The compatibility of admissible Minkowski decompositions

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The compatibility of admissible Minkowski decompositions

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Abstract

A recent study in deformation theory concerns toric deformations. This is related to the Minkowski decomposition of the cross cut of a cone that defines a toric variety. Instead of studying the deformations, the approach in this thesis is from a different perspective. Not the deformation theory, but the Minkowski decomposition will play a central role. These Minkowski decompositions will be described explicitly and investigated. The process of finding a decomposition of a given polytope in two summands is not described, as there is already a lot of literature concerning this problem. However, given two such decompositions a question that arises is whether these decompositions are compatible. First this question is answered in the particular case where the decompositions are actually decompositions of the same polytope. Once this problem is tackled, a more complex situation is discussed. This concerns the case where the decompositions come from two different cross cuts of a given cone.

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Loes Schenkeveld

Contents

Chapter 1

Introduction

Consider two polytopes P and Q. Their (Minkowski) sum is again a polytope.

$$R = P \oplus Q := \{p + q | p \in P, q \in Q\}$$

We call $P \oplus Q$ a decomposition of R. If a polytope R is not decomposable in different summands P and Q, then R is said to be *indecomposable*. Furthermore

$$P \oplus Q = P \oplus Q' \Longrightarrow Q = Q'.$$

Thus if R and one of the summands is known, the second summand is uniquely determined.

Consider a polytope P with

$$P = P_0 \oplus P_1 = Q_0 \oplus Q_1$$

such that $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ are two different decompositions of P.

The question that arises is whether the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ have a common refinement. This means that there exist polytopes S_i such that

$$\begin{aligned} P &= \bigoplus_{i=1}^{k} S_i \qquad \text{and} \\ P_0 &= \bigoplus_{i \in I_p} S_i \\ Q_0 &= \bigoplus_{i \in I_q} S_i \end{aligned}$$

with $I_p, I_q \subset \{1, \ldots, k\}$. If such a common refinement exists, the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ are said to be *compatible*. In this thesis a method is given to show whether two decompositions of the same polytope are compatible. The decision problem of whether a lattice polygon admits a Minkowski decomposition is *NP*-complete. Overall, doing computations with Minkowski sums is very complex, as described in [**?**]. Thus instead of computing the refinements of $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$, a different approach is used.

The second part of this thesis concerns a generalization of the concept of compatibility. Consider a cone σ. The intersection of σ and a hyperplane yields a polyhedron. Instead of comparing two decompositions of one polytope, the decompositions of two different polytopes are compared, where both polytopes are constructed as the intersection of σ and a hyperplane. Having a common refinement does not make sense for these decompositions. Two such decompositions are said to be compatible if there exists some isomorphism between both decompositions (or their refinement). A method for finding such an isomorphism is given in this thesis.

1.1 Background

An important application connected with this problem is multivariate polynomial factorization. For each multivariate polynomial f there exists a *Newton polytope* P_f. Let F be any field and let $f \in F[X_1, \ldots, X_n]$ be a multivariate polynomial with variables $X_1, \ldots, X_n$.

$$f = \sum c_{\boldsymbol{i}} \prod_{j=1}^{n} X_j^{i_j}$$

with $\boldsymbol{i} = (i_1, \ldots, i_n)$. The corresponding *Newton polytope* P_f is defined as the convex hull of all the points $\boldsymbol{i} \in \mathbb{Z}^n$ where $c_{i_1 \cdots i_n} \neq 0$. This polytope has integral vertices and is therefore an integral polytope, or a lattice polytope over lattice $\mathbb{Z}^n$.

$$P_f := \mathrm{Conv}(\{\boldsymbol{i} \in \mathbb{R}^n | c_{\boldsymbol{i}} \neq 0\})$$

Example 1.

Consider the polynomial

$$f = 1 + 2X + 3XY^2 + X^3Y$$

The corresponding Newton polygon P_f is given in Figure 1.1.
A first observation about Newton polytopes was made by Ostrowski in 1921 [14], [23].

Theorem 1.1 (Ostrowski)**.** *Let $f, g, h \in F[X_1, \ldots, X_n]$ such that $f = gh$. Then*

$$P_f = P_g \oplus P_h.$$

Proof. First observe that it is obvious that $P_f \subseteq P_g \oplus P_h$. Indeed, if $\boldsymbol{i}$ is a vertex of P_f then $c_{\boldsymbol{i}}(f) \neq 0$. We know that $f = gh$, thus there exist coefficients $c_{\boldsymbol{j}}(g)$ and $c_{\boldsymbol{k}}(h)$ that

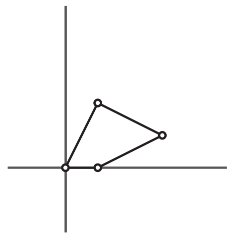

FIGURE 1.1: The Newton polytope P_f of f

are both nonzero with $\boldsymbol{j} + \boldsymbol{k} = \boldsymbol{i}$.

Conversely let $\boldsymbol{i}$ be a vertex of $P_g \oplus P_h$ and let $\boldsymbol{j} \in P_g$ and $\boldsymbol{k} \in P_h$ be points such that $\boldsymbol{i} = \boldsymbol{j} + \boldsymbol{k}$. To prove that this decomposition is unique, suppose that there exist points $\boldsymbol{j}' \in P_g$ and $\boldsymbol{k}' \in P_h$ such that

$$\boldsymbol{i} = \boldsymbol{j} + \boldsymbol{k} = \boldsymbol{j}' + \boldsymbol{k}'.$$

Then $\boldsymbol{i} = \frac{1}{2}(\boldsymbol{j}+\boldsymbol{k}) + \frac{1}{2}(\boldsymbol{j}'+\boldsymbol{k}') = \frac{1}{2}(\boldsymbol{j}+\boldsymbol{k}') + \frac{1}{2}(\boldsymbol{j}'+\boldsymbol{k})$. Hence $\boldsymbol{j}+\boldsymbol{k}'$ and $\boldsymbol{j}'+\boldsymbol{k} \in P_g \oplus P_h$ and $\boldsymbol{i}$ is a vertex of $P_g \oplus P_h$. This means that $\boldsymbol{j} + \boldsymbol{k}'$ and $\boldsymbol{j}' + \boldsymbol{k}$ must be equal. Thus

$$\begin{aligned} \boldsymbol{j} + \boldsymbol{k}' &= \boldsymbol{j}' + \boldsymbol{k} \quad \text{and} \\ \boldsymbol{j} + \boldsymbol{k} &= \boldsymbol{j}' + \boldsymbol{k}' \end{aligned}$$

which means that $\boldsymbol{k} = \boldsymbol{k}'$ and $\boldsymbol{j} = \boldsymbol{j}'$. Hence for every vertex $\boldsymbol{i}$ of $P_g \oplus P_h$ there exists a unique decomposition $\boldsymbol{j} + \boldsymbol{k}$ with $\boldsymbol{j} \in P_g$ and $\boldsymbol{k} \in P_h$. Furthermore, there is a unique term in the expansion of gh corresponding to $\boldsymbol{i} = \boldsymbol{j} + \boldsymbol{k}$. From the fact that $c_{\boldsymbol{j}}(g) \neq 0$ and $c_{\boldsymbol{k}}(h) \neq 0$ one can verify that $c_{\boldsymbol{i}} = c_{\boldsymbol{j}+\boldsymbol{k}} \neq 0$ in f, thus $\boldsymbol{i} \in P_f$. Hence, $P_f \supseteq P_g \oplus P_h$ and therefore we can conclude that $P_f = P_g \oplus P_h$. □

Theorem 1.2. *Let $f \in F[X_1, \ldots, X_n]$ be a nonzero polynomial not divisible by any X_i. If the Newton polytope is integrally indecomposable then f is absolutely irreducible.*

Proof. Let $f \in F[X_1, \ldots, X_n]$ be a nonzero polynomial not divisible by any X_i, with corresponding Newton polytope P_f. Suppose that P_f is integrally indecomposable. Moreover, let $f = gh$ be a nontrivial factorization of f. Since f is not divisible by any X_i, both factors g and h consist of at least two terms. From the previous theorem is known that there exist Newton polytopes P_g and P_h such that $P_f = P_g \oplus P_h$. But this is in contradiction with the assumption that P_f is integrally indecomposable. Thus, there does not exist a nontrivial factorization of $f = gh$, which means that f is absolutely irreducible. $\square$

The decision problem whether a polytope P_f is integrally decomposable is NP-complete, which is shown in [13]. In the literature some pseudo-polynomial time algorithms can be found for decomposing polygons. These algorithms are used for testing and factorizing polynomials via their Newton polynomials.

For a multivariate polynomial f consider the factorization. Let P_f be its Newton polytope and suppose that $P_f = P_{g_0} \oplus P_{h_0} = P_{g_1} \oplus P_{h_1}$. Assume that these two different decompositions are compatible and that g_0 and g_1 are factors of f and have a degree lower than h_0 respectively h_1. Now g_1 is a factor of h_0 and g_0 has a common factor with h_1.

These different compatible decompositions of a Newton polytope P_f give insight in the factorization of polynomial f, which is one of the motivations for this thesis.

1.2 Thesis outline

This thesis is about the compatibility of Minkowski decompositions of polytopes. The next chapter gives an introduction to the mathematics that is used in this thesis. Some definitions relating to polyhedra and geometry are given. Furthermore, the problem description that was the motivation of this thesis will be discussed.

In Chapter 3 the first case is discussed. The question how to determine whether two different decompositions of one polyhedron are compatible plays a central role in this chapter. In the first part this question is narrowed down to specific polytopes. Once a solution is found for this sub category, the main question can be answered.

This is elaborated in Chapter 4 where the definition of compatibility is extended to decompositions of different polytopes. A procedure is discussed how to determine this new notion of compatibility of two decompositions. In general this is very hard to determine. However, a lot of research is done for 2-dimensional case. This research is done on deformation theory, but in this chapter we will approach the topic from a combinatorial point. The fact that thorough research has been done for this topic is the reason that the 2dimensional case is treated in more detail.

Finally, Chapter 6 concerns the open problems related to this topic. The link to the toric varieties is made in this chapter and some recommendations on some interesting research topics are given.

Some definitions that will be used throughout this thesis and might lead to confusion are listed in the Appendix as a guidance.

Chapter 2

Preliminaries

In this section some definitions and theorems concerning the topic are discussed. These serve as a foundation for this thesis. Furthermore, some basic examples are treated as an introduction to the subject in order to get a good impression of the definitions. There is a lot of theory and background material. For more information, see [15], [16], [20], [25], [26], [27], [**?**], [32] or [34].

2.1 Preliminaries

Let $\mathbb{R}^d$ be the d-dimensional real vector space. For $\boldsymbol{x} \in \mathbb{R}^d$ we write $\boldsymbol{x} = (x_1, \ldots, x_d)$ but we will treat them as column vectors. It will be clear from the context when a vector should be treated as a row vector.

Definition 2.1. A set $S \subseteq \mathbb{R}^d$ is *convex* if for any two points $\boldsymbol{x}, \boldsymbol{y} \in S$ the line segment

$$[\boldsymbol{x}, \boldsymbol{y}] := \{\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y} | 0 \leq \lambda \leq 1\}$$

is contained in S.

The intersection of convex sets is again convex. The *convex hull* of a set S is the smallest convex set containing S and it is denoted as $\mathrm{Conv}(S)$. Hence, the line segment $[\boldsymbol{x}, \boldsymbol{y}]$ is the convex hull of $\boldsymbol{x}$ and $\boldsymbol{y}$.

Definition 2.2. The convex hull of a finite set of points S is called a *polytope*. A 2-dimensional polytope is sometimes referred to as a *polygon*.

Definition 2.3. A *hyperplane* is a set of points satisfying the equation $\boldsymbol{a}^\top \boldsymbol{x} = z$.

$$H := \{\boldsymbol{x} \in \mathbb{R}^d | \boldsymbol{a}^\top \boldsymbol{x} = z\}$$

for some vector $\boldsymbol{a} \in \mathbb{R}^d$ and $z \in \mathbb{R}$. The vector space $\mathbb{R}^d$ is the union of two *closed halfspaces* H^- and H^+ that are intersecting in H.

$$\begin{aligned} H^- &:= \{\boldsymbol{x} \in \mathbb{R}^d | \boldsymbol{a}^\top \boldsymbol{x} \leq z\} \\ H^+ &:= \{\boldsymbol{x} \in \mathbb{R}^d | \boldsymbol{a}^\top \boldsymbol{x} \geq z\} \end{aligned}$$

The sets satisfying the inequalities $\boldsymbol{a}^\top \boldsymbol{x} < z$ or $\boldsymbol{a}^\top \boldsymbol{x} > z$ are defined as open halfspaces.

Let $H \in \mathbb{R}^d$ be a hyperplane and $Q \in \mathbb{R}^d$ be a polytope. When Q is contained in one of the (closed) halfspaces (i.e. $Q \subseteq H^+$ or $Q \subseteq H^-$), then H is called a *supporting hyperplane* if

$$H \cap Q \neq \emptyset.$$

Definition 2.4. A *polyhedron* $Q \subseteq \mathbb{R}^d$ is the intersection of finitely many *closed halfspaces.*

$$Q = \{\boldsymbol{x} \in \mathbb{R}^d | A\boldsymbol{x} \leq \boldsymbol{z}\}$$

The rows $\boldsymbol{a}_i$ of the matrix A correspond to the halfspaces and $A\boldsymbol{x} \leq \boldsymbol{z}$ means that $\boldsymbol{a}_i \boldsymbol{x} \leq z_i$ for all i.

If a polyhedron Q is not bounded then there exist nonzero $\boldsymbol{y} \in \mathbb{R}^d$ such that for all $\boldsymbol{x} \in Q$ the set $\{\boldsymbol{x} + \lambda \boldsymbol{y} | \lambda \geq 0\}$ is contained in Q. This set of vectors is called a *ray* of Q.

A bounded polyhedron is the convex hull of finitely many points, hence it is a polytope as defined in Definition 2.2.

Definition 2.5. Let Q be a polyhedron. A set $F \subseteq Q$ is a *face* of Q if $F = \emptyset$ or $F = Q$ or if there exists a supporting hyperplane H such that

$$F = H \cap Q.$$

A nonempty face has dimension $0 \leq k \leq d$ and is referred to as a k-face of Q. The vertices of Q are its 0-faces and the set of all the vertices is denoted by $\mathrm{Vert}(Q)$. Moreover, if all vertices of a polyhedron are integral (so belong to lattice $\mathbb{Z}^d$), then it is called an *integral* or a *lattice polyhedron* with respect to the lattice $\mathbb{Z}^d$.

The addition of two convex sets can be determined and is defined as the *Minkowski sum.*

Definition 2.6. The Minkowski sum of two polyhedra P and Q is the set

$$P \oplus Q := \{\boldsymbol{x} + \boldsymbol{y} | \boldsymbol{x} \in P, \boldsymbol{y} \in Q\}.$$

Given this definition, the *Minkowski decomposition* of a polyhedron can be defined. As expected, this is the decomposition of a polyhedron Q in two (smaller) polyhedra Q_0

and Q_1 such that $Q_0 \oplus Q_1 = Q$. The polyhedra Q_0 and Q_1 are two *summands* of Q. Note that for a given integral polyhedron Q its summands are not necessarily integral. For example take Q as the unit square, which is the convex hull of the points $(0,0)$,$(1,0)$, $(0,1)$ and $(1,1)$. Now,

$$Q = \frac{1}{2}Q \oplus \frac{1}{2}Q.$$

Remark 1. If Q_0 and Q_1 are two summands of Q, then their Minkowski sum is a polyhedron

$$Q + t \cong Q,$$

such that t induces a shift over a vertex. To ensure that Q is equal to $Q_0 \oplus Q_1$ we will embed the summand in such way that $Q_0 = Q_0$ and $Q_1 = Q_1 + t$. Moreover, this means that $\mathbf{0} \in Q_1$. It will be clear from the context if a true summand Q_i or a polyhedron $Q_i + t$ is used and this will not be mentioned explicitly.

Definition 2.7. A Minkowski decomposition $Q = Q_0 \oplus Q_1$ in $\mathbb{R}^d$ is a *trivial decomposition* when one of the summ ands Q_0 or Q_1 is retrieved from Q by a shift over a vertex $\boldsymbol{x} \in \mathbb{R}^d$.

Theorem 2.8. *Let $Q_0 \oplus Q_1$ be a Minkowski decomposition of Q. Let F be a face of Q that is determined by the supporting hyperplane H of Q with $F \subset H^-$. Let F_i be a face of Q_i with supporting hyperplane $H_i//H$ such that $F_i = H_i \cap Q_i$ for $i = 0, 1$. Then $H = H_0 \oplus H_1$ and $F = F_0 \oplus F_1$. This is the unique decomposition of a face F into parts of Q_0 and Q_1.*

Proof. Let H be a supporting hyperplane of F and H_i the corresponding supporting hyperplanes of Q_i for $i = 0, 1$ such that the hyperplanes H_0, H_1 and H are parallel. Thus

$$\begin{aligned} H &= \{\boldsymbol{x} \in \mathbb{R}^d | \boldsymbol{a}^\top \boldsymbol{x} = z\} \\ H_i &= \{\boldsymbol{x} \in \mathbb{R}^d | \boldsymbol{a}^\top \boldsymbol{x} = z_i\} \end{aligned}$$

and $F = Q \cap H$ and $F_i = Q_i \cap H_i$ for $i = 0, 1$. Suppose that $Q \subset H^-$ which means that $\boldsymbol{a}^\top \boldsymbol{x} \geq z$ for all $\boldsymbol{x} \in Q$. Moreover, $\boldsymbol{a}^\top \boldsymbol{x}_i \geq z_i$ for all $\boldsymbol{x}_i \in Q_i$. Note that strict equality only holds if both $\boldsymbol{x} \in F$ and $\boldsymbol{x}_i \in F_i$. Observe that

$$\begin{aligned} z_0 + z_1 &= \boldsymbol{a}^\top \boldsymbol{y}_0 + \boldsymbol{a}^\top \boldsymbol{y}_1 \qquad \text{for } \boldsymbol{y}_i \in H_i \\ &= \boldsymbol{a}^\top (\boldsymbol{y}_0 + \boldsymbol{y}_1) \\ &= \boldsymbol{a}^\top \boldsymbol{y} \qquad \text{for } \boldsymbol{y} \in H \\ &= z \end{aligned}$$

The proof is split in two parts. In the first part we prove $F \subseteq F_0 \oplus F_1$. This is done by means of a proof by contradiction.

Suppose that there exists a point $\boldsymbol{u} \in F$ with $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1$ such that $\boldsymbol{u}_i \in Q_i$ but not both $\boldsymbol{u}_i \in F_i$. Without loss of generality, assume that $\boldsymbol{u}_0 \notin F_0$. Thus $z_0 > \boldsymbol{a}^\top \boldsymbol{u}_0$ and $z_1 \geq \boldsymbol{a}^\top \boldsymbol{u}_1$. This gives

$$z = z_0 + z_1 > \boldsymbol{a}^\top \boldsymbol{u}_0 + \boldsymbol{a}^\top \boldsymbol{u}_1 = \boldsymbol{a}^\top \boldsymbol{u}.$$

This implies that $\boldsymbol{u} \notin F$ which is a contradiction. Therefore the assumption was wrong and $F \subseteq F_0 \oplus F_1$.

We now prove that $F \supseteq F_0 \oplus F_1$. Let $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1$ such that the point $\boldsymbol{u}_i$ is contained in F_i for $i = 0, 1$. This gives,

$$z = z_0 + z_1 = \boldsymbol{a}^\top \boldsymbol{u}_0 + \boldsymbol{a}^\top \boldsymbol{u}_1 = \boldsymbol{a}^\top (\boldsymbol{u}_0 + \boldsymbol{u}_1) = \boldsymbol{a}^\top \boldsymbol{u}.$$

This implies that $\boldsymbol{u} \in H$ and since $\boldsymbol{u} \in Q$ we have that $\boldsymbol{u} \in F = QQ \cap H$. Thus $F \supseteq F_0 \oplus F_1$.

The combination of the two parts yields that $F = F_0 \oplus F_1$. □

In general there are infinitely many Minkowski decompositions $Q_0 \oplus Q_1$ of a polyhedron Q, but only finitely many of them are *admissible*. See [1] or [4] from Altmann about this definition.

Definition 2.9. A decomposition $Q = Q_0 \oplus Q_1$ is admissible if and only if for all the faces F of Q at most one of the summand faces F_i does not contain a lattice point.

Corollary 2.10. *It is enough to check admissibility on the vertices. Furthermore, if Q is a lattice polyhedron and $Q_0 \oplus Q_1$ is an admissible Minkowski decomposition, then the summands Q_0 and Q_1 are lattice polyhedra.*

From now on, each decomposition that is discussed refers to an admissible Minkowski decomposition. When a decomposition is not necessarily admissible, then this will be explicitly made clear in the context.

2.2 Admissible Minkowski decompositions

In order to get familiar with the Minkowski addition and decomposition, let $Q \subseteq \mathbb{R}^2$ be a polygon with n edges, which means that it also has n vertices. Thus we may write

$Q = \mathrm{Conv}(\boldsymbol{v}^0, \dots, \boldsymbol{v}^{n-1})$ where the vertices are numbered counterclockwise. Denote the edges of Q by the vector $\boldsymbol{d}^i = \boldsymbol{v}^{i+1} - \boldsymbol{v}^i$ for all $0 \leq i \leq n-1$ where i is taken modulo n.

Lemma 2.11. *Let D be the set of all the edge vectors of Q. The polygon Q is now completely determined by the (directed) edge set D together with an initial vertex $\boldsymbol{v}$ and initial direction $\boldsymbol{d} \in D$.*

Proof. To construct the polytope Q from its edge set, simply start in the initial vertex $\boldsymbol{v}$. Take the edges in their cyclic order starting with $\boldsymbol{d}$. This yields the corresponding convex polytope Q. □

If a vector $\boldsymbol{v} = (v_0, v_1)$ has $gcd(v_0, v_1) = 1$ then it is called a *primitive* vector. Denote the $gcd(d_0, d_1)$ of an edge $\boldsymbol{d}$ by δ. Now $\boldsymbol{f} = \frac{1}{\delta}\boldsymbol{d}$ is the primitive vector associated with the edge $\boldsymbol{d}$.

Given the edge set D, observe that Q is a polygon if and only if

$$\sum_{i=0}^{n-1} \boldsymbol{d}^i = \boldsymbol{0}.$$

In general, Q is a lattice polytope if and only if $\boldsymbol{v}^0 \in \mathbb{Z}^2$ and all $\boldsymbol{d} \in D$ are lattice vectors.

Minkowski addition Let P and Q be two polygons with n_P respectively n_Q edges (and vertices). The set of the edge vectors of the Minkowski sum $P \oplus Q$ is the union of the edge vectors of P and Q such that the vectors that are a positive multiple of each other are added. Obviously, if P and Q are integral polygons, then their Minkowski sum $P \oplus Q$ is an integral polygon as well.

Minkowski decomposition Let Q be a polygon with n edges and consider the edge set $D = \{\boldsymbol{d}^0, \dots, \boldsymbol{d}^{n-1}\}$. We know that $\sum_{i=0}^{n-1} \boldsymbol{d}^i = \boldsymbol{0}$. A polygon Q_0 is a summand of Q if and only if its edge set is $D_0 = \{\lambda_0 \boldsymbol{d}^0, \dots, \lambda_{n-1}\boldsymbol{d}^{n-1}\}$ such that $0 \leq \lambda_i \leq 1$ for all $i = 0, \dots, n-1$ and

$$\sum_{i=0}^{n-1} \lambda_i \boldsymbol{d}^i = \boldsymbol{0}.$$

Trivial summands are singletons and Q itself, with $\boldsymbol{\lambda} = \boldsymbol{0}$ respectively $\boldsymbol{\lambda} = \boldsymbol{1}$. Furthermore the Minkowski decomposition $\lambda Q \oplus (1-\lambda)Q$ is a trivial decomposition of Q for all $0 \leq \lambda \leq 1$. This corresponds to a vector $\boldsymbol{\lambda} = \lambda\boldsymbol{1}$ in the description above.

Definition 2.12. Let $Q \subseteq \mathbb{R}^d$ be a polyhedron for $d \geq 2$. A polyhedron Q is indecomposable if and only if for all the decompositions of $Q = Q_0 \oplus Q_1$ one of the summands equals λQ with $0 \leq \lambda \leq 1$.

