The numerical inversion of the Laplace transform

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DEDICATION

This masters thesis is dedicated to my parents (Johnson and Anna) and siblings (Glory, Joshua, Eunice and Nora) for their love and support.

Abstract

We consider the inverse Laplace transform $f(t) = \mathcal{L}^{-1}\{F(s)\}$. This problem is extremely ill-posed (in the sense of Hadamard's definition of well-posedness) in the case when the Laplace transform is measured, computed or known on the real positive axis only. The problem is difficult because a stable inversion formula does not exist. In such a case, numerical methods and possibly regularization must be used to reconstruct the inverse Laplace transform $f(t)$.

Consequently, we examine three different numerical methods for the inversion of the Laplace transform: the Gaver-Stehfest method, the Piessens methods (based on a Chebyshev polynomial expansion of the Laplace transform function) and we construct a new method, the regularized collocation method (based on Tikhonov regularization).

We show that the Gaver-Stehfest and the Piessens methods are suitable in the case of exact data $F(s)$, whereas the regularized collocation method is well suited to handle both cases, i.e., for exact and noisy data.

Finally, we implement these methods and test their applicability on a wide class of Laplace transform functions in the case of exact and noisy data. For noisy data, we examine the methods and compare them with respect to their stability.

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Contents

Chapter 1

Introduction

1.1 Introductive sentences

The Laplace transform is an important integral transform with several applications in physics and engineering. There, it is for example used in the solution and analysis of time-invariant systems such as electrical circuits, mechanical systems, optical devices, harmonic oscillators and several other applications.

At the moment, there exists an extensive list of literature on the Laplace transform. However, only a selected few are listed in this section. Widder [64] wrote a seminal monograph on the Laplace transform and inversion formulas in 1941. The book by Jaeger [32] written in 1961 provides an introduction to the Laplace transform with engineering applications. Spiegel's [55] Schaum's outline on the Laplace transform applications was published in 1965. Then in 1973, Stroud [58] completed his work on the Laplace transform containing several example problems. Bogart [9] examined the Laplace transform on a theoretical and experimental perspective in 1984. In 1996, Poularikas [52] gave an extensive review of the Laplace transform applied to ODEs and PDEs. More recently in 2003, Beerends et al. [6] considered the Fourier as well as the Laplace transform.

The Laplace transform of a function $f(t),\ t > 0$ is denoted by $\mathcal{L}\{f(t)\}$ and defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt \tag{1.1}$$

provided that the integral (1.1) exists. In Section 5.2, we give sufficient conditions for the existence of the Laplace transform $F(s)$.

In general, three different cases are encountered when a problem is being solved or a system is been analysed by means of the Laplace transform:

1. In the first case, the Laplace transform is obtained analytically. In such a case (and when the expression for the transform is simple), the inverse Laplace transform can be determined by the use of tables of Laplace transforms or by means of analytical methods.

2. In the case when the Laplace transform is only computed in the complex half-plane of convergence, then the inverse Laplace transform can be reconstructed by the use of the complex inversion integral (also known as the Bromwich Integral):

$$f(t) = \frac{1}{2\pi i} \lim_{\omega \to \infty} \int_{\sigma - i\omega}^{\sigma + i\omega} F(z) e^{zt} dz, \tag{1.2}$$

   where the integration is carried out along the vertical line $\Re(s) = \sigma$ in the complex plane such that σ is greater than all real part of all the singularities of $F(s)$.

3. Finally, in the case when the Laplace transform $F(s)$ is measured, computed or known only for $\Re(s),\ s > 0$, i.e., only on the real positive axis, the problem of reconstructing the original function $f(t)$ from $F(s)$ is extremely ill-posed. In this case, stable inversion formulas do not exist.

The ill-posed case, i.e., when the Laplace transform is only known at a finite number of points $F(s_i),\ i = 1, 2, \ldots, N$ for $\Re(s),\ s > 0$, is the focus and motivation of this masters thesis. As a result, we now consider some numerical methods for reconstructing the inverse Laplace transform.

1.2 Numerical inversion of the Laplace transform

In this section, we provide a brief review of some numerical methods for reconstructing the inverse Laplace transform.

The books by Krylov, Shoblya and Louvish [38] and Cohon [13] describe some numerical methods for the Laplace transform inversion. Miller and Guy, Jn [42] introduced an inversion method based on Jacobi polynomial, whereas Al-Shuaibi [1] used Legendre polynomials for his method. Essah and Delves [23] developed an inversion method using cross-validation Galerkin techniques, while Lee and Sheen's [40] method was based on the location of the poles in the Laplace transform $F(s)$. By using eigenfunctions and eigenvalues of the Laplace transform, Mcwhirter and Pike [41] developed a method for inverting the Laplace transform and similar Fredholm integral equations of the first kind. Ramm [54] considered an inversion method which uses only real values of $s > 0$ of the transform function $F(s)$. De Hoogs, Knight and Stokes [18] worked on an improved method for the numerical inversion of the Laplace transform by accelerating the convergence of the Fourier series obtained from the inversion integral by using a trapezoidal rule. A numerical method based on a Chebyshev polynomial expansion of the Laplace transform function was later introduced by Piessens [48], whereas Stehfest [56] expressed the inverse Laplace transform operator as coefficients, which depends only on the discretization parameter N. Finally, Coumo et al. [15] introduced a collocation method which uses real values of the Laplace transform.

We observe that in the above references, no regularization properties are built in into their methods. This implies that they become highly unstable in the presence of noisy data. Consequently, we examine some numerical methods, which have some form of regularization in them. In this direction, Al-Shuaibi [2] constructed a regularized displacement operator, while Ang, Lund and Stenger [3] introduced a complex variable and regularization method for the inversion of Laplace transform. Finally, Dong's [20] method was based on Tikhonov regularization method.

An extensive survey and comparison of numerical methods for the inversion of the Laplace transform was given be Davis and Martin [16].

1.3 Outline of thesis

The organization of this thesis is briefly described in this section.

In Chapter 2, we present the theory of integral equations and describe the classification scheme for them. We give some important definitions and results related to the solution of integral equations. We list some numerical methods for solving integral equations of the second kind. At the end, we give the conditions ensuring the existence of a solution to an integral equation of the first kind.

In Chapter 3, we examine the theory of inverse and ill-posed problems with respect to an operator equation $Kf = F$. We give some definitions and typical examples of inverse problems. We consider the relation between inverse and ill-posed problems. Finally, we present some important results related to the Moore-Penrose (generalized) inverse corresponding to the ill-posed operator K.

In Chapter 4, we consider regularization methods for ill-posed problems. We mention some iterative regularization methods and examine continuous regularization method. Regularization by *a-priori* and *a-posteriori* parameter choice rules are also examined. Lastly, we investigate the Tikhonov regularization method and its numerical realization.

In Chapter 5, we analyse the Laplace transform. We give sufficient conditions for the existence of the Laplace transform and prove several important properties. We consider the inverse Laplace transform and derive the complex inversion integral. For both, the direct Laplace transform and its inverse, we give some example problems.

In Chapter 6, we examine three numerical methods for the inversion of the Laplace transform. We begin by showing that the Laplace transform is indeed an ill-posed problem. We derive the Gaver-Stehfest method, Piessens method and we construct a regularized collocation method based on Tikhonov regularization.

In Chapter 7, we make use of MATLAB to implement the algorithms derived in Chapter 6. We test the algorithms on a wide class of transform functions in the case of exact data and noisy data. We make comparison between the methods and examine their numerical stability in the case of noisy data. Finally, we give a concise summary of the entire masters thesis and some concluding remarks.

In the appendix, we present some definitions for normed linear space, operators, differentiability and some other definitions that appear in this thesis.

Chapter 2

Integral equations

In this chapter, we briefly introduce the theory of integral equations. We classify integral equations, present some useful definitions and mention some numerical methods for solving certain type of integral equations. The motivation for including this chapter is that the Laplace transform is a typical example of a *Fredholm integral equations of the first kind.* This overview on the theory of integral equations is based on Engl [22] and Kress [37].

2.1 Introduction

An integral equation is a mathematical statement in which the unknown function is part of an integrand and it appears under an integral sign. Integral equations are an intriguing branch of applied mathematics because several problems with physical applications can be formulated as integral equations. This underscores its importance in mathematics, see Poularikas [52], physics, see Engl [22], and engineering, see Poljak and Tham [50], Dolezel, Karban and Solin [19], and Rahman [53].

2.2 Classification of integral equations

We classify integral equations according to its *type* and *kind*. Thus, we consider a general, one-dimensional, linear integral equation of the form:

$$a(s)f(s) - \int_{s_0}^{s_1} k(s,t)f(t)dt = F(s), \tag{2.1}$$

where $a(s)$ is a given function, $f(t)$ the unknown function, $k(s,t)$ the kernel function, and $F(s)$ is the given right hand side. We present the following classification scheme. (See, e.g., Engl [22]).

Classification by *kind*:

- If $a(s) = 0$ in (2.1), we have an integral equation of the first kind.
- If $a(s) = constant$ in (2.1), we obtain an integral equation of the second kind.
- If $a(s) = nonconstant\ function \neq 0$ in (2.1), we get an integral equation of the third kind.

Classification by *type*:

- If the limit of integration in (2.1) is fixed and not allowed to vary, we obtain a Fredholm type integral equation, e.g.,

  $$\int_0^{s_1} k(s,t)f(t)dt = F(s),$$

  with s_1 fixed, is a Fredholm integral equation of the first kind.
- If the limit of integration in (2.1) is allowed to depend on the free variable, we obtain a Voltera type integral equation, e.g.,

  $$\lambda f(s) - \int_0^{s} k(s,t)f(t)dt = F(s), \quad s \in G \subseteq \mathbb{R},$$

  where s is a free variable, is a Voltera integral equation of the second kind.

Relationship between integral equations of different *types* and *kinds*

A Voltera integral equation can be formally expressed as a Fredholm integral equation via the following relation

$$\int_0^s k(s,t)f(t)dt \;=\; \int_0^{s_1} k(s,t)\chi_{[0,s]}(t)f(t)dt, \quad s \in [0,s_1],$$

where

$$\chi_{[0,s]}(t) = \begin{cases} 0, & t \in [0,s] \\ +\infty, & t \notin [0,s] \end{cases}$$

is the characteristic function. This implies that the Voltera equations is a special case of the Fredholm integral equation.

Whereas, the third kind integral equation can be written as a second kind integral equation. For this, we consider a second kind integral equation given as:

$$a(s)f(s) - \int_{s_0}^{s_1} k(s,t)f(t)dt = F(s).$$

Assume $a(s) \neq 0$ for all s, we can write the above equation as a second kind integral equation in the form:

$$f(s) - \int_{s_0}^{s_1} \frac{k(s,t)}{a(s)} f(t)dt = \frac{F(s)}{a(s)}, \quad a(s) \neq 0.$$

In the following, we present some useful definitions and propositions related to the solution of integral equations.

2.3 Definitions and results

All the definitions and propositions that we present in this section are taken from Engl [22].

Definition 2.1. *(Degenerated kernel)*
The kernel function $k : G \times G \to \mathbb{R}$ is called degenerated if there exist functions: $\varphi_1, \ldots, \varphi_n$ and $\psi_1, \ldots, \psi_n \in L^2(G)$, $G \subseteq \mathbb{R}$, such that

$$k(s,t) = \sum_{i=1}^{n} \varphi_i(s)\psi_i(t)$$

almost everywhere.

Proposition 2.2. *Let $K : L^2(G) \to L^2(G)$ be an integral operator generated by the kernel function k. Then k is a degenerated kernel if and only if the range of K is finite dimensional.*

Definition 2.3. *(Weakly singular kernel)*
Let $G \subseteq \mathbb{R}^n$ be compact, and Jordan measurable. Then the kernel function $k : (G \times G) \backslash \{(s,s) : s \in G\} \to \mathbb{R}$ is weakly singular if $k \mid_{(G\times G)\backslash\{(s,s)\ :\ s\in G\}}$ is continuous and there exist constants $\beta > 0$ and $M > 0$ such that

$$|k(s,t)| \leq M\|s-t\|_N^{\beta-N}, \quad (s \neq t).$$

In the study of integral equations, compact linear operators are of great importance, since, as we have mentioned earlier, a lot of problems with physical applications can be formulated as integral equations. Under appropriate conditions (assumptions), integral operators are compact. It is known that compact operators do not have a continuous inverse, see Engl, Hanke and Neubauer [7], and as a result, integral equations of the first kind ($Kf = F$), having a compact operator K are standard examples of ill-posed problems, see Chapter 3 of this work.

In the light of the above, we now present the conditions ensuring the compactness of integral operators.

Proposition 2.4. *Let the integral operator K be generated by the kernel function $k \in C(G \times G)$. Then the operator K is compact between all the combinations of $L^2(G)$ and $C(G)$; that is*

$$\left.\begin{array}{rl} K : L^2(G) \to & L^2(G) \\ K : L^2(G) \to & C(G) \\ K : C(G) \to & L^2(G) \\ K : C(G) \to & C(G) \end{array}\right\} \text{ are all compact.}$$

Proposition 2.5. *Let $K : L^2(G) \to L^2(G)$ be an integral operator generated by the kernel function $k \in L^2(G \times G)$. Then the operator K is compact.*

Proposition 2.6. *Let the kernel function k be weakly singular. Then the integral operator $(Kf)(s) = \int_G k(s,t)f(t)dt$ exist and is well defined for $f \in C(G)$ and $K : C(G) \to C(G)$ is compact.*

Furthermore, the proposition given below contains some important properties of compact operators.

Proposition 2.7. *Let X, Y, Z be normed spaces. Let $K : X \to Y$ be a linear operator. Then,*

1. *if K is compact, then K is bounded,*
2. *linear combinations of compact operators are compact.*
3. *Let $K_1 \in L(X, Y)$ and $K_2 \in L(Y, Z)$, then $K_1 K_2$ is compact if either K_1 or K_2 is compact.*
4. *Let Y be a Banach space, for all $n \in \mathbb{N}$. Let $K_n : X \to Y$ be linear and compact. If $\lim_{n\to\infty} \|K - K_n\| = 0$, then K is compact.*
5. *If K is bounded and $\dim \mathcal{R}(K) < \infty$, then K is compact.*

We now present other useful definitions and results related to compact operators.

Definition 2.8. *(Adjoint operator)*
Let $K : X \to Y$ be an integral operator, where X, Y are Hilbert spaces. The operator K^* satisfying

$$\langle Kx, y\rangle_Y = \langle x, K^* y\rangle_Y \quad \forall\ x \in X, y \in Y$$

is called the adjoint operator of K.

As a side note, we mention that the adjoint operator does not always exist if X, Y are not Hilbert spaces and equipped with a general dual system.

Definition 2.9. *(Self-adjoint operator)*
Let X be a Hilbert space, then the integral operator $K : X \to X$ is said to be a self-adjoint operator ($K = K^*$) if

$$\langle Kx, y\rangle_Y = \langle x, Ky\rangle_Y \quad \forall\ x, y \in X.$$

Definition 2.10. *(Eigensystem)*
Let X be a Hilbert space, $K : X \to X$ a compact self-adjoint operator. Let $\lambda_1, \lambda_2, \ldots$ be the (non-zero) eigenvalues of K, where the $\lambda_i's$ are written $\dim \mathcal{N}(\lambda_i I - K)$ times. Suppose $x_1, x_2, \ldots$ form an orthonormal system with $\lambda_i x_i = K x_i$. Then the pair, $(\lambda_i, x_i)_{i\in\mathbb{N}}$ is called the eigensystem of the operator K.

Proposition 2.11. *For a compact, self-adjoint operator $K : X \to X$, between Hilbert space X, an eigensystem exists.*

Definition 2.12. *(Eigensystem expansion)*
Let K be a compact linear self-adjoint integral operator with eigensystem (λ_i, x_i), then the following eigensystem expansion exist for K,

$$Kf = \sum_{i=1}^{\infty} \lambda_i \langle f, x_i \rangle x_i \tag{2.2}$$

for $f \in X$.

Whereas, for non self-adjoint operators, an eigensystem expansion as given above is not possible. This is so because non self-adjoint operators do not necessarily have eigenvalues or an eigensystem. Thus, for such operators, a singular value expansion exists in place of an eigensystem expansion. As a result, we now give the definition of the *singular value expansion*.

Definition 2.13. *(Singular value expansion)*
Let $K : X \to Y$ be a self-adjoint compact linear operator, and $K^* : Y \to X$ the adjoint operator. Let $\{\sigma_i^2\}_{i\in\mathbb{N}}$ be the (non zero) eigenvalues of the self-adjoint operator K^*K (and also of KK^*) written in decreasing order and with multiplicity, $\sigma_i > 0$. The triad $(\sigma_i; x_i, y_i)$ defines the singular system, where $\{x_i\}_{i\in\mathbb{N}}$ is a complete orthonormal system of eigenvectors of K^*K, and where $\{y_i\}_{i\in\mathbb{N}}$ forms a complete orthonormal system of eigenvectors of KK^* and is define through

$$y_i := \frac{Kx_i}{\|Kx_i\|}.$$

From the above definition, we can write an expansion for non self-adjoint, compact, linear operators, given below in (2.3) and (2.4),

$$Kx_i = \sigma_i y_i, \quad K^* y_i = \sigma_i x_i$$

and

$$Kf = \sum_{i=1}^{\infty} \sigma_i \langle f, x_i \rangle y_i, \quad f \in X, \tag{2.3}$$

$$K^* g = \sum_{i=1}^{\infty} \sigma_i \langle g, y_i \rangle x_i, \quad g \in Y. \tag{2.4}$$

In Chapter 4 of this thesis, we examine the regularization of ill-posed problems and in doing so, we will make use of the *spectral family of a function.* As a result, we present the following definition.

Definition 2.14. *(Spectral family, E_λ)*
Let $(\sigma_i; x_i, y_i)$ be a singular system for the non self-adjoint compact linear operator K. Given the eigensystem $(\sigma_i^2; x_i)$ for the self-adjoint compact operator K^*K, we have the following expansion

$$K^*Kf = \sum_{i=1}^{\infty} \sigma_i^2 \langle f, x_i \rangle x_i. \tag{2.5}$$

The expression in (2.5) *can be written as an integral operator*

$$K^*Kf = \int \lambda dE_\lambda f,$$

where the spectral family *E_λ of f is defined through*

$$E_\lambda f := \sum_{\substack{i=1 \\ \sigma_i^2 < \lambda}}^{\infty} \langle f, x_i \rangle x_i + Pf \tag{2.6}$$

and $P : K^*K \to \mathcal{N}(K^*K)$ is an orthogonal projector, $\lambda > 0$.

For a more detailed explanation on the spectral theory as it applies to compact operators, see Engl [22].

Definition 2.15. *(Spectrum of K)*
Let X be a normed space, $K : X \to X$ be a linear operator. The spectrum of K is defined as

$$\sigma(K) = \{\lambda \in \mathbb{C} \mid \lambda I - K : X \to X \text{ has no continuous inverse on } X\},$$

where $\lambda \in \mathbb{C}$ is an eigenvalue of K if $\mathcal{N}(\lambda I - K) \neq \{0\}$.

Proposition 2.16. *Let X be a normed space, $K : X \to X$ a compact operator, then the following holds*

1. *If $\dim X = \infty$, then $0 \in \sigma(K)$.*

2. *If* $\lambda \in \sigma(K) \setminus \{0\}$*, then* λ *is an eigenvalue of* K *with finite geometrical multiplicity, i.e.,* $\dim \mathcal{N}(I - K) < \infty$.

3. $\sigma(K)$ *is at most countable, and* 0 *is the only accumulation point.*

We note that the Laplace transform (see Chapter 5 of this work) is a typical example of a Fredholm integral equation of the first kind and it has several important properties. However, at this point, we briefly introduce the Laplace transform as it relates to integral equation of the first kind.

