MASTER

Enkele aspekten van gravito-akoestische golven en geometrisch-optische straalberekeningen

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Enkele aspekten van gravito-akoestische golven en geometrisch-optische straalberekeningen

door

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Eindhoven, 28 oktober 1986.

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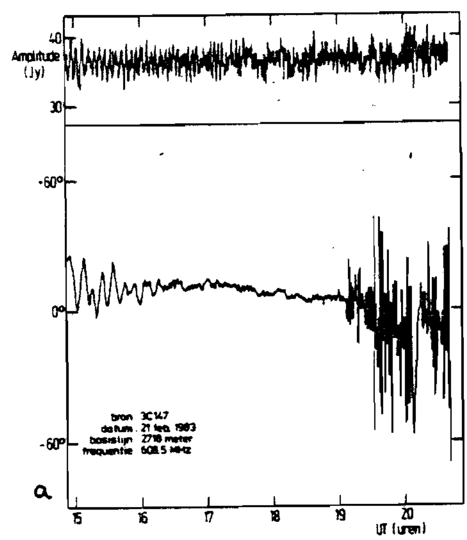
1. Abstract.

Density variations in the ionosphere cause the refractive index to vary and thus electromagnetic waves to be affected while traveling through the ionosphere. These density variations can be caused by ionospheric waves. For one of these, the gravito-acoustic wave , which causes radio astronomical observations to be distorted, two approximate dispersion relations have been derived. The first one is the limiting case for an unmagnetized flat ionosphere. The second one is valid for a magnetized ionosphere where for the f-layer the Alfvén velocity is far greater than the speed of sound and the phase velocity of the wave. The first dispersion relation has been derived before by E.A.P. Habraken [7], who used slightly different assumptions and started from a flat geometry. In this report the dispersion relation is obtained as the lowest order approximation in a power series expansion in the inverse of the radius of the earth. The method used here allows for higher order effects, i.e. that of the curvature of the earth, to be computed.

In the second part of the report ray path calculations are performed and correction terms for the ionospheric range error, i.e. the difference between optical phase path length and Euclidean distance are obtained. This difference is caused by an ionospheric refractive index that is slightly below 1, causing the optical phase path length, which is obtained in range measurements, to be somewhat less than the 'normal' distance. At first these calculations are done for an unmagnetized - and therefore isotropic - ionosphere and for the special case of east-west propagation in a magnetized ionosphere. Finally, the ray path calculations are performed for waves under oblique incidence upon a magnetized ionosphere, where Booker's quartic method has been used.

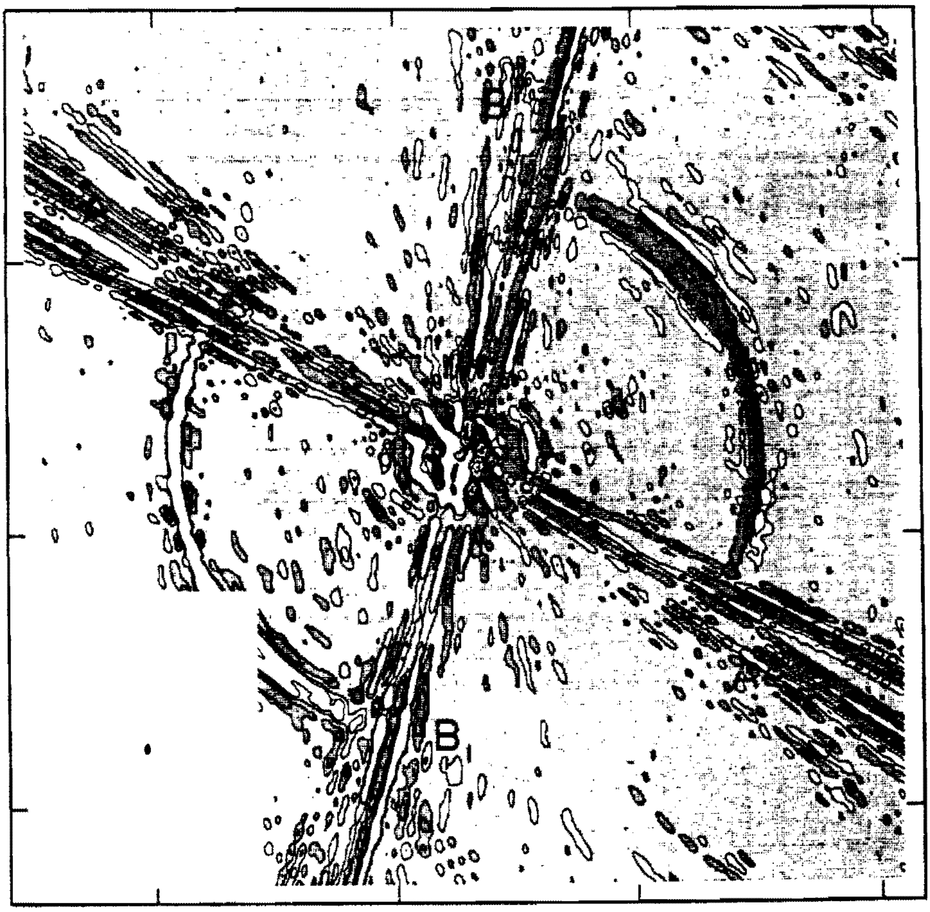
2. Preface.

Radio astronomical observations and earth-satellite links are affected by the ionosphere. The ionosphere is an inhomogeneous, anisotropic and time varying part of the atmosphere and extends from about 40 to several hundreds of kilometers. In the GHz range the effects produced by inhomogeneities are of major importance. One of the effects has been observed at the Westerbork Synthesis Radio Telescope (WSRT). It is the apparent occasional 'instability' in the position of an extra-terrestrial radio source on the celestial sphere. (See fig. 1 and 2)



Ionospheric irregularities observed on a baseline of 2.7 kilometer. The top figure shows the interferometer amplitude and the lower the phase as a function of time. During the first 1 ½ hours, slow wave phenomena distort the measurements and after 2 ½ hours of quiet conditions the last part is distorted by fast irregularities.

The position of the object seems to fluctuate periodically around a 'steady' position, with a period time of 5 to 30 minutes. A probable explanation for this phenomenon is refraction due to periodic changes in the electron density in the ionosphere, leading to periodic variations of the refractive index. This density variation has to be caused by ionospheric waves with appropriate period time, making gravito-acoustic waves the most likely candidate. They can exist because of the stratified character of the ionosphere, which is caused by the presence of the earth's gravitational field. In chapter 4 gravito-acoustic waves in a magnetized ionosphere is discussed in a spherical coordinate system.



Map of the field of radio source 3C147 from fig. 1. The spokes are the result of irregularities. The more horizontal ones are caused by the slow ionospheric waves and the relatively vertical ones by fast irregularities. The source itself, shown in the lower left corner, has been subtracted from the measurement to improve the clarity of the intensity picture shown. The perturbations have intensities of about 0.3% of the intensity of the source itself. The circular structure in the figure is a result of observational and computational methods and is irrelevant here.

Another effect produced in the ionosphere is the phase delay. Since the ionospheric medium is optically less dense in comparison with free space, it will take less time for a wave to traverse. This time period is the one measured on earth-satellite links. For distance measurements this is to be multiplied by the velocity of light, resulting in the optical phase path length. The distance thus found , however, is smaller than the Euclidean distance. The difference is the so-called ionospheric range error, that limits the accuracy of distance measurements. In chapter 5 the phase path for a wave will be derived and from that the optical phase path length and correction terms for this path are found. All this is done for an unmagnetized ionosphere.

In the next chapter a special case in the presence of a magnetic field is discussed. For propagation in east-west direction the refractive index tensor reduces to a diagonal tensor with the refractive index as element, allowing us to use the method derived for the unmagnetized case.

Chapter 7 deals with the propagation properties of waves obliquely incident to the geomagnetic field. It is based on the method derived by H.G. Booker.

I thank Prof. Dr. M.P.H. Weenink who coached me during my work and whose continuous interest, knowledge and insight have been a great help and stimulation.

Han van Bussel

Eindhoven, October 1986.

3. The ionosphere.

3.1. Introduction.

The earth is surrounded by a weakly ionized gaseous layer: *the atmosphere* The ionization is caused mainly by the sun's radiation. The earth's strong gravitational field keeps the ionosphere 'in place' and causes it to be almost horizontally stratified. The ionosphere is influenced by numerous factors, such as gravity, solar heating, photoionization, chemical reactions, meteors, vulcanic eruptions etc., making it a very complex but interesting subject of study.

3.2. Structure and nomenclature of the atmosphere.

As stated before the horizontal stratification is one of the most striking features. In the atmosphere the gravitational force and the pressure gradient balance. They are related via the hydrostatic equation:

$$dp = -\rho g dz \qquad 3.1$$

If we assume the atmosphere to be an ideal gas, we have:

$$p = nkT \qquad 3.2$$

and thus; in case of a constant temperature,

$$p = p_0 e^{\frac{-z}{H}} \qquad 3.3$$

where

$$H := \frac{kT}{mg}$$

is the vertical scale height.

More than one way exists to classify the atmosphere as a function of altitude , depending on the parameter involved. There are classifications based on thermal structure, electron density, composition etc. The *ionosphere* definition according to the IEEE standard (1969) is:

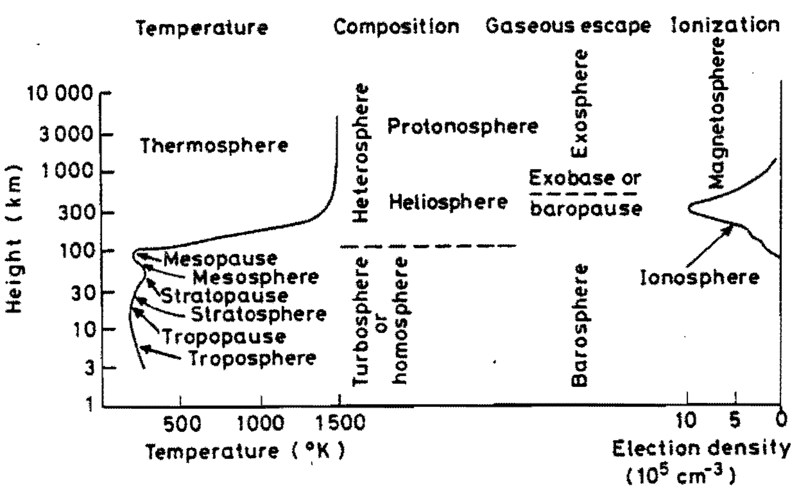
"that part of a planetary atmosphere where ions and electrons are present in quantities sufficient to effect the propagation of radio waves."

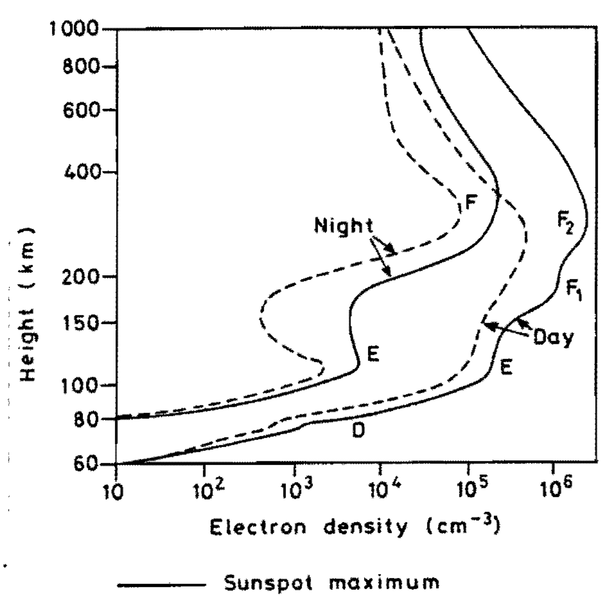
Usually it is divided into three regions. The lower one, extending from 40km to 90 km is the D-layer. The middle one is the E-layer between 90 km and 160 km and the upper layer, which is most strongly ionized, is the F-layer. It extends several hundreds of kilometers.

Historically, Heaviside and Kennely postulated the existence of a "conducting layer" in 1902 to explain the reflection of radio waves. The first experimental prove of the existance of the ionosphere is by Appleton and Barnett in 1925. This result was verified the next year by Breit and Tuve.

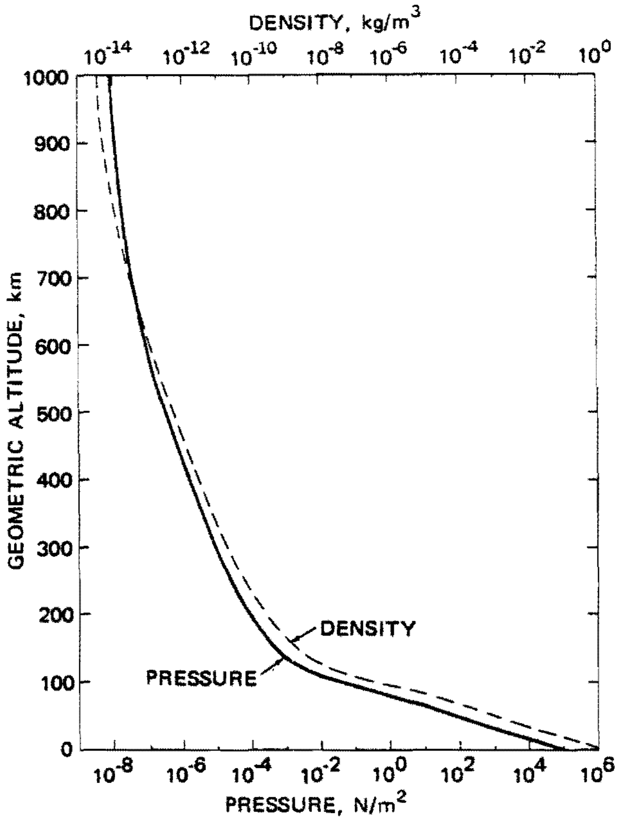
3.3. The standard atmosphere.

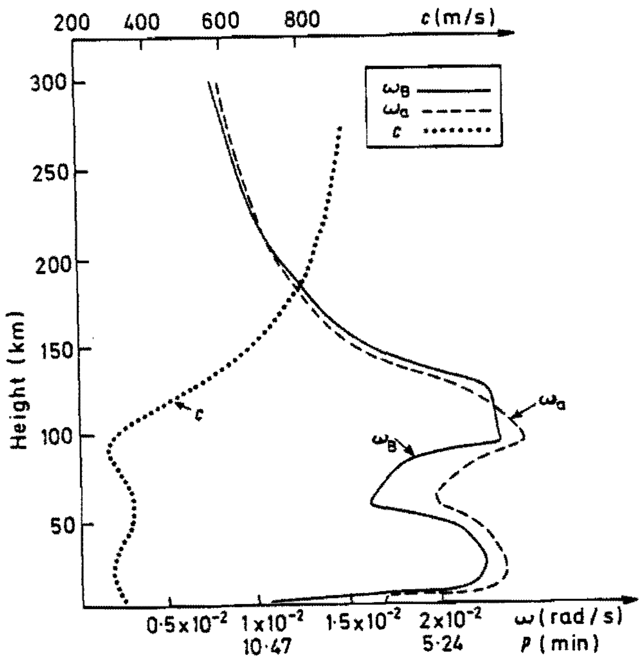
The standard atmosphere is shown below by a graphical representation of its major parameters as a function of altitude. For further information the interested reader is referred to the many textbooks on this subjest.