A polytope is *admissibly indecomposable* if it does not have a nontrivial decomposition that is admissible. A lattice polytope is sometimes called *integrally indecomposable* if it does not have a nontrivial admissible decomposition. Throughout this thesis, we will use the term indecomposable when a polytope is admissibly indecomposable, unless stated otherwise.

A decomposition $\bigoplus_{i=0}^{s-1} S_i$ of a polyhedron Q such that all the summands S_i of Q are indecomposable is called a *prime decomposition* of Q. In congruence with Remark 1 we have that $S_i = S_i + t$ for $i = 0, \dots, s-1$ to ensure that $Q = \bigoplus_{i=0}^{s-1} S_i$. Furthermore, each summand S_i should not be a multiple of Q.

In terms of the edge set D, the polygon Q is indecomposable if there does not exist any vector $\boldsymbol{\lambda} = (\lambda_0, \dots, \lambda_{n-1})$ such that $\sum_i \lambda_i \boldsymbol{d}^i = \mathbf{0}$ and $\boldsymbol{\lambda} \neq \lambda \mathbf{1}$ for $0 \leq \lambda \leq 1$. We will continue with some examples for the Minkowski summands of polygons. Polygons with a relatively small number of vertices n are treated below.

2.2.1 $n = 3$

Let $Q \in \mathbb{R}^2$ be a triangle. The edge set of Q is $D = \{\boldsymbol{d}^0, \boldsymbol{d}^1, \boldsymbol{d}^2\}$ and $\boldsymbol{d}^0 + \boldsymbol{d}^1 + \boldsymbol{d}^2 = \mathbf{0}$. Trivially every triangle is indecomposable, as the equations $\boldsymbol{d}^0 + \boldsymbol{d}^1 + \boldsymbol{d}^2 = \mathbf{0}$ and $\alpha\boldsymbol{d}^0 + \beta\boldsymbol{d}^1 + \gamma\boldsymbol{d}^2 = \mathbf{0}$ imply that $\alpha = \beta = \gamma$. Otherwise two of the edges (and hence all three) are pairwise dependent.

2.2.2 $n = 4$

Let Q have four vertices. There are three possibilities.

- $\boldsymbol{d}^0 = \alpha\boldsymbol{d}^2$ and $\boldsymbol{d}^1 = \beta\boldsymbol{d}^3$

  In this case the polytope Q is a parallelogram. Hence $\alpha = \beta = -1$ and the decomposition consists of two line segments.

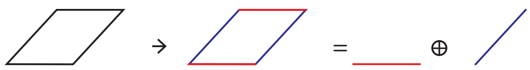

FIGURE 2.1: The decomposition of a parallelogram.

- $\boldsymbol{d}^0 = \alpha\boldsymbol{d}^2$ and $\boldsymbol{d}^1 \neq \beta\boldsymbol{d}^3$

  Without loss of generality let $\boldsymbol{d}^0 = \alpha\boldsymbol{d}^2$ and $\alpha < -1$. Thus Q is a trapezoid and it can be decomposed in a line segment and a triangle.

$$(\boldsymbol{v}^0, \boldsymbol{v}^1, \boldsymbol{v}^2, \boldsymbol{v}^3) = (\boldsymbol{v}^0, -\boldsymbol{v}^0) \oplus (\boldsymbol{v}^1, (\alpha+1)\boldsymbol{v}^2, \boldsymbol{v}^3)$$

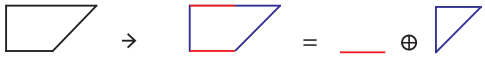

FIGURE 2.2: The decomposition of a trapezoid.

- $\boldsymbol{d}^0 \neq \alpha \boldsymbol{d}^2$ and $\boldsymbol{d}^1 \neq \beta \boldsymbol{d}^3$

  In the last case assume that the polytope Q has no parallel edges. Although there does not exist a line segment which is a summand of Q, there might be a decomposition, namely that of two triangles. Without loss of generality let $\boldsymbol{d}^0$ be the shortest edge. Now, let H_0 be the supporting hyperplane of $\boldsymbol{d}^0$ and H_0' the hyperplane parallel to H_1 such that $H_0' \cap Q$ has the same length as $\boldsymbol{d}^0$, which means that they have the same edge vector. This yields a triangle Q_0, which is a summand of Q. The other (parts of the) edges of Q that are not used in this triangle form the edge set of another triangle, which is a summand Q_0 such that $Q = Q_0 \oplus Q_1$. Figure 2.3 shows how this is done.

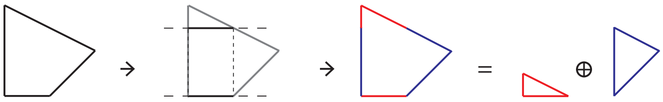

FIGURE 2.3: The decomposition of a polygon with 4 vertices without parallel edges.

Keep in mind that the decompositions that are described above are not always admissible decompositions. When Q is an integral polytope, the decomposition is admissible if and only if the summands Q_0 and Q_1 are integral polytopes.

Theorem 2.13. *Let Q be an integral polygon such that $Q_0 \oplus Q_1$ is a decomposition of Q. The summand Q_0 has an edge set $D_0 = \{\lambda_0 \boldsymbol{d}^0, \ldots, \lambda_{n-1} \boldsymbol{d}^{n-1}\}$ such that $\sum \lambda_i \boldsymbol{d}^i = \boldsymbol{0}$ and $\boldsymbol{\lambda} \neq \lambda \boldsymbol{1}$. Now $Q_0 \oplus Q_1$ is an admissible decomposition if and only if $\lambda_i \delta_i \in \mathbb{Z}$ for all $i = 0, \ldots, n-1$.*

Proof. By definition $Q_0 \oplus Q_1$ is an admissible decomposition if and only if both summands are integral summands. A summand Q_0 is integral if and only if the edges are integral. For each edge we have that $\lambda \boldsymbol{d} = \lambda \delta \boldsymbol{f}$ with $\delta = gcd(d_0, d_1)$ and $\boldsymbol{f} = \frac{1}{\delta} \boldsymbol{d}$. The vector $\boldsymbol{f}$ is primitive by definition, which means that $\lambda \delta \boldsymbol{f}$ is integral if and only if $\lambda \delta$ is integral. This proves the theorem. $\square$

2.2.3 Generalization of decompositions

Polygons with more than four vertices might be decomposable in more than two summands, which is denoted by $\bigoplus Q_i$. Let $Q = \bigoplus_{i=0}^{s-1} S_i$ be a prime decomposition of Q. Thus, every summand S_i is an indecomposable summand. Note that this decomposition does not have to be a unique prime decomposition. To see this, consider the (lattice)

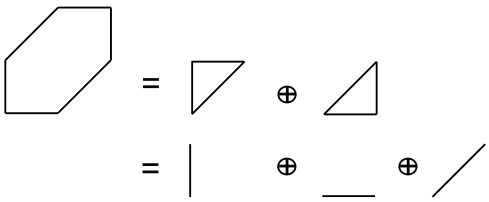

FIGURE 2.4: Two prime decompositions of a hexagon

hexagon that is shown in Figure 2.4. The hexagon has integral vertices and it is the Minkowski sum of two triangles. But the same hexagon can be decomposed in three line segments. As we saw earlier, every line segment and triangle is an indecomposable polygon. Hence, these two decompositions are different prime decompositions of the same polygon.

Definition 2.14. Two decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ are *compatible* if and only if both decompositions can be further decomposed in the same prime decomposition, i.e. if they have a *common refinement*.

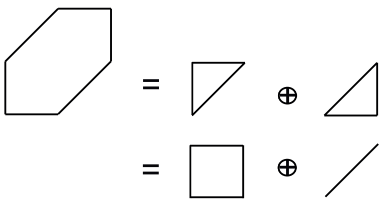

FIGURE 2.5: Two decompositions which are not compatible

Figure 2.5 shows an example of two decompositions of a hexagon that are not compatible. The first decomposition is actually a prime decomposition as it consists of two triangles. The second decomposition can be further decomposed in three line segments but there does not exist a refinement of this decomposition that exists of two triangles. Thus, both decompositions yield the same hexagon but they do not have a common refinement.

To see that there exist many polyhedra that do not have a unique prime decomposition, consider the following situation: Let Q be a polygon with $2n$ edges such that $P \oplus -P$ is an admissible decomposition of Q where P is a polygon with n edges. The polygon $Q = \mathrm{Conv}(\boldsymbol{v}^0, \dots, \boldsymbol{v}^{2n-1})$ is a centrally symmetric polygon. This means that the polygon P has n edges $\boldsymbol{d}^i$ for $i = 0, \dots, n-1$ and $-P$ is defined as the mirrored polygon with the edge set $\{-\boldsymbol{d}^i | i = 0, \dots, n-1\}$. The edge set of Q is the set $\{\boldsymbol{d}^0, \dots, \boldsymbol{d}^{n-1}, -\boldsymbol{d}^0, \dots, -\boldsymbol{d}^{n-1}\}$. A second prime decomposition can automatically be derived from this edge set:

$$Q = P \oplus -P = \bigoplus_{i=0}^{n-1} S_i$$

where the polygons S_i are the line segments $[\boldsymbol{v}^i, \boldsymbol{v}^{i+1}]$ for $0 \leq i \leq n-1$ counted modulo n. From this can be concluded that if P is not a centrally symmetric polygon, then Q has at least two different refinements.

2.3 Further definitions

We will continue with some more definitions. More background information can be found in [7], [8], [10], [11], [24], [31], [33] or [34].

Definition 2.15. Let $\boldsymbol{a}^0, \dots, \boldsymbol{a}^{k-1} \in \mathbb{Q}^d$ be k pairwise linearly independent vectors. A *convex polyhedral cone* σ is the *positive hull* of all $\boldsymbol{a}^i$.

$$\sigma := \{\sum_{i=0}^{k-1} \lambda_i \boldsymbol{a}^i | \lambda_i \geq 0\}$$

Moreover, a cone σ is a *strongly convex polyhedral cone* if $\boldsymbol{u} \in \sigma$ implies that $-\boldsymbol{u} \notin \sigma$.

For any two vectors $\boldsymbol{u}, \boldsymbol{v} \in \sigma$ and scalars $\alpha, \beta \geq 0$ the vector $\{\alpha \boldsymbol{u} + \beta \boldsymbol{v}\}$ is contained in σ. Every convex polyhedral cone has a unique minimal set of *generators* $\{\boldsymbol{a}^0, \dots, \boldsymbol{a}^{k-1}\}$ (modulo multiplication with a positive scalar). The set that consists of primitive generators is therefore unique. Moreover, the *Hilbert basis* E of σ is defined as the set of integer vectors such that every generator of σ is a nonnegative combination of these vectors. For strongly convex polyhedral cones this is a finite set ánd $\boldsymbol{a} \in E$ for each generator $\boldsymbol{a}$ of σ.

Example 2. *As an example consider the convex hull of the vectors* $(0,1)$ *and* $(2,1)$. *The (primitive) generators are exactly these two vectors, while the Hilbert basis consists of three vectors:* $(0,1)$, $(2,1)$ *and* $(1,1)$.

Throughout this report a *cone* refers to a (finitely generated) rational strongly convex polyhedral cone.

In Definition 2.4 a polyhedron Q is given in terms of inequalities. The vertex representation of Q is the Minkowski sum of the convex hull of the vertex set $\mathrm{Vert}(Q)$, which is a polytope, and the set of the rays of Q. This set of rays is the set of primitive generators and it is defined as the *characteristic cone* of the polyhedron Q.

Definition 2.16. The characteristic cone of a polyhedron Q is the set of all rays of Q.

$$\mathrm{Char}(Q) := \{\boldsymbol{y} \in \mathbb{R}^d | \boldsymbol{x} + \lambda \boldsymbol{y} \in Q \text{ for all } \boldsymbol{x} \in Q; \lambda \geq 0\}.$$

The vertex representation of Q is denoted by

$$Q = \mathrm{Conv}(\mathrm{Vert}(Q)) \oplus \mathrm{Char}(Q).$$

Corollary 2.17. *From Definition 2.9 and 2.7 it follows that a nontrivial Minkowski decomposition is admissible if and only if both summands have the same characteristic cone.*

Definition 2.18. Let σ be a cone that is generated by the vectors $\boldsymbol{a}^i$. The dual cone $\sigma^\vee$ of σ is the set

$$\sigma^\vee := \{\boldsymbol{r} \in \mathbb{Q}^d | \langle \sigma, \boldsymbol{r} \rangle \geq 0\}.$$

Observe that the dual cone is indeed a cone. Moreover, $(\sigma^\vee)^\vee = \sigma$. Hence, there are two ways of describing a cone; in terms of its generators and in terms of a set of inequalities.

As before, a polyhedron Q can be described as the intersection of finitely many halfspaces, $Q = \{\boldsymbol{x} \in \mathbb{R}^d | A\boldsymbol{x} \leq \boldsymbol{z}\}$. This can be rewritten as $(A| - \boldsymbol{z})\binom{\boldsymbol{x}}{1} \leq \boldsymbol{0}$. Now the same polyhedron can be given as the positive set of vectors $\boldsymbol{u}^0, \ldots, \boldsymbol{u}^{m-1}$.

$$Q = \{\sum_{i=0}^{m-1} \lambda_i \boldsymbol{u}^i | \lambda_i \geq 0\}$$

This is a polyhedral cone. Therefore a polyhedron Q can be described as a cone $\sigma(Q) \in \mathbb{R}^{d+1}$. Each vertex $\boldsymbol{v} \in Q$ corresponds to a vector $(\boldsymbol{v}, 1) \in \sigma(Q)$, while each ray $\boldsymbol{r} \in Q$ gives a corresponding vector $(\boldsymbol{r}, 0) \in \sigma(Q)$. The cone $\sigma(Q)$ is generated by the vectors $(\boldsymbol{v}, 1)$ and $(\boldsymbol{r}, 0)$. The initial polyhedron Q is exactly the intersection of $\sigma(Q)$ and the hyperplane $H_1 = \{x \in \mathbb{R}^{d+1} | (\boldsymbol{0}, 1)^\top \boldsymbol{x} = 1\}$.

The first decomposition of the hexagon that is given in Figure 2.5 is represented by the columns in Equation (2.1).

$$\sigma(Q_6) = \begin{pmatrix} 0 & 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.1}$$

where $\sigma(Q_6)$ is a hexagon at height If the polyhedron 1. When the polyhedron Q is bounded (i.e. it is a polytope and $\mathrm{Char}(Q) = \mathbf{0}$) then the cone $\sigma(Q)$ is generated by vectors of the form $(\boldsymbol{v}, 1)$ and the last row is usually omitted in the matrix representation.

Instead of using $\mathbb{Q}^d$ to embed the cones and polyhedra, we are interested in a *lattice* embedding. This means that not only the rays have primitive generators, but the vertices are all lattice points.

Definition 2.19. Let $\boldsymbol{b}^0, \dots, \boldsymbol{b}^{n-1} \in \mathbb{R}^n$ be linearly independent vectors. The set

$$N := \{\sum_{i=0}^{n-1} \lambda_i \boldsymbol{b}^i | \lambda_i \in \mathbb{Z}\}$$

is defined as a lattice with basis $\{\boldsymbol{b}^0, \dots, \boldsymbol{b}^{n-1}\}$.

The dual lattice is

$$N^* := M := \{\boldsymbol{u} \in \mathbb{R}^n | \boldsymbol{u}^\top \boldsymbol{b} \in \mathbb{Z} \ , \forall \boldsymbol{b} \in N\}.$$

The real vector space from a lattice N is denoted by $N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$. Furthermore $\mathbb{Z}^n \subset \mathbb{R}^n$ is known to be the *standard lattice*. Given a lattice N, a polytope Q is called a lattice polytope if and only if all vertices are lattice points. Moreover, we say that the cone is a lattice cone if it is generated by lattice vectors. This is denoted as $\sigma \subseteq N_\mathbb{R}$. Consequently, we have for the dual cone $\sigma^\vee \subseteq M_\mathbb{R}$.

As mentioned earlier one can describe a polyhedron $Q \in \mathbb{R}^d$ in terms of a cone in $\sigma(Q) \in \mathbb{R}^{d+1}$, where the intersection of $\sigma(Q)$ and the hyperplane H_1 yields the polyhedron Q. In general, the intersection of σ and a hyperplane yields a polyhedron.

Definition 2.20. Let $\sigma \subseteq N_\mathbb{R}$ be a cone and fix a vector $\boldsymbol{r} \in M$. Now

$$H_{\boldsymbol{r}} := \{\boldsymbol{x} \in N_\mathbb{R} | \boldsymbol{r}^\top \boldsymbol{x} = 1\}$$

is the hyperplane from vector $\boldsymbol{r}$ at height 1. The *cross cut* of σ is the intersection of σ and this hyperplane.

$$Q_{\boldsymbol{r}} := \sigma \cap \{\boldsymbol{x} \in N_\mathbb{R} | \boldsymbol{r}^\top \boldsymbol{x} = 1\} \tag{2.2}$$

Whenever the choice of $\boldsymbol{r}$ is clear from the context, we will use Q instead of $Q_{\boldsymbol{r}}$.

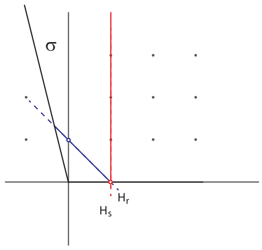

FIGURE 2.6: Construction of $Q_{\boldsymbol{r}}$ and $Q_{\boldsymbol{s}}$.

Theorem 2.21. *Fix a polyhedral cone $\sigma \subseteq \mathbb{R}^{d+1}$ and a hyperplane $H_{\boldsymbol{r}}$ as defined in Definition 2.20. Now $Q_{\boldsymbol{r}} = Q$ is the corresponding cross cut, which is a polyhedron. The cone σ can be retrieved from Q if and only if $\boldsymbol{r} \in \sigma^\vee \cap M$.*

Proof. For each vector $\boldsymbol{r} \in \sigma^\vee \cap M$ we have that $\boldsymbol{r}^\top \boldsymbol{b}^i \in \mathbb{Z}$ for all lattice generators $\boldsymbol{b}^i$. Moreover, $\boldsymbol{r}^\top \boldsymbol{a}^j \geq 0$ for all generators $\boldsymbol{a}^j$ of σ. The polyhedron Q can be decomposed as $\mathrm{Conv}(\mathrm{Vert}(Q)) \oplus \mathrm{Char}(Q)$, where $\mathrm{Vert}(Q)$ is the set of vertices of Q and $\mathrm{Char}(Q)$ is the set of its rays. The vertices of Q correspond to the rays $\boldsymbol{a}^j$ such that $\boldsymbol{r}^\top \boldsymbol{a}^j > 0$, which means that $\boldsymbol{r}^\top \boldsymbol{a}^j \geq 1$. Hence, the vertices of Q are $\boldsymbol{v}^j = \frac{\boldsymbol{a}^j}{\boldsymbol{r}^\top \boldsymbol{a}^j}$. Furthermore, the unbounded part of Q is exactly the set of rays for which $\boldsymbol{r}^\top \boldsymbol{a}^j = 0$.

$$\mathrm{Char}(Q) = \sigma \cap \{\boldsymbol{x} \in N_{\mathbb{R}} | \boldsymbol{r}^\top \boldsymbol{x} = 0\}$$

The generators $\boldsymbol{a}^j$ such that $\boldsymbol{r}^\top \boldsymbol{a}^j < 0$ do not contribute to any vertex or ray of Q. This means that the cone σ can only be retrieved from Q if and only if $\boldsymbol{r} \in \sigma^\vee$. $\square$

Consider a small example in the 2-dimensional standard lattice $\mathbb{Z}^2$. Let $\sigma = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 4 \end{smallmatrix}\right)$ and the dual cone $\sigma^\vee = \left(\begin{smallmatrix} 0 & 4 \\ 1 & 1 \end{smallmatrix}\right)$. This notation means that σ is generated by the rays $(1,0)$ and $(-1,4)$. Let $\boldsymbol{r} = (1,1) \in \sigma^\vee$ and $\boldsymbol{s} = (1,0) \notin \sigma^\vee$. The cross cut $Q_{\boldsymbol{r}}$ is the convex hull of the vertices $(-\frac{1}{3}, \frac{4}{3})$ and $(0,1)$, which is a line segment. The polyhedron $Q_{\boldsymbol{s}}$ consists of the vertex $(0,1)$ together with the ray $(0,1)$. The ray $(-1,4)$ of σ does not come back in $Q_{\boldsymbol{s}}$ as shown in Figure 2.6.

2.4 Problem description

Define $N \cong \mathbb{Z}^d$ as a lattice and let M be its dual lattice. Now $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$ and $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R}$ are real vector spaces of dimension d. A polyhedron Q is a lattice polytope if all its vertices $\boldsymbol{v}$ are lattice points. Throughout this section a polyhedron Q is considered to be the cross cut of a cone σ and a vector $\boldsymbol{r} \in \sigma^\vee \cap M$. Furthermore, the vector $\boldsymbol{r}$ is chosen to be a lattice vector such that it has no common divisor, i.e. $\boldsymbol{r}$ is a *primitive* vector. In general $Q \subseteq N_\mathbb{R}$ is not a lattice polyhedron. However, the corresponding cone $\sigma \subseteq N_\mathbb{R}$ has only lattice generators and is therefore a lattice cone.

Let $\sigma \subseteq N_\mathbb{R}$ be a polyhedral cone and fix two vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ such that they are both contained in $\sigma^\vee$. Determine the polyhedra $Q_{\boldsymbol{r}}$ and $Q_{\boldsymbol{s}}$ and fix the admissible decompositions $P_0 \oplus P_1$ for $Q_{\boldsymbol{r}}$ and $Q_0 \oplus Q_1$ for $Q_{\boldsymbol{s}}$. The questions that arise are

Are the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ compatible?
What does it mean for those decompositions to be compatible?

In order to answer these questions, it has to be subdivided into smaller problems. Mainly it consist of two cases. The first part treats the case where $\boldsymbol{r} = \boldsymbol{s}$, which means that the two decompositions are decompositions of the same polytope $Q_{\boldsymbol{r}} = Q_{\boldsymbol{s}}$. If $\boldsymbol{r} \neq \boldsymbol{s}$ then the problem becomes more complex. The notion of being compatible gets a different meaning, which will be discussed in the Chapter 4. The procedure to check compatibility for $\boldsymbol{r} = \boldsymbol{s}$ is described in the next chapter.

Chapter 3

Equal cross cuts

From now on let N be the standard lattice of degree d which is $N \cong \mathbb{Z}^d$ and therefore $M \cong \mathbb{Z}^d$. This means that $N_\mathbb{R} \cong \mathbb{R}^d$ and the lattice points are integral. Let $\sigma \subseteq N_\mathbb{R}$ be a cone generated by k lattice vectors $\boldsymbol{a}^i \in N$ for all $i = 0, \dots, k-1$. Fix a vector $\boldsymbol{r} \in \sigma^\vee \cap M$ and determine the cross cut $Q_{\boldsymbol{r}}$. This chapter concerns the compatibility of two admissible decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ of the polyhedron $Q_{\boldsymbol{r}}$. Before a solution is given, some useful sets are defined. Thereafter, the solution will be treated in three parts. In the first part $Q_{\boldsymbol{r}}$ is assumed to be a lattice polytope. After that a solution for general polytopes will be given. Finally, a general solution will be discussed.

3.1 The cross cut $Q_{\boldsymbol{r}}$ and its summands

Let $\sigma \subseteq N_\mathbb{R}$ be a cone such that the generators $\boldsymbol{a}^i$ are lattice vectors and fix some primitive vector $\boldsymbol{r} \in \sigma^\vee \cap M$ for which the cross cut $Q_{\boldsymbol{r}} = Q$ of σ is bounded. Let n be the number of edges of Q. An edge $[\boldsymbol{u}, \boldsymbol{v}]$ is denoted by the vector $\boldsymbol{d} = \boldsymbol{v} - \boldsymbol{u} \in N$. The edges of Q correspond to vectors $\boldsymbol{d}^i$ for $i = 0, \dots, n-1$. Define a sign vector $\boldsymbol{\varepsilon}(F) \in \{-1, 0, 1\}^n$ for every 2-face F of Q such that

$$\varepsilon_i(F) = \begin{cases} \pm 1 & \text{if } \boldsymbol{d}^i \text{ is an edge of } F \\ 0 & \text{else} \end{cases}$$

and the edges are oriented as a cycle along the boundary of F. There are two such orientations but they are each others negative. Both of them are a suitable choice. If the choice for the face F is obvious from the context, then we will leave this out and simply write $\boldsymbol{\varepsilon}$.

