

MASTER

One supplier-2 retailers inventory system with supply disruptions and transshipments problem

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Eindhoven, 17 August 2012

**One Supplier-2 Retailers
Inventory System with Supply
Disruptions and
Transshipments Problem**

by

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in partial fulfilment of the requirements for the degree of

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ABSTRACT

In this master thesis project, we investigate an inventory system consists of 1 supplier and 2 retailers that use a base-stock policy. The supply process at retailer 1 is always available but the supply process at retailer 2 might be disrupted. If necessary, transshipments (complete pooling policy) can be conducted from retailer 1 to retailer 2. The objective is to find an optimal base-stock level for each retailer that minimizes system wide holding, backorder and transshipments costs. Some numerical experiments are conducted in order to get insights about the sensitivity of each parameter and how the system benefits from transshipments.

PREFACE

This Master Thesis document presents the result of my graduation project for the Master of Science program in Operations Management and Logistics at Eindhoven University of Technology. This internal project was carried out from February 2012 to August 2012 at the University. This project gave a lot of meaningful experiences and I thank God for every guidance and wonderful blessings during my study.

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Eindhoven, August 2012

EXECUTIVE SUMMARY

Incorporating supply disruptions in an inventory system has gained a lot of interest due to several reasons such as recent high-profile disruptions events that harmed supply activities. Having supply disruptions modeled in an inventory system can protect companies from great loss caused by the risks of these events. Based on this, we examine an inventory system consists of a supplier and 2 retailers, where one of the retailers faces the possibility of supply disruptions. In order to improve system's performance, we allow transshipments between the retailers. The idea is that transshipments can reduce the number of backordered demand for the retailer that faces disruptions.

We assume that each retailer uses a base-stock inventory policy and the inventory is reviewed periodically. Orders attempt to the supplier are placed every period without fixed cost. The retailers face constant demand every period. The supply process to the disrupted retailer is assumed to follow a discrete time Markov chain (DTMC). The costs considered in the system are penalty costs, holding costs and transshipments costs, where all the costs are calculated in the end of the period. The objectives of this project are to find base-stock level for each retailer that minimizes the total costs and investigate about how the inventory system with supply disruptions can get benefit from transshipments.

We formulate a cost function for such system and prove that the cost function is piece-wise linear in the base-stock level of both retailers. Following that, we also prove that the cost function is convex. From those results, we built expressions to calculate the optimal base-stock levels for both retailers.

Several numerical experiments were conducted to get insights about the impact of each parameter on the optimal base-stock levels also to investigate the benefit of transshipments in a disrupted environment.

From the experiments, we find that each parameter has different influence on the optimal base-stock levels. Transshipments are beneficial in cases where inventory system has low holding cost at the undisrupted retailer, high holding cost at the disrupted retailer, high penalty cost at the disrupted retailer, low transshipments cost, high disruptions probability and low recovery probability. In these cases, having more inventory at the undisrupted retailer leads to lower cost as the backorders cost can be completely avoided. Numerical experiments also confirm our expectations that the cost resulted from policy that allow transshipments always lower or equal to the policy that does not allow transshipments. In addition, the benefit of transshipments is very obvious when the recovery probability is very low.

Related to the analysis regarding disruptions parameters, we find that in general changes in recovery probability provide more significant cost reductions compared to changes in disruptions probability. However, companies still need to look carefully at the value of each cost parameters when they want to minimize the cost by managing disruptions parameters since there are some cases where changing disruptions parameter does not change the cost or even make the cost higher.

The results in this thesis show that allowing transshipments between retailers in a disrupted environment is beneficial in terms of reducing the total cost. Through a well-planned base-stock levels value and transshipments decision, the risk of supply disruptions can be mitigated. The research presented here can be extended by relaxing some assumptions so that it can represent the real-life cases better such as make both retailers face supply disruptions, incorporate a positive lead time or consider a stochastic demand.

Table of Contents

ABSTRACT.....	iii
PREFACE	iv
EXECUTIVE SUMMARY	v
I. INTRODUCTION	1
1.1 Motivation	1
1.2 Problem Definition	2
1.3 Outline	2
II. LITERATURE REVIEW	3
2.1 Supply Disruptions	3
2.1.1 <i>Continuous Review</i>	4
2.1.2 <i>Periodic Review</i>	5
2.2 Transshipments Problems.....	6
2.2.1 <i>Continuous Review</i>	6
2.2.2 <i>Periodic Review</i>	7
2.3 Summary.....	8
2.3.1 <i>Summary of literature review</i>	8
2.3.2 <i>Contribution and Relevance</i>	9
III. SYSTEM INVESTIGATED	10
3.1 System Description.....	10
3.2 Series of Events	11
3.3 Notations	12
3.4 Pre-conditions	13
IV. COST FUNCTION FORMULATION	15
4.1 Explanation of Each Formula	15
4.1.1 <i>Formula 4.2</i>	15
4.1.2 <i>Formula 4.3</i>	16
4.1.3 <i>Formula 4.4</i>	16
4.1.4 <i>Formula 4.5</i>	16
4.1.5 <i>Formula 4.6</i>	16
4.1.6 <i>Formula 4.7</i>	17

4.1.7 Formula 4.8	20
4.2 Final Cost Function	22
V. COST FUNCTION ANALYSIS	23
5.1 The cost function is piece-wise linear in S_1 and S_2	25
5.2 Convexity of the cost Function	31
5.3 Optimal Base-stock Levels Solution	36
VI. NUMERICAL ANALYSIS	39
6.1 Base-case Experiment	39
6.2 Varying Parameter values from the base-case	40
6.2.1 Varying h_1	41
6.2.2 Varying h_2	43
6.2.3 Varying p_2	44
6.2.4 Varying c	45
6.2.5 Varying α	47
6.2.6 Varying β	48
6.2.7 Varying cost parameters in different situations	49
6.3 Investigate the sensitivity for α and β	50
6.3.1 All variations of α and β from the base-case cost parameters for 2 policies	50
6.3.2 α and β changes from various cases for policy 2	53
6.4 Conclusion from the experiments	56
VII. CONCLUSION AND FUTURE RESEARCH	59
7.1 Conclusion	59
7.2 Future Research	60
VIII. REFERENCES	61
IX. APPENDIX	63
9.1 Appendix Chapter 4	63
9.2 Appendix Chapter 5	65

List of Figures

Figure 1 System Investigated	10
Figure 2 Transition Diagram for the System	11
Figure 3 Example Cost Function.....	24
Figure 4 the contribution of each cost component when varying h_1 (cost with transshipments)	42
Figure 5 Comparison between the costs of 2 policies when varying h_1	42
Figure 6 the contribution of each cost component when varying h_2 (cost with transshipments)	43
Figure 7 Comparison between the costs of 2 policies when varying h_2	44
Figure 8 the contribution of each cost component when varying p_2 (cost with transshipments)	44
Figure 9 Comparison between the costs of 2 policies when varying p_2	45
Figure 10 the contribution of each cost component when varying c (cost with transshipments).....	46
Figure 11 Comparison between the costs of 2 policies when varying c	46
Figure 12 the contribution of each cost component when varying α (cost with transshipments).....	47
Figure 13 Comparison between the costs of 2 policies when varying α	48
Figure 14 the contribution of each cost component when varying β (cost with transshipments)	48
Figure 15 Comparison between the costs of 2 policies when varying β	49
Figure 16 Percentage of cost reduction in α for various values of β	52
Figure 17 Percentage of cost reduction in β for various values of α	52
Figure 18 Percentage of cost reduction for case 1 and case 2	54
Figure 19 Percentage of cost reduction for case 3 and case 4	55
Figure 20 Percentage of cost reduction for case 5 and case 6	55

List of Tables

Table 1 Examples of Supply Disruptions	1
Table 2 Experiment result with $c = 1$	23
Table 3 Experiment result with $c = 2$	23
Table 4 Experiment result with $c = 3$	24
Table 5 Experiment result with $c = 4$	24
Table 6 Parameters for the base-case problem.....	39
Table 7 the set of optimal base-stock level when varying h_1	41
Table 8 the set of optimal base-stock level when varying h_2	43
Table 9 the set of optimal base-stock level when varying p_2	44
Table 10 the set of optimal base-stock level when varying c	45
Table 11 the set of optimal base-stock level when varying α	47
Table 12 the set of optimal base-stock level when varying β	48
Table 13 The set of optimal base-stock level for policy 1.....	50
Table 14 The set of optimal base-stock level for policy 2.....	50
Table 15 The optimal cost for policy 1.....	51
Table 16 The optimal cost for policy 2.....	51
Table 17 Different cases for investigating the sensitivity of disruptions parameters	53
Table 18 Legend for figure 18, 19 and 20	54
Table 19 the impact of various parameters to the optimal base-stock levels.....	56

I. INTRODUCTION

1.1 Motivation

Incorporating supply disruptions in an inventory system has gained a lot of interest due to several reasons such as recent high-profile disruptions events that harmed supply activities (Snyder, 2008) or the popularity of just-in-time (JIT) practices that make the company become vulnerable to supply uncertainty events (Gullu et al., 1997). Having supply disruptions modeled in their inventory system can protect companies from great loss caused by the risks of these events. Disruptions can be caused by several problems and one of them is related with transportation problem. The example for this type of problems is presented in the table below:

No	Time	Disruption Source	Related Companies	Issue	Mitigations Strategy and Impact		Source
					Prepared	Unprepared	
1	Sept 2001	Terrorist attacks in U.S.	Daimler Chrysler, Continental Teves, Ford	the security at border crossing was very tight thereby delaying shipment of critical parts and components	Daimler Chrysler and Continental Teves (CT) used alternate transportation mode. CT also used their contingency relationship with transport firms	Ford was not prepared for alternate transportation strategy therefore it had to close five plants for several days	Griffy-Brown (2003)
2	Fall 2002	West-Coast Port Lockout	Playmates Toys, New United Motor Man-Inc. (NUMMI), Pontiac Vibe	10 days shut down of port facilities stopped the import and export activities of goods and parts/components.	Playmates toys mitigated the disruptions by Investing in inventory earlier in the year	NUMMI and Pontiac Vibe could only survive several days relying in their own inventory that were unmanaged for disruptions	Tomlin (2006) and Arabe (2002)
3	Aug-Sept 2005	Hurricane Katrina in the U.S. Gulf Coast Region	Chiquita, P&G, Wal-Mart	Major transportation hubs were clogged, the facilities around the affected area were not working	Wal-Mart opened a big distribution center outside Houston before this disaster happened as part of a direct-import strategy. The company could reduce the adverse impact of Katrina.	Chiquite rerouted its shipment to others possible ports, but some inventories still could not be distributed from the warehouse in New Orleans. P&G suffered a heavy loss of coffee production	Barriounevo and Deustch (2005)

Table 1 Examples of Supply Disruptions

In those events, disruptions had caused distribution activities in several links to stop functioning for an amount of time. For example, Hurricane Katrina in 2005 clogged major transportation hubs around US gulf coast region, the facilities around the affected area also were not working. These conditions made the affected companies could not receive material or stocks from its supplier. Thus, they should find a way to mitigate this situation in order to reduce the amount of stockouts such as by piling up inventory or having another source of stocks from other suppliers or from its own warehouse in other place that is not affected by disruptions. In this project, we interested in modeling those cases, that is, inventory model with supply disruptions in which the disruptions cause broken link between supplier and one of its branch/retailers.

In addition to supply disruptions, we also intended to incorporate the possibility of transshipments in the system. Most of the literature review about supply disruptions assumes that when disruptions occur between a supplier and a retailer, the supply process to that retailer is completely inoperative. In the

transportation disruptions, some distribution links in the system are disrupted and thus the retailers that are connected with the supplier through that link cannot receive stocks. We notice that transshipments activities between retailers allow them to improve their ability to satisfy customers' demand by stocks reallocation between retailers. In general some literature agree that transshipments allow the retailers or stocking points in the same echelon to pool their inventories, so that each retailer does not have to keep high level inventory, in order to decrease holding costs and yet still achieving the desired service levels.

1.2 Problem Definition

We are interested to examine an inventory system consists of a supplier and 2 retailers, where one of the retailers faces the possibility of supply disruptions. Another retailer is able to conduct transshipments when the retailer that faces disruptions experiences stockout. The idea is that transshipments can reduce the number of backordered demand for the retailer that faces disruptions.

Considering both problems (supply disruptions and transshipments), we would like to address the following research questions:

1. In an inventory system that allows transshipments between its retailers where supply disruptions risk is faced by one of the retailers, given that the retailers use a base-stock policy, how to calculate its parameter (base-stock level for each retailer) that minimizes the total costs for such system?
2. How the inventory system with supply disruptions benefited from transshipment?

1.3 Outline

The remainder of this thesis is outlined as follows: In Chapter 2 we review relevant literature in the field of supply disruptions and transshipments problem. In Chapter 3 we describe the system investigated in detail including the assumptions, series of events and constraints. In Chapter 4 we formulate the cost function for the system investigated. In Chapter 5 we analyze the cost function and show that the cost function is piece-wise linear in the base-stock level of retailer 1 and retailer 2. We also show the convexity of the cost function and the calculation regarding the optimal base-stock levels for the system. In chapter 6 we do some numerical analysis in order to gain some insights about how the system benefited from transshipments. We summarize our conclusion in chapter 7 and establish a future research direction.

II. LITERATURE REVIEW

In this chapter we present the important literature available about supply disruptions and transshipments. Supply disruptions are random events where the supply process is interrupted so that it is completely or partially not functioning (Snyder & Shen, 2006) while transshipments can be defined as a practice of transferring stock from a location with excess stock to a location that faces shortages. These stock movements are carried between locations in the same echelon.

We begin this chapter by a discussion about supply disruptions and then followed by a review about transshipments.

2.1 Supply Disruptions

The negative impacts of supply disruptions (due to interruptions in supply process) can be mitigated in 2 ways, strategic and tactical decisions. There are some strategic decisions to mitigate supply disruptions that are discussed in the literature. Tomlin (2006) proposed three strategies to tackle disruption problems:

- Mitigation tactics. These strategies are those in which the firm takes some action in advance of disruption. The examples of this tactic are business interruption insurance, flexible supply contracts (Tsay & Lovejoy, 1999), and multiple-supplier sourcing strategy.
- Contingency tactics. The methods here are those in which the firm takes an action only in the event a disruption occurs, such as temporarily increasing production at alternative suppliers/locations (rerouting) and the company also can shift the customer demand from high end to low end (demand management).
- Acceptance means that the company does nothing to mitigate disruptions and accept the risk caused by disruptions.

As for the mitigation strategies that focus on inventory decisions, we divided the discussion of these problems into two sub-sections, continuous review and periodic review. Most of the papers in this review model disruptions as a supply process that has 2 states:

- State when the supply functions normally, in this review we will refer it as an “ON” state. Usually on this state, the supplier has infinite capacity.
- State when disruptions occur, we will refer it as an “OFF” state.

The length of ON and OFF state is typically random. Some papers assume exponential distributions so that the supply process can form a two-state Markov chain. Some papers also discuss another type of distribution such as Bernoulli or Erlang distribution.

2.1.1 *Continuous Review*

One of the methods to deal with disruptions is to extend the EOQ model in a way that it incorporates disruptions process like the EOQD model that is first presented by Parlar and Berkin (1991). They used the renewal reward theorem to build an average cost objective function and optimized it to find the optimal value of the order quantity.

Berk & Arreola-Risa (1994) corrected the model by Parlar and Berkin (1991). They found 2 implicit assumptions that are invalid: (1) in every OFF interval, stockouts always occur and (2) the lost sales cost is incurred per unit per time instead of per unit. The authors derived a corrected model to calculate the expected annual cost as a function of the order quantity.

Snyder (2008) provided an approximation for the EOQD cost functions from Berk & Arreola-Risa (1994). The approximate cost function can be solved in a closed form. The author showed that when the ON period is significantly larger than the OFF period, the cost in this EOQD model approaches the cost in EOQ model.

Some papers also examine an inventory system where the reorder point is not zero (NZIO). Parlar and Perry (1995) extended the research from Parlar and Berkin (1991) by relaxing the zero order policy. They constructed a long run average cost functions for this system using the renewal reward theorem. Numerical solutions were shown and sensitivity analysis was conducted. In addition to deterministic yields, this paper also considers the case of yield uncertainty where the quantity received is a random function of the quantity ordered.

Gupta (1996) examined supply disruptions in an inventory system that has a continuous review, fixed order quantity and reorder point i.e. following a (Q, r) policy. The author assumed Poisson demand exponential distribution for the lengths of the supplier's ON and OFF periods. He constructed the expression for average cost for every situation and solved them numerically. Computational results show the cost function to be well behaved and show that ignoring supply uncertainty can be relatively very expensive, especially when the OFF periods are long or stockout penalty is high or both.

Arreola-Risa & DeCroix (1998) also investigated a continuous review inventory system where there is possibility that the supply is disrupted for a random amount of time. Different with the previous literature that have been discussed, this paper treats the demands in the system as partial backorders, where a fraction of demands that arrive during a stockout is backordered and the remaining fraction is lost. They explored the impact on the optimal values of the policy parameters of variations in the characteristics of supply disruptions and partial backorders. One of managerial insights from this paper is that to manage a change in the severity of supply disruptions, the inventory decision should be adjusted in the same direction as the change.

2.1.2 *Periodic Review*

Parlar et al. (1995) addressed a periodic-review setting with two types of setup costs: (1) setup cost which incurred whenever an order is placed and (2) setup cost that incurred only if the order is filled. The probability that the order is satisfied depends on the supplier's state in the previous period. They prove that an (s,S) inventory policy is optimal under this situation, where s depends on the state of the supplier in the last period, but S does not.

Song and Zipkin (1996) explored an inventory system with periodic review which has exogenous supply system, that is, the supply state is independent of the demands and orders from the retailer that is being investigated. They model the supply system as a discrete-time Markov process. They proved that when the fixed cost is positive, an (s,S) policy that depend on the current state of the supply system is optimal. In the linear cost case (with no fixed cost), the optimal policy become a base-stock policy with state-dependent base stock level.

Ozekici & Parlar (1999) examined a periodic-review inventory model that the parameters subject to a randomly changing environment. The state of the environment is modeled as a time-homogeneous Markov chain. They show that in a linear cost model where fixed ordering cost is omitted, an environment-dependent base-stock policy is optimal. In the presence of positive fixed ordering costs, the optimal policy is an environment-dependent (s, S) policy with some restriction of the fixed ordering costs.

Argon et al. (2001) investigated an inventory system with backorder dependent deterministic demand and supply uncertainty. They used a supply process where the number of periods between two supply realizations is random in order to capture the effects of changes in the demand process. Their objective functions is to maximize the expected profit per period and through numerical examples they found that the objective functions does not have a specific structure. However, they showed that finding an order-up-to level for the system that maximizes its profit is possible.

Schmitt et al. (2010) considered an inventory system that use a base-stock policy and has a possibility of supply disruptions. In addition to supply disruptions, the system also faces stochastic demand or yield uncertainty. The supply disruptions are modeled using an infinite-state discrete-time Markov chain and the demand is assumed to be normally distributed. They solve the problem using an approximation called Single Stochastic Period (SSP) and for the first case (combination of supply disruptions and stochastic demand) they found that in general, their approximation performed extremely well since the average percent error is very small.

2.2 Transshipments Problems

Transshipments provide an effective mechanism for correcting discrepancies between the locations' observed demand and their available inventory that is expected to cost reductions and improved service without increasing system-wide inventory (Ozdemir et al., 2006). The inventory management policy of this problem has been investigated in many different settings. Some authors consider a periodic review setting whereas others consider continuous setting. Models also differ in the number of stocking locations, types of ordering, and so on. Another feature is whether a transshipment policy is using complete pooling or partial pooling. The former is a general term attached to policies where the transshipping location is willing to share all of its stock, the latter is used when part of the stock is held back to cover future demand. The literatures presented here are divided into two sub-sections: continuous review and periodic review.

2.2.1 *Continuous Review*

Lee (1987) developed a continuous-review multi-echelon model for repairable items. The authors attempt to extend the basic METRIC framework developed by Sherbrooke (1968) to cases when emergency lateral transshipments are allowed and used. The system consists of one depot and M bases. The bases that are identical are pooled in one group. From numerical examples based on the approximate model it is shown that the use of emergency lateral transshipment can result in significant savings.

Ollson (2009) examined a single echelon inventory system with two identical locations where a given transshipment rules is applied. The author considered two approaches. First it is assumed that the locations apply (R, Q) policies for normal replenishment. Second, the assumption of (R, Q) policies is relaxed and the optimal replenishment policy using stochastic dynamic programming is derived. The author examines how well the commonly used (R, Q) policy performs compared to the optimal policy. From numerical results it is shown that the cost difference when using the (R, Q) policy compared to the optimal policy is very small when the demand rate is relatively small. However, when the demand rate is relatively large, the cost difference becomes significant.

The literatures reviewed above consider a complete pooling. Axsater (2003) considered a decision rule for partial pooling examining a single-echelon inventory system consisting of a number of parallel local warehouses facing compound Poisson demand. The warehouses have (R, Q) ordering policies when replenishing from the supplier. The transshipments take no time but incur additional costs. The purpose of the model is to find a suitable replenishment policy, which includes reorder points and batch quantities for normal replenishments as well as a decision rule for transshipments. The decision rule has been evaluated in a simulation study, and the results support that the rule performs very well. It

is also shown that the savings obtained by using lateral transshipments are larger when the stochastic demand variations are larger.

Evers (2001) considered similar model with Axsater (2003), but instead of backorder the author assume lost sales under (R, Q) policies. Two heuristics are developed to assist decision makers in determining when stock transfers should be made. The heuristics produce critical values for on-hand inventory levels above which transshipments minimize overall expected costs. The rule proposed considers the direct costs incurred by transshipping compared to the savings incurred before making a transshipment decision.

2.2.2 *Periodic Review*

Krishnan and Rao (1965) introduced the idea of transferring stock between identical retailers after demand is observed and they showed the advantage of transshipments (Ozdemir et al., 2006). Robinson (1990) examined the effects on the optimal ordering policy of allowing transshipment among retailers as recourse actions occurring after demands are realized but before they must be satisfied (reactive transshipment). The model used here is an extension of the problem considered by Krishnan and Rao (1965) by allowing for multiple periods, varying costs across the retailers, and more general distributions. The objective of the model is to determine the ordering and transshipment policies that minimize the expected net present value of current and future costs. The resultant savings in holding and shortage costs are balanced against the costs of transshipment. The author found that a base stock ordering policy is optimal for this model.

Jonsson & Silver (1987) investigated a two-level distributions system consisting of a central warehouse (CW) supplying several branch warehouses (BW's), which in turn, supply normally-distributed customer demands in a periodic-review environment. The CW replenishes system inventory using a base-stock replenishment policy and a predetermined order cycle of time periods. From numerical results, they show that for the same service level, the allocation method with transshipment gives a considerably smaller investment in inventory. Their results also showed that transshipment system become more advantageous in situations with high demand variability, a long planning horizon, many BW's, a high service level and short lead times.