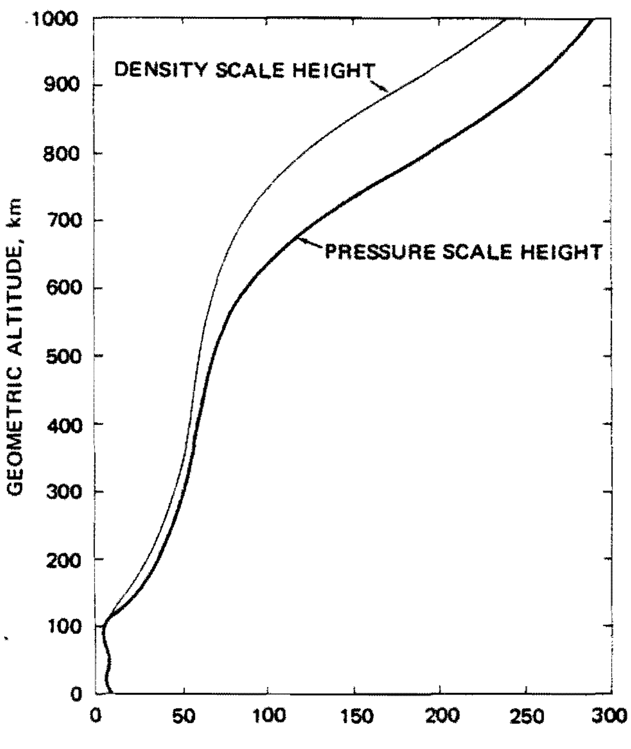


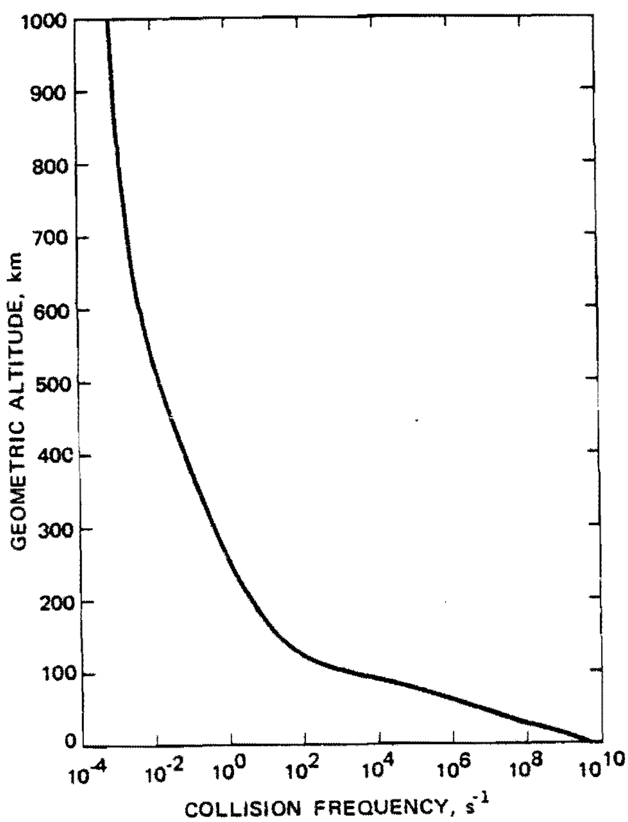


The standard atmosphere.

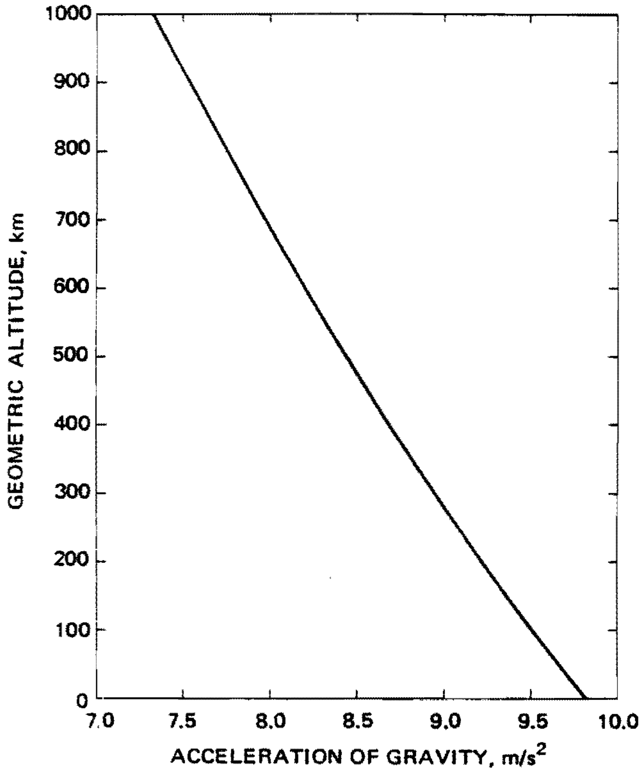


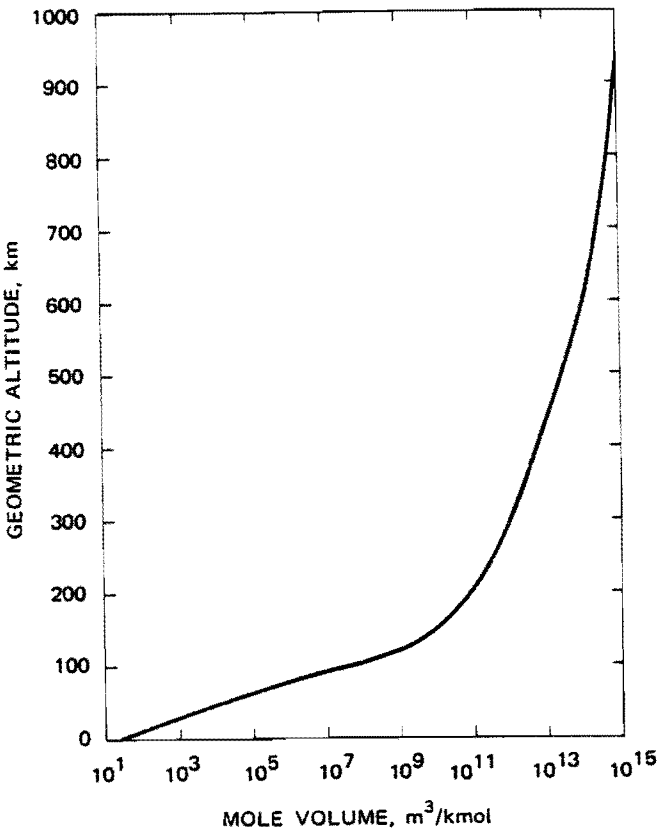


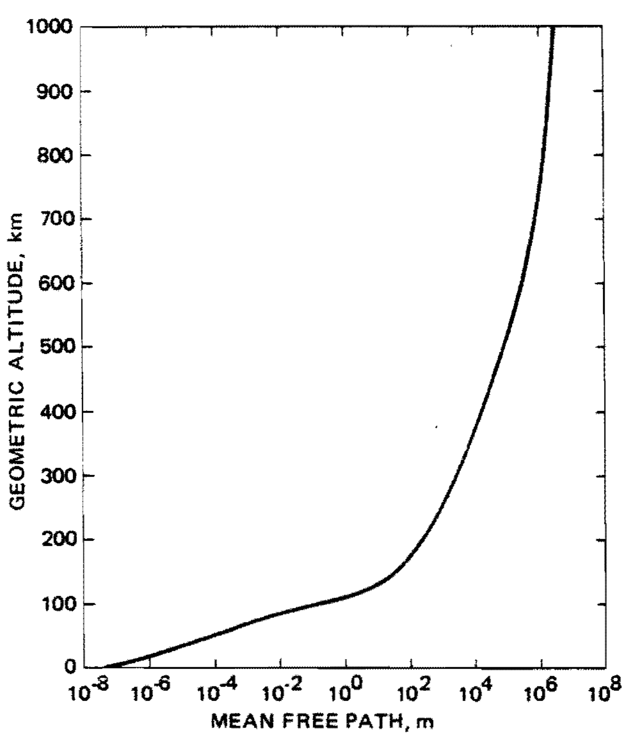


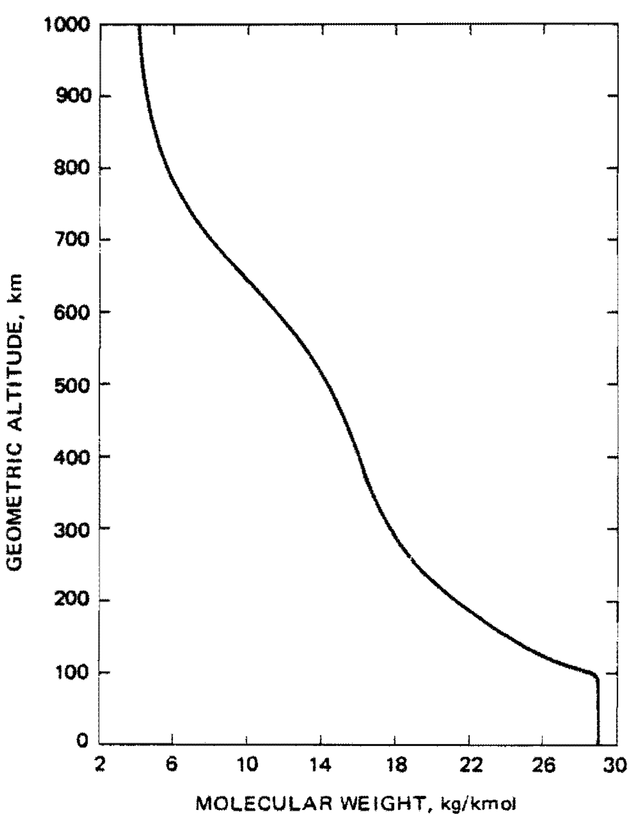


The standard atmosphere.









4. Equations.

4.1. Introduction.

In this chapter, some of the equations used in subsequent chapters will be derived. The derivations will be brief, since these can be found in most books on plasma physics and ionospheric waves.

4.2. Plasma equations.

We start the derivation of the plasma equations from the Vlasov equation, known also as the collisionless Boltzmann equation. This approximate equation - collisional effects are neglected - plays a major role in in a discussion in plasmas. It is:

$$\partial_t f_\alpha + \vec{v}_\alpha \cdot \nabla_{\vec{r}} f_\alpha + \vec{a}_\alpha \cdot \nabla_{\vec{v}} f_\alpha = 0 \qquad 4.1$$

where

$$f_\alpha = f(\vec{r}, \vec{v}, t) \qquad 4.2$$

is the generally anisotropic one particle density function for particle kind α and

$$\vec{a}_\alpha = \vec{g} + \frac{q_\alpha}{m_\alpha}\left(\vec{E} + \vec{v}_\alpha \times \vec{B}\right) \qquad 4.3$$

The influence of the gravitational field is taken into account, because of its importance in the discussion of gravito-acoustic waves.

Next we introduce some macroscopic variables:

the particle density

$$n_\alpha = \int f_\alpha d\vec{v} \qquad 4.4$$

the mass density

$$\rho_{m_\alpha} = m_\alpha \int f_\alpha d\vec{v} = m_\alpha n_\alpha \qquad 4.5$$

the charge density

$$\rho_{q_\alpha} = q_\alpha \int f_\alpha d\vec{v} = q_\alpha n_\alpha \qquad 4.6$$

the center of mass fluid flow velocity

$$\vec{v}_\alpha = \frac{1}{n_\alpha} \int \vec{v} f_\alpha d\vec{v} \qquad 4.7$$

the current density

$$\vec{J}_\alpha = q_\alpha \int \vec{v} f_\alpha d\vec{v} = q_\alpha n_\alpha \vec{v}_\alpha \tag{4.8}$$

and finally, the pressure tensor

$$\overline{\overline{p}}_\alpha = m_\alpha \int (\vec{v} - \vec{v}_\alpha)(\vec{v} - \vec{v}_\alpha) f_\alpha d\vec{v} \tag{4.9}$$

We now try to derive some relations between these variables by the method of moments, giving a more simple model for the plasma: the equations of magnetohydrodynamics. Define the n-th moment as:

$$M_n[\cdots]_\alpha = \frac{m_\alpha}{n!} \int \vec{v}^n [\cdots]_\alpha d\vec{v} \tag{4.10}$$

With n = 0 the *mass conservation equation* results:

$$\partial_t \rho_{m_\alpha} + \nabla \cdot (\rho_{m_\alpha} \vec{v}_\alpha) = 0 \tag{4.11}$$

For n = 1, we have the *momentum equation*

$$\rho_{m_\alpha} D_t \vec{v}_\alpha + \nabla \cdot \overline{\overline{P}} - \rho_{m_\alpha} \vec{g} - \rho_{q_\alpha} \vec{E} - \vec{J} \times \vec{B} = 0 \tag{4.12}$$

Higher order moments could be derived, i.e. n = 2 yields the equation of conservation of energy, but these are not used in subsequent chapters.

The effect of collisions on the momentum of the particles can be included through the addition of a collision term in equation 1.12. We obtain:

$$\rho_{m_\alpha} D_t \vec{v}_\alpha + \nabla \cdot \overline{\overline{P}} - \rho_{m_\alpha} \vec{g} - \rho_{q_\alpha} \vec{E} - \vec{J} \times \vec{B} = -\nu_\alpha \vec{v}_\alpha \tag{4.13}$$

where ν is the collision frequency.

In equation 1.12 the convective derivative has been used. It must be used if a time derivative is taken for any quantity corresponding to a particle moving in phase space. This particle in general has both a velocity and an acceleration and the change of position this causes in phase space gives rise to a time variation on top of the explicit time variation. Therefore the convective derivative is:

$$D_t = \partial_t + \vec{v} \cdot \nabla_{\vec{r}} + \vec{a} \cdot \nabla_{\vec{v}} \tag{4.14}$$

In cases where this derivative is used in this report, the quantities are independent of velocity, so the divergence vector in phase space is zero and equation 1.14 reduces to:

$$D_t = \partial_t + \vec{v} \cdot \nabla_{\vec{r}} \qquad 4.15$$

The pressure tensor $\overline{\overline{P}}_\alpha$ can also be written as:

$$\nabla \cdot \overline{\overline{P}}_\alpha = \nabla p_\alpha - \eta \left[\nabla^2 \vec{v}_\alpha + \frac{1}{3} \nabla \nabla \cdot \vec{v}_\alpha \right] \qquad 4.16$$

where p_α is the scalar pressure and η is the viscosity.