Observe that

$$\sum_{i=0}^{n-1} \varepsilon_i \boldsymbol{d}^i = \mathbf{0}$$

for every 2-face F of Q. This property is used to construct the vector space $\mathcal{V}(Q)$.

From [6] we now come to the following definition.

Definition 3.1. Let Q be a polytope that is contained in N. The vector space of the Minkowski summands $\mathcal{V}(Q)$ is defined as

$$\mathcal{V}(Q) = \{(t_0, \ldots, t_{n-1}) | \sum_{i=0}^{n-1} t_i \varepsilon_i \boldsymbol{d}^i = 0 \text{ for every 2-face } F \text{ of } Q\}. \tag{3.1}$$

Furthermore, the Minkowski summand cone

$$\mathcal{C}(Q) := \mathcal{V}(Q) \cap \mathbb{R}^n_{\geq 0} \tag{3.2}$$

is a cone in the vector space $\mathcal{V}(Q)$.

Theorem 3.2. *Every point $\boldsymbol{t} \in \mathcal{C}(Q)$ with $t_i \leq 1$ for all i corresponds to a Minkowski summand $Q_{\boldsymbol{t}}$ of Q which consists of the edges $t_i \boldsymbol{d}^i$. Moreover, every summand of Q has a corresponding vector in $\mathcal{C}(Q)$. Thus, $Q_{\boldsymbol{t}} \oplus Q_{\mathbf{1}-\boldsymbol{t}} = Q$.*

Proof. Let Q be a polytope with n edges such that $P_0 \oplus P_1$ is an admissible decomposition. Let $\boldsymbol{t} \in [0,1]^n$ be the vector where t_i is the fraction of the edge $\boldsymbol{d}^i$ that is used in the summand P_0. By definition the edges of P_0 are $t_i \boldsymbol{d}^i$ and therefore $\sum \varepsilon_i(F_0) t_i \boldsymbol{d}^i = \mathbf{0}$ for every 2-face F_0 of P_0. This yields that $\sum \varepsilon_i(F_0) \boldsymbol{d}^i = \mathbf{0}$ for every 2-face F_0 of P_0. But this yields $\sum \varepsilon_i(F) \boldsymbol{d}^i = \mathbf{0}$ for every 2-face F of Q. This is exactly the definition of the elements in $\mathcal{C}(Q)$.

Now let $\boldsymbol{t} \in \mathcal{C}(Q)$ such that $t_j \leq 1 \ \forall_i$. We want to show that there exists a corresponding polytope $Q_{\boldsymbol{t}}$ that is a summand of Q.

Without loss of generality one can assume that $\mathbf{0}$ is a vertex of Q. Let $\boldsymbol{v} \neq \mathbf{0}$ be a vertex of Q. There exists a walk from $\mathbf{0}$ to $\boldsymbol{v}$, using some edges of Q:

$$\boldsymbol{v} = \sum \lambda_i \boldsymbol{d}^i$$

with $\lambda_i \in \mathbb{Z} \ \forall_i$. The walk from $\mathbf{0}$ to $\boldsymbol{v}$ is a walk through the 1-skeleton of Q, which is the graph consisting of the vertices and edges of Q. Moreover, the polytope is embedded in $\mathbb{R}^d$ which means that all walks from $\mathbf{0}$ to $\boldsymbol{v}$ are homeomorphic. This means that for any walk from $\mathbf{0}$ to $\boldsymbol{v}$ there exists a continuous deformation to the chosen path. Therefore any walk can be chosen.

The walk can be split in two walks in the following way:

$$\begin{aligned}\boldsymbol{v} &= \sum \lambda_i \boldsymbol{d}^i \\ &= \sum \lambda_i (1 - t_i + t_i) \boldsymbol{d}^i \\ &= \sum \lambda_i t_i \boldsymbol{d}^i + \sum \lambda_i (1 - t_i) \boldsymbol{d}^i\end{aligned}$$

Define the vertices $\boldsymbol{v_t} := \sum \lambda_i t_i \boldsymbol{d}^i$ and $\boldsymbol{v}'_{\boldsymbol{t}} := \lambda_i (1 - t_i) \boldsymbol{d}^i$. Note that these vertices only depend on the vector $\boldsymbol{t}$ and not on the particular choice of the walk to $\boldsymbol{v}$. The edges $t_i \boldsymbol{d}^i$ are used to reach vertex $\boldsymbol{v_t}$. Since $\boldsymbol{t}$ is an element of the cone $\mathcal{C}(Q)$ we know that all its entries are nonnegative and smaller than or equal to 1. Consequently, the vector $\boldsymbol{1} - \boldsymbol{t}$ is also contained in $\mathcal{C}(Q)$. By construction we have that $\boldsymbol{v_t} + \boldsymbol{v}'_{\boldsymbol{t}} = \boldsymbol{v}$.

For each vertex $\boldsymbol{v}$ of Q such a walk can be constructed, together with the corresponding vertices $\boldsymbol{v_t}$ and $\boldsymbol{v}'_{\boldsymbol{t}}$. A polytope $Q_{\boldsymbol{t}}$ can be constructed as the convex hull of all the vertices $\boldsymbol{v_t}$. Similarly, the convex hull of the vertices $\boldsymbol{v}'_{\boldsymbol{t}}$ yields a polytope $Q'_{\boldsymbol{t}}$. The polytope $Q_{\boldsymbol{t}}$ has exactly all $t_i \boldsymbol{d}^i$ as its edges, with corresponding vector $\boldsymbol{t}$. Moreover, $Q'_{\boldsymbol{t}}$ corresponds to the vector $\boldsymbol{1} - \boldsymbol{t}$ and $Q = Q_{\boldsymbol{t}} \oplus Q'_{\boldsymbol{t}}$.
All the summands of Q can be constructed this way and the theorem is proven. □

Remark 2. The vector $\boldsymbol{1} \in \mathcal{C}(Q)$ corresponds to the polytope Q itself. For each summand P of Q there is a corresponding *edge vector* and this vector will be denoted by $\boldsymbol{t}_P \in \mathcal{C}(Q)$.

This yields the fact that the Minkowski addition now results in a vector summation in $\mathcal{V}(Q)$ or $\mathcal{C}(Q)$. Thus for two summands $Q_{\boldsymbol{t}}$ and $Q_{\boldsymbol{s}}$ we get

$$Q_{\boldsymbol{t}} \oplus Q_{\boldsymbol{s}} = Q_{\boldsymbol{t}+\boldsymbol{s}}.$$

3.2 The compatibility of lattice polytopes that have no interior lattice points on the edges

Let $\sigma \subseteq N_{\mathbb{R}}$ be a cone such that the cross cut $Q_{\boldsymbol{r}} = Q$ is a lattice polytope with n edges. Moreover, assume that no edge contains interior lattice points, i.e. the edges $\boldsymbol{d}^i \in \mathbb{Z}^d$ are primitive.
Hence Q is a lattice polytope and therefore the summands of any admissible decomposition are lattice polytopes. Thus $\boldsymbol{t}_P \in \{0,1\}^n$ for each admissible summand P of $Q_{\boldsymbol{r}}$.

Consider two decompositions of Q:

$$\begin{aligned}Q &= P_0 \oplus P_1 \text{ and} \\ Q &= Q_0 \oplus Q_1.\end{aligned}$$

Recall that these decompositions are compatible if they have a common refinement. Let Q have a prime decomposition which is denoted as

$$Q = S_0 \oplus \cdots \oplus S_q = \bigoplus_{i=0}^{q-1} S_i$$

where S_i is an indecomposable polytope fore $i = 0, \ldots, q-1$. Construct the vector space $\mathcal{V}(Q)$ as in defined in Definition 3.1.

$$\mathcal{V}(Q) = \left\{ \boldsymbol{t} \Big| \sum_{i=0}^{n-1} t_i \varepsilon_i \boldsymbol{d}^i = 0 \text{ for every 2-face } F < Q \right\}$$

Construct the cone $\mathcal{C}(Q) = \mathcal{V}(Q) \cap \mathbb{R}^d_{\geq 0}$. From Theorem 3.2 we know that each ray $\boldsymbol{t}$ of $\mathcal{C}(Q)$ is an edge vector of a (multiple of a) Minkowski summand of Q.

Lemma 3.3. *Let $\boldsymbol{t}_S \in \mathcal{C}(Q)$ be the edge vector of an indecomposable summand S. There does not exist any $\boldsymbol{t}'_S$ and $\boldsymbol{t}''_S$ that are both nonzero and not equal to $\boldsymbol{t}_S$ such that $\boldsymbol{t}_S = \boldsymbol{t}'_S + \boldsymbol{t}''_S$ for $\boldsymbol{t}'_S, \boldsymbol{t}''_S \in \mathcal{C}(Q)$.*

Proof. This directly follows from the fact that the summand S is indecomposable and therefore $\boldsymbol{t}_S$ is a primitive generator of $\mathcal{C}(Q)$. □

Definition 3.4. The *Schur product* (or Hadamard or pointwize product) of two vectors of the same length is obtained by the entry-wise multiplication. For two given vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ their Schur product is denoted by $\boldsymbol{x} \circ \boldsymbol{y} = (x_0 y_0, x_1 y_1, \ldots)$.

The edge vector of the polytope Q is by definition the all one vector $\mathbf{1}$. Denote the edge vectors of P_i and Q_i by $\boldsymbol{t}_{P_i}$ respectively $\boldsymbol{t}_{Q_i}$. Observe that $\boldsymbol{t}_{P_0} + \boldsymbol{t}_{P_1} = \mathbf{1}$ and $\boldsymbol{t}_{Q_0} + \boldsymbol{t}_{Q_1} = \mathbf{1}$.

Theorem 3.5. *Two decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ of Q (which is a lattice polytope without lattice points on the edges) have a common refinement if and only if $\boldsymbol{t}_{P_0} \circ \boldsymbol{t}_{Q_0} \in \mathcal{C}(Q)$. This means that*

$$\sum_{i=0}^{n-1} (t_{P_0})_i (t_{Q_0})_i \varepsilon_i \boldsymbol{d}^i = \mathbf{0} \tag{3.3}$$

for every 2-face F of Q.

One can easily verify that this does not work for a lattice polytope Q that does have interior lattice points on the edges.

Example 3. *To show that this theorem does not work for polyhedra that have lattice points on the edges, consider the polytope Q with the points* $(0,0), (3,0), (2,1)$ *and* $(0,1)$.

This polytope has the edge set

$$D = \{(3,0), (-1,1), (-2,0), (0,-1)\}.$$

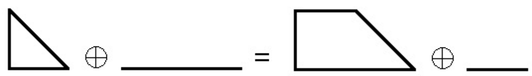

FIGURE 3.1: Two decompositions of Q

The polytope has two decompositions, given in figure 3.1. Furthermore, we have

$$\begin{aligned} t_{P_0} &= (\frac{1}{3}, 1, 0, 1) \\ t_{Q_0} &= (\frac{2}{3}, 1, \frac{1}{2}, 1). \end{aligned}$$

This leads to the following equation:

$$\begin{aligned} \sum_{i=0}^{3} (t_{P_0})_i (t_{Q_0})_i \varepsilon_i \boldsymbol{d}^i &= \frac{2}{9}(3,0) + (0,1) + (-1,-1) \\ &\neq \mathbf{0}. \end{aligned}$$

The proof of the theorem is given in the following section.

3.2.1 Proof of Theorem 3.5

Let $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ be two admissible decompositions of the lattice polytope Q. The polytopes P_0 and Q_0 are lattice polytopes with corresponding edge vectors, and therefore $\boldsymbol{t}_{P_0}, \boldsymbol{t}_{Q_0} \in \{0,1\}^n$. Consequently, the vector $t_{P_0} \circ t_{Q_0}$ only has binary entries, that are nonzero where $\boldsymbol{t}_{P_0}$ and $\boldsymbol{t}_{Q_0}$ both have nonzero entries.

Remark 3. If Equation (3.3) holds, then it holds for any pair $\boldsymbol{t}_{P_i}$ and $\boldsymbol{t}_{Q_j}$, as the Schur product $\boldsymbol{t}_{P_0} \circ \boldsymbol{t}_{Q_0}$ can be written in terms of any pair $\boldsymbol{t}_{P_i}, \boldsymbol{t}_{Q_j}$.

This is shown by the following computation.

$$\begin{aligned} \boldsymbol{t}_{P_0} \circ \boldsymbol{t}_{Q_0} &= \boldsymbol{t}_{P_0} \circ (\mathbf{1} - \boldsymbol{t}_{Q_1}) = \boldsymbol{t}_{P_0} - \boldsymbol{t}_{P_0} \circ \boldsymbol{t}_{Q_1} \\ &= \mathbf{1} - \boldsymbol{t}_{P_1} \circ \boldsymbol{t}_{Q_0} = \boldsymbol{t}_{Q_0} - \boldsymbol{t}_{P_1} \circ \boldsymbol{t}_{Q_0} \\ &= (\mathbf{1} - \boldsymbol{t}_{P_1}) \circ (\mathbf{1} - \boldsymbol{t}_{Q_1}) = \mathbf{1} - \boldsymbol{t}_{P_1} - \boldsymbol{t}_{Q_0} - \boldsymbol{t}_{P_0} \circ \boldsymbol{t}_{Q_1} \end{aligned}$$

Obviously $\mathbf{1}$, $\boldsymbol{t}_{P_0}$, $\boldsymbol{t}_{P_1}$, $\boldsymbol{t}_{Q_0}$ and $\boldsymbol{t}_{Q_1}$ are points in the cone $\mathcal{C}(Q)$. Moreover, $\boldsymbol{t}_{P_0} - \boldsymbol{t}_{P_0} \circ \boldsymbol{t}_{Q_1}$, $\boldsymbol{t}_{Q_0} - \boldsymbol{t}_{P_1} \circ \boldsymbol{t}_{Q_0}$, $\mathbf{1} - \boldsymbol{t}_{P_1} - \boldsymbol{t}_{Q_0} - \boldsymbol{t}_{P_0} \circ \boldsymbol{t}_{Q_1} \in \mathcal{C}(Q)$ if and only if $\boldsymbol{t}_{P_0} \circ \boldsymbol{t}_{Q_0} \in \mathcal{C}(Q)$.

For readability define $\boldsymbol{t}_P := \boldsymbol{t}_{P_0}$ and $\boldsymbol{t}_Q := \boldsymbol{t}_{Q_0}$.

Consider a prime decomposition of Q with q indecomposable summands.

$$Q = \bigoplus_{i=0}^{q-1} S_i$$

Denote the edge vector of an indecomposable summand S_i by $\boldsymbol{t}_i$. Now, $\mathbf{1} = \sum_{i=0}^{q-1} \boldsymbol{t}_i$. By definition there is a unique summand S_j that uses edge $\boldsymbol{d}^i$. Hence for each edge $\boldsymbol{d}^i$ there exists a unique edge vector $\boldsymbol{t}_j$ which is nonzero at position i. This yields

$$\boldsymbol{t}_i \circ \boldsymbol{t}_j = \begin{cases} \boldsymbol{t}_i & \text{if } i = j \\ 0 & \text{else.} \end{cases} \tag{3.4}$$

This follows from the fact that $\boldsymbol{t}_S \in \{0,1\}^n$ for every summand S.

3.2.1.1 A common refinement implies $\boldsymbol{t}_P \circ \boldsymbol{t}_Q \in \mathcal{C}(Q)$

The first thing to prove is that if $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ have a common refinement, then $\boldsymbol{t}_P \circ \boldsymbol{t}_Q \in \mathcal{C}(Q)$. Assume that $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ have a common refinement $\bigoplus_{i=0}^{q-1} S_i$ such that every S_i is indecomposable. Define I as the subset of $\{0, \ldots, q-1\}$ such that S_i is a summand of P_0 for $i \in I$. Equivalently, define subset J for polytope Q_0.

$$P_0 = \bigoplus_{i \in I} S_i \tag{3.5}$$

$$Q_0 = \bigoplus_{j \in J} S_j \tag{3.6}$$

The edge vectors $\boldsymbol{t}_P = \sum_{i \in I} \boldsymbol{t}_i$ and $\boldsymbol{t}_Q = \sum_{j \in J} \boldsymbol{t}_j$ are contained in $\mathcal{C}(Q)$. From Equation (3.4) we can compute the Schur product $\boldsymbol{t}_P \circ \boldsymbol{t}_Q$ in terms of the indecomposable summands.

$$\boldsymbol{t}_P \circ \boldsymbol{t}_Q = \sum_{i \in I} \boldsymbol{t}_i \circ \sum_{j \in J} \boldsymbol{t}_j = \sum_{i \in I \cap J} \boldsymbol{t}_i \tag{3.7}$$

Hence $\boldsymbol{t}_P \circ \boldsymbol{t}_Q$ is the sum of the edge vectors $\boldsymbol{t}_i$ with $i \in I \cap J$. These vectors correspond to the primitive summands that are contained in both summands P_0 and Q_0. The vector $\boldsymbol{t}_i$ is contained in $\mathcal{C}(Q)$ for all i and therefore the vector $\sum_{i \in I \cap J} \boldsymbol{t}_i$ is contained in $\mathcal{C}(Q)$ as well. This proves the first part of Theorem 3.5.

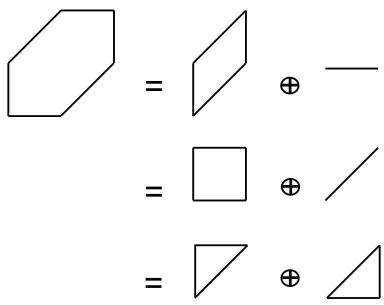

FIGURE 3.2: Three decompositions $P_0 \oplus P_1$, $Q_0 \oplus Q_1$ and $R_0 \oplus R_1$ of Q_6.

3.2.1.2 $\boldsymbol{t}_P \circ \boldsymbol{t}_Q \in \mathcal{C}(Q)$ implies compatibility

The next part is to show that if $\boldsymbol{t}_P \circ \boldsymbol{t}_Q \in \mathcal{C}(Q)$ then the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ of Q are compatible.

Let $\boldsymbol{t}_P \circ \boldsymbol{t}_Q \in \mathcal{C}(Q)$. From theorem 3.2 we know that this is equal to the edge vector of a summand Z_0 of Q. Moreover, as $\boldsymbol{t}_P \circ \boldsymbol{t}_Q$ is a strict subvector of both $\boldsymbol{t}_P$ and $\boldsymbol{t}_Q$, the polytope Z_0 is a summand of both P_0 and Q_0.

$$\begin{aligned} P_0 &= Z_0 \oplus Z_0^P \\ Q_0 &= Z_0 \oplus Z_0^Q \end{aligned}$$

We know that the polytopes Z_0^P and Z_0^Q do not have any edge in common. Similarly, we can construct a polytope Z_1 which is a summand of P_1 and Q_1. Now,

$$\begin{aligned} Q &= Z_0 \oplus Z_0^P \oplus Z_1 \oplus Z_1^P \\ &= Z_0 \oplus Z_0^Q \oplus Z_1 \oplus Z_1^Q. \end{aligned}$$

An immediate result is that $Z_0^P = Z_1^Q$ and $Z_0^Q = Z_1^P$. Consequently, the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ share a common refinement.

Example 4.

Let Q_6 be the hexagon that is discussed in the previous chapter. It is a polygon, hence it is its unique 2-face. Fix the vector $\boldsymbol{\varepsilon}$ such that the edges are oriented counterclockwise. The matrix D_6 lists the directed edges $\varepsilon_i \boldsymbol{d}^i$ as columns.

$$D_6 = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{pmatrix} \tag{3.8}$$

Figure 3.2 shows three decompositions of Q_6. The edge vectors of the summands P_0,

Q_0 and R_0 are given in (3.9), (3.10) and (3.11).

$$\boldsymbol{t}_P = (0,1,1,0,1,1) \tag{3.9}$$
$$\boldsymbol{t}_Q = (1,0,1,1,0,1) \tag{3.10}$$
$$\boldsymbol{t}_R = (0,1,0,1,0,1) \tag{3.11}$$

Observe that the vector $\mathbf{1}-\boldsymbol{t}_P$ is indeed the edge vector of the summand P_1 just as $\mathbf{1}-\boldsymbol{t}_Q$ is for Q_1 and $\mathbf{1}-\boldsymbol{t}_R$ is for R_1. The common refinement of the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ is the second prime decomposition in Figure 2.4 and consists of three line segments. The Schur product $\boldsymbol{t}_P \circ \boldsymbol{t}_Q = (0,0,1,0,0,1)$ is the edge vector of a line segment, which is a summand of Q. Therefore, $\boldsymbol{t}_P \circ \boldsymbol{t}_Q$ is contained in $\mathcal{C}(Q)$. On the other hand, the decomposition $R_0 \oplus R_1$ is a prime decomposition itself, consisting of two triangles. Thus, $R_0 \oplus R_1$ is not compatible to $P_0 \oplus P_1$ or $Q_0 \oplus Q_1$. We will compute $\boldsymbol{t}_P \circ \boldsymbol{t}_R$ and $\boldsymbol{t}_R \circ \boldsymbol{t}_Q$ to verify this.

$$\begin{aligned}\boldsymbol{t}_P \circ \boldsymbol{t}_R &= (0,1,0,0,0,1)\\ \boldsymbol{t}_R \circ \boldsymbol{t}_Q &= (0,0,0,1,0,1)\end{aligned}$$

These vectors do not correspond to a polytope, hence they are not vectors of $\mathcal{C}(Q)$. Moreover, Equation (3.3) gives

$$\begin{aligned}\sum_{i=0}^{5} (t_P)_i (t_Q)_i \varepsilon_i \boldsymbol{d}^i &= (0,1)+(0,-1)\\ &= \mathbf{0}\\ \sum_{i=0}^{5} (t_P)_i (t_R)_i \varepsilon_i \boldsymbol{d}^i &= (1,0)+(0,-1)\\ &= (1,-1)\\ \sum_{i=0}^{5} (t_R)_i (t_Q)_i \varepsilon_i \boldsymbol{d}^i &= (-1,0)+(0.-1)\\ &= (-1,-1).\end{aligned}$$

3.3 The compatibility of general lattice polytopes

Let Q be the cross cut of a cone $\sigma \subseteq N_{\mathbb{R}}$ such that Q is a lattice polytope. Now the edges are allowed to have interior lattice points and therefore an edge $\boldsymbol{d}^i$ is not necessarily primitive. Define $\boldsymbol{f}^i \in \mathbb{Z}^d$ as the corresponding primitive edge and let $\delta_i \in \mathbb{Z}$ be the nonnegative integer such that $\boldsymbol{d}^i = \delta_i \boldsymbol{f}^i$.
The entries of the edge vector $\boldsymbol{t} \in \mathbb{Q}^n$ are not binary anymore, but $\boldsymbol{t} \in [0,1]^n$, so $0 \le t_i \le 1$ for each i. Let $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ be two admissible decompositions of Q.

The edge vectors $\boldsymbol{t}_P$ and $\boldsymbol{t}_P$ are not necessarily disjoint. Therefore, the Schur product $\boldsymbol{t}_{P_0} \circ \boldsymbol{t}_{Q_0}$ will not directly show whether the decompositions have a common refinement. To be able to determine if a common refinement of $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ exists, define a vector $\tilde{\boldsymbol{t}}$.

Definition 3.6. Let $\boldsymbol{t} \in \mathbb{Q}^n$ be the edge vector of a given polytope Q. For each edge $\boldsymbol{d}^i$ let $\boldsymbol{f}^i$ be the corresponding primitive vector and $\delta_i \in \mathbb{Z}$ such that $\boldsymbol{d}_i = \delta_i \boldsymbol{f}^i$. Furthermore, let $\tilde{n} = \sum_{i=0}^{n-1} i\delta_i$. Now each edge $t_i \boldsymbol{d}^i$ of Q can be written as $t_i \delta_i \boldsymbol{f}^i$. Define

$$\tilde{\boldsymbol{t}} \in \mathbb{Z}^{\tilde{n}}$$

as the *primitive edge vector* of length $\tilde{n}$ such that Q is built from (primitive) edges $\tilde{t}_i \boldsymbol{f}^i$ for $0 \leq i \leq \tilde{n} - 1$. Every edge $\boldsymbol{d}^i$ is split into δ_i primitive edges $\boldsymbol{f}^i$.