Tagaras and Cohen (1992) examined an inventory system with 2 locations with positive replenishment lead times. The authors investigated different transshipment policies and compared the performance of those policies. The inventory decision (the order-up-to level) is calculated based on a heuristic algorithm. From numerical examples, it is shown that complete pooling outperforms partial pooling. The authors stress the important role of

transshipments in an environment with positive replenishment lead time since this practice can reduce the risk gained from uncertainty over future material availability.

Ozdemir et al. (2006) considered multi-location transshipment problem where transportation capacity are restricted. It means that transshipment quantities between locations are limited, depends on the capacity of transportation mode used. They developed a solution procedure based on infinitesimal perturbation analysis to solve the stochastic optimization problem where the objective is to minimize the total cost. By conducting experimental design, the authors showed that the capacity restriction changes the system's inventory distribution and increase the total cost.

2.3 Summary

2.3.1 Summary of literature review

From the supply disruptions papers discussed above that use continuous review, it is clear that one of the methods to deal with disruptions is to extend the EOQ model in a way that it incorporates disruptions process like the EOQD model that is presented by for example Berk & Arreola-Risa (1994) and Snyder (2008). In the EOQD model, the length of the ON and OFF period are commonly assumed exponentially distributed, hence the disruptions process can form a two-state continuous time Markov chain. To express the expected cost most of the papers use renewal reward theorem. When they could not prove the convexity, they solve the problem numerically and found that the results are consistent with the classic model EOQ. The order quantity from EOQD formula is higher than the order quantity calculated by classical EOQ model because more inventories are needed to buffer the risk of supply disruptions.

Some papers also examine an inventory system where the reorder point is not zero (NZIO). For this NZIO case, most of the papers consider an (s, S) policy. This policy means that when the inventory level reach s and the supply is available, the firm should order Q units so that the inventory level reach S . When the supply is unavailable (OFF period), the firm does not order. It is also shown that when disruptions are long and stock-out cost are high, the value of incorporating disruptions into inventory model is very high.

For periodic review problem, when there is no fixed cost, a base stock policy can be optimal. When we incorporate a positive fixed cost, some papers agree that an (s, S) policy is optimal. In this case, the optimal parameters depend on the current state of the supply system.

The review about transshipments problem has shown that this problem have been investigated in many different setting of inventory system with subtle differences between the models considered, such as how many locations considered, the pooling preferences, cost parameters and so on. The different setting will change the complexity of the problem and the nature of the transshipment policy. Transshipment can be used as a recourse action in order to minimize backorders (Lee, 1987)

therefore penalty cost can be decreased. In addition, for the same service level, allowing transshipment leads to smaller investment in inventory (Jonsson & Silver, 1987). However it should be noted that transshipments incur other costs such as additional transportation cost for the transshipment and the cost of issuing this request (transaction cost). The transshipments decisions usually include the trade-off between these costs.

2.3.2 Contribution and Relevance

To the best of our knowledge, the current literature is either focusing on just supply disruptions or just transshipments. In this project, we investigate an inventory system that incorporates both problems (supply disruptions and transshipments).

Some real-life cases that are relevant with our problem of interest are shown in table 1. From those cases, the most relevant case is case no 3 (Hurricane Katrina in the U.S. Gulf Coast Region). Before the disruption event happened, Wal-Mart already has several distribution centers including a big one in Houston. When the storm attacked, the supply process of two distribution centers were badly affected, but the Houston facility and Wal-Mart's many other distribution centers were survived. It means that the disrupted distribution centers or the customers under disrupted distribution centers can still receive goods from another distribution center that is not affected by the storm. In this case Wal-Mart mitigates supply disruptions by having more inventories and allowing stock movement between its distribution centers.

We realized that this project may not represent the real-life cases perfectly as we make some assumptions to reduce mathematical complexity. However, this project can be seen as the first step for academic community and companies to gain comprehensive insights into an inventory system that incorporates supply disruptions and transshipments.

III. SYSTEM INVESTIGATED

3.1 System Description

In this case, we consider 2 retailers where the supply process at retailer 1 is always available while the supply process at retailer 2 might be disrupted. Every period, both retailers face a constant demand equal to d . Transshipments can be conducted from retailer 1 to retailer 2. We use a complete pooling policy for the transshipment (see e.g. Axsater, 1990 or Olsson, 2009).

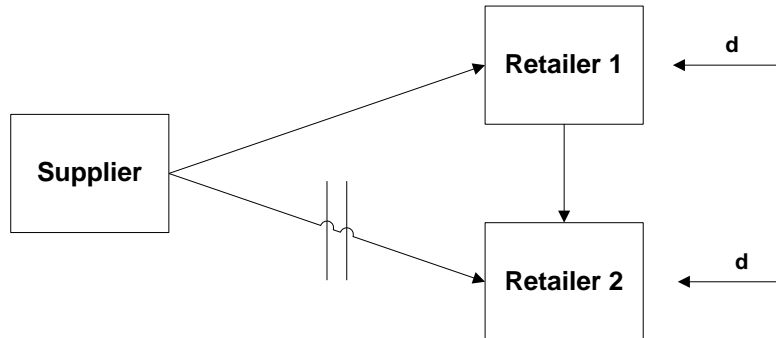


Figure 1 System Investigated

We have the following assumptions for the inventory system:

- a. Each retailer follows a base-stock inventory policy that is reviewed periodically. The inventory is observed each period so that it follows (1,S) policy. The reason why we assume a base-stock inventory policy is because some researchers have proven that this policy is optimal policy when there is no fixed ordering cost, for example:
 - Song and Zipkin (1996) showed that for a periodic-review inventory system with a Markovian supply process, a base-stock policy is optimal if there is no fixed cost.
 - Ozekici and Parlar (1999) examined a periodic-review inventory model that its parameters subject to a randomly changing environment. They also showed a similar result, that is, they show that where fixed ordering cost is omitted in their case, an environment-dependent base-stock policy is optimal.
- b. Orders attempt to the supplier are placed every period. We assume that there is no fixed order cost.
- c. The supply process to retailer 2 is assumed to follow a discrete time Markov chain (DTMC). The transition diagram of Markov chain of this process is depicted in the figure below:

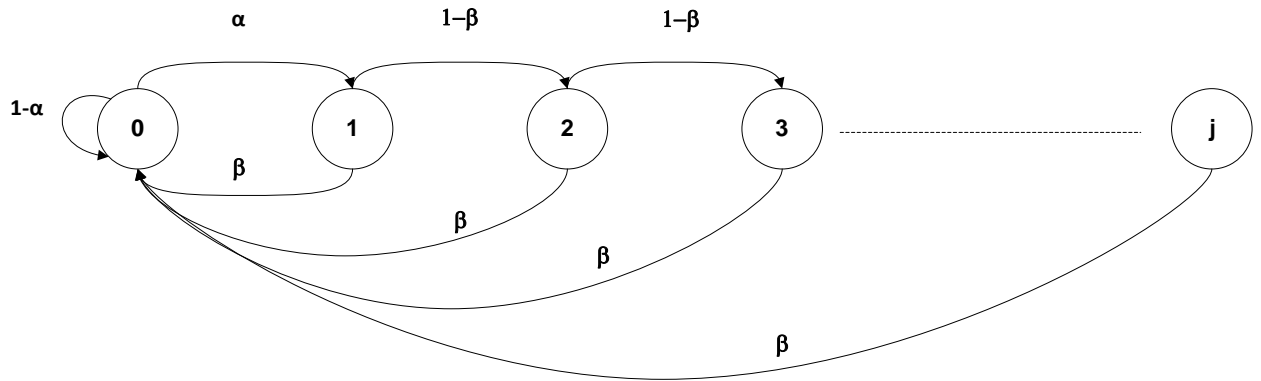


Figure 2 Transition Diagram for the System

State 0 until j (where j is a very large number) represents the number of consecutive periods that the supply process to retailer 2 has been disrupted. α represents the probability that the supply process to retailer 2 is disrupted in this period given that 1 period before this retailer was not disrupted (disruption probability) and β represents the probability that the supply process to retailer 2 is not disrupted in this period given that 1 period before this retailer was disrupted (recovery probability).

- d. The retailers face constant demand each period.
- e. The supplier's capacity is infinite.
- f. The lead times for the supply process and transshipment are zero.
- g. Unmet demands are fully backlogged.
- h. The costs considered in the system are penalty costs, holding costs and transshipments costs.

3.2 Series of Events

The system follows a series of events per period as described below:

1. At the beginning of a period, each retailer observes its inventory level.
2. Each retailer requests an order from the supplier to bring its inventory level to S (base stock level). Retailer 1 always manages to bring its inventory level to S_1 . For retailer 2, if the supply process for retailer 2 is ON, then its order is satisfied directly (zero lead time and infinite capacity) but if the supply process for this retailer is OFF, retailer 2 will not receive anything from the supplier.
3. Each retailer then tries to satisfy the demand using the on-hand inventory (if possible). Following this action, there are 2 possibilities:
 - a. Transshipment is not performed when retailer 2 can satisfy all of its demand in that period (i.e. retailer 2 does not have backorders). When retailer 2 does have backorders, transshipment still cannot be performed if retailer 1 does not have excess inventory.

- b. Transshipment is only performed when retailer 2 experiences shortage and retailer 1 has excess inventory. The amount transshipped from retailer 1 to retailer 2 is the minimum between the excess at retailer 1 and the shortage at retailer 2.
4. The remaining demand that cannot be satisfied at retailer 2 is backordered.
5. Finally, all costs are calculated at the end of period (holding cost due to on-hand inventory, penalty costs due to backorders and transshipments costs).

3.3 Notations

Below are some notations that are used in this document:

Index

Notation	Definition
j	Index representing the number of consecutive periods that retailer 2 has been disrupted, where $j \in \{0, 1, 2, \dots, \infty\}$
m	Index representing the retailer, where $m \in \{1, 2\}$

Parameters

Notation	Definition
c	Transshipment cost per item
h_m	Holding cost per item per period at retailer m
p_m	Penalty cost per item per period at retailer m
d	Constant demand at each retailer per period
I_m^j	The inventory level at retailer m at the end of a period when the system is in state (j)
B_m^j	The backorder level at retailer m at the end of a period when the system is in state (j)
α	Probability that the supply process to retailer 2 is disrupted in this period given that 1 period before this retailer was not disrupted (disruption probability)
β	Probability that the supply process to retailer 2 is not disrupted in this period given that 1 period before this retailer was disrupted (recovery probability)
π_j	Steady state probability of being in the j th state of disruptions at retailer 2
$F(j)$	Probability that retailer 2 is disrupted for j or less than j periods. $F(j) = \sum_{i=0}^j \pi_i$

Variables

Notation	Definition
S_m	Base-stock level for the inventory system at retailer m
X^j	Transshipment quantity from retailer 1 to retailer 2 in state (j)
TX^j	Total transshipments quantity from retailer 1 to retailer 2 up to state (j)
TX^{j-1}	Total transshipments quantity from retailer 1 to retailer 2 before state (j)

Mathematical Symbol

Notation	Definition
$[x]^+$	Maximum $\{0,x\}$
$[x]$	the largest integer less than or equal to x

3.4 Pre-conditions

1. Base stock level value for each retailer

We can limit the value of the base-stock level at each retailer as the following:

- a. Retailer 1

Since retailer 1 will never be disrupted and the demand is constant every period, retailer 1 needs at least to keep inventory equal to its own demand ($S_1 \geq d$) to avoid backorder in each period. If this condition is not satisfied, then every period retailer 1 always experiences backorder and this will lead to a very big expected cost per period.

The possibility of transshipments allow retailer 1 to save inventory more than its own demand. Since the demand at each retailer is the same, then at most retailer 1 will keep inventory equal to twice as much as the amount of the demand ($S_1 \leq 2d$) as an allocation to satisfy the demand from retailer 2 (if this retailer is disrupted and does not have enough inventory) by transshipping its inventory less or equal to d . If retailer 1 keeps inventory more than $2d$ every period, then even retailer 1 satisfy both retailers' demand (release $2d$ from its inventory), at the end of the period (after satisfying demand) retailer 1 will still keep inventory

and hence unnecessary holding cost will occur. This additional holding cost (due to keeping more inventory than needed) will lead to a very big expected cost per period.

In conclusion, the following expression holds for the optimal value of S_1 : $d \leq S_1 \leq 2d$. In the remaining analysis and numerical examples at chapter V and VI, this constraint will be used.

b. Retailer 2

Retailer 2 is the one that might face disruptions therefore it may need to keep more inventory than its periodic demand. At this point, we still cannot determine the maximum base-stock level value for retailer 2 (S_2), but we can say that S_2 should be bigger or equal to its demand ($S_2 \geq d$) under the assumption that the holding cost at retailer 2 is less than the sum of holding cost at retailer 1 plus transshipments cost ($h_1 + c > h_2$) which is a reasonable assumption. If the condition of S_2 is not satisfied ($S_2 < d$), then every period retailer 2 always experiences backorders.

2. Cost parameters

Cost parameters need to be carefully chosen so that the optimality of the complete pooling policy (where transshipment only take place from a retailer with excess inventory to a retailer with shortage until the excess inventory is depleted or the imminent shortage is completely avoided) is ensured. Therefore, while conducting numerical experiments, we choose our cost parameters according to the following conditions as suggested by Tagaras (1989): $h_1 + p_2 - c > 0$. If the condition does not hold, then one can already suggest that transshipments will not be beneficial.

IV. COST FUNCTION FORMULATION

For the system described above, an inventory system with 1 supplier and two retailers in the presence of supply disruptions risk and the possibility to conduct transshipments between the retailers, we are interested in formulating the cost function that involve holding and penalty costs at retailer 1 and retailer 2 as well as transshipments cost that may occur. We can determine the expected cost per unit of time for such system as the sum of the multiplication between the steady state probability of being in the j th state of disruptions (π_j) and the total costs (holding and penalty costs at retailer 1 and retailer 2 as well as transshipments) at j th state as written in the following formula:

$$C(S_1, S_2) = \sum_{j=0}^{\infty} \pi_j \left\{ h_1 E[I_1^j]^+ + p_1 E[B_1^j]^+ + h_2 E[I_2^j]^+ + p_2 E[B_2^j]^+ + c E[X^j] \right\} \quad (4.1)$$

Where:

$$I_1^j = S_1 - d - X^j \quad (4.2)$$

$$B_1^j = d - S_1 + X^j \quad (4.3)$$

$$I_2^j = S_2 - (j+1)d + TX^j \quad (4.4)$$

$$B_2^j = (j+1)d - S_2 - TX^j \quad (4.5)$$

$$X^j = \min \left\{ [S_1 - d]^+, [(j+1)d - S_2 - TX^{j-1}]^+ \right\} \quad (4.6)$$

$$TX^{j-1} = \left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+, 1 \right\} \right) \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) + [S_1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor - 1 \right]^+ \right) \quad (4.7)$$

$$TX^j = TX^{j-1} + X^j \quad (4.8)$$

Each formula is explained in detail hereafter.

4.1 Explanation of Each Formula

4.1.1 Formula 4.2

Formula 4.2 represents the inventory level at retailer 1 at the end of a period when the system is in state j . It equals to the base-stock level at retailer 1 (since in the beginning of a period retailer 1 always succeed to increase its inventory to its base-stock level) minus demand and transshipment quantity from retailer 1 to retailer 2 at state j .

4.1.2 Formula 4.3

Formula 4.3 represents the backorder level at retailer 1 at the end of a period when the system is in state j . It equals to the demand minus base-stock level at retailer 1 plus transshipment quantity from retailer 1 to retailer 2 at state j .

4.1.3 Formula 4.4

Formula 4.4 represents the inventory level at retailer 2 at the end of a period when the system is in state j . It equals to the base-stock level at retailer 2 minus demand that is multiplied by $(j+1)$ plus total transshipment quantity from retailer 1 to retailer 2 up to state j (TX^j). The demand in this formula has $(j+1)$ multiplier since in the j th period of a disruption, $j+1$ periods' worth of demand has taken place since the most recent replenishment.

4.1.4 Formula 4.5

Formula 4.5 represents the backorder level at retailer 2 at the end of a period when the system is in state j . It equals to the demand that is multiplied by $(j+1)$ minus the base-stock level at retailer 2 minus the total transshipment quantity from retailer 1 to retailer 2 up to state j . With the same reason as formula 4.4, the demand in this formula also has $(j+1)$ multiplier.

4.1.5 Formula 4.6

Formula 4.6 represents the transshipped quantity at state j . It can be found by finding the minimum value between the available inventory at retailer 1 ($[S_1-d]^+$) and the backorder in retailer 2 state j . The backorder at state j in a retailer that faces disruptions is $[(j+1)d-S_2]^+$ (Schmitt et al., 2010). Since now we allow transshipments between retailers, the backorders expression above should be reduced by the total transshipment quantity that has been received by retailer 2 before state j (TX^{j-1}).

Simplifying Formula 4.6

For ease of calculation, formula 4.6 can be replaced by:

$$X^j = \min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\} \quad (4.9)$$

The proof for above statement is as follows. The transshipped amount (X^j) can also be written in the following manner:

$$X^j = \begin{cases} 0 & , \text{for } j < \left\lfloor \frac{S_2}{d} \right\rfloor & \text{(a)} \\ \min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\} & , \text{for } j = \left\lfloor \frac{S_2}{d} \right\rfloor & \text{(b)} \\ [S_1 - d]^+ & , \text{for } j > \left\lfloor \frac{S_2}{d} \right\rfloor & \text{(c)} \end{cases}$$

To prove that formula 4.6 can be replaced by formula 4.9, we show that both formulas (formula 4.6 and 4.9) can satisfy each condition (a, b and c) for X^j as written above. Let j^* denotes the state of the system when the first transshipment occurs, that is when j is equal to the largest integer less than or equal to S_2/d .

- For condition (a)
Both formulas 4.6 and 4.9 hold for condition (a). In both cases, TX^{j-1} is always zero, because no transshipment happened before these states.
- For condition (b)
Both formulas 4.6 and 4.9 hold for condition (b). TX^{j-1} is always zero in this case since the first transshipment will occur in this stage, so there is no transshipment happened before this phase.
- For condition (c)
Both formulas 4.6 and 4.9 hold for condition (c). The reason why formula 4.9 also holds for this case is because the value of $[(j+1)d-S_2]^+$ will always be larger than or equal to $[S_1-d]^+$ for $j > S_2/d$ since previously we already constrained the value of S_1 ($d \leq S_1 \leq 2d$). Therefore finding the minimum value between $[S_1-d]^+$ and $[(j+1)d-S_2]^+$ will always yield to $[S_1-d]^+$ for $j > j^*$.

In conclusion, formula 4.9 holds for all values of j , hence it can be used to replace formula 4.6.

4.1.6 Formula 4.7

Formula 4.7 represents the total transshipment quantity that has been received by retailer 2 before state j . It means that before and up to state j^* , the value of TX^{j-1} is equal to 0. After j^* the value of TX^{j-1} is positive.

To build the expression for TX^{j-1} , we consider 3 cases:

1. TX^{j-1} when $j < j^*$

For all stages before j^* , there are no transshipments. Hence TX^{j-1} is equal to zero.

2. TX^{j-1} when $j = j^*$

At stage j^* , the first transshipment occurs, meaning that the total transshipments quantity before state j^* (TX^{j^*-1}) is zero since there are no transshipments before stage j^* .

3. TX^{j-1} when $j > j^*$

For every stage after j^* , the total transshipments quantity before state j (TX^{j-1}) is positive.

The general formula that holds for all 3 cases is shown below. For ease of explanation, the formula is divided into two parts:

$$TX^{j-1} = \underbrace{\left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+, 1 \right\} \right)}_A \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) + [S_1 - d]^+ \underbrace{\left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor - 1 \right]^+ \right)}_B$$

Part A of formula 4.7

We divide part A into two parts: A1 and A2 as shown below. Part A1 of formula 4.7 determines whether we should include the first transshipment quantity for calculating TX^{j-1} .

$$PartA = \underbrace{\left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+, 1 \right\} \right)}_{A1} \underbrace{\left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right)}_{A2}$$

The quantity of transshipped products at state j^* (the first transshipment) is represented by expression A2, that is, the minimum between the available products at retailer 1 and the backorder at retailer 2 at state j^* :

$$\left(\min \left\{ [S_1 - d]^+, (j^* + 1)d - S_2 \right\} \right) = \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right)$$

In order to make the expression general for every state j , we have to add another expression to make sure that up to state j^* , the value of part A is equal to zero. The expression that we need is formula A1 which then is multiplied with the transshipped quantity (A2). We distinguish 2 cases for A1:

- For $j \leq j^*$, the A1 expression for TX^{j-1} always returns 0, therefore part A is equal to 0 as well. Meaning that no transshipment occur for every state before state j^* .
- For $j > j^*$, the A1 expression for TX^{j-1} always returns 1, therefore part A is equal to A2. Meaning that for every state after j^* , part A is equal to the quantity of the first transshipment.

To sum up, the expression for the transshipment quantity when the system experiences the first transshipment activity at state (j) is:

$$\left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+, 1 \right\} \right) \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right)$$

Part B of formula 4.7

Part B represents the amount of transshipment quantity from the 2nd transshipment onward until state (j-1) without the first transshipment. If we do not constraint the value of S_1 , after the first transshipment, the transshipment quantity every period is:

$$\min \{ [S_1 - d]^+, d \}$$

Since we already constrained the value of S_1 to: $d \leq S_1 \leq 2d$, the value of $[S_1 - d]^+$ is always between zero and d ($0 < S_1 - d < d$), therefore finding the minimum value between $[S_1 - d]^+$ and d will always result in $[S_1 - d]^+$.

Therefore, if we constrained the value of S_1 as defined above, the amount of transshipment quantity from the 2nd transshipment onward is equal to $[S_1 - d]^+$.

The number of transshipments that has occurred including state (j-1) is:

$$[j - j^*]^+ = \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+$$

Since we already calculate the quantity of the first transshipment via part A, then we need to subtract 1 from the above formula. The number of transshipments occurred including state (j-1) without the first transshipment is:

$$\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor - 1 \right]^+$$

Therefore, the expression for the amount of transshipment quantity from the 2nd transshipment onward until state (j-1) without the first transshipment is the transshipment quantity every period multiplied by the number of transshipments occurred including state (j-1):

$$\left([S_1 - d]^+ \right) \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor - 1 \right]^+ \right)$$

All in all, the general expression for the total transshipments quantity from retailer 1 to retailer 2 before state j (state j is not included) is:

$$TX^{j-1} = \left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+, 1 \right\} \right) \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) + [S_1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor - 1 \right]^+ \right)$$

4.1.7 Formula 4.8

Formula 4.8 represents the total transshipments quantity from retailer 1 to retailer 2 including state j that equals to:

$$TX^j = TX^{j-1} + X^j$$

$$TX^j = TX^{j-1} + \min \left\{ [S_1 - d]^+, [(j+1)d - S_2 - TX^{j-1}]^+ \right\}$$

If we substitute the expression for TX^{j-1} , TX^j will be extremely long and complicated. Therefore we will simplify this expression that is explained hereafter.

