MASTER

Sharpness of the phase transition for percolation with quasi-transitive inhomogeneities

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SHARPNESS OF THE PHASE TRANSITION FOR PERCOLATION WITH QUASI-TRANSITIVE INHOMOGENEITIES

Master's Thesis

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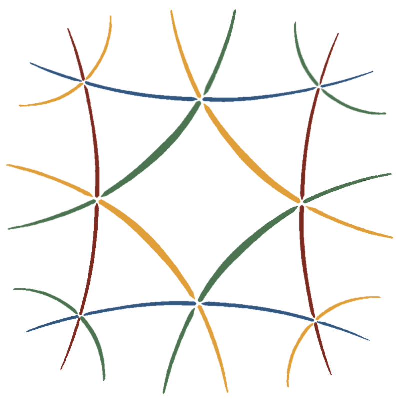

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ABSTRACT

We prove the sharpness of the phase transition for percolation on quasi-transitive graphs with quasi-transitive inhomogeneities. That is, we prove that the expected cluster size is finite in the subcritical regime. A quasi-transitive graph is an infinite graph with finitely many different types of vertices. Edges may be open with different probabilities, as long as the resulting graph is quasi-transitive. The proof is an extension of the proof for the sharpness of the phase transition for homogeneous percolation on vertex-transitive graphs by Duminil-Copin and Tassion [14]. The result in this thesis generalizes the result by Antunović and Veselić [2] for homogeneous percolation on quasi-transitive graphs to the inhomogeneous case and it generalizes the result by Menshikov [35] for inhomogeneous percolation on quasi-transitive graphs with sub-exponential growth to all quasi-transitive graphs.

Furthermore, we prove that the critical surface is Lipschitz continuous under some rather strong (possibly severe) restrictions on the parameters. From the Lipschitz continuity of the critical surface follows a specific lower bound in the supercritical regime on the probability that the origin is in an infinite open cluster.

We give an overview of the above mentioned proofs for the sharpness of the phase transition, as well as an overview of results in inhomogeneous percolation.

CONTENTS

1 INTRODUCTION

Imagine you are a fire fighter in California and you are trying the contain the biggest forest fire of the year. You didn't sleep in days, your feet hurt and your back aches from all the gear you have to carry all day. But more importantly you are completely frustrated because nothing you do stops the spread of the fire. The fire spreads quickly from tree to tree and before you now it the front of blazing trees is too large for you and your buddies to handle. But then the weather changes and the first rain in months falls from the sky. This is the helping hand that you need and after a couple of days the fire is finally under control. You come home and fall asleep on the couch with mixed feelings: you are content that the fire is finally extinguished but you know the wildfire claimed thousands of acres of forest. After sleeping no less than 18 hours you call your friend John for a well deserved beer. John, who is a mathematician, is not unfamiliar with the frustration caused by seemingly unsolvable problems. While enjoying your beer for breakfast you casually mention how the fire spread from tree to tree. And since the forest was dense enough, the entire area was ablaze sooner rather than later. Upon hearing this, John immediately recognizes this to be the same problem that he is working on: Percolation.

Percolation can be used to model forest fires, but, as with most things in mathematics, it has many more applications. The percolation model takes place on a large network consisting of nodes and links between these nodes. The network is called a graph, nodes are called vertices and links are called edges. The square lattice is commonly used as the graph in the percolation model. The square lattice is denoted by $\mathbb{Z}^2$ and is shown in Figure 1. This graph has infinitely many vertices and this is often the case for the graphs in a percolation model. Edges are randomly open or closed. We say that an edge is open with probability

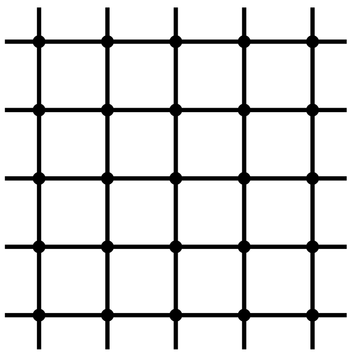
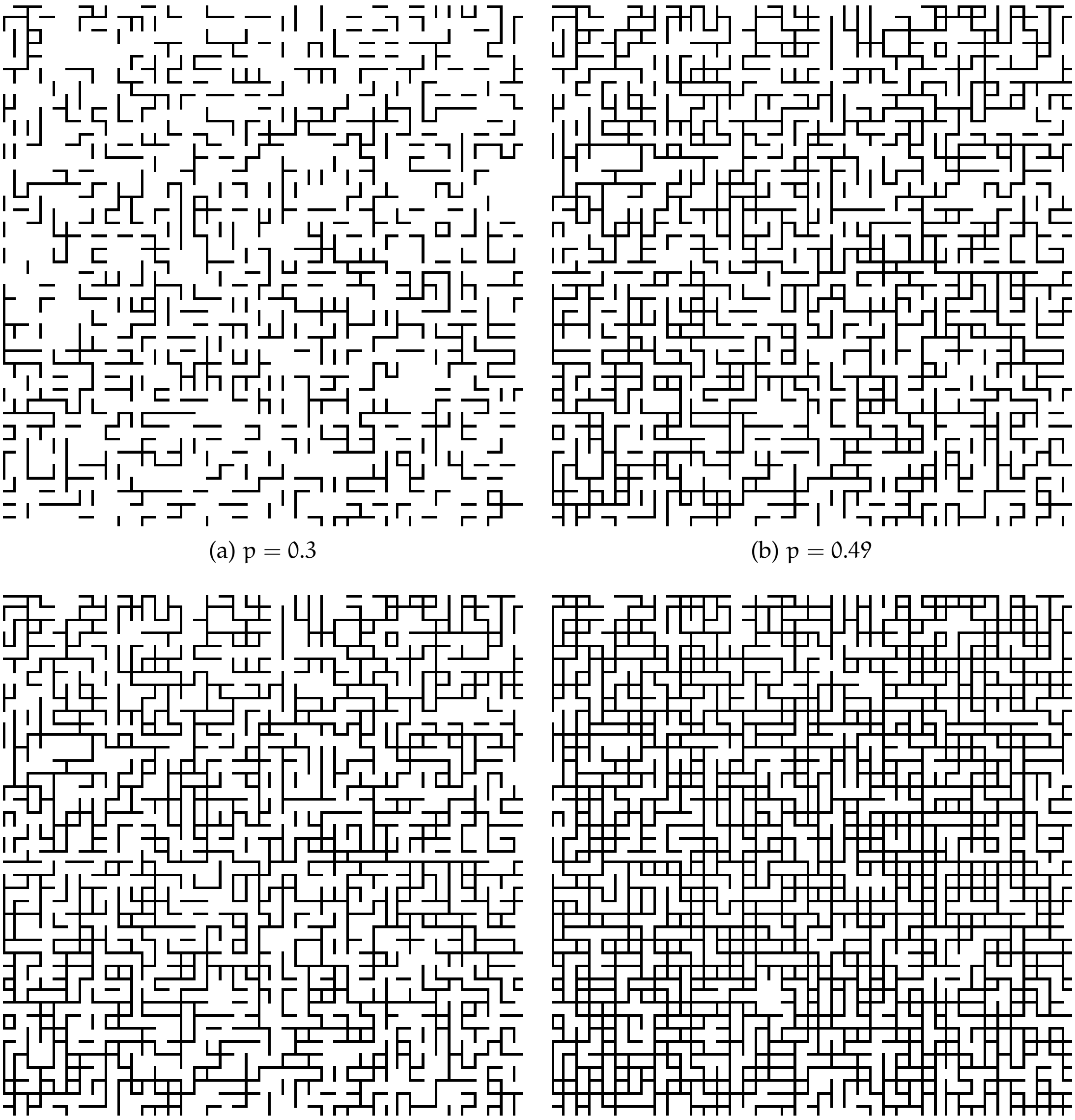


Figure 1: The square lattice

p and closed with probability $1-p$, independently of each other. This is the percolation model. If we only look at the open edges then we see a random subgraph of the original lattice. This subgraph is shown for the square lattice and for several values of p in Figure 2. Percolation theory deals with the properties of this random subgraph. When you look at



(a) $p = 0.3$ (b) $p = 0.49$

(c) $p = 0.51$ (d) $p = 0.7$

Figure 2: Percolation on $\mathbb{Z}^2$ for different values of p [16]

Figure 2 you might wonder whether there is a path of open edges from top to bottom. If you look closely you will see that such path does not exist for $p = 0.49$ and lower, but it does exist for $p = 0.51$. In fact, $p = 0.5$ turns out to be a very special value of the parameter for the square lattice. We will see this later. Remember however that we are working on an infinite graph and so the question whether there exists a path from top to bottom is not the right question to ask. We are better off asking if there exists an open path starting in a particular vertex that has infinite length. Or even better: what is the probability of having such a path and how does it depend on p? We will answer this question in Section 2.

Instead of letting the edges be open or closed we can also make the vertices open or closed. This gives a slightly different percolation model called site percolation. (The original model with open and closed edges is called bond percolation.) The site percolation model is well suited for the forest fire example. Open vertices can be associated with trees and closed vertices are associated with empty ground. If p is large there will be many open vertices and they will form large open clusters. If a tree is on fire then the entire cluster it belongs to will burn down as well. So in this case it is best to have a low value of p so that the clusters of trees are small. Does there exist a threshold value for p such that below this value there exists only small clusters of trees and above this threshold there is a large cluster of trees?

The bond percolation model on the other hand is well suited to model porosity of materials. It has been said that mathematicians convert coffee into theorems. So a good knowledge of brewing coffee is vital to mathematics, including knowledge of coffee filters. The filter has microscopic holes in random places. If we pour water onto the filter the holes on the upper side of the filter will get wet, but will the holes near the bottom of the filter get wet as well? In other words, will the water go through the filter? For this to happen there must be some path of holes from the top to the bottom of the filter. Such a path might exist if there are enough holes. Indeed, a hole corresponds to an open edge. If we assume that the holes are very small compared to the thickness of the filter, then we can model the filter as an infinite graph. So we arrive again at the problem of the existence of an infinite open path. This will depend on the density of the holes, which is modelled as the parameter p. Coffee filters are just an example of a porous material and many more porous media exist. The porosity of materials is for example also of interest in geology and building science.

Another application of percolation is found in polymers. Consider a solution of molecules, the monomers or building blocks of a polymer. These monomers will bond with each other and typically this will happen in a random manner. A single monomer can only bond with several other monomers, but together they can form a large cluster. Under the right conditions a very large cluster will be formed that extends through the entire solution. In that case a polymer (or gel) is formed. The transition from a solution of monomers to a very large cluster of monomers is called gelation. Percolation can be used to model this phenomenon and this has been done extensively in the literature. See for example [30] and [43].

Percolation is a half-century old branch of mathematics. The start of the field is commonly attributed to Broadbent and Hammersley [10] in 1957. They were the first to create the mathematical framework as well as introduce the name percolation. Their paper came forth from an interest in porosity as well as an intrinsic interest in the mathematical model itself.

CLASSICAL & NEW RESULTS 2

One of the main questions in percolation is for what values of the parameter there exists an infinite open cluster. Or equivalently, for what value of the parameter does there exist an infinite open path. To deal with this question we first have to give the formal definition of graphs as well as introduce some notation. The former is done in the next definition.

Definition 2.1. *Let V be a countably infinite set and let E be a set of 2-element subsets of V. The elements of V are called **vertices** and the elements of E are called **edges**. A graph G consists of these two sets: $G = (V, E)$. The **degree** of a vertex is the number of edges that the vertex is contained in.*

Throughout this thesis we assume that all graphs are locally finite, i.e., all vertices have finite degrees. We often fix a particular vertex of G that we call the origin or root of the graph. We denote this vertex by 0.

A percolation model consists of a graph $G = (V, E)$ and a percolation measure on G. In the case of the square lattice we have $V = \mathbb{Z}^2$ and E is the set of pairs of nearest neighbours of $\mathbb{Z}^2$. To formalize the probabilistic ideas we need a universe Ω along with a σ-algebra $\mathcal{F}$. We take $\Omega = \{0, 1\}^E$, so that a configuration $\omega \in \Omega$ sets each edge e to open, 1, or closed, 0, and this is denoted by $\omega(e)$. We then take the σ-algebra $\mathcal{F}$ to be the power set of Ω. Next we define the probability measure $\mathbb{P}$ for an event $A \in \mathcal{F}$. Let $p_e \in [0, 1]$ for all $e \in E$. This is the probability that the edge e is open. We define $\mathbb{P}$ to be the product measure:

$$\mathbb{P}(A) = \sum_{\omega \in A} \prod_{e \in E} p_e^{\omega(e)} (1 - p_e)^{1-\omega(e)}. \tag{2.1}$$

The product measure structure ensures that each edge is open or closed independently of each other. The measure $\mathbb{P}$ is a very general percolation measure. In this thesis we will study the special case where there are only finitely many different values for p_e. We divide the edge set into N disjoint subsets: $E = \cup E_i$. Each subset of E then has its own parameter p_i. We define $\mathbf{p} = (p_1, \ldots, p_{N-1})$, and $q = p_N$. The inhomogeneous percolation probability measure is now defined as

$$\mathbb{P}_{\mathbf{p},q}(A) = \sum_{\omega \in A} \prod_{i=1}^{N} \prod_{e \in E_i} p_i^{\omega(e)} (1 - p_i)^{1-\omega(e)}. \tag{2.2}$$

Another common special case of the percolation probability measure is the homogeneous measure $\mathbb{P}_p$, where each edge has the same probability of being open:

$$\mathbb{P}_p(A) = \sum_{\omega \in A} \prod_{e \in E} p^{\omega(e)} (1 - p)^{1-\omega(e)}. \tag{2.3}$$

This is the special case of the measure $\mathbb{P}_{,q}$ with $N = 1$. A percolation model with this measure is called homogeneous percolation. For example the homogeneous percolation model on the square lattice is the model $\mathscr{M} = (\mathbb{Z}^2, \mathbb{P}_p)$. Here we abuse notation and denote by $\mathbb{Z}^2$ both the vertex set, as well as the square lattice.

We use the notation $x \longleftrightarrow y$ for the event that there is a path of open edges from x to y. The event that there exists an infinite path of open edges starting in x is denoted by $x \longleftrightarrow \infty$. The percolation function $\theta(p)$ is given by

$$\theta(p) = \mathbb{P}_p(0 \longleftrightarrow \infty). \tag{2.4}$$

The origin is used in the definition of the percolation function, but this not crucial to the definition. For the square lattice for example any vertex could be substituted for the origin in the definition of $\theta(p)$, since every vertex of the square lattice plays the same role. This will be made more explicit in Section 3. The existence of an infinite open cluster is a tail event, since it depends on the state of infinitely many edges. So by applying the Kolmogorov zero-one law to this event we conclude that it must happen with probability zero or one. Furthermore if $\theta(p) > 0$, the probability of a having an infinite open cluster cannot be zero, so there exists an infinite open cluster almost surely. Similarly if $\theta(p) = 0$, then every vertex is almost surely not part of an infinite open cluster, so that an infinite open cluster exists with probability zero. So apparently $\theta(p) > 0$ and $\theta(p) = 0$ characterise two distinct cases in which the behaviour of the model is different. We say that percolation occurs in the former case, so whenever $\theta(p) > 0$.

Clearly $\theta(0) = 0$ and $\theta(1) = 1 > 0$. For larger values of p more edges will be open on average, so that the event $0 \longleftrightarrow \infty$ is more likely to occur. From this we conclude that $\theta(p)$ is increasing in p. (For a rigorous proof see [22, Theorem 2.1].) This implies that there exists some value of p, denoted by p_c such that

$$\theta(p) \begin{cases} = 0 & \text{if } p < p_c, \\ > 0 & \text{if } p > p_c. \end{cases} \tag{2.5}$$

We call this value of p the critical probability and it is defined as follows:

$$p_c = \inf\{p \in [0,1] : \theta(p) > 0\}. \tag{2.6}$$

2.1 SHARPNESS OF THE PHASE TRANSITION

The exact calculation of p_c turns out to be very difficult, and the exact value is only known on a few specific graphs. For percolation on a straight line $\mathbb{Z}$ there is a trivial critical probability $p_c = 1$, since for every $p < 1$ there will be infinitely many closed edges on either side of 0, so that no infinite open cluster exists. We can find more interesting behaviour on the square lattice $\mathbb{Z}^2$, in fact the critical probability on this graph equals $1/2$. Harris proved the fact that $p_c \geqslant 1/2$ in 1960 in [24], and twenty years later Kesten gave a proof for $p_c \leqslant 1/2$ in [28]. The proof makes use of several symmetries of the square lattice, which in general do not hold for other lattices.

So there is a non-trivial critical probability $0 < p_c < 1$ on the square lattice, and in fact it turns out that this is the case on $\mathbb{Z}^d$ for any $d \geqslant 2$. The square lattice $\mathbb{Z}^2$ is a subgraph of $\mathbb{Z}^d$ for any $d \geqslant 2$, so if an infinite open cluster exists almost surely on $\mathbb{Z}^2$ for some value of p, then it will also exist on $\mathbb{Z}^d$ almost surely for that value of p. So it must be the case that the critical probability on $\mathbb{Z}^d$, $p_c(\mathbb{Z}^d)$, satisfies $p_c(\mathbb{Z}^d) \leqslant p_c(\mathbb{Z}^2)$. On the other hand we can bound $p_c(\mathbb{Z}^d)$ from below by $1/2d$, by comparing the percolation process to a branching process: every vertex of $\mathbb{Z}^d$ has $2d$ neighbours. If we start at the origin and consider the neighbouring vertices that have an open edge to the origin to be the children of the origin, then we potentially have $2d$ children, each with probability p. Similarly, these children have at most $2d$ other children, and so on. So we find a branching process with critical probability $1/2d$, i.e., if $p > 1/2d$ then there exists a positive probability that the branching process has a total population of infinite size. However for percolation the analysis is not as simple, as vertices can be counted multiple times in this way. Nonetheless the population of the branching process will be at least as big as the size of the open cluster containing the origin. So this gives us the lower bound $1/2d$ for the critical probability of percolation on $\mathbb{Z}^d$. Combining this with the fact that $\mathbb{Z}^d \leqslant 1/2$ we see that it is indeed the case that there exists a non-trivial critical probability on $\mathbb{Z}^d$ for any $d \geqslant 2$.

From the non-trivial critical probability it follows that the model exhibits two distinct phases: a subcritical and a supercritical phase. For this reason we speak of a phase transition when p increases from below p_c to a value greater than p_c. Moreover, the critical case $p = p_c$ is a separate case from the subcritical and supercritical phases and also displays different behaviour.

Other properties of the percolation function have also been of interest, in particular continuity of $\theta(p)$. The percolation function is clearly continuous for $p < p_c$ and it has been proven by van den Berg and Keane that $\theta(p)$ is continuous on $p > p_c$ [7]. Furthermore $\theta(p)$ is right continuous everywhere. See [22, Chapter 8] for the these two results. So continuity of the percolation function boils down to the value of $\theta(p_c)$. If $\theta(p_c) = 0$ then the percolation function is continuous, otherwise it is not. This depends on the existence of an infinite of open cluster at criticality. For particular graphs this behaviour is known, in particular on $\mathbb{Z}^2$ and on $\mathbb{Z}^d$ for large d, but in general this is unknown. In fact this is often seen as the most important open problem in percolation theory.

Another way to characterize the behaviour of the system is by looking at the expected cluster size. Let $\mathscr{C}(x)$ be the open cluster that contains the vertex x, the cluster size is denoted by $|\mathscr{C}(x)|$. The susceptibility $\chi(p)$ is defined by

$$\chi(p) = \mathbb{E}_p|\mathscr{C}(0)|. \tag{2.7}$$

If $p > p_c$ it is clear that $\chi(p) = \infty$, but it is not immediately clear if the converse is true as well, i.e., does $p < p_c$ imply $\chi(p) < \infty$? We will investigate this further and define the critical value p_r for the susceptibility:

$$p_r = \inf\{p \in [0, 1] : \chi(p) = \infty\}. \tag{2.8}$$

From the above observation we get $p_r \leqslant p_c$. In fact, Aizenman and Barsky proved in 1987 in [1] that equality holds for percolation models with vertex-transitive graphs. A vertex-transitive graph has symmetry among all the vertices, the formal definition is given in Section 3. For example $\mathbb{Z}^d$ is a vertex-transitive graph, but there are many more vertex-transitive graphs.

Theorem 2.2 (Aizenman, Barsky, 1987)**.** *Let G be a vertex-transitive graph. The critical probability for $(G, \mathbb{P}_p)$ satisfies*

$$p_c = p_r. \tag{2.9}$$

This is known as the sharpness or uniqueness of the phase transition. A different proof was given by Menshikov in [35] for quasi-transitive graphs with sub-exponential growth. The concept of quasi-transitivity will be explained in Section 3, but in a nutshell it means that the graph has finitely many different types of vertices with respect to the neighbourhoods of these vertices. The proof by Menshikov is also valid for percolation models with the inhomogeneous measure $\mathbb{P}_{\mathbf{p},q}$.

Let the distance between two vertices $u, v \in V$ be denoted by $d(u, v)$, this is defined to be the number of edges in the shortest path from u to v in G. The set of vertices at distance at most k of u is denoted by Λ_k^u:

$$\Lambda_k^u := \{v \in V : d(u, v) \leqslant k\}. \tag{2.10}$$

The set of vertices at distance exactly k is then denoted by $\partial\Lambda_k^u := \Lambda_k^u \backslash \Lambda_{k-1}^u$. Menshikov proved the following theorem.

Theorem 2.3 (Menshikov, 1986)**.** *Let G be a quasi-transitive graph satisfying for all $k \in \mathbb{N}$: $|\Lambda_k^u| < c\exp(ak^\gamma)$, for some $c, a > 0$ and $0 < \gamma < 1$, and for any $u \in V$. Then the critical probability for $\left(G, \mathbb{P}_{\mathbf{p},q}\right)$ satisfies*

$$p_c = p_r. \tag{2.11}$$

The cubic lattice satisfies this sub-exponential growth condition, since for this graph $|\Lambda_k^u|$ grows polynomially in k. However trees for which vertices have a degree larger than 2 are excluded from the proof of Menshikov. For example the tree shown in Figure 3 for which every vertex has degree 4 has growth $|\Lambda_k^u| > 3^k$. There are many more examples of graphs

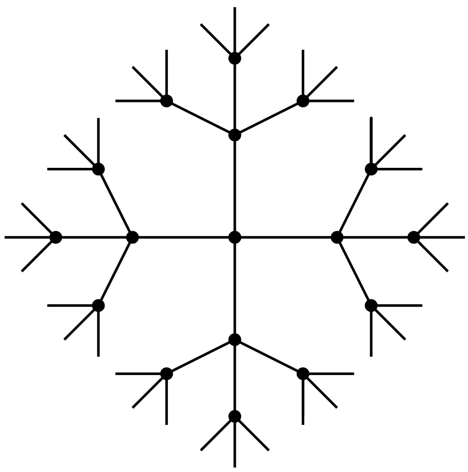


Figure 3: The 4-regular tree has exponential growth of $|\Lambda_k^u|$.

with exponential growth, we will discuss this in Section 4.