Let $\mathcal{C}(Q)$ be the cone as defined in Definition 3.1. This cone is generated by the primitive vectors $\boldsymbol{t}_i$, which correspond to the edge vectors of the primitive summands of Q. Now define $\tilde{\mathcal{C}}(Q)$ as the cone that is generated by the rays $\tilde{\boldsymbol{t}}_i$ which are the primitive edge vectors of the indecomposable summands of Q.

Theorem 3.7. *Every summand P of Q has a unique primitive edge vector $\tilde{\boldsymbol{t}}$ such that $\tilde{\boldsymbol{t}} \in \tilde{\mathcal{C}}(Q)$.*

Proof. Let Q be a polytope such that $\bigoplus S_i$ is a prime decomposition. From Definition 2.8 we know that for each face F of Q there exists a unique decomposition $F = \bigoplus_{i=0}^{q-1} F_{S_i}$, such that F_{S_i} is a face of the indecomposable summand S_i of Q. The edges of Q are its 1-faces, which means that there exists a unique prime decomposition for each edge. Moreover, Q is a lattice polytope and therefore the summands are lattice polytopes. Hence, every lattice segment of an edge of Q is a lattice segment of a uniquely determined indecomposable summand S_i. Thus there exists a unique primitive edge vector $\tilde{\boldsymbol{t}}_i$ corresponding to a summand S_i. Every summand $Q_{\boldsymbol{t}}$ of Q is the sum of indecomposable summands and therefore the primitive edge vector $\tilde{\boldsymbol{t}}$ of a summand Q_t is uniquely determined. $\square$

Corollary 3.8. *All the primitive edge vectors $\tilde{\boldsymbol{t}}$ have binary entries.*

Determine the primitive edge vectors $\tilde{\boldsymbol{t}}_P$ and $\tilde{\boldsymbol{t}}_Q$ and compute

$$\sum_{i=0}^{n-1} \sum_{j=0}^{\delta_i - 1} (\tilde{t}_P)_j (\tilde{t}_Q)_j \varepsilon_i \boldsymbol{d}^j \tag{3.12}$$

for each 2-face of Q. According to Theorem 3.5 the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ are compatible if and only if (3.12) evaluates to zero for every 2-face of Q.

Example 5.

Consider the polytope Q_5 which is the convex hull of the vertices of the matrix Q_5. It is a polygon and therefore it is its unique 2-face.

$$Q_5 = \begin{pmatrix} 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 2 & 3 & 2 \end{pmatrix}$$

Q_5 is a lattice polytope with edges that have interior lattice points. Moreover, it has

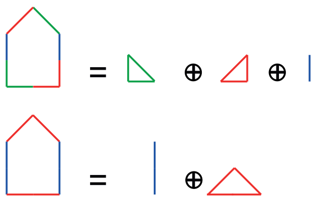

FIGURE 3.3: The two prime decompositions of Q_5.

two prime decompositions and both are given in Figure 3.3. Furthermore Figure 3.4

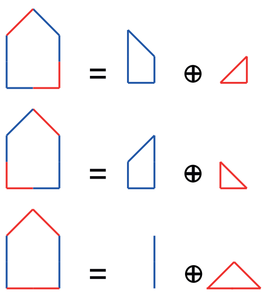

FIGURE 3.4: The decompositions $P_0 \oplus P_1$, $Q_0 \oplus Q_1$ and $R_0 \oplus R_1$ of Q_5.

shows three different decompositions of Q_5. The primitive edge vectors $\tilde{\boldsymbol{t}}$ of the first summand of each decomposition are given below.

$$\begin{aligned} \tilde{\boldsymbol{t}}_P &= (1,0,0,1,1,0,1,1) \\ \tilde{\boldsymbol{t}}_Q &= (0,1,1,1,0,1,1,0) \\ \tilde{\boldsymbol{t}}_R &= (0,0,1,1,0,0,1,1) \end{aligned}$$

The Schur products of these vectors can be computed.

$$\begin{aligned}\tilde{\boldsymbol{t}}_P \circ \tilde{\boldsymbol{t}}_Q &= (0,0,0,1,0,0,1,0)\\ \tilde{\boldsymbol{t}}_Q \circ \tilde{\boldsymbol{t}}_R &= (0,0,0,1,0,0,1,1)\\ \tilde{\boldsymbol{t}}_P \circ \tilde{\boldsymbol{t}}_R &= (0,0,1,1,0,0,1,0)\end{aligned}$$

For these vectors the sum in (3.12) evaluate to $(0,1)+(0,-1)=\mathbf{0}$, $(0,1)+(0,-1)+(0,-1)=(0,-1)$ respectively $(0,1)+(0,1)+(0,-1)=(0,1)$. From this can be concluded that the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ are compatible but the decompositions $P_0 \oplus P_1$ and $R_0 \oplus R_1$ and the decompositions $Q_0 \oplus Q_1$ and $R_0 \oplus R_1$ do not have a common refinement.

3.4 The compatibility of general lattice polyhedra

Let Q be a polyhedron with n bounded edges. This polyhedron is the Minkowski sum of a polytope and its characteristic cone.

$$Q = \mathrm{Conv}(\mathrm{Vert}(Q)) \oplus \mathrm{Char}(Q)$$

A decomposition $P_0 \oplus P_1$ of Q is admissible if and only if for every face F at most one of its summands F_i does not contain a lattice point. This means that if Q is not bounded, then both summands should have the same characteristic cone. Observe that $\mathrm{Conv}(\mathrm{Vert}(P_0)) \oplus \mathrm{Conv}(\mathrm{Vert}(P_1))$ is not necessarily equal to $\mathrm{Conv}(\mathrm{Vert}(Q))$ for an admissible decomposition of $Q = P_0 \oplus P_1$. An example is given in Figure 3.5.

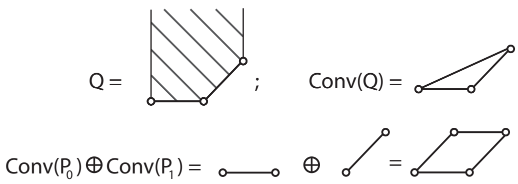

FIGURE 3.5: An example of a polyhedron Q and an admissible decomposition $P_0 \oplus P_1$ such that $\mathrm{Conv}(\mathrm{Vert}(Q)) \neq \mathrm{Conv}(\mathrm{Vert}(P_0))$.

If $\mathrm{Conv}(\mathrm{Vert}(P_0)) \oplus \mathrm{Conv}(V(P_1)) = \mathrm{Conv}(\mathrm{Vert}(Q_0)) \oplus \mathrm{Conv}(\mathrm{Vert}(Q_1)) = \mathrm{Conv}(\mathrm{Vert}(Q))$ then it automatically follows that the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ are compatible if and only if the decompositions $\mathrm{Conv}(\mathrm{Vert}(P_0)) \oplus \mathrm{Conv}(\mathrm{Vert}(P_1))$ and $\mathrm{Conv}(\mathrm{Vert}(Q_0)) \oplus \mathrm{Conv}(\mathrm{Vert}(Q_1))$ are compatible. These decompositions are decompositions of the polytope $\mathrm{Conv}(\mathrm{Vert}(Q))$, which means that the procedure that is described in the previous sections can be used to determine whether these decompositions are compatible.

From now on assume that at least one of the polytopes $\mathrm{Conv}(\mathrm{Vert}(P_0)) \oplus \mathrm{Conv}(\mathrm{Vert}(P_1))$ and $\mathrm{Conv}(\mathrm{Vert}(Q_0)) \oplus \mathrm{Conv}(\mathrm{Vert}(Q_1))$ is not equal to the polytope $\mathrm{Conv}(\mathrm{Vert}(Q))$. Let F be an unbounded 2-face of Q that has bounded edges $\boldsymbol{d}^i$ for $i \in I_F$ where I_F is a subset of $\{0, \ldots, n-1\}$. Observe that every decomposition of F has $\bigoplus \boldsymbol{d}^i (\oplus \mathrm{Char}(Q))$ as a refinement. Therefore we can say that every pair of decompositions have a common refinement regarding the unbounded 2-faces of Q. Furthermore define $\boldsymbol{\varepsilon}$ as the sign vector of a bounded 2-face B, such that $\varepsilon_i = \pm 1$ if the edge $\boldsymbol{d}^i$ is an edge of B and 0 otherwise. Now $\sum \varepsilon_i \boldsymbol{d}^i = \mathbf{0}$ for every bounded 2-face B. Equivalent to the previous section, the vector space $\mathcal{V}(Q)$ and the corresponding cone $\mathcal{C}(Q)$ can be constructed. The elements $\boldsymbol{t} \in \mathcal{C}(Q)$ correspond to the summands Q_t of Q such that it has bounded edges $t_i \boldsymbol{d}^i$. Every summands has the same characteristic cone, and therefore a summand Q_t does not have any bounded edges that is not a bounded edge of Q.

The bounded edge vector of a decomposition $P_0 \oplus P_1$ is denoted by $\boldsymbol{t}_P$. This is the edge vector of the summand P_0 and $\mathbf{1} - \boldsymbol{t}_P$ is the bounded edge vector of the summand P_1.

Theorem 3.9. *The decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ have a common refinement if and only if $\boldsymbol{t}_P \circ \boldsymbol{t}_Q \in \mathcal{C}(Q)$. This means that*

$$\sum_{i=0}^{n-1} (t_P)_i (t_Q)_i \varepsilon_i \boldsymbol{b}^i = \mathbf{0} \tag{3.13}$$

for every bounded 2-face B of Q.

The proof for this theorem is similar to the proof of Theorem 3.5.

This concludes the proof of this section. Now we have shown how to determine whether two decompositions of a polyhedron have a common refinement. One can generalize the theorem for polyhedra, combining the given proofs. The proof for this is not given explicitely, but the reader can see that a generalization is easily made.

Chapter 4

Unequal cross cuts

In the previous chapter the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ are decompositions of the polyhedron $Q_{\boldsymbol{r}}$. If the two decompositions have a common refinement, then the decompositions are compatible. This section concerns the case where the decompositions sum up to different, but related polyhedra. Given a cone $\sigma \subseteq N_{\mathbb{R}}$ and two vectors $\boldsymbol{r}, \boldsymbol{s} \in \sigma^{\vee}$, one can determine the polyhedra $Q_{\boldsymbol{r}}$ and $Q_{\boldsymbol{s}}$. The question that plays a central role in this chapter is whether two given decompositions of $Q_{\boldsymbol{r}}$ and $Q_{\boldsymbol{s}}$ are compatible. An answer can not be given directly. The summands should be embedded in a higher dimensional lattice in order to determine if two decompositions are compatible. The first step is to adjust the notion of compatibility for this particular case. Some basic steps are given which explain the approach that is used in this chapter. The procedure will be elaborated for the special case where σ is a 2-dimensional cone.

In [18] Nathan Ilten describes this problem for σ being a 2-dimensional cone in terms of Deformation Theory of toric varieties. The main objective of this chapter is to understand this combinatorically. One can see [1], [2], [4] from Altmann, or [12] [17] for more information about this Deformation theory and toric varieties.

4.0.1 The basic principle

To be able to understand the procedure that is used in this chapter, we show how the original polytope $Q_{\boldsymbol{r}}$ can be obtained from a higher dimensional polytope which is created from the summands P_0 and P_1 such that $Q_{\boldsymbol{r}} = P_0 \oplus P_1$. This is done by means of an example. Consider the hexagon $Q_6 = P_0 \oplus P_1$ and its decomposition in two triangles as in Figure 2.5. This decomposition is admissible and described in Equation (2.1). The 2-dimensional summands are embedded at different heights in the extended lattice $N \times \mathbb{Z}$. In general P_0 is embedded with $z = 1$ and P_1 at height $z = 0$, i.e. the vertices $\boldsymbol{v}$ of P_0 become $(\boldsymbol{v}, 1)$ and the vertices of P_1 are now $(\boldsymbol{v}, 0)$. This is in line with Remark 1 where we consider Q_6 as the polyhedral cone $\sigma(Q_6)$ such that the polyhedron Q_6 is embedded at height 1.

Consider the convex hull of these vertices. This yields a polytope with 6 vertices and 8 2-faces living in $N \times \mathbb{Z} \cong \mathbb{R}^3$. Its vertices are given below.

$$\mathrm{Conv}(P_0, P_1) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The original polytope Q_6 is now obtained by intersecting this polytope at $z = \frac{1}{2}$ as illustrated in Figure 4.1.

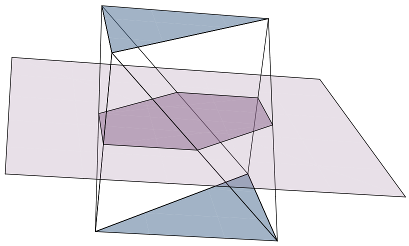

FIGURE 4.1: The convex hull of P_0 and P_1 with the intersection which yields $\frac{1}{2}Q_6$.

$$\begin{aligned} H_6 &:= \{\boldsymbol{a} \in \mathbb{R}^3 | \boldsymbol{a}^\top (0, 0, 1) = \frac{1}{2}\} \\ \frac{1}{2}Q_6 &= \mathrm{Conv}(P_0, P_1) \cap H_6 \end{aligned}$$

Instead of determining the polytope $\mathrm{Conv}(P_0, P_1)$ one can consider a cone $\sigma_{\boldsymbol{r}}$ that is embedded in $N \times \mathbb{Z}^2$. Its rays are constructed from the summands P_0 and P_1 such that the rays are $(\boldsymbol{v}_0, 1, 0)$ and $(\boldsymbol{v}_1, 0, 1)\}$ for the vertices $\boldsymbol{v}_i$ of P_i. Intersecting this cone yields the original cone σ. This principle of first lifting the summands and then intersecting the obtained polytope is used to determine whether cross cuts $Q_{\boldsymbol{r}}$ and $Q_{\boldsymbol{s}}$ are compatible if $\boldsymbol{r}$ is not equal to $\boldsymbol{s}$.

4.1 The procedure for $\boldsymbol{r} \neq \boldsymbol{s}$

As before, let $N \cong \mathbb{Z}^d$ be a lattice with M its dual lattice. Note that as $N \cong \mathbb{Z}^d$ we also have that $M \cong \mathbb{Z}^d$. Let $\sigma \subseteq N_{\mathbb{R}} \cong \mathbb{R}^d$ be a cone that is generated by k lattice vectors $\boldsymbol{a}^i$. Let $\sigma^\vee$ be the dual cone and fix two primitive vectors $\boldsymbol{r}, \boldsymbol{s} \in \sigma^\vee \cap M$. As defined in

Definition 2.20, the cross cut of σ and the hyperplane $H_{\boldsymbol{r}}$ gives the polyhedron

$$Q_{\boldsymbol{r}} = \sigma \cap \{\boldsymbol{x} \in N_{\mathbb{R}} | \boldsymbol{r}^\top \boldsymbol{x} = 1\}.$$

Similarly, the cross cut of the cone σ and the hyperplane $H_{\boldsymbol{s}}$ yields the polyhedron $Q_{\boldsymbol{s}}$. Fix two admissible decompositions $Q_{\boldsymbol{r}} = P_0 \oplus P_1$ and $Q_{\boldsymbol{s}} = Q_0 \oplus Q_1$.
The polytopes $Q_{\boldsymbol{r}}$ and $Q_{\boldsymbol{s}}$ are not equivalent for $\boldsymbol{r} \neq \boldsymbol{s}$ and therefore it is not straightforward to see what is meant by being compatible. This cannot be determined directly from the polytopes $Q_{\boldsymbol{r}}$ and $Q_{\boldsymbol{s}}$ and their decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$. Instead of comparing the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$, two new cones $\sigma_{\boldsymbol{r},\boldsymbol{s}}$ and $\sigma_{\boldsymbol{s},\boldsymbol{r}}$ are constructed in a higher dimensional lattice. These cones depend on the choice of the decompositions of the polytopes. If these cones satisfy some predefined conditions (which will be given later on), then the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ are compatible. The question why the compatibility can be derived from these two cones, will become clear when the construction of these cones is given.

Given the two decompositions $Q_{\boldsymbol{r}} = P_0 \oplus P_1$ and $Q_{\boldsymbol{s}} = Q_0 \oplus Q_1$, the cones $\sigma_{\boldsymbol{r}}$ and $\sigma_{\boldsymbol{s}}$ can be constructed such that the rays of these cones correspond to the vertices of the summands of $Q_{\boldsymbol{r}}$ respectively $Q_{\boldsymbol{s}}$. Observe that the cone σ can be obtained from $\sigma_{\boldsymbol{r}}$ and $\sigma_{\boldsymbol{s}}$ by taking $\sigma_{\boldsymbol{r}} \cap N$ or $\sigma_{\boldsymbol{s}} \cap N$ respectively.

Two vectors $\tilde{\boldsymbol{s}} \in \sigma_{\boldsymbol{r}}^\vee$ and $\tilde{\boldsymbol{r}} \in \sigma_{\boldsymbol{s}}^\vee$ can be constructed from the cones $\sigma_{\boldsymbol{r}}$ and $\sigma_{\boldsymbol{s}}$ such that they relate to the vectors $\boldsymbol{r}$ and $\boldsymbol{s}$. Now the cross cut of $\sigma_{\boldsymbol{r}}$ and $H_{\tilde{\boldsymbol{s}}} = \{\boldsymbol{x} \in \overline{N}_{\mathbb{R}} | \tilde{\boldsymbol{s}}^\top \boldsymbol{x} = 1\}$ yields the polyhedron $Q_{\boldsymbol{r},\boldsymbol{s}} = \sigma_{\boldsymbol{r}} \cap H_{\tilde{\boldsymbol{s}}}$, where $\overline{N}_{\mathbb{R}} \cong \mathbb{R}d + 2$. This polyhedron depends on both vectors $\boldsymbol{r}$ and $\boldsymbol{s}$. Similarly, the polyhedron $Q_{\boldsymbol{s},\boldsymbol{r}}$ is constructed.
Fix an admissible decomposition of the polyhedron $Q_{\boldsymbol{r},\boldsymbol{s}}$ and one for $Q_{\boldsymbol{s},\boldsymbol{r}}$. As before, two new cones $\sigma_{\boldsymbol{r},\boldsymbol{s}}$ and $\sigma_{\boldsymbol{s},\boldsymbol{r}}$ can be constructed such that the rays of these cones correspond to the vertices of the summands of $Q_{\boldsymbol{r},\boldsymbol{s}}$ respectively $Q_{\boldsymbol{s},\boldsymbol{r}}$. The polyhedra $Q_{\boldsymbol{r},\boldsymbol{s}}$ and $Q_{\boldsymbol{s},\boldsymbol{r}}$ are not equivalent and therefore their decompositions will not be comparable. However, if the original decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ are compatible, then the lifted cones $\sigma_{\boldsymbol{r},\boldsymbol{s}}$ and $\sigma_{\boldsymbol{s},\boldsymbol{r}}$ will have dual cones that are similar. This means that there exists a mapping between the cones $\sigma_{\boldsymbol{r},\boldsymbol{s}}$ and $\sigma_{\boldsymbol{s},\boldsymbol{r}}$.

Definition 4.1. If the dual cones $\sigma_{\boldsymbol{r},\boldsymbol{s}}^\vee$ and $\sigma_{\boldsymbol{s},\boldsymbol{r}}^\vee$ are row equivalent then the original decompositions of $Q_{\boldsymbol{r}}$ and $Q_{\boldsymbol{s}}$ are compatible.

We will clarify this definition with an example. Let σ be the cone that is generated by the rays $(0,0,1)$, $(0,1,1)$, $(1,0,1)$ and $(1,1,1)$. We compute three cross cuts of this cone, together with a decomposition and show whether they are compatible. The vectors $\boldsymbol{r} = (0,0,1)$, $\boldsymbol{s} = (1,0,1)$ and $\boldsymbol{t} = (0,1,1)$ are all elements in $\sigma^\vee$ and they are used to determine the polyhedra $Q_{\boldsymbol{r}}$, $Q_{\boldsymbol{s}}$ and $Q_{\boldsymbol{t}}$. This results in the three polytopes given in

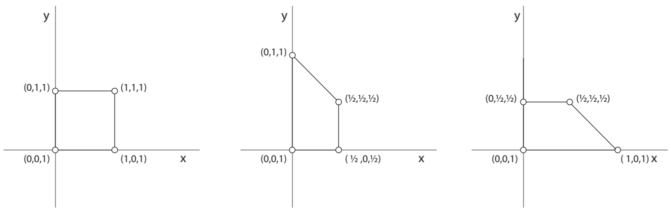

FIGURE 4.2: The polytopes $Q_{\boldsymbol{r}}$, $Q_{\boldsymbol{s}}$ and $Q_{\boldsymbol{t}}$.

Figure 4.2. The decompositions of these polytopes are given below. Remark that these decompositions are actually not admissible decompositions, but this will not affect our example. Intuitively, one can see that the decompositions of $Q_{\boldsymbol{s}}$ and $Q_{\boldsymbol{t}}$ are compatible,

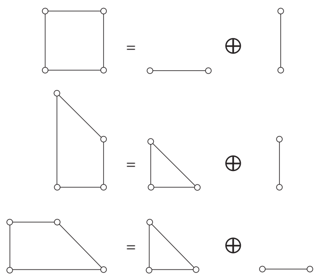

FIGURE 4.3: The polytopes $Q_{\boldsymbol{r}}$, $Q_{\boldsymbol{s}}$ and $Q_{\boldsymbol{t}}$ and their decompositions.

while both of these decompositions are not compatible with the decomposition of $Q_{\boldsymbol{r}}$. In this chapter we will elaborate on the construction of the cones $\sigma^{\vee}_{\boldsymbol{r},\boldsymbol{s}}$ and $\sigma^{\vee}_{\boldsymbol{s},\boldsymbol{r}}$ which shows the compatibility of the decompositions of $Q_{\boldsymbol{r}}$ and $Q_{\boldsymbol{s}}$.

4.1.1 Lifting the summands of $Q_{\boldsymbol{r}}$

Fix an admissible decomposition of the polyhedron $Q_{\boldsymbol{r}} = P_0 \oplus P_1$. As before the polytope $Q_{\boldsymbol{r}}$ has n edges. Define the extended lattice $\overline{N} = N \times \mathbb{Z}^2 \cong \mathbb{Z}^{d+2}$ from N and write $\overline{M}$ as its dual lattice. A new cone can be constructed in this extended lattice. This cone is constructed from the decomposition of $Q_{\boldsymbol{r}}$.

In line with Remark 1 we will embed the summand P_0 in $H_{\boldsymbol{r}} = \{\boldsymbol{x} \in N_{\mathbb{R}} | \boldsymbol{r}^{\top}\boldsymbol{x} = 1\}$ and the summand P_1 in $H^0_{\boldsymbol{r}} = \{\boldsymbol{x} \in N_{\mathbb{R}} | \boldsymbol{r}^{\top}\boldsymbol{x} = 0\}$. From this embedding we are able to construct the lifted cone $\sigma_{\boldsymbol{r}}$.

Definition 4.2. Let $Q_{\boldsymbol{r}} = P_0 \oplus P_1$ be an admissible decomposition. The cone $\sigma_{\boldsymbol{r}} \subseteq \overline{N}$ is generated by the rays $(\boldsymbol{v}_0, 1, 0)$, $(\boldsymbol{v}_1, 0, 1)$ and $(\boldsymbol{z}, 0, 0)$ such that each vertex $\boldsymbol{v}_0$ of P_0 corresponds to a ray $(\boldsymbol{v}_0, 1, 0)$ of $\sigma_{\boldsymbol{r}}$, each vertex $\boldsymbol{v}_1$ of P_1 corresponds to a ray $(\boldsymbol{v}_1, 0, 1)$ and each $\boldsymbol{z} \in \mathrm{Char}(Q)$ corresponds to a ray $(\boldsymbol{z}, 0, 0)$. Let k_i be the number of vertices of the summand P_i. Moreover let l be the number of rays of Q. The cone $\sigma_{\boldsymbol{r}}$ is generated by $k_0 + k_1 + l$ extremal rays.

$$\sigma_{\boldsymbol{r}} := \mathrm{Cone}\left\{ \left(\begin{smallmatrix} \boldsymbol{v}_0 \\ 1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \boldsymbol{v}_1 \\ 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \boldsymbol{z} \\ 0 \\ 0 \end{smallmatrix}\right) \middle| \boldsymbol{v}_0 \in P_0, \boldsymbol{v}_1 \in P_1, \boldsymbol{z} \in \mathrm{Char}(Q) \right\}$$

Remark 4. The original cone can be obtained from $\sigma_{\boldsymbol{r}}$ by $\sigma = \sigma_{\boldsymbol{r}} \cap N_{\mathbb{R}}$.