Simplifying formula 4.8

To simplify this formula, we will start by modifying TX^{j-1} formula into TX^j . Previously the formula for TX^{j-1} was the following:

$$TX^{j-1} = \underbrace{\left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+, 1 \right\} \right)}_A \underbrace{\left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right)}_A + [S_1 - d]^+ \underbrace{\left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor - 1 \right]^+ \right)}_B$$

So that the above formula can be changed into TX^j formula, the transshipped amount at state j (X^j) should be included. To make this happen, two adjustments are needed. Firstly, changes at part A:

$$PartA = \underbrace{\left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+, 1 \right\} \right)}_{A1} \underbrace{\left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right)}_{A2}$$

A

Part A in TX^{j-1} will be zero for every state $j \leq j^*$. But for TX^j , part A should be zero only for every state $j < j^*$. At state j^* , TX^j should have a value that is equal to the quantity of the first transshipment. In this case, part A1 needs to be adjusted as follow:

$$\left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right]^+, 1 \right\} \right)$$

The above expression ensures that for $j < j^*$, the A1 expression always returns 0, therefore part A is equal to 0 as well. Meaning that for every stage before j^* , there is no transshipment occur.

For $j \geq j^*$, the A1 expression always returns 1, therefore part A is equal to A2. Meaning that for every stage after and including j^* , part A is equal to the quantity of the first transshipment.

Hence, part A expression for TX^j is:

$$\left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right]^+, 1 \right\} \right) \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right)$$

The changes at part B is done by adjusting the number of transshipments that have occurred. Previously for TX^{j-1} the number of transshipments that has occurred before state j (state j is not included) and without the first transshipment was as follow:

$$\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor - 1 \right]^+$$

since now state (j) is included, the adjustment for part B is by subtracting the expression above with one and then multiply it with the transshipment quantity:

$$\left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) ([S_1 - d]^+)$$

Therefore, after adjusting part A and part B, the final general expression for TX^j is:

$$TX^j = \left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right]^+, 1 \right\} \right) \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) + [S_1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \quad (4.10)$$

4.2 Final Cost Function

Based on the arguments given above, the expected cost per period for the system is the following:

$$C(S_1, S_2) = \sum_{j=0}^{\infty} \pi_j \left\{ \begin{array}{l} h_1 [S_1 - d - X^j]^+ + p_1 [d - S_1 + X^j]^+ \\ + h_2 [S_2 - (j+1)d + TX^j]^+ + p_2 [(j+1)d - S_2 - TX^j]^+ \\ + c E[X^j] \end{array} \right\} \quad (4.11)$$

$$C(S_1, S_2) = \sum_{j=0}^{\infty} \pi_j \left\{ \begin{array}{l} h_1 [S_1 - d - \min\{[S_1 - d]^+, [(j+1)d - S_2]^+\}]^+ \\ + p_1 [d - S_1 + \min\{[S_1 - d]^+, [(j+1)d - S_2]^+\}]^+ \\ + h_2 \left[S_2 - (j+1)d + \left(\min\left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right\rfloor^+, 1 \right\} \right) \left(\min\left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) + [S_1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \\ + p_2 \left[(j+1)d - S_2 - \left(\min\left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right\rfloor^+, 1 \right\} \right) \left(\min\left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) + [S_1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \\ + c \left(\min\{[S_1 - d]^+, [(j+1)d - S_2]^+\} \right) \end{array} \right\} \quad (4.12)$$

V. COST FUNCTION ANALYSIS

To get insight in the behavior of the cost function, we conducted several numerical experiments. These experiments suggest that the cost function is piece wise linear in S_1 with breakpoints at integer multiple of d if S_2 is an integer multiple of d . If S_2 is not an integer multiple of d , then the cost function is piece-wise linear in S_1 with breakpoints where the sum of S_1 and S_2 is an integer multiple of d . The experiments also shows that the cost function is piece wise linear in S_2 with breakpoints at integer multiple of d if S_1 is an integer multiple of d . If S_1 is not an integer multiple of d , then the cost function is piece-wise linear in S_2 with breakpoints where the sum of S_1 and S_2 is an integer multiple of d . In addition, the experiments also suggest that the cost function is convex in S_1 and S_2 .

We present one of several numerical examples that support the above statements in the table below. The experiment is conducted with $d=4$, $h_1=h_2=1$, $p_1=p_2=5$, $\alpha=0.2$, and $\beta=0.8$. We vary the value for the transshipment cost (c) and the base-stock levels (S_1 and S_2).

The red highlight in the tables shows the optimal solution for the system, while the grey highlight shows the minimum cost when we fixed one of the base stock level. For example in table 2, if we do not specify the value of both S_1 and S_2 in advance, the set of the optimal base-stock levels is $S_1=8$ and $S_2=4$, with cost=4. If we fix $S_1 = 4$ (1st row), we can see that S_2 that minimizes the cost is 8 (cost = 4.2). Another example is if we fix $S_2 = 6$ (3rd row), then S_1 that minimizes the cost is 6 (cost =4.1). Table 3 until table 5 show other results with different values of c .

		S2				
		4	5	6	7	8
S1	4	5	4.8	4.6	4.4	4.2
	5	4.75	4.55	4.35	4.15	4.95
	6	4.5	4.3	4.1	4.9	5.7
	7	4.25	4.05	4.85	5.65	6.45
	8	4	4.8	5.6	6.4	7.2

Table 2 Experiment result with $c = 1$

		S2				
		4	5	6	7	8
S1	4	5	4.8	4.6	4.4	4.2
	5	4.95	4.75	4.55	4.35	4.99
	6	4.9	4.7	4.5	5.14	5.78
	7	4.85	4.65	5.29	5.93	6.57
	8	4.8	5.44	6.08	6.72	7.36

Table 3 Experiment result with $c = 2$

		S2				
		4	5	6	7	8
S1	4	5	4.8	4.6	4.4	4.2
	5	5.15	4.95	4.75	4.55	5.03
	6	5.3	5.1	4.9	5.38	5.86
	7	5.45	5.25	5.73	6.21	6.69
	8	5.6	6.08	6.56	7.04	7.52

Table 4 Experiment result with $c = 3$

		S2				
		4	5	6	7	8
S1	4	5	4.8	4.6	4.4	4.2
	5	5.35	5.15	4.95	4.75	5.07
	6	5.7	5.5	5.3	5.62	5.94
	7	6.05	5.85	6.17	6.49	6.81
	8	6.4	6.72	7.04	7.36	7.68

Table 5 Experiment result with $c = 4$

We tried other numerical examples and so far they deliver the same insight. If the conclusion (that the cost function is piece-wise linear) really holds for all cases, it will be beneficial for us in finding the optimal solution for the problem, since we can highly reduce the set of feasible solutions. Later we will build a theorem based on the above arguments and then prove it mathematically.

The graph below represents the behavior of the cost. The graph is taken from one of the numerical examples we have conducted (where $d=4$). The y axis is the expected cost per period and the x axis denotes the value of S_1 .



Figure 3 Example Cost Function

For different values of S_2 , the minimum cost also yields different values for S_1 . The breakpoint of each graph takes place when the sum of S_1+S_2 is a multiple of d . We also can see from the graph that the cost function is convex.

Based on the findings above, we take several steps in order to find the optimal base-stock levels for the problem as the following:

1. Show that the cost function is piece-wise linear in S_1 and S_2 with breakpoints at integer multiple of d . This step is taken in order to reduce the set of feasible solutions.
2. Show that the cost function is convex.
3. Find the optimal base-stock levels, that is, the base-stock level at retailer 1 and retailer 2 that minimizes the cost.

5.1 The cost function is piece-wise linear in S_1 and S_2

Based on the findings above, we build theorems about piece-wise linearity of the cost function as follows:

Theorem 1a: $C(S_1, S_2)$ is piece-wise linear in S_1 with breakpoints at integer multiple of d if S_2 is an integer multiple of d .

Theorem 1b: $C(S_1, S_2)$ is piece-wise linear in S_2 with breakpoints at integer multiple of d if S_1 is an integer multiple of d .

Theorem 2a: If S_2 is not an integer multiple of d , then $C(S_1, S_2)$ is piece-wise linear in S_1 with breakpoints where the sum of S_1 and S_2 is an integer multiple of d .

Theorem 2b: If S_1 is not an integer multiple of d , then $C(S_1, S_2)$ is piece-wise linear in S_2 with breakpoints where the sum of S_1 and S_2 is an integer multiple of d .

In order to prove the above theorems, it will be shown that the second order difference equation, $\Delta^2 C(S_1, S_2)$, equals to zero if the increment in the base-stock level is zero or less than an integer multiple of d . But $\Delta^2 C(S_1, S_2)$ is positive if the increment in the base-stock level for $\Delta^2 C(S_1, S_2)$ is an integer multiple of d . The further details about this will be explained later in this section.

The first order difference equation is defined as:

- with respect to S_1 :

$$\Delta C_{S_1}(S_1, S_2) = C(S_1 + 1, S_2) - C(S_1, S_2) \quad (5.1)$$

- with respect to S_2 :

$$\Delta C_{S_2}(S_1, S_2) = C(S_1, S_2 + 1) - C(S_1, S_2) \quad (5.2)$$

While the second order of difference equation is defined as:

- with respect to S_1 :

$$\Delta^2 C_{S_1}(S_1, S_2) = \Delta C_{S_1}(S_1 + 1, S_2) - \Delta C_{S_1}(S_1, S_2) \quad (5.3)$$

- with respect to S_2 :

$$\Delta^2 C_{S_2}(S_1, S_2) = \Delta C_{S_2}(S_1, S_2 + 1) - \Delta C_{S_2}(S_1, S_2) \quad (5.4)$$

Proof of theorem 1a and 2a

According to formula 5.1, the first order difference equation with respect to S_1 is the following:

$$\Delta C_{S_1}(S_1, S_2) = C(S_1 + 1, S_2) - C(S_1, S_2)$$

$$= \sum_{j=0}^{\infty} \pi_j \left\{ \begin{aligned} & h_1 \left(\left[S_1 + 1 - d - \min \{ [S_1 + 1 - d]^+, [(j+1)d - S_2]^+ \} \right]^+ - \left[S_1 - d - \min \{ [S_1 - d]^+, [(j+1)d - S_2]^+ \} \right]^+ \right) \\ & + p_1 \left(\left[d - S_1 - 1 + \min \{ [S_1 + 1 - d]^+, [(j+1)d - S_2]^+ \} \right]^+ - \left[d - S_1 + \min \{ [S_1 - d]^+, [(j+1)d - S_2]^+ \} \right]^+ \right) \\ & + h_2 \left(\begin{aligned} & \left[S_2 - (j+1)d + \left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right]^+, 1 \right\} \right) \left(\min \{ [S_1 + 1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \} \right) + [S_1 + 1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \right) \\ & - \left[S_2 - (j+1)d + \left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right]^+, 1 \right\} \right) \left(\min \{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \} \right) + [S_1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \right) \end{aligned} \right) \\ & + p_2 \left(\begin{aligned} & \left[(j+1)d - S_2 - \left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right]^+, 1 \right\} \right) \left(\min \{ [S_1 + 1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \} \right) + [S_1 + 1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \right) \\ & - \left[(j+1)d - S_2 - \left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right]^+, 1 \right\} \right) \left(\min \{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \} \right) + [S_1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \right) \end{aligned} \right) \\ & + c \left(\left(\min \{ [S_1 + 1 - d]^+, [(j+1)d - S_2]^+ \} \right) - \left(\min \{ [S_1 - d]^+, [(j+1)d - S_2]^+ \} \right) \right) \end{aligned} \right\} \quad (5.5)$$

Since the value of S_1 is never less than d and transshipments only take place from retailer with excess inventory to a retailer that faces shortage then it can be concluded that retailer 1 will never have

backorder, therefore we can omit the expression related to p_2 . The expression related to p_1 is the following:

$$p_1 \left(\left[d - S_1 - 1 + \min \left\{ [S_1 + 1 - d]^+, [(j+1)d - S_2]^+ \right\} \right]^+ - \left[d - S_1 + \min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\} \right]^+ \right)$$

In addition, since transshipment activities are only executed to reduce backorders at retailer 2, having one more item at retailer 1 will not give impact to the on-hand inventory at retailer 2 at the end of a period. It will only have impact to its backorder level. Hence, we can exclude the expression related to h_2 . The expression related to h_2 is the following:

$$h_2 \left(\left[S_2 - (j+1)d + \left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right]^+, 1 \right\} \right) \left(\min \left\{ [S_1 + 1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) + [S_1 + 1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \right. \\ \left. - \left[S_2 - (j+1)d + \left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right]^+, 1 \right\} \right) \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) + [S_1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \right) \right)$$

By erasing two expressions above, the first order difference equation with respect to S_1 is:

$$= \sum_{j=0}^{\infty} \pi_j \left\{ p_2 \left(\left[S_1 + 1 - d - \min \left\{ [S_1 + 1 - d]^+, [(j+1)d - S_2]^+ \right\} \right]^+ - \left[S_1 - d - \min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\} \right]^+ \right) \right. \\ \left. \left(\left[(j+1)d - S_2 - \left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right]^+, 1 \right\} \right) \left(\min \left\{ [S_1 + 1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) + [S_1 + 1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \right. \right. \\ \left. \left. - \left[(j+1)d - S_2 - \left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right]^+, 1 \right\} \right) \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) + [S_1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \right) \right) \right. \\ \left. + c \left(\left(\min \left\{ [S_1 + 1 - d]^+, [(j+1)d - S_2]^+ \right\} \right) - \left(\min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\} \right) \right) \right) \quad (5.6)$$

In formula 5.6 above, we have to determine the minimum between several expressions. To accommodate this situation, the first order difference equation with respect to S_1 is divided into 2 conditions:

– Condition 1, if:

$$[S_1 - d]^+ < \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2$$

- Condition 2, if:

$$[S_1 - d]^+ > \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2$$

The expression for the first order difference equation with respect to S_1 based on two conditions above is only different in the expression that is related to p_2 (derived in the appendix).

We can simplify the expression for the first order difference equation with respect to S_1 as shown below:

- For condition 1:

$$\Delta C_{S_1}(S_1, S_2) = h_1 \left(\sum_{j=0}^{\frac{S_1+S_2-2}{d}} \pi_j \right)^{(1)} + p_2 \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j \left(\left\lfloor \frac{S_1+S_2}{d} \right\rfloor - j - 2 \right) \right)^{(2)} + c \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j \right)^{(3)} \quad (5.7)$$

- For condition 2:

$$\Delta C_{S_1}(S_1, S_2) = h_1 \left(\sum_{j=0}^{\frac{S_1+S_2-2}{d}} \pi_j \right)^{(1)} + p_2 \left(\sum_{\frac{S_1+S_2}{d}}^{\infty} \pi_j \left(\left\lfloor \frac{S_1+S_2}{d} \right\rfloor - j - 1 \right) \right)^{(2)} + c \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j \right)^{(3)} \quad (5.8)$$

Following above results, we can calculate the second order difference equation with respect to S_1 as follows:

$$\Delta^2 C_{S_1}(S_1, S_2) = \Delta C_{S_1}(S_1 + 1, S_2) - \Delta C_{S_1}(S_1, S_2)$$

- Condition 1:

$$\begin{aligned} \Delta^2 C_{S_1}(S_1, S_2) = & \left(h_1 \left(\sum_{j=0}^{\frac{S_1+1+S_2-2}{d}} \pi_j \right) + p_2 \left(\sum_{\frac{S_1+1+S_2-1}{d}}^{\infty} \pi_j \left(\left\lfloor \frac{S_1+1}{d} + \frac{S_2}{d} \right\rfloor - j - 2 \right) \right) + c \left(\sum_{\frac{S_1+1+S_2-1}{d}}^{\infty} \pi_j \right) \right) - \\ & \left(h_1 \left(\sum_{j=0}^{\frac{S_1+S_2-2}{d}} \pi_j \right) + p_2 \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j \left(\left\lfloor \frac{S_1+S_2}{d} \right\rfloor - j - 2 \right) \right) + c \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j \right) \right) \end{aligned} \quad (5.9a)$$

- Condition 2:

$$\begin{aligned} \Delta^2 C_{S_1}(S_1, S_2) = & \left(h_1 \left(\sum_{j=0}^{\left\lfloor \frac{S_1+1+S_2-2}{d} \right\rfloor} \pi_j \right) + p_2 \left(\sum_{\frac{S_1+1+S_2}{d}}^{\infty} \pi_j \left(\left\lfloor \frac{S_1+1}{d} + \frac{S_2}{d} \right\rfloor - j - 1 \right) \right) + c \left(\sum_{\frac{S_1+1+S_2-1}{d}}^{\infty} \pi_j \right) \right) - \\ & \left(h_1 \left(\sum_{j=0}^{\left\lfloor \frac{S_1+S_2-2}{d} \right\rfloor} \pi_j \right) + p_2 \left(\sum_{\frac{S_1+S_2}{d}}^{\infty} \pi_j \left(\left\lfloor \frac{S_1+S_2}{d} \right\rfloor - j - 1 \right) \right) + c \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j \right) \right) \end{aligned} \quad (5.9b)$$

Since π_j is only defined for integer j values, equation 5.9a and 5.9b equals to zero for $d > 1$. Therefore, to make the above equations positive, the closest difference in S_1 should be d . Based on that argument, the expression for the second order equation can be replaced by:

$$\Delta^2 C_{S_1}(S_1, S_2) = \Delta C_{S_1}(S_1 + d, S_2) - \Delta C_{S_1}(S_1, S_2)$$

- Condition 1

$$\begin{aligned} \Delta^2 C_{S_1}(S_1, S_2) = & \left(h_1 \left(\sum_{j=0}^{\left\lfloor \frac{S_1+d+S_2-2}{d} \right\rfloor} \pi_j \right) + p_2 \left(\sum_{\frac{S_1+d+S_2-1}{d}}^{\infty} \pi_j \left(\left\lfloor \frac{S_1+d}{d} + \frac{S_2}{d} \right\rfloor - j - 2 \right) \right) + c \left(\sum_{\frac{S_1+d+S_2-1}{d}}^{\infty} \pi_j \right) \right) - \\ & \left(h_1 \left(\sum_{j=0}^{\left\lfloor \frac{S_1+S_2-2}{d} \right\rfloor} \pi_j \right) + p_2 \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j \left(\left\lfloor \frac{S_1+S_2}{d} \right\rfloor - j - 2 \right) \right) + c \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j \right) \right) \\ \Delta^2 C_{S_1}(S_1, S_2) = & h_1 \left(\pi_{\frac{S_1+S_2-1}{d}} \right) + p_2 \left(\sum_{\frac{S_1+S_2}{d}}^{\infty} \pi_j + \pi_{\frac{S_1+S_2-1}{d}} \right) - c \left(\pi_{\frac{S_1+S_2-1}{d}} \right) \end{aligned} \quad (5.10)$$

- Condition 2:

$$\begin{aligned} \Delta^2 C_{S_1}(S_1, S_2) &= \\ & \left(h_1 \left(\sum_{j=0}^{\left\lfloor \frac{S_1+d+S_2}{d} - 2 \right\rfloor} \pi_j \right) + p_2 \left(\sum_{\frac{S_1+d+S_2}{d}}^{\infty} \pi_j \left(\left\lfloor \frac{S_1+d}{d} + \frac{S_2}{d} \right\rfloor - j - 1 \right) \right) + c \left(\sum_{\frac{S_1+d+S_2}{d}-1}^{\infty} \pi_j \right) \right) - \\ & \left(h_1 \left(\sum_{j=0}^{\left\lfloor \frac{S_1+S_2}{d} - 2 \right\rfloor} \pi_j \right) + p_2 \left(\sum_{\frac{S_1+S_2}{d}}^{\infty} \pi_j \left(\left\lfloor \frac{S_1+S_2}{d} \right\rfloor - j - 1 \right) \right) + c \left(\sum_{\frac{S_1+S_2}{d}-1}^{\infty} \pi_j \right) \right) \\ \Delta^2 C_{S_1}(S_1, S_2) &= h_1 \left(\pi_{\frac{S_1+S_2}{d}-1} \right) + p_2 \left(\sum_{\frac{S_1+S_2}{d}+1}^{\infty} \pi_j + \pi_{\frac{S_1+S_2}{d}} \right) - c \left(\pi_{\frac{S_1+S_2}{d}-1} \right) \quad (5.11) \end{aligned}$$

Equation 5.10 and 5.11 show that the smallest increment of S_1 is d if S_2 is also an integer multiple of d , which supports theorem 1a.

Expression 5.10 can also be written as the following:

$$\Delta^2 C_{S_1}(S_1, S_2) = h_1 \left(\pi_{\frac{S_1+S_2}{d}-1} \right) + p_2 \left(\sum_{\frac{S_1+S_2}{d}}^{\infty} \pi_j + \pi_{\frac{S_1+S_2}{d}} \right) - c \left(\pi_{\frac{S_1+S_2}{d}-1} \right) \quad (5.12a)$$

And expression 5.11 can also be written as the following:

$$\Delta^2 C_{S_1}(S_1, S_2) = h_1 \left(\pi_{\frac{S_1+S_2}{d}-1} \right) + p_2 \left(\sum_{\frac{S_1+S_2}{d}+1}^{\infty} \pi_j + \pi_{\frac{S_1+S_2}{d}} \right) - c \left(\pi_{\frac{S_1+S_2}{d}-1} \right) \quad (5.12b)$$

Expression 5.12a and 5.12b show that if S_2 is not an integer multiple of d , the sum of S_1 and S_2 should be an integer multiple of d . This expression supports theorem 2a.

Following exactly the same approach, we can prove that the cost function is piece-wise linear in S_2 (theorem 1b and 2b).

5.2 Convexity of the cost Function

As stated in the numerical experiments above, the experiment suggests that the cost function is convex. Based on that observation, we propose the following theorem:

Theorem 3: The cost function $C(S_1, S_2)$ is convex in S_1 and S_2

In order to prove the convexity of $C(S_1, S_2)$ we use Hessian matrix. In this case, we need to show that the Hessian matrix is positive semi definite. To prove that the Hessian matrix is positive semi definite, we need to show that the determinant of its Hessian matrix is positive (Weir, Hass, & Giordano, 2005).