4.3. Gas equations.

The Earth's atmosphere here is considered to be a perfect or ideal gas. Therefore, from thermodynamics we have:

$$pV = NRT \qquad 4.17$$

and

$$U = \frac{f}{2} NRT \qquad 4.18$$

where N is the number of moles of gas, R is the gas constant, U is the total internal energy of the system and f is the number of degrees of freedom. For the atmosphere f is taken to be 5.

Furthermore we have the first law of thermodynamics:

$$dU = dQ - pdV \qquad 4.19$$

the heat capacity at constant volume:

$$c_v = \frac{1}{N} \left(\partial_T Q \right)_V = \frac{f}{2} \qquad 4.20$$

the heat capacity at constant volume:

$$c_p = \frac{1}{N} \left(\partial_T Q \right)_p = \frac{1}{N} \left[\left(\partial_T U \right)_p + p \left(\partial_T V \right)_p \right] = \frac{f+2}{2} R \qquad 4.21$$

$$\gamma = \frac{c_p}{c_v} = 1 + \frac{2}{f} \qquad 4.22$$

and finally

$$dS = \frac{dQ}{T} = \frac{dU}{T} + \frac{pdV}{T} = \frac{f}{2} NR \frac{dT}{T} + NR \frac{dV}{V} \qquad 4.23$$

where S is the total entropy in the system.

With the law of a perfect gas dS can be written as:

$$dS = Nc_v\, d\left(\ln\left[pV^{\gamma}\right]\right) \qquad 4.24$$

If the change of state for the ideal gas is assumed adiabatic, i.e. the gas doesn't exchange heat with its surroundings, we have $D_t Q = 0$ and $D_t S = 0$ and we have the adiabatic law:

$$D_t\left[pV^{\gamma}\right] = 0 \qquad 4.25$$

The volume of the gas is:

$$V = \frac{M}{\rho} = \frac{NmN_a}{\rho} \qquad 4.26$$

where M is the total mass, m the average particle mass and N_a is Avogadro's number. Inserting this, we find the *law for a perfect gas* and the *adiabatic law*

$$\frac{p}{\rho} = \frac{RT}{mN_a} = \frac{kT}{m} \qquad 4.27$$

$$D_t\left[p\rho^{-\gamma}\right] = 0 \qquad 4.28$$

4.4. Ray path equation.

One of the ways to describe the propagation of electromagnetic waves in an inhomogeneous but isotropic plasma is by the method of geometrical optics. It can be used if the scale on which the inhomogeneities vary, is large compared with the wavelength of the waves traveling through. This approximation is valid for electromagnetic wave propagation in the GHz range, that will be studied in subsequent chapters.

Consider an isotropic, inhomogeneous and source free medium. For monochromatic time variation, i.e. $e^{j\omega t}$, Maxwell's equations after Fourier transformation with respect to time are:

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \qquad 4.29$$

$$\nabla \times \vec{H} = j\omega\epsilon\mu\vec{E} \qquad 4.30$$

$$\nabla \cdot (\epsilon\vec{E}) = 0 \qquad 4.31$$

$$\nabla \cdot (\mu\vec{H}) = 0 \qquad 4.32$$

where $\epsilon = \epsilon(\vec{r},\omega)$, the permittivity and $\mu = \mu(\vec{r},\omega)$, the permeability contain the inhomogeneous properties of the medium.

In the homogeneous medium, the plane wave solutions of the above set are:

$$\vec{E} = \vec{e}e^{-j\vec{k}\cdot\vec{r}} \qquad 4.33$$
$$\vec{H} = \vec{h}e^{-j\vec{k}\cdot\vec{r}} \qquad 4.34$$

where

$$\vec{k} = \omega\sqrt{\epsilon\mu}\vec{s} \qquad 4.35$$

$\vec{s}$ is a unit vector in the direction of propagation and $\vec{e}$ and $\vec{h}$ are constant vectors.

If the medium is weakly inhomogeneous, i.e. the scale of the inhomogeneinity l ($\nabla\epsilon\rightarrow l, \nabla\mu\rightarrow l$) is large compared to the wavelength λ, we write the solution to Maxwell's equations as:

$$\vec{E} = \vec{e}(\vec{r},\omega)e^{-j\psi(\vec{r},\omega)} \qquad 4.36$$
$$\vec{H} = \vec{h}(\vec{r},\omega)e^{-j\psi(\vec{r},\omega)} \qquad 4.37$$

where $\psi(\vec{r},\omega)$ is a phase function varying on scale λ, $\vec{e}(\vec{r},\omega)$ and $\vec{h}(\vec{r},\omega)$ are amplitude functions varying on a scale l. From equations 1.29,1.30 and 1.36 we have:

$$\nabla\times(\nabla\times\vec{e}) - j\nabla\times(\nabla\psi\times\vec{e}) - j\nabla\psi\times(\nabla\times\vec{e}) - \nabla\psi\times(\nabla\psi\times\vec{e}) - k^2\vec{e} = 0 \qquad 4.38$$
$$\nabla\ln\epsilon\cdot\vec{e} + \nabla\cdot\vec{e} + \vec{e}\cdot\nabla\psi = 0 \qquad 4.39$$

where $k^2 = \omega^2\epsilon\mu$. The respective factors vary on a scale:

$\nabla\times(\nabla\times\vec{e})$: l^2

$\nabla\times(\nabla\psi\times\vec{e})$: λl

$\nabla\psi\times(\nabla\times\vec{e})$: λl

$\nabla\psi\times(\nabla\psi\times\vec{e})$: λ^2

$k^2\vec{e}$: λ^2

$\nabla\ln\epsilon\cdot\vec{e}$: l

$\nabla\cdot\vec{e}$: l

$\nabla\psi\cdot\vec{e}$: λ 4.40

Collecting factors varying in scale λ^2, we obtain:

$$\nabla\psi\times(\nabla\psi\times\vec{e}) + k^2\vec{e} = 0 \qquad 4.41$$
$$\vec{e}\cdot\nabla\psi = 0 \qquad 4.42$$

and therefore:

$$\nabla\psi\cdot\nabla\psi = k^2 \qquad 4.43$$

$\psi(\vec{r},\omega)$ is the so-called eikonal and equation 1.42 is the eikonal equation. It is a fundamental equation in geometrical optics.

Let $\vec{s}$ be the unit vector normal to planes of constant phase and $\vec{r}(\vec{s})$ a point on the ray. Then:

$$\vec{s} = \frac{\nabla \psi}{|\nabla \psi|} = \frac{\nabla \psi}{k} \qquad 4.44$$

$$d_s \vec{r} = \vec{s} \qquad 4.45$$

and with these:

$$\nabla \psi = k\vec{s} \qquad 4.46$$

$$\vec{s} \cdot \nabla \psi = k\vec{s} \cdot \vec{s} = k \qquad 4.47$$

$$\nabla (\vec{s} \cdot \nabla \psi) = \nabla k \qquad 4.48$$

$$\nabla (d_s \vec{r} \cdot k\vec{s}) = \nabla k \qquad 4.49$$

$$d_s (k d_s \vec{r}) = \nabla k \qquad 4.50$$

Introducing $k = k_0 n$, $n = \sqrt{\epsilon_r \mu_r}$ and $k_0 = \omega \sqrt{\epsilon_0 \mu_0}$ we obtain the *ray path equation:*

$$d_s (n d_s \vec{r}) = \nabla n \qquad 4.51$$

4.5. Optical phase path length.

Fermat's principle states that it takes a minimum time:

$$t = \int_A^B \frac{n}{c} ds \qquad 4.52$$

to travel from a point A to a point B along the actual ray. The distance a wave would travel in free space in this time period is called the *optical phase path length:*

$$s = ct = \int_A^B n \; ds \qquad 4.53$$

In measurements the time period t and thus, this optical path length is found. For a medium with refractive index under one, i.e. $n \leqslant 1$, as is the ionosphere, this leads to phase path lengths smaller than the Euclidian distance. In subsequent chapters, correction terms for the corresponding 'range error' will be computed.

5. Gravito-acoustic waves in a spherical atmosphere.

5.1. Introduction.

Gravito-acoustic waves are wave phenomena traveling in the atmosphere. They can exist because the atmosphere is stratified, which is caused by the presence of the gravitational field. One of the effects of gravito-acoustic waves is the refraction of electromagnetic waves, since they periodically change atmospheric properties such as density and pressure and thus the refractive index. This can be observed in radio astronomy, where radio stars appear to fluctuate with respect to their 'steady' position.

In his report E.A.P. Habraken described gravito-acoustic waves in an atmosphere with flat geometry. This chapter is concerned with the propagation of gravito-acoustic waves, perpendicular to the earth's magnetic field, in a spherical atmosphere.

5.2. Gravito-acoustic waves in a spherical atmosphere.

The following set of equations is valid:

$$\rho \frac{D\vec{v}}{Dt} + 2\vec{\omega}_{earth} \times \vec{v} = -\nabla \cdot \overline{\overline{P}} + \rho\vec{g} + \vec{J}\times\vec{B} + \rho_e \vec{E} \qquad 5.1$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho\vec{v} \right) = 0 \qquad 5.2$$

$$\frac{\partial}{\partial_t} \left(p\rho^{-\gamma} \right) + \vec{v}\cdot\nabla \left(p\rho^{-\gamma} \right) = 0 \qquad 5.3$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad 5.4$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \qquad 5.5$$

$$\vec{J} = \sigma \left(\vec{E} + \vec{v}\times\vec{B} \right) + \rho_e \vec{v} \qquad 5.6$$

They are the momentum eqn. for a collisionless plasma, the mass eqn., adiabatic law, the Maxwell eqns and Ohm's law.

Some assumptions are introduced:

* $\omega_{earth} = 0 \qquad \omega_{earth} << \omega_{gravito-acoustic}$. [1]

1 See table 1 at the end of this chapter.

* $\eta = 0$ no viscosity, so the pressure tensor reduces to a diagonal tensor with the scalar pressure as element.

* Stratifications exclusively along the radial coordinate.

* Isothermic equilibrium and ideal gas.

$$\frac{p_0}{\rho_0} = \frac{kT_0}{m}$$

* $v_0 = 0$, no wind.

* $\vec{B}_0 = B_0 \vec{e}_\theta$

* $\vec{g} = -g\vec{e}_r$, g assumed constant.

* $\frac{\partial \vec{E}}{\partial t} = 0$, quasi-static approximation.

* $\rho_e = 0$ quasi-neutral ionosphere.

* Wave propagation perpendicular to the direction of the geomagnetic field.

* $\sigma = \infty$; if a finite conductivity is assumed, it leads to attenuation terms in the dispersion relation and therefore to attenuation in time of the gravito acoustic wave. In reality this attenuation is often counteracted by a source term. Since the excitation mechanism is not included in the eqns. above, the assumption $\sigma = \infty$ is made to eliminate attenuation terms from the dispersion relation.

Next the set of eqns 1.1-6 is linearized, i.e. assume that each variable consists of an undisturbed part (index 0) and a small perturbation (index 1).

$p = p_0 + p_1$

$\vec{B} = \vec{B}_0 + \vec{B}_1$ etc.

The undisturbed part is:

$$0 = -\nabla p_0 - \rho_0 g \vec{e}_r \qquad 5.7$$

$$\frac{\partial \rho_0}{\partial t} = 0 \qquad 5.8$$

$$\frac{\partial p_0}{\partial t} = \gamma \frac{p_0}{\rho_0} \frac{\partial \rho_0}{\partial t} \qquad 5.9$$

$$J_0 = 0 \qquad 5.10$$

This leads to the undisturbed pressure and density; if we assume that g is a constant

$$p_0 = p_{00} e^{-\frac{r}{H}} \qquad 5.11$$

$$\rho_0 = \rho_{00} e^{-\frac{r}{H}} \qquad 5.12$$

where $H := \frac{kT_0}{mg}$ is the vertical scale height.

The first order set is:

$$\rho_0(r)\frac{dv}{dt} = -\nabla p + \rho\vec{g} + \vec{J}\times\vec{B}_0 \qquad 5.13$$

$$\frac{\partial\rho}{dt} = -\nabla\cdot\left(\rho_0\vec{v}\right) \qquad 5.14$$

$$\frac{\partial p}{\partial t} + \left(v\cdot\nabla\right)p_0 = \gamma\frac{p_0}{\rho_0}\left(\frac{\partial\rho}{\partial t} + v\cdot\nabla\rho_0\right) \qquad 5.15$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad 5.16$$

$$\nabla\times\vec{B} = \mu_0\vec{J} \qquad 5.17$$

$$\vec{E} + v\times\vec{B}_0 = 0 \qquad 5.18$$

in which all indices 1 have been left out for convenience.

Since all coefficients are independent of time. Fourier analysis with respect to time can be applied i.e. $\partial_t \rightarrow -i\omega$

After the introduction of spherical coordinates (r,θ,ϕ) the set of eqns becomes:

$$-i\omega\rho_0 v_r = -J_\phi B_0 - \rho(r)g - \partial_r p \qquad 5.19$$

$$-i\omega\rho_0 v_\theta = -\frac{1}{r}\partial_\theta p \qquad 5.20$$

$$-i\omega\rho_0 v_\phi = J_r B_0 - \frac{1}{r\sin(\theta)}\partial_\phi p \qquad 5.21$$

$$-i\omega\rho = -\frac{1}{r^2}\partial_r\left(r^2\rho_0(r)v_r\right) - \frac{1}{r\sin(\theta)}\partial_\phi\left(\rho_0(r)v_\phi\right)$$
$$-\frac{1}{r\sin(\theta)}\partial_\theta\left(\sin(\theta)\rho_0(r)v_\theta\right) \qquad 5.22$$

$$-i\omega\left(p - C_s{}^2\rho\right) = C_s{}^2 v_r\partial_r\rho_0 - v_r\partial_r p_0 \qquad 5.23$$

$$\mu_0 J_r = \frac{1}{r\sin(\theta)}\left(\partial_\theta\left(\sin(\theta)B_\phi\right) - \partial_\phi B_\theta\right) \qquad 5.24$$

$$\mu_0 J_\theta = \frac{1}{r}\left(\frac{1}{\sin(\theta)}\partial_\phi B_r - \partial_r\left(rB_\phi\right)\right) \qquad 5.25$$

$$\mu_0 J_\phi = \frac{1}{r}\left(\partial_r\left(rB_\theta\right) - \partial_\theta B_r\right) \qquad 5.26$$

$$i\omega B_r = \frac{1}{r\sin(\theta)}\left(\partial_\theta\left(sin(\theta)E_\phi\right) - \partial_\phi E_\theta\right) \qquad 5.27$$

$$i\omega B_\theta = \frac{1}{r}\left(\frac{1}{\sin(\theta)}\partial_\phi E_r - \partial_r\left(rE_\phi\right)\right) \qquad 5.28$$

$$i\omega B_\phi = \frac{1}{r}\left(\partial_r\left(rE_\theta\right) - \partial_\theta E_r\right) \qquad 5.29$$

$$\frac{1}{r^2}\partial_r\left(r^2 B_r\right) + \frac{1}{r\sin(\theta)}\partial_\theta\left(B_\theta \sin(\theta)\right) + \frac{1}{r sin(\theta)}\partial_\phi B_\phi = 0 \qquad 5.30$$

$$E_r - v_\phi B_0 = 0 \qquad 5.31$$

$$E_\theta = 0 \qquad 5.32$$

$$E_\phi + v_r B_0 = 0 \qquad 5.33$$

where $C_s{}^2 = \gamma \frac{p_0}{\rho_0}$ is the squared velocity of sound.