Antunović and Veselić later removed the sub-exponential growth condition for homogeneous percolation. By modifying the proof of Aizenman and Barsky in [2], they prove the following theorem.

Theorem 2.4 (Antunović, Veselić, 2008)**.** *Let* G *be a quasi-transitive graph. The critical probability for* $(G, \mathbb{P}_p)$ *satisfies*

$$p_c = p_r. \tag{2.12}$$

Recently Duminil-Copin and Tassion introduced a new proof for the sharpness of the phase transition in [15] for $\mathbb{Z}^d$ and in a more general setting in [14]. Their proof also gives insight in the behaviour of the system in the subcritical and supercritical phases.

Theorem 2.5 (Duminil-Copin, Tassion, 2015)**.** *If* G *is vertex-transitive the following three statements hold for* $(G, \mathbb{P}_p)$*:*

(a) If $p < p_c$*, then there exists a constant* $c > 0$ *such that for all* $k \in \mathbb{N}$

$$\mathbb{P}_p(0 \longleftrightarrow \partial\Lambda_k^0) \leqslant e^{-ck}. \tag{2.13}$$

(b) If $p < p_c$*, then the susceptibility* $\chi(p)$ *is finite, i.e.,*

$$\mathbb{E}_p|\mathscr{C}(0)| < \infty. \tag{2.14}$$

(c) If $p > p_c$*, then*

$$\mathbb{P}_p(0 \longleftrightarrow \infty) \geqslant \frac{p - p_c}{p(1 - p_c)}. \tag{2.15}$$

The sharpness of the phase transition follows immediately from Theorem 2.5(b). Note that the exponential decay in (2.13) is not enough to prove that the susceptibility is finite. The exponential decay of (2.13) only implies finite susceptibility on graphs with subexponential growth. The results proven by Duminil-Copin and Tassion in [14] are more general than the result stated above. They do not require the probability measure to be homogeneous, but they do require that each vertex has the same role. Edges can have different probabilities of being open, as long as the graph remains vertex-transitive. This is made more formal in 3.

The proof of Duminil-Copin and Tassion relies on a different characterisation of the critical point. For a finite $S \subset V$ we denote by ΔS the edge-boundary of S and we define

$$\phi_p(S) = p \sum_{\{y,z\} \in \Delta S} \mathbb{P}_p(0 \overset{S}{\longleftrightarrow} y), \tag{2.16}$$

where $x \overset{S}{\longleftrightarrow} y$ denotes the event that there exists a path from x to y using only open edges which have both endpoints in S. The alternative characterisation of the critical point is given by

$$\tilde{p}_c := \sup\left\{p \in [0,1] \,:\, \exists S \subset V \text{ with } 0 \in S,\ |S| < \infty \text{ and } \psi_{\mathbf{p},q}(x, S) < 1\right\}. \tag{2.17}$$

From the proof by Duminil-Copin and Tassion it follows that

$$p_c = \tilde{p}_c. \tag{2.18}$$

2.2 MAIN RESULTS

In this thesis we extend Theorem 2.5 to inhomogeneous percolation on quasi-transitive graphs. Thus we prove the sharpness of the phase transition for this model. This percolation model uses the probability measure $\mathbb{P}_{\mathbf{p},q}$. The details of this model are given in Section 6. Since we have multiple parameters we now have a critical surface instead of a critical point. This surface is defined as

$$q_c(\mathbf{p}) := \inf\{q \in [0,1] \, : \, \mathbb{P}_{\mathbf{p},q}(0 \longleftrightarrow \infty) > 0\}. \tag{2.19}$$

The following theorem is the main new contribution of this thesis.

Theorem 2.6. *Let* $G = (V, E)$ *be a quasi-transitive graph and let* $\{E_i\}_{i=1}^N$ *be a partition of* E. *Let* $p_i \in [0,1]$ *for all* $1 \leqslant i \leqslant N$ *and let* $\mathbb{P}_{\mathbf{p},q}$ *be as in (2.2). Then the following statements hold for* $(G, \mathbb{P}_{\mathbf{p},q})$*:*

(a) If $q < q_c(\mathbf{p})$*, then there exists a constant* $c > 0$ *such that for all* $v \in V$ *and for all* $k \in \mathbb{N}$

$$\mathbb{P}_{\mathbf{p},q}\left(v \longleftrightarrow \partial\Lambda_k^x\right) \leqslant e^{-ck}. \tag{2.20}$$

(b) If $q < q_c(\mathbf{p})$*, then the susceptibility is finite, i.e., for every* $v \in V$ *it holds that*

$$\mathbb{E}_{\mathbf{p},q}|\mathscr{C}(v)| < \infty. \tag{2.21}$$

Define the critical surface for the susceptibility

$$q_T(\mathbf{p}) = \inf\left\{q \in [0,1] \, : \, \exists\, v \in V \text{ with } \mathbb{E}_{\mathbf{p},q}|\mathscr{C}(v)| = \infty\right\}. \tag{2.22}$$

The sharpness of the phase transition for this model immediately follows from Theorem 2.6(b):

Corollary 2.7. *Consider the setting of Theorem 2.6. The critical probability for* $(G, \mathbb{P}_{\mathbf{p},q})$ *satisfies:*

$$q_c(\mathbf{p}) = q_T(\mathbf{p}). \tag{2.23}$$

The proof by Menshikov for the sharpness of the phase transition is also valid for inhomogeneous percolation, but as stated earlier only for graphs with sub-exponential growth. Our result does not rely on this condition; it is valid for all quasi-transitive graphs. So our proof extends the result of Duminil-Copin and Tassion, it generalises the result of Antunović and Veselić to inhomogeneous percolation, as well as generalise the result of Menshikov by removing the sub-exponential growth condition.

To summarise we give an overview of the different proofs for the sharpness of the phase transition in Table 1. We denote by $\mathscr{T}$ the set of vertex-transitive graphs, by $\mathscr{Q}$ the set of quasi-transitive graphs and by $\mathscr{Q}_s$ the set of quasi-transitive graphs with sub-exponential growth. A check mark denotes that the proof is valid for that particular percolation model.

Furthermore we prove that the critical surface q_c is Lipschitz continuous under some strong assumptions on the parameters. It is unclear if a percolation model exists that satisfies these assumptions. The Lipschitz continuity of q_c can be used to prove a lower bound

	$(\mathscr{T}, \mathbb{P}_p)$	$(\mathscr{Q}_s, \mathbb{P}_p)$	$(\mathscr{Q}, \mathbb{P}_p)$	$(\mathscr{Q}_s, \mathbb{P}_{\mathbf{p},q})$	$(\mathscr{Q}, \mathbb{P}_{\mathbf{p},q})$
Aizenman & Barsky	✓	×	×	×	×
Menshikov	✓	✓	×	✓	×
Antunović & Veselić	✓	✓	✓	×	×
Duminil-Copin & Tassion	✓	✓	✓	×	×
Thesis	✓	✓	✓	✓	✓

Table 1: Proofs for the sharpness of the phase transition

on $\mathbb{P}_{\mathbf{p},q}(0 \longleftrightarrow \infty)$ in the supercritical phase, similar to the bound in (2.15) for homogeneous percolation.

The concept of quasi-transitivity is explained in Section 3. Some background on percolation on different graphs is given in Section 4. We subsequently study the proofs of Duminil-Copin and Tassion, Aizenman and Barsky, and Menshikov in more detail in Section 8. Our results are then given in Section 6 and proofs can be found in subsequent sections. An overview of results in inhomogeneous percolation is given in Section 5.

In this section we take a short break from percolation and introduce a class of graphs that we use in percolation models later: quasi-transitive graphs. Informally speaking, these infinite graphs have a finite number of different types of vertices.

An automorphism of a graph is a permutation σ of the vertex set V such that $\{u, v\} \in E$ if and only if $\{\sigma(u), \sigma(v)\} \in E$. Because we are dealing with an infinite vertex set, the permutation σ should be seen as bijection from V to itself. So an automorphism permutes the vertices such that edges are preserved. And since the edges are preserved, the original graph and the graph obtained after applying the permutation are essentially the same. So if such an automorphism of G exists, then G exhibits some form of symmetry among the vertices. That is, there is symmetry between u and $\sigma(u)$. If this kind of symmetry exists between all the vertices of V we have a vertex-transitive graph. This is defined rigorously in the following definition.

Definition 3.1. *The graph* $G = (V, E)$ *is called* ***vertex-transitive*** *if for all* $u, v \in V$ *there exists a permutation* σ *of* V *such that* $\sigma(u) = v$ *and*

$$\{x, y\} \in E \iff \{\sigma(x), \sigma(y)\} \in E \qquad x, y \in V. \tag{3.1}$$

All vertices of a vertex-transitive graph are essentially the same. The cubic lattice, the triangular lattice and the hexagonal lattice are examples of infinite vertex-transitive graphs and these are shown in Figure 4.

We will now generalise the concept of vertex-transitivity to quasi-transitivity. We give each edge a colour from a set of N possible colours, with $N < \infty$. So E can be partitioned into N disjoint subsets:

$$E = \bigcup_{i=1}^{N} E_i \qquad \text{and} \qquad E_i \cap E_j = \emptyset \quad i \neq j. \tag{3.2}$$

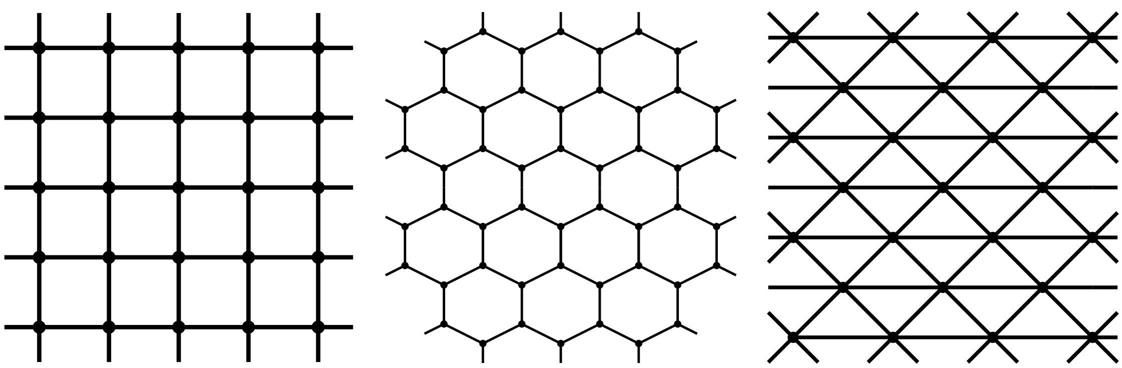


Figure 4: Examples of parts of vertex-transitive graphs

We define the edge-coloured graph

$$G_{\text{col}} = \big(V, \{E_1, \ldots, E_N\}\big). \tag{3.3}$$

For the edge-coloured graph we define the property of quasi-transitivity.

Definition 3.2. *The edge-coloured graph* G *is called* ***quasi-transitive*** *if there exists a finite subset* V_0 *of* V *such that*

$$\forall v \in V \quad \exists u \in V_0, \textit{ permutation } \sigma \textit{ of } V \textit{ such that } \sigma(v) = u \textit{ and } e \in E_i \iff \sigma(e) \in E_i, \tag{3.4}$$

where, with some abuse the notation, $\sigma(e) = \sigma(\{y, z\}) := \{\sigma(y), \sigma(z)\}$.

Quasi-transitivity can also be defined for monochrome graphs, simply by giving each edge the same colour. In that case our definition is equivalent to the standard definition of quasi-transitivity in the literature, see for example [5].

The property of quasi-transitivity can also be characterised in the following way.

Proposition 3.3. *The graph* G *is quasi-transitive if and only if there exists a* $k \in \mathbb{N}$ *and a partition of* V *into* k *disjoint subsets,* $V = \uplus_{i=1}^{k} V_i$ *such that*

$$\forall\, 1 \leqslant i \leqslant k,\ \forall\, u, v \in V_i \quad \exists \textit{ permutation } \sigma \textit{ of } V \textit{ s.t. } \sigma(u) = v \textit{ and } e \in E_i \iff \sigma(e) \in E_i. \tag{3.5}$$

Proof. Suppose G satisfies the condition in Definition 3.2. Fix the set $V_0 \subset V$. Define $k = |V_0|$ and order the elements of V_0 arbitrarily: $v_1, ..., v_k$. We can partition V in the following way. Define for $i = 1, \ldots, k$

$$V_i = \Big\{v \in V : \exists \text{ permutation } \sigma \text{ of } V \text{ s.t. } \sigma(v) = v_i \text{ and } e \in E_i \iff \sigma(e) \in E_i\Big\} \Big\backslash \bigcup_{j=1}^{i-1} V_j. \tag{3.6}$$

Then all V_i are disjoint and $V = \bigcup_{i=1}^{k} V_i$. Now let $u, v \in V_i$ for some i. Then there exist appropriate permutations σ_1 and σ_2 of V such that $\sigma_1(u) = v_i$ and $\sigma_2(v) = v_i$. Now define $\sigma = \sigma_2^{-1}\sigma_1$, then $\sigma(u) = v$ and $e \in E_i \iff \sigma(e) \in E_i$.

Now suppose G satisfies the condition in Proposition 3.3. Then we can partition V in disjoint subsets, $V = \bigcup_{i=1}^{k} V_i$, for some $k \in \mathbb{N}$. Define V_0 by taking an arbitrary element from each subset V_i, i.e., $V_0 = \bigcup_{i=1}^{k}\{v_i\}$ for some $v_i \in V_i$. Now let $v \in V$ be given, then $v \in V_j$ for some j and there exists a permutation σ of V such that $\sigma(v) = v_j \in V_0$ and $e \in E_i \iff \sigma(e) \in E_i$. □

Quasi-transitive graphs can thus be thought of as having a finite number of different types of vertices. The disjoint subsets of V in Proposition 3.3 are these different types. Note however that this partition of V is not unique. In the following example a graph is given that has clear symmetries, but is not quasi-transitive, precisely because it can be shown that it has infinitely many different types of vertices.

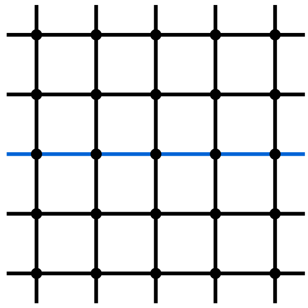


Figure 5: The cubic lattice with a lower-dimensional hyperplane

Example 3.4 (Hyperplane)**.** Consider the d-dimensional cubic lattice, $V = \mathbb{Z}^d$. In the lattice we embed an s-dimensional hyperplane $\mathbb{H}^s := \mathbb{Z}^s \times \{0\}^{d-s}$, for $s < d$. Let E_2 denote the edges in $\mathbb{H}^s$ and let E_1 be the set of all other edges. This graph is shown in Figure 5. We prove that this graph is not quasi-transitive. Suppose G is quasi-transitive, then we can use Proposition 3.3 to partition the vertices into k disjoint subsets $V = \bigcup_{i=1}^k V_i$. Let $e_d = (0, \dots, 0, 1)$ and define $\mathbb{I} := \{ie_d : i \in \mathbb{Z}\}$. For $u \in V$, denote by $\delta(u)$ the distance between u and the hyperplane. Then there exists a V_i such that there exist $u, v \in V_i \cap \mathbb{I}$ with $\delta(u) < \delta(v)$. Let σ be the permutation of V such that $\sigma(u) = v$ and $e \in E_i \iff \sigma(e) \in E_i$. So u moves away from the hyperplane under σ. If $u \in \mathbb{H}^s$ we find a contradiction, because the q-edges will not be preserved under σ. So assume $u \notin \mathbb{H}^s$. We can write $u = \delta(u)e_d$ and similarly $w := \big(\delta(u) - 1\big)e_d$, so that there is an edge between u and w. Since u moves away from the hyperplane, w has to move away from the hyperplane as well. Now if $w \in \mathbb{H}^s$ we again find a contradiction because the q-edges will not be preserved under σ. Otherwise we can repeat the argument to find a vertex $x \in \mathbb{H}^s$ that has to move away from the hyperplane under σ, which again gives a contradiction. We conclude that G is not quasi-transitive.

Vertices with different distances from the hyperplane are fundamentally different from each other. Each distance gives rise to a different type of vertex, which leads to infinitely many different types and so G cannot be quasi-transitive. This is also reflected in the proof. It does however suggest a way to alter this graph in such a way that it is quasi-transitive. This is shown in the next example.

Example 3.5 (Layers)**.** Consider again the d-dimensional cubic lattice, $V = \mathbb{Z}^d$. We now embed infinitely many parallel s-dimensional hyperplanes at distance K from neighbouring hyperplanes. The hyperplanes are indexed by $\mathbf{i} \in \mathbb{Z}^{d-s}$, denote them by $\mathbb{H}_{\mathbf{i}}^s := \mathbb{Z}^s \times \{i_1 K\} \times \cdots \times \{i_{d-s} K\}$. We again have two sets of edges, let E_2 denote the edges in $\mathbb{H}_{\mathbf{i}}^s$ for some $\mathbf{i}$ and let E_1 denote all other edges. A sketch of this graph is shown in Figure 6. We can show that this graph is quasi-transitive by showing that the graph satisfies the condition in Definition 3.2.

Define $V_0 := \{0\}^s \times \{0, 1, \dots, K-1\}^{d-s}$. Let $u = (u_1, \dots, u_d) \in V$ be given. Then $v = (v_1, \dots, v_d)$ defined by

$$v_i = \begin{cases} 0 & \text{if } i \leqslant s, \\ u_i \mod K & \text{if } i > s, \end{cases} \tag{3.7}$$

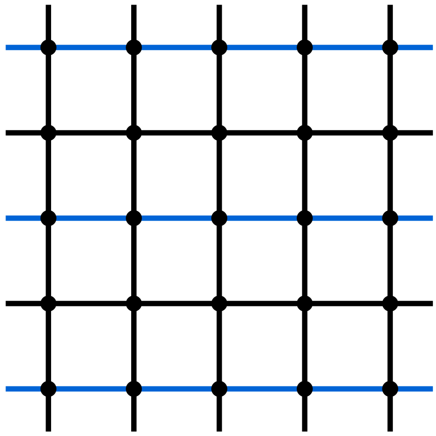


Figure 6: The cubic lattice with multiple lower-dimensional hyperplanes

is an element of V_0. Let σ be the permutation of ν that translates every vertex over $\nu - u$. Then $\sigma(u) = \nu$. Since the elements of $\nu - u$ are multiples of K, we have that for any $w \in \mathbb{H}^s_{\mathbf{i}}$ it holds that $\sigma(w) \in \mathbb{H}^s_{\mathbf{j}}$ for some $\mathbf{j}$. Furthermore if x and y are neighbours, $\sigma(x)$ and $\sigma(y)$ are neighbours as well. We conclude $e \in E_i \iff \sigma(e) \in E_i$ and thus the graph is quasi-transitive.

Several other examples of quasi-transitive graphs are shown in Figure 7.

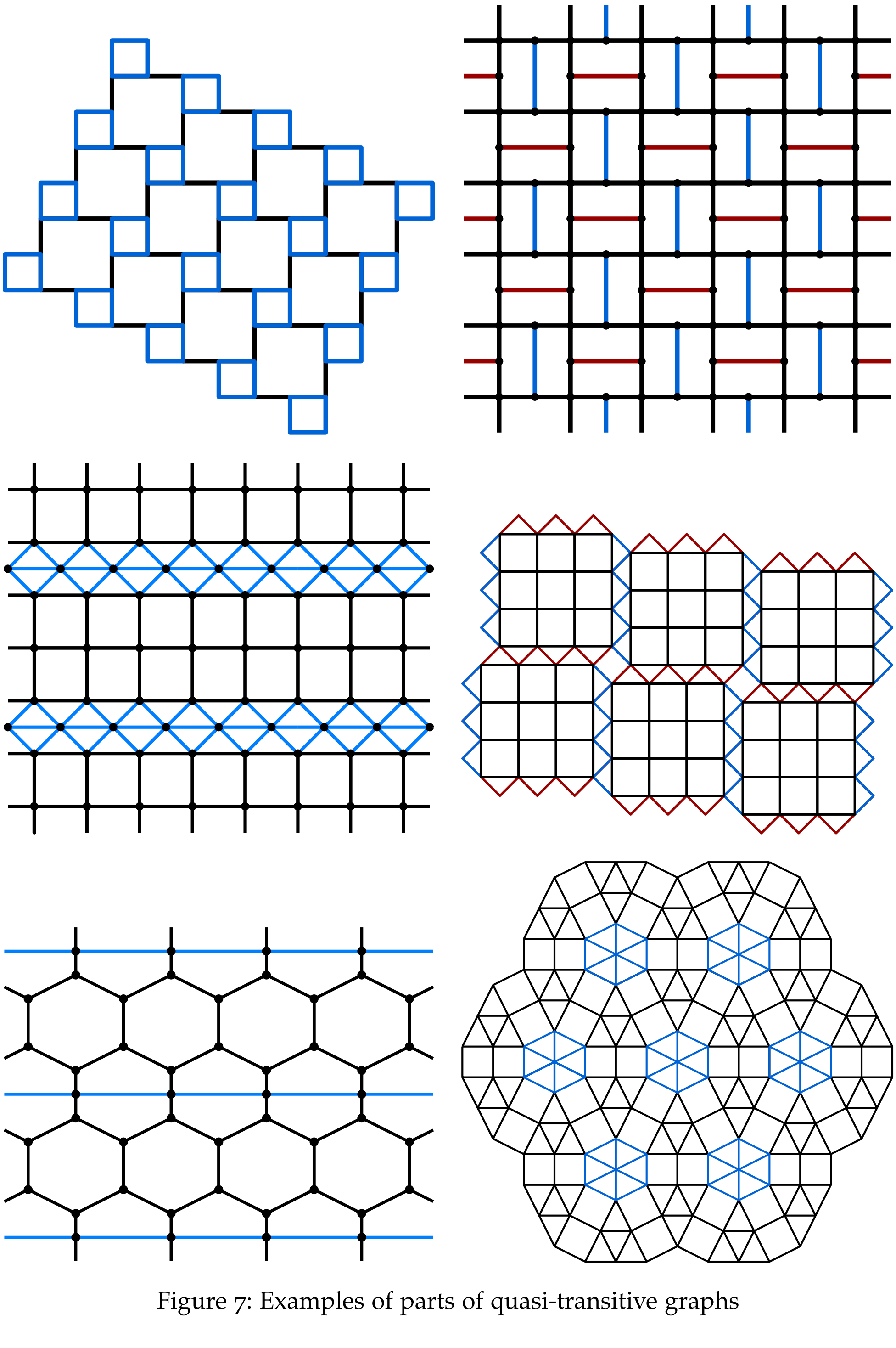


Figure 7: Examples of parts of quasi-transitive graphs

4 THE RELATION BETWEEN THE PERCOLATION MODEL AND ITS GRAPH

How does the structure of the graph affect the behaviour of the percolation model? In this section we will look at the relation between the percolation model and its graph.