The decompositions given in Figure 4.3 result in the following cones $\sigma_{\boldsymbol{r}}$, $\sigma_{\boldsymbol{s}}$ and $\sigma_{\boldsymbol{t}}$.

$$\sigma_{\boldsymbol{r}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \sigma_{\boldsymbol{s}} = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \qquad \sigma_{\boldsymbol{t}} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 1 & \frac{1}{2} & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

If the role of the summands P_0 and P_1 are swapped, then the resulting cone $\sigma_{\boldsymbol{r}}$ is different. However, there will exist an equivalency between the two cones $\sigma_{\boldsymbol{r},\boldsymbol{s}}$ and $\sigma_{\boldsymbol{s},\boldsymbol{r}}$. The reader can check that the decomposition of $Q_{\boldsymbol{r}}$ does not give a satisfying lifted cone $\sigma_{\boldsymbol{r},\boldsymbol{s}}$ or $\sigma_{\boldsymbol{r},\boldsymbol{t}}$, which means that they are not compatible.

4.1.2 Determining the cross cut of $\sigma_{\boldsymbol{r}}$ and $\tilde{\boldsymbol{s}}$

The cones $\sigma_{\boldsymbol{r}}$ and $\sigma_{\boldsymbol{s}}$ can be computed as explained in the previous section. The next step is to compute the lifted polyhedra $Q_{\boldsymbol{r},\boldsymbol{s}}$ and $Q_{\boldsymbol{s},\boldsymbol{r}}$. The polyhedron $Q_{\boldsymbol{r},\boldsymbol{s}}$ is defined as the cross cut of the cone $\sigma_{\boldsymbol{r}}$ and a vector $\tilde{\boldsymbol{s}} \in \sigma_{\boldsymbol{r}}^{\vee}$ that corresponds to the vector $\boldsymbol{s} \in \sigma^{\vee}$. Define a surjection $\phi_{\boldsymbol{r}} : \overline{N} \to N$ such that $\phi(\tilde{\boldsymbol{s}}) = \boldsymbol{s}$.

$$\phi_{\boldsymbol{r}} := \Big(I_d | \boldsymbol{r} | \boldsymbol{r} \Big) \tag{4.1}$$

with I_d the $d \times d$ identity matrix.
Moreover the matrix $\phi_{\boldsymbol{r}}^{\top}$ has the nice property that $\phi_{\boldsymbol{r}}^{\top}\sigma$ results in the cone $\sigma_{\boldsymbol{r}} \cap N$.

Theorem 4.3. *Let $\sigma_{\boldsymbol{r}} \subseteq \overline{N}_{\mathbb{R}}$ be the lifted cone and $\boldsymbol{r}, \boldsymbol{s} \in \sigma^{\vee}$. There exists a vector $\tilde{\boldsymbol{s}} \in \sigma_{\boldsymbol{r}}^{\vee} \cap \overline{M}$ such that $\phi_{\boldsymbol{r}}(\tilde{\boldsymbol{s}}) = \boldsymbol{s}$. Moreover, if $\boldsymbol{r} \neq \boldsymbol{s}$ then the cross cut $Q_{\boldsymbol{r},\boldsymbol{s}}$ is uniquely determined.*

Proof. The vectors $(\boldsymbol{r}, -1, 0)$ and $(\boldsymbol{0}, 1, -1)$ span the nullspace of $\phi_{\boldsymbol{r}}$.
By definition $\phi_{\boldsymbol{r}}((\boldsymbol{s}, 0, 0)) = \boldsymbol{s}$. This means that $\tilde{\boldsymbol{s}}$ can be written as

$$\tilde{\boldsymbol{s}} = (\boldsymbol{s}, 0, 0) + \alpha_0(\boldsymbol{r}, -1, 0) + \alpha_1(\boldsymbol{0}, 1, -1)$$

for some integers $\alpha_0, \alpha_1 \in \mathbb{Z}$. We show existence of a $\tilde{\boldsymbol{s}} \in \sigma_{\boldsymbol{r}}^{\vee} \cap \overline{M}$ by constructing such a vector. This vector $\tilde{\boldsymbol{s}}$ is an element of $\sigma_{\boldsymbol{r}}^{\vee}$ and therefore

$$\tilde{\boldsymbol{s}}^{\top}\boldsymbol{a} \geq 0 \text{ for all } \boldsymbol{a} \in \sigma_{\boldsymbol{r}}.$$

Recall that the extremal rays of $\sigma_{\boldsymbol{r}}$ are constructed from the Minkowski summands of $Q_{\boldsymbol{r}}$. Moreover, the summands are embedded such that $P_0 \subset H_{\boldsymbol{r}}$ and $P_1 \subset H_{\boldsymbol{r}}^0$. Therefore,

$$\begin{aligned} \boldsymbol{r}^{\top}\boldsymbol{v}_0 &= 1 \quad \text{and} \\ \boldsymbol{r}^{\top}\boldsymbol{v}_1 &= 0 \end{aligned}$$

where $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1$ and $\boldsymbol{v}_0 \in P_0, \boldsymbol{v}_1 \in P_1$. The rays $\boldsymbol{z}$ of $Q_{\boldsymbol{r}}$ are exactly the generators of σ such that $\boldsymbol{r}^{\top}\boldsymbol{z} = 0$. Thus

$$(\boldsymbol{r}, -1, 0)^{\top}\boldsymbol{a} = \begin{cases} (\boldsymbol{r}, -1, 0)^{\top}(\boldsymbol{v}_0, 1, 0) \\ (\boldsymbol{r}, -1, 0)^{\top}(\boldsymbol{v}_1, 0, 1) \\ (\boldsymbol{r}, -1, 0)^{\top}(\boldsymbol{z}, 0, 0) \end{cases}$$

all evaluate to 0. This means that the vector $(\boldsymbol{r}, -1, 0)$ does not contribute to a different cross cut $Q_{\boldsymbol{r},\boldsymbol{s}}$. From now on, assume that $\alpha_0 = 0$ and $\tilde{\boldsymbol{s}} = (\boldsymbol{s}, 0, 0) + \alpha_1(\boldsymbol{0}, 1, -1)$

Consider $\alpha_1(\boldsymbol{0}, 1, -1)^{\top}\boldsymbol{a}$ for all extremal rays $\boldsymbol{a} \in \sigma_{\boldsymbol{r}}$.

$$(\boldsymbol{0}, 1, -1)^{\top}\boldsymbol{a} = \begin{cases} (\boldsymbol{0}, 1, -1)^{\top}(\boldsymbol{v}_0, 1, 0) & = 1 \\ (\boldsymbol{0}, 1, -1)^{\top}(\boldsymbol{v}_1, 0, 1) & = -1 \\ (\boldsymbol{0}, 1, -1)^{\top}(\boldsymbol{z}, 0, 0) & = 0 \end{cases}$$

Thus $(\boldsymbol{0}, 1, -1) \notin \sigma_{\boldsymbol{r}}^{\vee}$. From the fact that $\boldsymbol{s} \in \sigma^{\vee}$ we know that $\boldsymbol{s}^{\top}\boldsymbol{v} \geq 0$ for each vertex $\boldsymbol{v}$ of Q. This yields $\boldsymbol{s}^{\top}\boldsymbol{v} = \boldsymbol{s}^{\top}\boldsymbol{v}_0 + \boldsymbol{s}^{\top}\boldsymbol{v}_1 \geq 0$ with $\boldsymbol{v}_0, \boldsymbol{v}_1$ a vertex of P_0 respectively P_1. Thus $(\boldsymbol{s}, 0, 0)^{\top}(\boldsymbol{v}_1, 0, 1) \geq -(\boldsymbol{s}, 0, 0)^{\top}(\boldsymbol{v}_0, 1, 0)$, which proves that there exists an integer α_1 such that the vector $\tilde{\boldsymbol{s}} = (\boldsymbol{s}, 0, 0) + \alpha_1(\boldsymbol{0}, 1, -1) \in \sigma_{\boldsymbol{r}}^{\vee}$. $\square$

To continue with our example, we will construct the lifted vector $\tilde{\boldsymbol{t}}$ which is an element in $\sigma_{\boldsymbol{s}}^{\vee}$. We have $\alpha = 0$ as $\sigma_{\boldsymbol{s}}$ only has nonnegative entries. Hence, $\tilde{\boldsymbol{t}} = (0, 1, 1, 0, 0)$. Similarly, the vector $\tilde{\boldsymbol{s}} = (1, 0, 1, 0, 0) \in \sigma_{\boldsymbol{t}}^{\vee}$ is constructed.

4.1.3 Lifting the summands of $Q_{\boldsymbol{r},\boldsymbol{s}}$

The next step is to determine the lifted polyhedron $Q_{\boldsymbol{r},\boldsymbol{s}}$ which is the cross cut of $\sigma_{\boldsymbol{r}}$ and the hyperplane $H_{\tilde{\boldsymbol{s}}} = \{\boldsymbol{x} \in \overline{N}_{\mathbb{R}} | \tilde{\boldsymbol{s}}^{\top} \boldsymbol{x} = 1\}$.

$$Q_{\boldsymbol{r},\boldsymbol{s}} = \sigma_{\boldsymbol{r}} \cap H_{\tilde{\boldsymbol{s}}}$$

Fix an admissible decomposition of this polyhedron.

$$Q_{\boldsymbol{r},\boldsymbol{s}} = \tilde{Q}_0 \oplus \tilde{Q}_1$$

These summands are lifted to a higher dimensional lattice $\overline{\overline{N}} := \overline{N} \times \mathbb{Z}^2 \cong \mathbb{Z}^{d+4}$, yielding the cone $\sigma_{\boldsymbol{r},\boldsymbol{s}}$. The summands of $Q_{\boldsymbol{r},\boldsymbol{s}}$ are embedded at different heights, which results in a second lifted cone $\sigma_{\boldsymbol{r},\boldsymbol{s}} \subseteq \overline{\overline{N}}_{\mathbb{R}}$. Each vertex $\tilde{\boldsymbol{v}}_0$ of the summand $\tilde{Q}_0$ corresponds to a ray $(\tilde{\boldsymbol{v}}_0, 1, 0)$ of $\sigma_{\boldsymbol{r},\boldsymbol{s}}$. Similarly, each vertex $\tilde{\boldsymbol{v}}_1$ of $\tilde{Q}_1$ corresponds to a ray $(\tilde{\boldsymbol{v}}_1, 0, 1)$. Moreover, the rays $\tilde{\boldsymbol{z}}$ of $Q_{\boldsymbol{r}}$ correspond to the rays $(\tilde{\boldsymbol{z}}, 0, 0)$. There exists a surjection from $\sigma_{\boldsymbol{r},\boldsymbol{s}}$ to σ and is induced by the following matrix.

$$\phi_{\boldsymbol{r},\boldsymbol{s}} = \begin{pmatrix} I_d & \boldsymbol{r} & \boldsymbol{r} & \boldsymbol{s} & \boldsymbol{s} \end{pmatrix} \tag{4.2}$$

where I_d is the d-dimensional identity matrix.

The cross cuts $\sigma_{\boldsymbol{s}} \cap H_{\tilde{\boldsymbol{t}}}$ and $\sigma_{\boldsymbol{t}} \cap H_{\tilde{\boldsymbol{s}}}$ yield the polyhedra $Q_{\boldsymbol{s},\boldsymbol{t}}$ and $Q_{\boldsymbol{t},\boldsymbol{s}}$. Observe that they both have four vertices and one ray, which is indicated by the last row.

$$Q_{\boldsymbol{s},\boldsymbol{t}} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 1 & 0 \\ 1 & \frac{2}{3} & 1 & 0 & 0 \\ 1 & \frac{2}{3} & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad Q_{\boldsymbol{t},\boldsymbol{s}} = \begin{pmatrix} 0 & \frac{1}{3} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & \frac{2}{3} & 1 & 0 & 0 \\ 1 & \frac{2}{3} & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

These polyhedra have a lattice shift as an admissible decomposition. These decompositions are fixed, which yields a lattice shift over the point $(0, 0, 1, 1, 0)$.

4.1.4 Comparing the cones $\sigma_{\boldsymbol{r},\boldsymbol{s}}$ and $\sigma_{\boldsymbol{s},\boldsymbol{r}}$

If the decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ of $Q_{\boldsymbol{r}}$ and $Q_{\boldsymbol{s}}$ respectively are compatible, then there exist cones $\sigma^{\vee}_{\boldsymbol{r},\boldsymbol{s}}$ and $\sigma^{\vee}_{\boldsymbol{s},\boldsymbol{r}}$ that are row equivalent. These cones depend on the choice of the decomposition of $Q_{\boldsymbol{r},\boldsymbol{s}}$ and $Q_{\boldsymbol{s},\boldsymbol{r}}$. The decompositions of $Q_{\boldsymbol{r},\boldsymbol{s}}$ and $Q_{\boldsymbol{s},\boldsymbol{r}}$ should be chosen correctly in order for the equivalence to exist. If a suitable admissible

decomposition of these lifted polyhedra does not exist, then the initial decompositions $P_0 \oplus P_1$ and $Q_0 \oplus Q_1$ are not compatible.
In our example the lifted cone $\sigma_{\boldsymbol{s},\boldsymbol{t}}$ is generated by six rays. Its dual cone $\sigma^{\vee}_{\boldsymbol{s},\boldsymbol{t}}$ is generated by six rays as well. The rays of both cones are given below.

$$\sigma_{\boldsymbol{s},\boldsymbol{t}} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 1 & 0 \\ 1 & 0 & -\frac{1}{3} & 0 & -1 & 0 \\ 1 & 0 & -\frac{1}{3} & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma^{\vee}_{\boldsymbol{s},\boldsymbol{t}} = \begin{pmatrix} 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & 1 & -1 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

From the construction of the polyhedra $Q_{\boldsymbol{s},\boldsymbol{t}}$ and $Q_{\boldsymbol{t},\boldsymbol{s}}$ can easily be seen that the cones $\sigma_{\boldsymbol{s},\boldsymbol{t}}$ and $\sigma_{\boldsymbol{t},\boldsymbol{s}}$ are generated by the same rays, except that the first two rows are swapped. Thus, the cones $\sigma^{\vee}_{\boldsymbol{t},\boldsymbol{s}}$ and $\sigma^{\vee}_{\boldsymbol{s},\boldsymbol{t}}$ are the same up to interchanging the first two rows, which means that these cones are indeed row equivalent. Therefore the decompositions of $Q_{\boldsymbol{s}}$ and $Q_{\boldsymbol{t}}$ are compatible.

4.2 The procedure for $\sigma \subseteq \mathbb{Z}^2$

In this section we will elaborate the specific case where $d = 2$. This is the specific case where that Ilten elaborates on in [18]. The result that is described in this section is in line with the results in this article, although derived with a different approach. Being a 2-dimensional cone, σ is embedded in the lattice $N \cong \mathbb{Z}^2$ and the polyhedra $Q_{\boldsymbol{r}}$ and $Q_{\boldsymbol{s}}$ are line segments. An admissible decomposition is defined according to Definition 2.9. However, a decomposition of the form $Q_{\boldsymbol{r}} = \lambda Q_{\boldsymbol{r}} \oplus (1-\lambda) Q_{\boldsymbol{r}}$ such that $0 < \lambda < 1$ is said to be a nontrivial admissible decomposition as well.

In order for a line segment to have an admissible decomposition, it has to contain at least one lattice point. Therefore, the choice for the vectors $\boldsymbol{r}$ for the polytope $Q_{\boldsymbol{r}}$ is limited. The line segment $\sigma \cap \{\boldsymbol{x} \in \mathbb{R}^2 | \boldsymbol{r}^\top \boldsymbol{x} = 1\}$ contains a lattice point if and only if $\boldsymbol{r}$ is a (primitive) vector that is an element of the Hilbert basis of the 2-dimensional cone $\sigma^{\vee}$. Thus, from now on we will assume that $\boldsymbol{r} \in E$ where E is the Hilbert basis of $\sigma^{\vee}$.

As one can see, the cross cut is a line segment that is embedded in a 2-dimensional lattice. There exists a projection such that this line segment is embedded in a 1-dimensional lattice. As each cross cut contains at least one interior lattice point (as $\boldsymbol{r}$ is a primitive vector and an element in the Hilbert basis), one can take this lattice point as the origin of this projected lattice. The notion of being compatible does not change due to this projection, it will only simplify the computations.

4.2.1 Constructing the cone σ_h

Theorem 4.4. *Let $N \cong \mathbb{Z}^2$ be the 2-dimensional standard lattice and let $M \cong \mathbb{Z}^2$ be its dual lattice. Let $\sigma \subseteq N_{\mathbb{R}}$ be a 2-dimensional cone that is generated by two (primitive) vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. Without loss of generality one can say that*

$$\sigma = \begin{pmatrix} 1 & -q \\ 0 & n \end{pmatrix}$$

where $det(\boldsymbol{u}, \boldsymbol{v}) = n$ and $q, n \in \mathbb{Z}$ are relatively prime with $0 \leq q < n$.

Proof. For all primitive generators $\boldsymbol{u} = (u_1, u_2)$ and $\boldsymbol{v} = (v_1, v_2)$ there exists a lattice automorphism mapping $\boldsymbol{u}$ to $(1, 0)$ and $\boldsymbol{v}$ to $(-q, n)$. If the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are independent, then the matrix σ is invertible. The group $GL(2, N)$ is exactly the set of 2×2 invertible matrices over the lattice N. Therefore there exists a (not necessarily unique) lattice automorphism $\phi : N \to N$ such that $\phi(\boldsymbol{u}) = (1, 0)$ and $\phi(\boldsymbol{u}) = (-q, n)$, where q and n with $0 \leq q < n$ are relatively prime. □

From now on assume that $\boldsymbol{u} = (1, 0)$ and $\boldsymbol{v} = (-q, n)$. The dual cone $\sigma^\vee$ is generated by two rays.

$$\sigma^\vee = \begin{pmatrix} 0 & n \\ 1 & q \end{pmatrix}$$

Furthermore, since the vectors $\boldsymbol{r}$ and $\boldsymbol{s}$ are elements of the Hilbert basis of $\sigma^\vee$ we are interested in this Hilbert basis. In the 2-dimensional case the Hilbert basis of $\sigma^\vee$ is defined as the set of vectors such that every ray in $\sigma^\vee$ is the sum of a nonnegative multiple of these vectors. Let $E = \{\boldsymbol{w}^0, \ldots, \boldsymbol{w}^{g+1}\}$ be the Hilbert basis of $\sigma^\vee$ such that $\boldsymbol{w}^0 = (1, 0)$ and $\boldsymbol{w}^{g+1} = (n, q)$. The vectors $\boldsymbol{w}^i$ with $1 \leq i \leq g$ can be determined from the (negative) continued fraction expansion:

$$\frac{n}{(n-q)} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \frac{1}{a_g}}}.$$

For more information about this continued fraction expansion, see [2] and [29].

The elements of the Hilbert basis are now obtained from $\boldsymbol{w}^{i-1} + \boldsymbol{w}^{i+1} = a_i \boldsymbol{w}^i$ for $i = 1, \ldots, g$.

Remark 5. The cross cut $Q_{\boldsymbol{r}}$ contains at least one interior lattice point and is bounded if an only if $\boldsymbol{r} = \boldsymbol{w}^h = (w_1^h, w_2^h)$ for some $1 \leq h \leq g$. This line segment will be denoted by Q_h.

$$Q_{\boldsymbol{r}} := Q_h = \left[\begin{pmatrix} \frac{-q}{nw_2^h - qw_1^h} \\ \frac{n}{nw_2^h - qw_1^h} \end{pmatrix}, \begin{pmatrix} \frac{1}{w_1^h} \\ 0 \end{pmatrix} \right]$$

Theorem 4.5. *Let Q_h be the intersection of σ and $\boldsymbol{w}^h$. If $1 < h < g$ then Q_h does not have lattice endpoints. Moreover, if $h \in \{1, g\}$ then Q_h has exactly one lattice endpoint.*

Proof. If $= 1/w_1^h \in N$ then we must have $w_1^h = 1$. This means that $\boldsymbol{w}^h = (w_1^h, w_2^h) = (1, 1) = \boldsymbol{w}^1$ since $0 < w_2^h \leq w_1^h$. This gives

$$\begin{pmatrix} -q/nw_2^h - qw_1^h \\ n/nw_2^h - qw_1^h \end{pmatrix} = \begin{pmatrix} -q/n - q \\ n/n - q \end{pmatrix} \notin N.$$

Moreover, $\frac{1}{nw_2^h - qw_1^h}\binom{-q}{n} \in N$ means that $nw_2^h - qw_1^h = 1$ since n and q are relatively prime. This implies that $\boldsymbol{w}^h = \boldsymbol{w}^g$. Thus, Q_h has exactly one lattice endpoint when $h \in \{1, g\}$ and no lattice endpoints otherwise. □

The polyhedron Q is embedded in $N \cong \mathbb{Z}^2$ and it always has an interior lattice point, say $\boldsymbol{v} \in N$. Therefore we can create the projected polyhedron Q'_h which is embedded in a lattice of one dimension less, which is referred to as $N^-(\cong \mathbb{Z})$. Every point $\boldsymbol{x} \in Q_h$ is projected to a specific $x \in Q'_h$:

$$\boldsymbol{x} = \boldsymbol{v} + x\boldsymbol{r}$$

where $\boldsymbol{r}$ is the vector that is perpendicular to $\boldsymbol{w}^h$ (i.e. $\boldsymbol{r} = (w_2^h, -w_1^h)$). In general this can be captured in the following theorem.

Theorem 4.6. *Let Q be a polyhedron that is embedded in an d-dimensional lattice N. Let $\boldsymbol{v} \in Q$ be a lattice vertex. Moreover, let $g^i \in N$ for $i = 1, \ldots, d-1$ be vectors such that $\boldsymbol{v}, \boldsymbol{g}^1, \ldots, \boldsymbol{g}^{d-1}$ is a generating set of N. Now Q' is the projected polyhedron of Q if for every $\boldsymbol{x} \in Q$ there exists a $\boldsymbol{x}' \in Q'$ such that*

$$\boldsymbol{x} = \boldsymbol{v} + \sum_{i=1}^{d-1} x'_i g^i. \tag{4.3}$$

Remark 6. The vertex $\boldsymbol{v} \in Q$ is projected to the origin of the lattice $(d-1)$-dimensional lattice N^- and every lattice point $\boldsymbol{u} \in N$ corresponds to a lattice point $\boldsymbol{u}' \in N^-$.

Furthermore, if Q is a polyhedron of dimension d, then there is at least one $x \in Q$ such that Equation 4.3 does not hold true.

As Q_h is a line segment and $N \cong \mathbb{Z}^2$ there exists a projected polyhedron Q'_h in $N^- \cong \mathbb{Z}$. From now on, we will simply denote this polyhedron as Q_h for readability. Thus Q_h is seen as the line segment in $\mathbb{Z}$ and say $Q_h = [a, b]$ where $a, b \in \mathbb{Q}$ and $a < 0 < b$. The end points a and b can be computed explicitly. The projection yields

$$\begin{aligned}
\frac{1}{nw_2^h - qw_1^h}(-q, n) &= \boldsymbol{v} + a(w_2^h, -w_1^h) \\
a &= \frac{\frac{-q}{nw_2^h - qw_1^h} - v_1}{w_2^h} \\
&= -\frac{nv_1 + qv_2}{nw_2^h - qw_1^h} \qquad (4.4)
\end{aligned}$$

$$\begin{aligned}
(\frac{1}{w_1^h}, 0) &= \boldsymbol{v} + b(w_2^h, -w_1^h) \\
b &= \frac{\frac{1}{w_1^h} - v_1}{w_2^h} \\
&= \frac{v_2}{w_1^h}. \qquad (4.5)
\end{aligned}$$

Lemma 4.7. *Let*

$$Q_h = [a, b] = [a, v] \oplus [0, b - v]$$

be a decomposition of Q_h such that $v < b$. It is an admissible decomposition if and only if

1. *$v \in N^-$ and $v \neq a$, or*
2. *$v \notin N^-$ but $(b - v) \in N^-$*

In the second case v might be equal to a. Note that $v = 0$ is one of the admissible decompositions of Q_h by construction.