The Laplace transform integral

We recall the definition of the Laplace transform of a function $f(t)$ given in (1.1), i.e.,

$$\int_0^\infty f(t)e^{-st}dt = F(s) \tag{2.7}$$

provided the integral (2.7) exists and the sufficient conditions given in Chapter 5 are satisfied. In operator notation, we define the Laplace transform as

$$Kf = F \tag{2.8}$$

where

$$\begin{aligned} K &\equiv \int_0^\infty \cdot\, e^{-st}dt \ \text{ is the Laplace transform operator} \\ f &\equiv f(t) \ \text{ is the unknown function} \\ F &\equiv F(s) \ \text{ is the data} \\ e^{-st} &\equiv \ \text{the kernel function associated with the integral operator } K. \end{aligned}$$

Numerical methods for solving certain class of integral equations are considered in the next section.

2.4 Solution methods for integral equations

In this section, we examine the solution methods for equations of the first kind and list some solution methods for equations of the second kind.

Methods for equations of the first kind

In the following theorem, we present the conditions ensuring the existence of a solution of equations of the first kind. The theorem can be found in Engl [1].

Theorem 2.17. *Let H be a Hilbert space and $K : H \to H$ a compact self-adjoint operator. Let (λ_i, x_i) be the corresponding eigensystem of K. Then the integral equation of the first kind*

$$\int_0^s k(s,t)f(t)dt = F(s) \quad \Rightarrow \quad Kf = F$$

is solvable if and only if:

1. *$F \in \mathcal{N}(K)^{\perp}$, and*
2. *$f = \sum_{i=1}^{\infty} \frac{|\langle F, x_i \rangle|^2}{\lambda_i^2} < \infty$ (Picard condition)*

are satisfied. In this case, the solution of the integral equation is given by

$$f = \sum_{i=1}^{\infty} \frac{\langle F, x_i \rangle}{\lambda_i} x_i + q \tag{2.9}$$

where $q \in \mathcal{N}(K)$.

Some useful references related to the solution of Fredholm integral equation of the first kind include Landweber [39], Baker et al. [5], Nashed and Wahba [45], Nair and Pereverzev [44], Wahba [62], and Hilgers [30] and their references.

For simplicity, we have stated Theorem 2.17 for self-adjoint operators K. This assumption is not needed, but a similar theorem also holds in the non self-adjoint case making use of the singular value decomposition.

Methods for equations of the second kind

Consider a Fredholm integral equation of the second kind given as

$$\lambda f(s) - \int_0^1 k(s,t)f(t)dt = F(s).$$

For the above equation, standard numerical methods for obtaining approximate solution are:

1. Degenerated kernel approximation,
2. Quadrature methods,
3. Projection methods,
4. Collocation methods.

For a detailed description of the above solution methods, see Kress [37] and Engl [22].

The numerical inversion of the Laplace transform is an inverse problem, and as we know, a mathematical formulation involving inverse problem often leads to ill-posedness. In the light of this, the next chapter is devoted to the study of inverse and ill-posed problems.

Chapter 3

Inverse and ill-posed problems

In this chapter, we introduce the theory of inverse and ill-posed problems, and we present this in terms of an operator equation. We consider some typical examples of inverse problems. Finally, we recall some results from Engl, Hanke and Neubauer [21].

3.1 Introduction

In the words of Keller [35], two problems are said to be inverse to each other if the formulation of one of the problems involves the other. In this sense (mainly for historical justification), one might call one of these problems (mostly the one which was first studied or the less difficult one) the direct problem, while the other one is called the inverse problem.

Throughout this chapter, we will always make reference to the operator equation:

$$Kf = F \tag{3.1}$$

where $K : X \to Y$ is a bounded linear operator between the Hilbert spaces X, Y.

A *direct problem* to (3.1) is: given information about the operator K and value(s) of f, we seek to compute corresponding values of the right hand side F.

Whereas, the *inverse problem*, is: given information about the operator K

and data F, we seek to reconstruct value(s) of f. The inverse problem is then the problem of finding f, given the data F and information on the forward problem.

For more introductory notes on the theory of inverse problems and regularization of integral equations of the first kind, see Groetsch [25].

3.2 Examples of inverse problems

In this section, we present some standard examples of inverse problems taken from Kabanikhin [33] and Engl, Hanke and Neubauer [21].

Signal and image processing

We consider a blurred image or a signal function which has been passed through a medium acting as a filter. An inverse problem is then to deblur or reconstruct an unblurred version of the image or the original signal function before it was filtered. This application is often important, e.g., in the field of telecommunications, where for example, telephone signals are distorted as they travel over several distances and the original signal needs to be reconstructed.

Parameter identification

In certain technical applications, the physical laws controlling a particular process are given, but the parameters associated with this process have to be determined. In such cases, the inverse problem is the determination of these parameters. A one-dimensional example of this is the determination of the diffusion coefficient a in the stationary heat equation:

$$\begin{aligned} -(a(s)u(s)_s)_s &= f(s), \quad s \in (0,1) \\ u(0) &= 0 = u(1), \end{aligned}$$

where $f \in L^2$ represent the internal heat source of the process, a stands for the heat conductivity of the process, and the subscript s represent a single derivative with respect to s.

Geophysics

An inverse problem that is frequently used by geophysicists is the determination of information from the interior of the earth from measurements taken at the surface of the earth. In this case, seismic waves are propagated into the interior of the earth and this is used, for example, to determine the density of rocks, location of natural resources (e.g., crude oil), to determine the location of earthquakes epicentre in the earth crust, and many other physical and industrial applications.

Numerical analysis

Inverse problem techniques are also used in the numerical solution of certain kind of integral equation (e.g., of the first kind):

$$\int_{s_0}^{s_1} k(s,t)f(t)dt = F(s),$$

where the right hand side $F(s)$ and the integral operator $K \equiv \int k(s,t) \cdot dt$ are known and we seek to reconstruct the unknown function $f(t)$. The numerical inversion of the Laplace transform is a typical example of this.

Radon inversion (X-ray tomography)

A well studied medical application of inverse problem is X-ray tomography. In this case, we seek to obtain transverse slices through the body of a patient in a non-destructive manner. Given a set of measurements of X-ray through the patient's body as well as a measurement of the total absorption along lines through the body, we seek to reconstruct the X-ray absorption as a function of the location in the body under examination.

Other references on the theory of inverse problems are Hofmann [31] and Kirsch [36].

Since the formulation of inverse problems often leads to ill-posedness, the next section is devoted to the study of ill-posed problems.

3.3 Ill-posed problems

We consider the inverse problem of recovering the function f from the operator equation,

$$Kf = F, \tag{3.2}$$

given the bounded linear operator $K : X \to Y$ and the data F. The inverse problem given above is assumed to be ill-posed in the sense of Hadamard's [26] definition of a well-posed problem.

Definition 3.1. *(Hadamard's definition of a well-posed problem)*
Let the operator equation $Kf = F$ satisfy the conditions given above for (3.2)*. The following three criteria should be fulfilled for it to be a well posed problem:*

$\mathcal{R}(K) = Y$, *i.e., existence of solution to* (3.2)

$\mathcal{N}(K) = \{0\}$, *i.e., uniqueness of solution to* (3.2)

$K^{-1} \in L(Y, X)$, *i.e., continuous dependence of solution f on the data F.*

If one of the conditions given in Definition 3.1 do not hold, we call the problem (3.2) ill-posed. This implies that in the presence of noisy data or perturbations of the exact data with some random noise, the solution of an ill-posed problem is unstable and thus a naive solution method has to be remedied. Special numerical methods designed to handle ill-posed problems in a numerically stable way are the so-called *regularization methods* (see Chapter 4 of this work for details).

In the ill-posed case, violating the existence condition sometimes does not pose any particular problem, e.g., if exact data are given. However, in general, for noisy data the lack of this condition does lead to problems, and hence regularization methods have to be used.

The second condition (i.e., the uniqueness condition) not being satisfied can be remedied by generalizing the notion of solution. If the problem is an ill-posed one, and it has several solutions, based on available information, one has to decide which of the solutions is most appropriate with respect to the particular problem in question.

A violation of the last condition (i.e., the continuous dependence of the solution on the data) leads to severe numerical problems, i.e., an instability of numerical methods. As a result, it therefore does not make sense to solve an ill-posed problem with *traditional* numerical methods. In such a case, regularization methods has to be used.

The existence and stability conditions are closely related (see Engl, Hanke and Neubauer [21]).

In the next section, we introduce the Moore-Penrose (generalized) inverse corresponding to the ill-posed operator K in Equation (3.2) and also present some definitions and useful results.

3.4 The Moore-Penrose (generalized) inverse

In this section, we present the definitions of a least-squares solution, best-approximate solution, and some useful results. For the definitions and proof of the theorems given in this section, see Engl, Hanke and Neubauer [21]. In the following, X, Y are Hilbert spaces.

Definition 3.2. *Let $K : X \to Y$ be a bounded linear operator as given in (3.2), then*

1. *$f \in X$ is called a least-squares solution of $Kf = F$ if*
   $\|Kf - F\| = \inf\{\|Ky - F\| \mid y \in X\}$

2. *$f \in X$ is called best-approximate solution of $Kf = F$ if f is a least-squares solution of $Kf = F$ and*
   $\|f\| = \inf\{\|y\| \mid y \text{ is a least-squares solution of } Kf = F\}$ *are satisfied.*

From the above definition, it is obvious that the best-approximate solution to the ill-posed problem in (3.2) is therefore defined as the least-squares solution with the smallest norm.

Definition 3.3. *The Moore-Penrose (generalized) inverse $K^\dagger$ of $K \in L(X, Y)$ is defined as the unique linear extension of $\tilde{K}^{-1}$ to*

$$D(K^\dagger) = \mathcal{R}(K) + \mathcal{R}(K)^\perp$$

with

$$\mathcal{N}(K^\dagger) = \mathcal{R}(K)^\perp$$

where

$$\tilde{K} := K\mid_{\mathcal{N}(K)^\perp} \;\; : \; \mathcal{N}(K)^\perp \to \mathcal{R}(K).$$

And finally, in the last two theorems, we give the relation between the Moore-Penrose (generalized) inverse and the least-squares solution. We also introduce the normal equation corresponding to the operator equation in (3.2).

Theorem 3.4. *Let $f \in D(K^\dagger)$, then $Kf = F$ has a unique best approximate solution, which is given by*

$$f^\dagger := K^\dagger F.$$

Theorem 3.5. *Let $f \in D(K^\dagger)$, then $f \in X$ is a least-squares solution of $Kf = F$ if and only if the normal equation*

$$K^*Kf = K^*F$$

is satisfied.

Theorem 3.6. *The Moore-Penrose inverse $K^\dagger$ is continuous if and only if $\mathcal{R}(K)$ is closed.*

This theorem relates the existence and the stability conditions in Definition 3.1.

As already mentioned in this chapter, attempting to solve an ill-posed problem with *traditional* numerical methods results in the instability of such methods. Instead, *regularization methods* should be used in the solution of ill-posed problems. This will be considered in the next chapter.

Chapter 4

Regularization of ill-posed problems

In this chapter, we construct a regularization operator for an ill-posed operator. We also discuss regularization by *a-priori* and *a-posteriori* parameter choice rules. Finally, we examine Tikhonov regularization and its numerical realization.

4.1 Introduction

In general, regularization of ill-posed problems is the approximation of such problems by a family of closely related well-posed problems. As a result of the inherent numerical instability associated with the solution of ill-posed problems (when they are being solved by *traditional* numerical methods) regularization methods must be used. However, it must be noted that a regularization method only serves as a partial remedy to the numerical instability of an ill-posed problem and it cannot completely recover the entire information lost due to the ill-posedness. The difficulty encountered in the solution of ill-posed problems is clearly explained in the works of Varah [61] and Tikhonov and Arsenin [60].

In the following, we consider the operator equation $Kf = F$ and seek to obtain the best-approximate solution $f^\dagger = K^\dagger F$ in the case when the exact data F are not known, but the *noisy data* F^δ are available. We assume also that the *noise level* $\|F^\delta - F\| \leq \delta$ is known.

With the above setting and in the case where $K^\dagger$ is an ill-posed operator, $K^\dagger F^\delta$ is for sure not an appropriate approximate solution since $K^\dagger$ is not continuously invertible, or $K^\dagger F^\delta$ might not even be defined. Thus, regularization techniques are used to seek for an approximate solution f_α^δ of $f^\dagger$. After regularization, it is expected that the solution f_α^δ will now depend continuously on the noisy data F^δ, and as the noise level δ tends to 0 and with a suitably chosen regularization parameter α, we can obtain the convergence $f_\alpha^\delta \to f^\dagger$.

An excellent reference on the regularization of ill-posed problems is the book by Engl, Hanke and Neubauer [21], and as a result, the results given in this chapter are taken from this book. Other useful references on regularization methods for ill-posed problems are Hanke and Hansen [28], Hein [29], Kaltenbacher, Neubauer and Scherzer [34], Tikhonov et al. [59], Wang, Yagola and Yang [63], and Zhdanov [66].

In the next section, we provide the definition and actual construction of a regularization operator.

4.2 Regularization operator

For the definition and proposition given in this section, see Engl, Hanke and Neubauer [21].

Definition 4.1. *Let $K : X \to Y$ be a bounded linear operator between Hilbert spaces X and Y, $\alpha_0 \in (0, +\infty]$. For every $\alpha \in (0, \alpha_0)$, let*

$$R_\alpha : Y \to X$$

be a continuous operator (which is in general nonlinear). The family $\{R_\alpha\}$ is known as a regularization or a family of regularization operators for $K^\dagger$, if for all $F \in D(K^\dagger)$, there exits a parameter choice rule $\alpha = \alpha(\delta, F^\delta)$ such that

$$\lim_{\delta \to 0} \sup\{\|R_{\alpha(\delta, F^\delta)} F^\delta - K^\dagger F\| \mid F^\delta \in Y,\ \|F^\delta - F\| \leq \delta\} = 0 \tag{4.1}$$

holds. In this case,

$$\alpha : \mathbb{R}^+ \times Y \to (0, \alpha_0)$$

is such that

$$\limsup_{\delta \to 0} \{\alpha(\delta, F^\delta) \mid F^\delta \in Y,\ \|F^\delta - F\| \leq \delta\} = 0. \tag{4.2}$$

For a particular $F \in D(K^\dagger)$, the pair (R_α, α) is called a convergent regularization method for solving $Kf = F$ provided that (4.1) *and* (4.2) *are both satisfied.*

From the above theorem, it is obvious that a convergent regularization method consists of a regularization operator R_α as well as a parameter choice rule $\alpha(\delta, F^\delta)$ such that if the regularization parameter α is chosen according to the rule, then convergence of the regularized solution towards the best approximate solution is obtained, i.e., $f_\alpha^\delta \to f^\dagger$ as the noise level δ tends to 0.

We now briefly distinguish between the two known main types of parameter choice rules, i.e., *a-priori* and *a-posteriori* parameter choice rules. If the regularization parameter α does not depend on F^δ but only on δ, then we have an *a-priori* parameter choice rule which is written as $\alpha = \alpha(\delta)$. On the other hand, we have an *a-posteriori* parameter choice rule if α depends both on the noise level δ and the noisy data F^δ. In this later case, we write $\alpha(\delta, F^\delta)$.

In the proposition below, we give the actual construction of a convergent regularization operator.

Proposition 4.2. *Let R_α be a continuous (and in general nonlinear) operator for all $\alpha > 0$. Then the family $\{R_\alpha\}$ is a regularization for $K^\dagger$ if*

$$R_\alpha \to K^\dagger \quad \textit{pointwise on} \quad D(K^\dagger), \quad \textit{as} \quad \alpha \to 0.$$

In this case, there exits, for every $F \in D(K^\dagger)$, an a-priori *parameter choice rule α such that (R_α, α) is a convergent regularization method for solving $Kf = F$.*

We now turn our attention to a consideration of suitable parameter choice rules and regularization operators.

4.3 Continuous regularization methods and parameter choice rules

In this section, we give the main idea for the construction of regularization methods as well as *a-prior* and *a-posteriori* parameter choice rules. We also give some convergence and stability results from Engl, Hanke and Neubauer [21].

4.3.1 Regularization by spectral filters and a-priori parameter choice rule

A regularization is constructed by the use of the spectral theory for self-adjoint linear operators. If we assume that the operator K^*K is continuously invertible, then we get $(K^*K)^{-1} = \int \frac{1}{\lambda} dE_\lambda$ where $\{E_\lambda\}$ is a spectral family for K^*K. We recall that the best-approximate solution corresponding to the operator equation $Kf = F$ is given as $f^\dagger = K^\dagger F$. Using the continuous invertibility of K^*K and the normal equation $K^*Kf = K^*F$, we can write the best-approximate solution formally as

$$f^\dagger = \int \frac{1}{\lambda} dE_\lambda K^* F, \tag{4.3}$$

provided that such an expression exists. We observe that the above result in Equation (4.3) is equivalent to that in Equation (2.9).

We note that in the ill-posed case, F might not be contained in $D(K^\dagger)$ and that $\mathcal{R}(K)$ might not be closed, and as such the integral (4.3) might not exists for all F because of a pole in 0 in the integrand $\frac{1}{\lambda}$. In such a case, the best-approximate solution might not exits. A remedy for this is to substitute the integrand $\frac{1}{\lambda}$ with a parameter-dependent family of functions $h_\alpha(\lambda)$ being piecewise continuous on the interval $[0, \|K\|^2]$. Hence, in place of (4.3), we can write

$$f_\alpha := \int h_\alpha(\lambda) dE_\lambda K^* F. \tag{4.4}$$

In this case, the family $\{R_\alpha\}$ such that $R_\alpha := \int h_\alpha(\lambda) dE_\lambda K^*$ can be made a regularization for the ill-posed operator $K^\dagger$. In the case of noisy data F^δ with noise level $\|F - F^\delta\| \leq \delta$, (4.4) becomes

$$f_\alpha^\delta := \int h_\alpha(\lambda) dE_\lambda K^* F^\delta. \tag{4.5}$$

To guarantee the existence of the solution f_α^δ and the convergence of the regularization, it is important that the parameter-dependent family $h_\alpha(\lambda)$ is carefully chosen such that the $\lim_{\alpha\to 0} h_\alpha(\lambda) = \frac{1}{\lambda}$, i.e., it should approximate the integrand $\frac{1}{\lambda}$ for all $\lambda \in (0, \|K\|^2]$.

For the difference between the regularized solution f_α with exact data (as given in (4.4)) and the best approximate solution $f^\dagger$, we have the following relation

$$\begin{aligned} f^\dagger - f_\alpha &= f^\dagger - h_\alpha(K^*K)K^*F = \big(I - h_\alpha(K^*K)K^*K\big)f^\dagger \\ &= \int \big(1 - \lambda h_\alpha(\lambda)\big) dE_\lambda f^\dagger. \end{aligned}$$

Hence, we can write the residual term as $f^\dagger - f_\alpha = r_\alpha(K^*K)f^\dagger$, where

$$r_\alpha(\lambda) := 1 - \lambda h_\alpha(\lambda). \tag{4.6}$$

With the regularization operators R_α already constructed and the expression for the regularized solution f_α known, it is appropriate that we also provide some convergence results of the regularized solution in the case of exact data F. This is the motivation for the next theorem.

Theorem 4.3. *Let for all $\alpha > 0$, $h_\alpha : [0, \|K\|^2] \to \mathbb{R}$ fulfill the following assumptions: h_α is piecewise continuous, and there is a constant $M > 0$ such that*

$$|\lambda h_\alpha(\lambda)| \le M$$

and

$$\lim_{\alpha\to 0} h_\alpha(\lambda) = \frac{1}{\lambda} \quad \forall\ \lambda \in (0, \|K\|^2].$$

Then for all $F \in D(K^\dagger)$

$$\lim_{\alpha\to 0} f_\alpha = \lim_{\alpha\to 0} h_\alpha(K^*K)K^*F = f^\dagger$$

holds with $f^\dagger = K^\dagger F$.

See Engl, Hanke and Neubauer [21].

Moreover, we present the stability of the regularized solution f_α in the next theorem.