4.1 PERCOLATION ON $\mathbb{Z}^d$

Many percolation theory results are known for the cubic lattice $\mathbb{Z}^d$. Furthermore $\mathbb{Z}^d$ is historically one of the first graphs to be studied in percolation theory. In this section we state some of the results that are known specifically for $\mathbb{Z}^d$. In fact, many results for the cubic lattice only hold for specific values of the dimension d.

Percolation on the line $\mathbb{Z}$ with nearest neighbour edges is the easiest case to consider, but not the most interesting one. If $p < 1$ there will be closed edges on either side of 0, so that 0 is not in an infinite cluster almost surely. Conversely if $p = 1$ all edges are open, so that 0 is in an open cluster. So we end up with a trivial critical probability $p_c = 1$. More interesting behaviour can be found in percolation models on $\mathbb{Z}$ with different edge sets. For example in long range percolation connection are possible between any two vertices.

The square lattice $\mathbb{Z}^2$ is arguably one of the most studied graph in percolation theory. The square lattice has some special properties and so a lot of results are known on $\mathbb{Z}^2$. Most notably, the square lattice is self-dual. The dual graph of a planar graph is obtained by taking a planar embedding of the graph. The dual has a vertex for every face of the planar embedding and an edge whenever two faces are separated by an edge in the original graph. The square lattice is a planar graph, so we can construct its dual in this way. This is shown in Figure 8. So we see that the dual of the square lattice is again the square lattice. Thus we say that the square lattice is self-dual. Self-duality is a very powerful property in percolation, because we can copy the percolation process to the dual. It can be used to prove $p_c = 1/2$, so that at criticality dual edges are closed with probability $1/2$ as well. This fact can then be used to prove the continuity of the percolation function: $\theta(p_c) = 0$.

These symmetry properties do not hold in higher dimensions. In fact not much is known for percolation on $\mathbb{Z}^d$ for $3 \leqslant d \leqslant 6$. This is the no man's land between the symmetry properties of the square lattice and the high-dimensional techniques for percolation on $\mathbb{Z}^d$ with $d \geqslant 7$.

On the cubic lattice with $d \geqslant 7$ the critical probability is low. Kesten proved in [29] that p_c is asymptotically equal to $1/2d$ as $d \to \infty$, i.e., $2dp_c \to 1$ as $d \to \infty$. So in high dimensions on average every vertex has about one open edge. This means that loops of open edges are unlikely, and loops are precisely what makes the analysis of percolation difficult. If there are no loops, than the model behaves as percolation on a tree, which is easier to analyse. Formalising this idea is a lot harder however. The existence of loops might have a low probability, but they are still possible. This leads to a perturbative analysis of the model. See [25] for an overview of results in high-dimensional percolation.

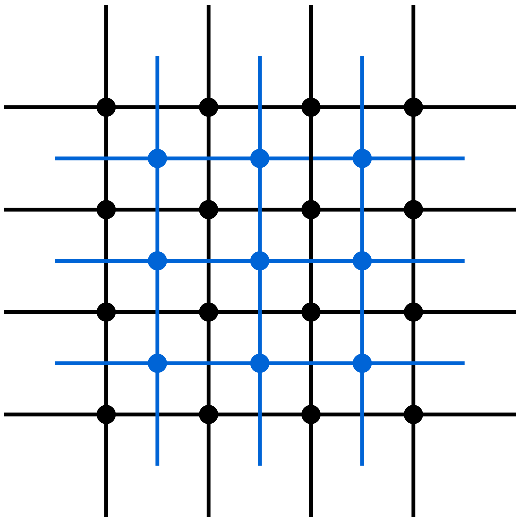


Figure 8: The square lattice and its dual in blue

4.2 PERCOLATION BEYOND $\mathbb{Z}^d$

The cubic lattice might be a commonly studied setting for percolation, but there exist many more possible percolation models that use different graphs. Some of these percolation models give rise to very different behaviour than the behaviour of percolation on $\mathbb{Z}^d$. In this section we will look at some of these graphs.

A graph is planar if it can be embedded in the plane such that no edges intersect each other. For example, a regular tree is planar because it can be drawn in the plane without edges crossing each other, as shown in Figure 3. Non-planar graphs are of interest as well, but less results are known for these graphs. Roughly speaking, every infinite, connected, planar, vertex-transitive graph is similar to exactly one of the following spaces: $\mathbb{Z}$, the tree with vertices of degree 3, $\mathbb{R}^2$, and the hyperbolic plane $\mathbb{H}^2$. This is made precise with the notion of quasi-isometry, see [3] and [6]. Two metric spaces are quasi-isometric if there exists a quasi-isometry between the two spaces.

Definition 4.1. *Let* (V_1, d_1) *and* (V_2, d_2) *be two metric spaces and let* f *be a function from* (V_1, d_1) *to* (V_2, d_2). *Then* f *is a* ***quasi-isometry*** *if there exists constants* $A \geqslant 1$ *and* $B, C \geqslant 0$ *such that*

$$\forall v, w \in V_1 : \frac{1}{A} d_1(v, w) - B \leqslant d_2\big(f(v), f(w)\big) \leqslant A d_1(v, w) + B, \tag{4.1}$$

and

$$\forall x \in V_2 \; \exists v \in V \text{ such that } d_2\big(x, f(v)\big) \leqslant C. \tag{4.2}$$

We use the graph distance as a metric on graphs. A quasi-isometry focusses on the large scale structure of the metric spaces and ignores the small scale structure. For example the square lattice is quasi-isometric to $\mathbb{R}^2$.

Definition 4.2. *A vertex-transitive graph* $G = (V, E)$ *has* ***one end*** *if for every finite* $S \subset V$ *there is precisely one infinite connected component in* $G \backslash S$.

The square lattice has one end, since the removal of a finite set of vertices can never break the graph into two or more unconnected infinite components. The removal of such a set can only disconnect a finite number of vertices. This is not true for $\mathbb{Z}$ however, because the removal of vertex will cause the graph to split into two infinite components, one on the left and one on the right of the removed vertex. Similarly a tree has more than one end, since the removal of some vertices might split the tree up into several infinite branches. Therefore an infinite connected planar vertex-transitive graph with one end can only be quasi-isometric to $\mathbb{R}^2$ or the hyperbolic plane $\mathbb{H}^2$.

Another important characteristic of graph is its growth rate: the speed at which $|\partial\Lambda_k|$ grows as k increases. The proof by Menshikov for the sharpness of the phase transition requires this speed to be sub-exponential. We will now make the notion or growth rate more formal. The Cheeger constant is defined as

$$H(G) := \inf_{\substack{S \subset V: \\ |S| < \infty}} \frac{|\partial S|}{|S|}, \tag{4.3}$$

where ∂V is the vertex boundary of V, i.e., the vertices in $V \backslash S$ that have a neighbour in S.

Definition 4.3. *An infinite graph G is* ***amenable*** *if $H(G) = 0$. Conversely, G is* ***nonamenable*** *if $H(G) > 0$.*

So this means that for large sets V in amenable graphs the boundary can become negligible. On the other hand the size of the boundary of a set V in nonamenable graphs is always larger than $c|V|$, for some constant $c > 0$. It immediately follows that for nonamenable graphs $|\partial\Lambda_k|$ grows exponential in k. The surface area of a ball in $\mathbb{R}^2$ grows quadratically in its radius, so a nonamenable graph cannot be quasi-isometric to $\mathbb{R}^2$. Therefore a nonamenable, connected, planar, vertex-transitive graph with one end has to be quasi-isometric to the hyperbolic plane $\mathbb{H}^2$. We call graphs quasi-isometric to $\mathbb{H}^2$ hyperbolic graphs.

The hyperbolic plane can be visualised using the unit disk. M.C. Escher has done this in an artistic manner in his Circle Limit Series. Circle Limit III is shown in Figure 9. Points are more spread out near the center and get more crowded towards the boundary of the disk. This is formalised in the Poincaré disk model. In fact the fish in Circle Limit III form a hyperbolic graph. This graph is vertex-transitive if we ignore the colour of the fish. However if we incorporate the colour of the fish into the graph we end up with a quasi-transitive coloured graph. There are four different types of vertices, since three different colours meet in every vertex. So the graph in Figure 9 is an example of a graph that is covered by the new result of this thesis, but is not covered by the existing proofs for the sharpness of the phase transition in the literature.

Percolation on trees is similar to a branching process, which makes it relatively easy to analyse. We will focus on hyperbolic graphs instead. Therefore the condition that the graph has one end is a natural requirement. We will now look at some results for these graphs. These results can be found in [6] and [31].

Nonamenable graphs have a non-trivial critical probability. In particular, Benjamini and Schramm proved the following theorem in [5].

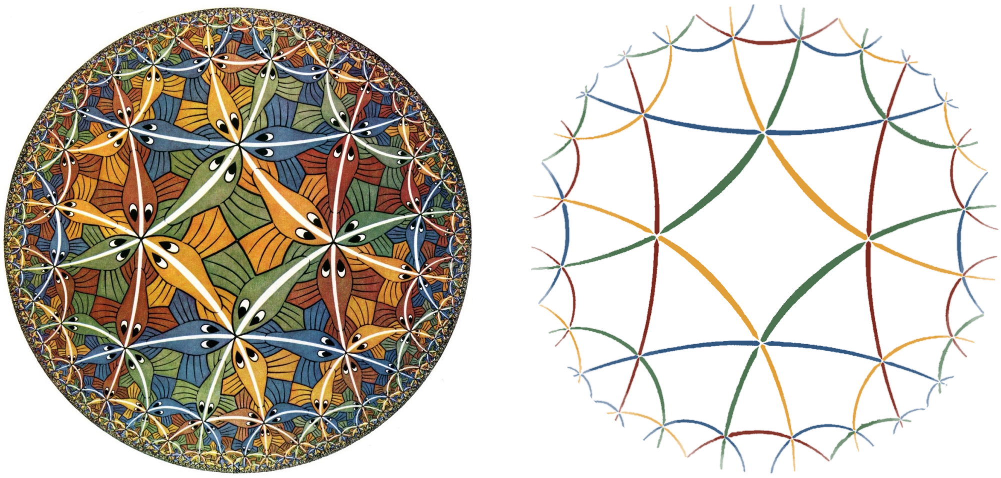


Figure 9: Circle Limit III by M.C. Escher and its underlying hyperbolic graph

Theorem 4.4 (Benjamini, Schramm). *Let* G *be a nonamenable graph. The critical probability for the model* $(G, \mathbb{P}_p)$ *satisfies*

$$p_c(G) \leqslant \frac{1}{1 + H(G)} < 1. \tag{4.4}$$

Burton and Keane showed in [11] that supercritical percolation on amenable vertex-transitive graphs has a unique infinite open cluster almost surely. It cannot be the case that there are two or more infinite open clusters in $\mathbb{Z}^d$. This contrasts with the result by Grimmett and Newman in [19] for the product of $\mathbb{Z}$ and some regular tree $\mathbb{T}$. They show that percolation on $\mathbb{T} \times \mathbb{Z}$ can have infinitely many infinite open clusters. In fact for any vertex-transitive graph there can only by 0, 1 or infinitely many infinite open clusters, as was shown by Newman and Schulman in [37]. Furthermore as p increases the model moves from no infinite open clusters to infinitely many infinite open clusters and finally to a unique infinite open cluster, possibly skipping the phase with infinitely many infinite open clusters. This follows from the result in [40] by Schonmann. Thus it is natural to define a new critical point p_u, the smallest value of p such that the model has a unique infinite open cluster almost surely:

$$p_u := \inf\{p \in [0, 1] \,:\, \text{there exists a unique infinite open cluster}\}. \tag{4.5}$$

On amenable vertex-transitive graphs there cannot be infinitely many infinite graphs, so that $p_c = p_u$ on these graphs. However Benjamini and Schramm show that for nonamenable planar vertex-transitive with one end this is not the case. They prove the following theorem in [6].

Theorem 4.5 (Benjamini, Schramm). *Let* G *be a planar, nonamenable, vertex-transitive graph with one end. Then the critical probabilities for the model* $(G, \mathbb{P}_p)$ *satisfy*

$$0 < p_c(G) < p_u(G) < 1. \tag{4.6}$$

Benjamini and Schramm further conjecture that $p_c < p_u$ holds for any nonamenable quasi-transitive graph.

The behaviour of percolation at the critical point has proven to be a hard problem. However for planar nonamenable vertex-transitive graphs with one end the following theorem is known.

Theorem 4.6 (Benjamini, Lyons, Peres, Schramm)**.** *Let* G *be a planar, nonamenable, vertex-transitive graph with one end. Then there is no infinite open cluster at* p_c*:*

$$\theta(p_c) = 0. \tag{4.7}$$

Benjamini, Lyons, Peres and Schramm prove the above theorem in greater generality in [4]. The behaviour at the other critical point $p = p_u$ is also known, see [6].

Theorem 4.7 (Benjamini, Schramm)**.** *Let* G *be a planar, nonamenable, vertex-transitive graph with one end and let* $p = p_u$*. Then there exists a unique infinite open cluster almost surely.*

It should be clear that different graphs can lead to very different behaviour of the associated percolation models. In the next section we will look at inhomogeneous percolation in some more detail.

INHOMOGENEOUS PERCOLATION 5

Inhomogeneous percolation is a generalisation of the percolation model in which edges do not all have the same probability of being open. This can be achieved in several ways and so there is not a single inhomogeneous percolation model. To stay in line with Section 3 on quasi-transitive graphs we give each edge a colour. This colour then determines the parameter of the edge, so the probability that the edge is open. In this way we can talk about the vertex-transitivity or quasi-transitivity of the coloured graph.

The parameters or colours of the edges can be chosen in such a way that the resulting coloured graph is vertex-transitive. For example by giving the horizontal edges of the square lattice one colour, and the vertical edges another. We will focus instead on models where the coloured graph is not vertex-transitive.

Why is inhomogeneous percolation of interest at all? As we have seen, percolation has many applications in physics. Physics deals with the real world and reality is not always as simple as we might like. The assumption that edges are open with the same probability everywhere might not be a very realistic, depending on the application. Recall that percolation theory can be used to model porous materials. It might be the case that the material is made up of two different substances, each with its own level of porosity. In that case an inhomogeneous percolation model would be more suitable than a homogeneous model.

As an example consider again the polymer gelation model. We can imagine that the solution of monomers is made up two different liquids that meet along a surface. Now the monomers might be more likely to bond along this surface than in one of the two liquids. So the corresponding percolation model would have a surface of edges which have a high probability of being open, while the bulk of the edges have a low probability of being open. In fact this is precisely the application and model described in [27].

Another application of inhomogeneous percolation might be found in superconductors. A lattice with layers such as in Example 3.5 may be used to model the BSCCO family of superconductors. The atomic structure of BSCCO is shown in Figure 10. The material consist of several layers of different molecules, and these layers appear in a regular fashion. A superconductor is a material that can conduct electricity without electrical resistance. They can be used to make powerful magnets, such as the ones found in MRI scanners or particle accelerators. Conventional superconductors work at temperatures below 30° K, or −243°C. However, BSCCO is a family of high-temperature superconductors and they work at temperatures around 100° K, or −173°C [34]. This makes cooling of the material much easier. Nitrogen has a boiling point of 77° K, so that liquid nitrogen can be used as a coolant of high-temperature superconductors. BSCCO is a brittle material, so this might be a reason to study a percolation model on this lattice. Alternatively, percolation theory on this lattice might be useful to gain a better understanding of the magnetic properties of the material. Percolation theory has strong ties with the Ising model, which is a model for magnetism.

There is one more reason to study inhomogeneous percolation other than applications in physics. The inhomogeneous model is thought to be useful for understanding critical

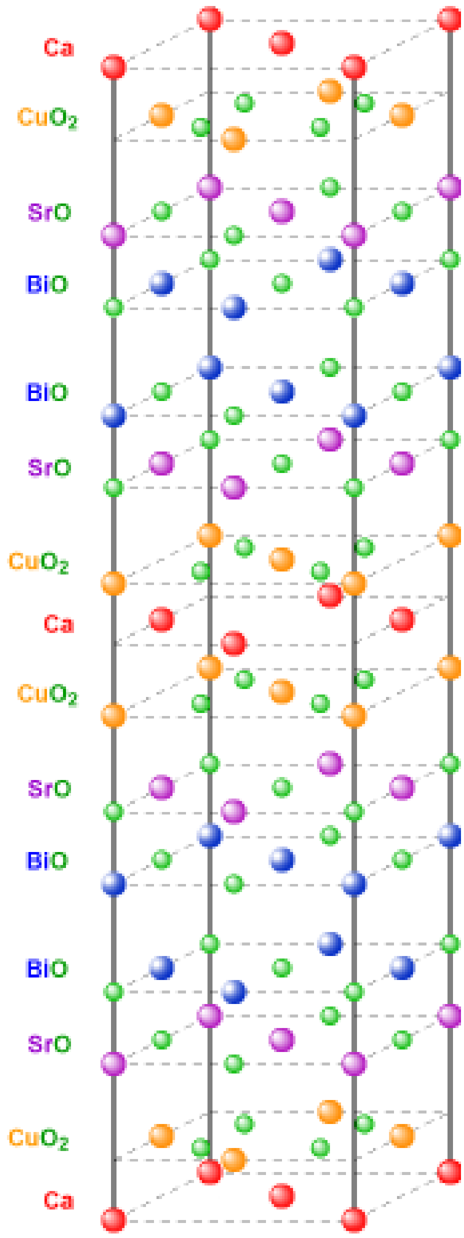


Figure 10: Atomic structure of the BSCCO superconductor [41]

homogeneous percolation. In particular it might be used to prove the existence of infinite open clusters at criticality, or lack thereof. This could give insight into the continuity of the percolation function defined in Section 2. So this is a purely theoretical motivation to look at inhomogeneous models.

In fact the first paper on inhomogeneous percolation, written by Chayes, Chayes and Durrett in 1987, had this motivation [13]. They studied percolation on $\mathbb{Z}^2$ with density $p(e) = p_c + |\bar{e}|^{-\lambda}$, where $\bar{e}$ is the midpoint of the two endpoints of the edge e. So that almost every edge had a different probability of being open. Furthermore the system is very close to criticality. They showed under some assumptions that there exists a critical value λ_c such that if $\lambda > \lambda_c$ there exists no infinite open cluster almost surely and if $\lambda < \lambda_c$ an infinite open cluster does exist almost surely. This gives some insight into the existence of an infinite open cluster at criticality, and the hope was to extend this proof to higher dimensions.

5.1 DEFECT PLANE PERCOLATION

In the subsequent years several other papers were published on inhomogeneous percolation and these publications share the same inhomogeneous model. This is bond percolation model on $\mathbb{Z}^d$ with two parameters. Similar to Example 3.4 we embed an s-dimensional hyperplane $\mathbb{H}^s := \mathbb{Z}^s \times \{0\}^{d-s}$ into the lattice. This hyperplane is often called a defect plane. The edge set is then divided into two separate sets: edges with both endpoints in $\mathbb{H}^s$ are open with probability q and all other edges with probability p. This is shown in Figure 11. We call this model the defect plane percolation model. This model was first

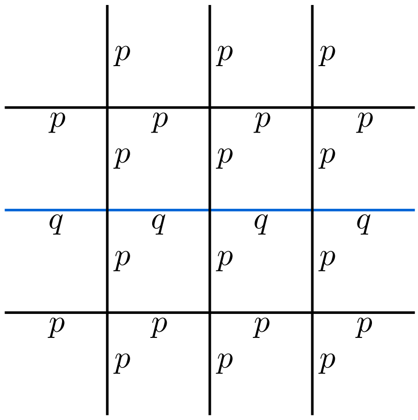


Figure 11: The inhomogeneous percolation model with a defect plane

introduced by Campanino and Klein in [12] in 1991, with $s = 1$, so with a line of defects. They prove the following theorem.

Theorem 5.1 (Campanino, Klein, 1991). *Consider defect plane percolation with* $s = 1$. *Then the following two statements hold for any* $d \geqslant 2$:

(a) If $p < p_c(\mathbb{Z}^d)$, *then* $\mathbb{P}_{p,q}(0 \longleftrightarrow \infty) = 0$, *for any* $q < 1$,

(b) If $p > p_c(\mathbb{Z}^d)$, *then* $\mathbb{P}_{p,q}(0 \longleftrightarrow \infty) > 0$, *for any* $q \geqslant 0$.

So percolation does not occur if $p < p_c$ for any $q < 1$, but if $p > p_c$ percolation does occur for every value of q. The case where $p = p_c$ was addressed by Zhang in [44] for $\mathbb{Z}^2$. He shows the following theorem.

Theorem 5.2 (Zhang, 1994). *Consider defect plane percolation with* $d = 2$ *and* $s = 1$ *and let* $p = p_c(\mathbb{Z}^2)$. *Then* $\mathbb{P}_{p,q}(0 \longleftrightarrow \infty) = 0$ *for any* $q < 1$.

This generalises the result that $\theta(p_c) = 0$ for $\mathbb{Z}^2$. Similarly if we can prove the same result for higher dimensions it would generalise $\theta(p_c) = 0$ in that dimension. Newman and Wu prove this for high dimensions in [38]:

Theorem 5.3 (Newman, Wu, 1997). *Consider defect plane percolation with* $d \geqslant 11$ *and* $s = 1$ *and let* $p = p_c$. *Then* $\mathbb{P}_{p,q}(0 \longleftrightarrow \infty) = 0$ *for any* $q < 1$.