These are all possible admissible decompositions, up to lattice shifts. Although these lattice shifts are admissible decompositions they will not be taken into consideration in this chapter, as their decompositions do not differ from the decompositions given here.

Define the extended lattice $N^+ \cong \mathbb{Z}^3$ and fix an admissible decomposition $[a, v] \oplus [0, b-v]$. A new cone $\sigma_h \subseteq N_{\mathbb{R}}^+$ is now constructed.

$$\sigma_h = \begin{pmatrix} a & v & 0 & b - v \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

If $v = a$ then the first two columns are identical, which means that the cone σ_h has three extremal rays. The dual cone $\sigma_h^\vee$ is generated by the following vectors.

$$\sigma_h^\vee = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -a & v & 1 & 0 \\ 0 & b-v & 0 & 1 \end{pmatrix} = \begin{pmatrix} nw_2^h - qw_1^h & -w_1^h & 0 & 0 \\ nv_1 + qv_2 & vw_1 & 1 & 0 \\ 0 & v_2 - vw_1 & 0 & 1 \end{pmatrix} \tag{4.6}$$

If $v = a$ then the ray $(0,0,1)$ is not an extremal ray of $\sigma_h^\vee$. It is however an element of the Hilbert basis of $\sigma_h^\vee$.
The Hilbert basis $E_h = \{\boldsymbol{w}_h^0, \ldots, \boldsymbol{w}_h^{g+1}, \tilde{\boldsymbol{w}}_h^h\}$ of $\sigma_h^\vee$ consists of $g+3$ vectors, with $\boldsymbol{w}_h^0 = (-w_1^h, vw_1^h, v_2 - vw_1^h)$, $\boldsymbol{w}_h^{g+1} = (nw_2^h - qw_1^h, nv_1 + qv_2, 0)$ and $\tilde{\boldsymbol{w}}_h^h = (0,0,1)$. There exists a surjection ϕ_h such that $\phi_h(\boldsymbol{w}_h^i) = \boldsymbol{w}^i$ for $i = 0, \ldots, g+1$ and $\phi_h(\tilde{\boldsymbol{w}}_h^h) = \boldsymbol{w}^h$. This mapping is induced by the following matrix:

$$\phi_h = \begin{pmatrix} v_2 & w_1^h & w_1^h \\ -v_1 & w_2^h & w_2^h \end{pmatrix} \tag{4.7}$$

where $\boldsymbol{v} = (v_1, v_2) \in N$ is the lattice point that is projected to the origin.

Example.

The statements in this section will be illustrated with an example. Let $n = 4$ and $q = 1$ and determine the corresponding cones σ and $\sigma^\vee$.

$$\sigma = \begin{pmatrix} 1 & -1 \\ 0 & 4 \end{pmatrix} \qquad \sigma^\vee = \begin{pmatrix} 0 & 4 \\ 1 & 1 \end{pmatrix}$$

The continued fraction expansion for $\frac{n}{n-q}$ gives us the factors $\{a_1, \ldots, a_g\}$ which are used to determine the Hilbert basis for $\sigma^\vee$.

$$\frac{n}{n-q} = \frac{4}{3} = 2 - \frac{1}{2 - \frac{1}{2}}$$

Therefore $\{a_1, a_2, a_3\} = \{2, 2, 2\}$ and

$$\begin{aligned} 2\boldsymbol{w}^1 &= (0,1) + \boldsymbol{w}^2 \\ 2\boldsymbol{w}^2 &= \boldsymbol{w}^1 + \boldsymbol{w}^3 \\ 2\boldsymbol{w}^3 &= \boldsymbol{w}^2 + (4,1) \end{aligned}$$

which results in $\{\boldsymbol{w}^0, \boldsymbol{w}^1, \boldsymbol{w}^2, \boldsymbol{w}^3, \boldsymbol{w}^4\} = \{(0,1), (1,1), (2,1), (3,1), (4,1)\}$. This is indeed the Hilbert basis of $\sigma^\vee$. Fix $\boldsymbol{r} = \boldsymbol{w}^1 = (1,1)$. This results in the cross cut $Q_h = Q_1$:

$$Q_1 = \sigma \cap H_1 = \left[(\frac{-1}{3}, \frac{4}{3}), (1,0) \right] \to [-\frac{1}{3}, 1]$$

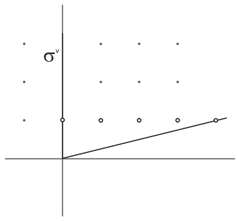

FIGURE 4.4: The dual cone $\sigma^\vee$ with its Hilbert basis.

This line segment has a unique admissible decomposition.

$$Q_1 = [-\frac{1}{3}, 0] \oplus [0, 1] \tag{4.8}$$

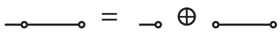

FIGURE 4.5: The admissible Minkowski decomposition of Q_1.

This leads to the cone σ_1 and its dual cone $\sigma_1^\vee$.

$$\sigma_1 = \begin{pmatrix} \frac{-1}{3} & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \sigma_1^\vee = \begin{pmatrix} 3 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \tag{4.9}$$

The Hilbert basis of $\sigma_1^\vee$ is given by the columns of E_1.

$$E_1 = \begin{pmatrix} -1 & 0 & 1 & 2 & 3 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrix $\phi_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ now induces the surjection $E_1 \to E : \phi_1(\tilde{\boldsymbol{w}}) = \boldsymbol{w}$ where E is the Hilbert basis of $\sigma^\vee$. Moreover, Figure 4.6 shows how to obtain the original cone σ from σ_1.

4.2.2 Determining the cross cut Q_{hk}

In order to construct the lifted polyhedron $Q_{\boldsymbol{r},\boldsymbol{s}} = Q_{hk}$ we are interested in a vector $\tilde{\boldsymbol{w}}^{\boldsymbol{k}} \in E_h$ such that $\phi_h(\tilde{\boldsymbol{w}}^{\boldsymbol{k}}) = \tilde{\boldsymbol{w}}^{\boldsymbol{k}}$ where E_h is the Hilbert basis of the cone $\sigma_h^\vee$.

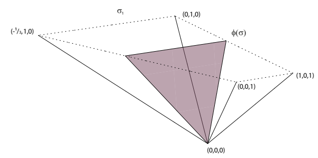

FIGURE 4.6: The intersection of σ_1.

Theorem 4.8. *Let Q_h be a polyhedron in N^- defined as the cross cut of σ and $\boldsymbol{w}^h$. For every $k \neq h$ and $1 \leq k \leq g$ there exists a unique $\tilde{\boldsymbol{w}}^{\boldsymbol{k}}$ in the Hilbert basis E_h of Q_h and the cross cut Q_{hk} is uniquely determined.*

Proof. From Theorem 4.3 we know that there exists a vector $\tilde{\boldsymbol{w}}^{\boldsymbol{k}} \in \sigma_h^\vee \cap M^+$. To see that this vector yields a unique cross cut Q_{hk} we will construct this vector. The matrix ϕ_h has $(0,1,-1)$ as its nullspace. Furthermore, for the vector $(w_1^k w_2^h - w_2^k w_1^h, v_1 w_1^k + v_2 w_2^k, 0)$ we have

$$\phi_h\big((w_1^k w_2^h - w_2^k w_1^h, v_1 w_1^k + v_2 w_2^k, 0)\big) = \boldsymbol{w}^k.$$

Thus,

$$\tilde{\boldsymbol{w}}^{\boldsymbol{k}} = (w_1^k w_2^h - w_2^k w_1^h, v_1 w_1^k + v_2 w_2^k, 0) + \alpha(0,1,-1)$$

for some $\alpha \in \mathbb{Z}$.

Now distinguish two cases, namely $h < k$ and $k < h$. First, let $h < k$. The cone is generated by the rays described in 4.6. Observe that

$$\begin{aligned} \phi_h\big((0,1,0)\big) &= \boldsymbol{w}^h \qquad \text{and} \\ \phi_h\big((n w_2^h - q w_1^h, n v_1 + q v_2, 0)\big) &= \boldsymbol{w}^{g+1} \end{aligned}$$

where ϕ_h is as described in 4.7. An immediate result is that $(w_1^k w_2^h - w_2^k w_1^h, v_1 w_1^k + v_2 w_2^k, 0) \in \sigma_h^\vee$, which means that $\alpha = 0$.

Consider the situation where $h > k$ which means that $(w_1^k w_2^h - w_2^k w_1^h, v_1 w_1^k + v_2 w_2^k, 0) \notin \sigma_h^\vee$ is not an element of the Hilbert basis E_h. For the generators of σ_h we get

$$\begin{aligned}
(0,0,1)^\top \tilde{\boldsymbol{w}}^{\boldsymbol{k}} &= -\alpha && \geq 0 \\
(b-v,0,1)^\top \tilde{\boldsymbol{w}}^{\boldsymbol{k}} &= \left(\tfrac{v_2}{w_1^h} - v\right)(w_1^k w_2^h - w_2^k w_1^h) - \alpha && \geq 0 \\
&= -v(w_1^k w_2^h - w_2^k w_1^h) + \tfrac{w_1^k}{w_1^h} v_2 w_2^h - v_2 w_2^k - \alpha && \geq 0 \\
&= -v(w_1^k w_2^h - w_2^k w_1^h) - (v_1 w_1^k + v_2 w_2^k) + \tfrac{w_1^k}{w_1^h} - \alpha && \geq 0 \\
(v,1,0)^\top \tilde{\boldsymbol{s}} &= v(w_1^k w_2^h - w_2^k w_1^h) + (v_1 w_1^k + v_2 w_2^k) + \alpha && \geq 0.
\end{aligned}$$

This yields

$$-v(w_1^k w_2^h - w_2^k w_1^h) - (v_1 w_1^k + v_2 w_2^k) \leq \alpha \leq -v(w_1^k w_2^h - w_2^k w_1^h) - (v_1 w_1^k + v_2 w_2^k) + \frac{w_1^k}{w_1^h}.$$

For $h > k$ the fraction $\frac{w_1^k}{w_1^h}$ is less than 1 and therefore there is a unique α such that it is a lattice point. This $\alpha \in N^-$ is defined as

$$\alpha = \begin{cases} 0 & \text{if } h < k \\ -v(w_1^k w_2^h - w_2^k w_1^h) - (v_1 w_1^k + v_2 w_2^k) & \text{if } h > k \text{ and } v \in N^- \\ \frac{w_1^k}{w_1^h} - v(w_1^k w_2^h - w_2^k w_1^h) - (v_1 w_1^k + v_2 w_2^k) & \text{if } h > k \text{ and } v \notin N^- \end{cases}$$

and Theorem 4.8 is proven. □

The polyhedron Q_{hk} is determined from the cone σ_h and $\tilde{\boldsymbol{w}}^{\boldsymbol{k}} \in \sigma_h^\vee \cap M^+$.

$$\begin{aligned} Q_{hk} &= \sigma_h \cap H_{\tilde{\boldsymbol{w}}^{\boldsymbol{k}}} \\ &= \begin{pmatrix} a & v & 0 & b-v \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \cap \{\boldsymbol{x} \in N_{\mathbb{R}}^+ | {\tilde{\boldsymbol{w}}^{\boldsymbol{k}}}^\top \boldsymbol{x} = 1\} \end{aligned}$$

We have $\tilde{\boldsymbol{w}}^{\boldsymbol{k}} = (w_1^k w_2^h - w_2^k w_1^h, v_1 w_1^k + v_2 w_2^k, 0) + \alpha(0, 1, -1)$ with $\alpha \in N^-$ as defined in Theorem 4.8.

For readability we will denote

$$\beta = (w_1^k w_2^h - w_2^k w_1^h).$$

The construction of this projection is described in the following section. The polyhedron Q_{hk} is embedded in the lattice N^+. However, there exists a projection polyhedron Q'_{hk} in N as described in Theorem 4.6.

4.2.2.1 Determining Q_{hk} for $h < k$.

First consider the case where $h < k$ which means that $\alpha = 0$, and therefore

$$\tilde{\boldsymbol{w}}^{\boldsymbol{k}} = (\beta, v_1 w_1^k + v_2 w_2^k, 0).$$

In this case $(0, 0, 1)^\top \tilde{\boldsymbol{w}}^{\boldsymbol{k}} = 0$ while $\boldsymbol{a}^\top \tilde{\boldsymbol{w}}^{\boldsymbol{k}} \neq 0$ for the generators $\boldsymbol{a} \neq (1, 0, 0)$ of Q_h. Hence Q_{hk} is a polyhedron built from one ray and three vertices. The vertices of Q_{hk}

are $\frac{\boldsymbol{a}}{\tilde{\boldsymbol{w}}^{\boldsymbol{k}\top}\boldsymbol{a}}$ for the generators $\boldsymbol{a} \neq (1,0,0)$ of σ_h.

$$Q_{hk} = \begin{pmatrix} -\frac{nv_1+qv_2}{nw_2^k-qw_1^k} & \frac{v}{v\beta+v_1w_1^k+v_2w_2^k} & \frac{1}{\beta} & 0 \\ \frac{nw_2^h-qw_1^h}{nw_2^k-qw_1^k} & \frac{1}{v\beta+v_1w_1^k+v_2w_2^k} & 0 & 0 \\ 0 & 0 & -\frac{1}{(v-\frac{v_2}{w_1^h})\beta} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \tag{4.10}$$

The last column represents the ray of the polyhedron. The polyhedron consists of two bounded edges combined with one ray. Moreover, the lattice vertex $\boldsymbol{u} = (\frac{1-(v_1w_1^k+v_2w_2^k)}{\beta}, 1, 0)$ is a vertex on the boundary of the polyhedron. This vertex is always a lattice point by construction, and from the following computation we know that it lies on the boundary of the polyhedron.

$$\begin{aligned} (\frac{1-(v_1w_1^k+v_2w_2^k)}{\beta}, 1, 0)^\top \tilde{\boldsymbol{w}}^{\boldsymbol{k}} &= (1-(v_1w_1^k+v_2w_2^k)) + v_1w_1^k + v_2w_2^k) \\ &= 1 \end{aligned}$$

where $\tilde{\boldsymbol{w}}^{\boldsymbol{k}}$ is taken as above. Taking the lattice vertex $\boldsymbol{u}$ as the projection vertex and the vectors $(v_1w_1^k + v_2w_2^k, -\beta, 0)$ and $(0,0,1)$ as generators yields the polyhedron Q'_{hk} in $N \cong \mathbb{Z}^2$.

This gives

$$\begin{pmatrix} -\frac{nv_1+qv_2}{nw_2^k-qw_1^k} - \frac{1-(v_1w_1^k+v_2w_2^k)}{\beta} \\ \frac{nw_2^h-qw_1^h}{nw_2^k-qw_1^k} - 1 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} v_1w_1^k + v_2w_2^k \\ -\beta \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

for the first vertex of Q_{hk}. From this equation we see that this vertex is projected to $\boldsymbol{x} = (x_1, x_2)$ with

$$\begin{aligned} x_2 &= 0 && \text{and} \\ x_1 &= \frac{1}{\beta}(1 - \frac{nw_2^h - qw_1^h}{nw_2^k - qw_1^k}). \end{aligned}$$

For the second vertex we can derive the projection as well:

$$\begin{pmatrix} \frac{v}{v\beta+v_1w_1^k+v_2w_2^k} - \frac{1-(v_1w_1^k+v_2w_2^k)}{\beta} \\ \frac{1}{v\beta+v_1w_1^k+v_2w_2^k} - 1 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} v_1w_1^k + v_2w_2^k \\ -\beta \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{thus}$$

$$\begin{aligned} x_2 &= 0 && \text{and} \\ x_1 &= \frac{1}{\beta}(1 - \frac{1}{v\beta + v_1w_1^k + v_2w_2^k}). \end{aligned}$$

The third vertex becomes

$$\begin{aligned}
\begin{pmatrix} \frac{1}{\beta} - \frac{1-(v_1 w_1^k + v_2 w_2^k)}{\beta} \\ -1 \\ -\frac{1}{\beta(v - \frac{v_2}{w_1^h})} \end{pmatrix} &= x_1 \begin{pmatrix} v_1 w_1^k + v_2 w_2^k \\ -\beta \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} && \text{thus} \\
x_2 &= -\frac{1}{\beta(v - \frac{v_2}{w_1^h})} && \text{and} \\
x_1 &= \frac{1}{\beta}.
\end{aligned}$$

Thus, the projected polyhedron is described as:

$$Q'_{hk} = \begin{pmatrix} \frac{1}{\beta}(1 - \frac{n w_2^h - q w_1^h}{n w_2^k - q w_1^k}) & \frac{1}{\beta}(1 - \frac{1}{v\beta + v_1 w_1^k + v_2 w_2^k}) & \frac{1}{\beta} & 0 \\ 0 & 0 & -\frac{1}{(v - \frac{v_2}{w_1^h})\beta} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{4.11}$$

4.2.2.2 Determining Q_{hk} for $h > k$.

Now determine Q_{hk} for $k < h$. There are two options for $\alpha \neq 0$. First consider the decomposition of Q_h as a decomposition of the first category as described in Lemma 4.7 which means that $\boldsymbol{v} \in N$. Thus $\alpha = -v\beta - (v_1 w_1^k + v_2 w_2^k) \in \mathbb{Z}$. The cross cut Q_{hk} is computed as the intersection of σ_h and

$$\tilde{\boldsymbol{w}}^{\boldsymbol{k}} = (\beta, v_1 w_1^k + v_2 w_2^k, 0) + \alpha(0, 1, -1) \tag{4.12}$$

with $\alpha = -v\beta - (v_1 w_1^k + v_2 w_2^k)$ and $\boldsymbol{v} \in N$.

Now $\tilde{\boldsymbol{w}}^{\boldsymbol{k}\top} \boldsymbol{a} > 0$ for the rays of σ_h that are not equal to $(v, 1, 0)$ and $\tilde{\boldsymbol{w}}^{\boldsymbol{k}\top}(v, 1, 0) = 0$. Hence, again Q_{hk} consists of three vertices $\frac{\boldsymbol{a}}{\tilde{\boldsymbol{w}}^{\boldsymbol{k}\top} \boldsymbol{a}}$ and one ray. Like the previous polyhedron, it consists of two bounded edges and one ray.

$$Q_{hk} = \begin{pmatrix} \frac{a}{\beta(a-v)} & 0 & (b-v)\frac{w_1^h}{w_1^k} & v \\ \frac{1}{\beta(a-v)} & 0 & 0 & 1 \\ 0 & \frac{1}{v\beta + v_1 w_1^k + v_2 w_2^k} & \frac{w_1^h}{w_1^k} & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \tag{4.13}$$

Here, a and b are as defined in Equations (4.4) and (4.5). If α is equal to 1, then the center vertex $(0, 0, -\frac{1}{\alpha})$ is a lattice point, hence it can be taken as the vertex $\boldsymbol{u}$ in Equation 4.7. Otherwise there exists a λ such that $\lambda\alpha \in N$ and $-\frac{1}{\alpha} + \lambda\beta \in N$. This gives a lattice point

$$\boldsymbol{u} = (0, 0, -\frac{1}{\alpha}) + \lambda(\alpha, 0, \beta)$$

that is contained in a bounded edge of the polyhedron. Taking $\boldsymbol{g}^1 = (-\alpha, 0, -\beta)$ and $\boldsymbol{g}^2 = (v, 1, 0)$ as generators, yields the projection of the vertices.

The first vertex is determined by the following.

$$\begin{aligned}
\begin{pmatrix} \frac{a}{\beta(a-v)} - \lambda\alpha \\ \frac{1}{\beta(a-v)} \\ \frac{1}{\alpha} - \lambda\beta \end{pmatrix} &= x_1 \begin{pmatrix} -\alpha \\ 0 \\ -\beta \end{pmatrix} + x_2 \begin{pmatrix} v \\ 1 \\ 0 \end{pmatrix} \\
x_2 &= \frac{1}{\beta(a-v)} \\
x_1 &= \lambda - \frac{1}{\alpha\beta}
\end{aligned}$$

The second vertex is retrieved by:

$$\begin{aligned}
\begin{pmatrix} -\lambda\alpha \\ 0 \\ -\frac{1}{v\beta + v_1 w_1^k + v_2 w_2^k} + \frac{1}{\alpha} - \lambda\beta \end{pmatrix} &= x_1 \begin{pmatrix} -\alpha \\ 0 \\ -\beta \end{pmatrix} + x_2 \begin{pmatrix} v \\ 1 \\ 0 \end{pmatrix} \\
x_2 &= 0 \\
x_1 &= \lambda
\end{aligned}$$

The third vertex is obtained from the following description.

$$\begin{aligned}
\begin{pmatrix} (b-v)\frac{w_1^h}{w_1^k} - \lambda\alpha \\ 0 \\ \frac{w_1^h}{w_1^k} + \frac{1}{\alpha} - \lambda\beta \end{pmatrix} &= x_1 \begin{pmatrix} -\alpha \\ 0 \\ -\beta \end{pmatrix} + x_2 \begin{pmatrix} v \\ 1 \\ 0 \end{pmatrix} \\
x_2 &= 0 \\
x_1 &= \lambda - \frac{1}{\beta}(\frac{w_1^h}{w_1^k} + \frac{1}{\alpha})
\end{aligned}$$

Therefore, the polyhedron is projected to the polyhedron:

$$Q'_{hk} = \begin{pmatrix} \lambda - \frac{1}{\alpha\beta} & \lambda & \lambda - \frac{1}{\beta}(\frac{w_1^h}{w_1^k} + \frac{1}{\alpha}) & 0 \\ \frac{1}{\beta(a-v)} & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{4.14}$$

Finally the third option for the polyhedron Q_{hk} has again $k < h$ but now $\alpha = -v\beta - (v_1 w_1^k + v_2 w_2^k) + \frac{w_1^k}{w_1^h}$. This gives

$$\tilde{\boldsymbol{w}}^{\boldsymbol{k}} = (\beta, -v\beta + \frac{w_1^k}{w_1^h}, v\beta + v_1 w_1^k + v_2 w_2^k - \frac{w_1^k}{w_1^h}) \tag{4.15}$$

The polyhedron Q_{hk} consists of two bounded edges (three vertices) and one ray:

$$Q_{hk} = \begin{pmatrix} \frac{a}{\beta(a-v)+\frac{w_1^k}{w_1^h}} & v\frac{w_1^h}{w_1^k} & 0 & b-v \\ \frac{1}{\beta(a-v)+\frac{w_1^k}{w_1^h}} & \frac{w_1^h}{w_1^k} & 0 & 0 \\ 0 & 0 & -\frac{1}{\alpha} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}. \tag{4.16}$$

Observe that $v = a$ indeed yields that the first vertex is the same as the second vertex of Q_{hk}. The vertex $(v\frac{w_1^h}{w_1^k}, \frac{w_1^h}{w_1^k}, 0)$ is a lattice vertex by definition, and therefore it can be taken as $\boldsymbol{u}$ for the projection. Together with generators $(v - \beta\frac{w_1^k}{w_1^h}, 1, 0)$ and $(b-v, 0, 1)$ we can compute the new vertices. The first vertex gives:

$$\begin{aligned} \begin{pmatrix} \frac{a}{\beta(a-v)+\frac{w_1^k}{w_1^h}} - v\frac{w_1^h}{w_1^k} \\ \frac{1}{\beta(a-v)+\frac{w_1^k}{w_1^h}} - \frac{w_1^h}{w_1^k} \\ 0 \end{pmatrix} &= x_1 \begin{pmatrix} v - \beta\frac{w_1^k}{w_1^h} \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} b-v \\ 0 \\ 1 \end{pmatrix} \\ x_2 &= 0 \\ x_1 &= \frac{1}{\beta(a-v)+\frac{w_1^k}{w_1^h}} - \frac{w_1^h}{w_1^k}. \end{aligned}$$

By definition, the second vertex is mapped to $\mathbf{0}$. The third vertex is mapped to:

$$\begin{aligned} \begin{pmatrix} -v\frac{w_1^h}{w_1^k} \\ -\frac{w_1^h}{w_1^k} \\ -\frac{1}{\alpha} \end{pmatrix} &= x_1 \begin{pmatrix} v - \beta\frac{w_1^k}{w_1^h} \\ 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} b-v \\ 0 \\ 1 \end{pmatrix} \\ x_2 &= -\frac{1}{\alpha} \\ x_1 &= -\frac{w_1^h}{w_1^k}. \end{aligned}$$

This yields the following polyhedron.

$$Q'_{hk} = \begin{pmatrix} \frac{1}{\beta(a-v)+\frac{w_1^k}{w_1^h}} - \frac{w_1^h}{w_1^k} & 0 & -\frac{w_1^h}{w_1^k} & 0 \\ 0 & 0 & -\frac{1}{\alpha} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \tag{4.17}$$

Example.

