MASTER

The relation of dominance

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The Relation of Dominance

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Declaration of Authorship

I, Ankur Pandey, declare that this thesis titled, 'The Relation of Dominance' and the work presented in it are my own. I confirm that this work was done wholly while in candidature for a Masters degree at Johannes Kepler University Linz. I have acknowledged all main sources of help.

Signed: ____

Date: ____

Abstract

The motivation to study dominance came from the framework of probabilistic metric spaces, where it turned out to be crucial for construction of Cartesian products of such spaces. In this early setting the operations considered with respect to dominance were mainly the triangle functions and the triangular norms. Later on researchers from the field of fuzzy logic proved that the concept of dominance plays an important role in a much wider class of problems related to construction of product-like structures. In particular it has been clarified how dominance allows to construct T-transitive fuzzy relations as Cartesian products of T-transitive factors.

In this thesis the notion of dominance has been reviewed from the most general viewpoint. We start with considering how the dominance arises and the instances where it occurs. A weaker notion of *weak dominance* has also been discussed. More attention has been paid to the class of t-norms, specifically continuous t-norms where several results have been presented to characterize the dominance property. We also present important results regarding preservation of T-transitivity with respect to fuzzy relations. Finally, we give a summary of some important parametric families of continuous T-norms where dominance has been shown to be a partial order.

The intention of the thesis is to be a self contained exposition on the relation of dominance and its relevance, and present the results in a constructive way.

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Contents

Chapter 1

Products of (Probabilistic) Metric Spaces

We start off with defining (probabilistic) metric spaces, and will examine the conditions under which their products also become metric spaces.

1.1 Metric Spaces

Definition 1.1.1 (Metric Space)**.** A metric space is an ordered pair (S, d) where S is a set and d is a metric on S, that is, a function

$$d\colon S \times S \to \mathbb{R}$$

such that for all x, y, $z \in S$

- $d(x, y) \geq 0$,
- $d(x, y) = 0 \Leftrightarrow x = y$,
- $d(x, y) = d(y, x)$,
- $d(x, y) + d(y, z) \geq d(x, z)$.

Definition 1.1.2 (Product Metric Space)**.** Suppose (S_1, d_1) and (S_2, d_2) are two metric spaces. If $p_1, q_1 \in S_1$ and $p_2, q_2 \in S_2$, and $d\colon\ (S_1 \times S_2) \times (S_1 \times S_2) \to \mathbb{R}$ given by,

$$d((p_1, p_2), (q_1, q_2)) = K(d_1(p_1, q_1), d_2(p_2, q_2))$$

is a metric on $S_1 \times S_2$ then $(S_1 \times S_2, d)$ is called a product metric space. where $K : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a suitable binary operation.

Lemma 1.1.1. *If $K(0, x) = K(x, 0) = x$ for all $x \in \mathbb{R}^+$, and $K(x, y) > 0$ when $x > 0$ or $y > 0$, then d satisfies, for $p = (p_1, p_2), q = (q_1, q_2)$*

- *$d(p, q) \geq 0$,*
- *$d(p, q) = 0 \Leftrightarrow p = q$,*
- *$d(p, q) = d(q, p)$.*

d is then called a semi metric space.

Proof. $d(p, q) = d((p_1, p_2), (q_1, q_2)) = K(d_1(p_1, q_1), d_2(p_2, q_2)) > 0$ when $d_1(p_1, q_1) > 0$ or $d_2(p_2, q_2) > 0$, and $d(p, q) = 0$ when $d_1(p_1, q_1) = d_2(p_2, q_2) = 0$ i.e. $p_1 = q_1$ and $p_2 = q_2$.
Clearly $K(x, y) = 0$ iff $x = y = 0$. Hence if $d((p_1, p_2), (q_1, q_2)) = K(d_1(p_1, q_1), d_2(p_2, q_2)) = 0$, then $d_1 = d_2 = 0$ or $p_1 = q_1$ and $p_2 = q_2$. Hence $p = (p_1, p_2) = (q_1, q_2) = q$.
Conversely if $p = (p_1, p_2) = (q_1, q_2) = q$, then $p_1 = q_1$ and $p_2 = q_2$. Hence $d_1 = d_2 = 0$, or $K = d = 0$.
$d(q, p) = d((q_1, q_2), (p_1, p_2)) = K(d_1(q_1, p_1), d_2(q_2, p_2)) = K(d_1(p_1, q_1), d_2(p_2, q_2))$, since d_1, d_2 are commutative. Hence $d(q, p) = d(p, q)$.

∎

Lemma 1.1.2. *If K is commutative, then d is isometric to the corresponding metric induced on $S_2 \times S_1$ under the natural map $M(p_1, p_2) = (p_2, p_1)$.*

Proof. If $p_1, q_1 \in S_1$ and $p_2, q_2 \in S_2$, then we can *define* a product metric space $(S_2 \times S_1, d')$ given by,

$$d'((p_2, p_1), (q_2, q_1)) = K(d_2(p_2, q_2), d_1(p_1, q_1))$$

Let K be commutative, i.e. $K(x, y) = K(y, x)$.
Then, $K(d_2(p_2, q_2), d_1(p_1, q_1)) = K(d_1(p_1, q_1), d_2(p_2, q_2))$.
Or $d'((p_2, p_1), (q_2, q_1)) = d((p_2, p_1), (q_2, q_1)) = d((p_1, p_2), (q_1, q_2))$. ∎

Now we examine the condition required on K such that the semi-metric space $d(p,q)$ becomes a metric space.

Theorem 1.1.3. *If K is subadditive, i.e. $K(x_1,y_1)+K(x_2,y_2) \geq K(x_1+x_2,y_1+y_2)$, and non-decreasing, i.e. $K(x_1,y_1) \geq K(x_2,y_2)$ for all$x_1,x_2,y_1,y_2 \in \mathbb{R}^+$ with $x_1 \geq x_2$ $y_1 \geq y_2$, then for $p,q,r \in S_1 \times S_2$, we have,*

$$d(p,q) + d(q,r) \geq d(p,r)$$

Proof.

$$\begin{aligned}
d(p,q) + d(q,r) &= d((p_1,p_2),(q_1,q_2)) + d((q_1,q_2),(r_1,r_2)) \\
&= K(d_1(p_1,q_1), d_2(p_2,q_2)) + K(d_1(q_1,r_1), d_2(q_2,r_2)) \\
&\geq K(d_1(p_1,q_1) + d_1(q_1,r_1), d_2(p_2,q_2) + d_2(q_2,r_2)) \\
&\geq K(d_1(p_1,r_1), d_2(p_2,r_2)) \\
&= d(p,r)
\end{aligned}$$

∎

Note 1.1.1. If we require K to be *associative*, we can define the product of three or more metric spaces in a canonical way.

1.2 Probabilistic Metric Spaces

We first need to define distance distribution functions and triangle functions in order to formulate the co-domain and products resp. in the case of probabilistic metric spaces.

Definition 1.2.1 (Distance Distribution Function)**.** A function $\tilde{F}\colon [0,\infty] \to [0,1]$, which is non-decreasing, left-continuous on $\mathbb{R}$, with $\tilde{F}(\infty) = 1$ and $\tilde{F}(0) = 0$ is called a distance distribution function. The set of all such functions is denoted by Δ^+.
The elements of Δ^+ are partially ordered following the usual pointwise order,

$$\tilde{F} \geq \tilde{G} \Leftrightarrow \tilde{F}(x) \geq \tilde{G}(x) \;\; \forall x \in \mathbb{R}$$

Also, $(\Delta^+, \leq, \varepsilon_\infty, \varepsilon_0)$ is a bounded lattice with smallest and greatest elements given by,

$$\varepsilon_\infty(x) = \begin{cases} 1 & \text{if } x = \infty, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \varepsilon_0(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.2.2 (Triangle Function)**.** A triangle function is a binary commutative and associative operator on Δ^+ (i.e. from $\Delta^+ \times \Delta^+ \to \Delta^+$), which is non-decreasing in each argument and has neutral element ε_0.

Corollary 1.2.1. *Let $\tilde{F}, \tilde{G} \in \Delta^+$, and let τ be a triangle function, then,*

$$\tau(\tilde{F}, \tilde{G}) \leq \tau(\tilde{F}, \varepsilon_0) = \tilde{F}$$
$$\textit{also,} \quad \tau(\tilde{F}, \tilde{G}) \leq \tau(\tilde{F}, \varepsilon_0) = \tilde{G}$$

Hence, $\tau(\tilde{F}, \tilde{G}) = \varepsilon_0$ if and only if $\tilde{F} = \tilde{G} = \varepsilon_0$

Definition 1.2.3 (Probabilistic Semi Metric Space)**.** A probabilistic semi-metric Space is an ordered pair (S, F), where S is a set, F is a mapping from $S \times S$ to Δ^+, such that for all $p, q \in S$,

- $F(p, q) = \varepsilon_0$ if and only if $p = q$,
- $F(p, q) = F(q, p)$.