Madras, Shinazi and Schonmann have also studied this model in [33] for the $s = 1$ case. However they prove a slightly different result: they prove that for a fixed $q > p_c$ the critical value for p remains unchanged. So if $p < p_c$ percolation does not occur, but for any $p > p_c$ an infinite open cluster exists.

Friedli, Ioffe and Velenik [18] study the rate of exponential decay along the defect line of $\mathbb{P}_{p,q}\big(0 \longleftrightarrow (n, 0, \dots, 0)\big)$ as n tends to infinity in the $s = 1$ case. In particular they studied how the rate of decay depends on q.

Newman and Wu also studied the $s \geqslant 2$ case [38]. They prove that in high dimensions and for $p = p_c$ there exists a non trivial critical value q_c for the edges in the defect plane. Define q_c by

$$q_c(p) = \inf\{q \in [0, 1] : \mathbb{P}_{p,q}(0 \longleftrightarrow \infty) > 0\}. \tag{5.1}$$

Newman and Wu prove:

Theorem 5.4 (Newman, Wu, 1997). *Consider defect plane percolation with* $d \geqslant 11$ *and* $2 \leqslant s \leqslant d - 3$ *and let* $p = p_c$. *Then the critical value* q_c *satisfies*

$$p_c(\mathbb{Z}^d) < q_c < p_c(\mathbb{Z}^s). \tag{5.2}$$

Recently Iliev, van Rensburg and Madras have also published results on this inhomogeneous percolation model with $s \geqslant 2$. They prove several properties of the critical probability $q_c(p)$. In particular they prove that $q_c(p)$ is a strictly decreasing function on the interval $[0, p_c]$, and they show that $q_c(p)$ is discontinuous at $p = p_c$. Furthermore they prove that whenever $q < q_c(p)$ that the susceptibility is finite:

Theorem 5.5 (Iliev, van Rensburg, Madras, 2015). *Consider defect plane percolation with* $d \geqslant 3$ *and* $2 \leqslant s < d$. *Suppose* $q < q_c$, *then there exist constant* $C, c > 0$ *such that*

$$\mathbb{P}_{p,q}(0 \longleftrightarrow \partial\Lambda_n^0) \leqslant Ce^{-cn}, \tag{5.3}$$

and

$$\mathbb{E}_{p,q}|\mathscr{C}(0)| < \infty. \tag{5.4}$$

So they prove the sharpness of the phase transition for this model. Conversely if $q > q_c(p)$, $\mathbb{P}_{p,q}\big(n \leqslant |\mathscr{C}(0)| < \infty\big)$ decays sub-exponentially. They give some numerical estimates of $q_c(p)$ as well.

PERCOLATION WITH QUASI-TRANSITIVE INHOMOGENEITIES 6

We now introduce our inhomogeneous percolation model and state our results for this model. These results are subsequently proven in the later sections. Most notably we prove the sharpness of the phase transition for this model.

6.1 THE INHOMOGENEOUS PERCOLATION MODEL

We first introduce the inhomogeneous percolation model. 'The setting is the graph $G = (V, E)$ as defined in Section 2. Thus V is countably infinite and G is locally finite. Furthermore we again give each edge a colour of a total of N possible colours and consider the coloured graph G_{col} as in (3.3). Lastly we require this graph to be quasi-transitive.

An edge $e \in E_i$ is open with probability p_i and closed with probability $1 - p_i$ independently of each other. So the colour of an edge determines the probability that that particular edge is open. This is our inhomogeneous bond percolation model. We write $\mathbf{p} = (p_1, \ldots, p_{N-1})$ and $q := p_N$. We denote a configuration of the edges by ω, which lies in the space $\Omega = \{0, 1\}^E$. We denote by $\omega(e)$ the state of edge e in the configuration ω. We take the σ-algebra $\mathcal{F}$ over Ω to be the power set of Ω. For an event $A \in \mathcal{F}$ we define the product measure $\mathbb{P}_{\mathbf{p},q}$ to be

$$\mathbb{P}_{\mathbf{p},q}(A) = \sum_{\omega \in A} \prod_{i=1}^{N} \prod_{e \in E_i} p_i^{\omega(e)} (1 - p_i)^{1-\omega(e)}. \tag{6.1}$$

We introduce some more notation that we will need later. For a set $S \subset V$ we write ΔS for its edge-boundary, i.e., $\{x, y\} \in \Delta S$ if $x \in S$ and $y \notin S$ and $\{x, y\} \in E$. Furthermore for $S \subset V$ we write $x \stackrel{S}{\longleftrightarrow} y$ whenever there is a path from x to y using only open edges which have both endpoints in S.

We are interested in the phase transition surface

$$q_c(\mathbf{p}) = \inf \left\{ q \in [0, 1] \,:\, \exists x \in V \text{ with } \mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \infty) > 0 \right\}. \tag{6.2}$$

The critical surface $q_c(\mathbf{p})$ is the generalisation of the critical probability p_c for homogeneous percolation defined in (2.6). Instead of just one critical value of the parameter, we now have a critical surface. We also define the susceptibility analogous to the homogeneous case:

$$\chi(\mathbf{p}, q) = \max_{x \in V} \mathbb{E}_{\mathbf{p},q} |\mathscr{C}(x)|. \tag{6.3}$$

This leads to the critical surface for the susceptibility:

$$q_T(\mathbf{p}) = \inf \left\{ q \in [0, 1] \,:\, \chi(\mathbf{p}, q) = \infty \right\}. \tag{6.4}$$

Since we require the coloured graph G_{col} to be quasi-transitive, our inhomogeneous percolation model does not generalise the defect plane percolation model given in Section 5.

This is because we have seen in Example 3.4 that $\mathbb{Z}^d$ with an embedded hyperplane is not a quasi-transitive graph. However if we embed infinitely many parallel equidistant hyperplanes into $\mathbb{Z}^d$ as in Example 3.5 we do end up with a quasi transitive graph. So our model generalises the $\mathbb{Z}^d$ model with layers, as well as percolation models on the graphs shown in Figure 7. Furthermore our model contains the inhomogeneous percolation models of Grimmett and Manolescu in [21].

6.2 SHARPNESS OF THE PHASE TRANSITION

Our aim is to show the sharpness of the phase transition, so to show that the two critical surfaces $q_c(\mathbf{p})$ and $q_r(\mathbf{p})$ are equal. In particular we aim to prove Theorem 2.6. Therefore we need insight into the behaviour of the model below the surface $q_c(\mathbf{p})$. This behaviour is established in the theorem below. The proof of the theorem relies on a different characterisation of the critical surface, which we will introduce first. We define for $x \in V$ and for a finite set $x \in S \subset V$

$$\psi_{\mathbf{p},q}(x,S) := \sum_{i=1}^{N-1} p_i \sum_{\substack{\{y,z\}\in \\ \Delta S\cap E_i}} \mathbb{P}_{\mathbf{p},q}\left(x \overset{S}{\longleftrightarrow} y\right) + q \sum_{\substack{\{y,z\}\in \\ \Delta S\cap E_N}} \mathbb{P}_{\mathbf{p},q}\left(x \overset{S}{\longleftrightarrow} y\right). \tag{6.5}$$

So $\psi_{\mathbf{p},q}$ is the analogue of ϕ_p defined in (2.16) for homogeneous percolation. Similarly the characterisation of the critical surface is analogous to (2.17):

$$\tilde{q}_c(\mathbf{p}) = \sup\left\{q \in [0,1] \,:\, \forall x \in V\, \exists S \subset V \text{ with } x \in S, |S| < \infty \text{ and } \psi_{\mathbf{p},q}(x,S) < 1\right\}, \tag{6.6}$$

or equivalently,

$$\tilde{q}_c(\mathbf{p}) = \sup_{x\in V} \inf_{\substack{S\subset V: \\ x\in S, |S|<\infty}} \sup\left\{q \in [0,1] \,:\, \psi_{\mathbf{p},q}(x,S) < 1\right\}. \tag{6.7}$$

The following theorem is the main new result of this thesis.

Theorem 6.1. *Let $G_{col} = \left(V, \{E_1, \dots, E_N\}\right)$ be an infinite, locally finite, quasi-transitive coloured graph with N colours, and let $\mathbf{p} \in [0,1]^{N-1}$. Suppose $q_c(\mathbf{p}) > 0$. Then for $(G_{col}, \mathbb{P}_{\mathbf{p},q})$ the following statements hold:*

(a) $q_c(\mathbf{p}) = \tilde{q}_c(\mathbf{p})$.

(b) If $q < q_c(\mathbf{p})$, then there exists a constant $c > 0$ such that for all $x \in V$ and for all $k \in \mathbb{N}$

$$\mathbb{P}_{\mathbf{p},q}\left(x \longleftrightarrow \partial\Lambda_k^x\right) \leqslant e^{-ck}. \tag{6.8}$$

(c) If $q < q_c(\mathbf{p})$, then the susceptibility $\chi(\mathbf{p},q)$ is finite, i.e., for every $x \in V$ it holds that

$$\mathbb{E}_{\mathbf{p},q}|\mathscr{C}(x)| < \infty. \tag{6.9}$$

From this theorem the sharpness of the phase transition immediately follows.

Corollary 6.2 (Sharpness of the phase transition). *Under the conditions of Theorem 6.1 it holds that*

$$q_c(\mathbf{p}) = q_r(\mathbf{p}). \tag{6.10}$$

Proof. Let $\mathbf{p} \in [0,1]^{N-1}$, for any $q > q_c(\mathbf{p})$ there exists some $x \in V$ with $\mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \infty) > 0$. It follows that $\chi(\mathbf{p},q) = \infty$ and hence $q_r(\mathbf{p}) \leqslant q_c(\mathbf{p})$. Conversely if $q > q_c(\mathbf{p})$, Theorem 6.1 (c) states that the susceptibility is finite, so that $q_r(\mathbf{p}) \geqslant q_c(\mathbf{p})$. □

The fact that the susceptibility is finite has to be proven separately; it does not follow from the exponential decay in (6.8) when G is nonamenable. If G is an amenable graph on the other hand, then finite susceptibility would follow from (6.8). In that case $|\partial\Lambda_k^x|$ would grow sub-exponential as k tends to infinity. So we obtain

$$\mathbb{E}_{\mathbf{p},q}|\mathscr{C}(x)| = \sum_{k=0}^{\infty} \sum_{v \in \partial\Lambda_k^x} \mathbb{P}_{\mathbf{p},q}(v \longleftrightarrow x) \leqslant \sum_{k=0}^{\infty} |\partial\Lambda_k^x| e^{-ck} < \infty. \tag{6.11}$$

6.3 FURTHER RESULTS

For particular values of $\mathbf{p}$ and q we can prove that the surface $q_c(\mathbf{p})$ is Lipschitz continuous. Let Δ be the maximum degree of G and denote by Δ_N the maximum degree of the graph (V, E_N).

Proposition 6.3 (Lipschitz Continuity). *Let $\delta_1, \delta_2, \varepsilon > 0$. The critical curve $q_c(\mathbf{p})$ is Lipschitz continuous on the set*

$$\mathscr{P} = \left\{ \mathbf{p} \in [0,1]^{N-1} \quad : \quad \forall i \; \delta_1 < p_i < \frac{1}{\Delta - 1} - \delta_2, \quad 0 < q_c(\mathbf{p}) < \frac{1}{\Delta_N} - \varepsilon \right\}, \tag{6.12}$$

i.e., there exists a constant C dependent on ε, δ_1 and δ_2 such that for any $i \in \{1, \dots, N-1\}$ and $\mathbf{p} \in \mathscr{P}$ we have

$$\left| \frac{\partial}{\partial p_i} q_c(\mathbf{p}) \right| \leqslant C. \tag{6.13}$$

The proof of this proposition is given in Section 10. We note that we have not been able to find an example for which we can prove that $\mathscr{P}$ is not empty.

In the case that the surface $q_c(\mathbf{p})$ is Lipschitz continuous we can give an explicit lower bound for $\mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \infty)$ in the supercritical regime. So in particular under the assumptions of Proposition 3.3 this lower bound holds.

Theorem 6.4. *Let $G_{col} = (V, \{E_1, \dots, E_N\})$ be an infinite, locally finite, quasi-transitive coloured graph with N colours and consider the percolation model $(G_{col}, \mathbb{P}_{\mathbf{p},q})$. Assume $q_c(\mathbf{p})$ is Lipschitz continuous on some domain $D \subset [0,1]^{N-1}$ with Lipschitz constant K. Let $\mathbf{p} \in D$ and suppose $q_c(\mathbf{p}) > 0$. Then if $q > q_c(\mathbf{p})$ it holds for all $x \in V$ that*

$$\mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \infty) \geqslant \frac{1}{4K+1}\big(q - q_c(\mathbf{p})\big). \tag{6.14}$$

6.4 DISCUSSION

There are several questions left unanswered about our inhomogeneous percolation model. These are some possible directions for future research. First of all, can we prove that $q_c(\mathbf{p})$ is Lipschitz continuous everywhere, instead of only on the set $\mathscr{P}$? This would imply the lower bound (6.14) for all parameter values in the supercritical phase.

The proof by Duminil-Copin and Tassion for the sharpness of the phase transition is also valid for the Ising model. This is a model for ferromagnetism. Since our proof is modification of their proof, it is reasonable to assume that we could adapt our proof to fit the Ising model.

We have seen that for homogeneous percolation on nonamenable graphs there can be an infinite number of infinite open clusters. It is likely that is also possible for inhomogeneous percolation on nonamenable graphs. In particular it is interesting to see how p_u, the smallest value of p such that there exists a unique infinite open cluster, would generalise to our inhomogeneous model. A generalisation could be

$$q_u(\mathbf{p}) := \inf\left\{q \in [0,1] \,:\, \text{there exists a unique infinite open cluster in } (G, \mathbb{P}_{\mathbf{p},q})\right\}. \tag{6.15}$$

How does this depend on $\mathbf{p}$, and how does this depend on the choice of the subset of the edges to take as q-edges?

7 SOME STANDARD TOOLS

Before we prove our new results we will look the existing proofs for the sharpness of the phase transition. Both the existing proofs and our new proofs rely on some standard techniques of percolation theory. We introduce three of these tools here. We start with a useful property of events that is a recurring theme in percolation theory. An event A is called increasing if it is more likely to happen whenever more edges are open. For configurations $\omega, \omega' \in \Omega$ we say that $\omega \leqslant \omega'$ whenever $\omega(e) \leqslant \omega'(e)$ for all $e \in E$, so that every open edge of ω is also open in ω'. The formal definition of an increasing event is as follows.

Definition 7.1. *An event A is called **increasing** if for all $\omega, \omega' \in \Omega = \{0,1\}^E$ with $\omega \leqslant \omega'$ it holds that $\mathbb{1}_A(\omega) \leqslant \mathbb{1}_A(\omega')$.*

From $\mathbb{1}_A(\omega) \leqslant \mathbb{1}_A(\omega')$ it follows that if A occurs for the configuration ω then it also occurs for ω'. Events such as the existence of an open path from a to b or the existence of an infinite open path from the origin are all increasing, so this property occurs often in percolation theory. We can be a bit more general by defining the same property for random variables.

Definition 7.2. *A random variable X is called **increasing** if $X(\omega) \leqslant X(\omega')$ whenever $\omega \leqslant \omega'$.*

We see right away that an event is increasing whenever its indicator function is increasing. The FKG inequality applies to increasing events, which states that these random variables are positively correlated. The inequality is named after Fortuijn, Kasteleyn and Ginibre who introduced the inequality in their 1971 paper [17].

Proposition 7.3 (FKG inequality)**.** *Let X and Y be two increasing random variables on a probability space with product measure $\mathbb{P}$. Suppose $\mathbb{E}[X^2] < \infty$ and $\mathbb{E}[Y^2] < \infty$, then*

$$\mathrm{Cov}(X, Y) \geqslant 0. \tag{7.1}$$

For the proof we refer to [22, p.35]. The inequality holds for all product measures, so in particular we can apply it the homogeneous measure $\mathbb{P}_p$ and the inhomogeneous measure $\mathbb{P}_{\mathbf{p},q}$. From the above inequality the next Corollary immediately follows.

Corollary 7.4 (Alternative form of the FKG inequality)**.** *Let A and B be two increasing events and let $\mathbb{P}$ be a product measure, then*

$$\mathbb{P}(A \cap B) \geqslant \mathbb{P}(A)\mathbb{P}(B). \tag{7.2}$$

Proof. The events A and B are increasing, so $\mathbb{1}_A$ and $\mathbb{1}_B$ are increasing random variables. Applying the FKG inequality on these indicators gives

$$\mathrm{Cov}(\mathbb{1}_A, \mathbb{1}_B) = \mathbb{E}[\mathbb{1}_A \mathbb{1}_B] - \mathbb{E}[\mathbb{1}_A]\mathbb{E}[\mathbb{1}_B] = \mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B) \geqslant 0. \tag{7.3}$$

□

The inequality in (7.2) is also known as the FKG inequality and this is a commonly used form in percolation theory. The FKG inequality gives a lower bound on the probability of the event $A \cap B$, but sometimes we require an upper bound instead. So a similar type of inequality that goes in the other direction would be useful. This gap is filled by the BK inequality, named after van den Berg en Kesten, who introduced the inequality in 1985 [8]. We cannot expect $\mathbb{P}_p(A \cap B) \leqslant \mathbb{P}_p(A)\mathbb{P}_p(B)$ to hold in general for increasing events, because combined with the FKG inequality this would imply that A and B are independent. Instead we will look at disjoint occurrences of A and B. Suppose A and B depend on the edges $e_1, \dots, e_n$ only. For a configuration $\omega \in \Omega$, define $K(\omega)$ by

$$K(\omega) = \{e_i : \omega(e_i) = 1, 1 \leqslant i \leqslant n\}, \tag{7.4}$$

the set of open edges in the configuration ω. Conversely a configuration ω' of the edges $e_1, \dots, e_n$ is uniquely determined by a set $K(\omega')$. We define $A \circ B$ to be the set of configurations ω for which we can partition $K(\omega)$ into two disjoint sets K' and K'', such that the configuration ω' determined by $K(\omega') = K'$ is an element of A and the configuration ω'' with $K(\omega'') = K''$ is an element of B. So we say that $A \circ B$ occurs whenever A and B occur on disjoint sets of edges. The BK inequality states the following about this event.

Proposition 7.5 (BK inequality). *Let A and B be two increasing events dependent on finitely many edges and let $\mathbb{P}$ be a product measure, then*

$$\mathbb{P}(A \circ B) \leqslant \mathbb{P}(A)\mathbb{P}(B). \tag{7.5}$$

The third standard tool that we introduce here is Russo's formula which also applies to increasing events. However, we first need to define the notion of a pivotal edge.

Definition 7.6. *An edge $e \in E$ is called* ***pivotal*** *for an event A and a configuration of the other edges $\omega \in \Omega$ whenever $\omega_o \in A$ and $\omega_c \notin A$, where ω_o and ω_c are the configurations obtained by taking the configuration ω and setting the edge e to open and closed, respectively.*

So an edge e is pivotal for A whenever its state decides the occurrence of the event A. Whether e is open or closed is irrelevant for e being pivotal. We now state the most general form of Russo's formula.

Proposition 7.7 (Russo's formula). *Let $f \in E$ and let $\mathbb{P}$ be the general percolation measure defined in (2.1). Consider the percolation model $(G, \mathbb{P})$. Let A be an increasing event dependent on finitely many edges. Then*

$$\frac{\partial}{\partial p_f}\mathbb{P}(A) = \mathbb{P}(f \textit{ is pivotal for } A). \tag{7.6}$$

Proof. Consider independent random variables $\{U(e)\}$ for every edge e, with uniform distribution on $[0, 1]$. We construct the configuration ω by setting an edge e to be open whenever $U(e) \leqslant p_e$, and to be closed otherwise. Let $f \in E$ and define

$$\mathbf{p}'(e) = \begin{cases} p'_f & \text{if } e = f, \\ p_e & \text{if } e \neq f. \end{cases} \tag{7.7}$$

The configuration ω' is constructed by setting an edge e to be open if and only if $U(e) \leqslant \tilde{\mathbf{p}}(e)$. Suppose $p'_f > p_f$. Then for an increasing event A we find

$$\begin{aligned}\mathbb{P}(A) - \mathbb{P}_{\mathbf{p}'}(A) &= \mathbb{P}_u(\omega \notin A, \omega' \in A)\\ &= (p'_f - p_f)\mathbb{P}(f \text{ is pivotal for } A) + O\big((p'_f - p_f)^2\big),\end{aligned} \tag{7.8}$$

where $\mathbb{P}_u$ is the measure induced by the random variables $\{U(e)\}$. By dividing by $p'_f - p_f$ and taking the limit $p'_f \to p_f$ we obtain

$$\frac{\partial}{\partial p_f}\mathbb{P}(A) = \mathbb{P}(f \text{ is pivotal for } A). \tag{7.9}$$

□

In our inhomogeneous percolation model multiple edges have the same parameter. This leads to an alternative form of Russo's formula.

Corollary 7.8 (Russo's Formula (Inhomogeneous case))**.** *Consider the percolation model* $(G, \mathbb{P}_{\mathbf{p},q})$*. Let* A *be an increasing event dependent only on edges in* Λ_n^v *for some* $n \in \mathbb{N}$ *and* $v \in V$*, then for any* $1 \leqslant i \leqslant N$ *we have*

$$\begin{aligned}\frac{\partial}{\partial p_i}\mathbb{P}_{\mathbf{p},q}(A) &= \sum_{e \in \Lambda_n^v \cap E_i} \mathbb{P}_{\mathbf{p},q}(e \textit{ is pivotal for } A)\\ &= \frac{1}{1-p_i}\sum_{e \in \Lambda_n^v \cap E_i} \mathbb{P}_{\mathbf{p},q}\big(\{e \textit{ is pivotal for } A\} \cap A^c\big).\end{aligned} \tag{7.10}$$

Proof. We can apply general form of Russo's formula to our inhomogeneous percolation model. Let A be an increasing event dependent only on the edges in Λ_n^x for some $n \in \mathbb{N}$ and $x \in V$. We use the Chain Rule to find for any $1 \leqslant i \leqslant N$,

$$\begin{aligned}\frac{\partial}{\partial p_i}\mathbb{P}_{\mathbf{p},q}(A) &= \sum_{e \in \Lambda_n^x} \frac{\partial}{\partial p_e}\mathbb{P}_{\tilde{\mathbf{p}}}(A)\frac{dp_e}{dp_i}\bigg|_{\tilde{\mathbf{p}}=(\mathbf{p},q)}\\ &= \sum_{e \in \Lambda_n^x \cap E_i} \frac{\partial}{\partial p_e}\mathbb{P}_{\tilde{\mathbf{p}}}(A)\bigg|_{\tilde{\mathbf{p}}=(\mathbf{p},q)}\\ &= \sum_{e \in \Lambda_n^x \cap E_i} \mathbb{P}_{\mathbf{p},q}(e \text{ is pivotal for } A)\\ &= \frac{1}{1-p_i}\sum_{e \in \Lambda_n^x \cap E_i} \mathbb{P}_{\mathbf{p},q}\big(\{e \text{ is pivotal for } A\} \cap A^c\big),\end{aligned} \tag{7.11}$$

where the final equality follows because an edge e is closed with probability $1 - p_i$, and if e is closed and pivotal for A, the event A will not happen. □

Finally we give a form of Russo's formula for the homogeneous percolation model. This form follows directly from the inhomogeneous case.

