MASTER

The diode laser with weak external phase conjugated feedback

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Abstract

The diode laser with weak optical phase conjugated feedback is characterized by rate-equations which are similar to the Lang & Kobayashi equations for normal feedback. The stationary state solutions are characterized by frequency- and phase locking. Another solution, the frequency difference solution, is found for the regime where stationary state solutions cannot exist. The effects of the feedback can be classified in three regimes. In regime I the laser exhibits the above mentioned stationary state behavior, while in regime II the frequency difference solution is found. This regime separates the stationary state regime I and the zero-feedback stationary solution. Finally, in regime III the relaxation oscillation undamps. Preliminary evidence is found for the existence of a fourth regime (IV) where chaotic behavior may be observed.

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Chapter 1

Introduction

In 1959 the research on diode lasers was initiated by a proposal of Basov *et al.* [1]. They conceived of a method to use a semiconductor structure as a laser (light amplification by stimulated emission of radiation). In 1962 the first diode laser based on a forward biased GaAs *p-n* junction was build, independently, by three groups, Hall *et al.*, Nathan *et al.* and Quist *et al.* [2,3,4]. Since then a lot of research with respect to diode lasers has been done.

A diode laser, see Figure 1.1, is a structure that basically consists of five parts. An active material, sandwiched between two bulk materials, which may be n- or p-doped and two electrodes, which are attached to the bulk material. The population inversion necessary for laser operation is obtained by a difference in band gap energy between the active and bulk materials and an applied current.

Diode lasers can be divided in two main groups: gain guided and index guided lasers. In gain guided diode lasers, the optical field is confined by the gain region of the structure. In index guided diode lasers the confinement is due to a difference in refractive index between the active and the bulk material.

A feature of diode lasers that was not immediately appreciated, is their extreme sensitivity to external signals. Due to this sensitivity, a lot of present day research on diode lasers concerns itself with the effects of external signals on their operating characteristics. External optical feedback is one of these research areas. In many

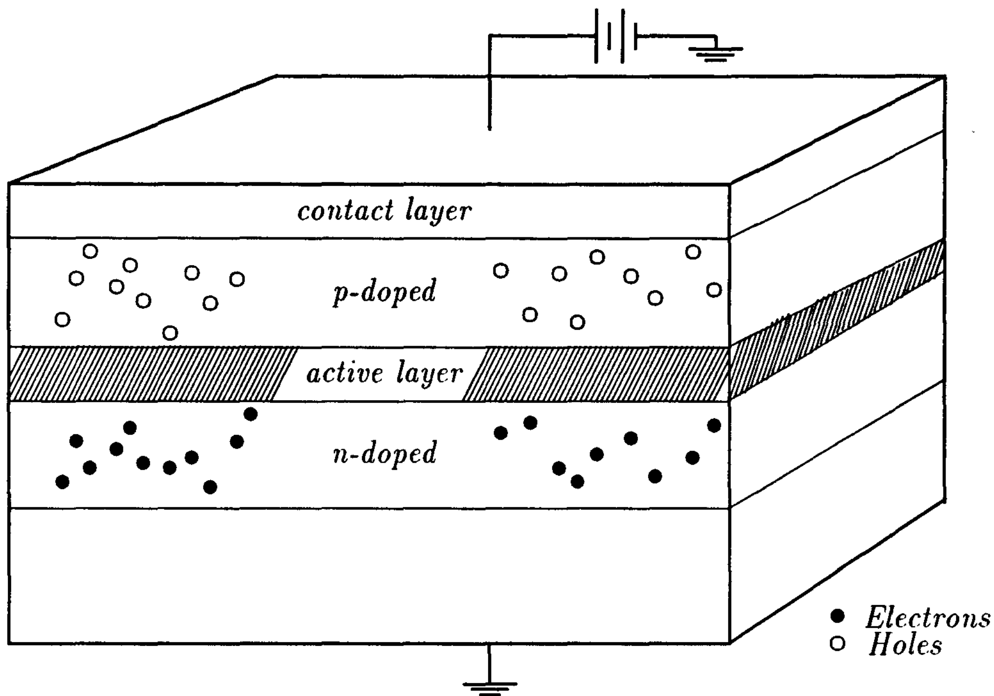

Figure 1.1: *Schematic representation of a laser diode*

cases feedback occurs when it is not desired such as when the light is coupled into an external device. However, inducing external optical feedback can, under certain conditions, reduce the linewidth of the laser output. The reduction of the linewidth is an important goal for various reasons, such as in optical communication applications or for narrow linewidth pump source applications. Therefore, a lot of present day research in optical feedback is aimed at controlling and reducing the linewidth of the laser output.

In 1980 Lang & Kobayashi [5] derived a set of rate-equations which form the basis for the theoretical analysis of the diode laser with external optical feedback. So far, these equations have been very successful in explaining phenomena observed in diode lasers subject to feedback.