Note 1.2.1. $F(p, q)$ can be denoted by $\tilde{F}_{p,q}$, i.e. a distance distribution function $\tilde{F}$ for a given pair of points p and q; and $\tilde{F}_{p,q}(x)$, its value at x is interpreted as the probabilty such that the distance between p and q is less than x.

Definition 1.2.4 (Probabilistic Metric Space)**.** A probabilistic metric space (S, F, τ) is a probabilistic semi-metric space (S, F) equipped with a triangle function τ such that for all $p, q, r \in S$,

$$F(p, q) \geq \tau(F(p, r), F(r, q))$$

Definition 1.2.5 (σ-Product)**.** Let (S_1, F_1) and (S_2, F_2) be probabilistic metric spaces and σ be a triangle function. Then $(S_1 \times S_2, F_1 \times_\sigma F_2)$ is called the σ Product Space, when, if $p_1, q_1 \in S_1$, and $p_2, q_2 \in S_2$, then,

$$F_1 \times_\sigma F_2(p, q) = \sigma(F_1(p_1, q_1), F_2(p_2, q_2))$$

where, $p = (p_1, p_2), q = (q_1, q_2)$.

Such product spaces are examined in more details by, for instance, Tardiff [1].

Lemma 1.2.2. *Let* (S_1, F_1) *and* (S_2, F_2) *be probabilistic metric spaces and* σ *be a triangle function. Then the* σ*-product space, i.e.* $(S_1 \times S_2, F_1 \times_\sigma F_2)$ *is a probabilistic semi-metric space.*

Proof. If $F_1 \times_\sigma F_2(p, q) = \varepsilon_0$, then $\sigma(F_1(p_1, q_1), F_2(p_2, q_2)) = \varepsilon_0$.
Hence, $F_1(p_1, q_1) = F_2(p_2, q_2) = \varepsilon_0$, i.e.

$$\tilde{F}_{1,(p_1,q_1)}(x) = \tilde{F}_{2,(p_2,q_2)}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Hence, $p_1 = q_1$ and $p_2 = q_2$, i.e.$p = (p_1, p_2) = (q_1, q_2) = q$
Now,

$$\begin{aligned} F_1 \times_\sigma F_2(q, p) &= \sigma(F_1(q_1, p_1), F_2(q_2, p_2)) \\ &= \sigma(F_1(p_1, q_1), F_2(p_2, q_2)) \quad \text{from Definition 1.2.3} \\ &= F_1 \times_\sigma F_2(p, q) \end{aligned}$$

∎

Lemma 1.2.3. $(S_1 \times S_2, F_1 \times_\sigma F_2)$ *is isometric to* $(S_2 \times S_1, F_2 \times_\sigma F_1)$ *under the natural map* $M(p_1, p_2) = (p_2, p_1)$.

Proof.

$$\begin{aligned} F_2 \times_\sigma F_1((p_2, p_1), (q_2, q_1)) &= \sigma(F_2(p_2, q_2), F_1(p_1, q_1)) \\ &= \sigma(F_1(p_1, q_1), F_2(p_2, q_2)) \\ &= F_1 \times_\sigma F_2((p_1, p_2), (q_1, q_2)) \\ &= F_1 \times_\sigma F_2(p, q) \end{aligned}$$

∎

We now examine the condition required such that σ-product becomes a probabilistic metric space.

Theorem 1.2.4. *If* (S_1, F_1, τ) *and* (S_2, F_2, τ) *are two probabilistic metric spaces under the same triangle function* τ*, then their* σ *Product* $(S_1 \times S_2, F_1 \times_\sigma F_2)$ *is a probabilistic* *metric space when,*

$$\sigma(\tau(F_1, G_1), \tau(F_2, G_2)) \geq \tau(\sigma(F_1, F_2), \sigma(G_1, G_2)) \quad \forall F_1, G_1, F_2, G_2 \in \Delta^+$$

Proof. From Lemma 1.2.1, $(S_1 \times S_2, F_1 \times_\sigma F_2)$ is a Probabilistic Semi Metric Space. Let $p = (p_1, p_2), q = (q_1, q_2), r = (r_1, r_2) \in S_1 \times S_2$, then,

$$\begin{aligned} F_1 \times_\sigma F_2(p, q) &= \sigma(F_1(p_1, q_1), F_2(p_2, q_2)) \\ &\geq \sigma(\tau(F_1(p_1, r_1) + F_1(r_1, q_1)), \tau(F_2(p_2, r_2) + F_2(r_2, q_2)) \\ &\geq \tau(\sigma(F_1(p_1, r_1) + F_1(p_2, r_2)), \sigma(F_2(r_1, q_1) + F_2(r_2, q_2)) \\ &= \tau(F_1 \times_\sigma F_2(p, r) + F_1 \times_\sigma F_2(r, q)) \end{aligned}$$

∎

Note 1.2.2. By requiring σ to be associative, we can extend the definition of σ product to more than two probabilistic metric spaces in a canonical way.

Chapter 2

Notion of Dominance

2.1 Dominance in Binary Operators

We are now ready to introduced the generalized definition of dominance. The notion was first introduced by Schweizer and Sklar [2].

Definition 2.1.1. Consider a partially ordered set $(P, \geq)$ and two binary operations f, g on P. Then f *dominates* g written as $f \gg g$ if for all $x, y, u, v \in P$,

$$f(g(x,y), g(u,v)) \geq g(f(x,u), f(y,v)) \tag{2.1}$$

It can be readily seen that the conditions in Theorem 1.1.3 and Theorem 1.2.4 can be written in terms of dominance operator, i.e.,

- When K is a suitable binary operation on $\mathbb{R}^+$,

$$K(x_1,y_1) + K(x_2,y_2) \geq K(x_1+x_2, y_1+y_2) \quad \text{from Theorem 1.1.3}$$

$$\text{i.e.} \quad \mathcal{A}(K(x_1,y_1), K(x_2,y_2)) \geq K(\mathcal{A}(x_1,x_2), \mathcal{A}(y_1,y_2)) \quad \text{where } \mathcal{A} \text{ is the addition operator}$$

$$\text{i.e.} \quad \mathcal{A} \gg K.$$

- When σ and τ are triangle functions on Δ^+

$$\sigma(\tau(F_1,G_1), \tau(F_2,G_2)) \geq \tau(\sigma(F_1,F_2) \quad \text{from Theorem 1.2.3}$$

$$\text{i.e.} \quad \sigma \gg \tau.$$

2.1.1 Examples

Many classical inequalities can be regarded as the special cases of generalized notion of dominance.

1. If $P = \mathbb{R}$ and f is addition operator, then Equation 2.1 becomes,

$$g(x, y) + g(u, v) \geq g(x + u, y + v)$$

   which says that g is *subadditive*.

2. If $P = \mathbb{R}$ and g is addition operator, then,

$$f(x + y, u + v) \geq f(x + u) + f(y + v)$$

   which says that f is *superadditive*.

3. If $f(x, y) = \frac{1}{2}(x + y)$, then,

$$\frac{1}{2}\left(g(x, y) + g(u, v)\right) \geq g\left(\frac{1}{2}(x + u), \frac{1}{2}(y + v)\right)$$

   g is then *midpoint convex*.

4. If $g(x, y) = \frac{1}{2}(x + y)$, then,

$$f\left(\frac{1}{2}(x + y), \frac{1}{2}(u + v)\right) \geq \frac{1}{2}\left(f(x, u) + g(y, v)\right)$$

   f is then *midpoint concave*.

5. If $P = \mathbb{R}$, f is addition operator and $g(x, y) = (x^p + y^p)^{\frac{1}{p}}$ where $p \geq 1$, then,

$$(x^p + y^p)^{\frac{1}{p}} + (u^p + v^p)^{\frac{1}{p}} \geq ((x + u)^p + (y + v)^p)^{\frac{1}{p}}$$

   is called *Minkowski inequality*.

6. In case when equality holds in Equation 2.1, i.e. when,

$$f(g(x, y), g(u, v)) = g(f(x, u), f(y, v))$$

   the equation is known as *generalised bisymmetry equation*.

7. If, moreover, $f = g$, i.e.,

$$f(f(x,y), f(u,v)) = f(f(x,u), f(y,v))$$

the equation is known as *classical bisymmetry equation*.

Theorem 2.1.1. *Consider a partial ordered set $(P, \geq)$ and two binary operations f, g on P. Let f and g have a common identity element e. Then,*

1. *$f \gg g$ implies $f \geq g$.*
2. *The dominance relation is antisymmetric (i.e. if $f \gg g$ and $g \gg f$ then $f = g$).*
3. *The dominance relation is reflexive (i.e. $f \gg f$) if and only if f is associative and commutative.*

Proof. 1. In Equation 2.1, set $y = u = e$, Then, $f(g(x,e), g(e,v)) \geq g(f(x,e), f(e,v))$, i.e. $f(x,v) \geq g(x,v)$.