To continue with Example 4.2.1 fix $\boldsymbol{w}^2 = (2, 1)$. From $h = 1 < 2 = k$ we know

$$\tilde{\boldsymbol{w}}^{\mathbf{2}} = (\beta, v_1 w_1^k + v_2 w_2^k, 0) = (1, 1, 0).$$

The intersection of $\sigma_1 = \begin{pmatrix} \frac{-1}{3} & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ and $H_{\tilde{\boldsymbol{w}}^2}$ results in $Q_{hk} = Q_{12}$.

$$\begin{aligned} Q_{12} &= \sigma_1 \cap H_{\tilde{\boldsymbol{w}}^2} \\ &= \begin{pmatrix} -\frac{1}{2} & 0 & 1 & 0 \\ \frac{3}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{aligned}$$

In order to construct the projected polyhedron Q'_h, consider the lattice point $\boldsymbol{u} = (\frac{1-(v_1 w_1^k + v_2 w_2^k)}{\beta}, 1, 0) = (0, 1, 0)$. This is the center vertex of the polyhedron Q_{12}. The projection can now be derived from the generators $(1, -1, 0)$ and $(0, 0, 1)$.

$$Q'_{12} = \begin{pmatrix} -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \tag{4.18}$$

The next step is to compute the polyhedron Q'_{21}. First determine Q'_2 and all its admissible decompositions.

$$\begin{aligned} Q_2 &= \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 2 \end{pmatrix} \to [-\frac{1}{2}, \frac{1}{2}] && (4.19) \\ &= [-\frac{1}{2}, 0] \oplus [0, \frac{1}{2}] && (4.20) \\ &= \begin{pmatrix} -\frac{1}{2} \\ 2 \end{pmatrix} \oplus [0, 1] && (4.21) \end{aligned}$$

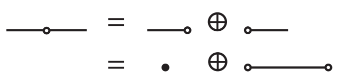

FIGURE 4.7: The admissible Minkowski decompositions of Q_2.

Consider the decomposition given in 4.20. The cone corresponding to this decomposition is denoted by σ_2^1.

$$\sigma_2^1 = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \text{and} \qquad \sigma_2^{1\vee} = \begin{pmatrix} 2 & 0 & 0 & -2 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Furthermore,

$$\begin{aligned} \alpha &= v\beta - (v_1 w_1^k + v_2 w_2^k) \\ &= -1 \end{aligned}$$

and therefore

$$\begin{aligned} \tilde{\boldsymbol{w}}^{\mathbf{1}} &= (\beta, v_1 w_1^k + v_2 w_2^k, 0) + \alpha(0,1,-1) \\ &= (-1,1,0) - 1(0,1,-1) \\ &= (-1,0,1). \end{aligned}$$

The polyhedron Q_{21}^1 is the intersection of σ_2^1 and the hyperplane corresponding to this $\tilde{\boldsymbol{w}}^{\mathbf{1}}$.

$$Q_{21}^1 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

By taking

$$\boldsymbol{u} = (0,0,-\frac{1}{\alpha}) + \lambda(\alpha,0,1) = (0,0,1)$$

together with generators $(1,0,1)$ and $(0,1,0)$ this polyhedron can be described in the two-dimensional lattice N.

$$Q_{21}^1 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \tag{4.22}$$

4.2.3 Constructing the cone σ_{hk} and check compatibility

If there exists an admissible decomposition of the polyhedron $Q_{hk} \in N$ then its summands can be embedded in $N_{\mathbb{R}}^+ \cong \mathbb{R}^3$.

The cone $\sigma_{hk} \subseteq N_{\mathbb{R}}^+$ can be constructed from this embedding. The polyhedron Q_{hk} is built from three vertices and one ray.

$$Q_{hk} = \begin{pmatrix} \boldsymbol{v}^0 & \boldsymbol{v}^1 & \boldsymbol{v}^2 & \boldsymbol{r} \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Assume that $\boldsymbol{v}^1$ is the center vertex, so $[\boldsymbol{v}^0, \boldsymbol{v}^1]$ and $[\boldsymbol{v}^1, \boldsymbol{v}^2]$ are the bounded edges of Q_{hk}. The vertex $\boldsymbol{v}^i$ might be, but not necessarily is, a lattice point.
There are three different descriptions for Q_{hk} possible, which all yield a different description of the cone σ_{hk}.

First consider the description given in (4.10), where $h < k$ and $\alpha = 0$. The vertices are

$$\begin{aligned} \boldsymbol{v}^0 &= \left(\frac{1}{\beta}(1 - \frac{nw_2^h - qw_1^h}{nw_2^k - qw_1^k}), 0\right) \\ \boldsymbol{v}^1 &= \left(\frac{1}{\beta}(1 - \frac{1}{v\beta + v_1 w_1^k + v_2 w_2^k}), 0\right) \qquad \text{and} \\ \boldsymbol{v}^2 &= \left(\frac{1}{\beta}, \frac{-1}{(v - \frac{v_2}{w_1^h})\beta}\right) \end{aligned}$$

The fact that $h < k$ implies that $\beta > 0$ and therefore the edge $[\boldsymbol{v}^1, \boldsymbol{v}^2]$ does not have interior lattice points. This directly follows from the first entry of both vertices. Thus, an admissible decomposition of Q_{hk} can be described as

$$Q_{hk} = \begin{pmatrix} \boldsymbol{v}^0 & \boldsymbol{u} & \boldsymbol{r} \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} \boldsymbol{0} & \boldsymbol{v}^1 - \boldsymbol{u} & \boldsymbol{v}^2 - \boldsymbol{u} & \boldsymbol{r} \\ 1 & 1 & 1 & 0 \end{pmatrix} \tag{4.23}$$

where $\boldsymbol{u} \in (\boldsymbol{v}^0, \boldsymbol{v}^1]$ is a lattice point. If the center vertex $\boldsymbol{v}^1$ is a lattice point and there exists a lattice point $\boldsymbol{u} \neq \boldsymbol{v}^1$ then there is another admissible decomposition:

$$Q_{hk} = \begin{pmatrix} \boldsymbol{v}^0 & \boldsymbol{u} & \boldsymbol{u} - \boldsymbol{v}^1 + \boldsymbol{v}^2 & \boldsymbol{r} \\ 1 & 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} \boldsymbol{0} & \boldsymbol{v}^1 - \boldsymbol{u} & \boldsymbol{r} \\ 1 & 1 & 0 \end{pmatrix} \tag{4.24}$$

For the other two descriptions of Q_{hk} that are given in (4.13) and (4.16) we have that $h > k$ and $\alpha \neq 0$. Take $\alpha = -(v_1 w_1^k + v_2 w_2^k)$ and consider the bounded edges of the polyhedron, that can be described by the following vertices:

$$\begin{aligned} \boldsymbol{v}^0 &= \left(\frac{1}{\alpha\beta} - \lambda, \frac{1}{\beta(a - v)}\right) \\ \boldsymbol{v}^1 &= (-\lambda, 0) \qquad \text{and} \\ \boldsymbol{v}^2 &= \left(\frac{1}{\beta}(\frac{w_1^k}{w_1^h} + \frac{1}{\alpha}) - \lambda, 0\right). \end{aligned}$$

The center vertex is $\boldsymbol{v}^1$ and the bounded edge $[\boldsymbol{v}^0, \boldsymbol{v}^1]$ does not have an interior lattice point. This can be seen from the first entries of $\boldsymbol{v}^0$ and $\boldsymbol{v}^1$ and the fact that $0 > \alpha \in \mathbb{Z}$. Therefore any admissible Minkowski decomposition of Q_{hk} is induced by a lattice point $\boldsymbol{u} \in [\boldsymbol{v}^1, \boldsymbol{v}^2)$.

$$Q_{hk} = \begin{pmatrix} \boldsymbol{v}^0 & \boldsymbol{v}^1 & \boldsymbol{u} & \boldsymbol{r} \\ 1 & 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} \boldsymbol{0} & \boldsymbol{v}^2 - \boldsymbol{u} & \boldsymbol{r} \\ 1 & 1 & 0 \end{pmatrix} \tag{4.25}$$

Furthermore, if the vertex $\boldsymbol{v}^1$ is a lattice point and there exists a lattice point $\boldsymbol{u} \neq \boldsymbol{v}^1$ then there is another admissible decomposition:

$$Q_{hk} = \begin{pmatrix} \boldsymbol{v}^0 & \boldsymbol{v}^1 & \boldsymbol{v}^1 - \boldsymbol{u} + \boldsymbol{v}^2 & \boldsymbol{r} \\ 1 & 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} \boldsymbol{0} & \boldsymbol{u} - \boldsymbol{v}^1 & \boldsymbol{r} \\ 1 & 1 & 0 \end{pmatrix} \tag{4.26}$$

For the polyhedron Q_{hk} with $h > k$ and $\alpha = \frac{w_1^k}{w_1^h} - (v_1 w_1^k + v_2 w_2^k) \in \mathbb{Z}$ the center vertex is $\boldsymbol{v}^1 = (0,0)$. Furthermore,

$$\begin{aligned}\boldsymbol{v}^0 &= \left(\frac{1}{\beta(a-v)+\frac{w_1^h}{w_1^k}}, 0\right)\\ \boldsymbol{v}^2 &= \left(-\frac{w_1^k}{w_1^h}, -\frac{1}{\alpha}\right)\end{aligned}$$

As $h > k$ we know that $\frac{w_1^k}{w_1^h} < 1$ which tells us that the edge $[\boldsymbol{v}^1, \boldsymbol{v}^2]$ does not have an interior lattice point. An admissible decomposition for this description of Q_{hk} is therefore induced by a lattice point $\boldsymbol{u} \in (\boldsymbol{v}^0, \boldsymbol{v}^1]$. The decomposition and the cone σ_{hk} is equivalent to the description in 4.23 and when applicable 4.24.

If for the decomposition of Q_h we have that $v = a$ then the decomposition of Q_{hk} consists of one bounded edge, as this results in $\boldsymbol{v}^0 = \boldsymbol{v}^1$. This bounded edge does not have any interior lattice points, which confirms the fact that there does not exist an admissible decomposition for this polyhedron for $h \neq k$.

Theorem 4.9. *A polytope Q_h has an admissible decomposition with $v = a$ if and only if $w_1^h = nw_2^h - qw_1^h$. Moreover, if such a decomposition exists, then there does not exist a $\boldsymbol{w}^k$ with $k \neq h$ with $0 < k < g$ such that Q_k has an admissible decomposition with $v = a$.*

Proof. In order for $a \oplus [0, b-a]$ to be an admissible decomposition, the line segment $[a,b]$ has to have integral length. Thus,

$$\begin{aligned}\frac{1}{w_1^h} + \frac{q}{nw_2^h - qw_1^h} = \frac{nw_2^h}{w_1^h(nw_2^h - qw_1^h)} &\in \mathbb{Z} \qquad \text{and}\\ \frac{n}{nw_2^h - qw_1^h} &\in \mathbb{Z} \quad .\end{aligned}$$

An immediate result is that $w_1^h | n$. Moreover, it means that either $nw_2^h - qw_1^h | w_1^h$ or $w_1^h | nw_2^h - qw_1^h$. Let $x \in \mathbb{Z}$ be an integer such that $xw_1^h = n$. The first case yields

$$\begin{aligned}nw_2^h - qw_1^h &\mid w_1^h\\ xw_1^h w_2^h - qw_1^h &\mid w_1^h\\ (xw_2^h - q)w_1^h &\mid w_1^h\end{aligned}$$

which implies that $nw_2^h - qw_1^h = w_1^h$.

In the second case we have that $w_1^h | (xw_2^h - q)w_1^h$. Let $y \geq 1$ be an integer such that

$nw_2^h - qw_1^h = yw_1^h$. This yields

$$\begin{aligned} xw_2^h - q &= y \\ xw_2^h &= q + y. \end{aligned}$$

Thus, $x|n$ and $x|q+y$. From the fact that $\frac{n}{nw_2^h - qw_1^h} \in \mathbb{Z}$ we know that $\frac{n}{yw_1^h} = \frac{x}{y}$ is a lattice point, and therefore $y|x$. This yields $y|n$ and $y|q+y$, and therefore $y|q$. The fact that q and n are relatively prime yields $y = 1$.

This means that Q_h has integral length if and only if $nw_2^h - qw_1^h = w_1^h$. The next thing to prove is that if there exists a $\boldsymbol{w}^h$ such that $nw_2^h - qw_1^h = w_1^h$, then this $\boldsymbol{w}^h$ is unique in the Hilbert basis of $\sigma^\vee$. To see this, let $\boldsymbol{w}^h$ be the smallest element of E such that $w_1^h = nw_2^h - qw_1^h$. For every $\boldsymbol{w}^k$ with $k > h$ we know that $w_1^k > w_1^h$ and $w_2^k \geq w_2^h$. Moreover, $nw_2^h - qw_1^h > nw_2^k - qw_1^k$. Thus,

$$nw_2^k - qw_1^k < nw_2^h - qw_1^h = w_1^h < w_1^k.$$

This shows that if there exists a $\boldsymbol{w}^h$ with $0 < h < g$ such that $w_1^h = nw_2^h - qw_1^h$, then this vector is unique. Therefore we can conclude that an admissible decomposition of Q_h such that $v = a$ is not compatible with any other admissible decomposition of Q_k with $k \neq h$. □

Example.

To continue with the example, decompose the polyhedra Q_{12} and Q_{21}^1. First consider

$$Q_{12} = \begin{pmatrix} -\frac{1}{2} & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Its bounded edges do not have any interior lattice points. Hence there is only one admissible decomposition with $\boldsymbol{u} = \boldsymbol{v}^1 = (0,0)$.

$$Q_{12} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Now consider Q_{21}^1. It is a polyhedron of the second form with $\alpha = -(v_1 w_1^k + v_2 w_2^k) =$

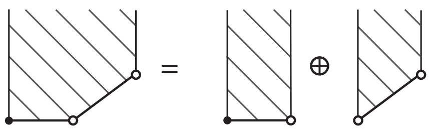

FIGURE 4.8: Minkowski decomposition of Q_{12}.

-1. The bounded edges do not have interior vertices, which yields a unique admissible decomposition with $\boldsymbol{u} = \boldsymbol{v}^1 = (0,0)$.

$$
\begin{aligned}
Q_{21}^1 &= \begin{pmatrix} -1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}
\end{aligned}
$$

The third polyhedron Q_{21}^2 is a line segment with no interior lattice points combined with

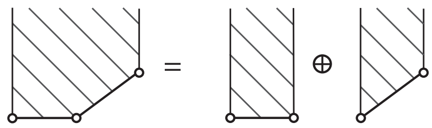

FIGURE 4.9: Minkowski decomposition of Q_{21}^1.

one ray. According to the definition of admissibility, there does not exist an admissible decomposition of Q_{21}^2, which means that it can not be lifted in order to obtain cone σ_{21}^2.

The computations of the admissible decompositions result in the cones $\sigma_{12}, \sigma_{12}^{\vee}$.

$$
\sigma_{12} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma_{12}^{\vee} = \begin{pmatrix} -1 & 2 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}
$$

Similarly, the cones σ_{21}^1 and $\sigma_{21}^{1\vee}$ can be computed from the polyhedron Q_{21}^1.

$$
\sigma_{21}^1 = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma_{21}^{1\vee} = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

There exists a morphism between the cones σ_{12} and σ_{21}^1. The matrix ϕ_{12} defines this morphism such that the matrix $\sigma_{21}^{\vee}$ can be written as $\phi_{12}\sigma_{12}^{\vee}$.

$$
\phi_{12} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.27}
$$

This means that there exists a common refinement for the cross cuts Q_1 and Q_2 with the corresponding decompositions.

4.3 More results for the example with $(n, q) = (4, 1)$

In the example of the previous section we showed that the first decomposition of cross cut Q_2 is compatible to the unique decomposition of Q_1. In this section all decompositions of the cross cuts Q_1, Q_2 and Q_3 are compared. This is done via the procedure which is described in the previous section.

4.3.1 Are the decompositions of Q_1 and Q_3 compatible?

4.3.1.1 Decomposing and lifting Q_{13}

In this case $\boldsymbol{w}^h = (1, 1)$ and $\boldsymbol{w}^k = (3, 1)$. This gives $\alpha = 0$ and $\tilde{\boldsymbol{w}}^{\mathbf{3}} = (2, 1, 0)$ .The polytope Q_1 has a unique decomposition and the corresponding cone σ_1 is computed in the previous section and described in (4.9). The intersection of σ_1 and $H_{\tilde{\boldsymbol{w}}^{\mathbf{3}}} = \{\boldsymbol{x} \in \mathbb{R}^3 | \tilde{\boldsymbol{w}}^{\mathbf{3}\top} \boldsymbol{x} = 1\}$ yields the polyhedron Q_{13}.

$$
\begin{aligned}
Q_{13} &= \sigma_1 \cap H_{\tilde{\boldsymbol{w}}^{\mathbf{3}}} \\
&= \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \cap H_{\tilde{\boldsymbol{w}}^{\mathbf{3}}} \\
&= \begin{pmatrix} -1 & 0 & \frac{1}{2} & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}
\end{aligned}
$$

where the generators $(1, -2, 0)$ and $(0, 0, 1)$ impose the projection. There is one admissible decomposition of Q_{13}, as the edges do not have any interior lattice points. This decomposition is

$$
\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \\ 1 & 1 & 0 \end{pmatrix}.
$$

The polytope Q_{13} gives the σ_{13}.

$$\sigma_{13} = \begin{pmatrix} -1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma_{13}^{\vee} = \begin{pmatrix} -1 & 1 & -2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

4.3.1.2 Decomposing and lifting Q_{31}

Now consider $\boldsymbol{w}^3 = (3,1)$ and determine the polyhedron which is the intersection of the hyperplane from $\boldsymbol{w}^3$ and σ. This yields

$$\begin{aligned} Q_3 &= \begin{pmatrix} \frac{1}{3} & -1 \\ 0 & 4 \end{pmatrix} \to [-1, \frac{1}{3}] \\ &= [-1, 0] \oplus [0, \frac{1}{3}]. \end{aligned} \tag{4.28}$$

The projection is imposed by the generators $(0,1)$ and $(1,-3)$. The given decomposition is the only admissible decomposition of Q_3. These summands yield the cone σ_3.

$$\sigma_3 = \begin{pmatrix} -1 & 0 & 0 & \frac{1}{3} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The decomposition is of the first category and therefore

$$\begin{aligned} \tilde{\boldsymbol{w}}^{\mathbf{1}} &= (\beta, -v\beta, v\beta + v_1 w_1^k + v_2 w_2^k) \\ &= (w_1^1 w_2^3 - w_2^1 w_1^3, 0, v_1 w_1^1 + v_2 w_2^1) \\ &= (1-3, 0, 1) = (-2, 0, 1). \end{aligned}$$

The intersection with σ_3 and $\tilde{\boldsymbol{w}}^{\mathbf{1}}$ results in polyhedron Q_{31}. It has a unique admissible decomposition.

$$\begin{aligned} Q_{31} &= \begin{pmatrix} -\frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -\frac{1}{2} & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{aligned}$$

The vectors $(1, 0, 2)$ and $(0, 1, 0)$ are the generators for the projection. This unique decomposition gives the following cone σ_{31}.

$$\sigma_{31} = \begin{pmatrix} -\frac{1}{2} & 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma_{31}^{\vee} = \begin{pmatrix} -1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

4.3.1.3 Comparing the cones σ_{13} and σ_{31}

There exists a matrix ϕ_{13} which maps the generators of σ_{13} to the generators of σ_{31}.

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

The existence of such a matrix shows that the two original decompositions of Q_1 and Q_3 are compatible. This matrix is defined by the mapping matrices that maps the vectors of σ_{hk} to σ:

$$\begin{pmatrix} (1,0) & \boldsymbol{w}^h & \boldsymbol{w}^k & \boldsymbol{w}^k \end{pmatrix} \tag{4.29}$$

where $(1, 0)$ is a vector in the Nullspace of $\boldsymbol{v}$ which is the vertex of $Q_h \in N$ that represents the origin in the projection of the polyhedron. One can see that ϕ_{13} defines a mapping between the two matrices.

$$\begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 & 3 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

Hence, the equivalency of the cones $\sigma_{13}^{\vee}$ and $\sigma_{31}^{\vee}$ confirms the compatibility of the (unique) decompositions of Q_1 and Q_3.

4.3.2 Are the decompositions of Q_2 and Q_3 compatible?

4.3.2.1 Decomposing and lifting Q_{23}

The polyhedron Q_2 has two admissible decompositions. The second decomposition is of the last category with $v = a$. According to Section 4.2.3 this leads to a polyhedron Q_{23} which does not have an admissible decomposition. Therefore the only decomposition of Q_2 that will be treated is given in 4.20.

The cone σ_2 is determined in the previous section. Furthermore $\alpha = 0$ yields $\tilde{\boldsymbol{w}}^{\mathbf{3}} = (1, 1, 0)$. The cross cut of σ_2 and $\tilde{\boldsymbol{w}}^{\mathbf{3}}$ yields the polyhedron Q_{23} which has a unique admissible decomposition.

$$
\begin{aligned}
Q_{23} &= \begin{pmatrix} -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \cap H_{\tilde{\boldsymbol{w}}^{\mathbf{3}}} \\
&= \begin{pmatrix} -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}
\end{aligned}
$$

The lifted cone $\sigma_{23}^1$is generated by the following rays.

$$
\sigma_{23}^1 = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma_{23}^{1\vee} = \begin{pmatrix} -2 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}
$$

4.3.2.2 Decomposing and lifting Q_{32}

The next step is to follow the same procedure for $Q_{\boldsymbol{s}} = Q_3$. The cone σ_3 is computed in Section 4.3.1.2. From $\alpha = -v\beta - (v_1 w_1^k + v_2 w_2^k) = -1$ one can compute $\tilde{\boldsymbol{w}}^{\mathbf{1}} = (-1, 0, 1)$. The cross cut yields the polyhedron Q_{32} which has a unique admissible decomposition.

$$
\begin{aligned}
Q_{32} &= \begin{pmatrix} -1 & 0 & 0 & \frac{1}{3} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \cap H_{\tilde{\boldsymbol{w}}^{\mathbf{1}}} \\
&= \begin{pmatrix} -1 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & \frac{3}{2} & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}
\end{aligned}
$$

This determines cone σ_{32}.

$$\sigma_{32} = \begin{pmatrix} -1 & 0 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma_{32}^{\vee} = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

4.3.2.3 Comparing the cones σ_{23}^{1} and σ_{32}

The same conclusion can be drawn as for the cones σ_{21}^{1} and σ_{12}. The first decomposition of Q_2 is compatible to the unique decomposition of Q_3, but the second decomposition of Q_2 is not.The mapping matrix is described as below:

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

4.3.3 Conclusion

To validate these answers, we go back to the original starting point, where this problem is described with Deformation theory and toric varieties. One can varify that a different approach does not lead to different results, as we see from [18]. This gives more insight in the problem that can be used in further research.

Chapter 5

example $(-q, n) = (-3, 8)$

5.1 The construction of σ

Let $N, M \cong \mathbb{Z}^2$ be dual lattices. Now consider the example with $(n, q) = (8, 3)$. Consider the cones that are generated by the following matrices.

$$\sigma = \begin{pmatrix} 1 & -3 \\ 0 & 8 \end{pmatrix} \qquad \sigma^\vee = \begin{pmatrix} 0 & 8 \\ 1 & 3 \end{pmatrix}$$

The Hilbert basis of $\sigma^\vee$ is computed from the continued fraction expansion of $\frac{n}{n-q}$.

$$\{\boldsymbol{w}^0, \ldots, \boldsymbol{w}^{g+1}\} = \left\{ \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 8 \\ 3 \end{smallmatrix}\right) \right\}$$

Each cross cut $Q_i = \sigma \cap H_i$ yields a polytope for $i = 1, \ldots, 3$. This polytope is a line segment embedded in N. In this section the compatibility of all possible decompositions of polytopes Q_i is verified, using the procedure described earlier.