Theorem 4.4. *Let h_α and M be as in Theorem* 4.3*, f_α and f_α^δ be defined as in* (4.4) *and* (4.5) *respectively. For $\alpha > 0$, let*

$$G_\alpha := \sup\{|h_\alpha(\lambda)| \mid \lambda \in [0, \|K\|^2]\}. \tag{4.7}$$

Then,

$$\|f_\alpha - f_\alpha^\delta\| \leq \delta\sqrt{MG_\alpha}.$$

See Engl, Hanke and Neubauer [21].

Finally, we present convergence rate estimates of the regularized solution f_α with an *a-priori* parameter choice rule. This estimate is given in terms of the residual functions in (4.6).

Theorem 4.5. *Let h_α fulfil the assumptions of Theorem* 4.3*, r_α be defined by* (4.6)*, $\mu > 0$. Let for all $\alpha \in (0, \alpha_0)$ and $\lambda \in [0, \|K\|^2]$,*

$$\lambda^\mu |r_\alpha(\lambda)| \leq m_\mu \alpha^\mu$$

hold for some $m_\mu > 0$ and assume that G_α as defined in (4.7) *fulfills*

$$G_\alpha = O(\alpha^{-1}) \quad as \quad \alpha \to 0.$$

If $f^\dagger$ satisfies the source condition

$$f^\dagger \in \mathcal{R}\big((K^*K)^\mu\big), \tag{4.8}$$

then, with the parameter choice rule

$$\alpha \sim \delta^{\frac{2}{2\mu+1}}, \tag{4.9}$$

we obtain the following estimate

$$\|f_\alpha^\delta - f^\dagger\| = O\big(\delta^{\frac{2\mu}{2\mu+1}}\big). \tag{4.10}$$

See Engl, Hanke and Neubauer [21].

Hence, with the *a-priori* parameter choice rule (4.9), the best possible convergence rate that can be obtained is given in (4.10), i.e., in the order of $\big(\frac{2\mu}{2\mu+1}\big)$. It can be shown that this is the best possible rate for any regularization under the source condition (4.8), i.e., the estimate (4.10) is sharp and cannot be improved.

4.3.2 Regularization by an a-posteriori parameter choice rule

In order to construct an *a-priori* parameter choice rule, information about the parameter $\mu > 0$ in the source condition (4.8) (i.e., the smoothness of the solution) has to be known. In general, this information is not always available. A way out of this is to construct an *a-posteriori* parameter choice rule. We will consider the well-known discrepancy principle according to Morozov [43]. Since we will be using Tikhonov regularization in this thesis, the work by Anzengruber and Ramlau [4] and Bonesky [10] on Morozov's discrepancy principle for Tikhonov-type functionals is of importance.

To begin, we first recall the parameter-dependent family of function $h_\alpha(\lambda)$ as given in Theorem 4.3 and the term $r_\alpha(\lambda)$ as given in (4.6). Besides, we assume that

$$\tau > \{|r_\alpha(\lambda)| \mid \alpha > 0,\ \lambda \in [0, \|K\|^2]\}. \tag{4.11}$$

With the above setting, we define the regularization parameter using the discrepancy principle as

$$\alpha(\delta, F^\delta) := \sup\{\alpha > 0 \mid \|Kf_\alpha^\delta - F^\delta\| \leq \tau\delta\}. \tag{4.12}$$

The regularization parameter (4.12) is selected by comparing the discrepancy $\|Kf_\alpha^\delta - F^\delta\|$ with the noise level δ.

We point out that there exists a reciprocal relationship between the regularization parameter and the numerical stability of the method. Now, given noisy data F^δ such that the bound $\|F - F^\delta\| \leq \delta$ is known, we seek to recover the solution f from $Kf = F$. As a result, it is not logical to look for an approximate solution, say $\tilde{f}$ of $f^\dagger$ with a residual $\|K\tilde{f} - F^\delta\| < \delta$ because a smaller regularization parameter $\alpha \to 0$ is equivalent to less stability. It therefore makes sense to select the biggest possible regularization parameter which leads to a discrepancy of the order of δ.

In the next theorem we will give the convergence rate result in the case of the *a-posteriori* parameter choice rule given by the discrepancy principle.

Theorem 4.6. *The regularization method (R_α, α), where the regularization parameter α is defined through the discrepancy principle given in* (4.12)*, is*

convergent for all $F \in \mathcal{R}(K)$ *and of optimal order in* $\mathcal{R}\big((K^*K)^\mu\big)$ *for* $\mu \in (0, \mu_0 - \frac{1}{2}]$, *i.e.,*

$$f^\delta_{\alpha(\delta,F^\delta)} - f^\dagger = O\big(\delta^{\frac{2\mu}{2\mu+1}}\big).$$

See Engl, Hanke and Neubauer [21].

Well known regularization methods in literature are Tikhonov and iterative regularization (such as the Conjugate gradient method, Landweber method, Accelerated Landweber method and Newton type methods). However, in this thesis, our focus is on Tikhonov regularization because of the nice properties it possesses. Hence, the next section, we consider the Tikhonov regularization method.

4.4 Tikhonov regularization

Tikhonov regularization method, named after Andrey Nikolayevich Tikhonov (a Russian mathematician), is the most commonly used regularization method for ill-posed problems. In order to derive this method, we begin by letting $Kf = F^\delta$ be an ill-posed problem and we seek to recover the unknown function f given the bounded linear operator $K : X \to Y$ (X, Y Hilbert spaces) and noisy data F^δ. Then, a standard procedure is known as *Linear Least Squares*, where we seek to minimize the residual $\|Kf - F^\delta\|^2$ where $\|\cdot\|$ is the norm in Y, in the discrete case usually a (weighted) Euclidean norm. In order to give preference to a specific solution with useful properties, the regularization term is added to the minimization, i.e.,

$$\|Kf - F^\delta\|^2 + \|\Gamma f\|^2$$

for some suitably selected Tikhonov matrix Γ. In many cases, the Tikhonov matrix Γ is chosen as the identity matrix $\Gamma = I$, thereby giving preference to solutions with smaller norms. The degree of regularization may be controlled by adding some parameter α (i.e., the Tikhonov regularization parameter) which serves as a scaling of the Tikhonov matrix, $\Gamma = \alpha I$. And in the case where $\alpha = 0$, this reduces to the unregularized least squares solution of $Kf = F^\delta$ provided the inverse $(K^*K)^{-1}$ exists.

The regularization process thus improves the conditioning of the ill-posed

problem, thereby enabling a stable numerical solution to exits. In the case of Tikhonov regularization, a precise expression for the regularized solution is given as

$$f_\alpha^\delta = (K^*K + \alpha I)^{-1}K^*F^\delta, \quad \text{where} \quad \Gamma = \alpha I. \tag{4.13}$$

We observe that the explicit form of the Tikhonov regularized solution can be characterized in a variational form. This is given in the theorem below.

Theorem 4.7. *Let f_α^δ be as in* (4.13)*. Then f_α^δ is the unique minimizer of the Tikhonov functional*

$$f \longmapsto \|Kf - F^\delta\|^2 + \alpha\|f\|^2. \tag{4.14}$$

See Engl, Hanke and Neubauer [21].

Finally, to ensure the convergence of the Tikhonov regularized solution, the following conditions given in the next theorem have to be be taken into consideration.

Theorem 4.8. *Let f_α^δ be defined by* (4.13)*, $F \in D(K^\dagger)$, $\|F - F^\delta\| \leq \delta$. If $\alpha = \alpha(\delta)$ is such that*

$$\lim_{\alpha\to 0} \alpha(\delta) = 0 \qquad \textit{and} \qquad \lim_{\alpha\to 0} \frac{\delta^2}{\alpha(\delta)} = 0$$

then

$$\lim_{\alpha\to 0} f_{\alpha(\delta)}^\delta = K^\dagger F.$$

See Engl, Hanke and Neubauer [21].

Numerical realization of Tikhonov regularization

Again, we consider the operator equation $Kf = F^\delta$, $K : X \to Y$ and we seek to recover the unknown function f given the bounded linear operator K and the data F^δ. We discretize the solution space X and the data space Y such that $X_n \subset X$ and $Y_m \subset Y$ with basis $X_n = span\{\varphi_1, \ldots, \varphi_n\}$, and $Y_m = span\{\psi_1, \ldots, \psi_m\}$.

In the discrete case, the approximate solution $f_n \in X_n$ is given by its coordinate vector $\mathbf{f} \in \mathbb{R}^n$ corresponding to the representation

$$f_n = \zeta_1\varphi_1, \ldots, \zeta_n\varphi_n, \quad \mathbf{f} = [\zeta_1, \ldots, \zeta_n]^T.$$

By minimizing the continuous form of the Tikhonov functional

$$f \longmapsto \|Kf - F^\delta\|^2 + \alpha\|f\|^2$$

over X_m, we can generate a finite-dimensional approximate solution f_n of the best approximate solution $f^\dagger$.

We also observe that the minimization problem given above is the same as minimizing the discrete form of the Tikhonov functional

$$f \longmapsto \|K_m f - F^\delta\|^2 + \alpha\|f\|^2$$

over the entire space X. In this case, the operator K_m is discretized such that $K_m := KP_m$ where $P_m : X \to X_m$ is the orthogonal projector on the subspace X_m. With this discretization scheme, the regularized solution corresponding to the Tikhonov minimization problem is then given by

$$f^\delta_{\alpha,m} = (K_m^* K_m + \alpha I)^{-1} K_m^* F^\delta.$$

Suppose we have an increasing sequence of finite-dimensional subspace of X with a dense union in X, our goal is to achieve the convergence

$$f \to f^\dagger \quad \text{as} \quad \alpha \to 0 \quad \text{and} \quad m \to \infty.$$

We note that achieving convergence of the approximate solution $f_{\alpha,m}$ towards the best approximate solution $f^\dagger$ as the regularization parameter $\alpha \to 0$, and the level of discretization $m \to \infty$ at the same time is to be expected only if α and m are suitably related. Thus for convergence of the regularized solution, the choice of α with respect to m is very important.

The choice of the discretization (X_n) is also important. There are several ways to do this. One possibility, with some advantages, is to use the dual projection method. Here, at first, the data space Y is discretized using the spaces Y_m. In this case, the discretization of the space X is given by $X_n = K^* Y_n$.

For the computer realization of the regularization parameter α_m^δ and the approximate solution $f_{\alpha,m}^\delta$ we recall the assumption that Y_m is spanned by $\{\psi_1, \ldots, \psi_m\}$. As a result, the approximate solution satisfies

$$f_{\alpha,m}^\delta = K_m^*(K_m K_m^* + \alpha I)^{-1} F =: K_m^* x_{\alpha,m}^\delta$$

and the following relation holds

$$\langle K_m^* x_{\alpha,m}^\delta, K_m^* \psi_i \rangle_X + \alpha \langle x_{\alpha,m}^\delta, \psi_i \rangle_Y = \langle F^\delta, \psi_i \rangle_Y, \quad i = 1, \ldots, m.$$

Employing the expansion for $x_{\alpha,m}^\delta$ and $f_{\alpha,m}^\delta$ through

$$x_{\alpha,m}^\delta = \sum_{i=1}^m \zeta_i \psi_i, \quad f_{\alpha,m}^\delta = \sum_{i=1}^m \zeta_i K^* \psi_i, \quad \mathbf{f} = [\zeta_1, \ldots, \zeta_m]^T,$$

then $\mathbf{f}$ solves the linear system

$$(M + \alpha H)\mathbf{f} = \mathbf{F}, \tag{4.15}$$

where $\quad M = [\langle K^* \psi_i, K^* \psi_j \rangle_X], \quad \mathbf{F} = [\langle F^\delta, \psi_i \rangle_Y] \quad$ and $\quad H = [\langle \psi_i, \psi_j \rangle_Y]$.

The implementation and numerical solution (4.15) will yield the Tikhonov regularized solution of the ill-posed problem $Kf = F^\delta$.

In the next chapter, we examine some properties of the Laplace transform and derive the complex inversion integral.

Chapter 5

The Laplace transform

In this chapter, we present sufficient conditions for the existence of the Laplace transform, and we also prove some important properties of the Laplace transform. Finally, we provide the derivation of the complex inversion integral.

5.1 Introduction

As already mentioned in Chapter 1 of this thesis, the Laplace transform is an integral transform that is frequently used in physics and engineering applications, where it is often used to solve differential and integral equations. It is named after a French mathematician and astronomer Pierre-Simon Laplace, who introduced the transform in his work on probability theory. It is represented using the notation $\mathcal{L}\{f(t)\} : f(t) \to F(s)$, i.e., it is a linear operator acting on a function $f(t)$ with a real argument t ($t \geq 0$), such that $f(t)$ is transformed into a new function $F(s)$ with a positive real or complex argument with positive real part, see Jaeger [32] and Beerends et al. [6]. In physics and engineering applications, the Laplace transform is often used in the analysis of linear time-invariant systems such as electrical circuits, mechanical systems, optical devices, harmonic oscillators. For other applications of the Laplace transforms, see Spiegel [55], Stroud [58] and Bogart [9]. Deakin [17] present an historical development of the Laplace transform.

It is important to mention that not all functions $f(t)$ are Laplace transformable. The Dirichlet conditions must be satisfied for any function to be

Laplace transformable. We consider this and more in the next section.

5.2 Definitions and examples

In this section, we give some important definitions needed for the proofs of the properties of the Laplace transform and we also solve some example problems.

Definition 5.1. *(Uniform convergence)*
Let E be a set and $f_n : E \to \mathbb{R}$ a real valued function for every natural number n. The sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent with limit $f : E \to \mathbb{R}$ if for every $\epsilon > 0$, an integer N can be found such that for $t \in E$ and all $n \geq N$, we have $|f_n(t) - f(t)| < \epsilon$.
A series of real valued functions $\sum f_n(t)$ is said to converge uniformly on E if the sequence $\{S_n\}$ of partial sums defined by

$$\sum_{i=1}^{n} f_i(t) = S_n(t)$$

converges uniformly on E.

Definition 5.2. *(Sectionally continuous)*
A function $f(t)$ is said to be sectionally continuous (or piecewise continuous) on an interval $t \in [a, b]$ if the interval can be subdivided into a finite number of intervals, such that the function is continuous and has finite right and left hand limits.

Definition 5.3. *(Laplace transformable)*
A function $f(t)$ is said to be Laplace transformable if it satisfies the Dirichlet conditions. These are:

1. *f(t) is sectionally continuous (only a finite number of discontinuities are allowed in the function).*

2. *f(t) is of exponential order, i.e., $|f(t)| \leq Me^{\alpha t}$ at $t \to \infty$, where M and α are both real positive constants.*

Definition 5.4. *(Laplace transform)*
Let $f(t)$ be a function that satisfies the Dirichlet conditions given in Definition 5.1*, then*

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt \quad \forall\ \Re(s) > \sigma, \tag{5.1}$$

is called the Laplace transform of $f(t)$ provided that the integral (5.1) *exists, and it is uniformly convergent.*

Definition 5.5. *(Abscissa of convergence)*
Suppose that for some $\sigma = \sigma_f > 0$, $F(s)$ converges for $s > -\sigma_f$ and diverges for $s < -\sigma_f$ and that $\sigma_f < \infty$. We call σ the abscissa of convergence of the Laplace transform $F(s)$.

See Hall, Teugels and Vanmarcke [27].

This definition is motivated from the fact that if $F(s)$ converges (exits) for some s, it will also converge (exits) for all values with real part larger than $\Re(s)$. See Widder [64].

Theorem 5.6. *Let $f(t)$ be Laplace transformable and hence of exponential order with constants α, M as in Definition* 5.3*. Then the Laplace transform converges (exits) uniformly for all s with $\Re(s) > \alpha$.*

Proof. See Widder [64]. □

In order to illustrate the procedure of finding the Laplace transform of a given function $f(t)$, we consider the following examples.

Example 5.7. *Find the Laplace transform of the function $f(t) = v(t)$, with*

$$v(t) = \begin{cases} 1, & t > 0, \\ 0, & t < 0. \end{cases} \tag{5.2}$$

From the definition given in (5.1)*, we have*

$$\mathcal{L}\{v(t)\} = F(s) = \int_0^\infty v(t)e^{-st}dt = \int_0^\infty e^{-st}dt = -\frac{e^{-st}}{s}\Big|_0^\infty = \frac{1}{s}.$$

See Poularikas [52].

Example 5.8. *Find the Laplace transform of the function* $f(t) = 2\sqrt{\frac{t}{\pi}}$.

We introduce the substitution $x = t^{\frac{1}{2}} \Rightarrow dx = \frac{1}{2}t^{-\frac{1}{2}} \Rightarrow dt = 2t^{\frac{1}{2}}dx = 2xdx$,

and from the tables of standard integral, we know that $\int_0^\infty x^2 e^{-sx^2}dx = \frac{\sqrt{\pi}}{4s^{3/2}}$.

Hence, we obtain the Laplace transform $\mathcal{L}\{f(t)\} = F(s) = \frac{1}{s^{3/2}}$.

See Poularikas [52].

Example 5.9. *Find the Laplace transform of the function* $f(t) = \sinh at$.

The exponential form of the function $f(t)$ *is given as*

$$\sinh at = \frac{e^{at} - e^{-at}}{2}.$$

Using the definition of the Laplace transform given in (5.1)*, we obtain*

$$\begin{aligned} \mathcal{L}\{\sinh at\} &= F(s) = \frac{1}{2}\int_0^\infty [e^{-(s-a)t} - e^{-(s+a)t}]dt \\ &= \frac{a}{s^2 - a^2}. \end{aligned}$$

See Poularikas [52].

In the next section, we examine some important properties of the Laplace transform.

5.3 Properties of the Laplace transform

In this section, we develop some important properties of the Laplace transform and also provide their proofs. These properties are a direct application of Definition 5.2. In order to develop these properties, it is necessary that we provide the following convention: $\lim_{t\to 0} f(t)$ denote the function $f(t)$ at $t = 0$ assumed from the positive direction. The *nth* derivative at $t = 0+$ is denoted as $f^{(n)}(0+)$. With these definition and notations, we can now rewrite Equation (5.1) as

$$\mathcal{L}\{\ f(t)\} = F(s) = \lim_{\substack{G\to\infty \\ b\to 0+}} \int_b^G f(t)e^{-st}dt, \quad G > 0, \quad b > 0. \tag{5.3}$$

In the following, we assume that s is always larger (or has larger real part) than the abscissa of convergence. Thus, the integral (5.3) will always be uniformly convergent. The theorems and proofs in this section can be found in Poularikas [52].

Theorem 5.10. *(Linearity)*
Let the functions $f(t)$ and $h(t)$ be Laplace transformable. Then the Laplace transform of the linear sum of the two functions $f(t) + h(t)$ is

$$\mathcal{L}\{f(t) + h(t)\} = F(s) + H(s).$$

Proof. Applying Equation (5.3), we can write

$$\mathcal{L}\{f(t) + h(t)\} = \int_0^\infty [f(t) + h(t)]e^{-st}dt = \int_0^\infty f(t)e^{-st}dt + \int_0^\infty h(t)e^{-st}dt.$$

Thus we obtain the desired result

$$\mathcal{L}\{f(t) + h(t)\} = F(s) + H(s).$$

We note that, for given constants β_1 and β_2, the above result can be extended to

$$\mathcal{L}\{\beta_1 f(t) + \beta_2 h(t)\} = \beta_1 F(s) + \beta_2 H(s).$$

□

Theorem 5.11. *(Differentiation)*
Assume the function $f(t)$ is continuous and with sectionally continuous derivatives $\frac{df(t)}{dt}$ in the interval $0 \leq t \leq T$. Assume also that the function $f(t)$ is of exponential order $e^{\alpha t}$ as $t \to 0$. Then

$$\mathcal{L}\Big\{\frac{df(t)}{dt}\Big\} = s\mathcal{L}\{f(t)\} - f(0+) = sF(s) - f(0+).$$

Proof. Again applying the definition (5.3), we obtain

$$\mathcal{L}\Big\{\frac{df(t)}{dt}\Big\} = \lim_{T\to\infty} \int_0^T \frac{df(t)}{dt} e^{-st}dt.$$

Splitting the above integral, we obtain

$$\int_0^T f^{(1)}(t)e^{-st} = \int_0^{t_1} [\ \ ]dt + \int_{t_1}^{t_2} [\ \ ]dt + \ \cdots\ + \int_{t_{n-1}}^{T} [\ \ ]dt.$$

Using integration by part on each of the above integrals, we obtain

$$\begin{aligned} u = e^{-st} &\implies du = -se^{-st}dt \\ dv = \frac{df}{dt} &\implies v = f. \end{aligned}$$

And as a result, we write

$$\begin{aligned} \int_0^T f^{(1)}(t)e^{-st}dt &= f(t)e^{-st}\Big|_0^{t_1} + f(t)e^{-st}\Big|_{t_1}^{t_2} + \cdots + f(t)e^{-st}\Big|_T^{t_{n-1}} \\ &+ s\int_0^T f(t)e^{-st}dt. \end{aligned}$$

Recall that $f(t)$ is continuous (by assumption), thus we obtain

$$\int_0^T f^{(1)}(t)e^{-st}dt = -f(0+) + e^{-sT}f(T) + s\int_0^T f(t)e^{-st}dt.$$

Taking the limit $\lim_{T\to\infty} f(T)e^{-sT} = 0$, we get the desired result.

