y_c - \psi_b) \\
&+ \int_\Omega [-\max_c(c^s\bar{\lambda} + c(\psi_a - y_c)) - \max_c(c^s\bar{\lambda})](y_c - \psi_a) \\
&+ \int_\Omega \max_c(c^s\bar{\lambda} + c(y_c - \psi_b))\psi_b + \int_\Omega \max_c(c^s\bar{\lambda})(y_c - \psi_b) \\
&- \int_\Omega \max_c(c^s\bar{\lambda} + c(\psi_a - y_c))\psi_a + \int_\Omega \max_c(c^s\bar{\lambda})(y_c - \psi_a)
\end{aligned}$$

By the monotonicity of $\max_c$,

$$\int_\Omega [\max_c(c^s\bar{\lambda} + c(y_c - \psi_b)) - \max_c(c^s\bar{\lambda})](y_c - \psi_b) \geq 0 \tag{4.11}$$

$$\int_\Omega [-\max_c(c^s\bar{\lambda} + c(\psi_a - y_c)) - \max_c(c^s\bar{\lambda})](y_c - \psi_a) \geq 0 \tag{4.12}$$

and the estimation of the last four integrals above (following the same procedure as we proved Theorem 3.8), applying these results we obtain

$$\frac{\langle A_2 y_c, y_c\rangle}{\|y_c\|_{H_0^1}} \to +\infty \quad \text{as} \quad \|y_c\|_{H_0^1} \to +\infty \quad \text{for any} \quad y_c \in K'.$$

This proves the claim that $B = A + A_2$ is coercive.

(iv) *Hemicontinuity of B*

We show that the operators A and A_2 are continuous so that this property then also holds for B and follows the claim. The operator A is linear and continuous and thus hemicontinuous. Referring section 4.3.3 in [8], one conclude that our Nemytskii operator $A_2 : L^2 \to L^2$ is continuous. Applying the embedding property $H_0^1 \hookrightarrow L^2 \hookrightarrow H^{-1}$ we get

$$\begin{aligned}
y_{c,n} \to y_c \text{ in } H_0^1(\Omega) &\Rightarrow y_{c,n} \to y_c \text{ in } L^2(\Omega) \\
&\Rightarrow A_2(y_{c,n}) \to A_2(y_c) \text{ in } L^2(\Omega) \\
&\Rightarrow A_2(y_{c,n}) \to A_2(y_c) \text{ in } H^{-1}(\Omega)
\end{aligned}$$

This proves that $A_2 : H_0^1 \to H^{-1}$ is continuous.

(v) Well-posedness of the solution

Existence and uniqueness of a weak solution $y_c \in H_0^1(\Omega)$ now follow directly from the main theorem on monotone operators. Since B is obviously strongly monotone, the asserted estimate also holds. Now to prove the estimate we take y_c itself as the test function to obtain

$$a(y_c, y_c) + \langle \max_c(c^s\bar{\lambda} + c(y_c - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y_c)), y_c\rangle = \int_\Omega u y_c dx.$$

$$\begin{aligned}
a(y_c, y_c) &= \int_\Omega u y_c dx - \int_\Omega (\max_c(c^s\bar{\lambda} + c(y_c - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y_c)))y_c dx \\
&\leq \int_\Omega |u y_c| dx + \int_\Omega (|c^s\bar{\lambda} + c(y_c - \psi_b)| + |c^s\bar{\lambda} + c(\psi_a - y_c)|)|y_c| dx \\
\nu\|y_c\|_{H^1}^2 &\leq \|u\|_{L^2}\|y_c\|_{L^2} + [c^s\|\bar{\lambda}\|_{L^\infty} + c(\|y_c\|_{L^2} + \|\psi_b\|_{L^2}) + c^s\|\bar{\lambda}\|_{L^\infty} \\
&+ c(\|\psi_a\|_{L^2} + \|y_c\|_{L^2})]\|y_c\|_{L^2} \\
&\leq \|u\|_{L^2}\|y_c\|_{H^1} + [c^s\|\bar{\lambda}\|_{L^\infty} + c(\|y_c\|_{H^1} + \|\psi_b\|_{L^2}) + c^s\|\bar{\lambda}\|_{L^\infty} \\
&+ c(\|\psi_a\|_{L^2} + \|y_c\|_{H^1})]\|y_c\|_{H^1} \\
\nu\|y_c\|_{H^1} &\leq \|u\|_{L^2} + c^s\|\bar{\lambda}\|_{L^\infty} + c(\|y_c\|_{H^1} + \|\psi_b\|_{L^2}) + c^s\|\bar{\lambda}\|_{L^\infty} + c(\|\psi_a\|_{L^2} + \|y_c\|_{H^1})
\end{aligned}$$

then the asserted estimate follows. This concludes the proof of the theorem. □

4.3 Elliptic regularity and continuity of solutions of regularized problems

In this section, we will prove the results concerning essential boundedness of the solution to the semilinear elliptic boundary value problem (4.7) in section 4.2.

Theorem 4.3. *Suppose that Assumption 2 hold, and let $r > N/2$. Then for any pair $u \in L^r(\Omega)$, we have $y_c \in L^\infty(\Omega)$.*

Proof. We follow the same technique as the proof of Theorem 3.9 for the unilateral case.

(i) Preliminaries

To this end, we will test the solution y_c to (4.7) in the variational formulation with the part of y_c that is larger than $k > 0$ in absolute value, and then show that this part vanishes for sufficiently large k. Integrability property of u was postulated in the statement of the theorem. Here, we denote the order of integrability by $\tilde{r}$. We thus have $u \in L^{\tilde{r}}(\Omega)$, where $\tilde{r} > N/2$.

We first assume $N \geq 3$ and explain at the end of the proof which modifications have to made for the case of $N = 2$. We fix some $\lambda \in (1, \frac{N-1}{N-2})$ sufficiently close to unity such that

$$\tilde{r} > r := \frac{N}{N - \lambda(N-2)}.$$

Since $N \geq 3$, and owing to the choice of λ, we obviously have $r > 1$. If we succeed in proving the result for r, then it will be valid for all $\tilde{r} > r$. The conjugate exponent r' for r is given by

$$\frac{1}{r'} = 1 - \frac{1}{r} = \lambda \frac{N-2}{N}, \tag{4.13}$$

Below, we will use the embedding estimate

$$\|v\|_{L^p(\Omega)} \leq \beta \|v\|_{H^1(\Omega)} \quad \text{for} \quad \frac{1}{p} = \frac{1}{2} - \frac{1}{N} = \frac{N-2}{2N} = \frac{1}{2\lambda r'}, \tag{4.14}$$

Since $2r' \leq p$, this implies that

$$\|v\|_{L^{2r'}(\Omega)} \leq \beta \|v\|_{H^1(\Omega)} \tag{4.15}$$

Next, we define for each $k > 0$ a function $v_k \in H_0^1(\Omega)$, such that

$$v_k(x) = \begin{cases} y_c(x) - k, & \text{if } \; y_c(x) \geq k \\ 0, & \text{if } \; |y_c(x)| < k \\ y_c(x) + k, & \text{if } \; y_c(x) \leq -k. \end{cases} \tag{4.16}$$

We aim to show that v_k vanishes almost everywhere for sufficiently large k, which then implies the boundedness of y_c. For the sake of brevity, we suppress the subscript k, writing v_k simply as v. We introduce the set

$$\Omega(k) = \{x \in \Omega : |y_c(x)| \geq k\}$$

(ii) _Convergence of monotonicity_

We claim that

$$\int_\Omega (\max_c(c^s\bar{\lambda} + c(y_c - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y_c)))v dx \geq 0 \tag{4.17}$$

On the set $\Omega_+(k) := \{x \in \Omega : y_c(x) > k\}$;

$$\int_{\Omega_+(k)} \max_c(c^s\bar{\lambda} + c(y_c - \psi_b))v dx = \int_{\Omega_+(k)} \max_c(c^s\bar{\lambda} + c(y_c - \psi_b))(y_c - k) dx \geq 0. \tag{4.18}$$

Since $-\max_c(c^s\bar{\lambda} + c(\psi_a - y_c))$ is monotonic increasing with respect to y_c, on the set $\Omega_+(k)$ we have

$$\begin{aligned} -\max_c(c^s\bar{\lambda} + c(\psi_a - y_c)) &\geq -\max_c(c^s\bar{\lambda} + c(\psi_a - k)) \\ &= 0 \qquad \text{if k is large enough.} \end{aligned}$$

implies

$$\int_{\Omega_+(k)} -\max_c(c^s\bar{\lambda} + c(\psi_a - y_c))vdx \geq 0 \tag{4.19}$$