Corollary 7.9 (Russo's Formula (Homogeneous case)). *Consider the percolation model* $(G, \mathbb{P}_p)$. *Let* A *be an increasing event dependent on finitely many edges, then*

$$\begin{aligned} \frac{d}{dp}\mathbb{P}_p(A) &= \sum_{e \in E} \mathbb{P}_p(e \textit{ is pivotal for } A) \\ &= \frac{1}{1-p} \sum_{e \in E} \mathbb{P}_p\big(\{e \textit{ is pivotal for } A\} \cap A^c\big). \end{aligned} \tag{7.12}$$

8 SHARPNESS OF THE PHASE TRANSITION FOR HOMOGENEOUS PERCOLATION

As a preparation of our original proofs we will look at several existing proofs in the literature for the sharpness of the phase transition

$$p_c := \inf\{p \in [0,1] \,:\, \theta(p) > 0\} = \inf\{p \in [0,1] \,:\, \chi(p) = \infty\} =: p_r. \tag{8.1}$$

We will look at on of the first proofs of this fact by Aizenman and Barsky, as well as at the proof given by Menshikov around the same time. The other proof we look at here is the recent proof by Duminil-Copin and Tassion.

8.1 THE AIZENMAN AND BARSKY PROOF

Aizenman and Barsky showed the sharpness of the phase transition for homogeneous percolation on vertex-transitive graphs in [1]. In this section we state the main ideas of their proof, so we prove Theorem 2.2. Let G be a vertex-transitive graph and consider the homogeneous percolation model $(G, \mathbb{P}_p)$.

The mean finite cluster size $\chi^f(p)$ is defined as

$$\chi^f(p) = \mathbb{E}_p\left[|\mathscr{C}(0)| \,\middle|\, |\mathscr{C}(0)| < \infty\right]. \tag{8.2}$$

If $p < p_c$, then $\mathbb{P}(|\mathscr{C}(0)| = \infty) = 0$, so that $\chi(p) = \chi^f(p)$. So to show the sharpness of the phase transition it suffices to show that $\chi^f(p) < \infty$ for all $p < p_c$.

The Aizenman and Barsky proof relies on the magnetization defined as

$$M(p,\gamma) := \mathbb{E}_p\left[1 - (1-\gamma)^{|\mathscr{C}(0)|}\right] = \sum_{k=1}^{\infty} \left(1 - (1-\gamma)^k\right) \mathbb{P}_p\left(|\mathscr{C}(0)| = k\right). \tag{8.3}$$

We can interpret this quantity as follows. We colour each vertex green with probability γ independently of each other. Then the magnetization is equal to the probability that $\mathscr{C}(0)$ contains at least one green vertex. We can see this by conditioning on the size of $\mathscr{C}(0)$. Denote by $\mathscr{G}$ the random set of green vertices, then

$$\begin{aligned}
\mathbb{P}_p^\gamma\left(\mathscr{C}(0) \cap \mathscr{G} \neq \emptyset\right) &= \sum_{k=1}^{\infty} \mathbb{P}_p\left(\mathscr{C}(0) \cap \mathscr{G} \neq \emptyset \,\middle|\, |\mathscr{C}(0)| = k\right) \mathbb{P}_p\left(|\mathscr{C}(0)| = k\right) \\
&= \sum_{k=1}^{\infty} \left(1 - (1-\gamma)^k\right) \mathbb{P}_p\left(|\mathscr{C}(0)| = k\right) \\
&= M(p,\gamma).
\end{aligned} \tag{8.4}$$

From this insight we see that $M(p,\gamma)$ is increasing in p and γ. Furthermore $M(p,0) = 0$ and $M(p,1) = 1$. This interpretation of the magnetization can also be used to prove a set of differential inequalities for the magnetization.

Lemma 8.1 (Aizenman and Barsky differential inequalities). *If* $0 < p < 1$ *and* $0 < \gamma < 1$, *then*

$$(1-p)\frac{\partial M}{\partial p} \leqslant 2d(1-\gamma)M\frac{\partial M}{\partial \gamma}, \tag{8.5}$$

$$M \leqslant \gamma\frac{\partial M}{\partial \gamma} + M^2 + pM\frac{\partial M}{\partial p}. \tag{8.6}$$

We only prove inequality (8.5) here. The proof for the other differential inequality is similar and can be found in [1], see also [25].

Proof. We write $\{x \longleftrightarrow \mathscr{G}\} := \{\mathscr{C}(x) \cap \mathscr{G} \neq \emptyset\}$. We start by using Russo's formula on this event, to find

$$\begin{aligned}(1-p)\frac{\partial}{\partial p}M(p,\gamma) &= (1-p)\sum_{e\in E}\mathbb{P}_p^\gamma(e \text{ is pivotal for } 0 \longleftrightarrow \mathscr{G}) \\ &= \sum_{e\in E}\mathbb{P}_p^\gamma(e \text{ is closed and pivotal for } 0 \longleftrightarrow \mathscr{G}).\end{aligned} \tag{8.7}$$

The edge $\{x,y\}$ is closed and pivotal for $0 \longleftrightarrow \mathscr{G}$ if and only if $0 \longleftrightarrow x$, $y \longleftrightarrow \mathscr{G}$, and $0 \not\longleftrightarrow \mathscr{G}$. Thus,

$$(1-p)\frac{\partial}{\partial p}M(p,\gamma) = \sum_{\{x,y\}\in E}\mathbb{P}_p^\gamma(0 \longleftrightarrow x, y \longleftrightarrow \mathscr{G}, 0 \not\longleftrightarrow \mathscr{G}). \tag{8.8}$$

Conditioning on $\mathscr{C}(0)$ gives

$$(1-p)\frac{\partial}{\partial p}M(p,\gamma) = \sum_{\{x,y\}\in E}\sum_{C}\mathbb{P}_p^\gamma(C\cap\mathscr{G}=\emptyset, y \longleftrightarrow \mathscr{G} \,|\, \mathscr{C}(0)=C)\mathbb{P}_p^\gamma(\mathscr{C}(0)=C), \tag{8.9}$$

where we sum over all $C \subset V$ with $0, x \in C$, $y \notin C$ and $\mathbb{P}_p^\gamma(\mathscr{C}(0) = C) \neq 0$. If we know $\mathscr{C}(0) = C$ for a set of vertices C that does not contain y, then the events $\{C \cap \mathscr{G} = \emptyset\}$ and $\{y \longleftrightarrow \mathscr{G}\}$ are independent, since the first event only depends on the colour of the vertices in C, and $\{\mathscr{C}(y) \cap \mathscr{G} \neq \emptyset\}$ depends on the status of the edges and vertices outside C. Furthermore, conditionally on the event $\mathscr{C}(0) = C$, the cluster of y cannot contain any vertices of C, so the probability that the cluster of y contains a green vertex cannot increase:

$$\mathbb{P}_p^\gamma(y \longleftrightarrow \mathscr{G} \,|\, \mathscr{C}(0)=C) \leqslant \mathbb{P}_p^\gamma(y \longleftrightarrow \mathscr{G}) = M(p,\gamma). \tag{8.10}$$

Combining this with the conditional independence of $\{C \cap \mathscr{G} = \emptyset\}$ and $\{y \longleftrightarrow \mathscr{G}\}$ gives

$$\begin{aligned}(1-p)\frac{\partial}{\partial p}M(p,\gamma) &\leqslant \sum_{\{x,y\}\in E}\sum_{\substack{0,x\in C,\\ y\notin C}}\mathbb{P}_p^\gamma(C\cap\mathscr{G}=\emptyset \,|\, \mathscr{C}(0)=C)\mathbb{P}_p^\gamma(\mathscr{C}(0)=C)M(p,\gamma) \\ &\leqslant \sum_{\{x,y\}\in E}\mathbb{P}_p^\gamma(0 \not\longleftrightarrow \mathscr{G}, 0 \longleftrightarrow x, 0 \not\longleftrightarrow y)M(p,\gamma) \\ &\leqslant \sum_{\{x,y\}\in E}\mathbb{P}_p^\gamma(0 \not\longleftrightarrow \mathscr{G}, 0 \longleftrightarrow x)M(p,\gamma).\end{aligned} \tag{8.11}$$

The above inequality is now independent of y, so we can sum over the vertices instead over the edges. Every vertex has $2d$ edges, so changing the sum gives a factor $2d$:

$$(1-p)\frac{\partial}{\partial p}M(p,\gamma) \leqslant 2d \sum_{x \in V} \mathbb{P}_p^{\gamma}(0 \not\leftrightarrow \mathscr{G}, 0 \longleftrightarrow x) M(p,\gamma). \tag{8.12}$$

Now we observe that

$$\sum_{x \in V} \mathbb{P}_p^{\gamma}(0 \not\leftrightarrow \mathscr{G}, 0 \longleftrightarrow x) = \mathbb{E}_{p,\gamma}\left[|\mathscr{C}(0)|\mathbb{1}_{\{0 \not\leftrightarrow \mathscr{G}\}}\right], \tag{8.13}$$

and

$$\begin{aligned} \mathbb{E}_{p,\gamma}\left[|\mathscr{C}(0)|\mathbb{1}_{\{0 \not\leftrightarrow \mathscr{G}\}}\right] &= \sum_{k=1}^{\infty} k\mathbb{P}_p^{\gamma}\left(|\mathscr{C}(0)| = k, 0 \not\leftrightarrow \mathscr{G}\right) \\ &= \sum_{k=1}^{\infty} k(1-\gamma)^k \mathbb{P}_p^{\gamma}\left(|\mathscr{C}(0)| = k\right) \\ &= (1-\gamma)\frac{\partial}{\partial \gamma} M(p,\gamma). \end{aligned} \tag{8.14}$$

Combining this with (8.12) proves the desired differential inequality. □

The Aizenman and Barsky differential inequalities can then be used to prove a lower bound on the magnetization.

Proposition 8.2. *Let $0 < p < 1$ such that $\chi^f(p) = \infty$ and let $0 < \gamma < 1$, then there exists a constant C such that*

$$M(p,\gamma) \geqslant C\sqrt{\gamma}. \tag{8.15}$$

This bound can be proven by inserting (8.5) into (8.6) and integrating the obtained inequality. For a complete proof we again refer to [1] and [25].

We will now prove the sharpness of the phase transition from the above results. Let $p < p_c$ and suppose $\chi^f(p) = \infty$, we will arrive at a contradiction by showing that $\theta(b) > 0$ for all $b > p$. We rewrite the differential inequality (8.6) as follows.

$$\frac{1}{M(p,\gamma)}\frac{\partial M(p,\gamma)}{\partial \gamma} + \frac{1}{\gamma}\frac{\partial}{\partial p}\left(pM(p,\gamma) - p\right) \geqslant 0. \tag{8.16}$$

Let $a < b < p_c$. We integrate the above inequality over the rectangle defined by $a \leqslant p \leqslant b$ and $\varepsilon \leqslant \gamma \leqslant \delta$, with $0 < \varepsilon < \delta$:

$$\begin{aligned} 0 &\leqslant \int_a^b \int_\varepsilon^\delta \frac{1}{M(p,\delta)}\frac{\partial M(p,\delta)}{\partial \gamma}\, d\gamma\, dp + \int_\varepsilon^\delta \int_a^b \frac{1}{\gamma}\frac{1}{\partial p}\left(pM(p,\delta) - p\right) dp\, d\gamma \\ &= \int_a^b \left(\log M(p,\delta) - \log M(p,\varepsilon)\right) dp + \int_\varepsilon^\delta \frac{1}{\gamma}\left(bM(b,\gamma) - b - aM(a,\gamma) + a\right) d\gamma. \end{aligned} \tag{8.17}$$

Since $M(p,\gamma)$ is increasing in both p and γ, it is maximal in the point (b,δ). Combining this with the above inequality gives

$$\begin{aligned} 0 &\leqslant (b-a)\big(\log M(b,\delta) - \log M(a,\varepsilon)\big) + \log(\delta/\varepsilon)\big(bM(b,\delta) - b - aM(a,\varepsilon) + a\big) \\ &= (b-a)\frac{\log M(b,\delta) - \log M(a,\varepsilon)}{\log\delta - \log\varepsilon} + \big(bM(b,\delta) - (b-a)\big). \end{aligned} \tag{8.18}$$

We now take the limit $\varepsilon \to 0$. By Propostion 8.2 we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0}\frac{\log M(b,\delta) - \log M(a,\varepsilon)}{\log\delta - \log\varepsilon} &\leqslant \lim_{\varepsilon\to 0}\frac{\log M(b,\delta) - \log C\sqrt{\varepsilon}}{\log\delta - \log\varepsilon} \\ &\leqslant \lim_{\varepsilon\to 0}\frac{\log M(b,\delta) - \log C - 1/2\log\varepsilon}{\log\delta - \log\varepsilon} = \frac{1}{2}, \end{aligned} \tag{8.19}$$

so that

$$0 \leqslant \frac{1}{2}(b-a) + \big(bM(b,\delta) - (b-a)\big) = \big(bM(b,\delta) - \frac{1}{2}(b-a)\big). \tag{8.20}$$

Finally we take the limit $\delta \to 0$ and claim that $M(b,\delta) \to \theta(b)$ as $\delta \to 0$. We obtain the contradiction

$$\theta(b) \geqslant \frac{b-a}{2b} > 0, \tag{8.21}$$

and we conclude $\chi^f(a) < \infty$.

It remains to show that $M(p,\gamma) \to \theta(p)$ as $\gamma \to 0$. We use (8.4) to write

$$M(p,\gamma) = 1 - \mathbb{P}_p^\gamma\big(\mathscr{C}(0) \cap \mathscr{G} = \emptyset\big). \tag{8.22}$$

Conditioning on the size of $\mathscr{C}(0)$ gives

$$\begin{aligned} M(p,\gamma) &= 1 - \sum_{k=1}^{\infty}\mathbb{P}_p\big(\mathscr{C}(0)\cap\mathscr{G} = \emptyset \mid |\mathscr{C}(0)| = k\big)\mathbb{P}_p\big(|\mathscr{C}(0)| = k\big) \\ &= 1 - \sum_{k=1}^{\infty}(1-\gamma)^k\mathbb{P}_p\big(|\mathscr{C}(0)| = k\big). \end{aligned} \tag{8.23}$$

We now take the limit $\gamma \to 0$ and obtain

$$\lim_{\gamma\to 0} M(p,\gamma) = 1 - \sum_{k=1}^{\infty}\mathbb{P}_p\big(|\mathscr{C}(0)| = k\big) = 1 - \mathbb{P}_p\big(|\mathscr{C}(0)| < \infty\big) = \theta(p). \tag{8.24}$$

□

8.2 THE MENSHIKOV PROOF

In this section we give a sketch of the proof of Menshikov for the sharpness of the phase transition. So we give a sketch of the proof of Theorem 2.3. This proof is arguably the most technical of the three proofs, and so we will not go into full detail. Menshikov's proof can be found in [35] and [36], see also [22, Chapter 5.2]. Let G be a quasi-transitive graph with sub-exponential growth. The proof is valid for the inhomogeneous percolation model $(G, \mathbb{P}_{p,q})$, but we will only look at the proof in the homogeneous case $(G, \mathbb{P}_p)$.

Menshikov proves the result for site percolation. This is a percolation where the vertices are open or closed instead of the edges. The sharpness of the phase transition for bond percolation immediately follows from this proof as well: consider the bond percolation process on $G = (V, E)$. We can construct an equivalent site percolation model by defining the graph $G' = (V', E')$ as follows. Create a vertex $v_e \in V'$ for every edge e in E. There is an edge in E' between vertices v_e and v_f whenever e and f have an endpoint in common in G. The site percolation process on G' is now equivalent to the bond percolation process on G, since an open/closed vertex in G' is associated with an open/closed bond in G. Furthermore if G is a quasi-transitive graph with sub-exponential growth, then so is G'. So without losing generality, we will only consider site percolation for the remainder of the proof. Note that the tools introduced in Section 7 also hold in an analogous form for site percolation.

The proof is split into two parts. We prove that for any quasi-transitive graph and for $p < p_c$ that

$$\mathbb{P}_p(v \longleftrightarrow \partial\Lambda_n^v) < \exp\left(-c\frac{n}{\log n}\right), \tag{8.25}$$

for some constant $c > 0$ and any $v \in V$. From this bound the finite susceptibility will follow for graphs with subexponential growth. We will prove this first. Let G be a graph satisfying for all $v \in V$ and n large enough:

$$|\Lambda_n^v| < \exp(n^\gamma), \tag{8.26}$$

for some constant $0 < \gamma < 1$. It follows that

$$\mathbb{P}_p(v \longleftrightarrow \partial\Lambda_n^v) \geqslant \mathbb{P}\big(|\mathscr{C}(v)| > |\Lambda_n^v|\big) \geqslant \mathbb{P}_p\big(|\mathscr{C}(v)| > \exp(n^\gamma)\big), \tag{8.27}$$

so that

$$\mathbb{P}_p\big(|\mathscr{C}(v)| > k\big) \leqslant \mathbb{P}_p\left(v \longleftrightarrow \partial\Lambda^v_{(\log k)^{1/\gamma}}\right). \tag{8.28}$$

Using the bound in (8.25) gives

$$\mathbb{P}_p\big(|\mathscr{C}(v)| > k\big) \leqslant \exp\left(-c\gamma\frac{(\log k)^{1/\gamma}}{\log\log k}\right). \tag{8.29}$$

We conclude

$$\mathbb{E}_p\left[|\mathscr{C}(v)|\right] = \sum_{k=0}^{\infty} \mathbb{P}_p\big(|\mathscr{C}(v)| > k\big) \leqslant \sum_{k=0}^{\infty} \exp\left(-c\gamma\frac{(\log k)^{1/\gamma}}{\log\log k}\right) < \infty, \tag{8.30}$$

since $1/\gamma > 1$. So it follows that $p_c = p_T$.

We will now look at the proof for the bound in (8.25). Let $\mathscr{N}(v)$ denote the number of pivotal vertices for the event $0 \longleftrightarrow \partial\Lambda_n^v$. Applying Russo's formula on this event gives

$$\begin{aligned}\frac{d}{dp}\mathbb{P}_p(v \longleftrightarrow \partial\Lambda_n^v) &= \sum_{v\in V} \mathbb{P}_p(v \text{ is pivotal for } v \longleftrightarrow \partial\Lambda_n^v) \\ &= \frac{1}{p}\sum_{v\in V} \mathbb{P}_p(\{v \text{ is pivotal for } v \longleftrightarrow \partial\Lambda_n^v\} \cap v \longleftrightarrow \partial\Lambda_n^v) \\ &= \frac{1}{p}\sum_{v\in V} \mathbb{P}_p(v \text{ is pivotal for } v \longleftrightarrow \partial\Lambda_n^v \mid v \longleftrightarrow \partial\Lambda_n^v)\mathbb{P}_p(v \longleftrightarrow \partial\Lambda_n^v) \\ &= \frac{1}{p}\mathbb{E}_p\left[\mathscr{N}(v) \mid v \longleftrightarrow \partial\Lambda_n^v\right]\mathbb{P}_p(v \longleftrightarrow \partial\Lambda_n^v). \end{aligned} \tag{8.31}$$

This gives

$$\frac{d}{dp}\log \mathbb{P}_p(v \longleftrightarrow \partial\Lambda_n^v) = \frac{1}{p}\mathbb{E}_p\left[\mathscr{N}(v) \mid v \longleftrightarrow \partial\Lambda_n^v\right]. \tag{8.32}$$

Integrating the above equation between a and b for some $0 < a < b < 1$ gives

$$\log \mathbb{P}_b(v \longleftrightarrow \partial\Lambda_n^v) - \log \mathbb{P}_a(v \longleftrightarrow \partial\Lambda_n^v) = \int_a^b \frac{1}{p}\mathbb{E}_p\left[\mathscr{N}(v) \mid v \longleftrightarrow \partial\Lambda_n^v\right]dp, \tag{8.33}$$

so that

$$\begin{aligned}\mathbb{P}_a(v \longleftrightarrow \partial\Lambda_n^v) &= \mathbb{P}_b(v \longleftrightarrow \partial\Lambda_n^v)\exp\left(-\int_a^b \frac{1}{p}\mathbb{E}_p\left[\mathscr{N}(v) \mid v \longleftrightarrow \partial\Lambda_n^v\right]dp\right) \\ &\leqslant \mathbb{P}_b(v \longleftrightarrow \partial\Lambda_n^v)\exp\left(-\int_a^b \mathbb{E}_p\left[\mathscr{N}(v) \mid v \longleftrightarrow \partial\Lambda_n^v\right]dp\right). \end{aligned} \tag{8.34}$$

The proof now comes down to a proper estimation of $\mathbb{E}_p\left[\mathscr{N}(v) \mid v \longleftrightarrow \partial\Lambda_n^v\right]$. We only give a sketch here. If $p < p_c$, then the event $v \longleftrightarrow \partial\Lambda_n^v$ is unlikely to occur. So if it does happen it is reasonable to assume that the connection between v and $\partial\Lambda_n^v$ is sparse. There might be multiple paths between v and $\partial\Lambda_n^v$, but these paths likely have many vertices in common. Therefore the number of pivotal edges is roughly equal to the length of a path from v to $\partial\Lambda_n^v$. It follows that the number of pivotal vertices is more or less linear in n. Formalising this observation and a proper choice of a and b gives the bound in (8.25).