By extension, we can write the form

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0+) - s^{n-2}f^{(1)}(0+) - \cdots - s^{n-1}f^{(n-1)}(0+).$$

□

Theorem 5.12. *(Integration)*
Let $f(t)$ be Laplace transformable. Then the function $\int_0^t f(\zeta)d\zeta$ is Laplace transformable, and its transform is given as

$$\mathcal{L}\Big\{\int_0^t f(\zeta)d\zeta\Big\} = \frac{F(s)}{s}.$$

Proof. Since $f(t)$ is Laplace transformable, it can be seen that $\int_0^t f(\zeta)d\zeta$ is Laplace transformable as well. Then its integral can be written as

$$\mathcal{L}\Big\{\int_0^t f(\zeta)d\zeta\Big\} = \int_0^\infty \Big[\int_0^t f(\zeta)d\zeta\Big]e^{-st}dt.$$

Using integration by parts, we obtain

$$\begin{aligned} u = \int_0^t f(\zeta)d\zeta &\implies du = f(\zeta)d\zeta = f(t)dt \\ dv = e^{-st}dt &\implies v = -\frac{1}{s}e^{-st}. \end{aligned}$$

Then we get

$$\begin{aligned}\mathcal{L}\Big\{\int_0^t f(\zeta)d\zeta\Big\} &= \Big[-\frac{e^{-st}}{s}\int_0^t f(\zeta)d\zeta\Big]\Big|_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt \\ &= \frac{1}{s}\int_0^\infty f(t)e^{-st}dt.\end{aligned}$$

Hence from the above, we obtain the desired result

$$\mathcal{L}\Big\{\int_0^t f(\zeta)d\zeta\Big\} = \frac{1}{s}F(s).$$

□

Corollary 5.13. *(Division by s)*
The division of the Laplace transform of a function $F(s)$ by s is equivalent to an integration of the function between the limits 0 and t. More precisely, the relations holds

$$\begin{aligned}\mathcal{L}^{-1}\Big\{\frac{F(s)}{s}\Big\} &= \int_0^t f(\zeta)d\zeta \\ \mathcal{L}^{-1}\Big\{\frac{F(s)}{s^2}\Big\} &= \int_0^t\int_0^\zeta f(\lambda)d\lambda d\zeta\end{aligned}$$

and in general, this result also hold for division by s^n, provided that the function $f(t)$ is Laplace transformable, and we assume that the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$ is defined.

Proof. The proof of the above theorem follows as an extension of that given above in Theorem 5.12. □

Theorem 5.14. *(Multiplication by t)*
Let $f(t)$ be a Laplace transformable function, then

$$\mathcal{L}\{tf(t)\} = -\frac{dF(s)}{ds},$$

and by extension,

$$\mathcal{L}\{t^n f(t)\} = (-1)^n\frac{d^nF(s)}{ds^n}.$$

Proof. From the assumption of the uniform convergence of the integral in (5.3), we obtain

$$\frac{\partial F(s)}{\partial s} = \int_0^\infty f(t)e^{-st}(-t)dt = \mathcal{L}\{-tf(t)\}.$$

Similarly, we have

$$\frac{\partial^2 F(s)}{\partial s^2} = \int_0^\infty f(t)e^{-st}(-t)^2dt = \mathcal{L}\{t^2 f(t)\}.$$

And in general, we obtain

$$\frac{\partial^n F(s)}{\partial s^n} = \int_0^\infty f(t)e^{-st}(-t)^n dt = \mathcal{L}\{(-1)^n t^n f(t)\}.$$

□

Corollary 5.15. *(Differentiation of a transform)*
The differentiation of the Laplace transform of a function $f(t)$ is equivalent to multiplying the function by $-t$, i.e.,

$$\frac{d^n F(s)}{ds^n} = F^{(n)}(s) = \mathcal{L}\{(-t)^n f(t)\}, \quad n = 1, 2, 3, \ldots$$

Proof. The proof of this theorem follows as an extension of the proof given in Theorem 5.13. □

Theorem 5.16. *(Complex integration)*
Let $f(t)$ be a Laplace transformable function and assume that the limit $\lim_{t\to 0+} \frac{f(t)}{t}$ exists, then the integral of the function $\int_s^\infty F(s)ds$ is equivalent to the Laplace transform of the division of the function by t, i.e.,

$$\mathcal{L}\Big\{\frac{f(t)}{t}\Big\} = \int_0^\infty F(s)ds.$$

Proof. Let the function $f(t)$ be piecewise continuous and of exponential order, then

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

is uniformly convergent with respect to s. Integrating both side of the equation, and for $\Re(s) > \alpha$ and any $b > \alpha$, we write

$$\int_s^b F(s)ds = \int_s^b \int_0^\infty f(t)e^{-st}dtds.$$

Rearranging the right hand side of the above integral, we obtain

$$\int_s^b F(s)ds = \int_0^\infty f(t) \int_s^b e^{-st}dsdt = \int_0^\infty \frac{f(t)}{t}(e^{-st} - e^{-bt})dt.$$

Letting $b \to \infty$ and assuming that the limit $\lim_{t\to\infty} \frac{f(t)}{t}$ exists, then the Dirichlet conditions are satisfied, and as a result the last integral is uniformly convergent with respect to b. Hence, we obtain the desired result

$$\int_s^\infty F(s)ds = \mathcal{L}\Big\{\frac{f(t)}{t}\Big\}.$$

□

Theorem 5.17. *(Time delay; real translation)*
Let $f(t)$ be a function with Laplace transform $\mathcal{L}\{f(t)\}$, then substituting $t-\lambda$ for the variable t is equivalent to multiplying the function $F(s)$ by $e^{-\lambda s}$, i.e.,

$$\mathcal{L}\{f(t-\lambda)v(t-\lambda)\} = e^{-\lambda s}F(s).$$

where $v(t)$ is defined in (5.2)

Proof. Consider the function $f(t)v(t)$ which is delayed by $t = \lambda, \ \ \lambda > 0$. By the definition of Laplace transform, we write

$$\mathcal{L}\{f(t-\lambda)v(t-\lambda)\} = \int_0^\infty f(t-\lambda)v(t-\lambda)e^{-st}dt.$$

Making use of the substitution $\gamma = t - \lambda$, we obtain

$$\mathcal{L}\{f(\gamma)v(\gamma)\} = e^{-s\lambda}\int_{-\lambda}^\infty f(\gamma)v(\gamma)e^{-s\gamma}d\gamma = e^{-s\lambda}\int_0^\infty f(\gamma)e^{-s\gamma}d\gamma = e^{-s\lambda}F(s)$$

where $v(\gamma) = 0$ for $-\lambda \le t \le 0$.

Similarly, we have $\mathcal{L}\{f(t+\lambda)v(t+\lambda)\} = e^{s\lambda}F(s)$. □

Theorem 5.18. *(Complex translation)*
Let $F(s)$ be the Laplace transform of the function $f(t)$, then the substituting $s+b$ for s (where b is real or complex) in the function $F(s+b)$, is equivalent to the Laplace transform of the product $e^{-bt}f(t)$.

Proof. From the definition of Laplace transform given in (5.3) and for $\Re(s) > \alpha - \Re(b)$ we obtain

$$\mathcal{L}\{e^{-bt}f(t)\} = \int_0^\infty e^{-bt}f(t)e^{-st}dt = \int_0^\infty f(t)e^{-(s+b)t}dt = F(s+b).$$

Similarly, we have

$$\mathcal{L}\{e^{bt}f(t)\} = F(s-b).$$

□

Theorem 5.19. *(Convolution)*
The multiplication of the transforms of the two Laplace transformable functions $f_1(t)$ and $f_2(t)$ is equivalent to the Laplace transform of the convolution of $f_1(t)$ and $f_2(t)$

$$F_1(s)F_2(s) = \mathcal{L}\{f_1(t) * f_2(t)\}.$$

Where the convolution of the two functions $f_1(t)$ and $f_2(t)$ is defined as

$$f_1(t) * f_2(t) = \int_0^t f_1(t-\gamma)f_2(\gamma)d\gamma = \int_0^t f_1(\gamma)f_2(t-\gamma)d\gamma.$$

Proof. From the definition given in (5.3), we have

$$\begin{aligned}
\mathcal{L}\{f_1(t) * f_2(t)\} &= \mathcal{L}\Big\{\int_0^t f_1(t-\gamma) * f_2(\gamma)d\gamma\Big\} \\
&= \int_0^\infty \Big[\int_0^t f_1(t-\gamma) * f_2(\gamma)d\gamma\Big]e^{-st}dt \\
&= \int_0^\infty f_2(\gamma)\int_\gamma^\infty f_1(t-\gamma)e^{-st}dtd\gamma,
\end{aligned}$$

where we used Fubini's theorem to change the order of integration. Introducing the change of variable in the above, i.e., $t-\gamma = \zeta \;\Rightarrow\; dt = d\zeta$, we obtain

$$\mathcal{L}\{f_1(t) * f_2(t)\} = \int_0^\infty f_2(\gamma)d\gamma\int_0^\infty f_1(\zeta)e^{-s(\zeta+\gamma)}d\zeta,$$

which leads to the desired result

$$\mathcal{L}\{f_1(t) * f_2(t)\} = F_1(s)F_2(s).$$

□

Corollary 5.20. *Let the functions $f_1(t), f_2(t), f_3(t)$ be Laplace transformable, then*

$$\mathcal{L}\{f_1(t) * f_2(t) * f_3(t)\} = \mathcal{L}\big\{f_1(t) * \{f_2(t) * f_3(t)\}\big\} = F_1(s)F_2(s)F_3(s).$$

Proof. The proof follows as a direct extension of Theorem 5.19. □

Theorem 5.21. *(Initial value theorem)*
Let the function $f(t)$ and its derivative $f^{(1)}(t)$ be Laplace transformable, then

$$\lim_{s\to\infty} sF(s) = \lim_{t\to 0+} f(t)$$

holds, provided the limit $\lim_{s\to 0} sF(s)$ exists.

Proof. From Theorem 5.11, we have

$$\mathcal{L}\Big\{\frac{df(t)}{dt}\Big\} = \int_0^\infty \frac{df(t)}{dt}e^{-st}dt = sF(s) - f(0+)$$

and as $s \to 0$, we can write

$$\lim_{s\to 0}\int_0^\infty \frac{df(t)}{dt}e^{-st}dt = \lim_{s\to 0}[sF(s) - f(0+)].$$

Since the integration (on the left hand side) above is independent of s, the calculation of the limit and the integration can be interchanged provided (assumption) the integral (on the left hand side) converges uniformly. Since the Laplace transform exits, then

$$\lim_{s\to 0}\frac{df(t)}{dt}e^{-st} = 0$$

hold. Hence, we get

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$$

□

We note that the initial value theorem does not hold if $f(t)$ has an impulse term.

Theorem 5.22. *(Final value theorem)*
Let the function $f(t)$ and its derivative $f^{(1)}(t)$ be Laplace transformable, then for $t \to \infty$,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

Proof. Again from Theorem 5.11, we have

$$\mathcal{L}\Big\{\frac{df(t)}{dt}\Big\} = \int_0^\infty \frac{df(t)}{dt} e^{-st} dt = sF(s) - f(0+)$$

and we evaluate the limit

$$\lim_{s\to 0} \int_0^\infty \frac{df(t)}{dt} e^{-st} dt = \lim_{s\to 0} [sF(s) - f(0+)].$$

Similarly, we can permute the sequence of limit and integration provided (assumption) the integral (on the left hand side) converges uniformly. Then, the result

$$\int_o^\infty \frac{df(t)}{dt} dt = \lim_{s\to 0} [sF(s) - f(0+)]$$

holds. And after integration, we get

$$\lim_{t\to\infty} f(t) - f(0+) = \lim_{s\to 0} [sF(s) - f(0+)].$$

Hence, we have that the required result holds, i.e.,

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

□

Here, we note that the final value theorem does not hold if $F(s)$ has imaginary axis poles, poles in the right half plane, or poles of higher order at the origin. However, a simple pole at the origin of $F(s)$ may occur and the result will still hold.

Theorem 5.23. *The Laplace transform is a continuous operator between $\mathcal{L} : L^2(0,\infty) \to L^2(0,\infty)$. Moreover, $\mathcal{L}$ is not compact.*

See Boumenir and Al-Shuaibi [11].

Remark 5.24. *We recall that the spectrum of compact operators have at most countably many eigenvalues. Now, since the spectrum of the operator* $\mathcal{L} \equiv \int_0^\infty \cdot\, e^{-st} dt$ *not countable as it was shown in Al-Shuaibi [11], therefore the Laplace transform operator is not compact. However, by the results of Chapter* 2*, the Laplace transform is compact of it is restricted to a finite interval* $\mathcal{L} : L^2(0,T) \to L^2(0,T)$*.*

5.4 The inverse Laplace transform

Given a function $f(t)$ with Laplace transform $F(s)$, the inverse Laplace transform corresponding to the Definition 5.4 is denoted as $\mathcal{L}^{-1}\{F(s)\}$. Thus, the equivalent relationship existing between the direct Laplace transform and its inverse is given as

$$F(s) = \mathcal{L}\{f(t)\}, \quad f(t) = \mathcal{L}^{-1}\{F(s)\}.$$

By definition, we note that the determination of the Laplace transform $F(s)$ for a given function $f(t)$ is unique. In a similar way, using the complex inversion integral, it can be shown also that for a given $F(s)$, there exists a unique $f(t)$. This implies that there exists a one-to-one equivalence between the Laplace transform and its inverse, justifying the notation $\mathcal{L}^{-1}$.

It should be noted that the study of the inverse Laplace transform is very important because many solutions of practical problems usually provide a known $F(s)$ from which $f(t)$ has to be reconstructed.

We now illustrate, by examples, the use of partial fractions in reconstructing the original function $f(t)$ from its Laplace transform $F(s)$.

Example 5.25. *Find the inverse Laplace transform of*

$$F(s) = \frac{s-3}{s^2+5s+6}.$$

By partial fractions, we can write

$$F(s) = \frac{s-3}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}. \tag{5.4}$$

To determine the constant A, we multiply (5.4) *by* $(s+2)$ *and set* $s=-2$. *This leads to*

$$A = F(s)(s+2)\Big|_{s=-2} = \frac{s-3}{s+3}\Big|_{s=-2} = -5.$$

Similarly, to determine B, we multiply (5.4) *by* $(s+3)$ *and set* $s=-3$

$$B = F(s)(s+3)\Big|_{s=-3} = \frac{s-3}{s+2}\Big|_{s=-2} = 6.$$

Using tables of Laplace transforms, we obtain

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} = -5\mathcal{L}^{-1}\Big\{\frac{1}{s+2}\Big\} + 6\mathcal{L}^{-1}\Big\{\frac{1}{s+3}\Big\} \\ &= -5e^{-2t} + 6e^{-3t}. \end{aligned}$$

Example 5.26. *Find the inverse Laplace transform of*

$$F(s) = \frac{s+1}{[(s+2)^2+1](s+3)}.$$

Again, by partial fractions, we can write

$$F(s) = \frac{s+1}{[(s+2)^2+1](s+3)} = \frac{A}{s+3} + \frac{Bs+C}{[(s+2)^2+1]}. \tag{5.5}$$

To obtain the value of the constant A, we multiply (5.5) *by* $(s+3)$ *and set* $s=-3$. *This leads to*

$$A = F(s)(s+3)\Big|_{s=-3} = \frac{-3+1}{(-3+2)^2+1} = -1.$$

To determine the values of B and C, we proceed as follows: merge the fractions in (5.5) *and using the value of* $A=-1$, *we obtain*

$$F(s) = \frac{-1[(s+2)^2+1] + (s+3)(Bs+C)}{[(s+2)^2+1](s+3)} = \frac{-3+1}{(-3+2)^2+1} = -1.$$

Rearranging in terms of the powers of s, we obtain

$$-(s^2+4s+5) + Bs^2 + (C+3B)s + 2C = s+1,$$

and from which we find

$$(-1+B)s^2+(-4+C+3B)s+(-5+3C)=s+1.$$

This implies: $-1+B=0,\ -4+C+3B=1, -5+3C=1,$ *and hence* $B=1, C=2.$

Using tables of Laplace transforms, we obtain the inverse Laplace transform

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\{F(s)\} = -\mathcal{L}^{-1}\Big\{\frac{1}{s+3}\Big\}+\mathcal{L}^{-1}\Big\{\frac{s+3}{(s+2)^2+1}\Big\} \\ &= -e^{-3t}+e^{-2t}\cos t, \quad t>0. \end{aligned}$$

A very important theorem useful in the inversion of the Laplace transform is the *Heaviside expansion theorem* which is given below.

Theorem 5.27. *Let $f(t)$ be a Laplace transformable function, and $F(s) = P(s)/Q(s)$ be the corresponding transform and a ratio of two polynomials in s such that $Q(s)$ has a higher degree. Assume also that $Q(s)$ consist of simple poles, the factor $s-s_k$ is not repeated. Then the inverse Laplace transform is given as*

$$\mathcal{L}^{-1}\{F(s)\} = \frac{P(s_k)}{Q^{(1)}s_k}e^{s_k t}.$$

An example will make the use of the above theorem clearer.

Example 5.28. *Using the Heaviside expansion theorem, we redo the Example* 5.26*, i.e., to obtain the inverse Laplace transform of*

$$F(s)=\frac{s+1}{[(s+2)^2+1](s+3)}.$$

Using the Heaviside expansion theorem, we can re-write the above problem as

$$F(s)=\frac{P(s)}{Q(s)}=\frac{s-3}{s^2+5s+6}=\frac{s+3}{(s+2)(s+3)}.$$

Differentiating the function $Q(s)$, we obtain

$$Q^{(1)}(s)=2s+5 \quad \Longrightarrow \quad Q^{(1)}(-2)=1,\ Q^{(1)}(-3)=-1.$$

Thus, we obtain the inverse Laplace transform

$$f(t)=\mathcal{L}^{-1}\{F(s)\}=-5e^{-2t}+6e^{-3t}.$$

The complex inversion integral

An integral formula named after Hjalmar Mellin, Joseph Fourier and Thomas John I'Anson Bromwich that is commonly used for reconstructing the inverse Laplace transform is called the Bromwich integral (also known as the Fourier-Mellin integral, and Mellin's inverse formula) and it is given by the contour integral. We present this in the next theorem.

Theorem 5.29. *(Bromwich inversion integral)*
Let $f(t)$ be a Laplace transformable function and $F(s)$ the corresponding transform function. The function $f(t)$ can be reconstructed from the contour integral

$$f(t) = \frac{1}{2\pi i} \lim_{\omega \to \infty} \int_{\sigma - i\omega}^{\sigma + i\omega} F(z) e^{zt} dz \tag{5.6}$$

where $i = \sqrt{-1}$.