In order to build Hessian matrix, we need to define the expression for the second order difference equation. Following from the theorem about piece-wise linearity of the cost function which stated that the smallest increment for S_1 and S_2 is d , we define difference equation as the following:

The first order difference equation:

$$\Delta C_{S_1}(S_1, S_2) = C(S_1 + d, S_2) - C(S_1, S_2)$$

$$\Delta C_{S_2}(S_1, S_2) = C(S_1, S_2 + d) - C(S_1, S_2)$$

The second order difference equation:

$$\Delta^2 C_{S_1, S_1}(S_1, S_2) = \Delta C_{S_1}(S_1 + d, S_2) - \Delta C_{S_1}(S_1, S_2)$$

$$\Delta^2 C_{S_1, S_2}(S_1, S_2) = \Delta C_{S_1}(S_1, S_2 + d) - \Delta C_{S_1}(S_1, S_2)$$

$$\Delta^2 C_{S_2, S_1}(S_1, S_2) = \Delta C_{S_2}(S_1 + d, S_2) - \Delta C_{S_2}(S_1, S_2)$$

$$\Delta^2 C_{S_2, S_2}(S_1, S_2) = \Delta C_{S_2}(S_1, S_2 + d) - \Delta C_{S_2}(S_1, S_2)$$

Let $H(C)$ denotes the Hessian matrix of $C(S_1, S_2)$ function, then,

$$H(C) = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} = \begin{bmatrix} \Delta^2 C_{S_1, S_1}(S_1, S_2) & \Delta^2 C_{S_1, S_2}(S_1, S_2) \\ \Delta^2 C_{S_2, S_1}(S_1, S_2) & \Delta^2 C_{S_2, S_2}(S_1, S_2) \end{bmatrix} \quad (5.13)$$

The first order difference equation with respect to S_1

The first order difference equation with respect to S_1 can be written as follow (using the same approach as we used for formula 5.7):

$$\Delta C_{S_1}(S_1, S_2) = h_1 \left(\sum_{j=0}^{\frac{S_1+S_2-2}{d}} \pi_j d \right) + p_2 \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j \left(\frac{S_1+S_2}{d} - j - 2 \right) d \right) + c \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j d \right) \quad (5.14)$$

The first order difference equation with respect to S_2

We found 2 different results for this expression based on the value of S_1 . The results are the following:

Condition 1: if $S_1=d$, then,

$$\begin{aligned} \Delta C_{S_2}(S_1, S_2) &= h_2 \left(\sum_{j=0}^{\frac{S_2-1}{d}} \pi_j d \right) + p_2 \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j (-d) \right) \\ \Delta C_{S_2}(S_1, S_2) &= h_2 \left(\sum_{j=0}^{\frac{S_2-1}{d}} \pi_j d \right) - p_2 \left(\sum_{\frac{S_2}{d}}^{\infty} \pi_j (d) \right) \end{aligned} \quad (5.15a)$$

Condition 2: if $S_1=2d$, then,

$$\begin{aligned} \Delta C_{S_2}(S_1, S_2) &= h_1 \pi_{\frac{S_1+S_2-2}{d}} d + h_2 \left(\sum_{j=0}^{\frac{S_2-1}{d}} \pi_j d \right) + c \pi_{\frac{S_1+S_2-2}{d}} (-d) \\ \Delta C_{S_2}(S_1, S_2) &= h_1 \pi_{\frac{S_2}{d}} d + h_2 \left(\sum_{j=0}^{\frac{S_2-1}{d}} \pi_j d \right) - c \pi_{\frac{S_2}{d}} d \end{aligned} \quad (5.15b)$$

Based on the above result, the expression of each part on the matrix is:

- Expression for h_{11}

$$\begin{aligned} \Delta^2 C_{S_1, S_1}(S_1, S_2) &= \Delta C_{S_1}(S_1 + d, S_2) - \Delta C_{S_1}(S_1, S_2) \\ &= h_1 \left(\sum_{j=0}^{\frac{S_1+S_2-1}{d}} \pi_j d \right) + p_2 \left(\sum_{\frac{S_1+S_2}{d}}^{\infty} \pi_j \left(\frac{S_1+S_2}{d} - j - 1 \right) d \right) + c \left(\sum_{\frac{S_1+S_2}{d}}^{\infty} \pi_j d \right) - \\ &\left(h_1 \left(\sum_{j=0}^{\frac{S_1+S_2-2}{d}} \pi_j d \right) + p_2 \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j \left(\frac{S_1+S_2}{d} - j - 2 \right) d \right) + c \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j d \right) \right) \end{aligned}$$

$$= \left(d\pi_{\frac{S_1+S_2-1}{d}} (h_1 + p_2 - c) \right) + p_2 \sum_{\frac{S_1+S_2}{d}}^{\infty} d\pi_j \quad (5.16)$$

- Expression for h_{12}

$$\begin{aligned} \Delta^2 C_{S_1, S_2} (S_1, S_2) &= \Delta C_{S_1} (S_1, S_2 + d) - \Delta C_{S_1} (S_1, S_2) \\ &= \left(\sum_{j=0}^{\frac{S_1+S_2-1}{d}} \pi_j d \right) + p_2 \left(\sum_{\frac{S_1+S_2}{d}}^{\infty} \pi_j \left(\frac{S_1+S_2}{d} - j - 1 \right) d \right) + c \left(\sum_{\frac{S_1+S_2}{d}}^{\infty} \pi_j d \right) - \\ &\left(h_1 \left(\sum_{j=0}^{\frac{S_1+S_2-2}{d}} \pi_j d \right) + p_2 \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j \left(\frac{S_1+S_2}{d} - j - 2 \right) d \right) + c \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j d \right) \right) \\ &= \left(d\pi_{\frac{S_1+S_2-2}{d}} (h_1 + p_2 - c) \right) + p_2 \sum_{\frac{S_1+S_2}{d}}^{\infty} d\pi_j \quad (5.17) \end{aligned}$$

- Expression for h_{21}

$$\Delta^2 C_{S_2, S_1} (S_1, S_2) = \Delta C_{S_2} (S_1 + d, S_2) - \Delta C_{S_2} (S_1, S_2)$$

Previously we had 2 conditions for the first difference equation of S_2 . Since h_{21} is based on the first difference equation of S_2 , therefore we also apply the same condition for the h_{21} expression.

Condition 1: if $S_1=d$, then,

$$\begin{aligned} \Delta^2 C_{S_2, S_1} (S_1, S_2) &= \left(h_2 \left(\sum_{j=0}^{\frac{S_2-1}{d}} \pi_j d \right) + p_2 \left(\sum_{\frac{S_1+S_2}{d}}^{\infty} \pi_j (-d) \right) \right) - \left(h_2 \left(\sum_{j=0}^{\frac{S_2-1}{d}} \pi_j d \right) + p_2 \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j (-d) \right) \right) \\ p_2 (-d) \left(\sum_{\frac{S_1+S_2}{d}}^{\infty} \pi_j - \sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j \right) &= p_2 (-d) \left(\sum_{\frac{S_1+S_2}{d}}^{\infty} \pi_j - \pi_{\frac{S_1+S_2-1}{d}} - \sum_{\frac{S_1+S_2}{d}}^{\infty} \pi_j \right) = p_2 d \pi_{\frac{S_2}{d}} \quad (5.18a) \end{aligned}$$

Condition 2: if $S_1=2d$, then,

$$\begin{aligned}\Delta^2 C_{S_2, S_1}(S_1, S_2) &= \left(h_1 \pi_{\frac{S_1+S_2-1}{d}} d + h_2 \left(\sum_{j=0}^{\frac{S_2-1}{d}} \pi_j d \right) + c \pi_{\frac{S_1+S_2-1}{d}}(-d) \right) - \left(h_1 \pi_{\frac{S_1+S_2-2}{d}} d + h_2 \left(\sum_{j=0}^{\frac{S_2-1}{d}} \pi_j d \right) + c \pi_{\frac{S_1+S_2-2}{d}}(-d) \right) \\ &= h_1 d \left(\pi_{\frac{S_2+1}{d}} - \pi_{\frac{S_2}{d}} \right) + c(-d) \left(\pi_{\frac{S_2+1}{d}} - \pi_{\frac{S_2}{d}} \right) \quad (5.18b)\end{aligned}$$

- Expression for h_{22}

$$\Delta^2 C_{S_2, S_2}(S_1, S_2) = \Delta C_{S_2}(S_1, S_2 + d) - \Delta C_{S_2}(S_1, S_2)$$

Using the same reason with h_{21} expression, we also apply 2 conditions for the h_{22} expression.

Condition 1: if $S_1=d$, then,

$$\begin{aligned}\Delta^2 C_{S_2, S_2}(S_1, S_2) &= h_2 \left(\sum_{j=0}^{\frac{S_2}{d}} \pi_j d - \sum_{j=0}^{\frac{S_2-1}{d}} \pi_j d \right) + p_2 \left(\sum_{\frac{S_1+S_2}{d}}^{\infty} \pi_j(-d) - \sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j(-d) \right) \\ &= h_2 \pi_{\frac{S_2}{d}} d + p_2 \pi_{\frac{S_2}{d}} d \quad (5.19a)\end{aligned}$$

Condition 2: if $S_1=2d$, then,

$$\begin{aligned}\Delta^2 C_{S_2, S_2}(S_1, S_2) &= h_1 d \left(\pi_{\frac{S_1+S_2-1}{d}} - \pi_{\frac{S_1+S_2-2}{d}} \right) + h_2 \left(\sum_{j=0}^{\frac{S_2}{d}} \pi_j d - \sum_{j=0}^{\frac{S_2-1}{d}} \pi_j d \right) + c(-d) \left(\pi_{\frac{S_1+S_2-1}{d}} - \pi_{\frac{S_1+S_2-2}{d}} \right) \\ &= h_1 d \left(\pi_{\frac{S_2+1}{d}} - \pi_{\frac{S_2}{d}} \right) + h_2 \pi_{\frac{S_2}{d}} d + c(-d) \left(\pi_{\frac{S_2+1}{d}} - \pi_{\frac{S_2}{d}} \right) \quad (5.19b)\end{aligned}$$

Since all expressions for the matrix are already defined, then we can define the Hessian matrix for our function as the following:

Condition 1 ($S_1=d$):

$$\mathbf{H(C)} = \begin{bmatrix} \left(d \pi_{\frac{S_1+S_2-1}{d}} (h_1 + p_2 - c) \right) + p_2 \sum_{\frac{S_1+S_2}{d}}^{\infty} d \pi_j & \left(d \pi_{\frac{S_1+S_2-1}{d}} (h_1 + p_2 - c) \right) + p_2 \sum_{\frac{S_1+S_2}{d}}^{\infty} d \pi_j \\ p_2 d \pi_{\frac{S_2}{d}} & h_2 \pi_{\frac{S_2}{d}} d + p_2 \pi_{\frac{S_2}{d}} d \end{bmatrix} \quad (5.20a)$$

Condition 2 ($S_1=2d$):

$$H(C) = \begin{bmatrix} \left(d\pi_{\frac{S_1+S_2}{d}-1} (h_1 + p_2 - c) \right) + p_2 \sum_{\frac{S_1+S_2}{d}}^{\infty} d\pi_j & \left(d\pi_{\frac{S_1+S_2}{d}-1} (h_1 + p_2 - c) \right) + p_2 \sum_{\frac{S_1+S_2}{d}}^{\infty} d\pi_j \\ h_1 d \left(\pi_{\frac{S_1+S_2}{d}-1} - \pi_{\frac{S_1+S_2}{d}-2} \right) + c(-d) \left(\pi_{\frac{S_2}{d}+1} - \pi_{\frac{S_2}{d}} \right) & h_1 d \left(\pi_{\frac{S_2}{d}+1} - \pi_{\frac{S_2}{d}} \right) + h_2 \pi_{\frac{S_2}{d}} d + c(-d) \left(\pi_{\frac{S_2}{d}+1} - \pi_{\frac{S_2}{d}} \right) \end{bmatrix} \quad (5.20b)$$

The determinant of H(C) is defined as:

$$|H(C)| = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} = (h_{11}h_{22}) - (h_{12}h_{21})$$

From the expression 5.20a and 5.20b above, we can see that:

- Both conditions only differ in h_{21} and h_{22}
- The expression for h_{11} and h_{12} are equal

Since h_{11} and h_{12} are equal, the determinant of H(C) can be written as:

$$|H(C)| = h_{11}(h_{22} - h_{21})$$

To show that $h_{11}(h_{22}-h_{21})$ is positive, then both h_{11} and $(h_{22}-h_{21})$ should be both positive or negative. The

expression for h_{11} is the following:

$$\left(d\pi_{\frac{S_1+S_2}{d}-1} (h_1 + p_2 - c) \right) + p_2 \sum_{\frac{S_1+S_2}{d}}^{\infty} d\pi_j$$

The relationship between h_1 , p_2 and c is already defined as the following: $h_1 + p_2 - c > 0$. This relationship

is stated in chapter III. Based on that, we can conclude that h_{11} is positive.

The expression for $(h_{22}-h_{21})$ is the following:

For condition 1 the expression is:

$$h_{22} - h_{21} = \left(h_2 \pi \frac{S_2}{d} d + p_2 \pi \frac{S_1+S_2-1}{d} d \right) - p_2 d \pi \frac{S_1+S_2-1}{d} = h_2 \pi \frac{S_2}{d} d$$

For condition 2 the expression is:

$$\begin{aligned} h_{22} - h_{21} &= \left(h_1 d \left(\pi \frac{S_1+S_2-1}{d} - \pi \frac{S_1+S_2-2}{d} \right) + h_2 \pi \frac{S_2}{d} d + c(-d) \left(\pi \frac{S_1+S_2-1}{d} - \pi \frac{S_1+S_2-2}{d} \right) \right) \\ &\quad - \left(h_1 d \left(\pi \frac{S_1+S_2-1}{d} - \pi \frac{S_1+S_2-2}{d} \right) + c(-d) \left(\pi \frac{S_1+S_2-1}{d} - \pi \frac{S_1+S_2-2}{d} \right) \right) \\ &= h_2 \pi \frac{S_2}{d} d \end{aligned}$$

Both conditions show that the expression for $(h_{22}-h_{11})$ has positive value. Since h_{11} is also positive then we can say that the determinant of $H(C)$ is positive. Therefore we can conclude that $C(S_1, S_2)$ is a convex function in S_1 and S_2 .

5.3 Optimal Base-stock Levels Solution

The optimal solution for above problem can be found by solving the expression for the first order difference equation. We can find the optimal base stock level for retailer 1 if the value of the base stock level for retailer 2 is fixed and vice versa. Let,

- $S_2^*(S_1=d)$ represents the optimal base stock level for retailer 2 when the value of the base stock level for retailer 1 is equal to d .
- $S_2^*(S_1=2d)$ represents the optimal base stock level for retailer 2 when the value of the base stock level for retailer 1 is equal to $2d$.
- $C(S_1=d, S_2^*)$ represents the cost when the base-stock level for retailer 1 is equal to d and base-stock level for retailer 2 is $S_2^*(S_1=d)$
- $C(S_1=2d, S_2^*)$ represents the cost when the base-stock level for retailer 1 is equal to $2d$ and base-stock level for retailer 2 is $S_2^*(S_1=2d)$

Since there are only 2 possibilities for the value of S_1 (either d or $2d$), we can find the optimal solution by executing the following steps:

1. Find the value for $S_2^*(S_1=d)$ and $S_2^*(S_1=2d)$
2. Calculate $C(S_1=d, S_2^*)$ and $C(S_1=2d, S_2^*)$
3. Compare $C(S_1=d, S_2^*)$ and $C(S_1=2d, S_2^*)$, if:
 - $C(S_1=d, S_2^*) < C(S_1=2d, S_2^*)$, then the optimal solution is $S_1=d$ and $S_2= S_2^*(S_1=d)$
 - $C(S_1=d, S_2^*) > C(S_1=2d, S_2^*)$, then the optimal solution is $S_1=2d$ and $S_2= S_2^*(S_1=2d)$

Step 2 and step 3 are quite straightforward. Hence we will only elaborate step 1, that is, how to find the value for $S_2^*(S_1=d)$ and $S_2^*(S_1=2d)$.

To find the value for $S_2^*(S_1=d)$ and $S_2^*(S_1=2d)$, we need to satisfy the following equality:

$$\Delta C_{S_2}(S_1, S_2) = 0$$

a. The value for $S_2^*(S_1=d)$

The first order difference with respect to S_2 if $S_1=d$ is:

$$\Delta C_{S_2}(S_1, S_2) = h_2 \left(\sum_{j=0}^{\frac{S_2-1}{d}} \pi_j d \right) - p_2 \left(\sum_{j=0}^{\frac{S_2}{d}} \pi_j(d) \right)$$

To find $S_2^*(S_1=d)$ we need to equate the above expression with zero.

$$h_2 \left(\sum_{j=0}^{\frac{S_2-1}{d}} \pi_j d \right) - p_2 \left(\sum_{j=0}^{\frac{S_2}{d}} \pi_j d \right) = 0$$

since $F(j) = \sum_0^j \pi_j$ and $\sum_{j=0}^{\infty} \pi_j = 1$, then,

$$dh_2 \left(F \left(\frac{S_2}{d} - 1 \right) \right) - dp_2 \left(1 - F \left(\frac{S_2}{d} - 1 \right) \right) = 0$$

$$h_2 F \left(\frac{S_2}{d} - 1 \right) - p_2 + p_2 F \left(\frac{S_2}{d} - 1 \right) = 0$$

$$F \left(\frac{S_2}{d} - 1 \right) = \frac{p_2}{h_2 + p_2}$$

$$\left(\frac{S_2}{d} - 1 \right) = F^{-1} \left(\frac{p_2}{h_2 + p_2} \right)$$

$$S_2^*(S_1 = d) = d \left(F^{-1} \left(\frac{p_2}{h_2 + p_2} \right) + 1 \right) \quad (5.21)$$

In another way, the optimal base stock level for retailer 2 can be written as follow:

$$S_2^*(S_1 = d) = d(k+1), \text{ where } k \text{ is the smallest integer such that } F(k) \geq \left(\frac{p_2}{h_2 + p_2} \right).$$

b. The value for $S_2^*(S_1=2d)$

The first order difference with respect to S_2 if $S_1=2d$ is:

$$\Delta C_{S_2}(S_1, S_2) = h_1 \pi \frac{S_2}{d} d + h_2 \left(\sum_{j=0}^{\frac{S_2-1}{d}} \pi_j d \right) - c \pi \frac{S_2}{d} d$$

To find $S_2^*(S_1=2d)$ we need to equate the above expression with zero.

$$dh_1 \pi \frac{S_2}{d} + dh_2 \left(F \left(\frac{S_2}{d} - 1 \right) \right) - dc \pi \frac{S_2}{d} = 0$$

$$(h_1 - c) \pi \frac{S_2}{d} + h_2 F \left(\frac{S_2}{d} - 1 \right) = 0$$

Since above expression is not a closed form expression, we changed the equality above to the inequality

below:

$$(h_1 - c) \pi \frac{S_2}{d} + h_2 F \left(\frac{S_2}{d} - 1 \right) \geq 0$$

$$F \left(\frac{S_2}{d} - 1 \right) \geq \frac{(c - h_1)}{h_2} \pi \frac{S_2}{d} \quad (5.22)$$

$S_2^*(S_1=2d)$ is the smallest S_2 to satisfy the inequality above (expression 5.22).

VI. NUMERICAL ANALYSIS

To conduct numerical analysis, the cost function that was developed is implemented as an Excel macro (using visual basic). The computation time for one set of experiment is around 1 minute. The results of the coding are validated by doing the following actions:

- We calculate the optimal cost and base-stock levels for several cases by hands and then compared them with the results provided from the computation tool. The results from both calculations are the same.
- We take several extreme cases such as the one where:
 - The value of α is extremely low and β is extremely high (e.g. the possibility of disruption is very small and the recovery probability is very high). In this case experiment shows that the behavior of the system is similar to the system where there are no disruptions and transshipments also are not conducted.
 - The penalty cost at retailer 2 is very high compared to other costs. As expected the experiment shows that transshipments are always conducted from retailer 1 to retailer 2.
 - The holding cost at retailer 1 or transshipment cost is very high compared to other costs. Again this experiment confirms our expectation, in those situations, no transshipments will be conducted.

After performing validation, we conduct numerical analysis in order to answer the following questions:

- a. What is the impact of each parameter on the optimal base-stock levels?
- b. What is the benefit of transshipments (compared to the situation where transshipments are not allowed)?
- c. How sensitive are the disruptions parameters (α and β) to the costs?

6.1 Base-case Experiment

We choose the following parameters as a base-case:

h1	p1	h2	p2	alpha	beta	c	d
5	10	5	10	0.5	0.5	5	3

Table 6 Parameters for the base-case problem

The values of the parameters are chosen as the base case due to the following reasons:

- Penalty cost is usually higher than holding cost
- They provide a wide range of possible changes for numerical analytical purposes.

The base-case experiment shows that the optimal base-stock levels are $S_1^*=6$ and $S_2^*=3$ and the cost for these set of base-stock levels is 15. We also calculate the optimal base-stock levels and its cost if we do not allow transshipment.

The expressions below shows the optimal base-stock levels for the system where no transshipments allowed:

- For retailer 1: $S_1^* = d$, it means that no cost will occur at retailer 1 since it always satisfies its demand every period.
- For retailer 2, the optimal base-stock level and its cost can be calculated as follows (Schmitt et al., 2010):

$$S_2^* = kd, \text{ where } k \text{ is the smallest integer such that } F(k-1) \geq \frac{p_2}{p_2+h_2}$$

And the cost for retailer 2 is:

$$E[C] = \sum_{j=0}^{\infty} \pi_j \left(h_2 [S_2 - (j+1)d]^+ + p_2 [(j+1)d - S_2]^+ \right)$$

The cost for the system is equal to the cost for retailer 2 since as explained before, no cost will occur at retailer 1. From the expressions above, the optimal base-stock levels for the base-case experiment is $S_1=3$ and $S_2=6$ and the cost is 22.5.

From this simple experiment we can see immediately the potential benefit of transshipments. The result shows that it is better to put extra inventory at retailer 1 so that the system can avoid penalty cost at retailer 2 by conducting transshipments when needed.

6.2 Varying Parameter values from the base-case

In order to see the impact of several parameters on the optimal base-stock level for retailer 1 and retailer 2 and on the optimal cost for the system as a whole, we conducted some experiments using the parameters from the base-case (Table 6) and in each experiment we vary the value of only one parameter (h_1 , h_2 , p_2 , c , α and β). We did not examine the various values of p_1 since for the optimal solution, the penalty cost for retailer 1 is always zero because retailer 1 is always keep inventory at least equal to its own demand (as discussed in chapter 3.4).