From (4.18) and (4.19) it follows that

$$\int_{\Omega_+(k)} (\max_c(c^s\bar{\lambda} + c(y_c - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y_c)))vdx$$

$$= \int_{\Omega_+(k)} (\max_c(c^s\bar{\lambda} + c(y_c - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y_c)))(y_c - k)dx \geq 0$$

On the set $\Omega_-(k) := \{x \in \Omega : y_c(x) < -k\}$;

$$\int_{\Omega_-(k)} -\max_c(c^s\bar{\lambda}+c(\psi_a-y_c))vdx = \int_{\Omega_-(k)} -\max_c(c^s\bar{\lambda}+c(\psi_a-y_c))(y_c+k)dx \geq 0. \tag{4.20}$$

Since $\max_c(c^s\bar{\lambda} + c(y_c - \psi_b))$ is monotonic increasing with respect to y_c, on the set $\Omega_-(k)$ we have

$$\begin{aligned} \max_c(c^s\bar{\lambda} + c(y_c - \psi_b)) &\leq \max_c(c^s\bar{\lambda} + c(-k - \psi_b)) \\ &\leq 0 \quad \text{if k is large enough.} \end{aligned}$$

implies

$$\int_{\Omega_-(k)} \max_c(c^s\bar{\lambda} + c(y_c - \psi_b))vdx \geq 0 \tag{4.21}$$

From (4.20) and (4.21) it follows that

$$\int_{\Omega_-(k)} (\max_c(c^s\bar{\lambda} + c(y_c - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y_c)))vdx$$

$$= \int_{\Omega_-(k)} (\max_c(c^s\bar{\lambda} + c(y_c - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y_c)))(y_c + k)dx \geq 0$$

This proves (4.17).
From the variational formulation for y_c, we infer that, with the bilinear form $a(y_c, v)$ defined in (3.1),

$$a(y_c, v) + \int_\Omega (\max_c(c^s\bar{\lambda} + c(y_c - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y_c)))vdx = \int_\Omega uvdx$$

hence from (4.17), we have

$$a(y_c, v) \leq \int_\Omega uvdx \tag{4.22}$$

(iii) <u>*Estimation of* $\|v\|_{H^1(\Omega)}$</u>

We claim that

$$a(v,v) \leq a(y_c, v) \tag{4.23}$$

Indeed, we obviously have $\frac{\partial y_c}{\partial x_i} = \frac{\partial v}{\partial x_i}$ in $\Omega(k)$ and $v = 0$ in $\Omega \backslash \Omega(k)$, hence it follows that

$$\int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial y_c}{\partial x_i} \frac{\partial v}{\partial x_j} dx = \int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} dx$$

$$\int_\Omega \sum_{i,j=1}^N b_i \frac{\partial y_c}{\partial x_i} v dx = \int_\Omega \sum_{i,j=1}^N b_i \frac{\partial v}{\partial x_i} v dx$$

Moreover, since $y_c - k > 0$ in $\Omega_+(k)$, $y_c + k < 0$ in $\Omega_-(k)$, and $v = 0$ in $\Omega \backslash \Omega(k)$,

$$\begin{aligned}
\int_\Omega c_0 y_c v dx &= \int_{\Omega_+(k)} c_0 y_c (y_c - k) dx + \int_{\Omega_-(k)} c_0 y_c (y_c + k) dx \\
&= \int_{\Omega_+(k)} c_0[(y_c - k)^2 + (y_c - k)k] dx + \int_{\Omega_-(k)} c_0[(y_c + k)^2 - (y_c + k)k] dx \\
&\geq \int_\Omega c_0 v^2 dx.
\end{aligned}$$

From (4.22) and (4.23) and the coercivity property of the elliptic boundary value problem, we conclude that, with some $\theta > 0$,

$$\theta \|v\|^2_{H^1(\Omega)} \leq \int_\Omega uv dx. \tag{4.24}$$

(iv) <u>*Estimation of both sides of* (4.24)</u>

Now recall the embedding inequality in (4.14) and Young's inequality $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$ for all $a, b \in \mathbb{R}$ and $\varepsilon > 0$. With some generic constant $\beta > 0$, and using Hölder's inequality, we can estimate the right-hand side as follows:

$$\begin{aligned}
|\int_\Omega uv dx| &\leq \|u\|_{L^r(\Omega)} \|v\|_{L^{r'}(\Omega)} \\
&\leq \|u\|_{L^r(\Omega)} [(\int_{\Omega(k)} |v|^{2r'} dx)^{\frac{1}{2}} (\int_{\Omega(k)} 1 dx)^{\frac{1}{2}}]^{\frac{1}{r'}} \\
&= \|u\|_{L^r(\Omega)} \|v\|_{L^{2r'}(\Omega)} |\Omega(k)|^{\frac{1}{2r'}} \\
&\leq \beta \|u\|_{L^r(\Omega)} \|v\|_{H^1(\Omega)} |\Omega(k)|^{\frac{1}{2r'}} \\
&\leq \beta \|u\|^2_{L^r(\Omega)} |\Omega(k)|^{\frac{1}{r'}} + \varepsilon \|v\|^2_{H^1(\Omega)} \\
&= \beta \|u\|^2_{L^r(\Omega)} |\Omega(k)|^{\lambda \frac{2}{p}} + \varepsilon \|v\|^2_{H^1(\Omega)}
\end{aligned}$$

The number $\varepsilon > 0$ is yet to be determined. Now, by choosing $\varepsilon := \theta/2$, we may absorb into the left-hand side of (4.24) the term $\varepsilon \|v\|^2_{H^1(\Omega)}$ occurring in the last

inequality.
Invoking (4.24) and the estimate given in (4.14), we find that

$$(\int_{\Omega(k)} |v|^p dx)^{\frac{2}{p}} \leq \beta \|v\|^2_{H^1(\Omega)},$$

hence, by the definition of v,

$$(\int_{\Omega(k)} (|y_c| - k)^p dx)^{\frac{2}{p}} \leq \beta \|v\|^2_{H^1(\Omega)}, \tag{4.25}$$

(v) *Application of Lemma* 3.10
Suppose that $h > k$. Then $\Omega(h) \subset \Omega(k)$, and thus $|\Omega(h)| \leq |\Omega(k)|$. Therefore,

$$\begin{aligned}(\int_{\Omega(k)} (|y_c| - k)^p dx)^{\frac{2}{p}} &\geq (\int_{\Omega(h)} (|y_c| - k)^p dx)^{\frac{2}{p}} \\ &\geq (\int_{\Omega(h)} (h - k)^p dx)^{\frac{2}{p}} \\ &= (h - k)^2 |\Omega(h)|^{\frac{2}{p}}.\end{aligned}$$

Finally, we infer from (4.25) and (4.24) that

$$\begin{aligned}(h - k)^2 |\Omega(h)|^{\frac{2}{p}} &\leq \beta \|u\|^2_{L^r(\Omega)} |\Omega(k)|^{\lambda \frac{2}{p}} \\ &\leq \beta \|u\|^2_{L^r(\Omega)} (|\Omega(k)|^{\frac{2}{p}})^{\lambda}\end{aligned}$$

Putting $\varphi(h) := |\Omega(h)|^{\frac{2}{p}}$, we therefore obtain the inequality

$$(h - k)^2 \varphi(h) \leq \beta \|u\|^2_{L^r(\Omega)} \varphi(k)^{\lambda},$$

for all $h > k \geq 0$. We now apply Lemma3.10 with the specifications

$$a = 2, \quad b = \lambda > 1, \quad k_0 = 0, \quad C = \beta \|u\|^2_{L^r(\Omega)}.$$

We obtain $\delta^2 = \tilde{c} \|u\|^2_{L^r(\Omega)}$, hence the assertion follows: in fact $\varphi(\delta) = 0$ means that $|y_c(x)| \leq \delta$ for almost every $x \in \Omega$.

(vi) *Modification for the case N=2*
Let $r > \frac{N}{2} = 1$ be the order of integrability of u assumed in the theorem. In the case of $N = 2$, the embedding inequality (4.14) is valid for all $p < \infty$. We therefore define p with $\lambda > 1$ by

$$\frac{1}{p} = \frac{1}{2\lambda r'}.$$

With this specification, all conclusions subsequent to (4.14) in the $N \geq 3$ case carry over to $N = 2$, resulting the validity of the assertion.