Introduction of a new variable

$$r = R + z \qquad 5.34$$

where R is the earths radius, allows for a series expansion in $\frac{1}{R}$ if $\frac{z}{R} << 1$:

$$\frac{1}{r} = \frac{1}{R\left(1 + \frac{z}{R}\right)} = \frac{1}{R}\left(1 - \frac{z}{R} + \frac{z^2}{R^2} - \cdots\right) \qquad 5.35$$

$$p = p^0 + \frac{1}{R}p^1 + \frac{1}{R^2}p^2 + \cdots \qquad 5.36$$

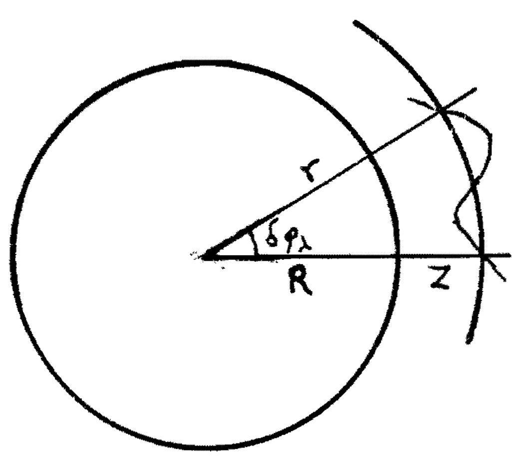
Note that the superscripts refer to the expansion in $\frac{1}{R}$, whereas the subscripts refer to the expansion in powers of the amplitude of the perturbation. Further, expressions for p_0 and ρ_0 i.e. eqns 1.11, 1.12 are inserted.

The derivation with respect to ϕ may be Fourier transformed i.e. $\partial_\phi \rightarrow im$ because of the ϕ independency of all coefficients. m is an indication for the number of waves that 'fit' around the earth

$$m\,\delta\phi_\lambda = 2\pi \qquad 5.37$$

in which $\delta\phi_\lambda$ is the angle that intersects the G.A.W. at two points with equal phase. From fig.1 we see that

$$r\,\delta\phi_\lambda = \lambda \qquad 5.38$$



so we can introduce a wavenumber k_ϕ by :

$$m = \frac{2\pi r}{\lambda} = k_\phi\left(R + z\right) = k_\phi R\left(1 + \frac{z}{R} + \cdots\right) \qquad 5.39$$

If we restrict our attention to waves with $\lambda << R$, we see that m appears to be of magnitude order R. With this, and the introduction of

$$\vec{V} := \rho_0(r)\vec{v} \qquad 5.40$$

the set of equations, up to first order in the $\frac{1}{R}$ expansion becomes:

$$-i\omega\left(V_r^0 + \frac{1}{R}V_r^1\right) = -\left(J_\phi^0 + \frac{1}{R}J_\phi^1\right)B_0 - \left(\rho^0 + \frac{1}{R}\rho^1\right)g - \partial_z\left(p^0 + \frac{1}{R}p^1\right) \qquad 5.41$$

$$-i\omega\left(V_\theta^0 + \frac{1}{R}V_\theta^1\right) = -\frac{1}{R}\left(1 - \frac{z}{R}\right)\partial_\theta\left(p^0 + \frac{1}{R}p^1\right) \qquad 5.42$$

$$-i\omega\left(V_\phi^0 + \frac{1}{R}V_\phi^1\right) = \left(J_r^0 + \frac{1}{R}J_r^1\right)B_0 - \frac{ik_\phi}{\sin(\theta)}\left(1 - \frac{z}{R}\right)\left(1 + \frac{z}{R}\right)\left(p^0 + \frac{1}{R}p^1\right) \qquad 5.43$$

$$\begin{aligned}-i\omega\left(\rho^0 + \frac{1}{R}\rho^1\right) = &\frac{-1}{R^2}\left(1 - \frac{2z}{R}\right)\left[\partial_z\left(R^2 + 2zR\right)\left(V_r^0 + \frac{V_r^1}{R}\right)\right] \\ &- i\frac{k_\phi}{\sin(\theta)}\left(1 - \frac{z}{R}\right)\left(1 + \frac{z}{R}\right)\left(V_\phi^0 + \frac{V_\phi^1}{R}\right) \\ &- \frac{1}{R\sin(\theta)}\left(1 - \frac{z}{R}\right)\partial_\theta\left(\sin(\theta)\left(V_\theta^0 + \frac{1}{R}V_\theta^1\right)\right)\end{aligned} \qquad 5.44$$

$$-i\omega\left[\left(p^0 + \frac{1}{R}p^1\right) - C_s^2\left(\rho^0 + \frac{1}{R}\rho^1\right)\right] = -\left(V_r^0 + \frac{V_r^1}{R}\right)\frac{(\gamma - 1)}{H}\frac{p_0(z)}{\rho_0(z)} \qquad 5.45$$

$$\begin{aligned}\mu_0\left(J_r^0 + \frac{1}{R}J_r^1\right) = &\frac{1}{R\sin(\theta)}\left(1 - \frac{z}{R}\right)\cdot \\ &\cdot\left[\partial_\theta\left(\sin(\theta)\left(B_\phi^0 + \frac{1}{R}B_\phi^1\right)\right) - ik_\phi R\left(1 + \frac{z}{R}\right)\left(B_\theta^0 + \frac{1}{R}B_\theta^1\right)\right]\end{aligned} \qquad 5.46$$

$$\begin{aligned}\mu_0\left(J_\theta^0 + \frac{1}{R}J_\theta^1\right) = &\frac{1}{R}\left(1 - \frac{z}{R}\right)\cdot \\ &\cdot\left[\frac{ik_\phi R}{\sin(\theta)}\left(1 + \frac{z}{R}\right)\left(B_r^0 + \frac{1}{R}B_r^1\right) - \partial_z\left(R\left(1 + \frac{z}{R}\right)\left(B_\phi^0 + \frac{1}{R}B_\phi^1\right)\right)\right]\end{aligned} \qquad 5.47$$

$$\mu_0\left(J_\phi^0 + \frac{1}{R}J_\phi^1\right) = \frac{1}{R}\left(1 - \frac{z}{R}\right)\left[\partial_z\left(\left(R + z\right)\left(B_\theta^0 + \frac{1}{R}B_\theta^1\right)\right) - \partial_\theta\left(B_r^0 + \frac{1}{R}B_r^1\right)\right] \qquad 5.48$$

$$\begin{aligned}i\omega\left(B_r^0 + \frac{1}{R}B_r^1\right) = &\frac{1}{R\sin(\theta)}\left(1 - \frac{z}{R}\right)\left[\partial_\theta\left(\sin(\theta)\left(E_\phi^0 + \frac{1}{R}E_\phi^1\right)\right)\right. \\ &\left. - ik_\phi R\left(1 + \frac{z}{R}\right)\left(E_\theta^0 + \frac{1}{R}E_\theta^1\right)\right]\end{aligned} \qquad 5.49$$

$$i\omega\left(B_\theta^0 + \frac{1}{R}B_\theta^1\right) = \frac{1}{R}\left(1 - \frac{z}{R}\right)\cdot \left[\frac{ik_\phi}{\sin(\theta)}R\left(1 + \frac{z}{R}\right)\left(E_r^0 + \frac{1}{R}E_r^1\right) - \partial_z\left(R\left(1 + \frac{z}{R}\right)\left(E_\phi^0 + \frac{1}{R}E_\phi^1\right)\right)\right] \qquad 5.50$$

$$i\omega\left(B_\phi^0 + \frac{1}{R}B_\phi^1\right) = -\frac{1}{R}\left(1 - \frac{z}{R}\right)\left(\partial_\theta\left(E_r^0 + \frac{1}{R}E_r^1\right)\right) + \frac{1}{R}\left(1 - \frac{z}{R}\right)\partial_z\left(R\left(1 + \frac{z}{R}\right)\left(E_\theta^0 + \frac{1}{R}E_\theta^1\right)\right) \qquad 5.51$$

$$\frac{1}{R^2}\left(1 - \frac{2z}{R}\right)\partial_z\left(R^2\left(1 + \frac{2z}{R}\right)\left(B_r^0 + \frac{1}{R}B_r^1\right)\right) + \frac{1}{R\sin(\theta)}\left(1 - \frac{z}{R}\right)\partial_\theta\left(\sin(\theta)\left(B_\theta^0 + \frac{1}{R}B_\theta^1\right)\right) + \frac{1}{R\sin(\theta)}\left(1 - \frac{z}{R}\right)ik_\phi R\left(1 + \frac{z}{R}\right)\left(B_\phi^0 + \frac{1}{R}B_\phi^1\right) = 0 \qquad 5.52$$

$$\rho_0(z)\left[E_r^0 + \frac{1}{R}E_r^1\right] - \left(V_\phi^0 + \frac{1}{R}V_\phi^1\right)B_0 = 0 \qquad 5.53$$

$$E_\theta^0 + \frac{1}{R}E_\theta^1 = 0 \qquad 5.54$$

$$\rho_0(z)\left[E_\phi^0 + \frac{1}{R}E_\phi^1\right] + \left(V_r^0 + \frac{1}{R}V_r^1\right)B_0 = 0 \qquad 5.55$$

The zeroth order set in $\frac{1}{R}$ represents the case of a flat earth. It is:

$$-i\omega V_r^0 = -J_\phi^0 B_0 - \rho^0 g - \partial_z p^0 \qquad 5.56$$

$$-i\omega V_\theta^0 = 0 \qquad 5.57$$

$$-i\omega V_\phi^0 = J_r^0 B_0 - \frac{ik_\phi}{\sin(\theta)}p^0 \qquad 5.58$$

$$-i\omega\rho^0 = \left(-\partial_z V_r^0 - \frac{ik_\phi}{\sin(\theta)}V_\phi^0\right) \qquad 5.59$$

$$-i\omega\left(p^0 - C_s^2\rho^0\right) = -\left(\gamma - 1\right)gV_r^0 \qquad 5.60$$

$$\mu_0 J_r^0 = \frac{-ik_\phi}{\sin(\theta)}B_\theta^0 \qquad 5.61$$

$$\mu_0 J_\theta^0 = \frac{ik_\phi}{\sin(\theta)}B_r^0 - \partial_z B_\phi^0 \qquad 5.62$$

$$\mu_0 J_\phi^0 = \partial_z B_\theta^0 \qquad 5.63$$

$$i\omega B_r^0 = 0 \qquad 5.64$$

$$i\omega B_\theta^0 = \frac{ik_\phi}{\sin(\theta)}E_r^0 - \partial_z E_\phi^0 \qquad 5.65$$

$$i\omega B_\phi^0 = 0 \qquad 5.66$$

$$\rho_0 E_r^0 - V_\phi^0 B_0 = 0 \tag{5.67}$$

$$E_\theta^0 = 0 \tag{5.68}$$

$$\rho_0 E_\phi^0 + V_r^0 B_0 = 0 \tag{5.69}$$

From this, it follows that $v_\theta^0 = 0$, $E_\theta^0 = 0$, $B_r^0 = 0$, $B_\phi^0 = 0$ and $J_\theta^0 = 0$

Since $\rho_0 = \rho_{00} e^{\frac{-z}{H}}$ a Fourier transformation with respect to z is not allowed. To solve the above set of eqns , it is reduced to one second order differential equation in V_r^0. The result is

$$\begin{aligned}
&\sin^2(\theta)\left[\left(C_s^2 + v_A^2\right)\sin^2(\theta) - \frac{1}{v_f^2}\left(C_s^2 + v_A^2\right)^2\right]\partial_{\zeta\zeta} V_r^0 + \\
&\sin^2(\theta)\left[\left(C_s^2 + 2v_A^2\right)\sin^2(\theta) - \frac{1}{v_f^2}\left(C_s^2 + v_A^2\right)^2\right]\partial_{\zeta} V_r^0 + \\
&\quad\left[\left(v_A^2 + v_f^2 k_\phi^2 H^2\right)\sin^4(\theta) + \frac{H^2 g^2(\gamma - 1)\sin^2(\theta)}{v_f^2} + \right. \\
&\quad\left.\left(\frac{H^2 g^2}{v_f^4} - 2H^2 k_\phi^2 \sin^2(\theta)\right)\left(C_s^2 + v_A^2\right) + \left(\frac{H^2 k_\phi^2}{v_f^2} - \frac{Hg}{v_f^4}\right)\left(C_s^2 + v_A^2\right)^2\right] V_r^0 = 0
\end{aligned} \tag{5.70}$$

where

$$v_A^2 := \frac{B_0^2}{\mu_0 \rho_0} \tag{5.71}$$

$$C_s^2 := \frac{\gamma p_0}{\rho_0} \tag{5.72}$$

$$v_f^2 := \frac{\omega^2}{k_\phi^2} \tag{5.73}$$

and

$$\zeta := \frac{z}{H} \tag{5.74}$$

where v_A is the Alfvén velocity.

The differential equation 1.70 has been derived for a flat earth geometry. Therefore we will use the more appropriate parameters k_x and v_{fx}, with:

$$k_x = \frac{k_\phi}{\sin\theta} \tag{5.75}$$

and

$$v_f^2 = \frac{\omega^2}{k_x^2} \tag{5.76}$$

Inserting 1.75 and 1.76 into equation 1.70, we obtain:

$$\left[\left(C_s^2 + v_A^2\right) - \frac{1}{v_f^2}\left(C_s^2 + v_A^2\right)^2\right]\partial_{\zeta\zeta}V_r^0 +$$
$$\left[\left(C_s^2 + 2v_A^2\right) - \frac{1}{v_f^2}\left(C_s^2 + v_A^2\right)^2\right]\partial_{\zeta}V_r^0 +$$
$$\left[\left(v_A^2 + v_f^2 k_x^2 H^2\right) + \frac{H^2 g^2(\gamma - 1)}{v_f^2} + \right.$$
$$\left.\left(\frac{H^2 g^2}{v_f^4} - 2H^2 k_x^2\right)\left(C_s^2 + v_A^2\right) + \left(\frac{H^2 k_x^2}{v_f^2} - \frac{Hg}{v_f^4}\right)\left(C_s^2 + v_A^2\right)^2\right]V_r^0 = 0 \qquad 5.77$$

If the magnetic field is taken zero in the above equation, thus setting $B_0 = 0$, we have the dispersion relation for the gravito-acoustic wave for a flat earth in absence of the earth's magnetic field:

$$\omega^4 - \omega^2 \gamma g H\left(k_z^2 + k_x^2 - j\frac{k_z}{H}\right) + (\gamma - 1)g^2 k_x^2 = 0 \qquad 5.78$$

This is in agreement with the results derived by E.A.P. Habraken [7] and by Yeh and Liu [14], who started from a flat earth geometry for the case without a magnetic field. Normalizing eqn 1.77 on the Brunt-Vaïsälä frequency:

$$\omega_b = \sqrt{\frac{\gamma - 1}{\gamma}\frac{g}{H}} \qquad 5.79$$

we obtain

$$\left(\frac{\omega}{\omega_b}\right)^4 - \frac{\gamma^2}{\gamma - 1}\left(k^2 H^2 - jk_z H\right)\left(\frac{\omega}{\omega_b}\right)^2 + \frac{\gamma^2}{\gamma - 1}k_x^2 H^2 = 0 \qquad 5.80$$

In a low frequency approximation, it reduces to:

$$\left(k^2 H^2 - jk_z H\right)\left(\frac{\omega}{\omega_b}\right)^2 = k_x^2 H^2 \qquad 5.81$$

For greater altitudes the Alfvén velocity is the dominant factor, due to its $e^{+\zeta}$ dependency i.e.