For a rigorous proof of the Bromwich inversion integral, see Widder [64].

In what follows however, we provide a heuristic derivation of Theorem 5.29. We recall the equivalence relation existing between the Laplace transform and its inverse (i.e., the Laplace transform pair):

$$F(s) = \mathcal{L}\{f(t)\}, \quad f(t) = \mathcal{L}^{-1}\{F(s)\}$$

and Cauchy's second integral theorem,

$$\oint \frac{F(z)}{s - z} dz = i2\pi F(s), \tag{5.7}$$

where the contour encloses the singularities at s. Let the Laplace transform $F(s)$ be analytic in the half-plane $\Re(s) \geq \alpha$. Taking the inverse Laplace transform of the functions in s on both sides of (5.7) and assuming that we can interchange the integral and the inverse Laplace transform operator $\mathcal{L}^{-1}$, we obtain

$$i2\pi \mathcal{L}^{-1}\{F(s)\} = \lim_{\omega \to \infty} \int_{\sigma - i\omega}^{\sigma + i\omega} F(z) \mathcal{L}^{-1}\Big\{\frac{1}{s - z}\Big\} dz.$$

Since $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and from the tables of Laplace transforms, we know that $\mathcal{L}^{-1}\Big\{\frac{1}{s-z}\Big\} = e^{zt}$, thus we get the desired result.

The Post inversion formula

Apart from the Complex inversion integral given in Theorem 5.29, in theory, the Post inversion formula (named after Emil Leon Post) can also be used to reconstruct the inverse Laplace transform. Although this inversion formula looks simple, in reality however, it is usually impractical for evaluating an inverse Laplace transform. We note that the complex inversion formula is more practical for reconstructing an inverse Laplace transform than the Post inversion formula. We formulate the Post inversion formula in the next theorem.

Theorem 5.30. *(Post inversion formula)*
Let $f(t)$ be a Laplace transformable function, then the inverse Laplace transform is given by

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \lim_{k\to\infty} \frac{(-1)^k}{k!}\Big(\frac{k}{t}\Big)^{k+1} F^{(k)}\Big(\frac{k}{t}\Big)$$

for $t > 0$, where $F^{(k)}$ is the $k-th$ derivative of F with respect to s.

See Post [51].

In the next theorem, we prove that $f(t)$ is uniquely determined by $F(s)$.

Theorem 5.31. *For a given Laplace transform function $F(s)$, there exists a unique $f(t)$.*

Proof. Let $f_1(t)$ and $f_2(t)$ be two Laplace transformable functions with the same transforms $F(s)$, i.e.,

$$\mathcal{L}\{f_1(t)\} = \mathcal{L}\{f_2(t)\} = F(s).$$

We introduce the definition, $\vartheta(t) := f_1(t) - f_2(t)$,

where the function $\vartheta(t)$ is also Laplace transformable. Then we get

$$\mathcal{L}\{\vartheta(t)\} = F(s) - F(s) = 0.$$

Moreover, from the complex inversion integral (5.6), we have

$$\vartheta(t) = \mathcal{L}^{-1}\{0\} = 0.$$

And from this, we obtain the desired result, i.e., $f_1(t) = f_2(t)$. This implies that the Laplace transform pair is unique. □

In the case where the Laplace transform is measured or computed only on the positive real axis, then the problem is extremely ill-posed. In such a case, a stable inversion formula does not exist, and we therefore cannot use analytical methods, Laplace transform tables or the complex inversion formula to reconstruct the inverse Laplace transform. As a result, numerical methods have to be used in reconstructing the function $f(t)$ from its transform.

We note that the complex inversion formula is not applicable for numerical implementation. This is so because, for it to be used, one has to know the analytic extension of $F(s)$ to the complex plane. Now the analytic extension is, however, an ill-posed problem, see Engl, Hanke, Neubauer [21] and Hadamard [26].

In the next chapter, we examine some numerical methods for the reconstruction of the inverse Laplace transform.

Chapter 6

Numerical inversion of the Laplace transform

In this chapter, we derive three numerical algorithms for the inversion of the Laplace transform: the Gaver-Stehfest method, the Piessens method, and the regularized collocation method.

6.1 Introduction

As a result of the importance of reconstructing the original function $f(t)$ from its transform $F(s)$, several numerical algorithms has been developed. The survey and comparison by Davis and Martin [16] tests 14 inversion algorithms on a set of 16 transform functions. Piessens [48] compiled an extensive bibliography up to 1975. The book by Bellman [8] presents a wide range of applications of numerical inversion. Krylov, Shoblya and Louvish's [38] work covers the theoretical basis of a number of inversion methods but do not include their implementation or present numerical results. Cohen's [13] work also contains some numerical methods for the Laplace transform inversion. For more references, see the review given in Chapter 1 of this thesis.

We begin by recalling the definition of the Laplace transform given in the previous chapter, i.e.,

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt. \tag{6.1}$$

The discrete form of (6.1) is given as

$$\mathcal{L}\{f(t)\} = F(s_i) = \int_0^\infty f(t)e^{-s_i t}dt, \quad i = 1, 2, \ldots, N, \tag{6.2}$$

where the data $F(s_i)$ and the Laplace transform operator $K \equiv \int_0^\infty \cdot\, e^{-st}dt$ are both given and we seek to reconstruct $f(t)$ or possibly only a finite number of functions values $\big(f(t_i)\big)_{i=1}^N$.

Now based on Chapter 2 of this thesis, we know that the Laplace transform is an integral equation of the first kind. Although it should be noted (see Boumenir and Al-Shuaibi [11]) that the Laplace transform operator is not a compact operator between L^2-spaces because the domain of integration is not compact. However, the inversion of the Laplace transform shares many similarities with integral of the first kind with a compact operator. It is well known that the task of reconstructing a real function $f(t),\ t \geq 0$ from (6.1) is an ill-posed problem in the sense of Hadamard's definition of well-posedness described in Section 3.3. As a result, the solution of the inverse Laplace transform is affected by numerical instability (see Mcwhirter and Pike [41] and Brianzi and Frontini [12]). In the following, we examine the ill-posedness of the problem.

Ill-posedness of the problem

We show here, that the problem of reconstructing the function $f(t)$ from (6.1) is ill-posed. To this end, we consider the Laplace transform as an operator $\mathcal{L}$ mapping from $C_{0\beta}$ into the space L^2, i.e., $\mathcal{L} : C_{0\beta} \to L^2$, where $\mathcal{L} \equiv K = \int_0^\infty \cdot\, e^{-st}dt$ and

$$\begin{aligned} C_{0\beta} &= \{f(t) \in C[0, +\infty) \mid \textit{supp } f \subset [0, \beta)\}, \\ L^2 &= L^2[0, +\infty), \quad \beta = \text{const.} > 0. \end{aligned}$$

With the above setting, we examine the conditions in Section 3.3 (i.e., Hadamard's conditions of well-posedness) with respect to the inverse Laplace transform operator:

1. $\mathcal{R}(K) \neq L^2$, i.e., the existence condition is not satisfied. This is true because it is possible that $F(s) \in L^2$ but $F(s) \notin \mathcal{L}\{C_{0\beta}\}$. (See Dong [20]).

2. $\mathcal{N}(K) = \{0\}$, i.e., the uniqueness condition is satisfied (see Theorem 5.31).

3. $K^{-1} \notin L(L^2, C_{0\beta})$, i.e., the solution $f(t)$ is not continuously dependent on the data $F(s)$, and this leads to an instability of the solution. To show this, we consider (see Dong [20]) the function

$$f_n(t) = \begin{cases} \sin nt, & 0 \le t \le 2\pi, \\ 0, & t > 2\pi. \end{cases}$$

In this case, we consider $f(t) = 0$ the exact solution of the equation (6.1). It is easy to see that $f_n(t), f(t) \in C_{0,\beta}$.

From tables of Laplace transforms, we know that $\mathcal{L}\{\sin nt\} = \frac{n}{s^2+n^2}$, and furthermore $\mathcal{L}\{f_n\} = \frac{n}{s^2+n^2} - \frac{ne^{-2\pi s}}{s^2+n^2}$, thus the L^2-norm is given as:

$$\begin{aligned} \|\mathcal{L}\{f_n(t)\} - \mathcal{L}\{f(t)\}\|_{L^2}^2 &\le \int_0^\infty \Big[\frac{n}{s^2+n^2} - \frac{ne^{-2\pi s}}{s^2+n^2}\Big]^2 ds \\ &= \int_0^\infty \Big[\frac{n - ne^{-2\pi s}}{s^2+n^2}\Big]^2 ds \\ &\le n^2 \int_0^\infty \frac{ds}{(s^2+n^2)^2} = \frac{\pi}{4n} \to 0 \quad (n \to \infty). \end{aligned}$$

Whereas, for the $C_{0\beta}$-norm, we have:

$$\|f_n(t) - f(t)\|_{C_{0\beta}}^2 = \max_{0 \le t \le \beta} |\sin nt| = 1$$

The instability is evident from the difference in the values of the L^2-norm and the $C_{0\beta}-$norm.

In the next section, we consider the Gaver-Stehfest method for the inversion of the Laplace transform.

6.2 The Gaver-Stehfest method

In this section, we present the Gaver-Stehfest method for the numerical inversion of the Laplace transform which was introduced by D.P. Gaver but

was later improved by H. Stehfest. The formula for the inversion as given by Stehfest [56] is

$$f_{num}(t) = \frac{\ln 2}{t} \sum_{i=1}^{N} G_i F\Big(\frac{\ln 2}{t} i\Big), \tag{6.3}$$

where N must be an even integer.

Derivation of the method

The method was initially introduced by Gaver [24], where he considered the expectation of the function $P(t)$ with respect to the probability density function

$$\begin{aligned} \rho_n(b,t) &= b\frac{(2n)!}{n!(n-1)!}(1-e^{-bt})^n e^{-nbt}, \quad b>0, \quad n=1,2,3,\ldots, \\ \bar{P}_n &= \int_0^\infty P(t)\rho_n(b,t)dt, \end{aligned} \tag{6.4}$$

where $\bar{P}_n$ is the probability that a particular process is in the state n when it is observed, and $P(t)$ is some function describing the process at time t.

Gaver based this method on the asymptotic expansion

$$\bar{P}_n \sim P\Big(\frac{\ln 2}{\eta}\Big) + \frac{\eta_1}{n} + \frac{\eta_2}{n^2} + \frac{\eta_3}{n^3} + \cdots$$

However, Stehfest [56] later improved the work done by Gaver for approximating $P(\frac{\ln 2}{\eta})$ using a linear combination of $\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_{N/2}$ and requiring that

$$\sum_{i=1}^{K} x_i(K) \frac{1}{(N/2+1-i)^j} = \delta_{j0}, \qquad j=0,1,\ldots,K-1, \qquad K \le \frac{N}{2},$$

where the function $x_i(K)$ is given as

$$x_i(K) = \frac{(-1)^{i-1}}{K!}\binom{K}{i} i(N/2+1-i)^{K-1}.$$

This implies that

$$\sum_{i=1}^{K} x_i(K)\bar{P}_{N/2+1-i} = P\Big(\frac{\ln 2}{\eta}\Big) + (-1)^{K+1}\eta_K \frac{(N/2-K)!}{(N/2)!} + o\Big(\frac{(N/2-K)!}{(N/2)!}\Big).$$

From equation (6.4), and setting $b = \frac{\ln 2}{t}$, $K = \frac{N}{2}$, we obtain the desired result

$$f_{num}(t) = \sum_{i=1}^{N/2} x_i(N/2)\bar{P}_{N/2+1-i} = \frac{\ln 2}{t}\sum_{i=1}^{N} G_i F\Big(\frac{\ln 2}{t} i\Big),$$

where the coefficient G_i present in (6.3) is given by the expression

$$G_i = (-1)^{N/2-i} \sum_{j=[\frac{i+1}{2}]}^{\min(i,N/2)} \frac{j^{N/2}(2j)!}{(N/2-j)!j!(j-1)!(i-j)!(2j-i)!}. \tag{6.5}$$

We note that some errors have crept into the original paper by Stehfest [56]. Significant among them is in the calculation of the coefficient G_i, in which 1 was mistakenly added to the index of a term in the numerator, i.e., $j^{N/2+1}(2j)!$, and this yields a bad approximation. However, Stehfest [57] later gave an erratum which contained the correct form, i.e., $j^{N/2}(2j)!$ which is given above in (6.5).

We note that in theory, the approximate solution $f_{num}(t)$ becomes more accurate as N increases, and $\lim_{N\to\infty} f_{num}(t) = f(t)$ for exact data. In reality however, this is not the case, because round-off errors begins to set in and thus worsen the approximation as N becomes too large. This is so because increasing the values of N implies that the absolute values of G_i begin to increase as well. Moreover, since the coefficients G_i appear with different signs, cancelation effects set in. This is to be expected since the inverse Laplace transform operator is not bounded and as a result the solution $f_{num}(t)$ will not depend continuously on the data $F(s)$ as N tends to ∞.

We observe that the Gaver-Stehfest method is only applicable in the case of exact data $F(s)$ and not suitable for dealing with noisy data $F^{\delta}(s)$. This is so because the method has no regularization properties.

Algorithm 6.1. *(Gaver-Stehfest method)*

1. *Choose a reasonable value for N (which must be an even integer).*
2. *Set up the vector $F\big(\frac{\ln 2}{t}i\big)$.*
3. *Compute the value of the coefficient G_i using* (6.5).
4. *Compute (i.e., reconstruct) the numerical approximation $f_{num}(t)$ using* (6.3).

In the next section, we examine the Piessens method.

6.3 The Piessens method

In this section, we present a numerical method for the inversion of the Laplace transform which was developed by Piessens [48] and is given as

$$f_{num}(t) = e^{\beta t}\frac{t^{a-1}}{\Gamma(a)}\sum_{k=0}^{\infty} c_k\phi_k\Big(\frac{bt}{2}\Big). \tag{6.6}$$

Derivation of the method

Piessens based his method on the fact that the Laplace transform $F(s)$ can be expanded using Chebyshev polynomials (see Definition 7.8 in the appendix) of the first kind. The computer realization of the algorithm is described by Piessens [46], and a convergence result of this method was presented by Cope [14].

To begin, we assume that the function $f(t)$ has a (one-sided) Laplace transform $F(s)$. We also assume that the function $F(s)$ is analytical for $\Re(s) > \beta$ and there exists a parameter $a > 0$ such that $s^aF(s)$ possesses a removable singularity at infinity. With this setting (assumptions), the Laplace transform can be expanded as

$$F(\beta + s) = s^{-a}{\sum_{k=0}^{\infty}}' c_kT_k\Big(1 - \frac{b}{s}\Big), \tag{6.7}$$

where $T_k(x)$ is the Chebyshev polynomials of the first kind and of degree k, and c_k's are the corresponding Chebyshev coefficients. The parameter b is positive and arbitrary, and the single prime in the summation (6.7) indicate that the first term is pre-multiplied by the factor $\frac{1}{2}$.

Taking the inverse Laplace transform on both sides of Equation (6.7), we obtain the inversion formular given above in (6.6) where

$$\phi_k(x) = \left[\begin{array}{cc} -k & k \\ & ; x \\ \frac{1}{2} & a \end{array} \right]$$

is a generalized hypergeometric function (see Definition 7.9 in the appendix).

For computer implementation, we truncate the infinite series in (6.6) after some $N+1$ terms, and we obtain

$$f_{num}(t) = e^{\beta t} \frac{t^{a-1}}{\Gamma(a)} \sum_{k=0}^{N} c_k^* \phi_k\Big(\frac{bt}{2}\Big) + \varepsilon(t) \tag{6.8}$$

where $\varepsilon(t)$ is the truncation error term. In order to determine the coefficients c_k^*'s of the truncated Chebyshev series expansion in (6.8), we solve the approximate system

$$\Big(\frac{b}{1-u}\Big)^a F\Big(\frac{b}{1-u} + \beta\Big) \simeq \sum_{k=0}^{N} c_k^* T_k(u). \tag{6.9}$$

It is obvious from (6.9) that the c_k^*'s are approximations of the Chebyshev coefficients $c_k, \quad k = 0, 1, \ldots, N$ in (6.8).

In order to determine the generalized hypergeometric function $\phi_k(x)$ in (6.8), we make use of the recurrence relation

$$\phi_n = (A_n + B_n x)\phi_{n-1} + (C_n + D_n x)\phi_{n-2} + E_n \phi_{n-3} \tag{6.10}$$

where

$$\begin{aligned} A_n &= \frac{3n^2 - 9n + an - 3a + 6}{(n + a - 1)(n - 3)} \\ B_n &= \frac{-4}{(n + a - 1)} \\ C_n &= -\frac{3n^2 - 9n - an + 6}{(n - 2)(n + a - 1)} \\ D_n &= -\frac{4(n - 1)}{(n + a - 1)(n - 2)} \\ E_n &= -\frac{(n - 1)(n - a - 2)}{(n + a - 1)(n - 2)} \end{aligned}$$

and the first three terms of the generalized hypergeometric function $\phi_k(x)$ in (6.10) are given as

$$\begin{aligned} \phi_0(x) &= 1 \\ \phi_1(x) &= 1 - \frac{2x}{a} \\ \phi_2(x) &= 1 - \frac{8x}{a} + \frac{8}{a(a+1)}x^2. \end{aligned}$$

Finally, we compute $\varepsilon^*(t)$ by an approximation of the truncated error term $\varepsilon(t)$. This estimate is given by the expression:

$$\varepsilon^*(t) = e^{\beta t}\frac{t^{a-1}}{\Gamma(a)}\sum_{k=N-2}^{N} c_k^* \phi_k\Big(\frac{bt}{2}\Big).$$

We note that, like the Gaver-Stehfest method, the Piessens method works best in the case of exact data. Since the method has no regularization property, it is not suitable for dealing with noisy data.

Algorithm 6.2. *(Piessens method)*

1. *Chose the parameters β, a, b, N:*
   - *(a) β such that $\Re(s) > \beta$*
   - *(b) $a > 0$ such that $s^a F(s)$ has a removable singularity at infinity*

(c) $b > 0$ *considering the analytic property of* $F(s)$

(d) N *is taken as the length of the sampling points of* $F(s)$

2. *Set up the vector* $F\Big(\frac{b}{1-u} + \beta\Big)$.

3. *Compute the truncated Chebyshev coefficients* c_k^* *by solving the approximate system in* (6.9).

4. *Compute the generalized hypergeometric function* $\phi_n(x)$ *using the recurrence relation in* (6.10).

5. *Compute the gamma* $\Gamma(a)$ *and exponential* $e^{\beta t}$ *terms.*

6. *Compute (i.e., reconstruct) the numerical approximation* $f_{num}(t)$ *using* (6.8).

In the next section, we construct the regularized collocation method.

6.4 The regularized collocation method

In this section, we construct a Tikhonov based regularized collocation method for the numerical inversion of the Laplace transform which is given as

$$f_{num}(t) = -\frac{1}{\alpha}\sum_{i=1}^{N} e^{-s_i t} c_i. \tag{6.11}$$

Derivation of the method

The method is based on the Tikhonov regularization method described in Section 4.4 of this thesis and by Engl, Hanke and Neubauer [21]. To begin, we recall the definition of the Laplace transform given in Equation (6.2), where it is evaluated at some distinct collocation points $s_i,\ i = 1, \ldots, N$, i.e.,

$$F(s_i) = \int_0^\infty f(t) e^{-s_i t} dt, \quad i = 1, 2, \ldots, N. \tag{6.12}$$

The Tikhonov minimization functional corresponding to (6.12) is given as

$$J(f) = \sum_{i=1}^{N} \Big[\int_0^\infty f(t) e^{-s_i t} dt - F(s_i) \Big]^2 + \alpha \|f(t)\|_{L^2}^2$$

and writing the regularization term as an integral, we get

$$J(f) = \sum_{i=1}^{N} \Big[\int_0^\infty f(t)e^{-s_i t}dt - F(s_i) \Big]^2 + \alpha \int_0^\infty f(t)^2 dt \tag{6.13}$$

where α is the Tikhonov regularization parameter.