For each experiment, we show the set of the optimal base-stock levels and its cost for 2 policies: policy 1 is when we do not allow transshipments and policy 2 is when we allow transshipments from retailer 1 to retailer 2. We also show the cost components for the system that allow transshipments (policy 2) to investigate the contribution of each cost to the total cost. We show the results by presenting 2 types of graph: (1) graph about the contribution of each cost component for policy 2 and (2) graph about the comparison between the costs from policy 1 and policy 2.

For the graph about the contribution of each cost component for policy 2, the cost components are abbreviated as follows:

- Inventories cost at retailer 1: Inv-R1
- Inventories cost at retailer 2: Inv-R2
- Backorders cost at retailer 2: BO-R2
- Transshipments cost: Transh

For the graphs that show the comparison between the cost for 2 policies, the following abbreviations are used:

- No Transh: cost for policy 1 (when we do not allow transshipments).
- With Transh: cost for policy 2 (when we allow transshipments).
- % difference: the % of cost difference between policy 1 and policy 2.

The primary y-axis in this type of graph refers to “No Transh” and “With Transh” while the secondary y-axis refers to “% difference”.

6.2.1 Varying h_1

In this experiment, we vary the value of h_1 from 1 to 15. We expect that for policy 2, low h_1 values are more beneficial for transshipments, while high h_1 may cause high inventory cost at retailer 1 due to extra inventories for transshipments.

The results of this experiment are presented below, started with the set of the optimal base stock levels (S_1^* , S_2^*) then followed by the graph showing the cost components for policy 2.

	h1 (holding cost at retailer 1)														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
No Transshipments	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)
With Transshipments	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)

Table 7 the set of optimal base-stock level when varying h_1

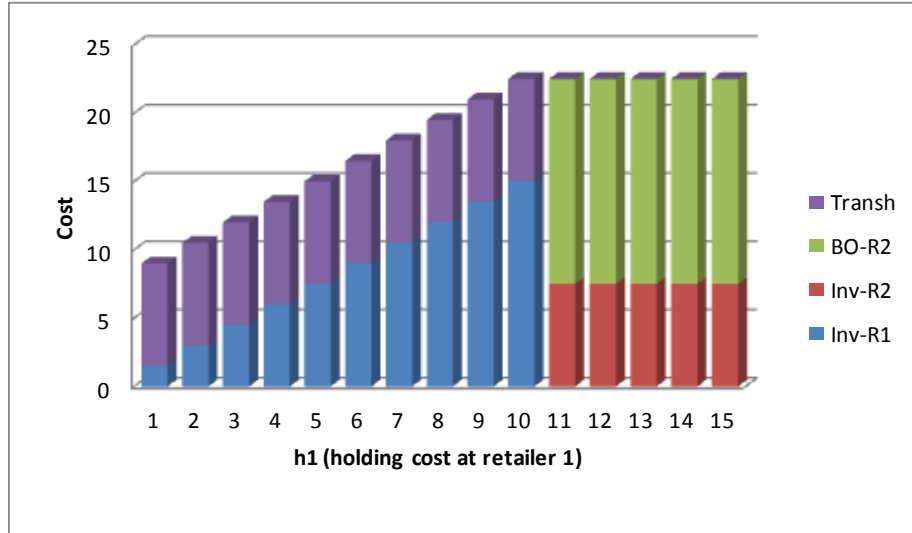


Figure 4 the contribution of each cost component when varying h_1 (cost with transshipments)

Figure 4 shows that in the system where transshipments are allowed, when h_1 is low, the cost components that are included are only holding cost at retailer 1 and transshipments cost since the policy tends to put more inventories at retailer 1. In this case, the base-stock level of retailer 1 equals to $2d$, hence retailer 2 will never experience backorders since transshipments will be conducted to avoid this. When h_1 is high, it is better to save more inventories at retailer 2 to avoid a high inventory cost at retailer 1 even there is a risk that retailer 2 may face a penalty if the disruptions holds on for a long periods of time. When this happens, the optimal base-stock levels solution and its cost will be the same as under the policy when no transshipments are allowed as can be seen in table 7 and figure 5. Figure 5 also shows that the smaller the h_1 value, the greater the benefit. These results support our expectation. From this experiment we can conclude that the system will benefit from transshipments when the holding cost at retailer 1 is low.

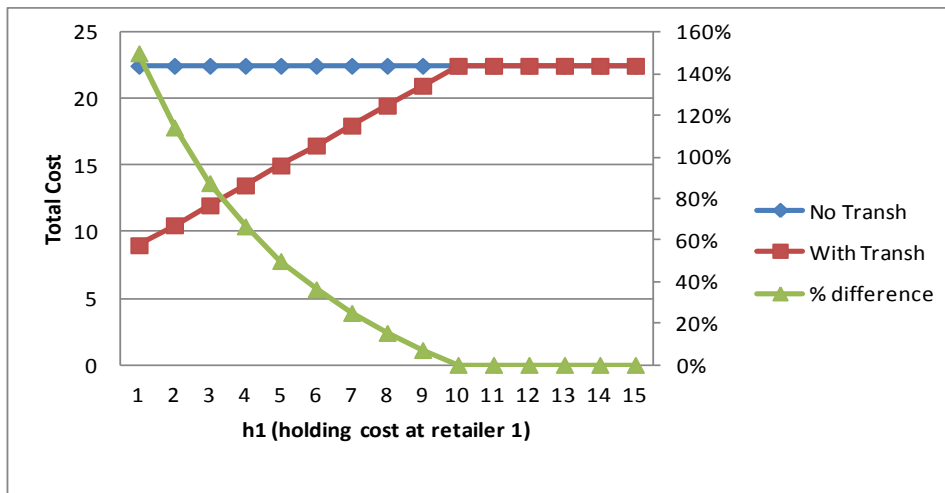


Figure 5 Comparison between the costs of 2 policies when varying h_1

6.2.2 Varying h_2

In this experiment, we vary the value of h_2 from 1 to 10. In the system where transshipments are allowed, if h_2 is high, it is expected that the system prefers to store more inventories at retailer 1 and conducts transshipments when needed to avoid high holding cost at retailer 2.

The set of the optimal base stock levels (S_1^* , S_2^*) from this experiment can be seen in the table below:

	h2 holding cost at retailer 2)									
	1	2	3	4	5	6	7	8	9	10
No Transshipments	(3,12)	(3,9)	(3,9)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,3)
With Transshipments	(3,12)	(3,9)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)

Table 8 the set of optimal base-stock level when varying h_2

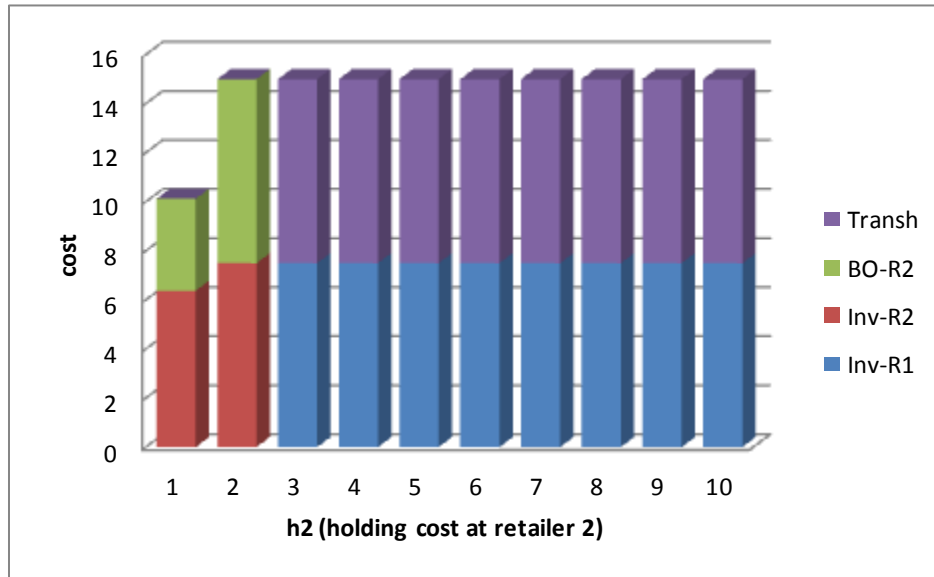


Figure 6 the contribution of each cost component when varying h_2 (cost with transshipments)

In the system where transshipments are allowed, when h_2 is low, the cost components that contribute to the total cost are only holding cost and backorders cost at retailer 2. In order to decrease or avoid backorders cost at retailer 2, it is cheaper to put more inventories at retailer 2 than to put more inventories at retailer 1 and generate extra cost for transshipments. Thus, the set of optimal base-stock levels and its cost are the same as the system that does not allow transshipments. When h_2 is getting higher, it is beneficial to do transshipment since the system will avoid high inventory cost at retailer 2 so the extra inventories will be kept at retailer 1. This phenomenon is shown in figure 7 below. The figure also shows that the bigger h_2 value, the cost difference between policy 2 and policy 1 also getting bigger.

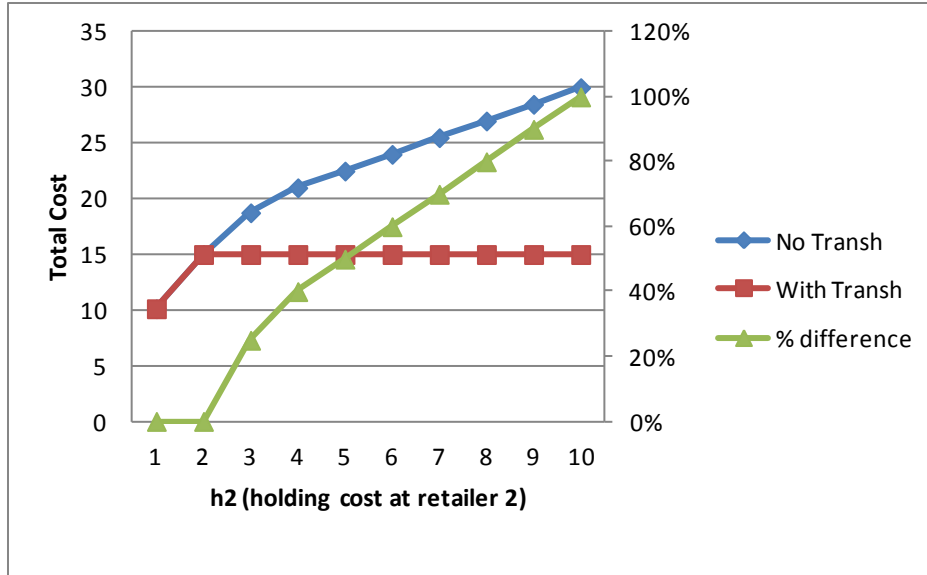


Figure 7 Comparison between the costs of 2 policies when varying h2

6.2.3 Varying p_2

In this experiment, we vary the value of p_2 from 1 to 15. We expect that the higher the penalty cost at retailer 2, the higher the possibility of conducting transshipments so that the penalty cost can be completely avoided.

The set of the optimal base stock levels (S_1^* , S_2^*) from this experiment is shown in the table below:

	p2 (penalty cost at retailer 2)														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
No Transshipments	(3,3)	(3,3)	(3,3)	(3,3)	(3,3)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)
With Transshipments	(3,3)	(3,3)	(3,3)	(3,3)	(3,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)

Table 9 the set of optimal base-stock level when varying p2

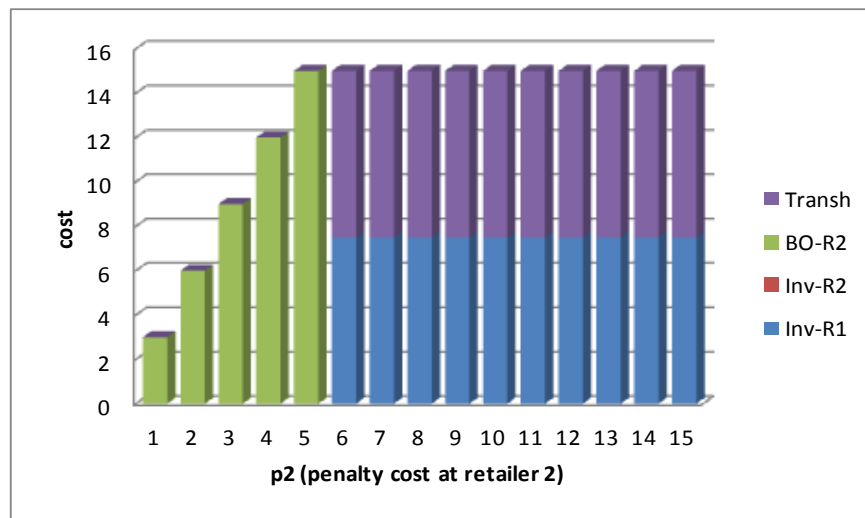


Figure 8 the contribution of each cost component when varying p2 (cost with transshipments)

In the system where transshipments are allowed, when p_2 is low, backorders cost at retailer 2 is the only one that composes the total cost. In this case, the system chooses to take the backorders risk since the penalty cost is low. By doing this, the system does not have to invest on other costs. As the penalty cost at retailer 2 increases, the system chooses to put extra inventories at retailer 1. Keeping more inventories at retailer 1 ($S_1 = 2d$) is a better solution to avoid backorders at retailer 2 rather than keeping more inventories at retailer 2 since it can completely avoid the backorders cost. This can happen because every time retailer 2 experience disruptions, retailer 1 can transship d amount of its products to retailer 2. Therefore, in the end of the period retailer 2 can always satisfies its demand. If we put the extra inventories at retailer 2 instead of retailer 1, in addition to causing high inventory cost at retailer 2, the risk of having backorders cost is still exists if the disruptions last for a long time period. The results from this experiment confirm our expectation. From figure 9 we can see that the cost-gap between the 2 policies (no transshipments and with transshipments) gets bigger as p_2 increases.

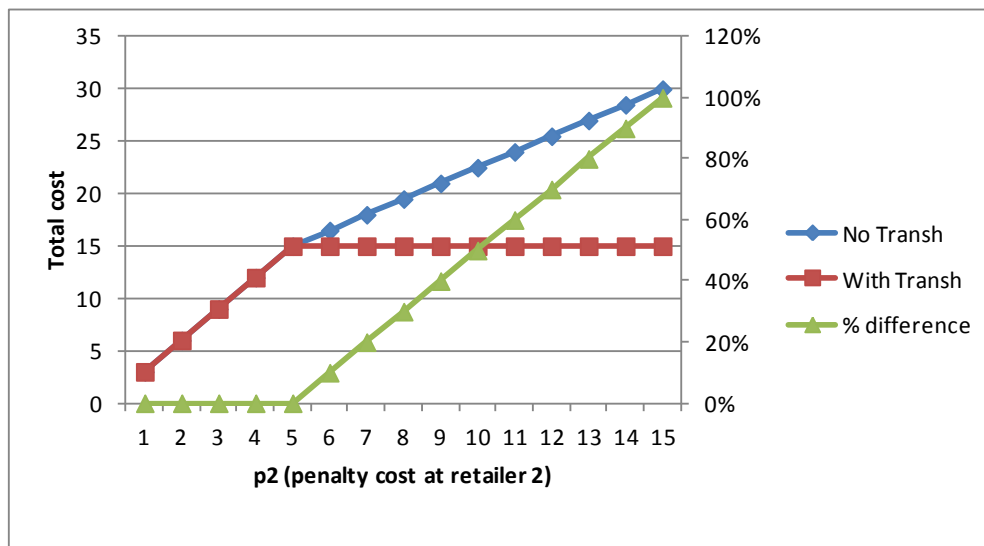


Figure 9 Comparison between the costs of 2 policies when varying p_2

6.2.4 Varying c

In this experiment, we vary the value of c from 1 to 15. We expect that the higher the transshipments cost, the lower the usefulness of conducting transshipments.

The set of optimal base stock levels (S_1^* , S_2^*) from this experiment can be seen in the table below:

	c (transshipment cost)														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
No Transshipments	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)
With Transshipments	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)

Table 10 the set of optimal base-stock level when varying c

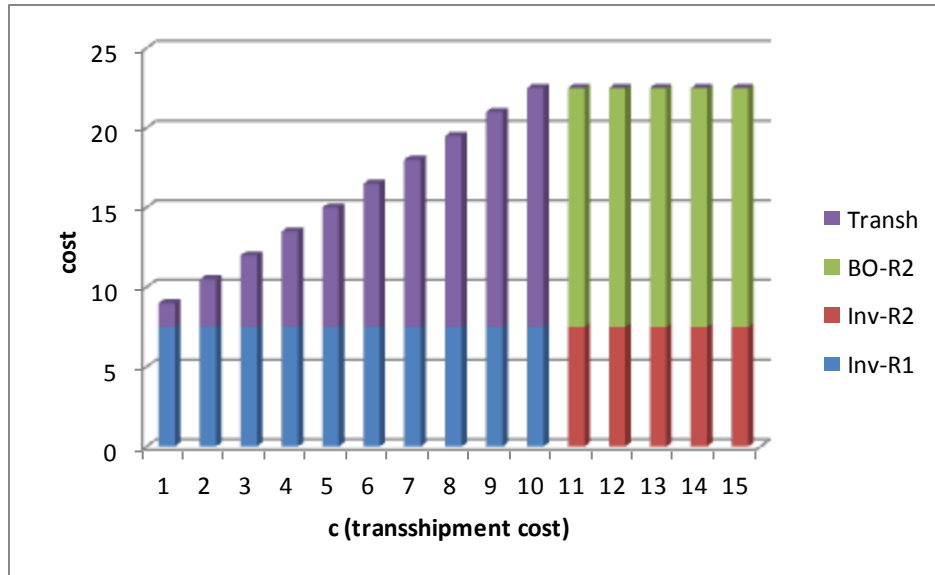


Figure 10 the contribution of each cost component when varying c (cost with transshipments)

The benefit of transshipment can be clearly seen when the transshipments cost is low. In this case, the system puts more inventories at retailer 1 and conducts transshipments when needed so the backorders at retailer 2 can be completely avoided. As c is getting higher, it is not beneficial to do transshipments and the system will put extra inventories at retailer 2 to reduce the backorder costs. In these situations the optimal base-stock levels and the cost from the 2 cases (no transshipments and with transshipments) is the same as can be seen in table 10 and figure 11. The results from this experiment justify our expectations.

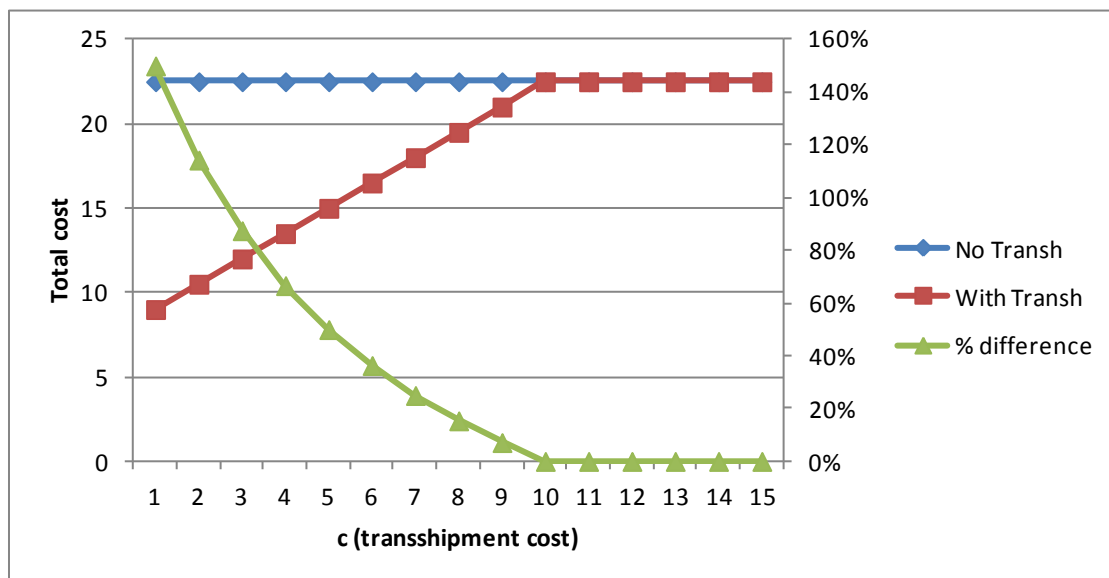


Figure 11 Comparison between the costs of 2 policies when varying c

6.2.5 Varying α

In this experiment, we vary the value of α from 0.1 to 0.9. The set of optimal base stock levels (S_1^* , S_2^*) is shown in the table below:

	alpha (Disruptions probability)								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
No Transshipments	(3,3)	(3,3)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)
With Transshipments	(3,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)

Table 11 the set of optimal base-stock level when varying alpha

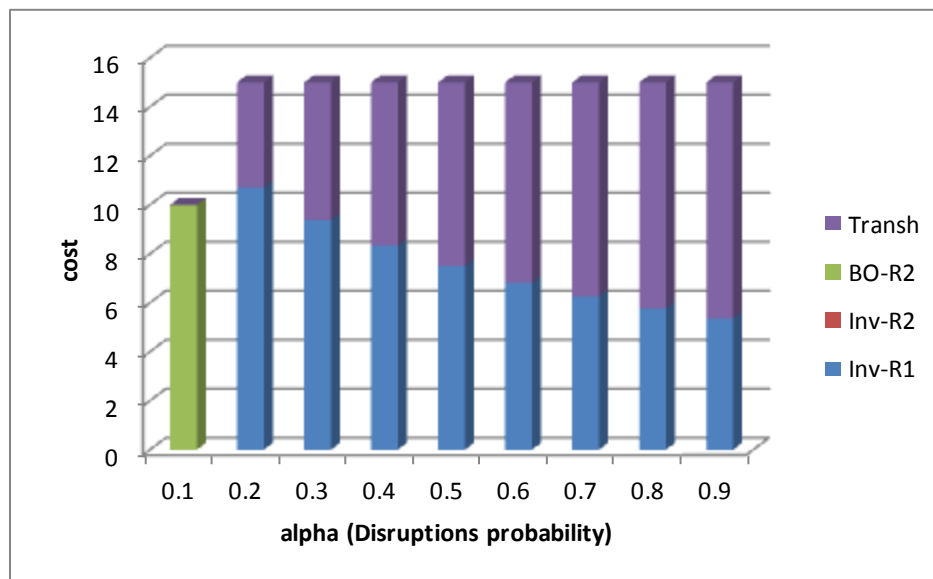


Figure 12 the contribution of each cost component when varying alpha (cost with transshipments)

In the system where transshipments are allowed, when α is low, the system chooses to take the risk of having backorders since it will not occur often. By doing this, each retailer only needs to keep inventories equal to d and in the same time transshipment activities cannot occur. As the value of α gets higher, the system prefer to keep more inventories at retailer 1 in order to allow transshipments to avoid backorders at retailer 2. The higher the value of α , the proportion of transshipments cost from the total cost is bigger and the proportion of inventory cost at retailer 1 is lower since transshipments occur more often. The benefit of transshipments is vivid when α is high, in this case, allowing transshipment can reduce the total cost because it gets rid of the backorders cost at retailer 2. When we do not allow transshipments, the backorders and inventory costs at retailer 2 still exist. Hence for high α , it is better to allow transshipments as shown in figure 13.