2. From part 1, $f \gg g$ implies $f \geq g$, and $g \gg f$ implies $g \geq f$. Hence if $f \gg g$ and $g \gg f$ then $f = g$.

3. Let $f \gg f$, then for $x, y, u, v \in S$, we have,

$$\begin{aligned} & f(f(x,y), f(u,v)) \geq f(f(x,u), f(y,v)) \\ \text{and,} \quad & f(f(x,u), f(y,v)) \geq f(f(x,y), f(u,v)) \\ \text{then,} \quad & f(f(x,y), f(u,v)) \geq f(f(x,u), f(y,v)) \geq f(f(x,y), f(u,v)) \\ \text{hence,} \quad & f(f(x,y), f(u,v)) = f(f(x,u), f(y,v)). \end{aligned}$$

Now let $u = e$, then,

$$f(f(x,y), v) = f(x, f(y,v)) \quad \textit{associativity}$$

From this,

$$\begin{aligned} & f(x,y) = f(f(e,x), f(y,e)) \geq f(f(e,y), f(x,e)) = f(y,x) \\ & f(y,x) = f(f(e,y), f(x,e)) \geq f(f(e,x), f(y,e)) = f(x,y) \quad \textit{commutativity} \end{aligned}$$

Now let f be associative and commutative. Then,

$$\begin{aligned} f(f(x,y),f(u,v)) &= f(f(f(x,y),u),v) = f(f(f(y,u)x),v) \\ &= f(f(f(u,y),x),v) = f(f(f(x,u),y),v) \\ &= f(f(x,u),f(y,v)) \end{aligned}$$

It follows that,
$f(f(x,y),f(u,v)) \geq f(f(x,u),f(y,v)) \geq f(f(x,y),f(u,v))$
Or, $f \gg f$ *reflexivity*.

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The dominance relation is reflexive and antisymmetric, but not transitive on set of binary operators, hence it is not a partial order.

Counterexample regarding transitivity

A counter example showing that dominance is not transitive is given due to H. Sherwood [3].

Let $P = \{0,1,2\}$ be lineraly ordered by $0 < 1 < 2$ and let f, g, h be binary operations on P defined by multiplication tables:

f	0	1	2
0	0	1	2
1	1	0	2
2	2	2	2

g	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

h	0	1	2
0	0	1	2
1	1	2	2
2	2	2	2

Then, f, g, h are commutative and associative operators on P. 0 is the common identity. Also $h \gg g$ and $g \gg f$. Assume that $h \gg f$.

Now $h \gg f$ implies $h(f(x,u),f(y,v)) \geq f(h(x,y),h(u,v))$ for all $x,y,u,v \in P$. Now let $x = 1$, $y = 1$, $u = 1$ and $v = 0$. Then $f(h(x,y),h(u,v)) = 2$ while $h(f(x,u),f(y,v)) = 1$ i.e.$1 \geq 2$. Hence the assumption $h \gg f$ is *contradictory*.

2.2 Weak Dominance in Binary Operators

A weaker case of dominance is *weak dominance*. The notion had been developed by Alsina, Frank, Schweizer [3].

Definition 2.2.1. Consider a partial ordered set $(P, \geq)$ and two binary operations f, g on P. Then f *weakly dominates* g written as $f \gg_w g$ if for all$x, y, u \in P$,

$$f(g(x,y),u)) \geq g(f(x,u),y) \tag{2.2}$$

Clearly, Equation 2.2 is a special case of Equation 2.1 by substituting identity element e in place of v (provided f and g have the same identity element).

Theorem 2.2.1. *Consider a partial ordered set $(P, \geq)$ and two binary operations f, g on P. Let f and g have a common identity element e. Then,*

1. *$f \gg_w g$ implies $f \geq g$ if at least one of f and g is commutative.*
2. *The weak-dominance relation is antisymmetric (i.e. if $f \gg_w g$ and $g \gg_w f$ then $f = g$).*
3. *The weak-dominance relation is reflexive (i.e. $f \gg_w f$) if and only if f is associative and commutative.*

Proof. 1. In Equation 2.1, set $x = e$, Then, $f(g(e,y),u) \geq g(f(e,u),y)$, i.e. $f(y,u) \geq g(u,y)$.
Now if f resp. g is commutative, we have $f(u,y) \geq g(u,y)$ resp. $f(y,u) \geq g(y,u)$.

2. From part 1, $f \gg_w g$ implies $f \geq g$, and $g >>_w f$ implies $g \geq f$. Hence if $f \gg_w g$ and $g \gg_w f$ then $f = g$.

3. Let $f \gg_w f$, then for $x, y, u \in S$, we have,

$$\begin{aligned} & f(f(x,y),u) \geq f(f(x,u),y) \\ \text{and,}\quad & f(f(x,u),y) \geq f(f(x,y),u) \\ \text{then,}\quad & f(f(x,y),u) \geq f(f(x,u),y) \geq f(f(x,y),u) \\ \text{hence,}\quad & f(f(x,y),u) = f(f(x,u),y) \end{aligned}$$

Now let $x = e$, then,

$$f(y, u) = f(u, y) \quad \textit{commutativity}$$

Rearranging by using commutativity,

$$f(u, f(x, y)) = f(f(u, x), y) \quad \textit{associativity}$$

Now let f be associative and commutative. Then,

$$\begin{aligned} f(f(x, y), u) &= f(f(y, x), u) = f(y, f(x, u)) \\ &= f(f(x, u), y) \end{aligned}$$

It follows that,
$f(f(x, y), u) \geq f(f(x, u), y) \geq f(f(x, y), u)$
Hence, $f \gg_w f$ *reflexivity*

∎

The weak dominance relation is not transitive on set of binary operators, hence it is not a partial order.

Counterexample regarding transitivity

A counter example showing that weak dominance is not transitive is given due to H. Sherwood [3].

Let $P = \{0, 1, 2\}$ be lineraly ordered by $0 < 1 < 2$ and let f, g, h be binary operations on P defined by multiplication tables:

f	0	1	2
0	0	1	2
1	1	0	2
2	2	2	2

g	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

h	0	1	2
0	0	1	2
1	1	2	2
2	2	2	2

Then, f, g, h are commutative and associative operators on P. 0 is the common identity. Also $h \gg_w g$ and $g \gg_w f$. Assume that $h \gg_w f$.

Now $h >>_w f$ implies $h(f(x,u),v) \geq f(h(x,y),u)$ for all $x,y,u \in P$. Now let $x = y = u = 1$. Then $f(h(x,y),u) = 2$ while $h(f(x,u),y) = 1$ i.e. $1 \geq 2$. Hence the assumption $h \gg_w f$ is *contradictory*.

2.3 Dominance in Aggregation Operators

The definition of dominance can be extended from the class of binary operators to *n-ary* operators called aggregation operators.

Definition 2.3.1 (Aggregation Operators)**.** An operator $A : \bigcup_{n\in\mathbb{N}}[0,1]^n \rightarrow [0,1]$ is called an *aggregation operator* if,

- $A(x_1,...,x_n) \leq A(y_1,...,y_n)$ whenever $x_i \leq y_i$ for all $i \in \{1,...,n\}$,
- $A(x) = x$ for all $x \in [0,1]$,
- $A(0,...,0) = 0$ and $A(1,...,1) = 1$.

Definition 2.3.2 (Dominance in Aggregation Operators)**.** We denote by $A_{(n)}$ an aggregation operator A having n arguments. Consider two Aggregation operators A and B. We say $A \gg B$ if $A_{(n)} \gg B_{(m)}$ for all $n,m \in \mathbb{N}$.

Definition 2.3.3 (Aggregation Operators on a bounded Lattice)**.** Consider two aggregation operators A, B on a bounded Lattice $(L, \geq, 0, 1)$, then $A \gg B$ if for all $n,m \in \mathbb{N}$ it holds that,

$$A(B(x_{1,1},...,x_{m,1}),...,B(x_{1,n},...,x_{m,n})) \geq B(A(x_{1,1},...,x_{1,n}),...,A(x_{m,1},...,x_{m,n}))$$

where $x_{i,j} \in L$ for all $i \in \{1,2,...,m\}$, $j \in \{1,2,...,n\}$.

A schematics of interpretaion of dominance in aggregation operators, as a two step aggregation process is presented. Here $a = B(a_1, ..., a_m)$ and $b = A(b_1, ..., b_n)$.

$$\begin{array}{cccccc|cc}
 & A & \rightarrow & & & & & \\
B & x_{1,1}, & . & . & ., & x_{1,n} & \rightarrow & a_1 \\
\downarrow & . & & & & . & & . \\
 & . & & & & . & & . \\
 & . & & & & . & & . \\
 & x_{m,1}, & . & . & ., & x_{m,n} & \rightarrow & a_m \\
\hline
 & \downarrow & & & & \downarrow & & \\
 & b_1 & & & & b_n & & b \geq a
\end{array}$$

We will now see an important result regarding the dominance in aggregation operators that will enable us to go back from the class of n-ary to binary operators.