From the definition of the Gateaux derivative (see Definition 7.11 in the appendix, and Behmardi and Nayeri [49]), we know that

$$J'(f;h) = \lim_{\varepsilon \to 0} \frac{J\big(f(t) - \varepsilon h(t)\big) - J\big(f(t)\big)}{\varepsilon} \tag{6.14}$$

provided the limit exists.

Thus by Gateaux differentiability, equation (6.13) becomes

$$J'(f;h) = 2 \int_0^\infty \Big[\Big(\sum_{i=1}^{N} f(t)e^{-s_i t}dt - F(s_i) \Big) e^{-s_i t} + \alpha f(t) \Big] h(t)dt. \tag{6.15}$$

Since the optimality condition corresponding to the minimization problem in (6.13) is $J'(f;h) = 0$, we can write

$$\sum_{i=1}^{N} \underbrace{\Big[\int_0^\infty f(t)e^{-s_i t}dt - F(s_i) \Big]}_{c_i} e^{-s_i t} + \alpha f(t) = 0$$

from which we get

$$-\frac{1}{\alpha} \sum_{i=1}^{N} e^{-s_i t} c_i = f(t), \tag{6.16}$$

where

$$c_i = \int_0^\infty f(t)e^{-s_i t}dt - F(s_i). \tag{6.17}$$

Substituting (6.16) into (6.17) yields

$$\frac{1}{\alpha} \sum_{i=1}^{N} \underbrace{\int_0^\infty e^{-(s_i + s_j)t}dt}_{M_{ij}} c_j + c_i = -F(s_i)$$

and the resulting system

$$\Big(\frac{1}{\alpha}M_{ij}c_j + c_i I\Big) = -F(s_i), \quad i = 1, 2, \ldots, N. \tag{6.18}$$

The coefficients c_i's given by (6.17) are obtained by solving the linear system

$$(\mathbf{M}\mathbf{c} + \alpha I\mathbf{c}) = -\alpha\mathbf{F}, \tag{6.19}$$

where

$$\begin{aligned} \mathbf{M} &= \int_0^\infty e^{-(s_i+s_j)t}dt & (6.20)\\ \mathbf{F} &= [F(s_1), F(s_2), \cdots, F(s_N)]^T & (6.21)\\ \mathbf{c} &= (\mathbf{M} + \alpha I) \setminus (-\alpha\mathbf{F}). & (6.22) \end{aligned}$$

We observe that the regularized collocation method is an excellent numerical method for the inversion of Laplace transform. This is so because, the method has an efficient regularization property that works very well even in the presence of noisy data.

Algorithm 6.3. *(regularized collocation method)*

1. *Set up the matrix M_{ij} and vector $F(s_i)$ using* (6.20) *and* (6.21) *respectively.*
2. *Calculate the coefficients c_i using* (6.22).
3. *Choose the Tikhonov regularization parameter α using the discrepancy principle given in* (4.12).
4. *Compute (i.e., reconstruct) the numerical approximation $f_{num}(t)$ using* (6.11).

We note that the matrix M_{ij} can be computed analytical, i.e., $M_{ij} = \frac{1}{s_i+s_j}$, and we use the left division (i.e., back slash) in MATLAB to calculate the coefficients $\mathbf{c}$ appearing in (6.22).

Remark 6.4. *From the three methods considered, we observe that the Gaver-Stehfest method need the samples of the Laplace transform $F(s)$ at the points $\big(\frac{\ln 2}{t}i\big)$, $i = 1, 2, \ldots, N$. Whereas the Piessens method need the samples at the points $\big(\frac{b}{1-u_i} + \beta\big)$, $i = 1, 2, \ldots, N$. For the regularized collation method, however, we need the samples at arbitrary points (s_i), $i = 1, 2, \ldots, N$.*

In the next chapter, we consider the implementation of the three algorithms described in this chapter.

Chapter 7

Numerical results

In this chapter, we consider the numerical implementation of the Gaver-Stehfest method, the Piessens method and the regularized collocation method. First, we give the parameter setup for the various methods. Then we described the implementation of these methods using MATLAB in the case of exact data. And for noisy data, we examine the stability of the methods and also give a comparison of the methods.

7.1 Introduction

We test the applicability of the algorithms using the functions given in Table 7.1. In choosing the functions used in comparing the methods, we consid-

Table 7.1: Functions used in comparing methods.

Laplace transform	Inverse Laplace transform
$F_1(s) = \frac{1}{s^5}$	$f_1(t) = \frac{1}{24}t^4$
$F_2(s) = \frac{1}{(s+1)^2}$	$f_2(t) = te^{-t}$
$F_3(s) = \frac{1}{\sqrt{s}}$	$f_3(t) = \frac{1}{\sqrt{\pi t}}$
$F_4(s) = \frac{s}{(s^2+1)^2}$	$f_4(t) = 0.5t\sin(t)$
$F_5(s) = \frac{1}{s}e^{-2s}$	$f_5(t) = \theta(t-2) = \begin{cases} 0, & t \leq 2 \\ 1, & t > 2 \end{cases}$
$F_6(s) = \mathrm{erf}(2/\sqrt{s})$	$f_6(t) = \frac{1}{\pi t}\sin(4\sqrt{t})$

ered different properties of such functions, such that we are able to classify

the numerical inversion methods according to the functions for which they are most suitable. For example, $f_1(t)$ is an increasing function, $f_2(t)$ is an exponentially decaying function, $f_3(t)$ is a function with singularities, $f_4(t)$ is an oscillating function with some increasing properties, $f_5(t)$ is a discontinuous function (i.e., not differentiable), and $f_6(t)$ is also an oscillating function with singularities.

Moreover, in order to observe the discrepancy between the numerical approximations and the exact solutions, we calculate the L^2-norm and the $L^\infty-$norm of the difference between between the exact solutions $f(t)$ and the numerical approximations $f_{num}(t)$.

7.2 Implementation

In this section, we consider the details of the implementation and the setting of the parameters appearing in the algorithms described in the previous chapter.

The Gaver-Stehfest method

The only parameter we have to determine here is the discretization parameter N. For the coefficient G_i in Equation (6.5) corresponding to the inverse Laplace transform operator to exits, N must be an even integer. And for stability of the Gaver-Stehfest method, the value of N cannot be allowed to be large. This is because, as already explained in Section 6.2, a large value of N leads to a greater absolute value of the coefficient G_i, which implies the unboundedness of the inverse Laplace transform operator. Stehfest [35] suggested that for $N = 18$ nice inversion results can be obtained.

Roughly speaking, N can be regarded as a regularization parameter since a discretization itself acts as a regularization, as it is well known (see Engl, Hanke and Neubauer [21] and Figure 7.16).

As a result, we compare the performance of the Gaver-Stehfest method for the numerical reconstruction of $te^{-t} = \mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\}$ using different values of N in the case of exact data (i.e., Table 7.2). Thus, from the results shown in Table 7.2, we observe that $N = 18$ is an optimal discretization parameter in

Table 7.2: Comparing the Gaver-Stehfest method for $N = 8, \ldots, 26$ (with exact data), for the numerical reconstruction of $te^{-t} = \mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\}$.

N	$\|f_{num} - f\|_2$
8	0.0167
10	0.0053
12	0.0016
14	4.3931×10^{-4}
16	1.0900×10^{-4}
18	2.4182×10^{-5}
20	2.3282×10^{-5}
22	6.4749×10^{-4}
24	0.0084
26	0.1873

the case of exact data since this choice yields the best approximate solution which is closest, in the L^2-norm, to the exact solution. So, for our numerical implementation, we fixed our discretization parameter at $N = 18$ since smaller values of N directly imply that we discretize $F(s)$ at very few points, which is not enough to obtain a good approximation of the inverse Laplace transform $f(t)$.

The Piessens method

In the case of the Piessens method, we have to set four parameters, i.e., β, a, b and N. And for this, Piessens [48] provided the following suggestion. The parameter β should be chosen such that $F(s)$ is analytical for $\Re(s) > \beta$. And to obtain good convergence of (6.7), the parameter a must be chosen such that $F(s) \sim s^{-a},\ s \to \infty$, i.e., $a > 0$ such that $s^a F(s)$ has a removable singularity at infinity. The parameter b determines the size of the interval for which the Chebyshev polynomial expansion (6.7) is valid. We note that the smallest value of b is determined by ensuring the analyticity of $F(s)$ for $\Re(s) > b/2$, in which case smaller values of b imply that the Laplace transform $F(s)$ has to be approximated on a larger interval, resulting in a slower convergence. The parameter N is chosen as the length of the sampling points of the Laplace transform $F(s)$.

For most of our numerical implementation, we used the values: $\beta = 0, a = 1$, $b = 1$ and we obtained good results. However, we also slightly vary these values for some functions and we still obtain good results.

The regularized collocation method

For the Tikhonov based regularized collocation method, we need to determine the regularization parameter α. And for this, we make use of Morozov's discrepancy principle given in Equation (4.12). Hence, we determine the regularization parameter α such that the relation $\|Kf^{\delta}_{\alpha(\delta,F^{\delta})} - F^{\delta}\| \leq \tau\delta$ holds, i.e., by comparing the discrepancy $\|Kf^{\delta}_{\alpha(\delta,F^{\delta})} - F^{\delta}\|$ to the assumed limit δ of the level. We determine the constant τ from the expression in Equation (4.11), and for our numerical implementation (in the case of noisy data) we take $\tau = 2$ or even $\tau = 1.5$.

In the following (i.e., Table 7.3 and Figure 7.1) we implement the discrepancy principle described above with a view to determining an optimal Tikhonov regularization parameter α for the numerical reconstruction of $te^{-t} = \mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\}$ with noise level δ, such that $\|F - F^{\delta}\| \leq 0.0024$.

Table 7.3: Determining the Tikhonov regularization parameter α, by the discrepancy principle for $\delta = 0.0024$.

α	$\|Kf^{\delta}_{\alpha} - F^{\delta}\|_2$	$\|f_{num} - f\|_2$
10^2	1.0194	1.5114
10	0.6241	1.1885
1	0.1827	0.7632
10^{-1}	0.0449	0.2775
10^{-2}	0.0061	0.1033
10^{-3}	0.0014	0.0397
10^{-4}	8.9180×10^{-4}	0.1149
10^{-5}	7.6943×10^{-4}	0.2074
10^{-6}	7.6548×10^{-4}	0.2240
10^{-7}	7.6283×10^{-4}	0.3306

From the implementation of the discrepancy principle (i.e., Table 7.3 and Figure 7.1), we observe that $\alpha = 10^{-3}$ is a suitable Tikhonov regularization

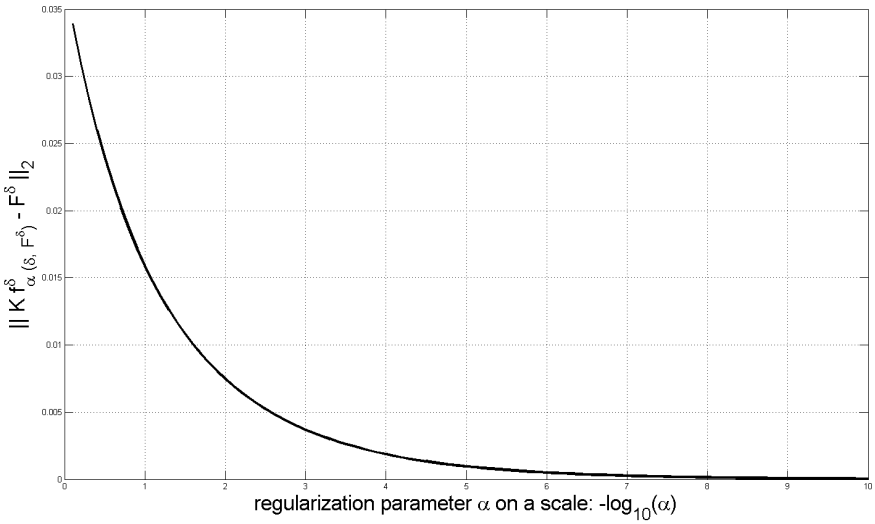

Figure 7.1: Determining the Tikhonov regularization parameter α, for $\delta = 0.0024$ using the discrepancy principle in the numerical reconstruction of $te^{-t} = \mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\}$.

parameter for the given noisy level $\delta = 0.0024$, since at this α the residual is for the first time below the noise level, $\delta = 0.0024$.

We also compare the convergence of the regularized solutions f^{δ}_{num} (with noisy data) towards the exact solution f for various values of α using the results from Table 7.3. We display this in Figure 7.2.

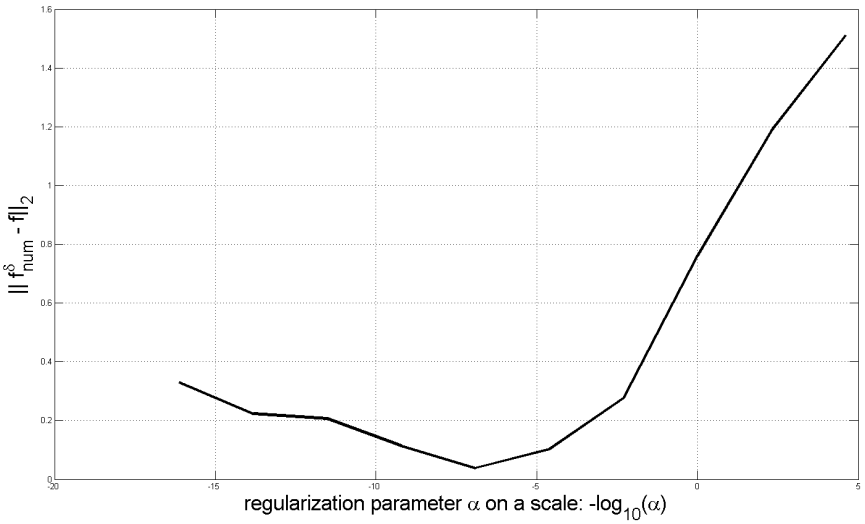

Figure 7.2: Comparing the convergence of the regularized solutions f^{δ}_{num} (with noisy data) towards the exact solution f for various values of α.

From Table 7.3, we observe that for $\alpha = 10^{-3}$ (i.e., using the discrepancy principle) with a noise level of $\delta = 0.0024$, we obtain the best convergence of the regularized solution, since $\|f_{num} - f\|_2 = 0.0397$ is the smallest at this α. This clearly shows the importance of the discrepancy principle when choosing the regularization parameter α. This also shows the typical behavior of regularization methods: for small α the problem is more unstable and the propagated data error dominates. For large α we have a bad approximation and the approximation error dominates. The "right" α is in the middle, a compromise between stability and approximation.

For exact data, we do not make use of the discrepancy principle in choosing the regularization parameter α. In this case, we used trial and error to detect the best α.

In the next section, we test the applicability of the three numerical methods described in Chapter 7 using the parameter setting described in Section 7.2.

7.2.1 Exact data $F(s)$

In this section, we implement the numerical methods and using the example problems given in Table 7.1, we test the the applicability of these methods. Here, we make use of exact values of the Laplace transform $F(s)$.

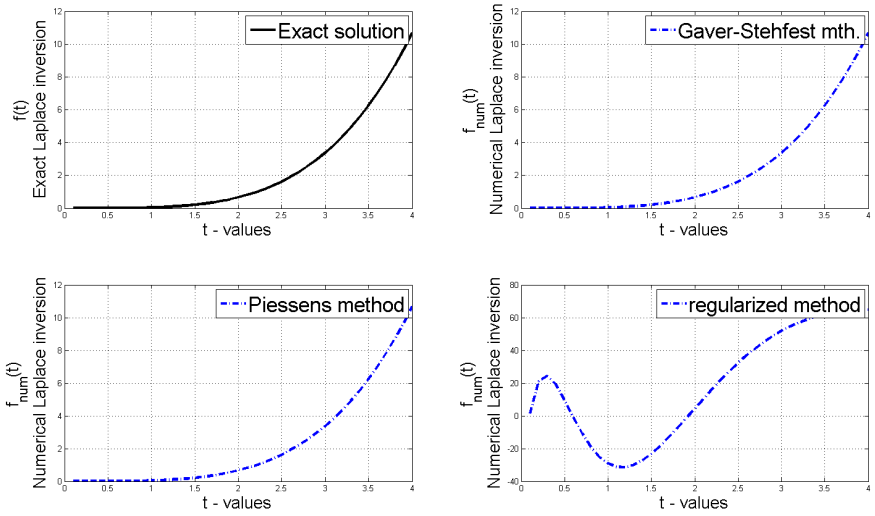

Figure 7.3: Numerical reconstruction of $\frac{1}{24}t^4 = \mathcal{L}^{-1}\{\frac{1}{s^5}\}$.

Table 7.4: Numerical reconstruction of $\frac{1}{24}t^4 = \mathcal{L}^{-1}\{\frac{1}{s^5}\}$.

	$\|f_{num}(t) - f(t)\|_2$	$\|f_{num}(t) - f(t)\|_\infty$
Gaver-Stehfest mth.	1.6813×10^{-4}	7.5576×10^{-5}
Piessens method	2.2126×10^{-9}	1.1982×10^{-9}
regularized method	222.2180	56.0845

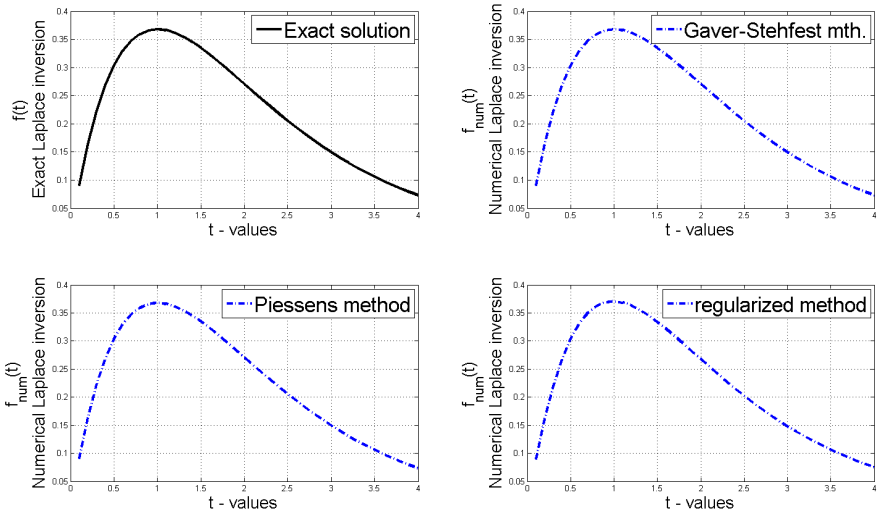

Figure 7.4: Numerical reconstruction of $te^{-t} = \mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\}$.

Table 7.5: Numerical reconstruction of $te^{-t} = \mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\}$.

	$\|f_{num}(t) - f(t)\|_2$	$\|f_{num}(t) - f(t)\|_\infty$
Gaver-Stehfest mth.	2.4182×10^{-5}	8.7517×10^{-6}
Piessens method	1.0732×10^{-4}	5.4516×10^{-5}
regularized method	0.0138	0.0035

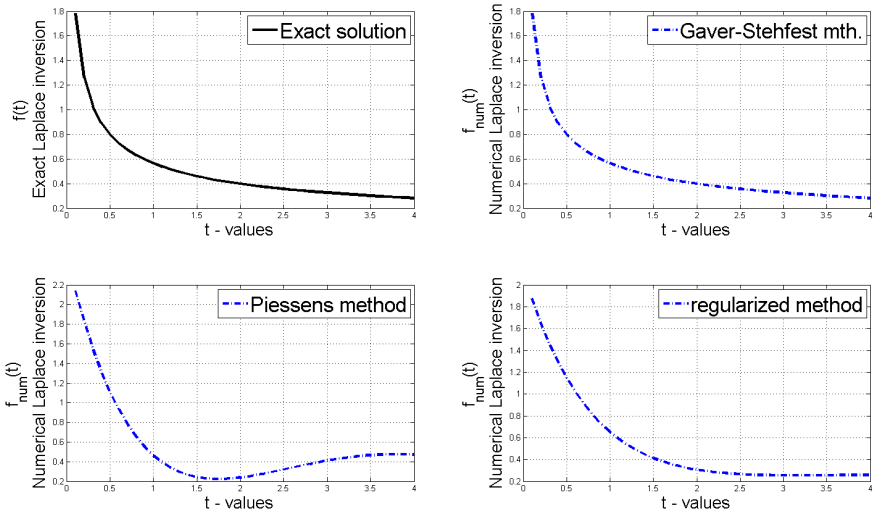

Figure 7.5: Numerical reconstruction of $\frac{1}{\sqrt{\pi t}} = \mathcal{L}^{-1}\{\frac{1}{\sqrt{s}}\}$.