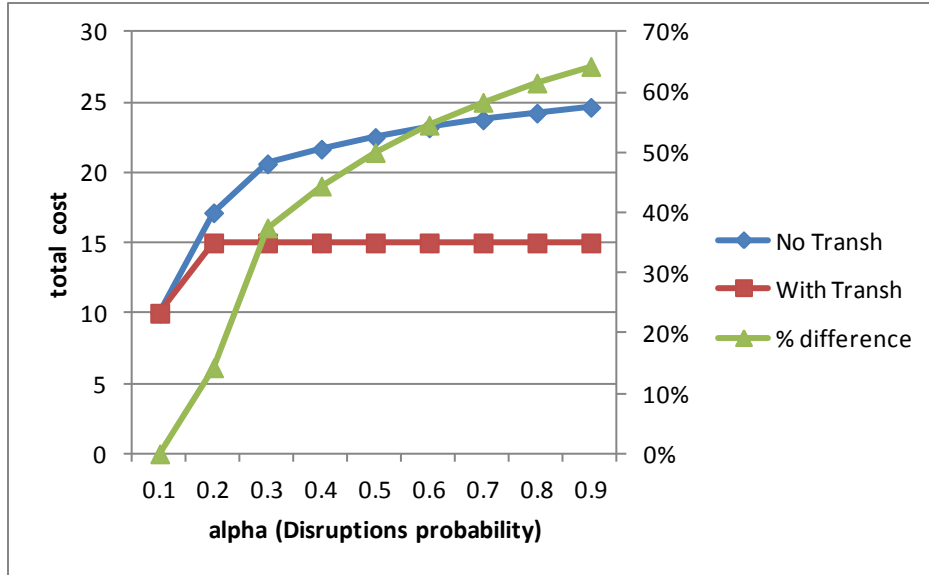


Figure 13 Comparison between the costs of 2 policies when varying alpha

6.2.6 Varying β

In this experiment, we vary the value of β from 0.1 to 0.9. The set of optimal base stock levels (S_1^* , S_2^*) is shown in the table below:

	beta (Recovery probability)									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	
No Transshipments	(3,30)	(3,15)	(3,9)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)
With Transshipments	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(3,6)	(3,6)	(3,6)	(3,6)

Table 12 the set of optimal base-stock level when varying beta

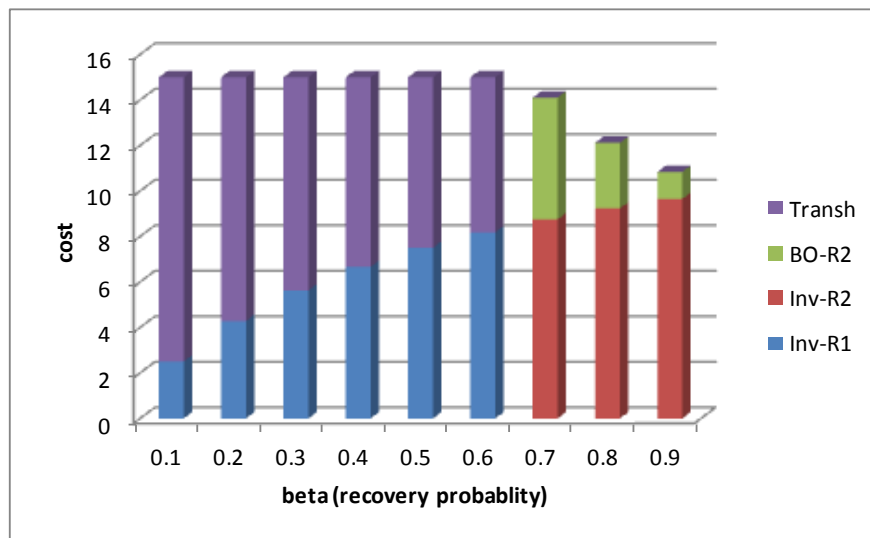


Figure 14 the contribution of each cost component when varying beta (cost with transshipments)

Figure 14 shows that where the value of β is high, the system is forced to take the risk of backorders at retailer 2 in order to reduce or even get rid of other costs (inventory and transshipments cost). When β is getting smaller, the system prefer to put extra inventories at retailer 1 ($S_1=2d$) so that it can give its products to retailer 2 whenever retailer 2 experiences backorders. The benefit of transshipment for these situations (when β is small) is shown in figure 15. When we do not allow transshipment, the system is forced to keep a lot of inventories to reduce the backorders cost at retailer 2. In contrast, for a system that allows transshipment, to tackle this problem (the low recovery probability) we only need to put extra inventories at retailer 1 and by doing this the backorders risk at retailer 2 can be completely avoided. Figure 15 also shows the significant different between policy 1 and policy 2 when β is small, for example when $\beta = 0.1$, the cost difference reaches 935%. This suggests that transshipments is very crucial when β is small.

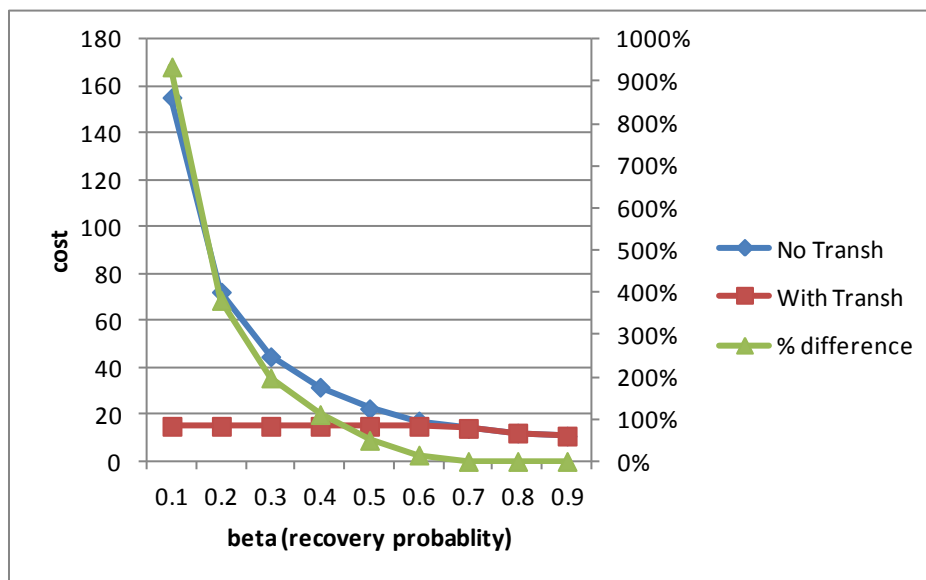


Figure 15 Comparison between the costs of 2 policies when varying beta

6.2.7 Varying cost parameters in different situations

All the experiments above that are related to cost parameters (experiment 6.2.1 until 6.2.4) were done under the condition where $\alpha = \beta = 0.5$. We try the same experiments (varying each cost parameter) with 2 different situations: (1) $\alpha > \beta$ and (2) $\alpha < \beta$. The results from those 2 new experiments are in general generate the same insights as the experiment where $\alpha = \beta$.

6.3 Investigate the sensitivity for α and β

6.3.1 All variations of α and β from the base-case cost parameters for 2 policies

For this experiment, we use the demand and cost parameters that we used earlier in the base case ($d=3$, $h_1=h_2=5$, $p_1=p_2=10$ and $c=5$). Unlike all previous experiments, in this experiment we try all possible values of α and β in order to get more insights on the impact of these parameters to the cost from 2 different policies (policy 1: without transshipments and policy 2: with transshipments). For each policy, we find the set of optimal base-stock level and its related cost for every pair of α and β as shown in the tables below.

		alpha								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
beta	0.1	(3,15)	(3,24)	(3,27)	(3,30)	(3,30)	(3,30)	(3,33)	(3,33)	(3,33)
	0.2	(3,3)	(3,9)	(3,12)	(3,15)	(3,15)	(3,15)	(3,15)	(3,15)	(3,18)
	0.3	(3,3)	(3,6)	(3,9)	(3,9)	(3,9)	(3,9)	(3,12)	(3,12)	(3,12)
	0.4	(3,3)	(3,3)	(3,6)	(3,6)	(3,6)	(3,9)	(3,9)	(3,9)	(3,9)
	0.5	(3,3)	(3,3)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)
	0.6	(3,3)	(3,3)	(3,3)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)
	0.7	(3,3)	(3,3)	(3,3)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)
	0.8	(3,3)	(3,3)	(3,3)	(3,3)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)
	0.9	(3,3)	(3,3)	(3,3)	(3,3)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)

Table 13 The set of optimal base-stock level for policy 1

		alpha								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
beta	0.1	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)
	0.2	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)
	0.3	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)
	0.4	(3,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)
	0.5	(3,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)
	0.6	(3,3)	(3,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)	(6,3)
	0.7	(3,3)	(3,3)	(3,3)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)
	0.8	(3,3)	(3,3)	(3,3)	(3,3)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)
	0.9	(3,3)	(3,3)	(3,3)	(3,3)	(3,6)	(3,6)	(3,6)	(3,6)	(3,6)

Table 14 The set of optimal base-stock level for policy 2

From the tables above we can see that for all values of α , transshipments are beneficial only when β is low (0.1 - 0.3). For medium β values (0.4 - 0.6), the decision whether it is better to use transshipment or not depends on the value of α . As expected, transshipments are mostly not needed when α is very low. For all high β values (0.7-0.9), retailer 1 only keep inventory equal to its own demand and retailer 2

stores more inventories to avoid backorders if needed (when α is high). In these cases (where β is high), the optimal solution is the same between policy 1 and policy 2.

		alpha								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
beta	0.1	132.6	148.5	152.8	154.5	155.3	155.9	156.0	156.1	156.2
	0.2	50.0	64.5	69.1	71.4	72.3	72.9	73.3	73.7	74.0
	0.3	25.0	37.0	41.8	43.4	44.7	45.7	46.0	46.1	46.1
	0.4	15.0	25.0	27.9	30.0	31.7	31.8	31.9	32.0	32.1
	0.5	10.0	17.1	20.6	21.7	22.5	23.2	23.8	24.2	24.6
	0.6	7.1	12.5	16.7	17.0	17.3	17.5	17.7	17.9	18.0
	0.7	5.4	9.5	12.9	14.2	14.1	14.0	13.9	13.9	13.8
	0.8	4.2	7.5	10.2	12.5	12.1	11.8	11.5	11.3	11.0
	0.9	3.3	6.1	8.3	10.3	10.8	10.3	9.9	9.5	9.2

Table 15 The optimal cost for policy 1

		alpha								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
beta	0.1	15	15	15	15	15	15	15	15	15
	0.2	15	15	15	15	15	15	15	15	15
	0.3	15	15	15	15	15	15	15	15	15
	0.4	15	15	15	15	15	15	15	15	15
	0.5	10	15	15	15	15	15	15	15	15
	0.6	7.1	12.5	15	15	15	15	15	15	15
	0.7	5.4	9.5	12.9	14.2	14.1	14.0	13.9	13.9	13.8
	0.8	4.2	7.5	10.2	12.5	12.1	11.8	11.5	11.3	11.0
	0.9	3.3	6.1	8.3	10.3	10.8	10.3	9.9	9.5	9.2

Table 16 The optimal cost for policy 2

It is shown from 2 tables above that for some cases, the costs of policy 2 are significantly lower than the cost of policy 1. Seeing this fact, we examine the benefit of transshipments by calculating the percentage of cost reduction that is generated if we compare policy 2 to policy 1 for all pairs of α and β . The results are shown in the figures below.

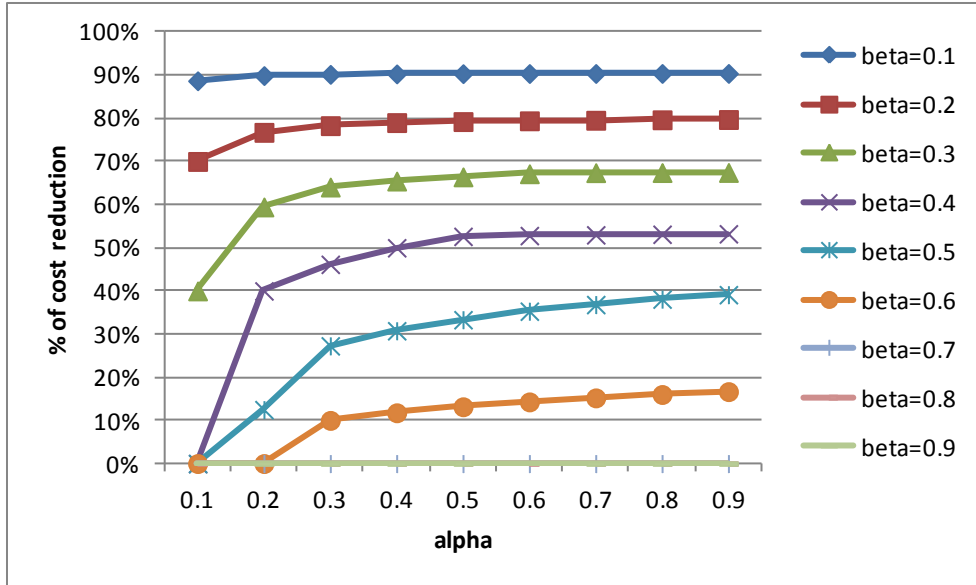


Figure 16 Percentage of cost reduction in alpha for various values of beta

Figure 16 shows that for all values of α , when β is lower, the advantage of having transshipments is higher. When the value of β is high (in this case from 0.7 until 0.9) it is better to keep more inventories at retailer 2 to reduce or avoid backorders rather than transship some products from retailer 1, therefore in this case transshipments is not needed. In general, when the value of β is constant, changes in α do not cause a significant difference in cost reduction from policy 1 to policy 2. This result implies that β gives more impact to the cost reduction rather than α . Figure 17 shows similar insights.

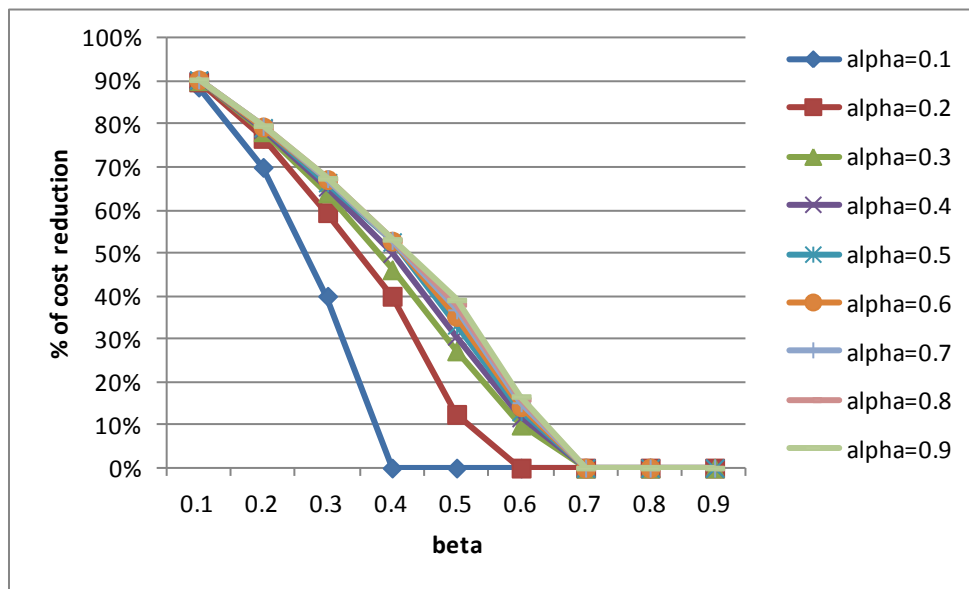


Figure 17 Percentage of cost reduction in beta for various values of alpha

From the figure above we can see that all α values except for the 2 lowest α ($\alpha=0.1$ and $\alpha=0.2$) show a similar pattern when it paired with different value of β . It means that for this range of α , the changes in β gives similar cost reduction.

6.3.2 α and β changes from various cases for policy 2

Experiments in section 6.3.1 above are done based on the base-case. We are interested in trying similar experiments with different cases since a company may have different set of parameter values. Another objective from this experiment is to gain insight for companies if they want to decrease their cost by dealing with the parameters that are related to disruptions (α and β). Companies can choose to decrease the disruptions probability or to increase recovery probability or even to do both in the same time. For above purposes, we investigate 6 cases as listed in the table below, all cases have $d=3$, $p_1=p_2=15$, $\alpha=0.5$ and $\beta=0.5$. As for h_1 , h_2 and c , we vary the value of those parameters. The reason why we choose those parameters to be varied is because the following reasons:

- From previous experiments, in cases where transshipments are conducted, the cost changes mostly involve inventory cost at retailer 1 and transshipments costs.
- Previously the holding cost and the penalty cost between retailer 1 and retailer 2 are always the same. Different holding cost between those retailers may yield different insights.
- Penalty costs are expected to be always higher than holding costs. Therefore we do not vary penalty cost.

Case	h_1	h_2	c	Remarks
1	5	5	5	$h_1 = h_2, h_1 = c$
2	5	5	10	$h_1 = h_2, h_1 < c$
3	5	10	5	$h_1 < h_2, h_1 = c$
4	5	10	10	$h_1 < h_2, h_1 < c$
5	10	5	5	$h_1 > h_2, h_1 = c$
6	10	5	10	$h_1 > h_2, h_1 < c$

Table 17 Different cases for investigating the sensitivity of disruptions parameters

From every case, we want to show the impact of either increasing recovery probability (β) or decreasing disruptions probability (α) on the optimal cost. We are interested to know whether one of the disruptions parameters is more sensitive compared to other and whether the behavior is the same for every case.

To see the sensitivity of the disruptions parameters, for each case, we change either α or β value. First with constant β (0.5), we decrease α by 20%, 40%, 60% and 80%. After that with constant α (0.5), we increased β by 20%, 40%, 60% and 80%. The result of these experiments is presented in 3 figures below. Table 18 shows the legend of the figures.

Case	Explanation	Case	Explanation
1a	Case 1: decrease α	1b	Case 1: increase β
2a	Case 2: decrease α	2b	Case 2: increase β
3a	Case 3: decrease α	3b	Case 3: increase β
4a	Case 4: decrease α	4b	Case 4: increase β
5a	Case 5: decrease α	5b	Case 5: increase β
6a	Case 6: decrease α	6b	Case 6: increase β

Table 18 Legend for figure 18, 19 and 20

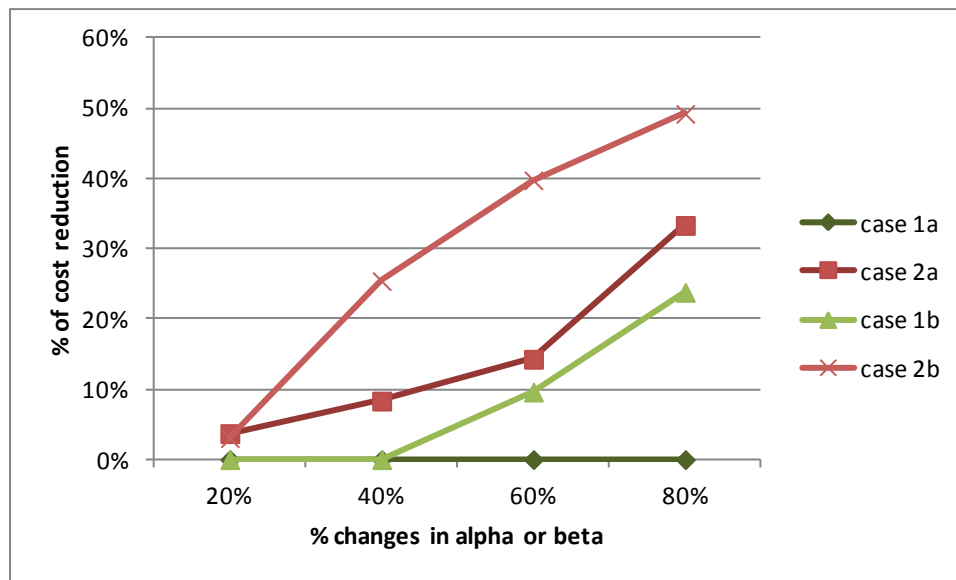


Figure 18 Percentage of cost reduction for case 1 and case 2

When the holding cost at retailer 1 and retailer 2 is the same, the changes in β gives more significant impact (bigger reduction cost) to changes in α . For the case where $h_1=c$ (case 1), the changes in α does not give any impact. The reason why there is no impact in case 1a, is because all the optimal solutions in this case have $S_1=6$ and $S_2=3$. It means that the costs are only inventory cost at retailer 1 and transshipments cost. For these cases, the higher the value of α , the lower the inventory cost at retailer 1 and the higher the transshipments cost. The cost reduction gained from the inventory cost at retailer 1 is equal with the increase of transshipment cost (since $h_1=c$), hence the total cost does not change. Figure 18 also shows that when c is higher, the cost reduction is also higher.

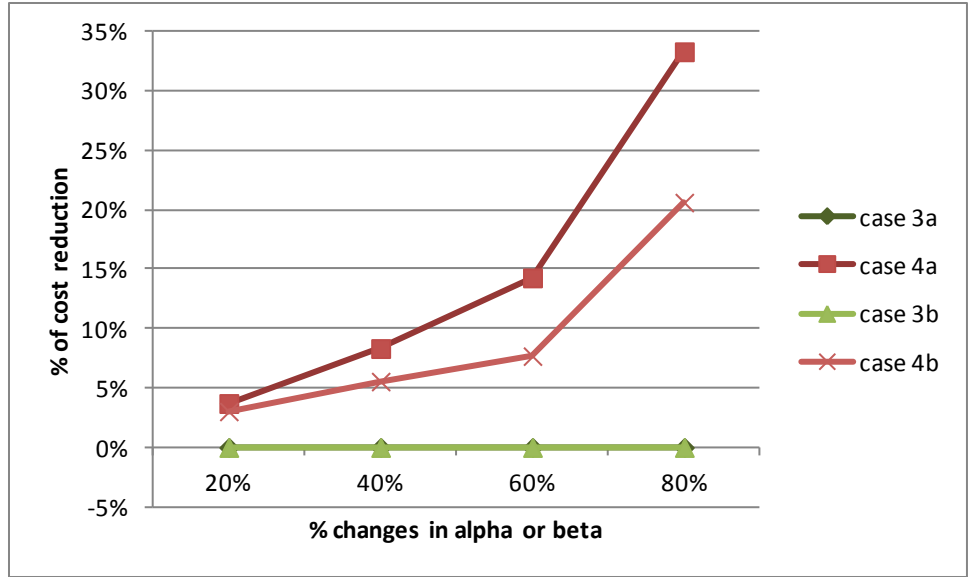


Figure 19 Percentage of cost reduction for case 3 and case 4

When the holding cost at retailer 1 is lower than retailer 2 and $h_1=c$, changing either the value of α or β does not reduce the total cost with the same reason as the one we just discussed above. For a high transshipments cost (case 4), the changes in α gives a slightly larger reduction cost rather than increasing β . For both cases (increasing β or decreasing α), mostly the system chooses to put more inventories at retailer 1 and conduct transshipments since $h_1 < h_2$.