This concludes the proof of the theorem. □

Remark 4.4. *: In particular, if we consider the case where $r = 2$, then we have $r = 2 > N/2$ which implies $N \leq 3$. In the above theorem, we have already proved for the case where $N = 2$ and $N = 3$. Therefore we conclude that if $u \in L^2(\Omega)$, then the weak solution $y_c \in H_0^1(\Omega)$ is actually essentially bounded(regular), i.e., $y_c \in L^\infty(\Omega)$.*

4.4 Feasibility of solutions of regularized problem for large c

For fixed $\bar{\lambda} \geq \inf \bar{\lambda} > 0$, $0 < s < 1/2$ and for large c sufficiently large the optimal states y_c of (P_c') are feasible, i.e., $\psi_a \leq y_c \leq \psi_b$.

We follow the proof given for unilateral case in Lemma 3.12 to deal with our bilateral case. Let us prove an estimate on the norm of the violation of the constraint $\psi_a \leq y \leq \psi_b$. First of all let us define for solutions y_c of the regularized equation:

$$\phi_c := \max(0, y_c - \psi_b),$$
$$\xi_c := \min(0, y_c - \psi_a).$$

Then from the definition of the max we have $\phi_c \geq 0$ and $\phi_c \in H_0^1(\Omega)$, and of min we have $\xi_c \leq 0$ and $\xi_c \in H_0^1(\Omega)$. Furthermore, we have the following Lemma.

Lemma 4.5. *Let y_c be a solution of the regularized equation (4.4) for a given right-hand side $u \in L^2(\Omega)$ and $\psi_b - \psi_a \geq \sigma > 0$ a.e. for $\sigma \in \mathbb{R}$. Then the violation of $\psi_a \leq y_c \leq \psi_b$ can be estimated by*

$$\|\phi_c\|_{H^1}^2 + c\|\phi_c\|_{L^2}^2 \leq Cc^{-1}\|\max(u - A\psi_b - c^s\bar{\lambda}, 0)\|_{L^2}^2, \tag{4.26}$$

$$\|\xi_c\|_{H^1}^2 + c\|\xi_c\|_{L^2}^2 \leq Cc^{-1}\|\max(A\psi_a - u - c^s\bar{\lambda}, 0)\|_{L^2}^2 \tag{4.27}$$

for c large enough, with a constant C independent of $c, \bar{\lambda}, u, y_c$.
Furthermore, if c is chosen large enough in the estimates (4.26) and (4.27), then the solutions y_c are feasible, i.e., $\psi_a \leq y_c \leq \psi_b$.

Proof. From the definition of ϕ_c,

$$\phi_c > 0 \Leftrightarrow y_c > \psi_b > \psi_a$$

Since $-\max_c(c^s\bar{\lambda} + c(\psi_a - y_c))$ is monotonic increasing with respect to y_c we obtain on the set $\{\phi_c > 0\}$ and for large $c > 0$

$$-\max_c(c^s\bar{\lambda} + c(\psi_a - y_c)) \geq -\max_c(c^s\bar{\lambda} + c(\psi_a - \psi_b)) = 0 \tag{4.28}$$

Testing the regularized equation

$$Ay_c + \max_c(c^s\bar\lambda + c(y_c - \psi_b)) - \max_c(c^s\bar\lambda + c(\psi_a - y_c)) = u$$

with $\phi_c = \max(0, y_c - \psi_b) \geq 0$ and using (4.28) we obtain

$$\langle Ay_c - A\psi_b, \phi_c\rangle + (\max_c(c^s\bar\lambda + c(y_c - \psi_b)), \phi_c) \leq \langle u - A\psi_b, \phi_c\rangle$$

Now using the fact that $\max_c(x) \geq \max(0, x)$ and the second term on the left-hand side of the above equation is positive, we get

$$\langle Ay_c - A\psi_b, \phi_c\rangle + (\max(0, c^s\bar\lambda + c(y_c - \psi_b)), \phi_c) \quad \leq \quad \langle u - A\psi_b, \phi_c\rangle$$

Since from the definition $\max(0, x) \geq x$ and $(y_c - \psi_b, \phi_c) = \|\phi_c\|^2_{L^2}$, it follows

$$\begin{aligned}(\max(0, c^s\bar\lambda + c(y_c - \psi_b)), \phi_c) \quad &\geq \quad (c^s\bar\lambda + c(y_c - \psi_b), \phi_c) \\ &= \quad (c^s\bar\lambda, \phi_c) + c\|\phi_c\|^2_{L^2}\end{aligned}$$

This implies

$$\langle Ay_c - A\psi_b, \phi_c\rangle + (c^s\bar\lambda, \phi_c) + c\|\phi_c\|^2_{L^2} \leq (u - A\psi_b, \phi_c)$$

and then (4.26) follows with

$$\begin{aligned}\langle A\phi_c, \phi_c\rangle + c\|\phi_c\|^2_{L^2} \quad &\leq \quad (u - A\psi_b - c^s\bar\lambda, \phi_c) \\ &\leq \quad (\max(u - A\psi_b - c^s\bar\lambda, 0), \phi_c) \\ &\leq \quad \|\max(u - A\psi_b - c^s\bar\lambda, 0)\|_{L^2}\|\phi_c\|_{L^2} \\ &= \quad \left(\frac{1}{\sqrt{c}}\|\max(u - A\psi_b - c^s\bar\lambda, 0)\|_{L^2}\right) . \left(\sqrt{c}\|\phi_c\|_{L^2}\right) \\ &\leq \quad \frac{1}{2c}\|\max(u - A\psi_b - c^s\bar\lambda, 0)\|^2_{L^2} + \frac{c}{2}\|\phi_c\|^2_{L^2}\end{aligned} \tag{4.29}$$

by the assumption on the elliptic operator A.
To prove (4.27) we follow the same idea as the proof given above. From the definition of ξ_c and $\psi_b - \psi_a \geq \sigma > 0$ a.e. for $\sigma \in \mathbb{R}$,

$$\xi_c < 0 \Leftrightarrow y_c < \psi_a < \psi_b$$

Since $\max_c(c^s\bar\lambda + c(y_c - \psi_b))$ is monotonic increasing with respect to y_c we obtain on the set $\{\xi_c < 0\}$ and for large $c > 0$

$$\max_c(c^s\bar\lambda + c(y_c - \psi_b)) \leq \max_c(c^s\bar\lambda + c(\psi_a - \psi_b)) = 0 \tag{4.30}$$

Testing the regularized equation

$$Ay_c + \max_c(c^s\bar\lambda + c(y_c - \psi_b)) - \max_c(c^s\bar\lambda + c(\psi_a - y_c)) = u$$

with $\xi_c = \min(0, y_c - \psi_a) \leq 0$ and using (4.30) we obtain

$$\langle Ay_c - A\psi_a, \xi_c\rangle + (-\max_c(c^s\bar{\lambda} + c(\psi_a - y_c)), \xi_c) = \langle u - A\psi_a, \xi_c\rangle$$

Note that the second term on the left hand side of the above equation is positive. Thus we rewrite it as

$$\langle Ay_c - A\psi_a, \xi_c\rangle + (\max_c(c^s\bar{\lambda} + c(\psi_a - y_c)), -\xi_c) = \langle u - A\psi_a, \xi_c\rangle$$

Now using the fact that $\max_c(x) \geq \max(0, x)$ we get

$$\langle Ay_c - A\psi_a, \xi_c\rangle + (\max(0, c^s\bar{\lambda} + c(\psi_a - y_c)), -\xi_c) \quad \leq \quad \langle u - A\psi_a, \xi_c\rangle$$

Since from the definition $\max(0, x) \geq x$ and $(\psi_a - y_c, -\xi_c) = \|\xi_c\|^2_{L^2}$, it follows

$$\begin{aligned}(\max(0, c^s\bar{\lambda} + c(\psi_a - y_c)), -\xi_c) &\geq (c^s\bar{\lambda} + c(\psi_a - y_c), -\xi_c) \\ &= (c^s\bar{\lambda}, -\xi_c) + c\|\xi_c\|^2_{L^2}\end{aligned}$$