Theorem 2.3.1. *Consider two aggregation operators A and B on a bounded Lattice $(L, \geq, 0, 1)$. Then,*

1. *If A resp. B are associative, then $A \gg B$ if and only if $A_2 \gg B$ resp.$A \gg B_2$. If both A and B are associative, then $A \gg B$ if and only $A_2 \gg B_2$.*
2. *Assume that A resp. B possess neutral elements e_A resp. e_B. Then $A \gg B$ implies $e_A \geq e_B$. If $e_A = e_B$ then $A \geq B$.*
3. *$A \gg A$ if and only if A is bisymmetric, i.e. if for all $n \in \mathbb{N}$ it holds that,*
$A(B(x_{1,1}, ..., x_{n,1}), ..., B(x_{1,n}, ..., x_{n,n})) = A(A(x_{1,1}, ..., x_{1,n}), ..., A(x_{n,1}, ..., x_{n,n}))$
where $x_{i,j} \in L$ for all $i, j \in \{1, 2, ..., n\}$.

For a proof see Saminger et al. [4].

Note 2.3.1. Hence, dominance in *associative* aggregation operators is characterized by dominance in binary aggregation operators. Under the assumptions of Theorem 2.3.1, dominance is a reflexive and antisymmetric operator.

Counterexample regarding transitivity

Dominance is not transitive on the set of aggregation operators. A counter example due to Saminger is given [4]. Let us define three aggregation operators given by,

$$A^{(w)}(x_1, ..., x_n) = \begin{cases} 1, & \text{if } x_1 = ... = x_n = 1, \\ 0, & \text{otherwise.} \end{cases}$$

$$A_{min}(x_1, ..., x_n) = \min(x_1, ..., x_n)$$
$$A_{mean}(x_1, ..., x_n) = \frac{(x_1 + ... + x_n)}{n}$$

Then $A^{(w)} \gg A_{min}$ and $A_{min} \gg A_{mean}$, but $A^{(w)} \not\gg A_{mean}$.

Chapter 3

Dominance in t-norms

3.1 t-norms

t-norms are an important type of binary aggregation operators which are used in the framework of fuzzy logic. They play a central role in generalizing the notion of conjunction from classical logic to fuzzy logic. See the works by Klement et al. for more details [5], [6], [7], [8].

Definition 3.1.1 (t-norm)**.** A binary aggregation operator $T\colon [0,1]^2 \to [0,1]$ is called a triangular norm (briefly t-norm) if it is associative, commutative and monotonically increasing with 1 as the neutral element.

Note 3.1.1. Since t-norms are a special class of binary aggregation operators, by Theorem 2.3.1 dominance on t-norms is reflexive and antisymmetric.

Corollary 3.1.1. *From the definition of t-norm, it follows that for all $x \in [0,1]$, we have,*

$$T(x,0) \leq T(1,0) = 0$$

Hence, $T(x,0) = 0$.

We will next see few important examples of t-norms, which occur frequently in context of fuzzy logic.

3.1.1 Examples

For all $x, y \in [0,1]$, we have

- Minimum t-norm,

$$T_M(x, y) = \min(x, y)$$

- Product t-norm,

$$T_P(x, y) = x.y$$

- Lukasiewicz t-norm,

$$T_L(x, y) = \max(x + y - 1, 0)$$

- Drastic t-norm,

$$T_D(x, y) = \begin{cases} x, & \text{if } y = 1, \\ y, & \text{if } x = 1, \\ 0, & \text{otherwise.} \end{cases}$$

We will now consider an important result which shows that T_M and T_D are the biggest and smallest t-norms resp. with respect to dominance.

Lemma 3.1.2. *For all t-norms T, we have,*

$$T_M \gg T \quad \textit{and} \quad T \gg T_D$$

Proof. Let $a, b, c, d \in [0, 1]$. Clearly $a \geq \min(a, c)$ and $b \geq \min(b, d)$. Now since T is monotonically increasing, we have,

$$\begin{aligned} T(a, b) &\geq T(\min(a, c), \min(b, d)) \\ \text{similarly,} \quad T(c, d) &\geq T(\min(a, c), \min(b, d)) \\ \text{hence,} \quad \min(T(a, b), T(c, d)) &\geq T(\min(a, c), \min(b, d)) \end{aligned}$$

Hence, $T_M \gg T$ for all T.
Now if we have $a = c = 1$, then,

$$T(T_D(a, b), T_D(c, d)) = T(b, d) = T_D(T(a, c), T(b, d))$$

Similarly, if $b = d = 1$, then,

$$T(T_D(a, b), T_D(c, d)) = T(a, c) = T_D(T(a, c), T(b, d))$$

In all other cases,

$$T(T_D(a,b), T_D(c,d)) \geq 0 = T_D(T(a,c), T(b,d))$$

Hence, $T \gg T_D$ for all T. ∎

We will now see the relation between the three partial orders, viz. dominance, weak-dominance and pointwise ordering on the class of t-norms.

Lemma 3.1.3. *For t-norms T_1 and T_2, if $T_1 \gg T_2$ then $T_1 \gg_w T_2$. If $T_1 \gg_w T_2$ then $T_1 \geq T_2$.*

Proof. For $a, b, c, d \in [0,1]$, $T_1 \gg T_2$ means,

$$T_1(T_2(a,b), T_2(c,d)) \geq T_2(T_1(a,c), T_1(b,d))$$

Choosing $d = 1$ gives,

$$T_1(T_2(a,b), c) \geq T_2(T_1(a,c), b)$$

Hence, $T_1 \gg_w T_2$. If now we choose $a = 1$, we obtain,

$$T_1(b,c) \geq T_2(c,b) = T_2(b,c)$$

Hence, $T_1 \geq T_2$ ∎

Note 3.1.2. The converse of the previous lemma does not hold. The counterexamples regarding this are given in subsection 3.4.3.

3.2 Continuous t-norms

Definition 3.2.1 (Continuous t-norm)**.** A t-norm T is continuous if for all convergent sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$, we have,

$$T(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = \lim_{n\to\infty} T(x_n, y_n)$$

Note 3.2.1. The monotonicity condition in the class of t-norms allows us to formulate a simpler definition of continuous t-norms. It follows that a t-norm $T(x,y)$ is continuous if and only if it is continuous in both arguments x and y.

3.2.1 Representation of Continuous t-norms

We will now consider the t-norms satisfying the *Archimedean property*, which will be used for the characterization of continuous t-norms. For proofs, see Klement et al. [6].

Definition 3.2.2 (Archimedean t-norm)**.** A t-norm T is called Archimedean if, for all $x, y \in (0, 1)$, there exists a $n \in \mathbb{N}$ such that,

$$T(x, T(x, T(x, ...)))_{n \text{ times}} < y$$

From a given family of t-norms T_i, we can generate a t-norm T termed as the ordinal sum of T_i's.

Definition 3.2.3 (Ordinal Sum)**.** Let $(a_i, b_i)_{i \in I}$ be a nonempty pairwise-disjoint open subinterval of $[0, 1]$ and let $\{T_i\}$ be a family of t-norm, then the t-norm $T = (\langle a_i, b_i, T_i \rangle)_{i \in I} : [0, 1]^2 \to [0, 1]$ defined by,

$$T(x, y) = \begin{cases} a_i + (b_i - a_i) T_i \left(\frac{x - a_i}{b_i - a_i}, \frac{y - a_i}{b_i - a_i} \right) & \text{if } (x, y) \in [a_i, b_i]^2, \\ T_M(x, y) & \text{otherwise.} \end{cases}$$

is called the ordinal sum of the summands $\langle a_i, b_i, T_i \rangle, i \in I$. T_i are called *summand operations*, and the intervals $[a_i, b_i]_{i \in I}$ are called *summand carriers*.

Definition 3.2.4 (Ordinally Irreducible t-norm)**.** A t-norm T which entertains $(\langle 0, 1, T \rangle)$ as the only ordinal sum representation is called ordinally irreducible t-norm.

Theorem 3.2.1. *A t-norm T is a continuous if and only if T is an ordinal sum of continuous Archimedean t-norms. Such a characterization of continuous t-norms as ordinal sum of continuous Archimedean t-norms is unique.*

3.2.2 Dominance in Continuous t-norms

As in the case of t-norms, dominance is reflexive and antisymmetric in the case of continuous t-norms.

Counterexample regarding transitivity

A counterexample by Sarkoci [9] shows that dominance is not transitive in the class of continuous t-norms.

We consider three ordinal sum t-norms, defined by,

$$
\begin{aligned}
T_1 &= (\langle 0, \frac{1}{2}, T_L \rangle, \langle \frac{1}{2}, 1, T_M \rangle), \\
T_2 &= (\langle 0, \frac{1}{2}, T_L \rangle, \langle \frac{1}{2}, 1, T_L \rangle), \\
T_3 &= T_L.
\end{aligned}
$$

Then $T_1 \gg T_2$ and $T_2 \gg T_3$, but $T_1 \not\gg T_3$.