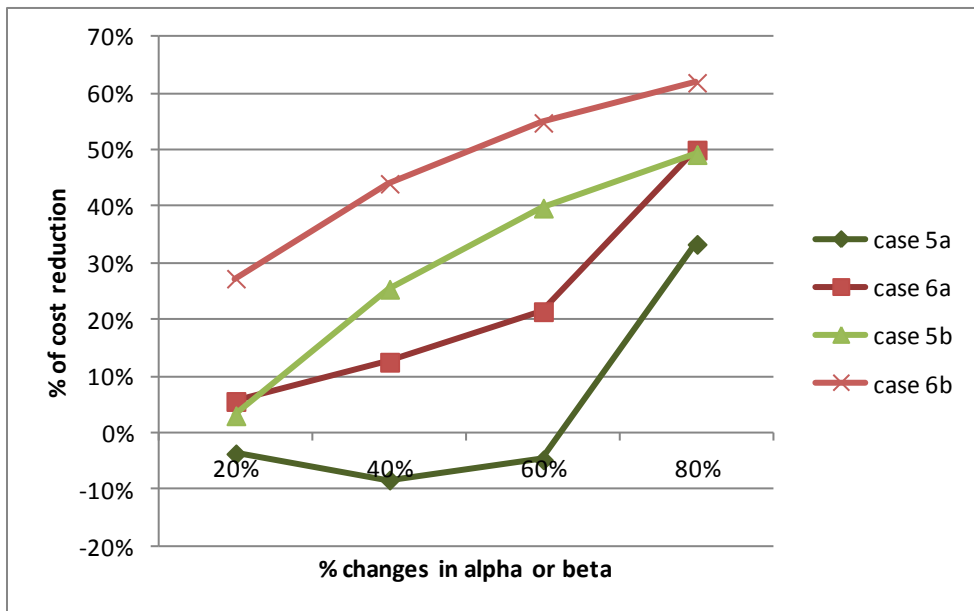


Figure 20 Percentage of cost reduction for case 5 and case 6

When the holding cost at retailer 1 is higher than retailer 2, increasing β gives bigger reduction cost than decreasing α in both cases (case 5 and case 6). The higher the transshipments cost, the higher the cost

reduces. For case 5a, at some points when we decrease the value of α (at 20%, 40% and 60%), the cost gets higher instead of lower. This can happen because when α gets lower, in the system where h_1 is quite higher than h_2 and c , the increase in inventory cost at retailer 1 is higher than the decrease in transshipments cost. When α is decreased by 80%, the system choose to take the penalty risk at retailer 2 so there is no transshipment and inventory cost at retailer 1.

6.4 Conclusion from the experiments

From several experiments above, we can infer the following points:

a. From experiment in section 6.2 (varying parameter values)

The values of each parameter have different consequences to the optimal base-stock level and its cost. In short, we can put the general conclusion of the experiments about varying parameters in the table below: (note that the determination whether a parameter is low or high is relative, it depends on the value of other parameters in the system)

Parameter		Transshipments		S1*	S2*
		Yes	No		
h_1	low	✓		2d	$S2^*(S1=2d)$
	high		✓	d	$S2^*(S1=d)$
h_2	low		✓	d	$S2^*(S1=d)$
	high	✓		2d	$S2^*(S1=2d)$
p_2	low		✓	d	$S2^*(S1=d)$
	high	✓		2d	$S2^*(S1=2d)$
c	low	✓		2d	$S2^*(S1=2d)$
	high		✓	d	$S2^*(S1=d)$
α	low		✓	d	$S2^*(S1=d)$
	high	✓		2d	$S2^*(S1=2d)$
β	low	✓		2d	$S2^*(S1=2d)$
	high		✓	d	$S2^*(S1=d)$

Table 19 the impact of various parameters to the optimal base-stock levels

From table 19, we can see the different impact of each parameter to the optimal base-stock levels. The transshipments activity is beneficial in the system that has low h_1 , high h_2 , high p_2 , low c , high α and low β . In these cases it is better to put more inventories at retailer 1 instead of retailer 2 in order to minimize the cost. If we put the extra inventories at retailer 1 ($S_1=2d$), every time retailer 2 experiences disruption, retailer 1 can always provide products to be transshipped to retailer 2 so the penalty cost can be completely avoided and in the same time we also reduce the inventory cost at retailer 2. On the other hand, if we put extra inventories at retailer 2, the inventory cost at retailer 2 will be high and we still have the risk of experiencing backorders if the disruptions stay for a long periods of time.

If we compare the 2 policies, in general the policy that allow transshipments performs better, the highest difference in cost between policy 1 and policy 2 are given when β is low, in this situation the benefit of transshipments is very obvious. In this case, transshipments activity can completely cut the inventory and penalty cost at retailer 2. In the system that does not allow transshipments, these 2 cost components (the inventory and penalty cost at retailer 2) are very high.

b. From experiment in section 6.3.1 (variations of α and β from the base-case)

From table 19 we can see what companies can do if they have low or high disruptions parameters (α and β). Experiment 6.3.1 confirms this conclusion by trying all possible pair of disruptions parameters and provides a more detail insights. When the value of β is high (0.7 - 0.9), then companies do not have to do transshipments but when the value of β is low (0.1 - 0.3), transshipments are needed. Additional insight that can be taken from this experiment is that for medium β values (0.4-0.6), the decision to do transshipments depends on α value. For these cases, transshipments most likely will be beneficial for medium to high α values.

Different with β , α does not give a certain pattern. But we can infer that when α is low, the system tends not to do transshipments except in the situation where β is very low. When α is high, the system tends to do transshipments except in the situation where β is very high.

From this experiment we also can see that in general, changes in β value provides more significant cost reduction from policy 1 to policy 2 compared to changes in α value. Hence β value is more influential in determining better policy.

c. From experiment in section 6.3.2 (α and β from various cases)

From this experiment, it is shown that in the policy where transshipments are allowed (policy 2), for the situation where $h_1 \geq h_2$, increasing recovery probability (β) gives bigger reduction costs compared to decreasing disruptions probability (α) with similar level of change (for example increasing β by 20% vs. decreasing α by 20%). In those cases, when the value of c is higher, the cost reduction is also higher. For the situation where $h_1 < h_2$, the changes in α gives a slightly larger reduction cost rather than increasing β except for some situations. The example of the situation is that when $h_1=c$, the changes either in α or β do not reduce the total cost because the holding cost at retailer 1 and transshipments cost are changing in the same amount, hence the total cost stays the same.

Another thing to be noted is that increasing β or decreasing α does not always generate lower cost. For example if we decrease disruptions probability, most probably the optimal solution will lead to the increase in inventory cost at retailer 1 and the decrease in transshipments cost. In this case the

total cost will be higher if the increase in inventory cost is higher than the decrease in transshipments cost. Therefore, companies need to consider and look carefully at the value of each cost parameters when they want to minimize the cost by managing disruptions parameters.

From various set of experiments that investigate the sensitivity of α and β , in general it is shown that the changes in β gives bigger reduction cost to the changes in α . This result implies that β is more crucial rather than α . One of possible reason for this is because as long as β is high, even the disruptions probability is also high, retailer 2 can always recover quickly every time it is disrupted and thus can avoid a high level of backorders. In the opposite situation, if the recovery probability is low, even the disruptions probability is also low, once retailer 2 is disrupted, the backorders level will quickly increase if retailer 2 does not get any product from retailer 1. In that sense, focusing on increasing the recovery probability gives more cost benefit rather than decreasing disruptions probability.

VII. CONCLUSION AND FUTURE RESEARCH

7.1 Conclusion

In this thesis, we investigated an inventory system with one supplier and 2 retailers where only one of the retailers may experience supply process disruptions. The supply process at another retailer is always available. Every period, both retailers face a constant demand. Transshipments can be conducted from the undisrupted retailer to the disrupted retailer in order to avoid backorders.

We derived a formula to calculate the expected cost per period for the system and showed that the cost function is convex in the base-stock level of both retailers. From those results, we built expressions to calculate the optimal base-stock levels for both retailers.

Several numerical experiments were conducted to get insights into the impact of each parameter on optimal base-stock levels. We found that each parameter value has a different influence on the optimal base-stock levels. Transshipments are beneficial in cases where an inventory system has low holding cost at undisrupted retailer, high holding cost at disrupted retailer, high penalty cost at disrupted retailer, low transshipments cost, high disruptions probability and low recovery probability. In these cases, having more inventories at the undisrupted retailer leads to lower cost as the backorders cost can be completely avoided. Compared to systems that do not allow transshipments, the benefit of transshipments is very obvious when the recovery probability is very low because transshipments completely cut the inventory and penalty cost at disrupted retailer.

We found some interesting results from the sensitivity analysis of the disruptions parameters:

- In general, increasing recovery probability provides more significant cost reduction compared to decreasing disruptions probability. Hence increasing recovery probability is more influential in decreasing the total cost.
- Increasing recovery probability or decreasing disruptions probability does not always lead to a lower cost, this action also can lead to the same cost or even higher cost. For example, when we decrease α value, if the optimal solution generates transshipment decision, then the lower the value of α , the higher the inventory cost at retailer 1 and the lower the transshipments cost. In the situation where the holding cost at retailer 1 is the same as the transshipments cost, the amount of cost reduction gained from the transshipment cost is equal with the increase of inventory cost at retailer 1, hence the total cost does not change. While in the situation where the holding cost at retailer 1 is higher than the transshipments cost, the amount of cost reduction gained from the transshipment cost is lower with the increase of inventory cost at retailer 1. Therefore the total cost is increased. In this sense, companies still need to look carefully at the value of each cost parameters when they want to

decrease the cost by managing disruptions parameters since there are some cases where changing disruptions parameter does not change the cost or even make the cost higher.

To conclude, the results in this thesis show that transshipments can reduce the cost of supply disruptions in the system. Through a well-planned base-stock levels value and transshipments decision, the risk of supply disruptions can be mitigated.

7.2 Future Research

The research presented here can be extended by relaxing some assumptions to better represent the real-life cases. The possible changes in the assumption are:

- a. Assume that both retailers face supply disruptions. The disruptions events are sometimes unexpected in terms of time and place, therefore assuming that both retailers may face disruptions risk are more relevant. In this case, we deal with a more complex DTMC. The challenge of this problem is to find the cost function for the system.
- b. Incorporate a positive lead time for supply process from the supplier or transshipments. For big companies like Wal-Mart, their distribution centers may be located in different areas so that conducting transshipments will need a significant amount of time that may delay their intention to satisfy their customers.
- c. Assume a stochastic demand for each retailer.
- d. Redefine the probability in DTMC. The DTMC presented here (figure 2) assumes that for $j \geq 1$, the probability that retailer 2 is disrupted again (the system moves from j to $j+1$) is always the same for all j (j represents the number of consecutive periods that retailer 2 has been disrupted). It seems more realistic if this probability is decreasing in j , for example the probability of moving from $j=3$ to $j=4$ is lower than the probability of moving from $j=2$ to $j=3$.
- e. Assuming a positive fixed ordering cost. For this case, we may want to investigate what is the inventory policy that suits the problem most since based on some papers, the base-stock inventory policy is optimal only for a system with no order cost. For a system with positive order cost, some researchers (e.g. Song and Zipkin, 1996) suggest that (s,S) policy is optimal.

Apart from relaxing some assumptions above, we may want to apply the result of this research to a real-life case and incorporate company demands and limitations to the problem. All in all, relaxing some assumptions or applying the result of the research to a specific problem in the real world can give more beneficial results to both researchers and practitioners.

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IX. APPENDIX

9.1 Appendix Chapter 4

Formulation for π_j

Related to the supply process, it is assumed that disruptions follow a discrete time Markov chain (DTMC). Let Y_t represent the state in the supply process in period t on state space $S = \{0, 1\}$ where $Y_t = 1$ if the supply process is ON and $Y_t = 0$ if the supply process is OFF. Defining transition probabilities as:

$$\alpha = P(Y_t = 0 | Y_{t-1} = 1) \text{ and } \beta = P(Y_t = 1 | Y_{t-1} = 0),$$

Then α can be defined as a disruption probability and β as a recovery probability. Since it is assumed that the disruptions follow a DTMC, the length of ON and OFF intervals are therefore geometrically distributed.

Define, $p_{i,j} = P(X_{t+1} = j | X_t = i)$

Then we can assign the following transition probability:

$$p_{0,0} = 1 - \beta; p_{0,1} = \beta; p_{1,0} = \alpha; p_{1,1} = 1 - \alpha$$

Let the steady state distribution of the state space is denoted by $G = [G_0, G_1]$, then the steady state probability should satisfy:

$$G_j = \sum_{i=0}^N G_i p_{i,j}, j \in S \quad (1)$$

and

$$\sum_{j=0}^N G_j = 1 \quad (2)$$

In this case G_0 is the steady-state probability that the supply process is disrupted and G_1 is the steady-state probability that the supply process is not disrupted. From equation (1) we can define:

$$G_0 = \sum_{i=0}^1 G_i p_{i,0} = G_0 p_{0,0} + G_1 p_{1,0}$$

$$G_0 = G_0(1 - \beta) + G_1(\alpha)$$

$$G_0 = \frac{\alpha}{\beta} G_1$$

$$G_1 = \sum_{i=0}^1 G_i p_{i,1} = G_0 p_{0,1} + G_1 p_{1,1}$$

$$G_1 = G_0(\beta) + G_1(1 - \alpha)$$

$$G_1 = \frac{\beta}{\alpha} G_0$$

From equation (2):

$$G_0 + G_1 = 1$$

$$G_0 + \frac{\beta}{\alpha} G_0 = 1$$

$$G_0 = \frac{\alpha}{\alpha + \beta}$$

$$G_1 = \frac{\beta}{\alpha} \left(\frac{\alpha}{\alpha + \beta} \right)$$

$$G_1 = \frac{\beta}{\alpha + \beta}$$

Since it is specified that the DTMC for the disruption states using two parameters: α and β , one can define the expression for π_j using those parameters as the following.

For $j = 0$,

π_0 = steady state probability of being in the 0 period of OFF interval, then π_0 is G_1

$$\pi_0 = \frac{\beta}{\alpha + \beta}$$

For $j > 0$,

π_1 = steady state probability of being in the 1st period of OFF interval

$$\pi_1 = p_{1,0} \pi_0 = \frac{\alpha \beta}{\alpha + \beta}$$

π_2 = steady state probability of being in the 2nd period of OFF interval

$$\pi_2 = p_{0,0} \pi_1 = (1 - \beta) \left(\frac{\alpha \beta}{\alpha + \beta} \right)$$

π_3 = steady state probability of being in the 3rd period of OFF interval

$$\pi_3 = p_{0,0} \pi_2 = (1 - \beta)^2 \left(\frac{\alpha \beta}{\alpha + \beta} \right)$$

and for the rest of j where $j \geq 2$: $\pi_i = p_{0,0}\pi_{j-1}$.

The definition above has a recurring pattern, therefore for $j \geq 1$, we can rewrite a general expression for π_j :

$$\pi_j = (1 - \beta)^{j-1} \left(\frac{\alpha\beta}{\alpha + \beta} \right)$$

9.2 Appendix Chapter 5

Derivation of Formula 5.7

Derivation of the first term (1):

The expression related to h_1 is the following,

$$h_1 \sum_0^{\infty} \pi_j \left(\left[S_1 + 1 - d - \min \{ [S_1 + 1 - d]^+, [(j+1)d - S_2]^+ \} \right]^+ - \left[S_1 - d - \min \{ [S_1 - d]^+, [(j+1)d - S_2]^+ \} \right]^+ \right) \quad (5.7a)$$

The above expression can be simplified by limiting the value for j , since after a specific value of j , formula 5.7a will be equal to zero. Simplifying the expression will be helpful for the next step of derivation (second order derivation).

To find the specific value for j which limits the expression, first we divide expression (5.7a) into two parts:

$$h_1 \sum_0^{\infty} \pi_j \left(\underbrace{\left[S_1 + 1 - d - \min \{ [S_1 + 1 - d]^+, [(j+1)d - S_2]^+ \} \right]^+}_A - \underbrace{\left[S_1 - d - \min \{ [S_1 - d]^+, [(j+1)d - S_2]^+ \} \right]^+}_B \right)$$

Part B in the formula above will have positive value if,

$$S_1 - d \geq \min \{ [S_1 - d]^+, [(j+1)d - S_2]^+ \}$$

The specific value of j that can satisfy above expression is:

$$(j+1)d - S_2 \leq S_1 - d$$

$$j \leq \frac{S_1 + S_2}{d} - 2$$

And part A will have positive value if,

$$S_1 - d + 1 \geq \min \left\{ [S_1 - d + 1]^+, [(j+1)d - S_2]^+ \right\}$$

$$(j+1)d - S_2 \leq S_1 - d + 1$$

$$jd \leq S_1 + S_2 - 2d + 1$$

$$j \leq \frac{S_1 + S_2 + 1}{d} - 2$$

Based on both results, for all $j > ((S_1+S_2)/d)-2$, both part A and B are always equal to 0, hence the value for expression (5.7a) is equal to 0.

Now, we will check the value of expression (5.7a) for all $j \leq ((S_1+S_2)/d)-2$. We find it by trying several values of j .

(1) For $j = ((S_1+S_2)/d)-2$

$$\begin{aligned} & h_1 \pi_j \left(\left[S_1 + 1 - d - \min \left\{ [S_1 + 1 - d]^+, [(j+1)d - S_2]^+ \right\} \right]^+ - \left[S_1 - d - \min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\} \right]^+ \right) \\ & h_1 \pi_j \left(\left[S_1 + 1 - d - \min \left\{ [S_1 + 1 - d]^+, \left[\left(\frac{S_1 + S_2}{d} - 1 \right) d - S_2 \right]^+ \right\} \right]^+ - \left[S_1 - d - \min \left\{ [S_1 - d]^+, \left[\left(\frac{S_1 + S_2}{d} - 1 \right) d - S_2 \right]^+ \right\} \right]^+ \right) \\ & = h_1 \pi_j \left(\left[S_1 + 1 - d - \min \left\{ [S_1 + 1 - d]^+, [S_1 - d]^+ \right\} \right]^+ - \left[S_1 - d - \min \left\{ [S_1 - d]^+, [S_1 - d]^+ \right\} \right]^+ \right) \\ & = h_1 \pi_j \left(\left[S_1 + 1 - d - [S_1 - d]^+ \right]^+ - \left[S_1 - d - [S_1 - d]^+ \right]^+ \right) = h_1 \pi_j = h_1 \pi_{\frac{S_1 + S_2}{d} - 2} \end{aligned}$$

(2) For $j = ((S_1+S_2)/d)-3$

$$\begin{aligned} & h_1 \pi_j \left(\left[S_1 + 1 - d - \min \left\{ [S_1 + 1 - d]^+, [(j+1)d - S_2]^+ \right\} \right]^+ - \left[S_1 - d - \min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\} \right]^+ \right) \\ & h_1 \pi_j \left(\left[S_1 + 1 - d - \min \left\{ [S_1 + 1 - d]^+, \left[\left(\frac{S_1 + S_2}{d} - 2 \right) d - S_2 \right]^+ \right\} \right]^+ - \left[S_1 - d - \min \left\{ [S_1 - d]^+, \left[\left(\frac{S_1 + S_2}{d} - 2 \right) d - S_2 \right]^+ \right\} \right]^+ \right) \\ & = h_1 \pi_j \left(\left[S_1 + 1 - d - \min \left\{ [S_1 + 1 - d]^+, [S_1 - 2d]^+ \right\} \right]^+ - \left[S_1 - d - \min \left\{ [S_1 - d]^+, [S_1 - 2d]^+ \right\} \right]^+ \right) \\ & = h_1 \pi_j \left(\left[S_1 + 1 - d - [S_1 - 2d]^+ \right]^+ - \left[S_1 - d - [S_1 - 2d]^+ \right]^+ \right) = h_1 \pi_j = h_1 \pi_{\frac{S_1 + S_2}{d} - 3} \end{aligned}$$

From two examples above expression (5.7a) equals to $h_1\pi_j$. This pattern continues for all $0 < j \leq (S_1+S_2)/d$
-2. This result is also applied for $j=0$ as shown below.

(3) For $j = 0$

$$\begin{aligned} & h_1\pi_j \left(\left[S_1 + 1 - d - \min \left\{ [S_1 + 1 - d]^+, [(j+1)d - S_2]^+ \right\} \right]^+ - \left[S_1 - d - \min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\} \right]^+ \right) \\ &= h_1\pi_j \left(\left[S_1 + 1 - d - \min \left\{ [S_1 + 1 - d]^+, [d - S_2]^+ \right\} \right]^+ - \left[S_1 - d - \min \left\{ [S_1 - d]^+, [d - S_2]^+ \right\} \right]^+ \right) \end{aligned}$$

Since $S_2 \geq d$, $[d - S_2]^+$ always results in 0. Hence,

$$= h_1\pi_j \left([S_1 + 1 - d]^+ - [S_1 - d]^+ \right) = h_1\pi_j = h_1\pi_0$$

Therefore expression (5.7a) can be rewritten as the following:

$$h_1 \left(\sum_{j=0}^{\frac{S_1+S_2}{d}-2} \pi_j \right)$$

Derivation of the second term (2):

The expression related to p_2 is the following,

$$\begin{aligned} & p_2 \sum_0^{\infty} \pi_j \left([(j+1)d - S_2 - TX^j]^+ - [(j+1)d - S_2 - TX^j]^+ \right) = \\ & p_2 \sum_0^{\infty} \pi_j \left(\left[(j+1)d - S_2 - \left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right]^+, 1 \right\} \right) \left(\min \left\{ [S_1 + 1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) + [S_1 + 1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \right. \right. \\ & \left. \left. - \left[(j+1)d - S_2 - \left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right]^+, 1 \right\} \right) \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) + [S_1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \right] \right) \right) \end{aligned} \quad (5.7b)$$

To analyze this expression, we first elaborated formula (5.7b) to various different values of j .