This implies

$$\langle Ay_c - A\psi_a, \xi_c\rangle + (c^s\bar{\lambda}, -\xi_c) + c\|\xi_c\|^2_{L^2} \leq (A\psi_a - u, -\xi_c)$$

and then (4.27) follows with

$$\begin{aligned}\langle A\xi_c, \xi_c\rangle + c\|\xi_c\|^2_{L^2} &\leq (A\psi_a - u - c^s\bar{\lambda}, -\xi_c) \\ &\leq (\max(A\psi_a - u - c^s\bar{\lambda}, 0), -\xi_c) \\ &\leq \|\max(A\psi_a - u - c^s\bar{\lambda}, 0)\|_{L^2}\|\xi_c\|_{L^2} \\ &= \left(\frac{1}{\sqrt{c}}\|\max(A\psi_a - u - c^s\bar{\lambda}, 0)\|_{L^2}\right) \cdot \left(\sqrt{c}\|\xi_c\|_{L^2}\right) \\ &\leq \frac{1}{2c}\|\max(A\psi_a - u - c^s\bar{\lambda}, 0)\|^2_{L^2} + \frac{c}{2}\|\xi_c\|^2_{L^2}\end{aligned} \tag{4.31}$$

by the assumption on the elliptic operator A. □

4.5 Convergence Analysis of regularized Equations

In this section we study on the convergence properties of the solutions of the semilinear elliptic regularized equation (4.4) and derive the rate of convergence of the family $\{y_c\}_{c>0}$ to y in $L^\infty(\Omega)$ as the regularization parameter $c \to \infty$:

$$Ay + \max_c(c^s\bar{\lambda} + c(y - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y)) = u,$$

and define

$$\lambda_c^b - \lambda_c^a = \max_c(c^s\bar{\lambda} + c(y - \psi_b)) - \max_c(c^s\bar{\lambda} + c(\psi_a - y)).$$

For the proof of the following Lemma we follow the same procedure as for proof of Lemma 3.15.

Proposition 4.6. *Let $\bar\lambda \in L^\infty(\Omega)$ be given. Let y and y_c denote the corresponding solutions of the variational inequality (4.1) and the regularized equation (4.4), respectively. Then*

$$\|y_c - y\|_{L^\infty(\Omega)} \leq \frac{\|\bar\lambda\|_{L^\infty}}{c^{1-s}} + \frac{1}{2c^2}.$$

for c large enough.

Proof. Let the test function be defined as

$$\phi_k := \begin{cases} y - y_c - k, & \text{if } \ y - y_c > k \\ y - y_c + k, & \text{if } \ y - y_c < -k \\ 0, & \text{else} \end{cases}$$

where $k \in \mathbb{R}^+$. Subtracting (4.4) from the first equation in (4.2) and testing with ϕ_k gives

$$a(y - y_c, \phi_k) + \langle \lambda^b - \max_c(c^s\bar\lambda + c(y_c - \psi_b)), \phi_k\rangle - \langle \lambda^a - \max_c(c^s\bar\lambda + c(\psi_a - y_c)), \phi_k\rangle = 0 \tag{4.32}$$

Since $\max(\phi_k, 0)$ and $\min(\phi_k, 0)$ belong to $H_0^1(\Omega)$, we can split $\phi_k = \max(\phi_k, 0) + \min(\phi_k, 0)$. We rewrite (4.32) as

$$\begin{aligned} &a(y - y_c, \phi_k) + \langle \lambda^b, \max(\phi_k, 0)\rangle + \langle \lambda^b, \min(\phi_k, 0)\rangle - (\max_c(c^s\bar\lambda + c(y_c - \psi_b)), \max(\phi_k, 0)) \\ &\quad -(\max_c(c^s\bar\lambda + c(y_c - \psi_b)), \min(\phi_k, 0)) - \langle \lambda^a, \max(\phi_k, 0)\rangle - \langle \lambda^a, \min(\phi_k, 0)\rangle \\ &\quad +(\max_c(c^s\bar\lambda + c(\psi_a - y_c)), \max(\phi_k, 0)) + (\max_c(c^s\bar\lambda + c(\psi_a - y_c)), \min(\phi_k, 0)) = 0 \end{aligned} \tag{4.33}$$

Since $\lambda^b, \lambda^a \geq 0$ we get $\langle \lambda^b, \max(\phi_k, 0)\rangle \geq 0$ and $-\langle \lambda^a, \min(\phi_k, 0)\rangle \geq 0$. By feasibility of y_c, i.e., $\psi_a \leq y_c \leq \psi_b$ we find

$$\begin{aligned} 0 \geq \min(\phi_k, 0) &= \min(y - y_c + k, 0) \geq \min(y - \psi_b + k, 0) \\ &\geq \min(y - \psi_b, 0) = y - \psi_b \end{aligned}$$

and

$$\begin{aligned} 0 \leq \max(\phi_k, 0) &= \max(y - y_c - k, 0) \leq \max(y - \psi_a - k, 0) \\ &\leq \max(y - \psi_a, 0) = y - \psi_a. \end{aligned}$$

By the complementarity condition this implies

$$0 \geq \langle \lambda^b, \min(\phi_k, 0)\rangle \geq \langle \lambda^b, y - \psi_b\rangle = 0$$

and

$$0 \leq \langle \lambda^a, \max(\phi_k, 0)\rangle \leq \langle \lambda^a, y - \psi_a\rangle = 0$$

On the set $\{\phi_k > 0\}$ since $y - y_c > k \Rightarrow \psi_b - y_c > y - y_c > k$, we get

$$\begin{aligned} 0 &\leq \max{}_c(c^s\bar{\lambda} + c(y_c - \psi_b)) \\ &\leq \max{}_c(c^s\bar{\lambda} + c(-k)) = 0, \quad \text{for} \quad k \geq \frac{\|\bar{\lambda}\|_{L^\infty}}{c^{1-s}} + \frac{1}{2c^2}, \end{aligned}$$

and on the set $\{\phi_k < 0\}$ since $y - y_c < -k \Rightarrow \psi_a - y_c < y - y_c < -k$, we get

$$\begin{aligned} 0 &\leq \max{}_c(c^s\bar{\lambda} + c(\psi_a - y_c)) \\ &\leq \max{}_c(c^s\bar{\lambda} + c(-k)) = 0, \quad \text{for} \quad k \geq \frac{\|\bar{\lambda}\|_{L^\infty}}{c^{1-s}} + \frac{1}{2c^2}, \end{aligned}$$

and hence

$$(\max{}_c(c^s\bar{\lambda} + c(y_c - \psi_b)), \max(\phi_k, 0)) = 0$$

and

$$(\max{}_c(c^s\bar{\lambda} + c(\psi_a - y_c)), \min(\phi_k, 0)) = 0$$

Since from the definition $\max_c(c^s\bar{\lambda} + c(y_c - \psi_b)) \geq 0$ and $\max_c(c^s\bar{\lambda} + c(\psi_a - y_c)) \geq 0$, we have

$$-(\max{}_c(c^s\bar{\lambda} + c(y_c - \psi_b)), \min(\phi_k, 0)) \geq 0.$$

and

$$(\max{}_c(c^s\bar{\lambda} + c(\psi_a - y_c)), \max(\phi_k, 0)) \geq 0.$$

Thus substituting the above results into (4.33), we get $a(y - y_c, \phi_k) \leq 0$ for $k \geq \frac{\|\bar{\lambda}\|_{L^\infty}}{c^{1-s}} + \frac{1}{2c^2}$.

Due to properties of the bilinear form a and the definition of ϕ_k, we have $a(y - y_c, \phi_k) \geq a(\phi_k, \phi_k)$, which implies $\phi_k = 0$. □

Chapter 5

Optimal control of semilinear elliptic regularized equation

In this chapter, we study an optimal control problem governed by elliptic semilinear regularized equation. Existence of a sequence of solutions to the optimal control problem with regularized equation is proved and weakly or strongly convergence of (P_c) to solutions of the original problem (P) is studied. The regularity of the adjoint state and the state constraint multiplier is also studied. A sharp optimality system for the original control problem is obtained as limit of the regularized optimality systems.

For convenience we repeat the problem formulation:

$$(P_c) \quad \begin{cases} \min J(y,u) = g(y) + j(u) \\ \textit{over} \quad u \in U_{ad}, \quad \text{subject to} \\ Ay + \max_c(c^s \bar{\lambda} + c(y-\psi)) = u, \quad y \in H_0^1(\Omega) \end{cases}$$

where the state y and the control u are coupled by the semilinear elliptic boundary value problem.