We give a simplified sketch here to make the upper bound in (8.25) plausible. Assume that $\mathbb{E}_p\left[\mathscr{N}(v) \mid v \longleftrightarrow \partial\Lambda_n^v\right] \geqslant n/\log n$. Then by setting $a = p$ and $b = 1$ we find

$$\mathbb{P}_p(v \longleftrightarrow \partial\Lambda_n^v) \leqslant \exp\left(-(1-p)\frac{n}{\log n}\right), \tag{8.35}$$

which is the same upper bound as in (8.25), so this completes the proof. □

8.3 THE DUMINIL-COPIN AND TASSION PROOF

In this section we introduce the proof of Duminil-Copin and Tassion for Theorem 2.5. From item (b) the sharpness of the phase transition immediately follows. Their proof can be found in [15] for $\mathbb{Z}^d$ and a more general proof in [14]. The latter proof is valid for percolation on any vertex-transitive graph and also holds for the Ising model. We restrict ourselves here to percolation on a general vertex-transitive graph.

The core idea of the proof is to characterise the critical probability in a new way. Before we can introduce this characterization we first need some notation. For a set $S \subset V$ we write ΔS for its edge-boundary, i.e., $\{x, y\} \in \Delta S$ if $x \in S$ and $y \notin S$ and $\{x, y\} \in E$. Furthermore for $S \subset V$ we write $x \overset{S}{\longleftrightarrow} y$ whenever there is a path from x to y using only open edges which have both endpoints in S. For a finite $S \subset V$ we define

$$\phi_p(S) = p \sum_{\{y,z\} \in \Delta S} \mathbb{P}_p(0 \overset{S}{\longleftrightarrow} y). \tag{8.36}$$

We now the define the alternative critical point

$$\tilde{p}_c := \sup\left\{p \in [0,1] \,:\, \exists S \subset V \text{ with } 0 \in S,\, |S| < \infty \text{ and } \phi_{p,q}(x, S) < 1\right\}. \tag{8.37}$$

If we can prove the three results of Theorem 2.5 with p_c replaced with $\tilde{p}_c$ we will immediately have shown that $\tilde{p}_c = p_c$, since item (a) ensures that $\theta(p) = 0$ for all $p < \tilde{p}_c$ and from item (c) it follows that $\theta(p) > 0$ for all $p > \tilde{p}_c$.

The first two items of Theorem 2.5 can be proven using the following lemma.

Lemma 8.3. *Let $G = (V, E)$, let $S \subseteq A \subseteq V$ and let $B \subseteq V$. Let $u \in S$ and suppose $B \cap S = \emptyset$. Then*

$$\mathbb{P}_p\left(u \overset{A}{\longleftrightarrow} B\right) \leqslant p \sum_{\{x,y\} \in \Delta S} \mathbb{P}_p\left(u \overset{S}{\longleftrightarrow} x\right) \mathbb{P}_p\left(y \overset{A}{\longleftrightarrow} B\right). \tag{8.38}$$

Proof. If there is a path of open edges from u to B then that path must use some edge from ΔS, since $u \in S$, but $B \cap S = \emptyset$. Let $\{x, y\}$ be the first edge of ΔS in this path. Then $u \overset{S}{\longleftrightarrow} x$ and $y \overset{A}{\longleftrightarrow} B$, and these two connections are disjoint. Furthermore the edge $\{x, y\}$ has to be open and this also occurs disjointly from these two connections. So the result follows from applying the union bound and the BK inequality twice. □

8.3.1 *Proof of Theorem 2.5 (a)*

We know that $p < \tilde{p}_c$, so there exists an $S \subset V$ with $0 \in S$ and $\phi(S) < 1 - \varepsilon$, for some $\varepsilon > 0$. We drop 0 from the notation: $\Lambda_n = \Lambda_n^0$. Let $L > 0$ be such that $S \subset \Lambda_{L-1}$. Applying Lemma 8.3 gives

$$\mathbb{P}_p\left(0 \longleftrightarrow \partial\Lambda_n\right) \leqslant p \sum_{\{x,y\} \in \Delta S} \mathbb{P}_p\left(0 \overset{S}{\longleftrightarrow} x\right) \mathbb{P}_p\left(y \longleftrightarrow \partial\Lambda_n\right). \tag{8.39}$$

If $y \longleftrightarrow \partial\Lambda_n$, it is connected to a vertex at distance at least $n - L$ of y, since y has distance at most L from 0. Therefore, by the by the vertex-transitivity of the graph we have

$$\begin{aligned} \mathbb{P}_p\left(0 \longleftrightarrow \partial\Lambda_n\right) &\leqslant p \sum_{\{x,y\} \in \Delta S} \mathbb{P}_p\left(0 \overset{S}{\longleftrightarrow} x\right) \mathbb{P}_p\left(0 \longleftrightarrow \partial\Lambda_{n-L}\right) \\ &= \phi_p(S) \mathbb{P}_p\left(0 \longleftrightarrow \partial\Lambda_{n-L}\right). \end{aligned} \tag{8.40}$$

Iteration gives the desired exponential decay.

8.3.2 *Proof of Theorem 2.5 (b)*

We have that $p < \tilde{p}_c$, so there exists an $S \subset V$ with $0 \in S$ and $\phi(S) < 1-\varepsilon$, for some $\varepsilon > 0$. Fix this set S. Let $u \in V$. Since the graph is vertex-transitive, there exists a graph automorphism sending 0 to u. Let S_u be the image of S under this automorphism. For a finite set $A \subset V$ define

$$\chi_p(A) := \max_{u \in V} \sum_{v \in V} \mathbb{P}_p\left(u \overset{A}{\longleftrightarrow} v\right). \tag{8.41}$$

Let $A \subset V$ be such that $S_u \subset A$ and $A \backslash S_u \neq \emptyset$. By Lemma 8.3 we know that

$$\mathbb{P}_p\left(u \overset{A}{\longleftrightarrow} v\right) \leqslant p \sum_{\{x,y\} \in \Delta S_u} \mathbb{P}_p\left(u \overset{S_u}{\longleftrightarrow} x\right) \mathbb{P}_p\left(y \overset{A}{\longleftrightarrow} v\right). \tag{8.42}$$

Summing this inequality over the vertices in $A \backslash S_u$ gives

$$\begin{aligned}\sum_{v \in A \backslash S_u} \mathbb{P}_p\left(u \overset{A}{\longleftrightarrow} v\right) &\leqslant \sum_{v \in A \backslash S_u} \sum_{\{x,y\} \in \Delta S_u} p\mathbb{P}_p\left(u \overset{S_u}{\longleftrightarrow} x\right) \mathbb{P}_p\left(y \overset{A}{\longleftrightarrow} v\right) \\ &\leqslant \phi_p(S) \sum_{u \in A \backslash S_u} \mathbb{P}_p\left(y \overset{A}{\longleftrightarrow} u\right) \\ &\leqslant \phi_p(S)\chi_p(A).\end{aligned} \tag{8.43}$$

We now add the vertices in S_u to the sum using the union bound:

$$\sum_{v \in A} \mathbb{P}_p\left(u \overset{A}{\longleftrightarrow} v\right) \leqslant |S| + \phi_p(S)\chi_p(A). \tag{8.44}$$

The right hand side of the above inequality is independent of u, so we can maximise over $u \in V$, keeping the bound intact:

$$\max_{u \in V} \sum_{v \in A} \mathbb{P}_p\left(u \overset{A}{\longleftrightarrow} v\right) \leqslant |S| + \phi_p(S)\chi_p(A). \tag{8.45}$$

Using the definition of $\chi_p(A)$ we conclude

$$\chi_p(A) \leqslant \frac{|S|}{1-\phi_p(S)} < \frac{|S|}{\varepsilon} < \infty. \tag{8.46}$$

Item (b) of Theorem 2.5 follows by letting A tend to V.

8.3.3 *Proof of Theorem 2.5 (c)*

Fix $n \in \mathbb{N}$. By Russo's formula we have that

$$\begin{aligned}\frac{d}{dp}\mathbb{P}_p(0 \longleftrightarrow \partial\Lambda_n) &= \sum_{e \in \Lambda_n} \mathbb{P}_p(e \text{ is pivotal for } 0 \longleftrightarrow \partial\Lambda_n) \\ &= \frac{1}{1-p} \sum_{e \in \Lambda_n} \mathbb{P}_p\left(\{e \text{ is pivotal for } 0 \longleftrightarrow \partial\Lambda_n\} \cap 0 \not\longleftrightarrow \partial\Lambda_n\right).\end{aligned} \tag{8.47}$$

We define the random set $\mathscr{S}$ as follows:

$$\mathscr{S} = \{v \in \Lambda_n : v \not\longleftrightarrow \partial\Lambda_n\}, \tag{8.48}$$

the set of vertices in Λ_n that are not connected to the boundary of Λ_n. If $0 \not\longleftrightarrow \partial\Lambda_n$ it is an element of $\mathscr{S}$. We can condition on the set $\mathscr{S}$ to obtain

$$\frac{d}{dp}\mathbb{P}_p(0 \longleftrightarrow \partial\Lambda_n) = \frac{1}{1-p} \sum_{\substack{S \subset \Lambda_n:\\ 0 \in S}} \sum_{e \in \Lambda_n} \mathbb{P}_p\big(\{e \text{ is pivotal for } 0 \longleftrightarrow \partial\Lambda_n\} \cap \mathscr{S} = S\big). \tag{8.49}$$

The edge boundary of $\mathscr{S}$ are closed edges of which the endpoint that is not in $\mathscr{S}$ is connected to $\partial\Lambda_n$ by a path of open edges (possibly of length 0). So the edges pivotal for $0 \longleftrightarrow \partial\Lambda_n$ are precisely the edges on the boundary of $\mathscr{S}$ of which the endpoint in $\mathscr{S}$ is connected to 0. Conditioning on $\mathscr{S} = S$ we obtain

$$\frac{d}{dp}\mathbb{P}_p(0 \longleftrightarrow \partial\Lambda_n) = \frac{1}{1-p} \sum_{\substack{S \subset \Lambda_n:\\ 0 \in S}} \sum_{\{x,y\} \in \Delta S} \mathbb{P}_p\left(0 \overset{S}{\longleftrightarrow} x, \mathscr{S} = S\right). \tag{8.50}$$

The occurrence of the event $\{\mathscr{S} = S\}$ can be determined from the configuration of the edges outside of S. This can be done by exploring from the boundary of Λ_n as follows. Set $\mathscr{T} = \emptyset$. Start from some vertex on the boundary of Λ_n and see which vertices inside Λ_n it is connected to using only edges in $\Lambda_n \backslash S$. Add these vertices to $\mathscr{T}$. We subsequently do this for every vertex on the boundary of Λ_n, so the set $\mathscr{T}$ contains vertices in Λ_n that are connected to the boundary of Λ_n. Now if $\mathscr{T}$ contains some vertex that is in S, then $\mathscr{S} \neq S$, and similarly if $\mathscr{T}$ does not contain some vertex that is not in S then $\mathscr{S} \neq S$. The only remaining case is $\mathscr{T} = \Lambda_n \backslash S$. We can split S into two disjoint subsets S_i and S_b, where S_i are the vertices in S that only have neighbours inside S and S_b is the set of vertices that have some neighbour outside S. If $\mathscr{T} = \Lambda_n \backslash S$, then all vertices in S_b are not connected to the boundary of Λ_n. Otherwise some vertex of S_b would have been added to $\mathscr{T}$ in the exploration process. Similarly all vertices in S_i are not connected to the boundary of Λ_n, otherwise $\mathscr{T}$ would contain some vertex of S_b. So if $\mathscr{T} = \Lambda_n \backslash S$, then $\mathscr{S} = S$. We conclude that the event $\{\mathscr{S} = S\}$ is determined by the configuration of the edges outside S, and is therefore independent of $\{0 \overset{S}{\longleftrightarrow} x\}$. We find

$$\begin{aligned}
\frac{d}{dp}\mathbb{P}_p(0 \longleftrightarrow \partial\Lambda_n) &= \frac{1}{1-p} \sum_{\substack{S \subset \Lambda_n:\\ 0 \in S}} \sum_{\{x,y\} \in \Delta S} \mathbb{P}_p\left(0 \overset{S}{\longleftrightarrow} x\right) \mathbb{P}_p(\mathscr{S} = S) \\
&= \frac{1}{p(1-p)} \sum_{\substack{S \subset \Lambda_n:\\ 0 \in S}} \phi_p(S) \mathbb{P}_p(\mathscr{S} = S) \\
&\geqslant \frac{1}{p(1-p)} \inf_{\substack{S \subset \Lambda_n:\\ 0 \in S}} \phi_p(S)\, \mathbb{P}_p(0 \not\longleftrightarrow \partial\Lambda_n).
\end{aligned} \tag{8.51}$$

Since $p > \tilde{p}_c$, we know that $\phi_p(S) \geqslant 1$ for all S, so

$$\frac{d}{dp}\mathbb{P}_p(0 \longleftrightarrow \partial\Lambda_n) \geqslant \frac{1}{p(1-p)}\big(1 - \mathbb{P}_p(0 \longleftrightarrow \partial\Lambda_n)\big), \tag{8.52}$$

and

$$-\frac{d}{dp}\log\left(1-\mathbb{P}_p(0\longleftrightarrow\partial\Lambda_n)\right)=\frac{\frac{d}{dp}\mathbb{P}_p(0\longleftrightarrow\partial\Lambda_n)}{1-\mathbb{P}_p(0\longleftrightarrow\partial\Lambda_n)}\geqslant\frac{1}{p(1-p)}=\frac{1}{p}+\frac{1}{1-p}. \quad (8.53)$$

We now integrate this inequality between $\tilde{p}_c$ and p to obtain

$$\log\left(1-\mathbb{P}_{\tilde{p}_c}(0\longleftrightarrow\partial\Lambda_n)\right)-\log\left(1-\mathbb{P}_p(0\longleftrightarrow\partial\Lambda_n)\right)\geqslant\log(p/\tilde{p}_c)-\log\left(\frac{1-p}{1-\tilde{p}_c}\right), \quad (8.54)$$

so that

$$-\log\left(1-\mathbb{P}_p(0\longleftrightarrow\partial\Lambda_n)\right)\geqslant\log(p/\tilde{p}_c)-\log\left(\frac{1-p}{1-\tilde{p}_c}\right). \quad (8.55)$$

It follows that

$$\frac{1}{1-\mathbb{P}_p(0\longleftrightarrow\partial\Lambda_n)}\geqslant\frac{p(1-\tilde{p}_c)}{\tilde{p}_c(1-p)}, \quad (8.56)$$

and finally

$$\mathbb{P}_p(0\longleftrightarrow\partial\Lambda_n)\geqslant 1-\frac{\tilde{p}_c(1-p)}{p(1-\tilde{p}_c)}=\frac{p-\tilde{p}_c}{p(1-\tilde{p}_c)}. \quad (8.57)$$

The final result follows from letting n tend to infinity. □

9 PROOF OF THE SHARPNESS OF THE PHASE TRANSITION FOR PERCOLATION WITH QUASI-TRANSITIVE INHOMOGENEITIES

In this section we prove Theorem 6.1. If we show Theorem 6.1 (b) for $q < \tilde{q}_c(\mathbf{p})$ as well as prove that for some $x \in V$

$$\mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \infty) > 0 \qquad \forall\, q > \tilde{q}_c(\mathbf{p}), \tag{9.1}$$

then we will immediately have shown that $\tilde{q}_c(\mathbf{p}) = q_c(\mathbf{p})$. Since if (6.8) holds for $q < \tilde{q}_c(\mathbf{p})$ then by taking the limit of k to infinity we see that $\mathbb{P}_{p,q}(0 \longleftrightarrow \infty) = 0$ and thus $\tilde{q}_c(\mathbf{p}) \leqslant q_c(\mathbf{p})$. Similarly, from (9.1) it follows that $\tilde{q}_c(\mathbf{p}) \geqslant q_c(\mathbf{p})$.

Furthermore, the condition $q_c(\mathbf{p}) > 0$ implies that $\tilde{q}_c(\mathbf{p}) > 0$. We show this by induction on N: Suppose $N = 2$ and $\tilde{q}_c(\mathbf{p}) = 0$. Then by setting $q = 0$ we obtain the homogeneous percolation model on a quasi-transitive graph. By extending the proof in [14] to quasi-transitive graphs it can be shown that percolation occurs in this model and thus we find $q_c(\mathbf{p}) = 0$. Now suppose $N > 2$ and $\tilde{q}_c(\mathbf{p}) = 0$. Then by setting $q = 0$ we find an inhomogeneous percolation model with $N - 1$ colours in which percolation occurs. By using the induction hypothesis and Theorem 6.1 we see that $q_c(\mathbf{p}) = \tilde{q}_c(\mathbf{p}) = 0$. So $q_c(\mathbf{p}) > 0$ indeed implies $\tilde{q}_c(\mathbf{p}) > 0$.

9.1 PROOF OF EXPONENTIAL DECAY

We now prove Theorem 6.1 (b). Suppose $q < \tilde{q}_c(\mathbf{p})$. Then for any $x \in V$ there exists a finite set $S_x \subset V$ with $x \in S_x$ such that $\psi_{\mathbf{p},q}(x, S_x) < 1$. Furthermore, since G is quasi-transitive there are only finitely many different types of vertices so we can find an $L \in \mathbb{N}$ such that $S_x \subset \Lambda_L^x$ for any $x \in V$. Now let $x \in V$ be given and fix the set S with $x \in S$ and $\psi_{\mathbf{p},q}(x, S) < 1$. Since G is quasi-transitive there exists a $u \in V$ such that

$$\min_{\substack{u \in S \subset V: \\ S \subset \Lambda_L^u}} \psi_{\mathbf{p},q}(u, S) = \sup_{v \in V} \min_{\substack{v \in S \subset V: \\ S \subset \Lambda_L^v}} \psi_{\mathbf{p},q}(v, S). \tag{9.2}$$

The above step is where the proof would fail if the graph was not quasi-transitive. Because of the quasi-transitivity the above supremum is really a maximum that is attained for some $u \in V$. Without quasi-transitivity it is not clear if the supremum would be attained in some vertex, or at infinity.

Since $q < \tilde{q}_c(\mathbf{p})$ we have for some $\varepsilon > 0$ that

$$\min_{\substack{u \in S \subset V: \\ |S| < \Lambda_L^u}} \psi_{\mathbf{p},q}(u, S) = 1 - \varepsilon. \tag{9.3}$$

Define the random set

$$\mathscr{C} := \{z \in S : x \overset{S}{\longleftrightarrow} z\}. \tag{9.4}$$

Now if we let $k \in \mathbb{N}$ and we suppose $x \longleftrightarrow \partial\Lambda^x_{kL}$, then we know that there exists an edge $\{y,z\} \in \Delta S$ such that $x \overset{S}{\longleftrightarrow} y$, $\{y,z\}$ is open and $z \overset{\mathscr{C}^c}{\longleftrightarrow} \partial\Lambda^x_{kL}$. So by summing over all possible edges in ΔS and over all possible values of $\mathscr{C}$ we obtain

$$\mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \partial\Lambda^x_{kL}) \leqslant \sum_{C \subset S} \sum_{\{y,z\} \in \Delta S} \mathbb{P}_{\mathbf{p},q}\left(\left\{x \overset{S}{\longleftrightarrow} y, \mathscr{C} = C\right\}, \{y,z\} \text{ open}, z \overset{C^c}{\longleftrightarrow} \partial\Lambda^x_{kL}\right). \tag{9.5}$$

The three events in the above inequality depend on disjoint sets of edges: $\{x \overset{S}{\longleftrightarrow} y, \mathscr{C} = C\}$ only depends on edges in C, the event $z \overset{C^c}{\longleftrightarrow} \partial\Lambda^x_{kL}$ depends on edges in C^c and the edge $\{y,z\}$ is neither in C nor in C^c. So these events are independent and we obtain

$$\begin{aligned}\mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \partial\Lambda^x_{kL}) \leqslant{}& \\ & q \sum_{C \subset S} \sum_{\substack{\{y,z\} \in \\ \Delta S \cap E_N}} \mathbb{P}_{p,q}\left(x \overset{S}{\longleftrightarrow} y, \mathscr{C} = C\right) \mathbb{P}_{\mathbf{p},q}\left(z \longleftrightarrow \partial\Lambda^x_{kL}\right) \\ & + \sum_{i=1}^{N-1} p_i \sum_{C \subset S} \sum_{\substack{\{y,z\} \in \\ \Delta S \cap E_i}} \mathbb{P}_{p,q}\left(x \overset{S}{\longleftrightarrow} y, \mathscr{C} = C\right) \mathbb{P}_{\mathbf{p},q}\left(z \longleftrightarrow \partial\Lambda^x_{kL}\right).\end{aligned} \tag{9.6}$$

Since $z \in \Lambda^x_L$, we know that the distance between x and z is at most L. So if $z \longleftrightarrow \partial\Lambda^x_{kL}$, z is connected to a vertex at distance kL from x. Therefore, by the triangle inequality, z is connected to a vertex at distance at least $kL - L$ from z. So we find

$$\mathbb{P}_{\mathbf{p},q}\left(z \longleftrightarrow \partial\Lambda^x_{kL}\right) \leqslant \mathbb{P}_{\mathbf{p},q}\left(z \longleftrightarrow \partial\Lambda^z_{(k-1)L}\right). \tag{9.7}$$

Furthermore since G is quasi-transitive there exists a $w \in V$ such that

$$\mathbb{P}_{\mathbf{p},q}\left(w \longleftrightarrow \partial\Lambda^w_{(k-1)L}\right) = \sup_{v \in V} \mathbb{P}_{\mathbf{p},q}\left(v \longleftrightarrow \partial\Lambda^v_{(k-1)L}\right) \geqslant \mathbb{P}_{\mathbf{p},q}\left(z \longleftrightarrow \partial\Lambda^z_{(k-1)L}\right). \tag{9.8}$$

So that for this $w \in V$

$$\mathbb{P}_{\mathbf{p},q}\left(z \longleftrightarrow \partial\Lambda^x_{kL}\right) \leqslant \mathbb{P}_{\mathbf{p},q}\left(w \longleftrightarrow \partial\Lambda^w_{(k-1)L}\right). \tag{9.9}$$

This bound can now be used in (9.6) to obtain

$$\begin{aligned}\mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \partial\Lambda^x_{kL}) \leqslant{}& q \sum_{C \subset S} \sum_{\substack{\{y,z\} \in \\ \Delta S \cap E_N}} \mathbb{P}_{p,q}\left(x \overset{S}{\longleftrightarrow} y, \mathscr{C} = C\right) \mathbb{P}_{\mathbf{p},q}\left(w \longleftrightarrow \partial\Lambda^w_{(k-1)L}\right) \\ & + \sum_{i=1}^{N-1} p_i \sum_{C \subset S} \sum_{\substack{\{y,z\} \in \\ \Delta S \cap E_i}} \mathbb{P}_{p,q}\left(x \overset{S}{\longleftrightarrow} y, \mathscr{C} = C\right) \mathbb{P}_{\mathbf{p},q}\left(w \longleftrightarrow \partial\Lambda^w_{(k-1)L}\right) \\ \leqslant{}& \psi_{\mathbf{p},q}(x,S)\mathbb{P}_{p,q}(w \longleftrightarrow \partial\Lambda^w_{(k-1)L}) \\ \leqslant{}& (1-\varepsilon)\mathbb{P}_{p,q}(w \longleftrightarrow \partial\Lambda^w_{(k-1)L}).\end{aligned} \tag{9.10}$$