We will now look into the properties of summands of continuous t-norms T_1 and T_2, whenever $T_1 \gg T_2$. For proofs and related discussions, see the paper by Saminger et al. [10].

Lemma 3.2.2. *Consider $T_1 = (\langle a_i, b_i, T_{1,i} \rangle)_{i \in I}$ and $T_2 = (\langle a_i, b_i, T_{2,i} \rangle)_{i \in I}$, i.e. both t-norms having identical summand carriers. Then $T_1 \gg T_2$ if and only if $T_{1,i} \gg T_{2,i}$ for all $i \in I$.*

Theorem 3.2.3. *Consider $T_1 = (\langle a_i, b_i, T_{1,i} \rangle)_{i \in I}$ and $T_2 = (\langle a_j, b_j, T_{2,j} \rangle)_{j \in J}$ with ordinally irreducible summands $T_{1,i}$ and $T_{2,j}$, with different summand carriers. Then $T_1 \gg T_2$ if and only if,*

- $\cup_{j \in J} I_j = I, \quad \textit{with} \quad I_j = \{i \in I : [a_i, b_i] \subseteq [a_j, b_j]\},$
- $\forall j \in J : T_i^j \gg T_{2,j}$

$$
\begin{aligned}
\textit{where} \quad & T_i^j = (\langle \phi_j(a_i), \phi_j(b_i), T_{1,i} \rangle)_{i \in I_j}, \\
& \phi_j : [a_j, b_j] \to [0, 1], \quad \phi_j(x) = \frac{x - a_j}{b_j - a_j}
\end{aligned}
$$

Apart from the summand operations, the condition for dominance in ordinal sum t-norms can also be formulated in terms of the set of idempotent elements of the t-norms.

Definition 3.2.5 (Set of idempotent elements)**.** For a t-norm T and $x \in [0, 1]$, define,

$$\mathcal{I}(T) = \{x | T(x, x) = x\}$$

Theorem 3.2.4. *Consider* $T_1 = (\langle a_i, b_i, T_{1,i}\rangle)_{i\in I}$ *and* $T_2 = (\langle a_j, b_j, T_{2,j}\rangle)_{j\in J}$ *with ordinally irreducible summands* $T_{1,i}$ *and* $T_{2,j}$*. If* $T_1 \gg T_2$ *then,*

$$\mathcal{I}(T_2) \subseteq \mathcal{I}(T_1),$$
$$\textit{and,}\quad \mathcal{I}(T_1) \textit{ is closed under } T_2.$$

Corollary 3.2.5. *In the special case when* $T_{1,i} = T_{2,j} = T_L$*, i.e.* $T_1 = (\langle a_i, b_i, T_L\rangle)_{i\in I}$ *and* $T_2 = (\langle a_j, b_j, T_L\rangle)_{j\in J}$*, we have the result that* $T_1 \gg T_2$ if and only if,

$$\mathcal{I}(T_2) \subseteq \mathcal{I}(T_1),$$
$$\textit{and,}\quad \mathcal{I}(T_1) \textit{ is closed under } T_2.$$

3.3 Continuous Archimedean t-norms

In the previous section, we saw that the dominance between continuous t-norms can be characterized by considering the dominance between their summand operations, which are given by a unique family of continuous Archimedean t-norms. Hence it becomes important to consider the properties of continuous Archimedean t-norms. See the paper by Klement et al. [7].

3.3.1 Representation of Continuous Archimedean t-norms

Definition 3.3.1 (Additive Generator Representation)**.** Let $f\colon [0,1] \to [0,+\infty]$ be a strictly decreasing function such that $f(1) = 0$ for all $x, y \in [0,1]$. Then the function $T : [0,1]^2 \to [0,1]$ defined as,

$$T(x,y) = \min(f(0), f^{-1}(f(x) + f(y)))$$

is a t-norm. f is called the additive generator of T.

Theorem 3.3.1. *A t-norm* T *which is generated by an additive generator* f *is necessarily Archimedean. It is continuous if and only if* f *is continuous.*

Note 3.3.1. All continuous Archimedean t-norms are uniquely characterized by additive generator representation, i.e. there exists a unique additive generator f for a given continuous Archimedean t-norm T.

3.3.2 Dominance in Continuous Archimedean t-norms

The dominance between continuous Archimedean t-norms T_1 and T_2 can be characterized by their additive generators f_1 and f_2. For proofs and related discussions, see the paper by Saminger-Platz et al. [11]. We have the following theorem.

Theorem 3.3.2. *Consider two continuous Archimedean t-norms T_1 and T_2, generated by f_1 and f_2 resp. Consider a function $h\colon [0,\infty] \to [0,\infty]$ given by $h = f_1 \circ f_2^{-1}$. Then $T_1 \gg T_2$ if and only if, for all $a, b, c, d \in [0, f_2(0)]$, it holds that,*

$$h^{(-1)}(h(a) + h(c)) + h^{(-1)}(h(b) + h(d)) \geq h^{(-1)}(h(a+b) + h(c+d)) \tag{3.1}$$

where the function $h^{(-1)}\colon [0,\infty] \to [0,\infty]$ is the pseudo-inverse *of h, given by $h^{(-1)} = f_2 \circ f_1^{-1}$.*

Note 3.3.2. If the function h satisfies equation for all $a, b, c, d \in [0,\infty]$, we say that h satisfies the *generalized Mulholland inequality*.

Corollary 3.3.3. *The function h fulfills $h(0) = 0$, and $h(x) = f_1(0)$ for all $x \in [f_2(0), \infty]$. Furthermore, h is* superadditive *on $[0, f_2(0)]$.*

It is natural to examine the properties of the composite generator h for the conditions when it satisfies the inquality 3.1. We will formulate the conditions in terms of definitions given as follows.

Definition 3.3.2 (Convexity)**.** A function $h\colon [0,\infty) \to [0,\infty)$ is called *convex* on an interval (a,b) if for any $k \in (0,1)$, we have, for all $x, y \in (a,b)$,

$$h(kx + (1-k)y) \leq kh(x) + (1-k)h(y)$$

It implies that if h is a differentiable function than h is convex if and only if $h''(x) > 0$ for all $x \in (a,b)$.

Definition 3.3.3 (Geometric Convexity)**.** A function $h\colon [0,\infty) \to [0,\infty)$ is called geometric-convex (*geo-convex*) on an interval (a,b) if, we have, for all $x, y \in (a,b)$,

$$h(\sqrt{xy}) \leq \sqrt{h(x)h(y)}$$

Definition 3.3.4 (Logarithmic Convexity)**.** A function $h\colon [0,\infty) \to [0,\infty)$ is called logarithmic-convex (*log-convex*) on an interval (a,b) if, the function $log \circ h\colon [0,\infty) \to [-\infty,\infty)$ is convex on (a,b).

The following lemma shows the relation between the three preceding definitions.

Lemma 3.3.4. *If a function h such that $h\left((0,\infty)\right) \subseteq (0,\infty)$ is continuous, then,*

- *h is geo-convex on $(0,t)$ if and only if the function $log \circ h \circ exp$ is convex on $(-\infty, log(t))$ (hence equivalent to function $h \circ exp$ log-convex on $(-\infty, log(t))$).*
- *If $h(0) = 0$, then geo-convexity holds for $[0,t)$.*
- *If h is strictly increasing, its log-convexity on $(0,t)$ implies its geo-convexity on $(0,t)$.*

We now define a function $g\colon [0,\infty] \to [0,\infty]$ and $H\colon [0,\infty]^2 \to [0,\infty]$ by,

$$g(x) = \begin{cases} h^{-1}(x) & \text{if } x \in [0, h(t)], \\ t & \text{otherwise.} \end{cases}$$

$$H(x,y) = g(h(x) + h(y))$$

Based on the conditions on h given by the preceding definitions, we have the following theorems.

Lemma 3.3.5. *Consider a function $h\colon [0,\infty] \to [0,\infty]$ and a constant $t \in (0,\infty)$. We have the following results,*

- *If h is continuous, strictly increasing, convex and geo- convex on $(0,t)$, h is increasing elsewhere, and $h(0) = 0$, then,*

$$H(a+b, c+d) \leq H(a,c) + H(b,d) \quad \textit{for all } a,b,c,d \in [0,\infty]$$

- *If h is continuous, strictly increasing, convex on $(0,t)$ and h' is geo- convex on $(0,t)$, h is increasing elsewhere, and $h(0) = 0$, then,*

$$H(a+b, c+d) \leq H(a,c) + H(b,d) \quad \textit{for all } a,b,c,d \in [0,\infty]$$

- *If h is continuous and strictly increasing on $(0,t)$, h is increasing elsewhere, and $h(0) = 0$, then,*

$$H \textit{ is convex if} \quad H(a+b, c+d) \leq H(a,c) + H(b,d) \quad \textit{for all } a,b,c,d \in [0,\infty]$$

For continuous Archimedean t-norms T_1 and T_2, generated by f_1 and f_2 resp., and $h = f_1 \circ f_2^{-1}$, we have $H(x, y) = h^{(-1)}(h(x) + h(y))$. From this and Lemma 3.3.4, we can restate the results from previous lemma in terms of dominance between Archimedean continuous t-norms. Thus we have the follwoing theorem.