5.1 Existence of solutions to (P_c)

Definition 5.1. **(i)** *A control $\bar{u} \in U_{ad}$ is said to be* ***optimal*** *if it satisfies, together with the associated optimal state $\bar{y} = y(\bar{u})$, the inequality*

$$J(y(\bar{u}), \bar{u}) \leq J(y(u), u) \quad \forall u \in U_{ad}.$$

(ii) *A control $\bar{u} \in U_{ad}$ is said to be* ***locally optimal*** *in the sense of $L^2(\Omega)$ if there exists some $\rho > 0$ such that the above inequality holds for all $u \in U_{ad}$ such that $\|u - \bar{u}\|_{L^2(\Omega)} \leq \rho$.*

In this section we prove the existence of at least one solution $(y_c(u_c), u_c)$ with $y_c = y_c(u_c)$ to (P_c).

Proposition 5.2. *Let $j : L^2(\Omega) \to \mathbb{R}$ be weakly lower semi-continuous. For every $c > 0$, there exists a solution $(y_c, u_c) \in H_0^1(\Omega) \times L^2(\Omega)$ to (P_c).*

Proof. Since j is radially unbound and g is bounded below, every minimizing sequence $\{(y_c(u_n), u_n)\}$ to (P_c), i.e., $J(y_c(u_n), u_n)$ is monotonicity decreasing and

$$\lim_{n\to\infty} J(y_c(u_n), u_n) = \inf J(y_c(u_c), u_c) \text{ subject to } (3.13), \tag{5.1}$$

has a weakly convergent subsequence, denoted by the same symbol, with weak limit $u_c \in L^2(\Omega)$.
By Lemma 3.17 we find $y_c(u_n) \to y_c(u_c)$ strongly in $H_0^1(\Omega)$. Weak lower semi-continuity of j and continuity of $g : H_0^1(\Omega) \to \mathbb{R}$ imply that $(y_c(u_c), u_c)$ is a solution to (P_c). □

5.2 Necessary Optimality Conditions for (P_c)

Since the control problems governed by nonlinear equations are nonsmooth and non-convex optimization problems, the standard methods for deriving necessary conditions of optimality are inapplicable here. To cope with this problem, the idea is to approximate the given problem (P) by a family of smooth optimization problems (P_c), and then to tend to the limit in the corresponding optimality equations [22].

Suppose g and j are C^1-regular. Next a formal derivation of the first order necessary optimality conditions for regularized optimal control problem (P_c) is given. Suppose that there exists a Lagrange multiplier $p_c \in H_0^1(\Omega) \times L^\infty(\Omega)$ (adjoint state) satisfies the Lagrangian functional given by

$$L(y_c, u_c, p_c) = g(y_c) + j(u_c) + \langle Ay_c + \max_c(c^s\bar{\lambda} + c(y_c - \psi)) - u_c, p_c\rangle$$

We find

$$\begin{cases} L_{y_c} = 0: & A^*p_c + c\,\mathrm{sgn}_c(c^s\bar{\lambda} + c(y_c - \psi))p_c + g'(y_c) = 0 \\ L_{u_c} = 0: & j'(u_c) - p_c = 0 \\ L_{p_c} = 0: & Ay_c + \max_c(c^s\bar{\lambda} + c(y_c - \psi)) - u_c = 0 \end{cases}$$

Then we obtain formally the necessary optimality system for (P_c):

$$(OC) \quad \begin{cases} Ay_c + \max_c(c^s\bar{\lambda} + c(y_c - \psi)) = u_c, \\ A^*p_c + c\,\mathrm{sgn}_c(c^s\bar{\lambda} + c(y_c - \psi))p_c + g'(y_c) = 0, \\ (j'(u_c) - p_c, u - u_c) \geq 0 \end{cases}$$

In the following proposition we will address convergence of the solutions of the regularized optimal control problem (P_c) to those of the original problem (P).

Proposition 5.3. *Let $j : L^2(\Omega) \to \mathbb{R}$ be weakly lower semi-continuous.*
For every subsequence of controls $\{u_{c_n}\}$ converging weakly in $L^2(\Omega)$ to some u^*, the corresponding states $y_{c_n} = y(u_{c_n})$ converge strongly in $H_0^1(\Omega)$ to $y^* = y(u^*)$, and (y^*, u^*) is a global solution to (P). Moreover
$\lambda_{c_n} = max_c(c^s\bar{\lambda} + c_n(y_{c_n} - \psi)) \rightharpoonup \lambda(u^*)$ weakly in $H^{-1}(\Omega)$.

In addition, in the feasible case with $y_{c_n} \le \psi$ for all n, $\{(p_{c_n}, \mu_{c_n})\}$ converge weakly in $H_0^1(\Omega)$ and weakly star in $L^\infty(\Omega)^*$ to $(p^*, \mu^*) \in H_0^1(\Omega) \times L^\infty(\Omega)^*$ satisfying (3.20)-(3.24).

Proof. By proposition 5.2 we have a family of solutions $\{(y_c, u_c)\}$ to (P_c).
Let $y_c(0)$ denote the solution to the equality constraint in (P_c) with $u = 0$, and note that $\{y_c(0)\}_{c\ge1}$ is bounded in $H_0^1(\Omega)$. Hence $\{g(y_c(0))\}_{c\ge1}$ is bounded as well.
Then $(y_c(0), 0)$ is a feasible pair for (P_c) for every $c > 0$, and $J(y_c, u_c) \le J(y_c(0), 0)$. Thus $\{j(u_c)\}_{c\ge1}$ is bound and radial unboundedness of j implies that $\{u_c\}_{c\ge1}$ is bounded in $L^2(\Omega)$. Since every bounded sequence in a reflexive Banach space contains a weakly convergent subsequence, consequently there exists a weakly convergent subsequence u_{c_n} in $L^2(\Omega)$ with weak limit $u^* \in L^2(\Omega)$. By Lemma 3.17 the sequence $y_{c_n} = y(u_{c_n}) \to y(u^*)$ strongly in $H_0^1(\Omega)$. Moreover by Lemma 3.14 $\lambda_{c_n} = \max_{c_n}(c^s\bar{\lambda} + c_n(y_{c_n} - \psi)) \to \lambda(y^*)$ strongly in $H^{-1}(\Omega)$, where $Ay^* + \lambda(y^*) = u^*$. Now passing to the limit in (P_{c_n}) as $n \to \infty$ and obtain that (y^*, u^*) is a solution to (P), with associated Lagrange multiplier $\lambda(y^*)$. By Theorem 3.4 there exists an associated adjoint state $p^* \in H_0^1(\Omega)$ and $\mu^* \in H^{-1}(\Omega) \cap L^\infty(\Omega)^*$ satisfying (3.20)-(3.24).
Taking the inner product of the second equation of (OC) by p_c, we find

$$a(p_c, p_c) + c(\operatorname{sgn}_c(c^s\bar{\lambda} + c(y_c - \psi))p_c, p_c) = -\langle g'(y_c), p_c\rangle.$$

Since $\operatorname{sgn}_c(c^s\bar{\lambda} + c(y_c - \psi)) \ge 0$ and a is coercive on $H_0^1(\Omega)$, there exists a constant M_1 independent of $c \ge 1$ such that

$$\|p_c\|_{H_0^1}^2 + c(\operatorname{sgn}_c(c^s\bar{\lambda} + c(y_c - \psi))p_c, p_c) \le M_1. \tag{5.2}$$

Next, we show that $\mu_c = c\operatorname{sgn}_c(c^s\bar{\lambda} + c(y_c - \psi))p_c$ is bounded in $L^1(\Omega)$ uniformly with respect to $c \ge 1$. For $\varepsilon > 0$ define the function ρ_ε by

$$\rho_\varepsilon(x) = \begin{cases} 1, & x \ge \varepsilon; \\ \frac{x}{\varepsilon}, & |x| \le \varepsilon; \\ -1, & x \le -\varepsilon, \end{cases} \tag{5.3}$$

and note that $0 \le \rho'_\varepsilon(x) \le \frac{1}{\varepsilon}$ on $\mathbb{R}$. Taking the duality product of the second equation in (OC) with $\rho_\varepsilon(p_c)$ we obtain

$$\langle A^* p_c, \rho_\varepsilon(p_c)\rangle + (c\operatorname{sgn}_c(c^s\bar{\lambda} + c(y_c - \psi))p_c, \rho_\varepsilon(p_c)) = -\langle g'(y_c), \rho_\varepsilon(p_c)\rangle.$$