Table 7.6: Numerical reconstruction of $\frac{1}{\sqrt{\pi t}} = \mathcal{L}^{-1}\{\frac{1}{\sqrt{s}}\}$.

	$\|f_{num}(t) - f(t)\|_2$	$\|f_{num}(t) - f(t)\|_\infty$
Gaver-Stehfest mth.	2.0900×10^{-5}	8.5744×10^{-6}
Piessens method	1.3278	0.5657
regularized method	0.9854	0.4338

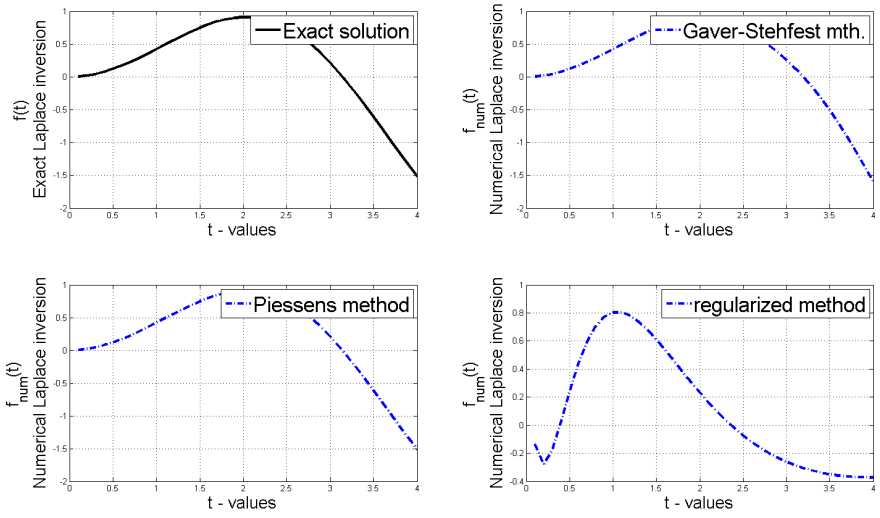

Figure 7.6: Numerical reconstruction of $0.5t\sin(t) = \mathcal{L}^{-1}\{\frac{s}{(s^2+1)^2}\}$.

Table 7.7: Numerical reconstruction of $0.5t\sin(t) = \mathcal{L}^{-1}\{\frac{s}{(s^2+1)^2}\}$.

	$\|f_{num}(t) - f(t)\|_2$	$\|f_{num}(t) - f(t)\|_\infty$
Gaver-Stehfest mth.	0.2495	0.1020
Piessens method	0.0101	0.0037
regularized method	3.3836	1.1420

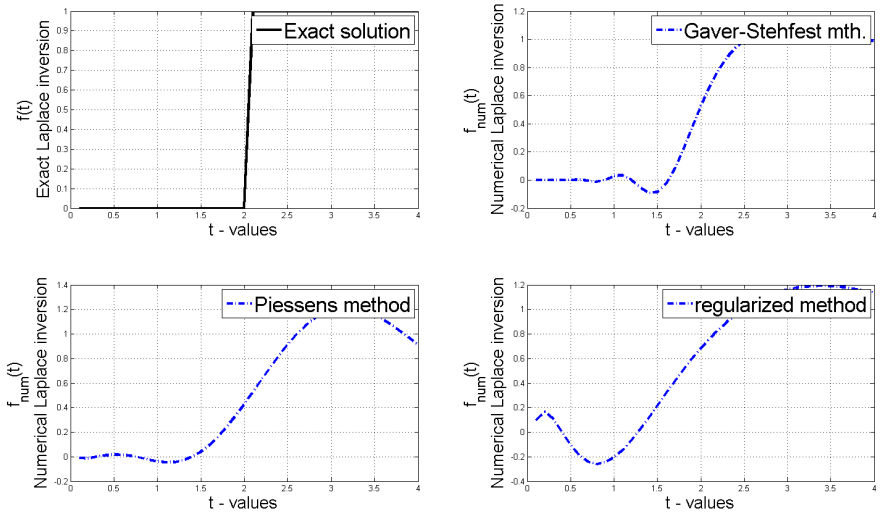

Figure 7.7: Numerical reconstruction of $\theta(t-2) = \mathcal{L}^{-1}\{\frac{1}{s}e^{-2s}\}$.

Table 7.8: Numerical reconstruction of $\theta(t-2) = \mathcal{L}^{-1}\{\frac{1}{s}e^{-2s}\}$.

	$\|f_{num}(t) - f(t)\|_2$	$\|f_{num}(t) - f(t)\|_\infty$
Gaver-Stehfest mth.	0.8318	0.5251
Piessens method	1.0881	0.4718
regularized method	1.4980	0.6832

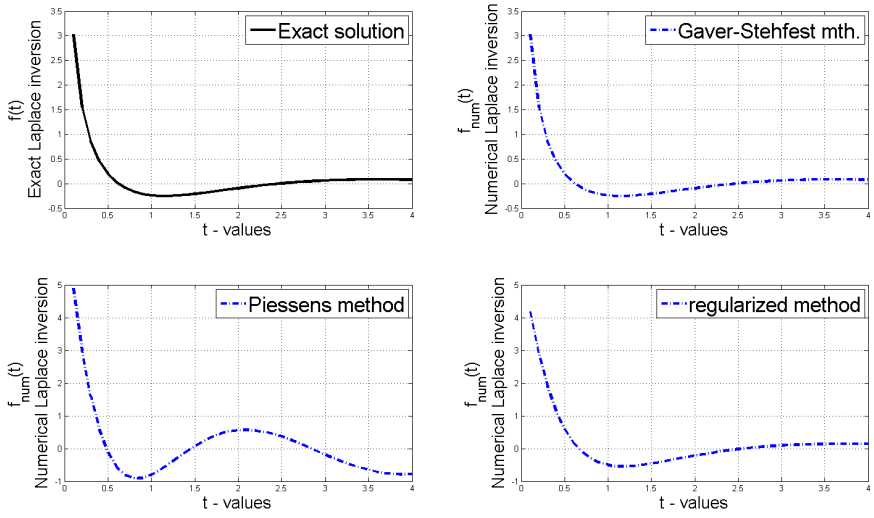

Figure 7.8: Numerical reconstruction of $\frac{1}{\pi t}\sin(4\sqrt{(}t)) = \mathcal{L}^{-1}\big\{\operatorname{erf}(2/\sqrt{s})\big\}$.

Table 7.9: Numerical reconstruction of $\frac{1}{\pi t}\sin(4\sqrt{(}t)) = \mathcal{L}^{-1}\big\{\operatorname{erf}(2/\sqrt{s})\big\}$.

	$\|f_{num}(t) - f(t)\|_2$	$\|f_{num}(t) - f(t)\|_\infty$
Gaver-Stehfest mth.	6.8897×10^{-5}	3.2812×10^{-5}
Piessens method	4.1449	1.8519
regularized method	2.3567	1.3334

Exact data: Explaining the results and plots

For the reconstruction of the first function, $f_1(t) = \frac{1}{24}t^4$ (i.e., increasing function), the Gaver-Stehfest method and the Piessens method performed very well with respect to the shape and values of the original function. Whereas the regularized collocation method was able to partially reconstruct the shape and values of the function. It seems that the regularized collocation method is not so good for reconstructing an increasing function. This can be explained since the regularization penalizes the large values at large t and therefore only yields a suboptimal reconstruction.

In the case of the second method, $f_2(t) = te^{-t}$ (i.e., exponentially decaying function), all the three methods performed very well in reconstructing the shape of the original function. As for the values of the function, the performance of the Gaver-Stehfest and the Piessens methods were better than that of the regularized collocation method, although it also performed well.

For the third function, $f_3(t) = \frac{1}{\sqrt{\pi t}}$ (i.e., function with singularities), all the three methods performed very well in reconstructing the shape of the original function. In reconstructing the values of the function, the Piessens and the regularized collocation method performed well. However, the Gaver-Stehfest method was better.

In reconstructing the fourth function, $f_4(t) = 0.5t\sin(t)$ (i.e., oscillating function), all three methods performed well with respect to the shape and values of the of the original function.

For the fifth function, $f_5(t) = \theta(t-2)$ (i.e., discontinuous function), all three methods were able to partially reconstruct the shape and values of the original function.

For the last function, $f_6(t) = \frac{1}{\pi t}\sin(4\sqrt{t})$ (i.e., oscillating function with singularities), all three methods reconstructed the shape and values of the original function well. However, the performance of the Gaver-Stehfest method was the best.

7.2.2 Noisy data $F(s^\delta)$

In this section, we repeat the numerical inversion of the Laplace transform using the same example problems given in Table 1.1, but this time with noisy data $F(s^\delta)$. Hence, we added some noise to the data $F(s)$ and examined the stability of the numerical methods. For all three methods, we added noise of magnitude $10^{-3} * rand(1,1)$ in the vector containing the Laplace transform values $\big(F(s_i)\big)_{i=1}^{N}$.

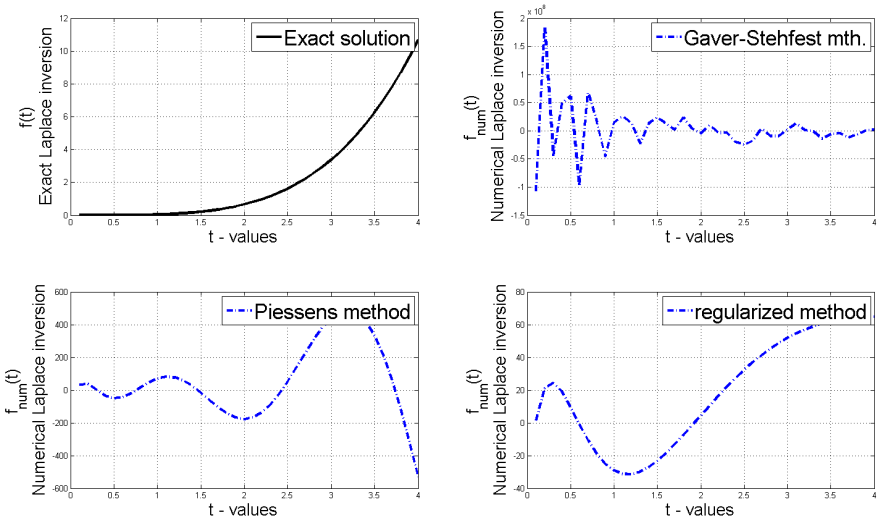

Figure 7.9: Numerical reconstruction of $\frac{1}{24}t^4 = \mathcal{L}^{-1}\{F_1(s^\delta)\}$.

Table 7.10: Numerical reconstruction of $\frac{1}{24}t^4 = \mathcal{L}^{-1}\{F_1(s^\delta)\}$.

	$\|f_{num}(t) - f(t)\|_2$	$\|f_{num}(t) - f(t)\|_\infty$
Gaver-Stehfest mth.	2.7487×10^8	1.844×10^8
Piessens method	1.4476×10^3	542.9727
regularized method	222.2357	56.0877

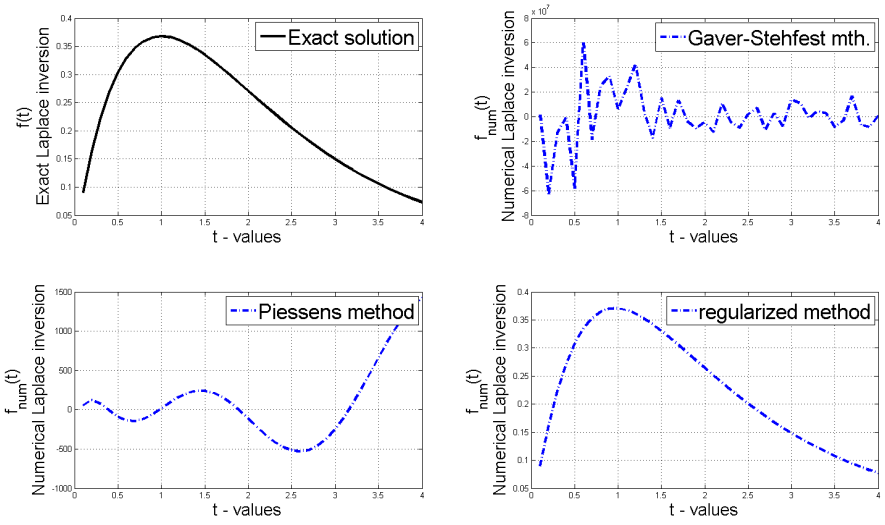

Figure 7.10: Numerical reconstruction of $te^{-t} = \mathcal{L}^{-1}\{F_2(s^\delta)\}$.

Table 7.11: Numerical reconstruction of $te^{-t} = \mathcal{L}^{-1}\{F_2(s^\delta)\}$.

	$\|f_{num}(t) - f(t)\|_2$	$\|f_{num}(t) - f(t)\|_\infty$
Gaver-Stehfest mth.	1.3449×10^8	6.3673×10^7
Piessens method	3.1524×10^3	1.4205×10^3
regularized method	0.0244	0.0064

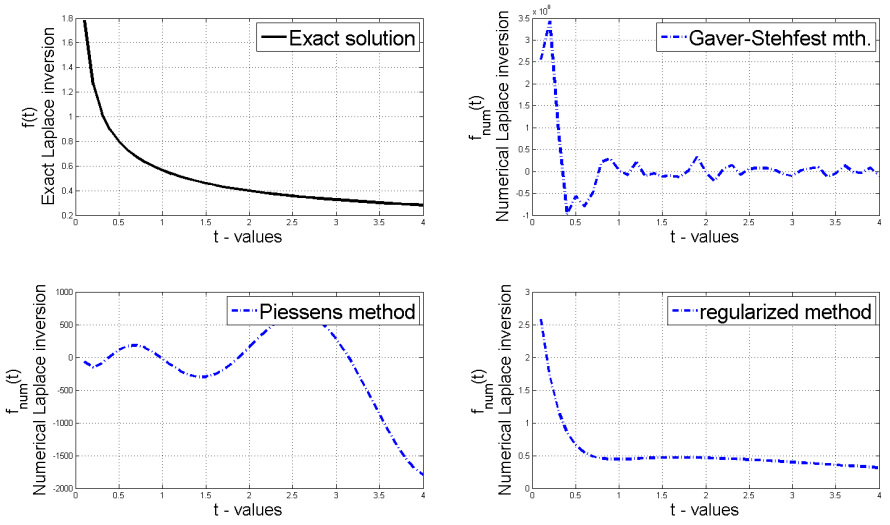

Figure 7.11: Numerical reconstruction of $\frac{1}{\sqrt{\pi t}} = \mathcal{L}^{-1}\{F_3(s^\delta)\}$.

Table 7.12: Numerical reconstruction of $\frac{1}{\sqrt{\pi t}} = \mathcal{L}^{-1}\{F_3(s^\delta)\}$.

	$\|f_{num}(t) - f(t)\|_2$	$\|f_{num}(t) - f(t)\|_\infty$
Gaver-Stehfest mth.	4.6764×10^8	3.4338×10^8
Piessens method	4.0262×10^3	1.7946×10^3
regularized method	1.0652	0.7938

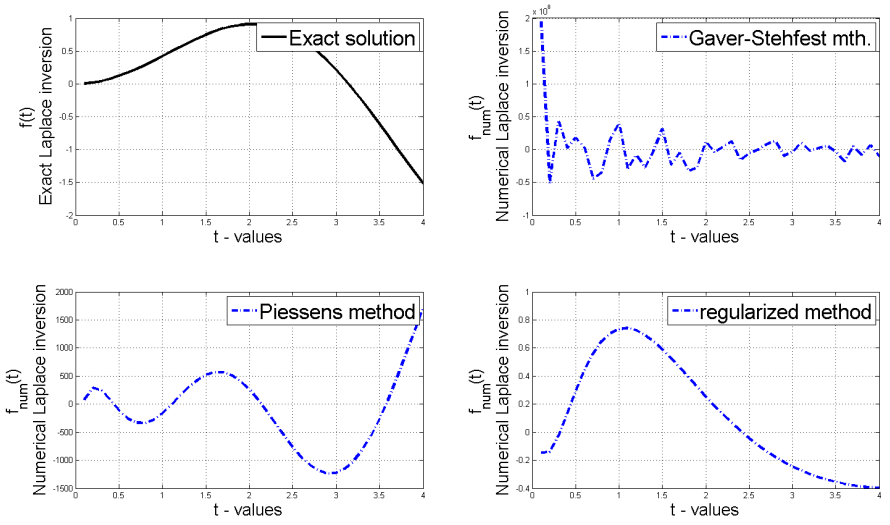

Figure 7.12: Numerical reconstruction of $0.5t\sin(t) = \mathcal{L}^{-1}\{F_4(s^\delta)\}$.

Table 7.13: Numerical reconstruction of $0.5t\sin(t) = \mathcal{L}^{-1}\{F_4(s^\delta)\}$.

	$\|f_{num}(t) - f(t)\|_2$	$\|f_{num}(t) - f(t)\|_\infty$
Gaver-Stehfest mth.	2.3400×10^8	1.9503×10^8
Piessens method	4.3474×10^3	1.7168×10^3
regularized method	3.2410	1.1155

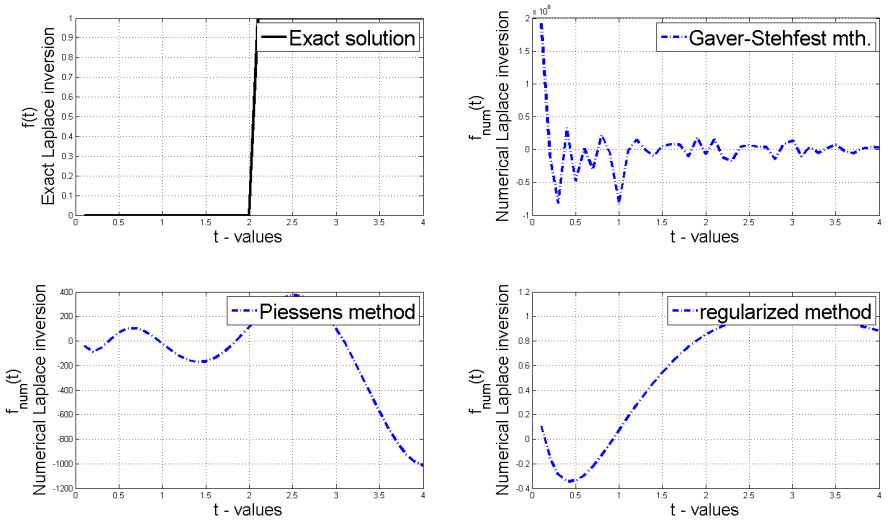

Figure 7.13: Numerical reconstruction of $\theta(t-2) = \mathcal{L}^{-1}\{F_5(s^\delta)\}$.

Table 7.14: Numerical reconstruction of $\theta(t-2) = \mathcal{L}^{-1}\{F_5(s^\delta)\}$.