Theorem 3.3.6. *Consider two continuous Archimedean t-norms T_1 and T_2, generated by f_1 and f_2 resp. Consider a function $h\colon [0, \infty] \to [0, \infty]$ given by $h = f_1 \circ f_2^{-1}$. Then the following holds,*

1. *If h is convex on $(0, f_2(0))$, and log- or geo- convex on $(0, f_2(0))$, then $T_1 \gg T_2$.*
2. *If h is differentiable and convex on $(0, f_2(0))$, and if h' is log- or geo- convex on $(0, f_2(0))$, then $T_1 \gg T_2$.*
3. *If $T_1 \gg T_2$, then h is convex on $(0, f_2(0))$.*

3.3.3 Easy to check conditions

Since the function h is defined in terms of f_1 and f_2, we need to check conditions on f_1 and f_2 whenever h satisfies convexity, geo- or log- convexity. We give the relationship of f_1, f_2 to the required conditions on h.

Theorem 3.3.7. *For $h = f_1 \circ f_2^{-1}$,*

1. *The function h is convex on $(0, f_2(0))$ if and only if for all $x \in (0, 1)$, we have,*

$$f_1'(x)f_2''(x) - f_2'(x)f_1''(x) \geq 0$$

2. *The function h is log- convex on $(0, f_2(0))$ if and only if for all $x \in (0, 1)$, we have,*

$$f_1'^2(x)f_2'(x) - f_1(x)\left(f_1'(x)f_2''(x) - f_2'(x)f_1''(x)\right) \geq 0$$

3. *The function h is geo- convex on $(0, f_2(0))$ if and only if for all $x \in (0, 1)$, we have,*

$$\frac{f_1'^2(x) - f_1'(x)f_1''(x)}{f_1(x)f_1'(x)} \geq \frac{f_2'^2(x) - f_2'(x)f_2''(x)}{f_2(x)f_2'(x)}$$

4. *The function h' is log- convex on $(0, f_2(0))$ if and only if for all $x \in (0,1)$, we have,*

$$f_1'^2(x)\left(2f_2''^2(x) - f_2'(x)f_2'''(x)\right) \geq f_2'^2(x)\left(2f_1''^2(x) - f_1'(x)f_1'''(x)\right) + f_1'(x)f_1''(x)f_2'(x)f_2''(x)$$

5. *The function h' is geo- convex on $(0, f_2(0))$ if and only if for all $x \in (0,1)$, we have,*

$$\begin{aligned} &f_1'(x)f_2'(x)(f_1'''(x)f_2'(x) - f_2'''(x)f_1'(x)) - (f_1''(x)f_2'(x) - f_2''(x)f_1'(x)) \\ &\quad - (2f_1''(x)f_2'(x) + f_2''(x)f_1'(x)) \geq f_2^{-1}(x)\left(f_1'(x)f_2'^2(x)\left(f_2''(x)f_1'(x) + f_1''(x)f_2'(x)\right)\right) \end{aligned}$$

3.4 Strict t-norms

Definition 3.4.1 (Strict t-norms)**.** A t-norm $T(x,y)$ is *strict* if it is continuous for all $x, y \in [0,1]$, and strictly increasing in each place for all $x, y \in (0,1]$.

We now give an equivalent definition of a strict t-norm in terms of additive generator f.

Lemma 3.4.1. *If $f\colon [0,1] \to [0,\infty]$ be a continuous additive generator of a continuous Archimedean t-norm T, then T is* strict *if and only if $f(0) = \infty$.*

Note 3.4.1. A t-norm T is called *nilpotent* if it is continuous and if each $x \in (0,1)$ is a nilpotent element of of T, i.e. if there exists a $n \in \mathbb{N}$ such that

$$T(x, T(x, T(x, ...)))_{n \text{ times}} = 0$$

An equivalent condition for nilpotence is that a t-norm T having an additive generator f is nilpotent if and only if $f(0) < \infty$. Clearly, a continuous Archimedean t-norm can only be either strict or nilpotent, and that depends solely on the additive generator f.

An important result is that each continuous t-norm can be approximated by some continuous Archimedean t-norm with arbitrary precision. Since a continuous Archimedean t-norm can either be strict or nilpotent, we have the following theorem, see the paper by Klement et al. [7].

Theorem 3.4.2. *Let T be a continuous t-norm. Then for each $\varepsilon > 0$ we have a strict t-norm T_1 and a nilpotent t-norm T_2 such that for all $x, y \in [0, 1]$,*

$$|T(x,y) - T_1(x,y)| < \varepsilon,$$
$$|T(x,y) - T_2(x,y)| < \varepsilon$$

Since strict t-norms are also continuous Archimedean t-norms, Theorem 3.3.2 and Corollary 3.3.3 holds for them. We have certain additional results for dominance and weak dominance in their case. For details, see the book by Alsina, Frank, Schweizer [3].

3.4.1 Transitivity of Dominance in Strict t-norms

Lemma 3.4.3. *If under the assumptions of Theorem 3.3.2, the t-norm T_2 is strict, then necessarily T_1 is strict. Furthermore h is a strictly increasing bijection, and $h^{(-1)}$ is the standard inverse h^{-1} of h.*

Note 3.4.2. Under the assumptions of Lemma 3.4.3, the generalized Mulholland inequality 3.1 is called *classical Mulholland inequality*.

Theorem 3.3.6 which characterizes dominance in the class the continuous Archimedean t-norms also holds for the strict t-norms. Moreover, we can show that under certain conditions dominance is *transitive* on the class of strict t-norms.

Lemma 3.4.4. *If T_1 and T_2 are strict t-norms with additive generators f_1 and f_2 resp. with h given by $h = f_1 \circ f_2^{-1}$ such that $T_1 \gg T_2$. Then h is convex on $[0, \infty)$.*

This lemma can be generalized to the following theorem due to Tardiff [3].

Theorem 3.4.5. *Consider a subset $\mathcal{D}$ of strict t-norms, then for any two strict t-norms T_1 and $T_2 \in \mathcal{D}$ generated by f_1 and f_2 resp., and a function $h\colon [0, \infty] \to [0, \infty]$ given by $h = f_1 \circ f_2^{-1}$, the following holds,*

1. *If h is geo- convex on $(0, f_2(0))$, then $T_1 \gg T_2$ for all $T_1, T_2 \in \mathcal{D}$ Moreover for all $T_1, T_2, T_3 \in \mathcal{D}$, if $T_1 \gg T_2$ and $T_2 \gg T_3$, then $T_1 \gg T_3$.*
2. *If h is at least thrice differentiable and if h' geo- convex on $(0, f_2(0))$, then $T_1 \gg T_2$ for all $T_1, T_2 \in \mathcal{D}$. Moreover for all $T_1, T_2, T_3 \in \mathcal{D}$, if $T_1 \gg T_2$ and $T_2 \gg T_3$, then $T_1 \gg T_3$.*

3.4.2 Transitivity of Weak dominance in Strict t-norms

Lemma 3.4.4 for the case of dominance in strict t-norms can be strengthened for weak dominance. We have

Lemma 3.4.6. *Let T_1 and T_2 be strict t-norms with additive generators f_1 and f_2 resp. with $h = f_1 \circ f_2^{-1}$. Then $T_1 \gg_w T_2$ if and only if h is convex.*

Lemma 3.4.7. *Let T_1 and T_2 be strict t-norms with additive generators f_1 and f_2 resp. with $h_{1,2} = f_1 \circ f_2^{-1}$, $h_{2,3} = f_2 \circ f_3^{-1}$, and $h_{1,3} = f_1 \circ f_2^{-3}$. If $h_{1,2}$ and $h_{2,3}$ are increasing and convex on $[0, \infty)$, then $h_{1,3}$ is increasing and convex on $[0, \infty)$.*

From the preceding two lemma we can immediately deduce the transitivity of weak dominance on the set of strict t-norms.

Theorem 3.4.8. *Let T_1, T_2 and T_3 be strict t-norms. Then if $T_1 \gg_w T_2$ and $T_2 \gg_w T_3$, then $T_1 \gg_w T_3$, i.e. the weak dominance relation is transitive on the set of all strict t-norms.*

3.4.3 Distinction between Relations

As we saw in Lemma 3.1.3, dominance implies weak dominance and weak dominance implies pointwise ordering for t-norms. We will now provide a counterexample showing that the converse does not hold. Hence dominance, weak dominance and pointwise order are distinct relations.