Since $a(\rho_\varepsilon(p_c), p_c) \ge 0$, by the definitions of 5.3 and 3.1, we have

$$(c\operatorname{sgn}_c(c^s\bar{\lambda} + c(y_c - \psi))p_c, \rho_\varepsilon(p_c)) \le \|g'(y_c)\|_{L^1}.$$

Moreover, $\rho_\varepsilon(p_c)p_c \to |p_c|$ a.e in Ω as $\varepsilon \to 0$ and $\text{sgn}_c(c^s\bar{\lambda} + c(y_c - \psi))p_c \le 1$ and thus by Lebesgue's dominated convergence theorem $\|\mu_c\|_{L^1} \le \|g'(y_c)\|_{L^1}$.
Hence there exists a subsequence $\{c_n\}$ of $\{c\}$ and $p^* \in H_0^1(\Omega)$ and $\mu^* \in (L^\infty(\Omega))^*$ such that
$p_{c_n} \to p^*$ weakly in $H_0^1(\Omega)$,
$\mu_{c_n} \to \mu^*$ weakly star in $(L^\infty(\Omega))^*$. □

5.3 Existence of approximating families

In this section we will introduce a family of regularized problems (P_c) which asymptotically approximate (P) as $c \to \infty$. Let (y^*, u^*) be strictly locally optimal pair for (P). Then by definition 5.1 there exists $\rho > 0$ such that

$$J(y^*, u^*) < J(y, u) \ \ \forall (y, u) \ \text{ satisfying } \ (3.9) \ \text{ and } \ 0 < \|u - u^*\|_{L^2} < \rho \tag{5.4}$$

In the next theorem we will show that there is a family of local solutions (y_c, u_c) of (P_c) for each strictly optimal pair (y^*, u^*) of the original problem (P) that converges strongly to (y^*, u^*) in $H_0^1(\Omega) \times L^2(\Omega)$.

Theorem 5.4. *Let j be weakly lower semi-continuous. Moreover, we require for j that*

$$u_n \rightharpoonup u \ \ in \ \ L^2(\Omega) \ \ , and \ \ j(u_n) \to j(u) \ \ imply \ \ u_n \to u \ \ in \ \ L^2(\Omega). \tag{5.5}$$

Let (y^*, u^*) be a strictly locally optimal pair for (P). Then there exists a family of local solutions (y_c, u_c) of (P_c) that converges strongly to (y^*, u^*) in $H_0^1(\Omega) \times L^2(\Omega)$.

Proof. Let ρ be given by (5.4) and take ρ' with $0 < \rho' < \rho$. Consider the auxiliary problem

$$(P_c^{\rho'}) \quad \begin{cases} \min J(y, u) = g(y) + j(u) \\ over \ \ u \in U_{ad}, \ \text{ with } \ \|u - u^*\|_{L^2} \le \rho' \ \text{ and subject to} \\ Ay + \max_c(c^s\bar{\lambda} + c(y - \psi)) = u. \end{cases}$$

Then by Proposition 5.2 the optimal control problem $(P_c^{\rho'})$ is solvable for every $c > 0$. Let u_c denote a global solution of $(P_c^{\rho'})$. By construction, the set $\{u_c\}_{c>0}$ is bounded, which yields weak convergence of a subsequence $u_{c_n} \rightharpoonup \tilde{u}$ in $L^2(\Omega)$ with $\|\tilde{u} - u^*\|_{L^2} \le \rho'$. By Lemma 3.17 we find that $y_{c_n} \to \tilde{y}$ strongly in $H_0^1(\Omega)$, where $(\tilde{y}, \tilde{u})$ is a solution of the original optimal control problem (P) . In the assumption since (y^*, u^*) is a strictly locally optimal pair for (P), then it follows $J(y^*, u^*) < J(\tilde{y}, \tilde{u})$.

Let $y_c(u^*)$ denotes the solution of the regularized equation to the control u^*, we have $J(y_c, u_c) \le J(y_c(u^*), u^*)$. By Lemma 3.14, we have $y_c(u^*) \to y^*$ in $H_0^1(\Omega)$ as $c \to \infty$. By (5.4), the optimality and convergence properties above, we obtain

$$J(y^*, u^*) < J(\tilde{y}, \tilde{u})$$

$$\begin{aligned}\Rightarrow g(y^*) + j(u^*) &\leq g(\tilde{y}) + j(\tilde{u}) \leq \lim g(y_{c_n}) + \liminf j(u_{c_n}) \\ &\leq \lim g(y_{c_n}) + \limsup j(u_{c_n}) \\ &\leq \lim g(y_{c_n}(u^*)) + j(u^*) \\ &= g(y^*) + j(u^*).\end{aligned} \tag{5.6}$$

Then follows $J(y^*, u^*) = J(\tilde{y}, \tilde{u})$ and since (y^*, u^*) is strict local optimality of (P), then we get $u^* = \tilde{u}$. Moreover, it follows that $\lim j(u_{c_n}) = j(\tilde{u})$ which yields $u_{c_n} \to u^*$ in $L^2(\Omega)$ by (5.5). Since u^* is the unique local solution in the L^2-neighborhood of u^* of radius ρ', the whole family u_c converges u^*.
Convergence of $u_c \to u^*$ also implies the existence of c_0 such that $\|u_c - u^*\|_{L^2} < \rho'$ for all $c > c_0$. Consequently, if $c > c_0$, then (y_c, u_c) is locally optimal for (P_c). □

Now we show the convergence of adjoint and multipliers of (P_c) to (P).

Corollary 5.5. *Let (y_c, u_c) be a family of local solutions of (P_c) converging strongly in $H_0^1(\Omega) \times L^2(\Omega)$ to (y^*, u^*). Let (y^*, u^*) solve the variational inequality and satisfy together with (λ^*, p^*, μ^*) the first-order optimality system (3.20)-(3.25) given by Theorem 3.4. Then we have*

$$\lambda_c \to \lambda^* \quad \text{and} \quad \mu_c \rightharpoonup \mu^* \quad \text{in} \quad H^{-1}(\Omega), \quad p_c \rightharpoonup p^* \quad \text{in} \quad H_0^1(\Omega)$$

where (λ_c, p_c, μ_c) are the corresponding multipliers and adjoint state for (P_c), see (OC).

Proof. Since y_c, u_c are strongly converge, the strong convergence of λ_c follow immediately

$$\lambda_c = u_c - Ay_c \to u^* - Ay^* = \lambda^* \quad \text{in} \quad H^{-1}(\Omega).$$

Testing the second equation in (OC) by p_c gives

$$C_c \|p_c\|_{H^1}^2 \leq \|g'(y_c)\|_{L^2} \|p_c\|_{L^2},$$

which gives boundedness of $\{p_c\}$ in $H_0^1(\Omega)$. Then, we get a subsequence $\{p_{c_n}\}$ which converges weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$ to $\tilde{p}$. Hence, the last equation in (OC) implies

$$j'(u_{c_n}) = p_{c_n} \to j'(u^*) = \tilde{p},$$

which gives $\tilde{p} = p^*$ by optimality condition (3.24). Since the adjoint state p^* is uniquely determined by (y^*, u^*), the whole family p_c converges weakly in $H_0^1(\Omega)$ to p^*. Arguing as above, we find for μ_c

$$\mu_c = -A^* p_c - g'(y_c) \rightharpoonup -A^* p^* - g'(y^*) = \mu^* \quad \text{in} \quad H^{-1}(\Omega),$$

which finishes the proof. □

Chapter 6

Obstacle problems

The study of variational inequalities occupies a central position in calculus of variations and in the applied sciences [24]. Variational inequalities form an important family of nonlinear problems. Some of the more complex physical processes are described by variational inequalities.

The obstacle problem is typical of a class of inequality problems known as elliptic variational inequalities (EVIs) of the first kind in our case (see for e.g., [5]). They are of interest both for their intrinsic beauty and for the wide range of applications they describe in subjects from physics to finance. Many important problems can be formulated by transformation to an obstacle problem, e.g., the filtration dam problem, the Stefan problem, the subsonic flow problem, American options pricing model, etc (see Yongmin Zhang [12]).

In this chapter, we formulate the obstacle problems for which the solution algorithms are developed in the forthcoming sections. A regularized problem and iterative second-order algorithms for its solution are analyzed in infinite dimensional function spaces. Motivated from the dual formulation, a primal-dual active set strategy and a semismooth Newton method for a regularized problem are presented and their close relation is analyzed.