	$\|f_{num}(t) - f(t)\|_2$	$\|f_{num}(t) - f(t)\|_\infty$
Gaver-Stehfest mth.	2.4097×10^8	1.9210×10^8
Piessens method	2.3890×10^3	1.0138×10^3
regularized method	2.0268	0.8503

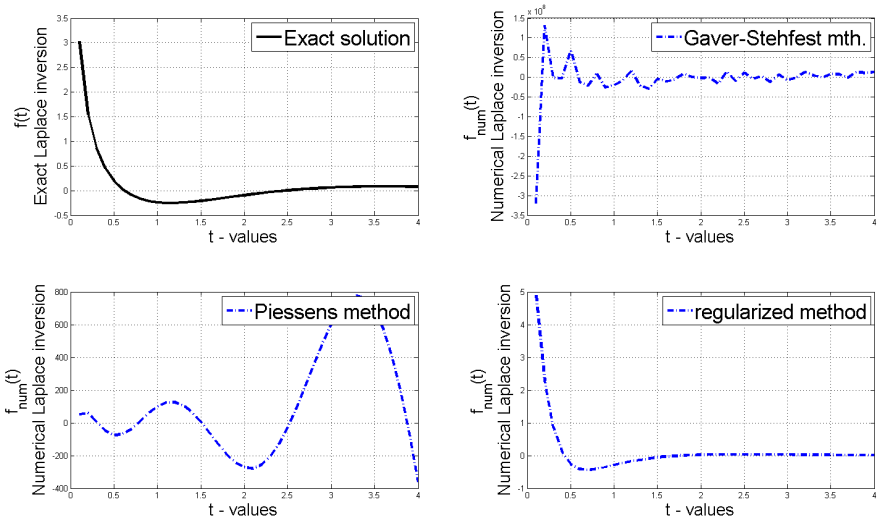

Figure 7.14: Numerical reconstruction of $\frac{1}{\pi t}\sin(4\sqrt{(}t)) = \mathcal{L}^{-1}\{F_6(s^\delta)\}$.

Table 7.15: Numerical reconstruction of $\frac{1}{\pi t}\sin(4\sqrt{(}t)) = \mathcal{L}^{-1}\{F_6(s^\delta)\}$.

	$\|f_{num}(t) - f(t)\|_2$	$\|f_{num}(t) - f(t)\|_\infty$
Gaver-Stehfest mth.	3.6019×10^8	3.2015×10^8
Piessens method	2.1448×10^3	775.6237
regularized method	2.2183	1.8530

Noisy data: Explaining the results and plots

In the case of noisy data, the Gaver-Stehfest method is extremely unstable for all six functions: $f_1(t), \ldots, f_6(t)$ and as such fails to reconstruct both the shape or the values of the original function. The Piessens method also failed for all functions as well. However, the instability is more pronounced for the Gaver-Stehfest method.

As a result of its regularization properties, the regularized collocation method performed very well in reconstructing the shape and values of the original function in the case of noisy data.

7.3 Discussion

In general, numerical algorithms for the inversion of the Laplace transform begin to oscillate and therefore become unstable in the presence of noisy data.

A consideration of the numerical results and plots shown in Sections 7.2.1 and 7.2.2 showed that the Gaver-Stehfest method is indeed an excellent method for the numerical inversion of several class of the Laplace transform. In the case of exact data, the method produced better numerical approximation when compared to the Piessens and the regularized collocation methods. However, the Piessens method gave better results than the regularized collocation method in the case of exact data.

In the case of noisy data, the regularized collocation method remained stable for appropriately chosen regularization parameter (according to the discrepancy principle) and hence produced nice approximate solution. The Piessens method did not do well in this regard, while the Gaver-Stehfest method completely failed to reconstruct both the shape or the values of the original function.

For all implementation above, we have carried out the Laplace transform inversion on the interval, $t \in [0, 4]$. However, for a larger interval, say $t \in [0, 20]$, we observe that the Piessens methods becomes highly unstable and it even fails to produce a good approximation. This is true even in the case of exact data for all the functions in Table 7.1 except for the increasing function, i.e., $f_1(t)$. Whereas for smaller t−interval, the Piessens method performed well, and as a result, we conclude that the Piessens method works best at smaller t−interval. On the other hand, the Gaver-Stehfest, and the regularized collocation method still remained stable and thus produce nice approximate solutions at larger t−interval in the case of exact data.

The regularized collocation method is based on Tikhonov regularization which minimizes the Tikhonov functional in Equation (4.14). This obviously explains why the method did not perform well for reconstructing the increasing function, i.e., $f_1(t)$. But for the exponentially decaying function and those with singularities, it performed well.

Again for all our implementation so far, we have sampled the Laplace trans-

form at 17 data points, i.e., $s_i,\ i = 1, \ldots, 17$ and the results for this are shown in Sections 7.2.1 and 7.2.2. However, for more data points, say $s_i,\ i = 1, \ldots, 161$, the Piessens method completely failed and became extremely unstable. To illustrate this, we reconsider the numerical reconstruction of the exponentially decaying function $te^{-t} = \mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\}$ using an increased number of data points, i.e., $\big(F(s_i)\big)\ i = 1, \ldots, 161$.

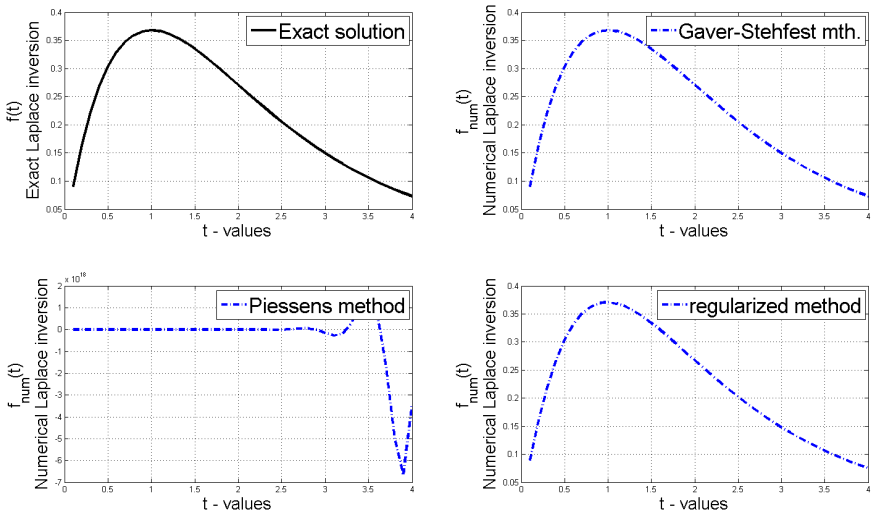

Figure 7.15: Numerical reconstruction of $te^{-t} = \mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\}$ for $s_i,\ i = 1, \ldots, 161$.

Table 7.16: Numerical reconstruction of $te^{-t} = \mathcal{L}^{-1}\{\frac{1}{(s+1)^2}\}$ for $s_i,\ i = 1, \ldots, 161$.

	$\|f_{num}(t) - f(t)\|_2$	$\|f_{num}(t) - f(t)\|_\infty$
Gaver-Stehfest mth.	2.4182×10^{-5}	8.7517×10^{-6}
Piessens method	9.3557×10^{18}	6.6610×10^{18}
regularized method	0.0144	0.0036

From the implementation in Table 7.16 and Figure 7.15, it is easy to see that the Piessens method completely fails when the data points of the Laplace transform is increased.

The Piessens, and the regularized methods did not produce good approximation for the reconstruction of $f_{num}(t)$ near $t = 0$. Whereas, the Gaver-Stehfest method performed very well in this case.

As we have already mentioned at the outset of this chapter, the discretization parameter N in the Gaver-Stehfest can be seen (roughly speaking) as a regularization parameter, which becomes useful in the case of noisy data. In the following therefore, we revisit the numerical reconstruction of $\frac{1}{\pi t}\sin(4\sqrt{t}) = \mathcal{L}^{-1}\big\{\operatorname{erf}(2/\sqrt{s})\big\}$ in the case of noisy data, and we try to improve the result by using $N = 4$ as the discretization parameter for the Gaver-Stehfest method. In the first plot (i.e., Figure 7.16), we compare the performance of the Gaver-Stehfest method for $N = 2, \ldots, 20$ with a view to determining the optimal discretization parameter N in the case of noisy data. From the implementation (i.e., Table 7.17 and Figure 7.17), we ob-

Table 7.17: Comparing the Gaver-Stehfest method for $N = 2, \ldots, 20$ (with noisy data), for the numerical reconstruction of $\frac{1}{\pi t}\sin(4\sqrt{t}) = \mathcal{L}^{-1}\big\{F_6(s^\delta)\big\}$.

N	$\|f_{num} - f\|_2$
2	1.2669
4	0.5092
6	2.8674
8	52.8888
10	1.3347×10^3
12	3.3329×10^4
14	5.0353×10^5
16	1.1117×10^6
18	3.0183×10^8
20	4.5764×10^9

serve that $N = 4$ gave the best result for the Gaver-Stehfest method in the case of noisy data. Thus in Figure 7.17, we make use of $N = 4$ to reconstruct $\frac{1}{\pi t}\sin(4\sqrt{t})e^{-t} = \mathcal{L}^{-1}\big\{F_6(s^\delta)\big\}$.

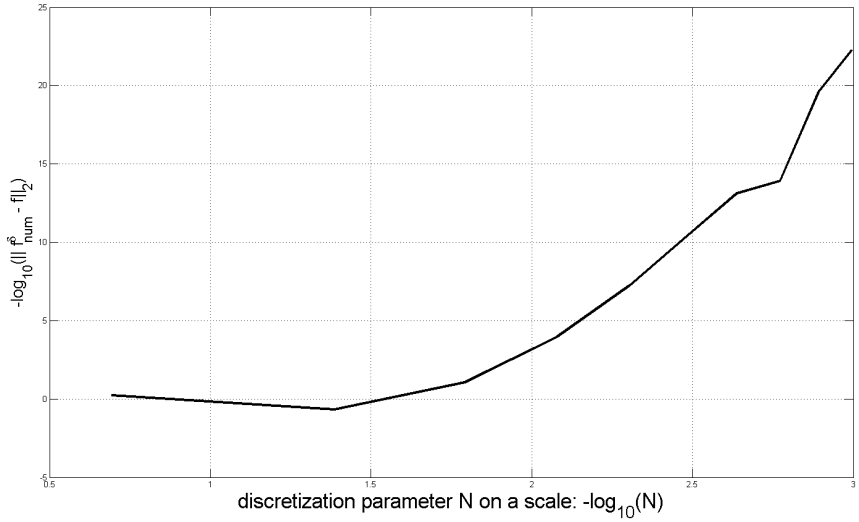

Figure 7.16: Gaver-Stehfest method for $N = 2, \ldots, 20$ (with noisy data) for the numerical reconstruction of $\frac{1}{\pi t}\sin(4\sqrt{t}) = \mathcal{L}^{-1}\{F_6(s^\delta)\}$

Table 7.18: Numerical reconstruction of $\frac{1}{\pi t}\sin(4\sqrt{t})e^{-t} = \mathcal{L}^{-1}\{F_6(s^\delta)\}$, where $N = 4$ serves as a regularization parameter for the Gaver-Stehfest method.

	$\|f_{num}(t) - f(t)\|_2$	$\|f_{num}(t) - f(t)\|_\infty$
Gaver-Stehfest mth.	0.5345	0.1439
Piessens method	2.1729×10^3	783.4614
regularized method	1.9724	1.5428

Again, we mention here that the shape of the plot in Figure 7.16 is the typical behavior of regularization methods (i.e., stability versus approximation). Comparing the result obtained in Figure 7.17 to the one in Figure 7.14, we observe a significant improvement in the Gaver-Stehfest method both in the reconstruction of the shape and values of the original function when the discretization parameter is reduced from $N = 18$ to $N = 4$. However, it should be noted that this is a poor approximation of the original function since $N = 4$ does not contain enough points to obtain a good approximation of the inverse Laplace transform $f(t)$. Thus, the discretization parameter N in the Gaver-Stehfest can only be intuitively considered as a regularization

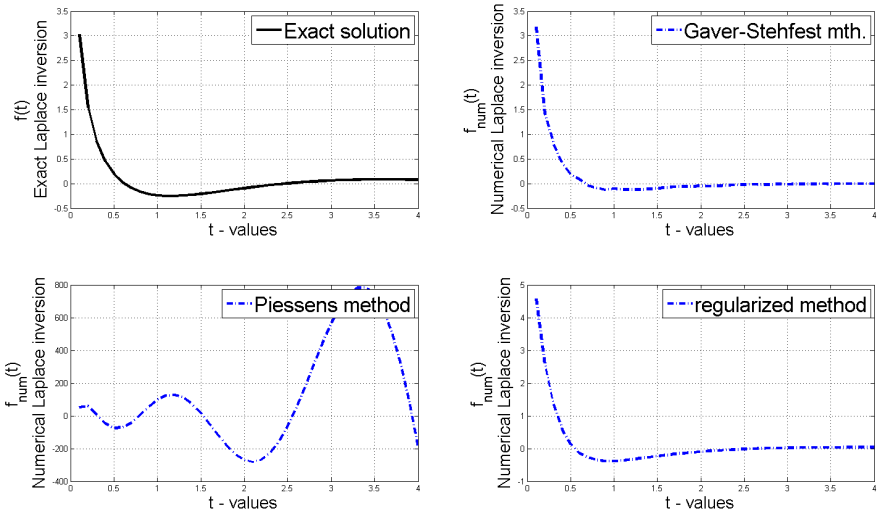

Figure 7.17: Numerical reconstruction of $\frac{1}{\pi t}\sin(4\sqrt{t})e^{-t} = \mathcal{L}^{-1}\{F_6(s^\delta)\}$, where $N = 4$ serves as a regularization parameter for the Gaver-Stehfest method.

parameter and not in a practically useful sense.

7.4 Summary

There are several known algorithms for the numerical inversion of the Laplace transform. In this thesis, we have considered three different methods: the Gaver-Stehfest method, the Piessens method and we constructed the regularized collocation method based on Tikhonov regularization.

For the Gaver-Stehfest method, the inverse Laplace transform operator is given as a coefficient G_i which depends only on the discretization parameter N. In the presence of exact data, the Gaver-Stehfest method gave very nice approximate solution for a wide range of functions. However, it completely failed in the presence of noisy data. Roughly speaking, the discretization parameter N served as a regularization parameter for this method, and when this parameter is significantly reduced, the method still performed well in the presence of noisy data. In this case, it is only able to reconstruct the

shape of the original function but not the values.

The Piessens method is based on a Chebyshev polynomial expansion of the Laplace transform $F(s)$, and it also performed well in the case of exact data $F(s)$ for several functions. Since the method has no regularization, it became unstable in the presence of noisy data. However, the instability was less compared with that obtained from the Gaver-Stehfest method in the presence of noisy data.

In the case of the regularized collocation method, good approximate solutions were also obtained in the case of exact data. However the method did not do so well for reconstructing an increasing function. The method is evidently most suitable for dealing with noisy data as a result of the inherent regularization property it possesses.

From the numerical implementation, we conclude that there exists no single algorithm that is able to invert all classes of Laplace transform functions $F(s)$ in a numerically stable way. Therefore, a given numerical algorithm for the inversion of Laplace transform is most effective when it is applied to a specific class or classes of function for which it is most suitable.

Appendix

Definition 7.1. *(Norm−Banach space)*
Let X be a real vector space.

(i) *The mapping $\|\cdot\| : X \to [0,\infty)$ is called norm on X, if*

- **a)** $\|x\| \geq 0 \quad \forall\, x \in X$
- **b)** $\|x\| = 0 \Leftrightarrow x = 0$
- **c)** $\|\lambda x\| = |\lambda|\|x\| \quad \forall\, x \in X, \quad \lambda \in \mathbb{R}$ *(positive homogeneity)*
- **d)** $\|x+y\| \leq \|x\| + \|y\| \quad \forall\, x, y \in X$ *(triangle inequality)*

Then $(X, \|\cdot\|)$ is known as a real(normed) space.

(ii) *A normed real vector space X is called* Banach space*, if it is complete, i.e., if every Cauchy-sequence converges in X, thus a limit $x \in X$ exists.*

Definition 7.2. *(Inner product−Hilbert space)*
Let H be a real vector space.

(i) *A mapping $(\cdot,\cdot) : H \times H \to \mathbb{R}$ is called inner product on H, if*

- **a)** $(x,y) = (y,x) \quad \forall\, x, y \in H$
- **b)** *For every $y \in H$ the mapping $x \in H \mapsto (x,y)$is linear*
- **c)** $(x,x) \geq 0 \quad \forall\, x \in H$ *and* $(x,x) = 0 \Leftrightarrow x = 0$.

(ii) *A vector space H with an inner product $(\cdot,\cdot)$ and a related norm*

$$\|x\| := \sqrt{(x,x)}$$

is called a pre-Hilbert space.

(iii) *A Pre-Hilbert space $(H, (\cdot, \cdot))$ is called a* Hilbert space*, if it is complete with respect to its norm $\|x\| := \sqrt{(x, x)}$.*

Definition 7.3. *(Linear operator)*
A mapping $K : X \to Y$ is called a linear operator*, if*

$$\begin{aligned} K(x+y) &= Kx + Ky \quad \forall \quad x, y \in X \\ K(\lambda x) &= \lambda K x \quad \forall\ x \in H \quad \textit{and} \quad \forall\ \lambda \in \mathbb{R}. \end{aligned}$$

Definition 7.4. *(Bounded linear operator)*
A linear operator $K : X \to Y$ is called bounded*, if there exists a $c > 0$ such that*

$$\|Kx\|_Y \leq c\|x\|_X \quad \forall\ x \in X.$$

Definition 7.5. *(Compact operators)*
Let X, Y be normed spaces, $K : X \to Y$ be linear. K is compact if for every bounded set $B \subseteq X$, the set $\overline{K(B)}$ is compact.

Definition 7.6. *(Compact operators)*
Let X, Y be normed spaces, $K : X \to Y$ be linear. K is compact if for every bounded sequence $\{x_n\} \subset X$, the sequence $\{Kx_n\}$ is compact.

Definition 7.7. *A subset X_n of X (metric space) is compact if for every sequence in X_n, there exists a convergent subsequence with limit in X_n.*

Definition 7.8. *(Chebyshev polynomials)*
The Chebyshev polynomials $T_n(x)$ of the first kind are defined by the recurrence relation

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \end{aligned}$$

where the conventional generating function *for T_n is given as:*

$$\sum_{n=0}^{\infty} T_n(x)t^n = \frac{1-tx}{1-2tx+t^2},$$

and the exponential generating function *for T_n is given as:*

$$\sum_{n=0}^{\infty} T_n(x)\frac{t^n}{n!} = \frac{1}{2}\Big[e^{(x-\sqrt{x^2-1})t} + e^{(x+\sqrt{x^2-1})t}\Big].$$

Definition 7.9. *(Generalized hypergeometric function)*
A generalized hypergeometric function ${}_pH_q(a_1,\ldots,a_p;b_1,\ldots,b_p;x)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series $\sum_{k=0}^{\infty} c_k$ for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)}x.$$

Definition 7.10. *(Directional derivative)*
Let X, Y be normed linear spaces. Let $x \in X, h \in X$ and assume the limit $\lim_{t\to 0}\frac{1}{t}\big(f(x+th)-f(x)\big)$ exits. Then $f : X \longrightarrow Y$ is called directional differentiable *at x in the direction h, and we write*

$$f'(x;h) = \lim_{t\to 0}\frac{\big(f(x+th)-f(x)\big)}{t}.$$

Definition 7.11. *(Gáteaux derivative)*
Let X, Y be normed linear spaces, and let $f : X \longrightarrow Y$ be directional differentiable at x. If $f'(x;h) = Ah$ with $A \in \mathcal{L}(X,Y)\ \forall\ h \in X$, then f is called Gáteaux differentiable at x. If f is Gáteaux differentiable, then $f'(x;h) = f'(x)h$.

Definition 7.12. *(Frechét derivative)*
Let X, Y be normed linear spaces, and let $f : X \longrightarrow Y$ be Gáteaux differentiable at x. f is called Frechét differentiable at x if and only if

$$\lim_{\|h\|_X\to 0}\frac{\|f(x+h)-f(x)-f'(x)h\|_Y}{\|h\|_X} = 0.$$

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