Iteration gives the desired exponential decay:

$$\mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \partial\Lambda^x_{kL}) \leqslant (1-\varepsilon)^k = \exp\left(\log(1-\varepsilon)k\right) = \exp(-ck). \tag{9.11}$$

□

9.2 PROOF OF INEQUALITY (9.1) (SUPERCRITICAL PHASE)

If $\tilde{q}_c(\mathbf{p}) = 1$, the theorem is automatically satisfied. So suppose $\tilde{q}_c(\mathbf{p}) < 1$ and let $q > \tilde{q}_c(\mathbf{p})$. The characterisation of the critical curve

$$\tilde{q}_c(\mathbf{p}) = \sup_{x \in V} \inf_{\substack{S \subset V: \\ x \in S, |S| < \infty}} \sup \left\{ q \in [0,1] \,:\, \psi_{\mathbf{p},q}(x,S) < 1 \right\}. \tag{9.12}$$

as given in 6.7 is equivalent to

$$\tilde{q}_c(\mathbf{p}) = \max_{x \in V} \inf_{k \in \mathbb{N}} \min_{\substack{S \subset V \cap \Lambda_k^x: \\ x \in S}} \sup \left\{ q \in [0,1] \,:\, \psi_{\mathbf{p},q}(x,S) < 1 \right\}, \tag{9.13}$$

since the supremum over $x \in V$ is attained in some $x_0 \in V$, because G is quasi-transitive, and the infimum over $S \subset V$ can be split into an infimum and a minimum by conditioning on the size of S. So for every $k \in \mathbb{N}$ the above minimum is attained in some $S_k \subset V \cap \Lambda_k^x$ with $x \in S_k$. We know that for any $x \in V$ and for any $S \subset V$, $\psi_{\mathbf{p},q}(x,S)$ is increasing in q. Furthermore, for fixed $\mathbf{p}$, $\psi_{\mathbf{p},q}(x_0,S_k)$ is a polynomial in q and for large k this polynomial has degree at least 1, since $\tilde{q}_c(\mathbf{p}) > 0$. Combining these observations we conclude that $\psi_{\mathbf{p},q}(x_0,S_k)$ is strictly increasing in q. So $\sup\{q \in [0,1] \,:\, \psi_{\mathbf{p},q}(x_0,S_k) < 1\}$ can be seen as the inverse in the q-variable of $\psi_{\mathbf{p},q}(x_0,S_k)$ at 1. This justifies the notation

$$\psi_{\mathbf{p}}^{\leftarrow}(1; x_0, S_k) := \sup \left\{ q \in [0,1] \,:\, \psi_{\mathbf{p},q}(x_0,S_k) < 1 \right\}. \tag{9.14}$$

Furthermore, since $\psi_{\mathbf{p},q}(x_0,S_k)$ is a polynomial of order at most k, we conclude that $\psi_{\mathbf{p}}^{\leftarrow}(1; x_0, S_k)$ is continuous in $\mathbf{p}$ for all k. So the critical curve can be written as

$$\tilde{q}_c(\mathbf{p}) = \lim_{k \to \infty} \psi_{\mathbf{p}}^{\leftarrow}(1; x_0, S_k). \tag{9.15}$$

The inverse $\psi_{\mathbf{p}}^{\leftarrow}(1; x_0, S_k)$ is decreasing in k, so $\tilde{q}_c(\mathbf{p})$ is a decreasing limit of continuous functions and is therefore upper semi-continuous. We use this property to define an auxiliary point $(\hat{\mathbf{p}}, \hat{q})$ in the $(\mathbf{p}, q)$-space. Let $\varepsilon = \big(q - \tilde{q}_c(\mathbf{p})\big)/2$. Then there exists a $\delta > 0$ such that

$$\tilde{q}_c(\mathbf{p}') < \tilde{q}_c(\mathbf{p}) + \varepsilon \qquad \text{for all} \quad \|\mathbf{p}' - \mathbf{p}\| \leqslant \delta, \tag{9.16}$$

where $\|\cdot\|$ denotes the 2-norm on $\mathbb{R}^{N-1}$. We define $\hat{\mathbf{p}} = \mathbf{p} - (\delta/\sqrt{N-1}, \ldots, \delta/\sqrt{N-1})$, so that $\|\hat{\mathbf{p}} - \mathbf{p}\| = \delta$. We take $\hat{q} = \tilde{q}_c(\hat{\mathbf{p}})$, so that $q - \hat{q} \geqslant \varepsilon$. Consider the line segment from $(\hat{\mathbf{p}}, \hat{q})$ to $(\mathbf{p}, q)$. We parametrise this line segment in the following way:

$$\mathbf{r}(t) = \begin{pmatrix} \hat{\mathbf{p}} + t(\mathbf{p} - \hat{\mathbf{p}}) \\ \hat{q} + t(q - \hat{q}) \end{pmatrix} \qquad t \in [0,1]. \tag{9.17}$$

Since $\tilde{q}_c(\hat{\mathbf{p}}) = \hat{q}$ and since G is quasi-transitive, there exists an $x \in V$ such that

$$\inf_{\substack{S \subset V: \\ x \in S, |S| < \infty}} \psi_{\hat{\mathbf{p}},\hat{q}}(x,S) = 1. \tag{9.18}$$

Moreover, since $\psi_{\mathbf{p},q}(x,S)$ is increasing in $\mathbf{p}$ and q, we have for any $0 \leqslant t \leqslant 1$ and for the same $x \in V$ that

$$\inf_{\substack{S \subset V: \\ x \in S, |S| < \infty}} \psi_{\mathbf{r}(t)}(x,S) \geqslant 1. \tag{9.19}$$

Fix $n \in \mathbb{N}$. We apply Russo's Formula to $x \longleftrightarrow \partial\Lambda_n^x$, to find

$$\frac{\partial}{\partial q}\mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \partial\Lambda_n^x) = \frac{1}{1-q} \sum_{e \in \Lambda_n^x \cap E_N} \mathbb{P}_{\mathbf{p},q}(e \text{ pivotal}, x \not\longleftrightarrow \partial\Lambda_n^x), \tag{9.20}$$

and a similar expression for the derivative to p_i:

$$\frac{\partial}{\partial p_i}\mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \partial\Lambda_n^x) = \frac{1}{1-p_i} \sum_{e \in \Lambda_n^x \cap E_i} \mathbb{P}_{\mathbf{p},q}(e \text{ pivotal}, x \not\longleftrightarrow \partial\Lambda_n^x), \tag{9.21}$$

so we also have an expression for $\nabla\mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \partial\Lambda_n)$. Specifically for $(\mathbf{p},q) = \mathbf{r}(t)$ the gradient equals:

$$\nabla\mathbb{P}_{\mathbf{r}(t)}(x \longleftrightarrow \partial\Lambda_n) = \begin{pmatrix} \frac{1}{1-\hat{p}_1-t(p_1-\hat{p}_1)} \sum_{e \in \Lambda_n^x \cap E_1} \mathbb{P}_{\mathbf{p},q}(e \text{ pivotal}, x \not\longleftrightarrow \partial\Lambda_n^x) \\ \frac{1}{1-\hat{p}_2-t(p_2-\hat{p}_2)} \sum_{e \in \Lambda_n^x \cap E_2} \mathbb{P}_{\mathbf{p},q}(e \text{ pivotal}, x \not\longleftrightarrow \partial\Lambda_n^x) \\ \vdots \\ \frac{1}{1-\hat{p}_{N-1}-t(p_{N-1}-\hat{p}_{N-1})} \sum_{e \in \Lambda_n^x \cap E_{N-1}} \mathbb{P}_{\mathbf{p},q}(e \text{ pivotal}, x \not\longleftrightarrow \partial\Lambda_n^x) \\ \frac{1}{1-\hat{q}-t(q-\hat{q})} \sum_{e \in \Lambda_n^x \cap E_N} \mathbb{P}_{\mathbf{p},q}(e \text{ pivotal}, x \not\longleftrightarrow \partial\Lambda_n^x) \end{pmatrix}, \tag{9.22}$$

and we can integrate this in the $(\mathbf{p},q)$-space along the straight line segment starting in $(\hat{\mathbf{p}},\hat{q})$ and ending in $(\mathbf{p},q)$. We use the Gradient Theorem to obtain

$$\begin{aligned} \int_0^1 \nabla\mathbb{P}_{\mathbf{r}(t)}(x \longleftrightarrow \partial\Lambda_n^x) d\mathbf{r}(t) &= \mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \partial\Lambda_n^x) - \mathbb{P}_{\hat{\mathbf{p}},\hat{q}}(x \longleftrightarrow \partial\Lambda_n^x) \\ &\leqslant \mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \partial\Lambda_n^x). \end{aligned} \tag{9.23}$$

On the other hand we can use the expression for $\nabla \mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \partial\Lambda_n^x)$ and the definition of the line integral along with the parametrization $\mathbf{r}(t)$ to obtain

$$\begin{aligned}
&\int_0^1 \nabla \mathbb{P}_{\mathbf{r}(t)}(x \longleftrightarrow \partial\Lambda_n^x) d\mathbf{r}(t) \\
&\quad = \int_0^1 \nabla \mathbb{P}_{\mathbf{r}(t)}(x \longleftrightarrow \partial\Lambda_n^x) \cdot \mathbf{r}'(t)\, dt \\
&\quad = \sum_{i=1}^{N-1} \int_0^1 \frac{p_i - \hat{p}_i}{1 - \hat{p}_i - t(p_i - \hat{p}_i)} \sum_{e \in \Lambda_n^x \cap E_i} \mathbb{P}_{\mathbf{r}(t)}(e \text{ pivotal}, x \not\longleftrightarrow \partial\Lambda_n^x)\, dt \\
&\qquad + \int_0^1 \frac{q - \hat{q}}{1 - \hat{q} - t(q - \hat{q})} \sum_{e \in \Lambda_n^x \cap E_N} \mathbb{P}_{\mathbf{r}(t)}(e \text{ pivotal}, x \not\longleftrightarrow \partial\Lambda_n^x)\, dt \\
&\quad \geqslant \sum_{i=1}^{N-1} \int_0^1 (p_i - \hat{p}_i)(p_i + t(p_i - \hat{p}_i)) \sum_{e \in \Lambda_n^x \cap E_i} \mathbb{P}_{\mathbf{r}(t)}(e \text{ pivotal}, x \not\longleftrightarrow \partial\Lambda_n^x)\, dt \\
&\qquad + \int_0^1 (q - \hat{q})(q + t(q - \hat{q})) \sum_{e \in \Lambda_n^x \cap E_N} \mathbb{P}_{\mathbf{r}(t)}(e \text{ pivotal}, x \not\longleftrightarrow \partial\Lambda_n^x)\, dt. \qquad (9.24)
\end{aligned}$$

We now define the random subset of Λ_n^x

$$\mathscr{S} = \{ y \in \Lambda_n^x \text{ such that } y \not\longleftrightarrow \partial\Lambda_n^x \}, \qquad (9.25)$$

The boundary of $\mathscr{S}$ are the vertices of Λ_n^x for which all neighbours that are not in $\mathscr{S}$ are connected to $\partial\Lambda_n^x$. If $x \not\longleftrightarrow \partial\Lambda_n^x$, then $x \in \mathscr{S}$, so if we sum over all possible values of $\mathscr{S}$ we find

$$\begin{aligned}
&\mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \partial\Lambda_n^x) \\
&\quad \geqslant \sum_{i=1}^{N-1} \int_0^1 (p_i - \hat{p}_i)(p_i + t(p_i - \hat{p}_i)) \sum_{\substack{S \subset \Lambda_n^x: \\ x \in S}} \sum_{\substack{e \in \\ \Lambda_n^x \cap E_i}} \mathbb{P}_{\mathbf{r}(t)}(e \text{ pivotal}, \mathscr{S} = S)\, dt \\
&\qquad + \int_0^1 (q - \hat{q})(q + t(q - \hat{q})) \sum_{\substack{S \subset \Lambda_n^x: \\ x \in S}} \sum_{\substack{e \in \\ \Lambda_n^x \cap E_N}} \mathbb{P}_{\mathbf{r}(t)}(e \text{ pivotal}, \mathscr{S} = S)\, dt. \qquad (9.26)
\end{aligned}$$

When we know $\mathscr{S} = S$ we know that the pivotal edges for the event $\{x \longleftrightarrow \partial\Lambda_n^x\}$ are exactly the edges $\{y, z\}$ on the edge-boundary ΔS of S that are connected to x, i.e., $y \in S$, $z \notin S$ and $x \longleftrightarrow y$. We can sum over these edges to obtain

$$\begin{aligned}
&\mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \partial\Lambda_n^x) \\
&\quad \geqslant \sum_{i=1}^{N-1} \int_0^1 (p_i - \hat{p}_i)(p_i + t(p_i - \hat{p}_i)) \sum_{\substack{S \subset \Lambda_n^x: \\ x \in S}} \sum_{\substack{\{y,z\} \in \\ \Delta S \cap E_i}} \mathbb{P}_{\mathbf{r}(t)}\left(x \overset{S}{\longleftrightarrow} y, \mathscr{S} = S\right) dt \\
&\qquad + \int_0^1 (q - \hat{q})(q + t(q - \hat{q})) \sum_{\substack{S \subset \Lambda_n^x: \\ x \in S}} \sum_{\substack{\{y,z\} \in \\ \Delta S \cap E_N}} \mathbb{P}_{\mathbf{r}(t)}\left(x \overset{S}{\longleftrightarrow} y, \mathscr{S} = S\right) dt. \qquad (9.27)
\end{aligned}$$

The occurrence of the event $\{\mathscr{S} = S\}$ can be determined from the configuration of the edges outside of S. This can be done by exploring from the boundary of Λ_n^x similar to the exploration process preceding (8.51). We conclude that the event $\{\mathscr{S} = S\}$ is determined by the configuration of the edges outside S, and is therefore independent of $\{x \overset{S}{\longleftrightarrow} y\}$. We find

$$\begin{aligned}
&\mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \partial\Lambda_n^x) \\
&\geqslant \sum_{i=1}^{N-1} \int_0^1 (p_i - \hat{p}_i)(p_i + t(p_i - \hat{p}_i)) \sum_{\substack{S \subset \Lambda_n^x: \\ x \in S}} \sum_{\substack{\{y,z\} \in \\ \Delta S \cap E_i}} \mathbb{P}_{\mathbf{r}(t)}\left(x \overset{S}{\longleftrightarrow} y\right) \mathbb{P}_{\mathbf{r}(t)}(\mathscr{S} = S)\, dt \\
&\quad + \int_0^1 (q - \hat{q})(q + t(q - \hat{q})) \sum_{\substack{S \subset \Lambda_n^x: \\ x \in S}} \sum_{\substack{\{y,z\} \in \\ \Delta S \cap E_N}} \mathbb{P}_{\mathbf{r}(t)}\left(x \overset{S}{\longleftrightarrow} y\right) \mathbb{P}_{\mathbf{r}(t)}(\mathscr{S} = S)\, dt. \qquad (9.28)
\end{aligned}$$

By (9.19) we have that $\psi_{\mathbf{r}(t)}(x, S_x) \geqslant 1$ for all $x \in S_x \subset \mathbb{Z}^d$. Using bounds on $p_i - \hat{p}_i$ and $q - \hat{q}$ we find

$$\begin{aligned}
\mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \partial\Lambda_n^x) &\geqslant \min\left\{\frac{\delta}{\sqrt{N-1}}, \varepsilon\right\} \int_0^1 \sum_{\substack{S \subset \Lambda_n^x: \\ x \in S}} \psi_{\mathbf{r}(t)}(x, S_x) \mathbb{P}_{\mathbf{r}(t)}(\mathscr{S} = S)\, dt \\
&\geqslant \min\left\{\frac{\delta}{\sqrt{N-1}}, \varepsilon\right\} \int_0^1 \mathbb{P}_{\mathbf{r}(t)}(x \not\longleftrightarrow \partial\Lambda_n^x)\, dt \\
&\geqslant \min\left\{\frac{\delta}{\sqrt{N-1}}, \varepsilon\right\} \mathbb{P}_{\mathbf{p},q}(x \not\longleftrightarrow \partial\Lambda_n^x). \qquad (9.29)
\end{aligned}$$

Finally we obtain

$$\begin{aligned}
\mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \partial\Lambda_n^x) &\geqslant \frac{\min\left\{\frac{\delta}{\sqrt{N-1}}, \varepsilon\right\}}{1 + \min\left\{\frac{\delta}{\sqrt{N-1}}, \varepsilon\right\}} \\
&> 0, \qquad (9.30)
\end{aligned}$$

and the final result follows by letting n tend to infinity. □

9.3 PROOF OF FINITE SUSCEPTIBILITY IN THE SUBCRITICAL PHASE

We prove Theorem 6.1 (c). Suppose again $q < q_c(\mathbf{p})$, so that also $q < \tilde{q}_c(\mathbf{p})$. Therefore, for any $x \in V$ there exists a finite set $S_x \subset V$ with $x \in S_x$ and $\psi_{\mathbf{p},q}(x, S_x) < 1$. For a finite set $A \subset V$ define

$$\chi(A) = \max_{u \in V} \sum_{v \in A} \mathbb{P}_{\mathbf{p},q}\left(u \overset{A}{\longleftrightarrow} v\right). \qquad (9.31)$$

Let $x \in V$ and let $A \subset V$ be such that $A \backslash S_x \neq \emptyset$. Now suppose that the event $x \overset{A}{\longleftrightarrow} u$ holds for some $u \in A \backslash S_x$. Then there exists an open edge $e = \{y, z\}$ on the boundary ΔS_x such that $x \overset{S_x}{\longleftrightarrow} y$ and $z \overset{A}{\longleftrightarrow} u$ using disjoint paths. We can use this observation together with the BK inequality to bound the probability of $x \overset{A}{\longleftrightarrow} u$:

$$\mathbb{P}_{\mathbf{p},q}\left(x \overset{A}{\longleftrightarrow} u\right) \leqslant \sum_{i=1}^{N} \sum_{\substack{\{y,z\}\in \\ \Delta S_x \cap E_i}} p_i \mathbb{P}_{\mathbf{p},q}\left(x \overset{S_x}{\longleftrightarrow} y\right) \mathbb{P}_{\mathbf{p},q}\left(z \overset{A}{\longleftrightarrow} u\right). \tag{9.32}$$

Summing the above inequality over all $u \in A \backslash S_x$ gives:

$$\sum_{u\in A\backslash S_x} \mathbb{P}_{\mathbf{p},q}\left(x \overset{A}{\longleftrightarrow} u\right) \leqslant \sum_{u\in A\backslash S_x} \sum_{i=1}^{N} \sum_{\substack{\{y,z\}\in \\ \Delta S_x \cap E_i}} p_i \mathbb{P}_{\mathbf{p},q}\left(x \overset{S_x}{\longleftrightarrow} y\right) \mathbb{P}_{\mathbf{p},q}\left(z \overset{A}{\longleftrightarrow} u\right) \tag{9.33}$$

$$\leqslant \psi_{\mathbf{p},q}(x, S_x) \sum_{u\in A\backslash S_x} \mathbb{P}_{\mathbf{p},q}\left(z \overset{A}{\longleftrightarrow} u\right) \tag{9.34}$$

$$\leqslant \psi_{\mathbf{p},q}(x, S_x)\chi(A). \tag{9.35}$$

We subsequently add the vertices in S_x and use the trivial bound $\mathbb{P}_{\mathbf{p},q}\left(x \overset{A}{\longleftrightarrow} u\right) \leqslant 1$:

$$\sum_{u\in A} \mathbb{P}_{\mathbf{p},q}\left(x \overset{A}{\longleftrightarrow} u\right) \leqslant |S_x| + \psi_{\mathbf{p},q}(x, S_x)\chi(A) \tag{9.36}$$

$$\leqslant \max_{x\in V}\left\{|S_x| + \psi_{\mathbf{p},q}(x, S_x)\chi(A)\right\}. \tag{9.37}$$

The above inequality holds for any $x \in V$, so in particular it holds for the vertex which maximizes the left hand side. This vertex exists because the graph is quasi-transitive. We find

$$\max_{x\in V} \sum_{u\in A} \mathbb{P}_{\mathbf{p},q}\left(x \overset{A}{\longleftrightarrow} u\right) \leqslant \max_{x\in V}\left\{|S_x| + \psi_{\mathbf{p},q}(x, S_x)\chi(A)\right\}, \tag{9.38}$$

so that

$$\chi(A) \leqslant \max_{x\in V}\left\{|S_x| + \psi_{\mathbf{p},q}(x, S_x)\chi(A)\right\}. \tag{9.39}$$

We conclude

$$\chi(A) \leqslant \frac{\max_{x\in V}|S_x|}{1 - \max_{x\in V}\psi_{\mathbf{p},q}(x, S_x)}, \tag{9.40}$$

so that $\chi(A)$ is uniformly bounded from above. Therefore the final result follows by letting A tend to V. □

Let $i \in \{1, \dots, N-1\}$ be given. If we can prove Lipschitz continuity for $\tilde{q}_c(\mathbf{p})$ we immediately have proved the same property for $q_c(\mathbf{p})$, since $\tilde{q}_c(\mathbf{p}) = q_c(\mathbf{p})$ by Theorem 6.1. Similar to (9.13), the critical curve $\tilde{q}_c(\mathbf{p})$ can be written as

$$\tilde{q}_c(\mathbf{p}) = \max_{x \in V} \inf_{k \in \mathbb{N}} \min_{\substack{S \subset V \cap \Lambda_k^x: \\ x \in S}} \sup\{q \in [0,1] \,:\, \psi_{\mathbf{p},q}(x, S) < 1\}. \tag{10.1}$$

So we obtain again

$$\psi_{\mathbf{p}}^{\leftarrow}(1; x_0, S_k) := \sup\{q \in [0,1] \,:\, \psi_{\mathbf{p},q}(x_0, S_k) < 1\}, \tag{10.2}$$

as in (9.14). Since $\tilde{q}_c(\mathbf{p}) < 1$ we have for k large enough that $\psi_{\mathbf{p}}^{\leftarrow}(1; x_0, S_k) < 1$. This immediately leads to

$$\psi_{\mathbf{p}, \psi_{\mathbf{p}}^{\leftarrow}(1; x_0, S_k)}(x_0, S_k) = 1. \tag{10.3}$$