6.1 Problem Formulation

Suppose the bilinear form a, a second order linear elliptic partial differential operator A associated to a and the closed convex subset K of $H_0^1(\Omega)$ be the same as we defined in chapter 3. We then consider the approximation of obstacle problems of the following type:

$$\begin{cases} \text{Find } \; y \in K \; \text{ such that} \\ a(y, v-y) \geq (u, v-y) \quad \forall v \in K. \end{cases} \tag{6.1}$$

It was shown by Lions and Stampacchia in [6] that, under slightly weaker conditions (symmetry is not needed), there exists a unique solution to Problem 6.1. Furthermore, (with symmetry) Problem 6.1 is equivalent to
Problem. Find $y \in K$ such that

$$\mathcal{J}(y) = \inf_{v \in K} \mathcal{J}(v) \tag{6.2}$$

with the closed, convex, and non-empty set K,

$$K = \{v \in H_0^1(\Omega) | v \le \psi \ \text{ a.e. in } \ \Omega\}.$$

and the energy functional $\mathcal{J}$,

$$\mathcal{J}(v) = \frac{1}{2} a(v, v) - (u, v), \tag{6.3}$$

is induced by a symmetric, H-elliptic, bilinear form $a(\cdot, \cdot)$.
Concerning the existence and uniqueness of solutions of problem (6.2), we have the following classical results from Stampacchia [6]:

Theorem 6.1. *If the function $v \to \mathcal{J}(v)$ satisfies*

$$\mathcal{J}(v) \to +\infty \quad \textit{when} \quad \|v\|_{H^1} \to \infty, \quad v \in K. \tag{6.4}$$

then there exists a solution y of (6.2).

A sufficient condition for (6.4) to be satisfied is that

$$\begin{cases} \text{there exists } \ C_c \ge 0 \ \text{ such that} \\ a(v, v) \ge C_c \|v\|_{H^1}^2 \quad \forall v \in K. \end{cases} \tag{6.5}$$

Theorem 6.2. *If the function $v \to \mathcal{J}(v)$ is strictly convex, then problem (6.2) admits at most one solution.*

Since the obstacle problem is nonlinear, the computation of approximate solutions can be difficult and expensive. A major difficulty in solving the problem (6.1) numerically is the treatment of the non-differentiable term. Semismooth Newton methods and primal-dual active set methods are efficient methods for coping with nondifferentiable functionals in infinite dimensional spaces; see, e.g., [2],[13], [14], [16].

We shall now study some constructive methods for the infinite dimensional approximation of the solution y of the obstacle problems (6.1).

6.2 Infinite-dimensional Approximation Methods

An infinite dimensional analysis gives more insight into the problem, which is also of significant practical importance since the performance of a numerical algorithm is closely related to the infinite dimensional problem structure [6] .

A finite dimensional approach misses important features as, for example, the regularity of Lagrange multipliers and its consequences as well as smoothing and uniform definiteness properties of the involved operators. It is well accepted that these properties significantly influence the behavior of numerical algorithms [4].

The approximate solution of obstacle problems is usually solved by variable projection method, for example, the relaxation method [5], multilevel projection method [12], multigrid method [10], and projection method [23] for nonlinear complementarity problems.
In the next subsections we will consider some approaches for the iterative solution of obstacle problems ; see, for instance, [13], [14], [15] and the references given there.

6.2.1 Penalization Method

The method of penalization consists in substituting the obstacle problem by a family of nonlinear boundary value problems and demonstrating that their solutions converge to the solution of the obstacle problem. Penalty methods based on straightforward regularization are popular in the field of engineering.

The idea of penalization consists of approximating (6.1) by equations in which the relation appearing in (6.1), which expresses the fact that y belongs to K, is replaced by a penalization term which becomes progressively larger as the solution moves awayfrom K and thus forces the limit of the approximate solutions to belong to K.

Using this technique the obstacle problem is approximated by a series of nonlinear boundary value problems. Specifically, we introduce a penalization operator π which has the following properties [15]:

$$\begin{cases} \pi \ \text{ is Lipschitz continuous,} \\ Ker(\pi) = K \\ \pi \ \text{ is monotone,} \end{cases} \tag{6.6}$$

the obstacle problem (6.1) can be approximated by the penalized equation

$$a(y_\varepsilon, v) + \frac{1}{\varepsilon}\langle \pi(y_\varepsilon), v\rangle = (u, v) \quad \forall v \in H_0^1(\Omega), \tag{6.7}$$

with $\varepsilon > 0$ being the penalty parameter. Due to the monotonicity of the nonlinear operator π, equation (6.7) has a unique solution y_ε (see, e.g., [6]).

In [15], M. Hintermüller and I. Kopacka considered

$$\pi(v) := -max(0, -v) \quad \forall v \in H_0^1(\Omega)$$

as the penalty operator where the max −operation is to be understood point-wise. As the $\max(0, \cdot)$-function is not differentiable at the origin we introduce different regularizations yielding C^2-approximations of $\max(0, \cdot)$.
The following result describes the approximation properties of the regularized penalized equations (cf. Stampacchia [6]).

Theorem 6.3. *As $\varepsilon \to 0$, $y_\varepsilon \to y$ in $H_0^1(\Omega)$, y being the solution of (6.1).*

6.2.2 The primal-dual active set method

In this section a regularized problem and iterative second-order algorithms for its solution are analyzed in infinite dimensional function spaces. Motivated from the dual formulation, a primal-dual active set strategy for a regularized problem is presented.

This method is an efficient way to solve discrete obstacle problems which is given by active set strategies [10]. Basic iteration of active set strategies consists of two steps. In the first phase, the (mesh) domain is decomposed into active and inactive parts, based on a criterion specifying a certain active set method. In the second phase, a reduced linear system associated with the inactive set is solved.

Recall that in chapter 3 we approximated the obstacle problem (6.1) by a semilinear equation (3.13): For convenience we repeat the problem formulation

$$\begin{cases} a(y, v) + (\lambda_c, v) = (u, v) \quad \forall v \in H_0^1(\Omega) \\ \lambda = \max(0, c^s\bar{\lambda} + c(y - \psi)) \end{cases} \tag{6.8}$$

which is also equivalent to

$$Ay + \max(0, c^s\bar{\lambda} + c(y - \psi)) = u \tag{6.9}$$

where $A \in \mathcal{L}(H_0^1, H^{-1})$ and $\langle Ay, v\rangle = a(y, v)$.
Note that for each $c > 0$

$$y \to \max(0, c^s\bar{\lambda} + c(y - \psi))$$

is Lipschitz continuous and monotone from $H_0^1(\Omega)$ to $H_0^1(\Omega)$. Thus by the monotone theory we have proved the existence of a unique solution $(y_c, \lambda_c) \in H_0^1(\Omega) \times L^2(\Omega)$ to (6.8).
The primal-dual active set strategy for (6.8) is given next. We introduce $\chi_{\mathcal{A}_{k+1}}$, the characteristic function of the set $\mathcal{A}_{k+1} \subseteq \Omega$.

Primal-dual active set(PDAS) algorithm

(i) Choose $c > 0$, (y_0, λ_0); set $k = 0$.

(ii) Set $\mathcal{A}_{k+1} = \{x : (c^s\bar{\lambda} + c(y_k - \psi))(x) > 0\}$, $\mathcal{I}_{k+1} = \Omega \backslash \mathcal{A}_{k+1}$.

(iii) Solve for $y_{k+1} \in H_0^1(\Omega)$:

$$a(y_{k+1}, v) + (c^s\bar{\lambda} + c(y_{k+1} - \psi), \chi_{A_{k+1}} v) = (u, v) \quad \forall v \in H_0^1(\Omega). \tag{6.10}$$

(iv) Set

$$\lambda_{k+1} = \begin{cases} 0 & \text{on} \quad \mathcal{I}_{k+1}, \\ c^s\bar{\lambda} + c(y_{k+1} - \psi) & \text{on} \quad \mathcal{A}_{k+1}. \end{cases}$$

(v) Stop or $k = k + 1$, goto (ii).

The iterates λ_{k+1} as assigned in step (iv) are not necessary for the algorithm but they are useful in the convergence analysis.

Remark 6.4. *1. The primal-dual active set method discussed above is equivalent to the semi-smooth Newton method (see; e.g., [14], [16])*

2. For every $c > 0$ we have $\lim_{k\to\infty}(y_k, \lambda_k) = (y_c, \lambda_c)$ in $H_0^1(\Omega) \times L^2(\Omega)$. This guarantees global convergence of a semi-smooth Newton method, i.e., the algorithm converges for any initial condition.