$C_s^2 << v_A^2$ and

$v_f^2 << v_A^2$. [2]

2 See tables 1 and 2 at the end of this chapter.

Let

$$\epsilon := \frac{C_s^2}{v_A^2} << 1 \qquad 5.82$$

$$\delta := \frac{v_f^2}{v_A^2} << 1 \qquad 5.83$$

and

$$V_r^0 := V_{r0} + \epsilon V_{r1} + \cdots \qquad 5.84$$

then eqn 1.70 becomes

$$\left[\frac{\delta^2}{\epsilon^2}\epsilon^2\left(1+\epsilon\right) - \frac{\delta}{\epsilon}\epsilon\left(1+2\epsilon-\epsilon^2\right)\right]\partial_{\zeta\zeta}\left(V_{r0}+\epsilon V_{r1}\right) +$$
$$\left[\frac{\delta^2}{\epsilon^2}\epsilon^2\left(2+\epsilon\right) - \frac{\delta}{\epsilon}\epsilon\left(1+2\epsilon-\epsilon^2\right)\right]\partial_{\zeta}\left(V_{r0}+\epsilon V_{r1}\right) +$$
$$\left[\left(\frac{\delta^2}{\epsilon^2}\epsilon^2 + \frac{\delta^3}{\epsilon^3}\epsilon^3 k_x^2 H^2\right) + \frac{\delta\epsilon^3}{\epsilon\gamma^2}(\gamma-1) + \right.$$
$$\left.\left(\frac{\epsilon^2}{\gamma^2} - 2\frac{\delta^2}{\epsilon^2}\epsilon^2 H^2 k_x^2\right)(1+\epsilon) + \left(\frac{\delta}{\epsilon}\epsilon H^2 k_x^2 - \frac{\epsilon}{\gamma}\right)\left(1+2\epsilon+\epsilon^2\right)\right]$$
$$\left(V_{r0}+\epsilon V_{r1}\right) = 0 \qquad 5.85$$

Since $\frac{\delta}{\epsilon} = \frac{v_f^2}{C_s^2}$ terms of this kind can be considered as being of order 0 in ϵ.[3] First order terms in ϵ give

$$-\frac{\delta}{\epsilon}\partial_{\zeta\zeta}V_{r0} - \frac{\delta}{\epsilon}\partial_{\zeta}V_{r0} + \left(\frac{\delta}{\epsilon}H^2k_x^2 - \frac{1}{\gamma}\right)V_{r0} = 0 \qquad 5.86$$

All coefficients are independent of ζ so

$$\partial_\zeta \rightarrow jk_\zeta = jk_z H \qquad 5.87$$

$$\partial_{\zeta\zeta} \rightarrow -k_\zeta^2 = -k_z^2 H^2 \qquad 5.88$$

and the dispersion relation is obtained

$$\omega^2\left[\left(k_z^2 - \frac{jk_z}{H}\right) + k_x^2\right] - \frac{g}{H}k_x^2 = 0 \qquad 5.89$$

The dispersion relation for a gravito acoustic wave in general is a quartic. However, the assumption that $C_s^2 << v_a^2$ causes the acoustic branch to disappear and therefore the equation to become a quadratic. This dispersion relation, normalized on ω_b is:

3 See tables 1 and 2 at the end of this chapter.

$$\left(\frac{\omega}{\omega_b}\right)^2\left(k^2H^2 - jk_zH\right) = \frac{\gamma}{\gamma-1}k_x^2H^2 \qquad 5.90$$

This differs from the low frequency approximation dispersion relation in absence of the Earth's magnetic field, i.e. eqn 1.81, by a factor $\frac{\gamma}{\gamma-1}$ in the right hand side of the dispersion relation.

Second order terms in ϵ

$$-\frac{\delta}{\epsilon}\partial_{\zeta\zeta}\left(V_{r1}\right) - \frac{\delta}{\epsilon}\partial_{\zeta}\left(V_{r1}\right) + \left(H^2k_x^2\frac{\delta}{\epsilon} - \frac{1}{\gamma}\right)\epsilon V_{r1} =$$
$$-\epsilon\left[\left(\frac{\delta^2}{\epsilon^2} - 2\frac{\delta}{\epsilon}\right)\partial_{\zeta\zeta}V_{r0} + \left(2\frac{\delta^2}{\epsilon^2}\right)\partial_{\zeta}V_{r0} +\right.$$
$$\left.\left(\frac{\delta^2}{\epsilon^2} + \frac{1}{\gamma^2} - 2\frac{\delta^2}{\epsilon^2}H^2k_x^2 + \frac{\delta}{\epsilon}2H^2k_x^2 - \frac{2}{\gamma}\right)V_{r0}\right] \qquad 5.91$$

The right hand side of this equation is known from eqns 1.85 and 1.86. It is the solution of the first order ϵ part and can be considered as a 'source' for the second order equation. With eqn 1.82 it follows that

$$\partial_{\zeta}\epsilon = -\epsilon \qquad 5.92$$

and

$$\partial_{\zeta\zeta}\epsilon = +\epsilon \qquad 5.93$$

If we set

$$V_{r1} = Ce^{ik_{\zeta}\zeta} \qquad 5.94$$

then

$$\partial_{\zeta}V_{r1} = Cik_{\zeta}e^{ik_{\zeta}\zeta} \qquad 5.95$$

and

$$\partial_{\zeta\zeta}V_{r1} = -k_{\zeta}^2Ce^{ik_{\zeta}\zeta} \qquad 5.96$$

By substituting this in eqn 1.90 the constant C is determined:

$$C = \frac{\left[\left(\frac{\delta^2}{\epsilon^2} - 2\frac{\delta}{\epsilon}\right)k_{\zeta}^2 - 2i\left(\frac{\delta^2}{\epsilon^2} - \frac{\delta}{\epsilon}\right)k_{\zeta} - \left(\frac{\delta^2}{\epsilon^2} - 2\frac{\delta^2}{\epsilon^2}H^2k_x^2 + \frac{\delta}{\epsilon}2H^2k_x^2 + \frac{1}{\gamma^2} - \frac{2}{\gamma}\right)\right]}{\frac{\delta}{\epsilon}k_{\zeta}^2 + \frac{\delta}{\epsilon}ik_{\zeta} + \frac{\delta}{\epsilon}H^2k_{\zeta}^2 - \frac{1}{\gamma}}V_{r0} \qquad 5.97$$

Thus, for v_r correct until first ϵ order

$$v_r = \frac{V_r}{\rho_0} = \frac{V_{r0} + \epsilon V_{r1}}{\rho_0} =$$

$$\frac{1}{\rho_{00}} e^{(ik_\zeta + 1)\zeta} \left[1 + \frac{C_s^2}{v_A^2} |_{\zeta=0} e^{-\zeta} \left\{ \frac{\left[\left(\frac{\delta^2}{\epsilon^2} - 2\frac{\delta}{\epsilon} \right) k_\zeta^2 - 2i \left(\frac{\delta^2}{\epsilon^2} - \frac{\delta}{\epsilon} \right) k_\zeta - \left[\left(\frac{\delta^2}{\epsilon^2} - 2\frac{\delta^2}{\epsilon^2} H^2 k_x^2 + \frac{\delta}{\epsilon} 2H^2 k_x^2 + \frac{1}{\gamma^2} - \frac{2}{\gamma} \right) \right] \right]}{\frac{\delta}{\epsilon} k_\zeta^2 + \frac{\delta}{\epsilon} i k_\zeta + \frac{\delta}{\epsilon} H^2 k_\zeta^2 - \frac{1}{\gamma}} \right\} \right] V_{r0} \qquad 5.98$$

where V_{r0} is an arbitrary constant.

Note that inthe dispersion relations 1.89 and 1.90 the strength of the magnetic field does not show up. One has to go to first order in the expansion in ϵ to find its effect in the dispersion relation.

5.3. Data.

To justify some of the approximations used and to give an idea of the order of magnitude of the parameters in this chapter, the tables below are presented. They give a rough estimate of the main parameters.

Since gravito-acoustic waves have long wavelengths, so networks of observation stations are necessary to measure the parameters, few data is available. The material for the tables has been taken from a report by E.A.P. Habraken [7].

Table 1 : gravito-acoustic waves.		
parameter	range	unit
τ	20-40	min.
ω	$2.5-5\cdot 10^{-3}$	$\frac{1}{s}$
k_x	$2.5-6\cdot 10^{-5}$	$\frac{1}{m}$
k_z	$1-3\cdot 10^{-5}$	$\frac{1}{m}$
v_f	$1-3\cdot 10^{2}$	$\frac{m}{s}$

Table 2		
parameter	range	unit
ω_{earth}	$7.3\cdot 10^{-5}$	$\frac{1}{s}$
R_{earth}	$6.38\cdot 10^{6}$	m
C_s	$4-8\cdot 10^{2}$	$\frac{m}{s}$
v_A above 200 km.	$\geqslant 10^{4}$	$\frac{m}{s}$

6. Ionospheric range correction.

6.1. Introduction.

One of the limiting factors in the accuracy of range measurements from terrestrial points to satellites or astronomical radio sources is the refractive effect of the ionosphere. In this chapter approximations are derived for the ray trajectory and for the correction term of the optical (phase) path length as compared to the Euclidian distance. The influence of the geomagnetic field has not been taken into account.

6.2. Ray-path equation.

Assuming that the wavelength of the electro-magnetic waves is small compared to the scale of the inhomogeneities of the ionosphere, the geometric optical approximation can be used to determine the ray-path of an e.m. wave. The general formula is:

$$\nabla n = \frac{d}{ds}\left(n\frac{d\vec{r}_b}{ds}\right) \qquad 6.1$$

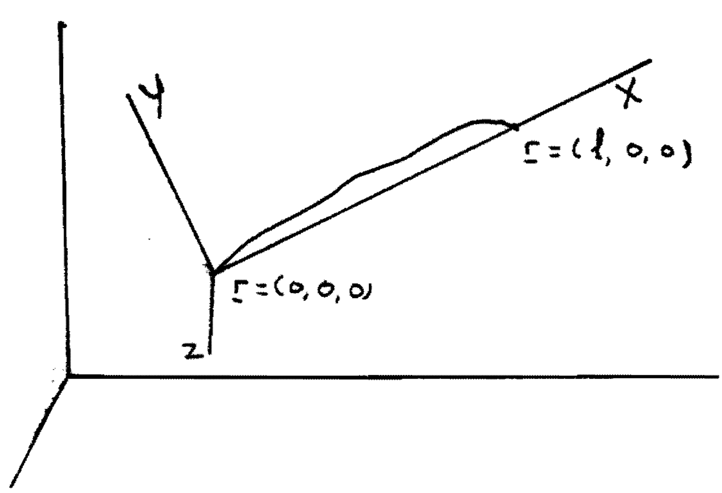
where

n : refractive index,

$\vec{r}_b$: ray path,

s : length as measured along the path from the starting point.

Since the refractive index in the ionosphere is approximately one, the path of an electro-magnetic wave of (relatively) high frequency, will resemble a straight line. We choose the X-axis of the carthesian coordinate-system along this line, so the starting point of the ray is (x,y,z) = (0,0,0) and the end-point (x,y,z) = (1,0,0). See fig. 1.



We assume that n can be written as

$$n = 1+\epsilon N(x,y,z) \quad 0<\epsilon<<1 \tag{6.2}$$

$$\vec{r}_b = x\vec{e}_x+y\vec{e}_y+z\vec{e}_z \tag{6.3}$$

and we expand y(x) and z(x) in power series:

$$y(x) = y_0(x)+\epsilon y_1(x)+\epsilon^2 y_2(x)+\cdots \tag{6.4}$$

$$z(x) = z_0(x)+\epsilon z_1(x)+\epsilon^2 z_2(x)+\cdots \tag{6.5}$$

Inserting 1.2 - 1.5 with $\epsilon=0$ in 1.1 gives

$$y_0 = 0, \quad z_0 = 0 \tag{6.6}$$

From fig.1. we have

$$(ds)^2 = (dx)^2+(dy)^2+(dz)^2 = (dx)^2\left[1+\frac{(dy)^2}{(dx)^2}+\frac{(dz)^2}{(dx)^2}\right] \tag{6.7}$$

so

$$ds = \sqrt{1+(y')^2+(z')^2}dx \tag{6.8}$$

and

$$\frac{d}{ds} = \frac{1}{\sqrt{1+(y')^2+(z')^2}}\frac{d}{dx} \tag{6.9}$$

From 1.4, 1.5

$$y'(x) = \epsilon y'_1(x)+\epsilon^2 y'_2(x)+\cdots \tag{6.10}$$

$$z'(x) = \epsilon z'_1(x)+\epsilon^2 z'_2(x)+\cdots \tag{6.11}$$

so

$$\left[1+(y')^2+(z')^2\right]^{-½} = 1-½(y')^2-½(z')^2+O(\epsilon^3) \tag{6.12}$$

The refractive index n can be expanded in a Taylor-series around the X-axis :

$$n(x,y_b,z_b) = n(x,0,0)+y_b\frac{\partial n(x,y,z)}{\partial y}|_{y=0}+z_b\frac{\partial n(x,y,z)}{\partial z}|_{z=0}+\cdots \tag{6.13}$$