Differentiating both sides to p_i gives

$$\left.\frac{\partial \psi_{\mathbf{p},q}(x_0, S_k)}{\partial p_i}\right|_{q=\psi_{\mathbf{p}}^{\leftarrow}(1;x_0,S_k)} + \left.\frac{\partial \psi_{\mathbf{p},q}(x_0, S_k)}{\partial q}\right|_{q=\psi_{\mathbf{p}}^{\leftarrow}(1;x_0,S_k)} \frac{\partial \psi_{\mathbf{p}}^{\leftarrow}(1; x_0, S_k)}{\partial p_i} = 0, \tag{10.4}$$

so that

$$\left|\frac{\partial \psi_{\mathbf{p}}^{\leftarrow}(1; x_0, S_k)}{\partial p_i}\right| = \left| \frac{\left.\frac{\partial \psi_{\mathbf{p},q}(x_0,S_k)}{\partial p_i}\right|_{q=\psi_{\mathbf{p}}^{\leftarrow}(1;x_0,S_k)}}{\left.\frac{\partial \psi_{\mathbf{p},q}(x_0,S_k)}{\partial q}\right|_{q=\psi_{\mathbf{p}}^{\leftarrow}(1;x_0,S_k)}} \right|. \tag{10.5}$$

Note that the denominator is not equal to zero for k large enough, because $\psi_{\mathbf{p},q}(x_0, S_k)$ is a polynomial in q of degree at least 1. We will give an upper bound for this fraction by computing upper and lower bounds for the numerator and denominator respectively. Computing the derivative gives

$$\begin{aligned} &\left.\frac{\partial \psi_{\mathbf{p},q}(x_0, S_k)}{\partial q}\right|_{q=\psi_{\mathbf{p}}^{\leftarrow}(1;x_0,S_k)} \\ &= \sum_{i=1}^{N} p_i \sum_{e \in \Delta S_k \cap E_i} \left.\frac{\partial}{\partial q} \mathbb{P}_{\mathbf{p},q}\left(x_0 \xleftrightarrow{S_k} e\right)\right|_{q=\psi_{\mathbf{p}}^{\leftarrow}(1;x_0,S_k)} + \mathbb{P}_{\mathbf{p},\psi_{\mathbf{p}}^{\leftarrow}(1;x_0,S_k)}\left(x_0 \xleftrightarrow{S_k} e\right). \end{aligned} \tag{10.6}$$

If $x_0 \in e$ for some $e \in \Delta S_k$, then $\mathbb{P}_{\mathbf{p},\psi_{\mathbf{p}}^{\leftarrow}(1;x_0,S_k)}(x_0 \xleftrightarrow{S_k} e) = 1$ and the above derivative is bounded from below by $\min_{0<i\leqslant N} p_i$. Now assume that there is no $e \in \Delta S_K$ such that $x_0 \in e$. Then we have

$$\left.\frac{\partial \psi_{\mathbf{p},q}(x_0, S_k)}{\partial q}\right|_{q=\psi_{\mathbf{p}}^{\leftarrow}(1;x_0,S_k)} \geqslant \sum_{i=1}^{N} p_i \sum_{e \in \Delta S_k \cap E_i} \left.\frac{\partial}{\partial q} \mathbb{P}_{\mathbf{p},q}\left(x_0 \xleftrightarrow{S_k} e\right)\right|_{q=\psi_{\mathbf{p}}^{\leftarrow}(1;x_0,S_k)}. \tag{10.7}$$

and we can use an inequality on the above derivative. We follow the proof of Grimmett in [22, Theorem 2.36b]. Let $A = A_k$ be the event that $x_0 \overset{S_k}{\longleftrightarrow} e$, then A depends on the edges in $\Lambda^{x_0}_{k+1}$ only. Let m_i be the number of i-edges in $\Lambda^{x_0}_{k+1}$ and let $O_i(\omega)$ be the number of open i-edges for a configuration ω of the edges in $\Lambda^{x_0}_{k+1}$. Then we find

$$\mathbb{P}_{\mathbf{p},q}(A) = \sum_{\omega} \mathbb{1}_A(\omega) \prod_{i=1}^{N} p_i^{O_i(\omega)} (1-p_i)^{m_i - O_i(\omega)}, \tag{10.8}$$

so that

$$\begin{aligned}
\frac{\partial}{\partial q}\mathbb{P}_{\mathbf{p},q}(A) &= \sum_{\omega} \mathbb{1}_A(\omega) \prod_{i=1}^{N} p_i^{O_i(\omega)} (1-p_i)^{m_i - O_i(\omega)} \left(\frac{O_N(\omega)}{q} - \frac{m_N - O_N(\omega)}{1-q} \right) \\
&= \frac{1}{q(1-q)} \sum_{\omega} \mathbb{1}_A(\omega) \prod_{i=1}^{N} p_i^{O_i(\omega)} (1-p_i)^{m_i - O_i(\omega)} (O_N(\omega) - m_N q) \\
&= \frac{1}{q(1-q)} \sum_{\omega} \prod_{i=1}^{N} p_i^{O_i(\omega)} (1-p_i)^{m_i - O_i(\omega)} \big(O_N(\omega) - m_N q\big)\big(\mathbb{1}_A(\omega) - \mathbb{P}_{\mathbf{p},q}(A)\big),
\end{aligned} \tag{10.9}$$

since $\mathbb{E}_{\mathbf{p},q}[O_N] = m_N q$. We decompose the sum over ω into two sums as follows: let $\Omega_{\mathbf{p}}$ be the set of configurations for the edges in $E \backslash E_N$ and let $\omega_{\mathbf{p}} \in \Omega_{\mathbf{p}}$. We denote by $\Omega(\omega_{\mathbf{p}})$ the set of configurations $\omega \in \Omega$ that satisfy $\omega(e) = \omega_{\mathbf{p}}(e)$ for all $e \in E \backslash E_N$. Then we obtain

$$\begin{aligned}
\frac{\partial}{\partial q}\mathbb{P}_{\mathbf{p},q}(A) = \frac{1}{q(1-q)} \sum_{\omega_{\mathbf{p}} \in \Omega_{\mathbf{p}}} \sum_{\omega \in \Omega(\omega_{\mathbf{p}})} \prod_{i=1}^{N} p_i^{O_i(\omega)} (1-p_i)^{m_i - O_i(\omega)} \\
\cdot \big(O_N(\omega) - m_N q\big)\big(\mathbb{1}_A(\omega) - \mathbb{P}_{\mathbf{p},q}(A)\big)
\end{aligned} \tag{10.10}$$

So we get

$$\frac{\partial}{\partial q}\mathbb{P}_{\mathbf{p},q}(A) = \frac{1}{q(1-q)} \sum_{\omega_{\mathbf{p}} \in \Omega_{\mathbf{p}}} \operatorname{Cov}\big(O_N, \mathbb{1}_A \,\big|\, \Omega(\omega_{\mathbf{p}})\big). \tag{10.11}$$

The number of open q-edges O_N is an increasing random variable, and we can show that conditioned on $\Omega(\omega_{\mathbf{p}})$ the random variable $O_N - \mathbb{1}_A$ is increasing as well: let ω and ω' be two configurations in $\Omega(\omega_{\mathbf{p}})$ satisfying $\omega \leqslant \omega'$. Then if $\mathbb{1}_A(\omega) = \mathbb{1}_A(\omega')$ we get $O_N(\omega) - \mathbb{1}_A(\omega) \leqslant O_N(\omega') - \mathbb{1}_A(\omega')$. Since A is an increasing event, the only remaining option is that $\mathbb{1}_A(\omega) = 0$ and $\mathbb{1}_A(\omega') = 1$. In that case there is at least one extra open q-edge in ω' compared to ω, so that $O_N(\omega) - \mathbb{1}_A(\omega) \leqslant O_N(\omega') - \mathbb{1}_A(\omega')$. So $\Omega(\omega_{\mathbf{p}})$ is indeed an increasing random variable conditioned on $\Omega(\omega_{\mathbf{p}})$. Note that the measure $\mathbb{P}_{\mathbf{p},q}\big(\,\cdot\,\big|\,\Omega(\omega_{\mathbf{p}})\big)$ is a product measure. We obtain

$$\begin{aligned}
\operatorname{Cov}\big(O_N, \mathbb{1}_A \,\big|\, \Omega(\omega_{\mathbf{p}})\big) &= \operatorname{Cov}\big(\mathbb{1}_A, \mathbb{1}_A \,\big|\, \Omega(\omega_{\mathbf{p}})\big) + \operatorname{Cov}\big(O_N - \mathbb{1}_A, \mathbb{1}_A \,\big|\, \Omega(\omega_{\mathbf{p}})\big) \\
&\geqslant \operatorname{Var}\big(\mathbb{1}_A \,\big|\, \Omega(\omega_{\mathbf{p}})\big),
\end{aligned} \tag{10.12}$$

where we used the FKG inequality to conclude

$$\mathrm{Cov}\big(O_N - \mathbb{1}_A, \mathbb{1}_A \,\big|\, \Omega(\omega_{\mathbf{p}})\big) \geqslant 0. \tag{10.13}$$

This gives the inequality

$$\begin{aligned}\frac{\partial}{\partial q}\mathbb{P}_{\mathbf{p},q}(A) &\geqslant \frac{1}{q(1-q)} \sum_{\omega_{\mathbf{p}} \in \Omega_{\mathbf{p}}} \mathrm{Var}\big(\mathbb{1}_A \,\big|\, \Omega(\omega_{\mathbf{p}})\big) \\ &\geqslant \frac{1}{q(1-q)} \mathbb{P}_{\mathbf{p},q}(A)\big(1 - \mathbb{P}_{\mathbf{p},q}(A)\big).\end{aligned} \tag{10.14}$$

Combining this with (10.6) we find

$$\begin{aligned}&\left.\frac{\partial \psi_{\mathbf{p},q}(x_0, S_k)}{\partial q}\right|_{q=\psi_{\mathbf{p}}^{\leftarrow}(1;x_0,S_k)} \geqslant \frac{1}{q(1-q)} \\ &\quad \cdot \sum_{i=1}^{N} p_i \sum_{e \in \Delta S_k \cap E_i} \mathbb{P}_{\mathbf{p},\psi_{\mathbf{p}}^{\leftarrow}(1;x_0,S_k)} \left(x_0 \xleftrightarrow{S_k} e\right) \left(1 - \mathbb{P}_{\mathbf{p},\psi_{\mathbf{p}}^{\leftarrow}(1;x_0,S_k)} \left(x_0 \xleftrightarrow{S_k} e\right)\right).\end{aligned} \tag{10.15}$$

Since $x_0 \notin e$, we can now bound the last factor with the probability that x_0 is an isolated vertex. We find

$$\begin{aligned}&\left.\frac{\partial \psi_{\mathbf{p},q}(x_0, S_k)}{\partial q}\right|_{q=\psi_{\mathbf{p}}^{\leftarrow}(1;x_0,S_k)} \\ &\quad \geqslant \frac{1}{q(1-q)} \sum_{i=1}^{N} p_i \sum_{e \in \Delta S_k \cap E_i} \mathbb{P}_{\mathbf{p},\psi_{\mathbf{p}}^{\leftarrow}(1;x_0,S_k)} \left(x_0 \xleftrightarrow{S_k} e\right) (1 - p_{\max} \vee q)^{\Delta} \\ &\quad = \frac{(1 - p_{\max} \vee q)^{\Delta}}{q(1-q)} \psi_{\mathbf{p},\psi_{\mathbf{p}}^{\leftarrow}(1;x_0,S_k)}(x_0, S_k) \\ &\quad = \frac{(1 - p_{\max} \vee q)^{\Delta}}{q(1-q)},\end{aligned} \tag{10.16}$$

so that the derivative to q is bounded away from zero uniformly in k.

We can find an upper bound for the derivative to p_i as follows:

$$\begin{aligned}\frac{\partial}{\partial p_i}\psi_{\mathbf{p},q}(x, S) &= \sum_{j=1}^{N} p_j \sum_{\substack{\{y,z\} \in \\ \Delta S \cap E_j}} \frac{\partial}{\partial p_i} \mathbb{P}_{\mathbf{p},q}\left(x \xleftrightarrow{S} y\right) + \sum_{\substack{\{y,z\} \in \\ \Delta S \cap E_i}} \mathbb{P}_{\mathbf{p},q}\left(x \xleftrightarrow{S} y\right) \\ &\leqslant \Delta \sum_{y \in S} \frac{\partial}{\partial p_i} \mathbb{P}_{\mathbf{p},q}\left(x \xleftrightarrow{S} y\right) + \frac{1}{p_i}\psi_{\mathbf{p},q}(x, S).\end{aligned} \tag{10.17}$$

We use Russo's formula on the first term to find

$$\begin{aligned}\frac{\partial}{\partial p_i}\psi_{\mathbf{p},q}(x, S) &\leqslant \Delta \sum_{y \in S} \frac{1}{p_i} \mathbb{E}_{\mathbf{p},q}\left[\#\text{ of open edges in } E_i \text{ pivotal for } \left\{ x \xleftrightarrow{S} y \right\}\right] \\ &\quad + \frac{1}{p_i}\psi_{\mathbf{p},q}(x, S).\end{aligned} \tag{10.18}$$

If an open edge is pivotal for $\{ x \overset{S}{\longleftrightarrow} y \}$ then there is a path of open edges in S from x to y. Let $W_l(x, y, e)$ denote the set of paths in S of length l from x to y that contain the edge e. Then we can sum over all edges e, all $l > 0$ and all of these paths to find

$$\sum_{y\in S} \mathbb{E}_{\mathbf{p},q}\left[\# \text{ of open edges in } E_i \text{ pivotal for } \left\{ x \overset{S}{\longleftrightarrow} y \right\}\right]$$
$$\leqslant \sum_{y\in S} \sum_{e\in E_i} \sum_{l=d(x,y)}^{\infty} \sum_{\substack{w\in \\ W_l(x,y,e)}} \prod_{i=1}^{N} p_i^{|w\cap E_i|}. \tag{10.19}$$

Let m be the number of q-edges on the path. The number of paths of length l is less than or equal to $\Delta(\Delta-1)^{l-1-m}\Delta_N^m$, since there are Δ possible edges for the first edge in the path and $\Delta - 1$ possibilities for subsequent $\mathbf{p}$-edges. Similarly there are at most Δ_N possibilities for each q-edge. Furthermore, a path of length l has at most l edges in E_i, so that the path gets counted at most l times when we sum over e. Using these observations we find

$$\sum_{y\in S} \mathbb{E}_{\mathbf{p},q}\left[\# \text{ of open edges in } E_i \text{ pivotal for } \left\{ x \overset{S}{\longleftrightarrow} y \right\}\right]$$
$$\begin{aligned} &\leqslant \sum_{l=1}^{\infty} \sum_{m=0}^{l} l\Delta(\Delta-1)^{l-1-m} p_{max}^{l-m} q^m \\ &= \frac{\Delta}{\Delta-1} \frac{\Delta_N q - (\Delta-1)p_{max}(2\Delta_N q - 1)}{(1-(\Delta-1)p_{max})^2 (1-\Delta_N q)^2} \\ &\leqslant \frac{\Delta}{(\Delta-1)^3\Delta_N^2} \left(\frac{2}{\delta_2^2\varepsilon^2} - \frac{\Delta_N}{\delta_2^2\varepsilon} - \frac{(\Delta-1)}{\delta_2\varepsilon^2} \right). \end{aligned} \tag{10.20}$$

It follows that

$$\frac{\partial}{\partial p_i} \psi_{\mathbf{p},q}(x, S) \leqslant \frac{1}{\delta_1} \left(\frac{\Delta}{(\Delta-1)^3\Delta_N^2} \left(\frac{2}{\delta_2^2\varepsilon^2} - \frac{\Delta_N}{\delta_2^2\varepsilon} - \frac{(\Delta-1)}{\delta_2\varepsilon^2} \right) + 1 \right) \leqslant C'. \tag{10.21}$$

Combining (10.5) with (10.16) and (10.21) we conclude that

$$\left| \frac{\partial}{\partial p_i} \min_{\substack{S\subset V\cap\Lambda_k^x: \\ x\in S}} \sup\{q \in [0, 1] \,:\, \psi_{\mathbf{p},q}(x, S) < 1\} \right| \leqslant C, \tag{10.22}$$

for some constant C independent of k and all $\mathbf{p} \in \mathscr{P}$. Write

$$\eta_k(\mathbf{p}) = \min_{\substack{S\subset V\cap\Lambda_k^x: \\ x\in S}} \sup\{q \in [0, 1] \,:\, \psi_{\mathbf{p},q}(x, S) < 1\}, \tag{10.23}$$

then the functions $\eta_k(\mathbf{p})$ are all Lipschitz continuous with the same Lipschitz constant. By the Arzelà-Ascoli Theorem (see for example [39, Theorem 11.28]) there exists a subsequence k_n such that

$$\lim_{n\to\infty} \eta_{k_n}(\mathbf{p}) = \eta(\mathbf{p}), \tag{10.24}$$

for some $\eta(\mathbf{p})$ which is again Lipschitz continuous with the same Lipschitz constant C. Furthermore, $\big(\eta_k(\mathbf{p})\big)$ is a decreasing sequence in k. We conclude that

$$\inf_{k \in \mathbb{N}} \eta_k(\mathbf{p}) = \lim_{k \to \infty} \eta_k(\mathbf{p}) = \lim_{n \to \infty} \eta_{k_n}(\mathbf{p}) = \eta(\mathbf{p}). \tag{10.25}$$

We conclude that

$$\left| \frac{\partial}{\partial p_i} \tilde{q}_c(\mathbf{p}) \right| \leqslant C, \tag{10.26}$$

for all $\mathbf{p} \in \mathscr{P}$. □

11 LOWER BOUND ON THE PERCOLATION FUNCTION: PROOF OF THEOREM 6.4

This proof is analogous to the proof for inequality (9.1), however, the auxiliary point $(\hat{\mathbf{p}}, \hat{q})$ is defined differently. Let $\hat{q} = (q + \tilde{q}_c(\mathbf{p}))/2$, so that the point $(\mathbf{p}, \hat{q})$ lies above the surface $\tilde{q}_c(\mathbf{p})$. We have assumed that $q_c(\mathbf{p})$ is Lipschitz continuous, so by Theorem 6.1 we know that $\tilde{q}_c(\mathbf{p})$ is Lipschitz continuous as well. Thus there exists a unique $0 \leqslant \alpha < 1$ such that $\tilde{q}_c(\alpha\mathbf{p}) = \hat{q}$. Define $\hat{\mathbf{p}} = \alpha\mathbf{p}$ and consider the line segment from $(\hat{\mathbf{p}}, \hat{q})$ to $(\mathbf{p}, q)$. We parametrise this line segment in the following way:

$$\mathbf{r}(t) = \begin{pmatrix} \hat{\mathbf{p}} + t(\mathbf{p} - \hat{\mathbf{p}}) \\ \hat{q} + t(q - \hat{q}) \end{pmatrix} \qquad t \in [0, 1]. \tag{11.1}$$

Furthermore, by Proposition 6.3 we know that $\tilde{q}_c(\mathbf{p})$ is Lipschitz continuous so that it holds for some $K > 0$ that

$$|\tilde{q}_c(\mathbf{p}) - \tilde{q}_c(\hat{\mathbf{p}})| \leqslant K|\mathbf{p} - \hat{\mathbf{p}}|, \tag{11.2}$$

so that

$$|\mathbf{p} - \hat{\mathbf{p}}| \geqslant \frac{1}{2K}|q - \tilde{q}_c(\mathbf{p})|. \tag{11.3}$$

We can now repeat the proof of Theorem 6.1 to get the same inequality as in (9.28):

$$\begin{aligned} &\mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \partial\Lambda_n^x) \\ &\geqslant \sum_{i=1}^{N-1} \int_0^1 (p_i - \hat{p}_i)(p_i + t(p_i - \hat{p}_i)) \sum_{\substack{S \subset \Lambda_n^x: \\ x \in S}} \sum_{\substack{\{y,z\} \in \\ \Delta S \cap E_i}} \mathbb{P}_{\mathbf{r}(t)}\left(x \overset{S}{\longleftrightarrow} y\right) \mathbb{P}_{\mathbf{r}(t)}(\mathscr{S} = S)\, dt \\ &\quad + \int_0^1 (q - \hat{q})(q + t(q - \hat{q})) \sum_{\substack{S \subset \Lambda_n^x: \\ x \in S}} \sum_{\substack{\{y,z\} \in \\ \Delta S \cap E_N}} \mathbb{P}_{\mathbf{r}(t)}\left(x \overset{S}{\longleftrightarrow} y\right) \mathbb{P}_{\mathbf{r}(t)}(\mathscr{S} = S)\, dt. \end{aligned} \tag{11.4}$$

By (9.19) we have that $\psi_{\mathbf{r}(t)}(x, S_x) \geqslant 1$ for all $x \in S_x \subset \mathbb{Z}^d$. Using (11.3) and multiplying the second term in (9.28) by $1/2K$ gives

$$\begin{aligned} \mathbb{P}_{\mathbf{p},q}(x \longleftrightarrow \partial\Lambda_n^x) &\geqslant \frac{1}{2K}(q - \hat{q}) \int_0^1 \sum_{\substack{S \subset \Lambda_n^x \\ x \in S}} \psi_{\mathbf{r}(t)}(x, S_x) \mathbb{P}_{\mathbf{r}(t)}(\mathscr{S} = S)\, dt \\ &\geqslant \frac{1}{2K}(q - \hat{q}) \int_0^1 \mathbb{P}_{\mathbf{r}(t)}(x \not\longleftrightarrow \partial\Lambda_n^x)\, dt \\ &\geqslant \frac{1}{2K}(q - \hat{q}) \mathbb{P}_{\mathbf{p},q}(x \not\longleftrightarrow \partial\Lambda_n^x). \end{aligned} \tag{11.5}$$

Finally we obtain

$$\mathbb{P}_{p,q}(x \longleftrightarrow \partial\Lambda_n^x) \geqslant \frac{q-\hat{q}}{2K+(q-\hat{q})} = \frac{q-\tilde{q}_c}{4K+(q-\tilde{q}_c)} \geqslant \frac{1}{4K+1}(q-\tilde{q}_c), \tag{11.6}$$

and the final result follows by letting n tend to infinity. □

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