5.2 The cross cuts and their decompositions

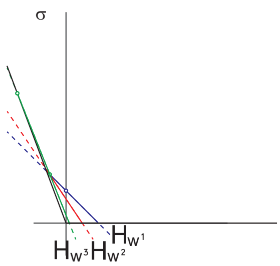

FIGURE 5.1: Cone σ and the intersections Q_1, Q_2 and Q_3

First determine the cross cuts Q_1, Q_2 and Q_3. Figure 5.1 shows the cone σ and these intersections. The line sement Q_1 has a unique decomposition.

$$\begin{aligned} Q_1 &= \begin{pmatrix} 1 & -\frac{3}{5} \\ 0 & \frac{8}{5} \end{pmatrix} \to [-\frac{3}{5}, 1] \\ &= [-\frac{3}{5}, 0] \oplus [0, 1] \end{aligned} \tag{5.1}$$

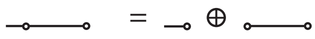

FIGURE 5.2: Decomposition of Q_1

Now consider the polytope derived from $\boldsymbol{w}^2 = (2, 1)$. It does not have a unique admissible decomposition, but four of them which are given below.

$$Q_2 = \begin{pmatrix} -\frac{3}{2} & \frac{1}{2} \\ 4 & 0 \end{pmatrix} \to [-\frac{3}{2}, \frac{1}{2}]$$

$$= [-\frac{3}{2}, -\frac{1}{2}] \oplus [0, 1] \tag{5.2}$$

$$= [-\frac{3}{2}, 0] \oplus [0, \frac{1}{2}] \tag{5.3}$$

$$= [-\frac{3}{2}, -1] \oplus [0, \frac{3}{2}] \tag{5.4}$$

$$= \{-\frac{3}{2}\} \oplus [0, 2] \tag{5.5}$$

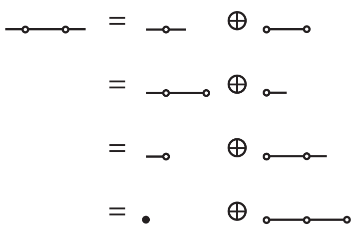

FIGURE 5.3: the decompositions of Q_2

Similarly, the unique decomposition of Q_3 can be determined.

$$\begin{aligned} Q_3 &= \begin{pmatrix} \frac{1}{5} & -3 \\ 0 & 8 \end{pmatrix} \to [-1, \frac{3}{5}] \\ &= [-1, 0] \oplus [0, \frac{3}{5}] \end{aligned} \tag{5.6}$$

FIGURE 5.4: Decomposition of Q_3

5.3 Are the decompositions of Q_1 and Q_2 compatible?

The first decompositions that are treated are those of polytopes Q_1 and Q_2. The first polytope has a unique decomposition which is given in Equation 5.2. This decomposition is lifted and embedded in $\overline{N} = N \times \mathbb{Z}^2$ yielding the cone $\sigma_{12} \subset \overline{N}$ and its dual cone $\sigma_{12}^\vee \subset \overline{M}$. On the other hand, Q_2 has more than one admissible decomposition. These decompositions are treated separately. Finally the cone $\sigma_{12}^\vee$ is compared with the cones $\sigma_{21}^{i\vee}$ to verify the compatibility.

5.3.1 Compute Q_{12}

The decomposition of Q_1 is given in Equation 5.2. The lifted cone σ_1 can be determined from this decomposition, together with its dual cone $\sigma_1^\vee$.

$$\sigma_1 = \begin{pmatrix} -\frac{3}{5} & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \sigma_1^\vee = \begin{pmatrix} 5 & 0 & -1 & 0 \\ 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Now ϕ_1 induces a mapping such that $\phi_1(\boldsymbol{w}_1^i) = \boldsymbol{w}^i$ for all $i = 0, \dots, 4$ where $\boldsymbol{w}_1^i \in \sigma_1^\vee$ is an element of the Hilbert basis E_1 and $\boldsymbol{w}^i$ is an element of the Hilbert basis of $\sigma^\vee$.

$$\phi_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Moreover, the Hilbert basis of $\sigma_1^\vee$ consists of the columns $(\boldsymbol{w}_1^0, \dots, \boldsymbol{w}_1^4, \tilde{\boldsymbol{w}}_1^1)$ of matrix E_1 such that $\tilde{\boldsymbol{w}}_1^1$ maps to $\boldsymbol{w}^1$.

$$E_1 = \begin{pmatrix} -1 & 0 & 1 & 3 & 5 & 0 \\ 0 & 1 & 1 & 2 & 3 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The cross cut of σ_1 with $\boldsymbol{w}_1^2 = (1,1,0)$ yields the following polyhedron and the corresponding decompositions. The first two decompositions are as described in 4.23 and the latter one is admissable as the vertex $\boldsymbol{v}^1 = (0,0,1)$ is a lattice point.

$$
\begin{aligned}
Q_{12} &= \begin{pmatrix} -\frac{3}{2} & 0 & 1 & 0 \\ \frac{5}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -\frac{3}{2} & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} && (5.7) \\
&= \begin{pmatrix} -\frac{3}{2} & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} && (5.8) \\
&= \begin{pmatrix} -\frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} && (5.9)
\end{aligned}
$$

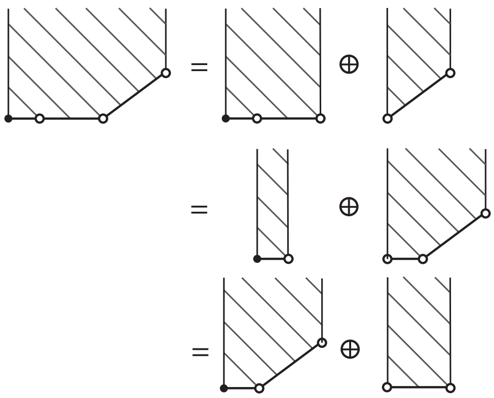

Figure 5.5: Polyhedron Q_{12}

There exist three admissible decompositions of polyhedron Q_{12} which are given above. These decompositions lead to three different cones σ_{12}.

$$\sigma_{12}^1 = \begin{pmatrix} -\frac{3}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma_{12}^{1\vee} = \begin{pmatrix} -1 & 2 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_{12}^2 = \begin{pmatrix} -\frac{3}{2} & -1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma_{12}^{2\vee} = \begin{pmatrix} -1 & 2 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 3 & -1 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_{12}^3 = \begin{pmatrix} -\frac{3}{2} & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma_{12}^{3\vee} = \begin{pmatrix} -1 & 2 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The dual cones $\sigma_{12}^{i\vee}$ all have their Hilbert basis E_{12}^i which maps to E using the matrix ϕ_{12}, which does not depend on the decomposition.

$$\phi_{12} = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

5.3.2 Compute Q_{21}

The same procedure is followed to compute σ_{21}. The polytope Q_2 has more than one admissible decomposition. A first remark is that the fourth decomposition is a decomposition such that $\boldsymbol{v} = \boldsymbol{a}$ and therefore it is not compatible with the decomposition of Q_1. For the other decompositions it is not straightforward to see whether they are compatible to Q_1, thus the cones σ_{21} are computed for the other three admissible decompositions of Q_2. The morphisms ϕ_2 and ϕ_{21} do not depend on the choice of the decomposition, but are the same for all decompositions.

$$\phi_2 = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\phi_{21} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

First consider the decomposition described in Equation 5.2. It yields cone σ_2^1, where the upper indicator stands for the choice of the decomposition.

$$\sigma_2^1 = \begin{pmatrix} -\frac{3}{2} & -\frac{1}{2} & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \sigma_2^{1\vee} = \begin{pmatrix} -1 & -2 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 1 & 2 & 0 & 1 \end{pmatrix}$$

The vertex $\boldsymbol{v} = (-\frac{1}{2}, 2)$ is not a lattice point, thus $\alpha = \frac{w_1^k}{w_1^h} - (v_1 w_1^k + v_2 w_2^k) = \frac{1}{2} - (\frac{3}{2}) = -1$. This yields $\tilde{\boldsymbol{w}}_1 = (-1,1,0) - (0,1,-1) = (-1,0,1)$. The intersection of $\tilde{\boldsymbol{w}}_1$ and σ_2^1 gives the polyhedron Q_{21}^1. This polyhedron has a unique admissible decomposition which is given below.

$$\begin{aligned} Q_{21}^1 &= \begin{pmatrix} -1 & -1 & 0 & 1 \\ \frac{2}{3} & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -\frac{1}{3} & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & -\frac{1}{3} & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{aligned}$$

The cone σ_{21}^1 can be constructed from this decomposition.

$$\sigma_{21}^1 = \begin{pmatrix} -1 & -\frac{1}{3} & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma_{21}^{1\vee} = \begin{pmatrix} -1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The second decomposition as described in Equation 5.3, gives cone σ_2^2. From the vertex $\boldsymbol{v} = (0,1)$ we get $\alpha = -(v_1 w_1^k + v_2 w_2^k) = -1$. This gives $\tilde{\boldsymbol{w}}_1 = (-1,1,0) - (0,1,-1) = (-1,0,1)$ which results in the cross cut Q_{21}^2. It has a unique admissible decomposition, which is given below.

$$
\begin{aligned}
\sigma_2^2 &= \begin{pmatrix} -\frac{3}{2} & 0 & 0 & \frac{1}{2} \\ 4 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \\
Q_{21}^2 &= \begin{pmatrix} -1 & 0 & 1 & 0 \\ \frac{2}{3} & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & 0 \\ \frac{2}{3} & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & 0 & 0 \\ \frac{2}{3} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\sigma_{21}^2 = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ \frac{2}{3} & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma_{21}^{2\vee} = \begin{pmatrix} -1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

The third decomposition is described by Equation 5.4. Now $\alpha = -(v_1 w_1^k + v_2 w_2^k) = -2$ yields $\tilde{\boldsymbol{w}}_1 = (-1,1,0) - 2(0,1,-1) = (-1,-1,2)$. The cross cut of σ_2^3 yields the polyhedron Q_{21}^3.

$$
\begin{aligned}
Q_{21}^3 &= \begin{pmatrix} -3 & 0 & 3 & -1 \\ 2 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 2 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & -\frac{1}{2} & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -1 & -\frac{1}{2} & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}
\end{aligned}
$$

This unique decomposition leads to the cone σ_{21}^3 and its dual cone.

$$
\sigma_{21}^3 = \begin{pmatrix} -1 & -\frac{1}{2} & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma_{21}^{3\vee} = \begin{pmatrix} -1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

5.3.3 Conclusion

Although there exists one decomposition for Q_1, the lifted polyhedron Q_{12} has three admissible decompositions. These decompositions lead to three different cones σ_{12} and their dual cones $\sigma_{12}^\vee$. The first decomposition of Q_{12} yields the cone σ_{12}^1. This cone can be mapped to the cone σ_{21}^2, which is induced by the second decomposition of Q_2.
There exists a similar morphism between the cones σ_{12}^2 and σ_{21}^3, which shows compatibility of the second decomposition of Q_2 and the unique admissible decomposition of Q_1.
Moreover, the first decomposition of Q_2 is compatible with the unique admissible decomposition of Q_1. This is shown by the cones σ_{12}^3 and σ_{21}^1.

$$
\begin{aligned}
\pi_1 &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
\pi_1 \sigma_{12}^1 &= \sigma_{21}^2 \\
\pi_2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
\pi_2 \sigma_{12}^2 &= \sigma_{21}^3 \\
\pi_3 &= \begin{pmatrix} 2 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
\pi_3 \sigma_{12}^3 &= \sigma_{21}^1
\end{aligned}
$$

5.4 Are the decompositions of Q_1 and Q_3 compatible?

Consider the cross cuts that are retrieved from the rays $\boldsymbol{w}^1 = (1, 1)$ and $\boldsymbol{w}^3 = (5, 2)$. The polyhedra Q_1 and Q_3 have one admissible decomposition which are given in Equations 5.2 and 5.6.

5.4.1 Compute Q_{13}

The decomposition of Q_1 is induced by $\boldsymbol{a} = (0,1)$. Moreover, $\alpha = 0$ and $\boldsymbol{w}_1^3 = (3,2,0)$. The polyhedron Q_{13} is computed, together with its decomposition.

$$\begin{aligned} Q_{13} &= \begin{pmatrix} -3 & 0 & \frac{1}{3} & 0 \\ 5 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & \frac{1}{2} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \frac{1}{2} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{aligned}$$

Embedding these summands at different heights yields the cone σ_{13}.

$$\sigma_{13} = \begin{pmatrix} -1 & 0 & 0 & \frac{1}{2} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma_{13}^{\vee} = \begin{pmatrix} -2 & 1 & -3 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{pmatrix}$$

The Hilbert basis of $\sigma_{13}^{\vee}$ to the elements of E by matrix $\phi_{13} = \begin{pmatrix} 3 & 1 & 5 & 5 \\ 1 & 1 & 2 & 2 \end{pmatrix}$

5.4.2 Compute Q_{31}

Polytope Q_3 has a unique decomposition which is given in Equation 5.6.

This yields the following cones.

$$\sigma_3 = \begin{pmatrix} -1 & 0 & 0 & \frac{3}{5} \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \qquad \sigma_3^{\vee} = \begin{pmatrix} -5 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix}$$

Furthermore, $\boldsymbol{u} = (-1,3)$ and $\alpha = -(v_1 w_1^k + v_2 w_2^k) = -(-1+3) = -2$ determine $\tilde{\boldsymbol{w}}_1 = (-3,2,0) - 2(0,1,-1) = (-3,0,2)$. The cross cut of σ_3 and $\boldsymbol{w}_1^3$ yields the polyhedron

Q_{31}.

$$
\begin{aligned}
Q_{31} &= \begin{pmatrix} -\frac{1}{3} & 0 & 3 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 5 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -\frac{2}{3} & -\frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} -\frac{2}{3} & -\frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}
\end{aligned}
$$

This is a unique admissible decomposition. The next step is to compute cone σ_{31}.

$$
\sigma_{31} = \begin{pmatrix} -\frac{2}{3} & -\frac{1}{2} & 0 & 0 & 1 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma_{31}^{\vee} = \begin{pmatrix} -1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
$$

5.4.3 Conclusion

To check compatibility of the unique decompositions of Q_1 and Q_3, cones $\sigma_{13}^{\vee}$ and $\sigma_{31}^{\vee}$ are compared. The given matrix π_{13} describes the mapping that verifies the compatibiliy of the two decompositions.

$$
\begin{aligned}
\pi_{13} &= \begin{pmatrix} 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
\pi_{13}\sigma_{13} &= \sigma_{31}
\end{aligned}
$$

5.5 Are the decompositions of Q_2 and Q_3 compatible?

Consider the decompositions of Q_2 and Q_3. As discussed earlier, the fourth decomposition of Q_2 is induced by $\boldsymbol{v} = \boldsymbol{a}$. From Theorem 4.9 we know that this decomposition is not compatible to any other decomposition of Q_k with $k \neq h$. Thus to check compatibility, only the first three decompositions of Q_2 are discussed.

5.5.1 Compute Q_{23}

There are three polyhedra Q_{21} that can be computed. These will be discussed successively. Matrix ϕ_{23} does not depend on the choice of decomposition, but solely on $\boldsymbol{w}^2$ and $\boldsymbol{w}^3$. They induce the mapping to $\sigma^\vee$.

$$\phi_{21} = \begin{pmatrix} 1 & 2 & 5 & 5 \\ 0 & 1 & 2 & 2 \end{pmatrix}$$

The cones σ_2 are computed in Section 5.3.2. As $2 > 1$ have $\alpha = 0$ and $\tilde{\boldsymbol{w}}^{\mathbf{3}} = (1, 2, 0)$ for all its decompositions. The first decomposition of Q_2 as described in Equation 5.2 gives the polyhedron Q_{23}^1.

$$\begin{aligned} Q_{23}^1 &= \sigma_2^1 \cap H_{\tilde{\boldsymbol{w}}^{\mathbf{3}}} \\ &= \begin{pmatrix} -3 & -1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 0 & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{2}{3} & 1 \\ 1 & 1 & 0 \end{pmatrix} \end{aligned} \tag{5.10}$$

This is the unique decomposition of Q_{23}^1 which gives cone σ_{23}^1 and its dual.

$$\sigma_{23}^1 = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & \frac{2}{3} & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma_{23}^{1\vee} = \begin{pmatrix} -2 & 1 & -1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

The second decomposition is induced by $\boldsymbol{v} = (0, 1)$ and is given in Equation 5.3. The intersection of σ_2^2 and $\tilde{\boldsymbol{w}}^3 = (1, 2, 0)$ gives the polyhedron Q_{23}^2 which has a unique admissible decomposition. The cross cut of the cone σ_2^2 yields Q_{23}^2.

$$\begin{aligned} Q_{23}^2 &= \begin{pmatrix} -3 & 0 & 1 & 0 \\ 2 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{aligned}$$

Cone σ_{23}^2 is constructed from polyhedron Q_{23}^2.

$$\sigma_{23}^2 = \begin{pmatrix} -1 & 0 & 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma_{23}^{2\vee} = \begin{pmatrix} -4 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Now consider the third decomposition described in Equation 5.4. Cross cut $Q_{23}^3 = \sigma_2^3 \cap H_{\tilde{\boldsymbol{w}}^3}$ has a unique admissible decomposition.

$$\begin{aligned} Q_{23}^3 &= \begin{pmatrix} -3 & -\frac{1}{3} & 1 & 0 \\ 2 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & \frac{1}{3} & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{aligned} \tag{5.11}$$

This leads to cone σ_{23}^3.

$$\sigma_{23}^3 = \begin{pmatrix} -1 & 0 & 0 & \frac{1}{3} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \qquad \sigma_{23}^{3\vee} = \begin{pmatrix} -3 & 1 & -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

5.5.2 Compute Q_{32}

The next step is to compute σ_{32}. The cone σ_3 is computed in Section 5.4.2. Moreover, $\alpha = -(v_1 w_1^k + v_2 w_2^k) = -(-2+3) = -1$ gives $\tilde{\boldsymbol{w}}^{\mathbf{2}} = (-1,2,0) - 2(0,1,-1) = (-1,0,1)$. This results in the polyhedron Q_{32}.

$$Q_{32} = \begin{pmatrix} -1 & 0 & \frac{3}{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{5}{2} & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & \frac{3}{2} & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

This polyhedron has three admissible decomposition that all result in a different cone σ_{32}.

$$
\begin{aligned}
Q_{32} &= \begin{pmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \frac{3}{2} & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\
\sigma_{32}^{1} &= \begin{pmatrix} -1 & 0 & 0 & \frac{3}{2} & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \\
\sigma_{32}^{1\vee} &= \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Another admissible decomposition of Q_{32} yields the cone σ_{32}^{2}.

$$
\begin{aligned}
Q_{32} &= \begin{pmatrix} -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\
\sigma_{32}^{2} &= \begin{pmatrix} -1 & 0 & 1 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \\
\sigma_{32}^{2\vee} &= \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Finally, the last decomposition of Q_{32} gives the cone σ^3_{32}.

$$
\begin{aligned}
Q_{32} &= \begin{pmatrix} -1 & 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \\
\sigma^3_{32} &= \begin{pmatrix} -1 & 0 & \frac{1}{2} & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \\
\sigma^{3\vee}_{32} &= \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

5.5.3 Conclusion

Consider the mapping matrix

$$
\pi_{23} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{5.12}
$$

and the matrix descriptions of the cones σ_{23} and σ_{32}. We now have the following equations.

$$\pi_{23}\sigma_{23}^1 = \sigma_{32}^1$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 & -1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\pi_{23}\sigma_{23}^2 = \sigma_{32}^2$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -4 & 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\pi_{23}\sigma_{23}^3 = \sigma_{32}^3$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -3 & 1 & -1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

These show that all the decomposition of Q_2 are compatible with the unique decomposition of Q_3.

5.6 Conclusion

We now have verified that all the admissable decompositions are compatible, except for the decomposition that is constructed from $\boldsymbol{a} = \boldsymbol{v}$. This decompostion is however compatible with another decomposition of Q_2, namely the decomposition as described in 5.2. This decomposition is of the second type, where $\boldsymbol{v} \notin N$ and $\boldsymbol{v} = -\frac{1}{2}$. This decomposition is not discussed in this chapter, as these decompositions are of the same polytope. Although the compatibility can be verified equivalently with the process that is described in this chapter, it can also be derived as described in the previous chapter. As that works more efficiently, the decompositions that come from the same polytope are not verified in this chapter. However, the reader can easily verify which decompositions are compatible.

Overall, the results that come from the computations in this thesis are according to the results in [18]. Thus as a conclusion, the combinatorial approach gives equivalent results for these decompositions, which strengthen our results.

Chapter 6

Discussion

The previous chapter shows an example of how the compatibilty of two decompositions is determined for degree = 2. The case where the decompositions come from the same line segment is not treated in that chapter, as it is more convenient to use the other procedure for this. It is possible to determine this with the two computed cones $\sigma_{hh}^{\vee}$ as well, but as one can see that is much more complex. As mentioned earlier, this example is treated in [18] with respect to toric varieties. It is interesting to see that two different approaches lead to the same result. A possible next step is to combine both approaches that might lead to new results.

Another possible opportunity for further research is to generalize the mapping between the two cones σ_{hk} and σ_{kh}. This thesis ended by showing the morphism, but one can describe this in terms of the given factors ($\boldsymbol{w}_k, \boldsymbol{w}_h, \alpha$ etcetera).

When such a description of π is found, then it is possible to extend the example towards a d-dimensional solution. Although from examples we have seen that such a solution most likely does exist, we did not succeed to give a generic description for the higher dimensional lattices. However, this thesis gives a good starting point for research.

The computational approach that is taken in this thesis might help the notion of compatibility in other research areas. Many of the available literature concerning this topics make the link to toric varieties and deformation theory. It would be interesting to combine these results in order to find a generic description. Although several steps were taken in this direction, we were not able to combine the results of the two research area. However, with this thesis we were able to determine some promising descriptions.

We will not elaborate on the deformation theory in detail, as that would require a whole new set of defitions and prerequisites. Refer to the existing literature like [3], [2] and [5] as background information.

In terms of versal deformations, one of the main objectives of this thesis was to combine the versal deformation in degree $-R$ and $-S$ to a deformation that is versal in both degrees, in which we succeeded by taking a different approach. [6] describes the problem description with respect to the deformation theory.

Appendix A

Admissible Minkowski decomposition

The decomposition of a polyhedron

$$Q = \bigoplus_{i=0}^{q-1} Q_i = Q_0 \oplus Q_1 \oplus \ldots \oplus Q_{q-1}$$

is admissible if and only if for all the faces F of Q at most one of the summand faces F_i does not contain a lattice point.

Compatible decompositions

Two admissible Minkowski decompositions are compatible when they can be further decomposed in the same prime decomposition. This is defined as the common refinement of the decompositions.

Cross cut

The cross cut is the intersection of a cone σ and a hyperplane H. The resulting intersection is a polyhedron.

$$Q := \sigma \cap H.$$

In this thesis the hyperplane H is taken as the set $\{\boldsymbol{x} \in N_{\mathbb{R}} | \boldsymbol{r}^\top \boldsymbol{x} = 1\}$ for a given vector $\boldsymbol{r}$. Therefore the obtained polytope is sometimes referred to as $Q_{\boldsymbol{r}}$.

Edge vector

The edge vector $\boldsymbol{t} \in \mathbb{Q}^n$ of a polyhedron Q is a vector such that $0 \leq t_i \leq 1$ for each entry t_i $(i = 0, \ldots, n-1)$ where n is the size of the edge set of Q. This edge set is the set of primitive vectors that relate to the edges and therefore $t_i d_i$ is an edge of Q for all i.

Indecomposable polyhedron

When a polyhedron Q is called indecomposable, this means that the polyhedron is not decomposable in a nontrivial way. This means that every decomposition of Q is a trivial one. Thus, the decomposition is of the type $Q = \lambda Q \oplus (1-\lambda)Q$ for $0 \leq \lambda \leq 1$.

Prime decomposition

A prime decomposition is an admissible Minkowski decomposition of a polyhedron Q such that all its summands are admissibly indecomposable polyhedra.

Polyhedral cone

A cone in this thesis is actually a strongly convex polyhedral cone that is convex and finitely generated. This means that it is the positive hull of a finite set of generators, and if $\boldsymbol{u}$ is such a generator, then by definition $-\boldsymbol{u}$ is not. Moreover, these generators are rational vectors, i.e. $\boldsymbol{u} \in \mathbb{Q}^d$.

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