With eqn. 1.2 this gives

$$n = 1+\epsilon N(x,0,0)+\epsilon y(x)\frac{\partial N}{\partial y}|_{y=0}+\epsilon z(x)\frac{\partial N}{\partial z}|_{z=0}+\cdots \tag{6.14}$$

and

$$\nabla n = \frac{\partial n}{\partial x}\vec{e}_x+\frac{\partial n}{\partial y}\vec{e}_y+\frac{\partial n}{\partial z}\vec{e}_z =$$

$$\vec{e}_x \left[\epsilon \frac{\partial N}{\partial x}|_{y=0,z=0} + \epsilon^2 y_1(x) \frac{\partial}{\partial x}\frac{\partial N}{\partial y}|_{y=0} + \epsilon^2 z_1(x)\frac{\partial}{\partial x}\frac{\partial N}{\partial z}|_{z=0}\right] +$$
$$\vec{e}_y \left[\epsilon \frac{\partial N}{\partial y}|_{y=0} + \epsilon^2 y_1 \frac{\partial^2 N}{\partial y^2}\right] +$$
$$\vec{e}_z \left[\epsilon \frac{\partial N}{\partial z}|_{z=0} + \epsilon^2 z_1 \frac{\partial^2 N}{\partial z^2}\right] + O(\epsilon^3) \qquad 6.15$$

Inserting the above in eqn 1.1 yields

$$\vec{e}_x \left[\epsilon \frac{\partial N}{\partial x}|_{y=0,z=0} + \epsilon^2 y_1(x) \frac{\partial}{\partial x}\frac{\partial N}{\partial y}|_{y=0} + \epsilon^2 z_1(x)\frac{\partial}{\partial x}\frac{\partial N}{\partial z}|_{z=0}\right] +$$
$$\vec{e}_y \left[\epsilon \frac{\partial N}{\partial y}|_{y=0} + \epsilon^2 y_1 \frac{\partial^2 N}{\partial y^2}\right] +$$
$$\vec{e}_z \left[\epsilon \frac{\partial N}{\partial z}|_{z=0} + \epsilon^2 z_1 \frac{\partial^2 N}{\partial z^2}\right] =$$
$$\left[1-\tfrac{1}{2}\epsilon^2(y'_1)^2-\tfrac{1}{2}\epsilon^2(z'_1)^2\right]\frac{d}{dx}\left[\left[1-\tfrac{1}{2}\epsilon^2(y'_1)^2-\tfrac{1}{2}\epsilon^2(z'_1)^2\right]\right.$$
$$\left[1+\epsilon N(x,0,0)+\epsilon^2 y_1(x)\frac{\partial N}{\partial y}|_{y=0}+\epsilon^2 z_1(x)\frac{\partial N}{\partial z}|_{z=0}\right]$$
$$\left.\frac{d}{dx}\left\{x\vec{e}_x + \left[\epsilon y_1(x)+\epsilon^2 y_2(x)\right]\vec{e}_y + \left[\epsilon z_1(x)+\epsilon^2 z_2(x)\right]\vec{e}_z\right\}\right] =$$
$$\vec{e}_x \left[\epsilon \frac{\partial N}{\partial x}|_{y=0,z=0} + \epsilon^2 y_1(x)\frac{\partial}{\partial x}\frac{\partial N}{\partial y}|_{y=0} + \epsilon^2 z_1(x)\frac{\partial}{\partial x}\frac{\partial N}{\partial z}|_{z=0} + \right.$$
$$\left. - y'_1(x)y''_1(x) + y'_1(x)\frac{\partial N}{\partial y}|_{y=0} - z'_1(x)z''_1(x) + z'_1(x)\frac{\partial N}{\partial z}|_{z=0}\right] +$$
$$\vec{e}_y \left[\epsilon y''_1(x) + \epsilon^2 y''_2 + \epsilon^2 y''_1 N(x,0,0) + \epsilon^2 y'_1 \frac{\partial N}{\partial x}\right] +$$
$$\vec{e}_z \left[\epsilon z''_1(x) + \epsilon^2 z''_2 + \epsilon^2 z''_1 N(x,0,0) + \epsilon^2 z'_1 \frac{\partial N}{\partial x}\right] + O(\epsilon^3) \qquad 6.16$$

Comparison of equal ϵ order terms yields

* $\vec{e}_x$ component :

the first order term vanishes and a set of two differential equations is obtained :

$$y'_1(x)\left[y''_1(x) - \frac{\partial N}{\partial y}|_{y=0}\right] = 0 \qquad 6.17$$

$$z'_1(x)\left[z''_1(x) - \frac{\partial N}{\partial z}|_{z=0}\right] = 0 \qquad 6.18$$

Solutions are

$$z_1(x) = c, \quad y_1(x) = c \tag{6.19}$$

or

$$y_1(x) = y'_1(0)x + \int_0^x dx' \int_0^{x'} \frac{\partial N(x'',y,z)}{\partial y}|_{y=0} dx'' \tag{6.20}$$

$$z_1(x) = z'_1(0)x + \int_0^x dx' \int_0^{x'} \frac{\partial N(x'',y,z)}{\partial z}|_{z=0} dx'' \tag{6.21}$$

Using boundary conditions on x = l i.e. y(l) = z(l) = 0 the constants above can be determined:

$$y_1(x) = 0, \quad z_1(x) = 0 \tag{6.22}$$

or

$$y'_1(0) = -\frac{1}{l}\int_0^l dx' \int_0^{x'} \frac{\partial N(x'',y,z)}{\partial y}|_{y=0} dx'' \tag{6.23}$$

$$z'_1(0) = -\frac{1}{l}\int_0^l dx' \int_0^{x'} \frac{\partial N(x'',y,z)}{\partial z}|_{z=0} dx'' \tag{6.24}$$

* $\vec{e}_y$ component:

First order ϵ terms:

$$\frac{\partial N}{\partial y}|_{y=0} = y''_1(x) \tag{6.25}$$

* $\vec{e}_z$ component:

$$\frac{\partial N}{\partial z}|_{z=0} = z''_1(x) \tag{6.26}$$

This implies that the second pair of solutions satisfies the ray path- equation. The total solution is:

$$\begin{aligned}\vec{r}_b = x\vec{e}_x + \epsilon &\left[\int_0^x dx' \int_0^{x'} \frac{\partial N}{\partial y}|_{y=0} dx'' - \frac{x}{l}\int_0^l dx' \int_0^{x'} \frac{\partial N}{\partial y}|_{y=0} dx''\right]\vec{e}_y + \\ \epsilon &\left[\int_0^x dx' \int_0^{x'} \frac{\partial N}{\partial z}|_{z=0} dx'' - \frac{x}{l}\int_0^l dx' \int_0^{x'} \frac{\partial N}{\partial z}|_{z=0} dx''\right]\vec{e}_z + O(\epsilon^2)\end{aligned} \tag{6.27}$$

6.3. Optical path length.

The optical (phase) path length in absence of the geomagnetic field between two points on a ray, as defined by

$$\bar{s} = \int_0^s n(\bar{r}(s'))ds' \qquad 6.28$$

can be expanded in a power-series:

$$\bar{s} = s_0 + \epsilon s_1 + \epsilon^2 s_2 + \cdots \qquad 6.29$$

in which s_0 is the distance between begin and end point of the ray as measured along a straight line. Expressions for the first and second order correction terms will be now be derived.

$$\begin{aligned}
\bar{s} &= \int_0^s n \; ds = \\
&\int_0^l n(x, y_b(x), z_b(x))\sqrt{1 + y'^2 + z'^2}dx = \\
&\int_0^l \left[1 + \epsilon N(x,0,0) + \epsilon^2 y_1 \frac{\partial N}{\partial y}|_{y=0} + \epsilon^2 z_1 \frac{\partial N}{\partial z}|_{z=0}\right] \left[1 + \tfrac{1}{2}\epsilon^2 y'^2_1 + \tfrac{1}{2}\epsilon^2 z'^2_1\right] dx + O(\epsilon^3) \\
&\int_0^l \left[1 + \epsilon N(x,0,0) + \epsilon^2 \left[y_1 \frac{\partial N}{\partial y} + z_1 \frac{\partial N}{\partial z} + \tfrac{1}{2}y'^2_1 + \tfrac{1}{2}z'^2_1\right]\right] dx + O(\epsilon^3) = \\
&\int_0^l dx + \epsilon \int_0^l N(x,0,0)dx + \\
&\epsilon^2 \int_0^l \left[y_1 \frac{\partial N}{\partial y} + \tfrac{1}{2}y'^2_1 + z_1 \frac{\partial N}{\partial z} + \tfrac{1}{2}z'^2_1\right] dx + O(\epsilon^3)
\end{aligned} \qquad 6.30$$

So the optical path length correct to second order can be written as :

$$\begin{aligned}
\bar{s} &= s + \epsilon \int_0^l N(x,0,0)dx + \\
&\epsilon^2 \int_0^l \left[y_1 \frac{\partial N}{\partial y} + \tfrac{1}{2}y'^2_1 + z_1 \frac{\partial N}{\partial z} + \tfrac{1}{2}z'^2_1\right] dx
\end{aligned} \qquad 6.31$$

7. The effect of the earth's magnetic field for east-west propagation.

7.1. Introduction.

With the inclusion of the effect of the earth's magnetic field the ionosphere becomes an anisotropic medium. Some of the main effects introduced thus are:

- production of ordinary and extra-ordinary waves, i.e. the ionosphere is doubly-refracting.
- a difference between the direction of the ray-direction i.e. the energy flow and the phase normal.
- dependency of the refractive index on the angle of incidence of the ray with respect to the earth magnetic field.

This implies the use of more complex techniques to determine the ray path.

7.2. Propagation perpendicular to the earth's magnetic field.

For waves traveling in the East-West direction, the refractive index remains a scalar quantity. Thus, the theory derived in the previous chapter remains valid. The magnetic field however splits an incoming arbitrarily polarized electromagnetic wave in general into an ordinary and an extra-ordinary wave , that have to be dealt with separately.

* For the ordinary wave the refractive index is:

$$
\begin{aligned}
n^2 &= 1 - \frac{\omega_p^2}{\omega^2} \\
&= 1 - \frac{\omega_{p0}^2}{\omega^2}\frac{\omega_p^2}{\omega_{p0}^2} \qquad 7.1
\end{aligned}
$$

where ω_{p0} is the maximum plasma frequency in the considered profile.

$$
\begin{aligned}
n &= 1 + \left(\frac{\omega_{p0}^2}{2\omega^2}\right)\left(\frac{-\omega_p^2(r)}{\omega_{p0}^2}\right) + O(\epsilon^2) \\
&= 1 + \epsilon N \qquad 7.2
\end{aligned}
$$

where $\epsilon := \frac{\omega_{p0}^2}{2\omega^2}$. The ordinary wave is not influenced by the magnetic field.

* For the extra-ordinary wave, the refractive index is:

$$n^2 = 1 - \frac{\omega_p^2\left(\omega^2 - \omega_p^2\right)}{\omega^2\left(\omega^2 - \omega_p^2 - \omega_c^2\right)}$$

$$= 1 - \frac{\omega_p^2}{\omega^2}\left(1 - \frac{\omega_p^2}{\omega^2}\right)\left(1 + \frac{\omega_p^2}{\omega^2} + \frac{\omega_c^2}{\omega^2}\right) + O(\epsilon^3) =$$

$$= 1 - \frac{\omega_{p0}^2}{\omega^2}\frac{\omega_p^2(r)}{\omega_{p0}^2}\left(1 - \frac{\omega_{p0}^2}{\omega^2}\frac{\omega_p^2(r)}{\omega_{p0}^2}\right)\left(1 + \frac{\omega_{p0}^2}{\omega^2}\frac{\omega_p^2(r)}{\omega_{p0}^2} + \frac{\omega_{p0}^2}{\omega^2}\frac{\omega_c^2}{\omega_{p0}^2}\right) + O(\epsilon^3)$$

$$= 1 + \epsilon\left(\frac{-2\omega_p^2(r)}{\omega_{p0}^2}\right) + \epsilon^2\left(\frac{-4\omega_c^2\omega_p^2}{\omega_{p0}^4}\right) + O(\epsilon^3) \qquad 7.3$$

where ω_c is the gyro frequency and $\frac{\omega_c^2}{\omega_{p0}^2} = O(1)$ which is reasonable for the ionosphere.

$$n = 1 + \epsilon\left(-\frac{\omega_p^2(r)}{\omega_{p0}^2}\right) + \epsilon^2\left(\frac{-2\omega_c^2\omega_p^2}{\omega_{p0}^4} - \frac{\omega_p^4(r)}{2\omega_{p0}^4}\right)$$

$$= 1 + \epsilon N_1 + \epsilon^2 N_2 \qquad 7.4$$

Since the method of the previous chapter cannot handle the second order part, which partly represents the influence of the magnetic field, it's region of validity has to be enlarged. Therefore, let

$$n(x,y,z) = 1 + \epsilon N_1(x,y,z) + \epsilon^2 N_2(x,y,z) \qquad 7.5$$

In a way similar to the one used before we have, to second order in ϵ :

$$\begin{aligned}\nabla n &= \partial_x n(x,y,z)\vec{e}_x + \partial_y n(x,y,z)\vec{e}_y + \partial_z n(x,y,z)\vec{e}_z = \\ &= \left[\epsilon\partial_x N_1 + \epsilon^2\left(y_1\partial_{xy}^2 N_1 + z_1\partial_{xz}^2 N_1 + \partial_x N_2\right)\right]\vec{e}_x + \\ &\quad \left[\epsilon\partial_y N_1 + \epsilon^2\left(\partial_y N_2 + y_1\partial_{yy}^2 N_1\right)\right]\vec{e}_y + \\ &\quad \left[\epsilon\partial_z N_1 + \epsilon^2\left(\partial_z N_2 + z_1\partial_{zz}^2 N_1\right)\right]\vec{e}_z\end{aligned} \qquad 7.6$$

and

$$\begin{aligned}d_s(n d_s\vec{r}) = \\ &\left[\epsilon\partial_x N_1 + \epsilon^2\left(-y'_1 y''_1 - z'_1 z''_1 + \right.\right. \\ &\left.\left. y'_1\partial_y N_1 + z'_1\partial_z N_1 + y_1\partial_{xy}^2 N_1 + z_1\partial_{xz}^2 N_1 + \partial_x N_2\right)\right]\vec{e}_x + \\ &\left[\epsilon y''_1 + \epsilon^2\left(y''_2 + y''_1 N_1 + y'_1\partial_x N_1\right)\right]\vec{e}_y +\end{aligned}$$

$$\left[\epsilon z''_1 + \epsilon^2\left[z''_2 + z''_1 N_1 + z'_1\partial_x N_1\right]\right]\vec{e}_z \qquad 7.7$$

Comparing equal orders of ϵ for the three coordinates, we obtain:

* $\vec{e}_x$ First order: an identity.