Counterexample showing Weak dominance does not imply Dominance

Let T_1 and T_2 be strict t-norms with additive generators f_1 and f_2 resp. given by,

$$f_1(x) = x^{-1} - 1,\ f_2(x) = \begin{cases} (2x)^{-1}, & 0 < x \leq \frac{1}{2}, \\ f_1(x), & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Now since $f_1 \circ f_2^{-1}$ is convex, hence from Lemma 3.4.6 $T_1 \gg_w T_2$. But, for $h = f_1 \circ f_2^{-1}$ and $a = c = d = 1$, $b = \frac{1}{2}$, we immediately see that

$$h^{-1}(h(a)+h(c))+h^{-1}(h(b)+h(d)) \not\geq h^{-1}(h(a+b)+h(c+d))$$

So by Theorem 3.3.2 $T_1 \not\gg T_2$.

Counterexample showing Pointwise ordering does not imply Weak dominance

We first need a lemma in order to proceed with the counterexample.

Lemma 3.4.9. *Let T_1 and T_2 be continuous Archimedean t-norms, not necessarily strict, with additive generators f_1 and f_2 resp. Then $T_1 \geq T_2$ if and only if $f_2 \circ f_1^{(-1)}$ is subadditive.*

Let T_1 and T_2 be strict t-norms with additive generators f_1 and f_2 resp. such that $f_2 = \varphi \circ f_1$, with φ given by,

$$\varphi(x) = \begin{cases} 1-(1-x)^2, & 0 \leq x \leq 1, \\ 2-(2-x)^2, & 1 \leq x \leq 2, \\ x, & 2 \leq x. \end{cases}$$

Now since $\varphi = f_2 \circ f_1^{-1}$ is subadditive, hence from the previous lemma it follows that $T_1 \geq T_2$. But since φ is not concave, $h^{-1} = f_1 \circ f_2^{-1}$ is not convex, and hence by Lemma 3.4.6 $T_1 \not\gg_w T_2$.

Chapter 4

Preservation of T-transitivity

4.1 Fuzzy Equivalence

In this section we will define the notion of equivalence for the elements of a fuzzy set.

Definition 4.1.1 (Fuzzy Set)**.** A fuzzy set is a pair (X, μ) where X is a non-empty set and $\mu \colon X \to [0, 1]$ is a *membership function* which maps each element from X to the unit interval.

Definition 4.1.2 (Fuzzy Relation)**.** Let X and Y be two non-empty sets. A fuzzy subset R of the cartesian product $X \times Y$ is called a *fuzzy relation* from X to Y. For $(x, y) \in R$ for some pair (x, y), the degree to which x is R-related to y is denoted by $R(x, y)$, where $R(x, y) \in [0, 1]$.

Note 4.1.1. If $X = Y$, i.e. R is a fuzzy subset of $X \times X$, we say that R is a *binary fuzzy relation* on X.

Definition 4.1.3 (T-transitive Relation)**.** Consider a binary fuzzy relation R on some set X and an arbitrary t-norm T. R is called *T-transitive* if and only if for all $x, y, z \in X$ the following holds,

$$T(R(x, y), R(y, z)) \leq R(x, z)$$

Definition 4.1.4 (Fuzzy Equivalence Relation)**.** Consider a t-norm T. A binary fuzzy relation R on some set X is called a *T-equivalence* relation (also called an equality relation) on X if and only if it is reflexive, symmetric and T-transitive, i.e. for all $x, y, z \in X$ the following holds,

- $R(x, x) = 1$,
- $R(x, y) = R(y, x)$,
- $T(R(x, y), R(y, z)) \leq R(x, z)$.

Note 4.1.2. The T-equivalence relation on a fuzzy set generalizes the property of relation on a crisp set. If $x, y \in X$ are T-equivalent, then it signifies the *closeness* of the two elements with respect to a given t-norm T.

4.2 Preservation of T-transitivity

As we saw in the last section, T-transitivity is the defining property of T-equivalence. If we have two sets such that the relations between the corresponding elements of the sets are T-equivalent, it is natural to require that the the aggregation of these relations is also a T-equivalent relation. This situation desires that T-transitivity be preserved after the aggregation step. For proofs and more details, see the paper by Saminger et al. [4].
This motivates the following definition.

Definition 4.2.1 (Preservation of T-transitivity)**.** An aggregation operator A *preserves T-transitivity* if for all $n \in \mathbb{N}$ and for all binary T-transitive fuzzy relations R_i on a set X_i with $i \in \{1, ..., n\}$, the aggregated relation $\tilde{R} = A(R_1, ..., R_n)$ on a cartesian product of all X_i given by,

$$\tilde{R}(A, B) = \tilde{R}((a_1, ...a_n), (b_1, ..., b_n)) = A(R_1(a_1, b_1), ..., R_n(a_n, b_n))$$

is also T-transitive, i.e. for all $A, B, C \in \prod_{i=1}^{n} X_i$,

$$T(\tilde{R}(A, B), \tilde{R}(B, C)) \leq \tilde{R}(A, C)$$

For a given t-norm, we define,

Definition 4.2.2 (Dominating class of Aggregation Operators)**.** We denote the class of aggregation operators A which dominates the t-norm T by,

$$\mathcal{D}_T = \{A | A \gg T\}$$

The preservation of T-transitivity has been shown to be related to the dominance of the chosen aggregation operator and the given t-norm. We have the following result.

Theorem 4.2.1. *Let $X_i = X$ for all $i \in \{1, ..., n\}$, and let $|X| > 3$. Then, an aggregation operator A preserves the T-transitivity of fuzzy relations on X for an arbitrary t-norm T if and only if $A \in \mathcal{D}_T$.*

4.2.1 Isomorphic Operators

Definition 4.2.3 (Isomorphic Aggregation Operators)**.** Consider an aggregation operator $A\colon [a,b]^{\mathbb{N}} \to [a,b]$ and a monotone bijection $\varphi\colon [c,d] \to [a,b]$. Then the operator $A_\varphi\colon [c,d]^{\mathbb{N}} \to [c,d]$ defined by,

$$A_\varphi(x_1, ..., x_n) = \varphi^{-1}(A(\varphi(x_1), ...\varphi(x_n)))$$

is an aggregation operator *isomorphic* to A.

Note 4.2.1. In an anologues way, we can define T_φ, a t-norm *isomorphic* to T.

For continuous Archimedean t-norms, we have the following result.

Lemma 4.2.2. *A t-norm T is strict if and only if it is isomorphic to the product T_P. It is nilpotent if and only if it is isomorphic to the Lukasiewicz t-norm T_L.*

The problem of finding a class of aggregation operators dominating a t-norm can be reformulated in their respective isomorphic operators.

Theorem 4.2.3. *Consider an aggregation operator A and a given t-norm T. Then $A \in \mathcal{D}_T$ if and only if $A_\varphi \in \mathcal{D}_{T_\varphi}$ for all strictly increasing bijections $\varphi\colon [0,1] \to [0,1]$.*

4.2.2 Ordinal sum

Next, we have a result relating the *lower ordinal sum* of a dominating class of aggregation operators to the ordinal sum of the corresponding t-norms.

Definition 4.2.4 (Lower Ordinal Sum of Aggregation Operators)**.** If $(A_\alpha)_{\alpha \in I}$ be a family of aggregation operators, and $(a_\alpha, e_\alpha)_{|\alpha \in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$, then the *lower ordinal sum* of the family $A^{(w)} = (\langle a_\alpha, e_\alpha, A_\alpha \rangle)_{\alpha \in I}$ is given by,

$$A^{(w)}(x_1,...,x_n) = \begin{cases} \sup_{\alpha\in I}\{A^*_\alpha(\min(x_1,e_i),...,\min(x_n,e_i))|a_\alpha < u\} & \text{if} \quad u<1, \\ 1 & \text{otherwise.} \end{cases}$$

with $\sup\{\phi\} = 0$, $u = \min(x_1, ..., x_n)$ and,

$$A^*_\alpha(x_1,...,x_n) = a_\alpha + (e_\alpha - a_\alpha)\cdot A_\alpha\left(\frac{x_1 - a_\alpha}{e_\alpha - a_\alpha},...,\frac{x_n - a_\alpha}{e_\alpha - a_\alpha}\right)$$

Theorem 4.2.4. *Let $(T_\alpha)_{\alpha\in I}$ be a family of t-norms, $(A_\alpha)_{\alpha\in I}$ be a family of aggregation operators, and $(a_\alpha, e_\alpha)_{|\alpha\in I}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. If $A_\alpha \in \mathcal{D}_{T_\alpha}$ for all $\alpha \in I$, then $A^{(w)} \in \mathcal{D}_T$ for $T = (\langle a_\alpha, e_\alpha, T_\alpha\rangle)_{\alpha\in I}$.*

4.2.3 Domination of basic t-norms

For t-norms T_M and T_D, we have the following results for the class of dominating aggregation operators.