3. If $\lambda_0 \in L^2(\Omega)$ and $\|\lambda_0 - \lambda_c\|_{L^2(\Omega)}$ is sufficiently small, then $(y_k, \lambda_k) \to (y_c, \lambda_c)$ super-linearly in $H_0^1(\Omega) \times L^2(\Omega)$.

In chapter 4, we have seen that a class of optimization problems with bilateral constraints $\psi_a \le y \le \psi_b$ can be expressed as

$$\begin{cases} Ay + \lambda - u = 0, \\ \\ \lambda = \max(0, \lambda + c(y - \psi_b)) - \max(0, \lambda + c(\psi_a - y)) \end{cases} \tag{6.11}$$

The primal-dual active set strategy applied to (6.11) can be express as

$Ay_{k+1} - u + \lambda_{k+1} = 0,$

$y_{k+1} = \psi_b$ in $A_k^+ = \{x : \lambda_k(x) + c(y_k(x) - \psi_b(x)) > 0\},$

$\lambda_{k+1} = 0$ in $I_k = \{x : \lambda_k(x) + c(y_k(x) - \psi_a(x)) \le 0 \le \lambda_k(x) + c(y_k(x) - \psi_b(x))\},$

$y_{k+1} = \psi_a$ in $A_k^- = \{x : \lambda_k(x) + c(y_k(x) - \psi_a(x)) < 0\}.$

Chapter 7

Summary

In the optimal control problem of a variational inequality the main difficulty comes from the fact that the mapping between the control and the state (control-to-state operator) is not differentiable but only Lipschitz-continuous. As a consequence of this, to get sharp optimality conditions and build numerical algorithms are difficult tasks, overcoming this difficulty was a major motivation of our study.

In this dissertation we have investigated the analytical background of an optimal control problem subject to elliptic variational inequalities of the first kind with unilateral and bilateral obstacle problems and studied a regularization method (i.e. to approximate the nondifferentiable ones depending on ($c \geq 0 \quad c \to \infty$)) for solving a nondifferentiable minimization problem.

Convergence properties of the optimal solutions of the regularized problems (P_c) towards the solution of the original problem (P) are proven and an $L^\infty-$ error estimate for the convergence is obtained.

Using a local smoothing of the max function, we derived a first order optimality conditions for the regularized problems and first order necessary optimality system for the original control problem is obtained as limit of the regularized optimality systems, i.e., the regularity of the adjoint state and the state constraint multipliers of the the original control problem is obtained as limit of the regularized optimality systems.

A regularized problem and iterative second-order algorithms for its solution are analyzed in infinite dimensional function spaces. Motivated from the dual formulation, a primal-dual active set strategy and a semismooth Newton method for a regularized problem are presented and their close relation is analyzed.

Appendix

Normed linear Spaces

Definition 7.1. *(Norm-Banach space)*
Let X be a real vector space.

(i) *The mapping $\|\cdot\| : X \to [0, \infty)$ is called norm on X, if*

- **a)** $\|u\| \geq 0 \;\; \forall u \in X$
- **b)** $\|u\| = 0 \Leftrightarrow u = 0$
- **c)** $\|\lambda u\| = |\lambda| \|u\| \;\; \forall u \in X, \;\; \lambda \in \mathbb{R}$ *(positive homogeneity)*
- **d)** $\|u + v\| \leq \|u\| + \|v\| \;\; \forall u, v \in X$ *(triangle inequality)*

Then $\{X, \|\cdot\|\}$ is known as a real(normed) space.

(ii) *A normed real vector space X is called* ***Banach Space****, if it is complete, i.e. if every Cauchy-sequence converges in X, thus a limit $u \in X$ exists.*

Definition 7.2. *(Inner product- Hilbert space)*
Let H be a real vector space.

(i) *A mapping $(\cdot, \cdot) : H \times H \to \mathbb{R}$ is called inner product on H, if*

- **a)** $(u, v) = (v, u) \;\; \forall u, v \in H$
- **b)** *For every $v \in H$ the mapping $u \in H \mapsto (u, v)$is linear*
- **c)** $(u, u) \geq 0 \;\; \forall u \in H$ *and* $(u, u) = 0 \Leftrightarrow u = 0.$

(ii) *A vector space H with an inner product $(\cdot, \cdot)$ and a related norm*

$$\|u\| := \sqrt{(u, u)}$$

is called Pre-Hilbert space.

(iii) *A Pre-Hilbert space $(H, (\cdot, \cdot))$ is called a Hilbert space, if it is complete with respect to its norm $\|u\| := \sqrt{(u, u)}$.*

Theorem 7.3. *(Cauchy-Schwarz-inequality)*
Let H be a Pre-Hilbert space. Then the ***Cauchy-Schwarz-inequality***

$$|(u,v)| \leq \|u\|\|v\| \qquad \forall u,v \in H$$

holds. $\square$

Definition 7.4. *(Linear operator)*
A mapping $A : X \to Y$ is called ***linear operator****, if*

$$A(x+y) = Ax + Ay \qquad \forall x,y \in X$$

$$A(\lambda x) = \lambda Ax \qquad \forall x \in H \quad \text{and} \quad \forall \lambda \in \mathbb{R}.$$

Definition 7.5. *(Bounded linear operator)*
A linear operator $A : X \to Y$ is called ***bounded****, if there exists a $c > 0$ such that*

$$\|Ax\|_Y \leq c\|x\|_X \quad \forall x \in X$$

Definition 7.6. *(Linear functionals and dual space)*

(i) *Let X be a Banach space. A bounded (continuous) linear operator $u^* : X \to \mathbb{R}$ denoted by $u^* \in \mathcal{L}(X,\mathbb{R})$ is called a* ***linear functional*** *on X.*

(ii) *The space of the bounded (continuous) linear functionals on X is called the dual space $X^* = \mathcal{L}(X,\mathbb{R})$ of X.*

(iii) *For an application of a linear, continuous functional $u^* \to X^*$ to an element $x \in X$ we often write the expression*

$$\langle u^*, x\rangle_{X^*,X} = \langle u^*, x\rangle \underset{=}{def} u^*(x).$$

We call the application u^* on x ***dual pairing****.*

Definition 7.7. *(Operator norm)*
We denote by

$$\|A\|_{\mathcal{L}(X,Y)} \underset{=}{def} \sup_{\|x\|=1} \|Ax\|_Y = \sup_{x\neq 0} \frac{\|Ax\|_Y}{\|x\|_X}$$

the ***operator norm*** *of A.*

Definition 7.8. *(Adjoint operator)*
*Let G and H be Hilbert spaces and $B \in \mathcal{L}(G,H)$. A mapping $B^* : H \to G$ is called the (****Hilbert space****)-****adjoint operator*** *associated to B, if*

$$(Bg,h)_H = (g,B^*h)_G$$

for all $g \in G$ and $h \in H$. In the case that $G = H, B$ is called ***self-adjoint****, if $B^* = B$.*

Definition 7.9. *(Bilinear form)*
A bilinear form $a(\cdot,\cdot)$ on a linear space X is a mapping $a : X \times X \to \mathbb{R}$ such that each of the maps $v \mapsto a(v, w)$ and $w \mapsto a(v, w)$ is a linear form.

(i) *a is called* bounded *if, there exists $C_b \geq 0$ such that*

$$|a(v, w)| \leq C_b \|v\| \|w\| \quad \textit{for each} \quad (v, w) \in X^2.$$

(ii) *a is called* symmetric *if, and only if,*

$$a(v, w) = a(w, v) \quad \textit{for each} \quad (v, w) \in X^2.$$

(iii) *a is called* skew-symmetric *or* alternating *if, and only if,*

$$a(v, w) = -a(w, v) \quad \textit{for each} \quad (v, w) \in X^2.$$

(iv) *a is called* positive semidefinite *if, and only if,*

$$a(v, v) \geq 0 \quad \textit{for each} \quad v \in X.$$

(v) *a is called* positive definite *if, and only if, a is positive semidefinite and*

$$(a(v, v) = 0 \Leftrightarrow v = 0) \quad \textit{for each} \quad v \in X.$$

(vi) *a is called* coercive *or* elliptic *if, there exists $C_c > 0$ such that*

$$a(v, v) \geq C_c \|v\|^2 \quad \textit{for each} \quad v \in X.$$

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Linz, July 2011

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