Second order:

$$y'_1\left[y''_1 - \partial_y N_1\right] = 0$$

$$z'_1\left[z''_1 - \partial_z N_1\right] = 0 \qquad 7.8$$

* $\vec{e}_y$ First order:

$$\partial_y N_1 = y''_1 \qquad 7.9$$

Second order:

$$\partial_y N_2 + y_1\partial^2_{yy}N_1 = y''_2 + y''_1 N_1 + y'_1\partial_x N_1 \qquad 7.10$$

* $\vec{e}_z$ First order:

$$\partial_z N_1 = z''_1 \qquad 7.11$$

Second order:

$$\partial_z N_2 + z_1\partial^2_{zz}N_1 = z''_2 + z''_1 N_1 + z'_1\partial_x N_1 \qquad 7.12$$

The first order path is identical with the one derived before. The second order path follows from:

$$y''_2 = \; - y''_1 N_1 - y'_1\partial_x N_1 + \partial_y N_2 + y_1\partial^2_{yy}N_1 \qquad 7.13$$

$$z''_2 = \; - z''_1 N_1 - z'_1\partial_x N_1 + \partial_z N_2 + z_1\partial^2_{zz}N_1 \qquad 7.14$$

So

$$y'_2(x) = \int_0^x\left[-y''_1 N_1 - y'_1\partial_x N_1 + \partial_y N_2 + y_1\partial^2_{yy}N_1\right]dx' + y'_2(0) \qquad 7.15$$

$$y_2(x) = \int_0^x dx'\int_0^{x'}\left[-y''_1 N_1 - y'_1\partial_x N_1 + \partial_y N_2 + y_1\partial^2_{yy}N_1\right]dx'' + y'_2(0)\cdot x \qquad 7.16$$

The constant $y'_2(0)$ is determined by $y_2(l) = 0$:

$$y'_2(0) = \frac{-1}{l}\left\{\int_0^l dx'\int_0^{x'}\left[-y''_1 N_1 - y'_1\partial_x N_1 + \partial_y N_2 + y_1\partial^2_{yy}N_1\right]dx''\right\} \qquad 7.17$$

The first and second derivative of the first order path and the path itself are known from the previous chapter. Inserting these gives:

$$y_2(x) = \int_0^x dx' \int_0^{x'} \Bigg[-\partial_y N_1 \cdot N_1 - \int_0^{x''} \partial_y N_1 dx''' \cdot \partial_x N_1 + \frac{1}{l}\int_0^l dx \int_0^x \partial_y N_1 dx' \cdot \partial_x N_1 + \partial_y N_2 +$$
$$\left[\int_0^{x''} dx''' \int_0^{x'''} \partial_y N_1 dx'''' - \frac{x}{l}\int_0^l dx' \int_0^{x'} \partial_y N_1 dx'' \right] \partial_{yy}^2 N_1 \Bigg] dx'' +$$
$$\frac{-1}{l}\Bigg\{ \int_0^l dx' \int_0^{x'} \Bigg[-\partial_y N_1 \cdot N_1 - \int_0^{x''} \partial_y N_1 dx''' \cdot \partial_x N_1 + \frac{1}{l}\int_0^l dx \int_0^x \partial_y N_1 dx' \cdot \partial_x N_1 + \partial_y N_2 +$$
$$\left[\int_0^{x''} dx''' \int_0^{x'''} \partial_y N_1 dx'''' - \frac{x}{l}\int_0^l dx' \int_0^{x'} \partial_y N_1 dx'' \right] \partial_{yy}^2 N_1 \Bigg] dx'' \Bigg\} \cdot x \qquad 7.18$$

For z_2 an analogous approach can be used, that results in:

$$z_2(x) = \int_0^x dx' \int_0^{x'} \Bigg[-\partial_z N_1 \cdot N_1 - \int_0^{x''} \partial_z N_1 dx''' \cdot \partial_x N_1 + \frac{1}{l}\int_0^l dx \int_0^x \partial_z N_1 dx' \cdot \partial_x N_1 + \partial_z N_2 +$$
$$\left[\int_0^{x''} dx''' \int_0^{x'''} \partial_z N_1 dx'''' - \frac{x}{l}\int_0^l dx' \int_0^{x'} \partial_z N_1 dx'' \right] \partial_{zz}^2 N_1 \Bigg] dx'' +$$
$$\frac{-1}{l}\Bigg\{ \int_0^l dx' \int_0^{x'} \Bigg[-\partial_z N_1 \cdot N_1 - \int_0^{x''} \partial_z N_1 dx''' \cdot \partial_x N_1 + \frac{1}{l}\int_0^l dx \int_0^x \partial_z N_1 dx' \cdot \partial_x N_1 + \partial_z N_2 +$$
$$\left[\int_0^{x''} dx''' \int_0^{x'''} \partial_z N_1 dx'''' - \frac{x}{l}\int_0^l dx' \int_0^{x'} \partial_z N_1 dx'' \right] \partial_{zz}^2 N_1 \Bigg] dx'' \Bigg\} \cdot x \qquad 7.19$$

The path of the extra-ordinary wave can now be determined as being:

$$r = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z =$$
$$x\vec{e}_x + \epsilon\left[y_1\vec{e}_y + z_1\vec{e}_z \right] + \epsilon^2\left[y_2\vec{e}_y + z_2\vec{e}_z \right] + O(\epsilon^3) \qquad 7.20$$

in which the effect of the magnetic field is contained in the second order path.

7.3. Optical path length.

The optical path length, as defined by

$$\bar{s} = \int_0^s n ds \qquad 7.21$$

is independent of y_2 and z_2 if we consider only terms until second order in it's ϵ power series expansion. Thus, the optical (phase) path length in this approximation is not influenced by the effect of the earth's magnetic field.

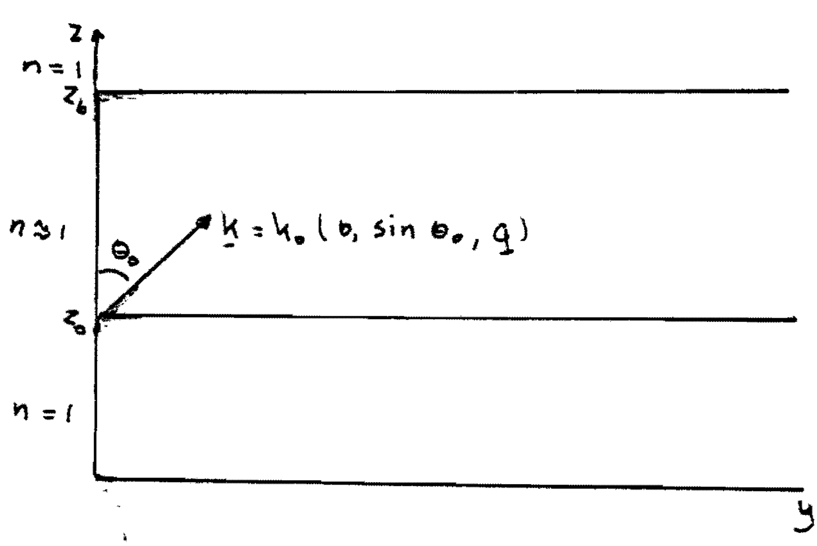
8. Oblique ray propagation, including the earth's magnetic field.

8.1. Introduction.

Except for the case of East-West propagation, dealt with in the previous chapter , it is impossible to study oblique ray propagation in an - as a result of the presence of the earth's magnetic field - anisotropic ionosphere with the ray equation used before. Most methods for finding the ray path deal with the group path. Some of the better known are the graphical construction by Poeverlein, the method of Titheridge for simple refractive indices and the method developed by Haselgrove, that is often used as the basis for computer implemented ray tracing. Since we are interested in the phase path, a method developed by Booker will be followed. Booker developed a quartic equation, the solution of which describes the phase propagation of the four magneto-ionic modes, two upgoing and two downgoing, in a stratified anisotropic medium.

8.2. Derivation of Booker's quartic.

We suppose a plane wave is obliquely incident upon the ionosphere. The ionosphere is assumed to be a horizontally stratified medium. We choose the coordinate system thus, that z is the vertical direction and that the yz plane is the plane of incidence. The wave enters the ionosphere , making an angle θ_0 with the z axis. A two dimensional description is sufficient for the treatment of phase propagation, since in contrast with group propagation, the phenomenon of *lateral deviation* , i.e. the deviation of the ray out of the plane of incidence, does not occur.



The vector wave function in a stratum can be written as

$$E = E_0 e^{+j\left[\omega t - \vec{k}\cdot\vec{r}\right]} \qquad 8.1$$

where the wavenumber $\vec{k}$ depends on the usual magneto-ionic parameters, familiar i.e. from the well-known Appleton-Hartree equation, including the angle it makes with the vertical. From fig 1. we see that

$$\vec{k}\cdot\vec{r} = k_0 n\left[y\sin(\theta) + z\cos(\theta)\right] \qquad 8.2$$

Using Snel's law for a plane stratified medium, i.e.

$$n(\theta)\sin(\theta) = \sin(\theta_0) \qquad 8.3$$

the vector wave function is

$$E = E_0 e^{+j\left[\omega t - k_0\left[y\sin(\theta_0) + qz\right]\right]} \qquad 8.4$$

where

$$q = n(\theta)\cos(\theta) \qquad 8.5$$

is a function, depending on the ionospheric parameters that characterizes the behavior of the wave in different strata, i.e. as a function of height.

In the ionosphere the wave function has to satisfy both Maxwell's equations and the equation of motion simultaneously. From Maxwell's equations

$$\nabla\times\vec{E} = -\partial_t\vec{B} \qquad 8.6$$
$$\nabla\times\vec{H} = \partial_t\vec{D} \qquad 8.7$$

and the constitutive relations

$$\vec{B} = \mu_0\vec{H} \qquad 8.8$$
$$\vec{D} = \epsilon_0\vec{E} + \vec{P} \qquad 8.9$$

we may obtain taking the curl of equation 1.6 :

$$\begin{aligned}\nabla\times\nabla\times\vec{E} &= -j\omega\mu_0\nabla\times\vec{H}\\ &= -j\omega\mu_0\left[j\omega\vec{P} + j\omega\epsilon_0\vec{E}\right]\\ &= \frac{k_0^2}{\epsilon_0}\vec{P} + k_0^2\vec{E}\end{aligned} \qquad 8.10$$

The left hand side can be determined from equation 1.4, resulting in a set of equations for the three components:

$$k_0^2\left|\sin^2\theta_0 + q^2\right|E_x = \frac{k_0^2}{\epsilon_0}P_x + k_0^2 E_x \qquad 8.11$$

$$k_0^2 q^2 E_y - k_0^2 \sin\theta_0 q E_z = \frac{k_0^2}{\epsilon_0}P_y + k_0^2 E_y \qquad 8.12$$

$$k_0^2 \sin^2\theta_0 E_z - k_0^2 \sin\theta_0 q E_y = \frac{k_0^2}{\epsilon_0}P_z + k_0^2 E_z \qquad 8.13$$

Solving this set yields:

$$\epsilon_0 E_x = -\frac{P_x}{\cos^2\theta_0 - q^2} \qquad 8.14$$

$$\epsilon_0 E_y = \frac{-P_y\cos^2\theta_0}{\cos^2\theta_0 - q^2} - \frac{P_z \sin\theta_0 q}{\cos^2\theta_0 - q^2} \qquad 8.15$$

$$\epsilon_0 E_z = \frac{P_y \sin\theta_0 q}{\cos^2\theta_0 - q^2} - \frac{P_z\left|1 - q^2\right|}{\cos^2\theta_0 - q^2} \qquad 8.16$$

A second set of relations can be determined from the equation of motion. Since the ionosphere consists of light electrons and comparatively heavy particles, only the equation of motion for the electrons will be considered.

$$m\,\partial_t \vec{v} = e\vec{E} - m\nu\vec{v} - \mu_0 e\left|\vec{H}\times\vec{v}\right|, \quad e < 0 \qquad 8.17$$

where ν is the collision frequency and $\vec{H}$ represents the influence of the Earth's magnetic field. Taking the time derivative of the polarization:

$$\vec{P} = Ne\vec{r} \qquad 8.18$$

where N is the electron density, we obtain an equation for the velocity:

$$\vec{v} = \frac{j\omega\vec{P}}{Ne} \qquad 8.19$$

that can be substituted in equation 1.17. The result is a vector equation for $\epsilon_0\vec{E}$:

$$\epsilon_0\vec{E} = -\omega^2\frac{m\epsilon_0}{Ne^2}\vec{P} + j\omega^2\frac{m\epsilon_0}{Ne^2}\frac{\nu}{\omega}\vec{P} + j\omega^2\frac{m\epsilon_0}{Ne^2}\frac{e\mu_0}{\omega m}\vec{H}\times\vec{P} \qquad 8.20$$

For convenience some new variables are introduced. Let

$$X := \frac{1}{\omega^2}\frac{Ne^2}{m\epsilon_0} = \frac{\omega_p^2}{\omega^2} \qquad 8.21$$

$$Z := \frac{\nu}{\omega} \qquad 8.22$$

and finally

$$Y_i := -\frac{H_i e\mu_0}{m\omega} = -\frac{\omega_c}{\omega} \qquad i = x, y, z. \qquad 8.23$$

Equation 1.20 written in its components is:

$$\epsilon_0 E_x = -\frac{1}{X}\left(1 - jZ\right)P_x + j\left(\frac{Y_y}{X}P_z - \frac{Y_z}{X}P_y\right) \qquad 8.24$$

$$\epsilon_0 E_y = -\frac{1}{X}\left(1 - jZ\right)P_y + j\left(\frac{Y_z}{X}P_x - \frac{Y_x}{X}P_z\right) \qquad 8.25$$

$$\epsilon_0 E_z = -\frac{1}{X}\left(1 - jZ\right)P_z + j\left(\frac{Y_x}{X}P_y - \frac{Y_y}{X}P_x\right) \qquad 8.26$$

Next the two sets of equations 1.14-1.16 and 1.24-1.26 are combined. Before the result is given, some further definitions are introduced:

$$\rho := \frac{1}{q^2 - \cos^2\theta_0} \qquad 8.27$$

$$u := 1 - jZ \qquad 8.28$$

$$\rho_0 := -\frac{u}{X} \qquad 8.29$$

and

$$\rho_i := -\frac{Y_i}{X} \qquad i = x, y, z. \qquad 8.30$$

Combination of the two sets gives:

$$\left(\rho - \rho_0\right)P_x - j\rho_z P_y + j\rho_y P_z = 0 \qquad 8.31$$

$$-j\rho_z P_x - \left(\cos^2\theta_0\rho - \rho_0\right)P_y + \left(\sin\theta_0 q\rho + j\rho_x\right)P_z = 0 \qquad 8.32$$

$$j\rho_y P_x + \left(\sin\theta_0 q\rho - j\rho_x\right)P_y + \left(q^2\rho - \rho + \rho_0\right)P_z = 0 \qquad 8.33$$

With $(q^2 - 1)\rho = 1 - \sin^2\theta_0\rho$ the determinant of this set can be written as

$$\begin{matrix} \rho-\rho_0 & -j\rho_z & j\rho_y \\ -j\rho_z & \rho_0-\cos^2\theta_0\rho & \sin\theta_0 q\rho+j\rho_x \\ j\rho_y & \sin\theta_0 q\rho-j\rho_x & \rho_0-\sin^2\theta_0\rho+1 \end{matrix}$$

and a solution can be found if this determinant equals zero. Expansion of this determinant leads to a quartic equation in q. The so called **Booker's quartic**

$$aq^4 + bq^3 + cq^2 + dq + e = 0 \qquad 8.34$$

where

$$a = u\left(u^2 - Y^2\right) - X\left(u^2 - Y_z^2\right) \qquad 8.35$$

$$b = 2\sin\theta_0 XY_y Y_z \qquad 8.36$$

$$c = 2u\left[\cos^2\theta_0 Y^2 - (u - X)(\cos^2\theta_0 u - X)\right] - X\left[Y_x^2 + \cos^2\theta_0 Y_y^2 + (1 + \cos^2\theta_0)Y_z^2\right] \quad 8.37$$

$$d = -2\cos^2\theta_0 \sin\theta_0 X Y_y Y_z \quad 8.38$$

and finally

$$e = (u - X)\left[(\cos^2\theta_0 u - X)^2 - Y_y^2\cos^4\theta_0\right] - \cos^2\theta_0(\cos^2\theta_0 u - X)(Y_x^2 + Y_z^2) \quad 8.39$$

In general this equation yields four distinct roots q, each depending on the magneto-ionic parameters. These represent the upgoing and downgoing ordinary and extra-ordinary waves.