Lemma 4.2.5. *For any $n \in \mathbb{N}$, the class of all n-ary aggregation operators $A_{(n)}$ dominating the t-norm T_M is given by,*

$$\begin{aligned} \mathcal{D}^{(n)}_{T_M} = \{\min_{\mathcal{F}} |\mathcal{F} =& (f_1,...,f_n) \\ & f_i\colon [0,1]\to[0,1], \quad \textit{non-decreasing, with} \\ & \quad f_i(1) = 1 \quad \textit{for all } i\in\{1,...,n\}, \\ & \quad f_i(0) = 0 \quad \textit{for at least one } i\in\{1,...,n\}\}, \end{aligned}$$

where $\min_{\mathcal{F}}(x_1,...,x_n) = \min(f_1(x_1),...,f_n(x_n))$.

Lemma 4.2.6. *For any $n \in \mathbb{N}$, consider an n-ary aggregation operators $A_{(n)}$. Then $A_{(n)} \gg T_D$ if and only if there exists a non-empty subset $I = \{k_1,...,k_m\} \subseteq \{1,...,n\}, k_1 < ... < k_m$ and a non-decreasing mapping $B\colon [0,1]^m \to [0,1]$ satisfying,*

1. $B(0,...,0) = 0$,
2. $B(u_1,...,u_m) = 1$ *if and only if* $u_1 = ... = u_m = 1$,

such that $A(x_1,...,x_n) = B(x_{k_1},...,x_{k_m})$.

Appendix A

Transitivity of Dominance on Parametric Families of t-norms

Although dominance has been shown to be not transitive in general on continuous t-norms, it nevertheless exhibits transitivity on several important parametric families of continuous t-norms. We now enlist some of such families where dominance is transitive and hence a partial order; and also give the criteria for dominance in these families in terms of their respective parameters.
For analysis and further discussions, see [10], [11], [12].

A.1 Ordinal sum t-norms

Dubois- Prade family

The Dubois- Prade t-norms family is given by,

$$(T_\lambda^{\mathbf{DP}})_{\lambda \in [0,1]} = (\langle 0, \lambda, T_P \rangle) \quad \text{for } \lambda \in [0,1]$$

Then, $T_{\lambda_1}^{\mathbf{DP}} \gg T_{\lambda_2}^{\mathbf{DP}}$ if and only if either $\lambda_1 = 0$ or $\lambda_1 = \lambda_2$.

Mayor- Torrens family

The Mayor- Torrens t-norms family is given by,

$$(T_\lambda^{\mathbf{MT}})_{\lambda\in[0,1]} = (\langle 0, \lambda, T_L\rangle) \quad \text{for } \lambda \in [0,1]$$

Then, $T_{\lambda_1}^{\mathbf{MT}} \gg T_{\lambda_2}^{\mathbf{MT}}$ if and only if either $\lambda_1 = 0$ or $\lambda_1 = \lambda_2$.

Modified Mayor- Torrens family

The Modified Mayor- Torrens t-norms family is given by,

$$(T_\lambda^{\mathbf{MMT}})_{\lambda\in[0,1]} = (\langle \lambda, 1, T_L\rangle) \quad \text{for } \lambda \in [0,1]$$

Then, $T_{\lambda_1}^{\mathbf{MMT}} \gg T_{\lambda_2}^{\mathbf{MMT}}$ if and only if $\lambda_1 \geq \lambda_2$.

A.2 Continuous Archimedean t-norms

We present several families of continuous Archimedean t-norms in which dominance is a transitive relation.

Schweizer- Sklar family

The Schweizer- Sklar t-norms family is given by,

$$(T_\lambda^{\mathbf{SS}}(x,y))_{\lambda\in[-\infty,\infty]} = \begin{cases} T_M(x,y) & \text{if } \lambda = -\infty, \\ T_P(x,y) & \text{if } \lambda = 0, \\ T_D(x,y) & \text{if } \lambda = \infty, \\ \max(x^\lambda + y^\lambda - 1, 0)^{\frac{1}{\lambda}} & \text{if } \lambda = (-\infty, 0) \cup (0, \infty) \end{cases}$$

Then, $T_{\lambda_1}^{\mathbf{SS}} \gg T_{\lambda_2}^{\mathbf{SS}}$ if and only if $\lambda_1 \leq \lambda_2$.

Frank family

The Frank t-norms family is given by,

$$(T_\lambda^{\mathbf{F}}(x,y))_{\lambda\in[-\infty,\infty]} = \begin{cases} T_L(x,y) & \text{if } \lambda = -\infty, \\ T_P(x,y) & \text{if } \lambda = 0, \\ T_M(x,y) & \text{if } \lambda = \infty, \\ -\frac{1}{\lambda} log\left(1 + \dfrac{(e^{-\lambda x}-1)(e^{-\lambda y}-1)}{e^{-\lambda}-1}\right) & \text{if } \lambda = (-\infty,0)\cup(0,\infty) \end{cases}$$

Then, $T_{\lambda_1}^{\mathbf{F}} \gg T_{\lambda_2}^{\mathbf{F}}$ if and only if either $\lambda_1 = \infty$ or $\lambda_1 = \lambda_2$, or $\lambda_2 = -\infty$.

Hamacher family

The Hamacher t-norms family is given by,

$$(T_\lambda^{\mathbf{H}}(x,y))_{\lambda\in[-\infty,1]} = \begin{cases} T_D(x,y) & \text{if } \lambda = -\infty, \\ 0 & \text{if } \lambda = 1 \; x = y = 0, \\ \dfrac{xy}{1-\lambda(1-x)(1-y)} & \text{otherwise} \end{cases}$$

Then, $T_{\lambda_1}^{\mathbf{H}} \gg T_{\lambda_2}^{\mathbf{H}}$ if and only if either $\lambda_1 = 1$ or $\lambda_1 = \lambda_2$, or $\lambda_2 = -\infty$.

Sugeno- Weber family

The Sugeno- Weber t-norms family is given by,

$$(T_\lambda^{\mathbf{SW}}(x,y))_{\lambda\in[0,\infty]} = \begin{cases} T_P(x,y) & \text{if } \lambda = 0, \\ T_D(x,y) & \text{if } \lambda = \infty, \\ \max(0,(1-\lambda)xy, \lambda(x+y-1)) & \text{if } \lambda \in (0,\infty) \end{cases}$$

Then, $T_{\lambda_1}^{\mathbf{SW}} \gg T_{\lambda_2}^{\mathbf{SW}}$ if either $\lambda_1 \leq \min(1,\lambda_2)$ or $1 < \lambda_1 \leq \lambda_2 \leq 6.00914$. On the other hand if $T_{\lambda_1}^{\mathbf{SW}} \gg T_{\lambda_2}^{\mathbf{SW}}$, then $\lambda_1 \leq \lambda_2$.

A.3 Dominance between different families of continuous t-norms

We present three families of continuous t-norms, in which dominance between the members of different families can be characterized by the relationship between their parameters.

Dombi family

The Dombi t-norms family is given by,

$$(T_\lambda^{\mathbf{D}}(x,y))_{\lambda\in[0,\infty]} = \begin{cases} T_D(x,y) & \text{if } \lambda = 0, \\ T_M(x,y) & \text{if } \lambda = \infty, \\ \dfrac{1}{1+\left((\frac{1-x}{x})^\lambda + (\frac{1-y}{y})^\lambda\right)^{\frac{1}{\lambda}}} & \text{if } \lambda = (0,\infty) \end{cases}$$

Yager family

The Yager t-norms family is given by,

$$(T_\lambda^{\mathbf{Y}}(x,y))_{\lambda\in[0,\infty]} = \begin{cases} T_D(x,y) & \text{if } \lambda = 0, \\ T_M(x,y) & \text{if } \lambda = \infty, \\ \max\left(1-((1-x)^\lambda + (1-y)^\lambda)^{\frac{1}{\lambda}}, 0\right) & \text{if } \lambda = (0,\infty) \end{cases}$$

Aczel- Alsina family

The Aczel- Alsina t-norms family is given by,

$$(T_\lambda^{\mathbf{AA}}(x,y))_{\lambda\in[0,\infty]} = \begin{cases} T_D(x,y) & \text{if } \lambda = 0, \\ T_M(x,y) & \text{if } \lambda = \infty, \\ e^{-\left((-logx)^\lambda + (-logy)^\lambda\right)^{\frac{1}{\lambda}}} & \text{if } \lambda = (0,\infty) \end{cases}$$

We have the following Lemma regarding the dominance between members of preceding families.

Lemma A.3.1. *Let* $T_{\lambda_1} \in \{T^{\boldsymbol{D}}_{\lambda_1}, T^{\boldsymbol{Y}}_{\lambda_1}, T^{\boldsymbol{AA}}_{\lambda_1}\}$*, and* $T_{\lambda_2} \in \{T^{\boldsymbol{D}}_{\lambda_2}, T^{\boldsymbol{Y}}_{\lambda_2}, T^{\boldsymbol{AA}}_{\lambda_2}\}$*. Then,* $T_{\lambda_1} \gg T_{\lambda_2}$ *if and only if* $\lambda_1 \geq \lambda_2$*.*

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