(1) For $j < \frac{S_2}{d}$

In this case, we can omit TX^j since no transshipment has occurred before this state.

$$\pi_j p_2 \left([(j+1)d - S_2]^+ - [(j+1)d - S_2]^+ \right) = 0$$

$$(2) \text{ For } j \geq \frac{S_2}{d}$$

For this range of j , expression (5.7b) can be written as follow:

$$p_2 \sum_0^{\infty} \pi_j \left(\begin{array}{c} \left[\underbrace{(j+1)d - S_2 - \left(\min \left\{ [S_1 + 1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} + [S_1 + 1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]}_A \right]^+ \\ - \left[\underbrace{(j+1)d - S_2 - \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} + [S_1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]}_B \right]^+ \end{array} \right)$$

Since in part A and part B we have to determine the minimum between two expressions, we will analyze expression (5.7b) with 2 conditions.

(a) **Condition 1**, if:

$$[S_1 - d]^+ < \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2$$

$$[S_1 - d]^+ < (j^* d + d) - S_2$$

$$S_1 + S_2 - 2d < j^* d$$

$$j^* > \frac{S_1 + S_2}{d} - 2$$

Now, we will check the value of expression (5.7b) for $j > ((S_1 + S_2)/d) - 2$.

$$(1) \text{ For } j = \frac{S_1 + S_2}{d} - 1$$

$$\begin{aligned} & \pi_j p_2 \left(\left[\left(\frac{S_1 + S_2}{d} \right) d - S_2 - \left(\min \left\{ [S_1 + 1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) \right]^+ - \left[\left(\frac{S_1 + S_2}{d} \right) d - S_2 - \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) \right]^+ \right) \\ &= \pi_j p_2 \left(\left[S_1 - [S_1 + 1 - d]^+ \right]^+ - \left[S_1 - [S_1 - d]^+ \right]^+ \right) = -\pi_{\frac{S_2}{d}} p_2 \end{aligned}$$

$$(2) \text{ For } j = \frac{S_1 + S_2}{d}$$

$$\begin{aligned}
& \pi_j p_2 \left(\left[\begin{aligned} & \left(\frac{S_1 + S_2}{d} + 1 \right) d - S_2 - \left(\min \left\{ [S_1 + 1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) - [S_1 + 1 - d]^+ \left(\left[\frac{S_1 + S_2}{d} - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \\ & - \left[\left(\frac{S_1 + S_2}{d} + 1 \right) d - S_2 - \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) - [S_1 - d]^+ \left(\left[\frac{S_1 + S_2}{d} - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \end{aligned} \right) \right) \\
& = \pi_j p_2 \left(\left[(S_1 + d) - [S_1 + 1 - d]^+ - [S_1 + 1 - d]^+ \right]^+ - \left[(S_1 + d) - [S_1 - d]^+ - [S_1 - d]^+ \right]^+ \right) \\
& = \pi_j p_2 \left([3d - S_1 - 2]^+ - [3d - S_1]^+ \right) = -2\pi_j p_2
\end{aligned}$$

(2) For $j = \frac{S_1 + S_2}{d} + 1$

$$\begin{aligned}
& \pi_j p_2 \left(\left[\begin{aligned} & \left(\frac{S_1 + S_2}{d} + 2 \right) d - S_2 - \left(\min \left\{ [S_1 + 1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) - [S_1 + 1 - d]^+ \left(\left[\frac{S_1 + S_2}{d} + 1 - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \\ & - \left[\left(\frac{S_1 + S_2}{d} + 2 \right) d - S_2 - \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) - [S_1 - d]^+ \left(\left[\frac{S_1 + S_2}{d} + 1 - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \end{aligned} \right) \right) \\
& = \pi_j p_2 \left(\left[(S_1 + 2d) - [S_1 + 1 - d]^+ - 2[S_1 + 1 - d]^+ \right]^+ - \left[(S_1 + 2d) - [S_1 - d]^+ - 2[S_1 - d]^+ \right]^+ \right) \\
& = \pi_j p_2 \left([5d - 2S_1 - 3]^+ - [5d - 2S_1]^+ \right) = -3\pi_j p_2
\end{aligned}$$

For every j that is bigger than $((S_1 + S_2)/d) - 2$, the result for formula 16 has the same pattern, that is =

$$p_2 \pi_j \left(\left\lfloor \frac{S_1 + S_2}{d} \right\rfloor - j - 2 \right)$$

(b) **Condition 2**, if,

$$[S_1 - d]^+ > \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2$$

Using the same approach as what we have done in condition (1), we found the result for expression (5.7b) under condition 2 as the following:

$$p_2 \pi_j \left(\left\lfloor \frac{S_1 + S_2}{d} \right\rfloor - j - 1 \right)$$

Based above results, expression 5.7b can be replaced by:

For condition 1:

$$\sum_{\frac{S_1+S_2}{d}-1}^{\infty} p_2 \pi_j \left(\left\lfloor \frac{S_1+S_2}{d} \right\rfloor - j - 2 \right)$$

For condition 2:

$$\sum_{\frac{S_1+S_2}{d}}^{\infty} p_2 \pi_j \left(\left\lfloor \frac{S_1+S_2}{d} \right\rfloor - j - 1 \right)$$

Derivation of the third term (3):

The expression related to c (transshipment cost) is as the following,

$$c \sum_0^{\infty} \pi_j \left(\underbrace{\min \left\{ [S_1 + 1 - d]^+, [(j+1)d - S_2]^+ \right\}}_A \right) - \left(\underbrace{\min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\}}_B \right) \quad (15c)$$

To analyze this expression, we examined the result of expression (5.7c) in three different ranges of j , that are for $j < j^*$, $j = j^*$ and $j > j^*$ (j^* is the stage when the first transshipment occurs).

Expression (5.7c) is always equal to 0 for $j < j^*$. Since no transshipment occurs in those stages both for part A and B.

For $j > j^*$, part A is equal to $[S_1 + 1 - d]^+$ and part B is equal to $[S_1 - d]^+$, therefore expression (5.7c) generates result as the following:

$$\begin{aligned} & \pi_j c \left(\min \left\{ [S_1 + 1 - d]^+, [(j+1)d - S_2]^+ \right\} \right) - \left(\min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\} \right) \\ &= \pi_j c \left([S_1 + 1 - d]^+ - [S_1 - d]^+ \right) \\ &= \pi_j c \end{aligned}$$

As for $j = j^*$, Expression (5.7c) also depends on S_1 because at this stage if S_1 is bigger than d and S_2 is not an integer multiple of d , then at j^* both part A and part B are equal to $[(j^*+1)d - S_2]^+$ thus expression (5.7c) is equal to 0. But, for other conditions, expression (5.7c) can have positive value bigger than 0.

In order to find the lower bound for j where expression (5.7c) can have positive value bigger than 0, we tried to include S_1 in the definition of j^* . There are two conditions for this:

1. If $\left\lfloor \frac{S_1}{d} \right\rfloor = 1$, then j^* can be rewritten as:

$$j^* = \left\lfloor \frac{S_2}{d} \right\rfloor + \left(\left\lfloor \frac{S_1}{d} \right\rfloor - 1 \right) = \left\lfloor \frac{S_1 + S_2}{d} \right\rfloor - 1$$

2. If $\left\lfloor \frac{S_1}{d} \right\rfloor = 2$, then j^* can be rewritten as:

$$j^* = \left\lfloor \frac{S_2}{d} \right\rfloor + \left(\left\lfloor \frac{S_1}{d} \right\rfloor - 2 \right) = \left\lfloor \frac{S_1 + S_2}{d} \right\rfloor - 2$$

Now, we check the value for each condition,

(1) For $j = \frac{S_1 + S_2}{d} - 1$

$$\begin{aligned} & c \sum_0^{\infty} \pi_j \left(\min \left\{ [S_1 + 1 - d]^+, [(j+1)d - S_2]^+ \right\} \right) - \left(\min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\} \right) \\ &= c \sum_0^{\infty} \pi_j \left(\min \left\{ [S_1 + 1 - d]^+, [S_1 + S_2 - S_2]^+ \right\} \right) - \left(\min \left\{ [S_1 - d]^+, [S_1 + S_2 - S_2]^+ \right\} \right) \\ &= c \pi_j (S_1 - d + 1 - S_1 + d) = c \pi_j \end{aligned}$$

(2) For $j = \frac{S_1 + S_2}{d} - 2$

$$\begin{aligned} & c \sum_0^{\infty} \pi_j \left(\min \left\{ [S_1 + 1 - d]^+, [(j+1)d - S_2]^+ \right\} \right) - \left(\min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\} \right) \\ & c \sum_0^{\infty} \pi_j \left(\min \left\{ [S_1 + 1 - d]^+, [S_1 - d + S_2 - S_2]^+ \right\} \right) - \left(\min \left\{ [S_1 - d]^+, [S_1 - d + S_2 - S_2]^+ \right\} \right) \\ &= c \pi_j (S_1 - d - S_1 + d) = 0 \end{aligned}$$

Hence, the lower bound for j is when $j = \left\lfloor \frac{S_1 + S_2}{d} \right\rfloor - 1$

Based on above results, expression (5.7c) can be rewritten as follow:

$$c \sum_{j^*}^{\infty} \pi_j = c \sum_{\frac{S_1+S_2}{d}-1}^{\infty} \pi_j$$

Derivation of Formula 5.14

The derivation of this formula is similar with formula (5.7).

$$\Delta C_{S_1}(S_1, S_2) = h_1 \left(\sum_{j=0}^{\frac{S_1+S_2}{d}-2} \pi_j d \right)^{(1)} + p_2 \left(\sum_{\frac{S_1+S_2}{d}-1}^{\infty} \pi_j \left(\frac{S_1+S_2}{d} - j - 2 \right) d \right)^{(2)} + c \left(\sum_{\frac{S_1+S_2}{d}-1}^{\infty} \pi_j d \right)^{(3)} \quad (5.14)$$

Derivation of the first term (1):

In the first order derivation of $C(S_1, S_2)$, the expression related to h_1 is the following,

$$h_1 \sum_0^{\infty} \pi_j \left(\left[S_1 + d - d - \min \left\{ [S_1 + d - d]^+, [(j+1)d - S_2]^+ \right\} \right]^+ - \left[S_1 - d - \min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\} \right]^+ \right) \quad (5.14a)$$

To find the limit value for j , first we divide above expression into two parts:

$$h_1 \sum_0^{\infty} \pi_j \left(\underbrace{\left[S_1 - \min \left\{ [S_1]^+, [(j+1)d - S_2]^+ \right\} \right]^+}_A - \underbrace{\left[S_1 - d - \min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\} \right]^+}_B \right)$$

Part B in the formula above will have positive value if,

$$\begin{aligned} S_1 - d &\geq \min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\} \\ (j+1)d - S_2 &\leq S_1 - d \\ j &\leq \frac{S_1 + S_2}{d} - 2 \end{aligned}$$

And part A will have positive value if,

$$\begin{aligned} S_1 &\geq \min \left\{ [S_1]^+, [(j+1)d - S_2]^+ \right\} \\ j &\leq \frac{S_1 + S_2}{d} - 1 \end{aligned}$$

Based on both results, for all $j > ((S_1+S_2)/d)-2$, both part A and B are always equal to 0, hence the value for expression (5.14a) is equal to 0.

Now, we will check the value of expression (5.14a) for all $j \leq ((S_1+S_2)/d)-2$. We find it by trying several values of j .

(1) For $j = ((S_1+S_2)/d)-2$

$$\begin{aligned} & h_1 \pi_j \left(\left[S_1 - \min \{ [S_1]^+, [(j+1)d - S_2]^+ \} \right]^+ - \left[S_1 - d - \min \{ [S_1 - d]^+, [(j+1)d - S_2]^+ \} \right]^+ \right) \\ &= h_1 \pi_j \left(\left[S_1 - \min \{ [S_1]^+, [S_1 - d]^+ \} \right]^+ - \left[S_1 - d - \min \{ [S_1 - d]^+, [S_1 - d]^+ \} \right]^+ \right) \\ &= h_1 \pi_j \left([S_1 - S_1 + d]^+ - [S_1 - d - [S_1 - d]^+]^+ \right) = h_1 \pi_j d \end{aligned}$$

(2) For $j = ((S_1+S_2)/d)-3$

$$\begin{aligned} & h_1 \pi_j \left(\left[S_1 - \min \{ [S_1]^+, [(j+1)d - S_2]^+ \} \right]^+ - \left[S_1 - d - \min \{ [S_1 - d]^+, [(j+1)d - S_2]^+ \} \right]^+ \right) \\ &= h_1 \pi_j \left(\left[S_1 - \min \{ [S_1]^+, [S_1 - 2d]^+ \} \right]^+ - \left[S_1 - d - \min \{ [S_1 - d]^+, [S_1 - 2d]^+ \} \right]^+ \right) \\ &= h_1 \pi_j \left([S_1 - [S_1 - 2d]^+]^+ - [S_1 - d - [S_1 - 2d]^+]^+ \right) = h_1 \pi_j d \end{aligned}$$

From two examples above expression (5.14a) equals to $h_1 \pi_j d$. This pattern continues for all $0 < j \leq (S_1+S_2)/d - 2$.

Therefore expression (5.14a) can be rewritten as the following:

$$h_1 \left(\sum_{j=0}^{\frac{S_1+S_2}{d}-2} \pi_j d \right)$$

Derivation of the second term (2):

The expression related to p_2 is the following,

$$p_2 \sum_0^{\infty} \pi_j \left([(j+1)d - S_2 - TX^j]^+ - [(j+1)d - S_2 - TX^j]^+ \right) =$$

$$p_2 \sum_0^{\infty} \pi_j \left(\begin{array}{l} \left[(j+1)d - S_2 - \left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right\rfloor^+, 1 \right\} \right) \left(\min \left\{ [S_1 + d - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) + [S_1 + d - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \\ - \left[(j+1)d - S_2 - \left(\min \left\{ \left[j - \left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right\rfloor^+, 1 \right\} \right) \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) + [S_1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right]^+ \end{array} \right) \quad (5.14b)$$

To analyze this expression, we first elaborated formula (5.14b) to various different values of j .

(3) For $j < \frac{S_2}{d}$

In this case, we can omit TX^j since no transshipment has occurred before this state.

$$\pi_j p_2 \left(\left[(j+1)d - S_2 \right]^+ - \left[(j+1)d - S_2 \right]^+ \right) = 0$$

(4) For $j \geq \frac{S_2}{d}$

For this range of j , expression (16) can be written as follow:

$$p_2 \sum_0^{\infty} \pi_j \left(\begin{array}{l} \left[\underbrace{(j+1)d - S_2 - \left(\min \left\{ [S_1]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) + [S_1]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right)}_A \right]^+ \\ - \left[\underbrace{(j+1)d - S_2 - \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) + [S_1 - d]^+ \left(\left[j - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right)}_B \right]^+ \end{array} \right)$$

Since the value of expression 5.14b is also depends on S_1 , we try several value of j that also includes S_1 .

(1) For $j = \frac{S_1 + S_2}{d} - 1$

$$\begin{aligned} & \pi_j p_2 \left(\left[\left(\frac{S_1 + S_2}{d} \right) d - S_2 - \left(\min \left\{ [S_1]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) \right]^+ - \left[\left(\frac{S_1 + S_2}{d} \right) d - S_2 - \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) \right]^+ \right) \\ & = \pi_j p_2 \left(\left[S_1 - d - S_1 \left(\frac{S_1}{d} - 1 \right) \right]^+ - \left[S_1 - [S_1 - d]^+ - [S_1 - d]^+ \left([S_1 - d]^+ \right) \right]^+ \right) = -\pi_j p_2 d \end{aligned}$$

(2) For $j = \frac{S_1 + S_2}{d}$

$$\pi_j p_2 \left(\left[\left(\frac{S_1 + S_2}{d} + 1 \right) d - S_2 - \left(\min \left\{ [S_1]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) - [S_1]^+ \left(\left[\frac{S_1 + S_2}{d} - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right] \right. \\ \left. - \left[\left(\frac{S_1 + S_2}{d} + 1 \right) d - S_2 - \left(\min \left\{ [S_1 - d]^+, \left(\left\lfloor \frac{S_2}{d} \right\rfloor + 1 \right) d - S_2 \right\} \right) - [S_1 - d]^+ \left(\left[\frac{S_1 + S_2}{d} - \left\lfloor \frac{S_2}{d} \right\rfloor \right]^+ \right) \right] \right) \right) \\ = \pi_j p_2 \left([S_1 - S_1(S_1/d)]^+ - [2d - (S_1 - d)]^+ \right) = (-2d) \pi_j p_2$$

For every j that is bigger than $((S_1 + S_2)/d) - 2$, the result for formula 5.14b has the same pattern, that is =

$$p_2 \pi_j \left(\frac{S_1 + S_2}{d} - j - 2 \right)$$

Therefore, expression 5.14b can be rewritten as:

$$\sum_{\frac{S_1 + S_2}{d} - 1}^{\infty} dp_2 \pi_j \left(\frac{S_1 + S_2}{d} - j - 2 \right)$$

Derivation of the third term (3):

The expression related to c (transshipment cost) is as the following,

$$c \sum_0^{\infty} \pi_j \left(\underbrace{\min \left\{ [S_1 + d - d]^+, [(j+1)d - S_2]^+ \right\}}_A \right) - \left(\underbrace{\min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\}}_B \right) \quad (5.14c)$$

To analyze this expression, we examined the result of expression (5.14c) in three different ranges of j , that are for $j < j^*$, $j = j^*$ and $j > j^*$ (j^* is the stage when the first transshipment occurs).

Expression (5.14c) is always equal to 0 for $j < j^*$. Since no transshipment occurs in those stages both for part A and B.

For $j > j^*$, part A is equal to $[S_1]^+$ and part B is equal to $[S_1 - d]^+$, therefore expression (5.14c) generates result as the following:

$$\pi_j c \left(\min \left\{ [S_1]^+, [(j+1)d - S_2]^+ \right\} \right) - \left(\min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\} \right) \\ = \pi_j c \left([S_1]^+ - [S_1 - d]^+ \right) \\ = \pi_j c d$$

As for $j = j^*$, Expression (5.14c) also depends on S_1 because at this stage if S_1 is bigger than d and S_2 is not an integer multiple of d , then at j^* both part A and part B are equal to $[(j^*+1)d - S_2]^+$ thus expression (5.14c) is equal to 0. But, for other conditions, expression (5.14c) can have positive value bigger than 0.

In order to find the lower bound for j where expression (5.14c) can have positive value bigger than 0, we tried to include S_1 in the definition of j^* . There are two conditions for this:

3. If $\frac{S_1}{d} = 1$, then j^* can be rewritten as:

$$j^* = \frac{S_2}{d} + \left(\frac{S_1}{d} - 1 \right) = \frac{S_1 + S_2}{d} - 1$$

4. If $\frac{S_1}{d} = 2$, then j^* can be rewritten as:

$$j^* = \frac{S_2}{d} + \left(\frac{S_1}{d} - 2 \right) = \frac{S_1 + S_2}{d} - 2$$

Now, we check the value for each condition,

$$(3) \text{ For } j = \frac{S_1 + S_2}{d} - 1$$

$$\begin{aligned} & c \sum_0^{\infty} \pi_j \left(\min \left\{ [S_1]^+, [(j+1)d - S_2]^+ \right\} \right) - \left(\min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\} \right) \\ & = c \pi_j (S_1 - S_1 + d) = cd \pi_j \end{aligned}$$

$$(4) \text{ For } j = \frac{S_1 + S_2}{d} - 2$$

$$\begin{aligned} & c \sum_0^{\infty} \pi_j \left(\min \left\{ [S_1]^+, [(j+1)d - S_2]^+ \right\} \right) - \left(\min \left\{ [S_1 - d]^+, [(j+1)d - S_2]^+ \right\} \right) \\ & = c \pi_j (S_1 - d - S_1 + d) = 0 \end{aligned}$$

Hence, the lower bound for j is when $j = \frac{S_1 + S_2}{d} - 1$

Based on above results, expression 5.14c can be rewritten as follow:

$$c \sum_{j^*}^{\infty} \pi_j = c \sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j(d)$$

The derivation for formula 5.15a and 5.15b can be done using the same approach as above.

Derivation of Formula 5.16

$$\begin{aligned} \Delta^2 C_{S_1, S_1}(S_1, S_2) &= \Delta C_{S_1}(S_1 + d, S_2) - \Delta C_{S_1}(S_1, S_2) \\ &= h_1 \left(\sum_{j=0}^{\frac{S_1+S_2-1}{d}} \pi_j d \right) + p_2 \left(\sum_{\frac{S_1+S_2}{d}}^{\infty} \pi_j \left(\frac{S_1+S_2}{d} - j - 1 \right) d \right) + c \left(\sum_{\frac{S_1+S_2}{d}}^{\infty} \pi_j d \right) - \\ &\quad \left(h_1 \left(\sum_{j=0}^{\frac{S_1+S_2-2}{d}} \pi_j d \right) + p_2 \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j \left(\frac{S_1+S_2}{d} - j - 2 \right) d \right) + c \left(\sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j d \right) \right) \\ &= h_1 \left(\sum_{j=0}^{\frac{S_1+S_2-2}{d}} \pi_j d + \pi_{\frac{S_1+S_2-1}{d}} - \sum_{j=0}^{\frac{S_1+S_2-2}{d}} \pi_j d \right) \\ &\quad + p_2 \left\{ \left(\pi_{\frac{S_1+S_2}{d}}(-d) + \pi_{\frac{S_1+S_2+1}{d}}(-2d) + \pi_{\frac{S_1+S_2+2}{d}}(-3d) + \dots \right) \right. \\ &\quad \left. - \left(\pi_{\frac{S_1+S_2-1}{d}}(-d) + \pi_{\frac{S_1+S_2}{d}}(-2d) + \pi_{\frac{S_1+S_2+1}{d}}(-3d) + \dots \right) \right\} \\ &\quad \left(+ c \left(\sum_{\frac{S_1+S_2}{d}}^{\infty} \pi_j d - \sum_{\frac{S_1+S_2-1}{d}}^{\infty} \pi_j d - \pi_{\frac{S_1+S_2-1}{d}} d \right) \right) \\ &= h_1 d \pi_{\frac{S_1+S_2-1}{d}} - c d \pi_{\frac{S_1+S_2-1}{d}} + p_2 d \pi_{\frac{S_1+S_2-1}{d}} + p_2 \sum_{\frac{S_1+S_2}{d}}^{\infty} d \pi_j \\ &= \left(d \pi_{\frac{S_1+S_2-2}{d}} (h_1 + p_2 - c) \right) + p_2 \sum_{\frac{S_1+S_2}{d}}^{\infty} d \pi_j \end{aligned}$$

The derivation for formula 5.17 is using a similar approach as used for the derivation of formula 5.16, and in the end it provides the same results.