8.3. Approximate solution to Bookers quartic.

In order to solve Bookers quartic equation, we expand both the coefficients a,b,c,d and e and the function q into series of a parameter ϵ. Since the effect of collisions is very small, it will be neglected hereafter, thus Z = 0 and therefore u = 1. For the parameters a-e we obtain:

$$\begin{aligned}
a &= 1 - \frac{\omega_c^2}{\omega^2} - \frac{\omega_p^2}{\omega^2}\left(1 - \frac{\omega_{cz}^2}{\omega^2}\right) \\
&= 1 - \epsilon\left(\frac{\omega_c^2}{\omega_{p0}^2}\right) - \epsilon\left(\frac{\omega_p^2}{\omega_{p0}^2}\right)\left(1 - \epsilon\frac{\omega_{cz}^2}{\omega_{p0}^2}\right) \\
&= 1 - \epsilon\left(\frac{\omega_c^2}{\omega_{p0}^2} + \frac{\omega_p^2}{\omega_{p0}^2}\right) + \epsilon^2\left(\frac{\omega_{cz}^2\omega_p^2}{\omega_{p0}^4}\right) \\
&= 1 + \epsilon a_1 + \epsilon^2 a_2 \qquad 8.40
\end{aligned}$$

where

$$\epsilon := \frac{\omega_{p0}^2}{\omega^2} \quad 8.41$$

and ω_{p0} is the maximum of the plasma frequency in the considered medium. It is not necessary to use different expansion parameters for factors containing the plasma and the gyro frequency, since their respective magnitudes are of the same order. Typical values are:

$\omega_c \approx 0.9 \cdot 10^7$

$\omega_p \approx 5 \cdot 10^7$ at daytime in the F layer.

In the same manner it can be shown that

$$b = \epsilon^2 2\sin\theta_0 \frac{\omega_p^2 \omega_{cy} \omega_{cz}}{\omega_{p0}^4}$$

$$= \epsilon^2 b_2 \qquad 8.42$$

$$c = -2\cos^2\theta_0 + \epsilon \left[2\cos^2\theta_0 \frac{\omega_c^2}{\omega_{p0}^2} + 2\cos^2\theta_0 \frac{\omega_p^2}{\omega_{p0}^2} + 2\frac{\omega_p^2}{\omega_{p0}^2} \right]$$

$$+ \epsilon^2 \left[-2\frac{\omega_p^4}{\omega_{p0}^4} - \frac{\omega_p^2 \omega_{cx}^2}{\omega_{p0}^4} - \frac{\omega_p^2 \omega_{cy}^2 \cos^2\theta_0}{\omega_{p0}^4} - \frac{\omega_p^2 \omega_{cz}^2 (1 + \cos^2\theta_0)}{\omega_{p0}^4} \right]$$

$$= c_0 + \epsilon c_1 + \epsilon^2 c_2 \qquad 8.43$$

$$d = -\epsilon^2 2\cos^2\theta_0 \sin\theta_0 \frac{\omega_p^2 \omega_{cy} \omega_{cz}}{\omega_{p0}^4}$$

$$= \epsilon^2 d_2 \qquad 8.44$$

$$e = \cos^4\theta_0 + \epsilon \left[-\frac{\omega_p^2 \cos^4\theta_0}{\omega_{p0}^2} - 2\cos^2\theta_0 \frac{\omega_p^2}{\omega_{p0}^2} - \frac{\omega_c^2 \cos^4\theta_0}{\omega_{p0}^2} \right]$$

$$+ \epsilon^2 \left[+2\cos^2\theta_0 \frac{\omega_p^4}{\omega_{p0}^4} + \frac{\omega_p^4}{\omega_{p0}^4} + \cos^2\theta_0 \frac{\omega_p^2 \left[\omega_{cx}^2 + \omega_{cz}^2 \right]}{\omega_{p0}^4} + \cos^4\theta_0 \frac{\omega_p^2 \omega_{cy}^2}{\omega_{p0}^4} \right] - \epsilon^3 \frac{\omega_p^6}{\omega_{p0}^6}$$

$$= e_0 + \epsilon e_1 + \epsilon^2 e_2 + \epsilon^3 e_3 \qquad 8.45$$

The function q also is expanded in an ϵ series:

$$q = q_0 + \epsilon q_1 + \epsilon^2 q_2 + \epsilon^3 q_3 + \cdots \qquad 8.46$$

Substituting these series into equation 1.34 we have:

$$\left[a_0 + \epsilon a_1 + \epsilon^2 a_2\right] \left[q_0 + \epsilon q_1 + \epsilon^2 q_2 + \cdots\right]^4 + \epsilon^2 b_2 \left[q_0 + \epsilon q_1 + \epsilon^2 q_2 + \cdots\right]^3 +$$
$$\left[c_0 + \epsilon c_1 + \epsilon^2 c_2\right] \left[q_0 + \epsilon q_1 + \epsilon^2 q_2 + \cdots\right]^2 + \epsilon^2 d_2 \left[q_0 + \epsilon q_1 + \epsilon^2 q_2 + \cdots\right] +$$
$$\left[e_0 + \epsilon e_1 + \epsilon^2 e_2 + \epsilon^3 e_3\right] = 0 \qquad 8.47$$

Collecting zeroth order terms in ϵ yields:

$$q_0^4 - 2\cos^2\theta_0 q_0^2 + \cos^4\theta_0 = 0 \qquad 8.48$$

This is a quadratic in q_0^2 and can be solved easily. The two coinciding solutions are:

$$q_0^2 = \cos^2\theta_0 \qquad 8.49$$

or

$$q_0 = \pm\cos\theta_0 \qquad 8.50$$

Collecting first order terms, we obtain:

$$4a_0q_1q_0^3 + a_1q_0^4 + 2c_0q_1q_0 + c_1q_0^2 + e_1 = 0 \qquad 8.51$$

or

$$q_1 = \frac{a_1q_0^4 + c_1q_0^2 + e_1}{4a_0q_0^3 + 2c_0q_0} \qquad 8.52$$

However, if a_0, c_0, a_1, c_1, e_1 and q_0 are substituted into this equation , a zero by zero division results. This does not present any problem as will be shown now. We consider the function

$$y(q,\epsilon) = a(\epsilon)q^4 + b(\epsilon)q^3 + c(\epsilon)q^2 + d(\epsilon)q + e(\epsilon) \qquad 8.53$$

which reduces to Booker's quartic for y = 0. The parameters a–e are:

$$a(\epsilon) = a_0 + \epsilon a_1 + \epsilon^2 a_2 + \cdots =$$
$$a(0) + \epsilon \partial_\epsilon a\,|_{\epsilon=0} + \tfrac{1}{2}\epsilon^2 \partial^2_{\epsilon\epsilon} a\,|_{\epsilon=0} + \cdots \quad etc. \qquad 8.54$$

Assuming that

$$q = q_0 + \epsilon q_1 + \epsilon^2 q_2 + \cdots \qquad 8.55$$

is a solution to equation 1.53 with $y = 0$, then, after substitution of the series expansion of parameters a–e into equation 1.53 with y = 0 we obtain:

$$a(0)q_0^4 + b(0)q_0^3 + c(0)q_0^2 + d(0)q_0 + e(0) = 0 \qquad 8.56$$

Solving this equation gives two real zeros: $q_0 = \pm\cos\theta_0$. Collecting first order terms in ϵ we find:

$$a_1q_0^4 + b_1q_0^3 + c_1q_0^2 + d_1q_0 + e_1 + q_1\left[4a_0q_0^3 + 3b_0q_0^2 + 2c_0q_0 + d_0\right] = 0 \qquad 8.57$$

or, solving for q_1:

$$q_1 = \frac{a_1q_0^4 + b_1q_0^3 + c_1q_0^2 + d_1q_0 + e_1}{4a_0q_0^3 + 3b_0q_0^2 + 2c_0q_0 + d_0} \qquad 8.58$$

which is in agreement with 1.52, and as seen before, results in a zero by zero division. The denominator being equal to zero implies that

$$\partial_q y\,|_{q_0}(\epsilon = 0) = 0 \qquad 8.59$$

whereas according to 1.56 also holds

$$y(q_0, 0) = 0 \qquad 8.60$$

Since the denominator of the right hand side of equation 1.58 is the derivative of the left hand side of equation 1.56, i.e. q_0 is a double real root of equation 1.56, equation 1.57 can be written alternatively as:

$$\partial_\epsilon y \mid_{\epsilon=0,q=q_0} + \partial_q y \mid_{\epsilon=0,q=q_0} dq = 0 \qquad 8.61$$

where $dq = \epsilon q_1$. Therefore the surface y (q, ϵ) is tangent to the y = 0 surface in q = q_0, $\epsilon = 0$.

Introduce

$$x = q - q_0 = \epsilon q_1 \qquad 8.61$$

In small surroundings $0 < |x| \leqslant \epsilon$ of (q_0 , 0) we can write y (q , ϵ) as:

$$\begin{aligned} y(q,\epsilon) = {} & y(q_0,0) + (q - q_0)\partial_q y \mid_{q_0,0} + \epsilon\partial_\epsilon y \mid_{q_0,0} + \tfrac{1}{2}(q-q_0)^2\partial^2_{qq} y \mid_{q_0,0} \\ & + (q-q_0)\epsilon\partial^2_{q\epsilon} y \mid_{q_0,0} + \tfrac{1}{2}\epsilon^2\partial^2_{\epsilon\epsilon} y \mid_{q_0,0} + O(\epsilon^3) = \\ & 0 + x\cdot 0 + \epsilon\cdot 0 + \tfrac{1}{2}x^2\partial^2_{qq} y \mid_{q_0,0} + \epsilon x\, \partial^2_{q\epsilon} y \mid_{q_0,0} + \tfrac{1}{2}\epsilon^2\partial^2_{\epsilon\epsilon} y \mid_{q_0,0} \end{aligned} \qquad 8.62$$

Setting y = 0 in equation 1.62 , we obtain a quadratic equation in x, that can be solved easily. We find:

$$x = \frac{-\epsilon\partial^2_{q\epsilon} y \pm \sqrt{\epsilon^2\left(\partial^2_{q\epsilon} y\right)^2 - 4\cdot\frac{1}{4}\epsilon^2\partial^2_{qq} y\, \partial^2_{\epsilon\epsilon} y}}{\partial^2_{qq} y} \qquad 8.63$$

Inserting the derivatives of y and recalling that $x = \epsilon q_1$ we find:

$$q_1 = \frac{-\left(4a_1 q_0^3 + 2c_1 q_0\right)}{12a_0 q_0^2 + 2c_0} \pm \frac{\sqrt{\left(4a_1 q_0^3 + 2c_1 q_0\right)^2 - 2\left(12a_0 q_0^2 + 2c_0\right)\left(a_2 q_0^4 + b_2 q_0^3 + c_2 q_0^2 + d_2 q_0 + e_2\right)}}{12a_0 q_0^2 + 2c_0} \qquad 8.64$$

where equations 1.40-1.45 have been used.

Higher order q_i can be found iteratively. The wave vector thus is:

$$\vec{k} = \left(0, k_0 \sin\theta_0, k_0 \sum \epsilon^i q_i\right) \qquad 8.65$$

8.4. Ray path.

Since the wavevector is tangent to the ray trajectory, it is possible now to find the ray path:

So

$$z = \int_{y_0}^{y} \frac{k_z}{k_y} dy$$

$$= \frac{1}{\sin\theta_0} \int_{y_0}^{y} \sum \epsilon^i q_i \, dy \qquad 8.67$$

In lowest order of ϵ we have:

$$z = cotan\, \theta_0 y \qquad 8.68$$

This is the unaffected ray, as we expected.

9. Symbols.

Symbols are defined as they occur in the text, but those frequently used are summarized here for reference.

$\vec{B}$	magnetic induction vector
$\vec{B}_0$	earth magnetic field
C_s	velocity of sound
c_p	heat capacity at constant pressure
c_v	heat capacity at constant volume
d_t	time derivative
∂_t	partial derivative
D_t	convective derivative
$\vec{D}$	electric displacement vector
$\vec{E}$	electric field strength
$\vec{e}$	unit vector
f_α	particle distribution function
$\vec{g}$	gravitational constant
H	vertical scale height
$\vec{H}$	magnetic field vector
i,j	imaginary unit ,$i^2 = -1$
$\vec{J}$	current density
k	wavenumber, Boltzmann's constant
$\vec{k}$	propagation vector
m_α	particle mass
n_α	particle density
n	refractive index
N_A	Avogadro's number

$\overline{\overline{P}}$	pressure tensor
p	scalar pressure
q_α	particle charge
q	Booker's parameter
Q	total heat
R	gas constant, earth radius
$\vec{r}$	position vector
s	optical path length
S	total entropy
s	ray path parameter
T	absolute temperature
U	internal energy
V	volume
$\vec{v}$	velocity
$\vec{V}$	normalized velocity
v_A	Alfvén velocity
v_f	phase velocity

γ	ratio of specific heats
δ	expansion parameter
ϵ	expansion parameter
$\bar{\bar{\epsilon}}$	dielectric tensor
ϵ_0	permittivity of free space
ϵ_r	relative permittivity
η	viscosity
λ	wavelength
$\bar{\bar{\mu}}$	permeability tensor
μ_0	permeability of free space
μ_r	relative permeability
ν	collision frequency
ρ	density
ρ_m	mass density
ρ_q	charge density
ω	angular frequency
ω_p	angular plasma frequency
ω_c	angular gyro frequency
ω_b	Brunt-Väisälä frequency
ω_{earth}	angular earth frequency

indices	
α	particle kind
e	electron
i	ion
n	neutral
0	equilibrium
1	perturbation

coordinate systems	
carthesian	x,y,z
spherical	r, θ, ϕ

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1. Contents.