In this report a diode laser subject to weak external phase conjugated feedback will be studied. The concept of phase conjugation, which has long been considered a mathematical trick to simulate time reversal, was for the first time experimentally observed in 1967. In that year Gerritsen [6] performed an experiment which produced a grating in a material due to the interference between two beams. Subsequently, the beams diffracted off their own grating, thus producing a phase conjugated wave. Since that first experiment, a lot of research has been done with respect to phase conjugation. Nowadays, there are many different ways in which phase conjugated waves can be generated.

An often used means of conjugation is self-pumping, a case of degenerate four-wave mixing (DFWM). This has been used to generate phase conjugated feedback for diode lasers by Y. Champagne *et al.* [7], K. Vahala *et al.* [8], M. Cronin-Golomb *et al.* [9], R. Stephens *et al.* [10] and M. Segev *et al.* [11].

In 1991, Agrawal & Klaus [12] have adapted the Lang & Kobayashi equations so that they could describe a diode laser subject to reflections from a DFWM phase conjugating reflector. They describe the effect of phase conjugated feedback on the dynamical response of the laser, showing that the stationary state exists only for certain, well defined, values of the *phase* of the intracavity optical field. Furthermore, depending on the amount of feedback, the stationary state becomes unstable through two, independent, instabilities, referred to as fold and Hopf instabilities. The fold instability is solely due to the phase conjugated nature of the feedback and does not occur in normal feedback. In the instability region, the laser output is found to become chaotic by following a period doubling or a quasi periodic route to chaos, depending on the amount of feedback.

In this thesis, another way of creating phase conjugated feedback, i.e. nearly degenerate four-wave mixing (NDFWM), is analyzed. In NDFWM, the frequency of the pump-waves is *different* from the frequency of the laser output. Due to this frequency mismatch the laser behaves quite differently from the way it does in the case of DFWM. One of the features of the stationary state for NDFWM phase conjugation feedback is that the frequency locks to the pump frequency. This may provide a way to *externally* set the operating frequency of the diode laser.

The analysis of the laser with phase conjugated feedback naturally divides in two parts: the theory of phase conjugation and the theory of a diode laser with feedback.

In chapter 2 the principle of phase conjugation will be addressed. There, the interaction of optical fields in a Kerr-like medium, i.e. far from an optical resonance, will be analyzed. Furthermore, it is shown how a phase conjugated wave can be created in the NDFWM scheme.

In chapter 3 the diode laser with normal feedback will be analyzed. To that end the Lang & Kobayashi equations will be introduced. Furthermore, some theoretical results obtained from these equations will be rederived. The stationary state solutions and their stability will be analyzed. This will serve as a convenient starting point for chapter 4, where the diode laser subject to phase conjugated external feedback will be studied. A set of equations, similar to the Lang & Kobayashi equations, will be derived, to our knowledge for the first time. It will be shown that these equations reduce to those used by Agrawal & Klaus if the DFWM scheme is assumed. The stationary state solutions, which are characterized by frequency- and phase locking, will be derived. Besides this solution, another, a frequency difference solution, is suggested for a regime where the stationary state solutions do not exist. As in the case of normal feedback, the stability of the stationary state solutions will be analyzed. The equations are also solved numerically, confirming and extending the analytical findings.

In chapter 5 the results of the theoretical and numerical analysis of the laser with phase conjugated feedback will be compared with those for the laser with normal feedback.

The results of this report will be presented at the IPR- and IQEC conferences [13,14] and a publication is in progress [15]. Furthermore, the research will be continued as a combined experimental and theoretical Ph.D. research in Amsterdam.

Chapter 2

Optical phase conjugation

2.1 Introduction

In optical phase conjugation both the phase factor and the direction of propagation of a wave are reversed due to nonlinear optical effects. Therefore, the material in which this occurs acts as a peculiar kind of mirror. Due to the peculiarities of this so called phase conjugation mirror (PCM), the "reflected" beam of light retraces its original path. Figure 2.1 illustrates this difference in reflection between a conventional mirror and a phase conjugation mirror.

Nonlinear optical effects depend on the strength of the optical fields, while positive- and negative frequency terms "mix" with each other. Let a plane wave $E(x,t)$ be represented as a complex field, with an angular frequency ω and a wavenumber $k = n_0\omega/c$, where n_0 is the linear index of refraction of the medium. Let the field have a complex amplitude $\mathcal{E}(x,t)$, that may still be slowly varying in time and place. The wave can then be written as:

$$E(x,t) = \frac{1}{2}\mathcal{E}(x,t)e^{i(kx-\omega t)} + c.c. \tag{2.1}$$

where the abbreviation *c.c.* denotes the complex conjugate. If (2.1) is phase conjugated, all terms, *except* for the term with the angular frequency ω, are complex conjugated. So, the phase conjugated wave corresponding to (2.1) is given by:

$$E_c(x,t) = \frac{1}{2}\mathcal{E}^*(x,t)e^{i(-kx-\omega t)} + c.c. \tag{2.2}$$

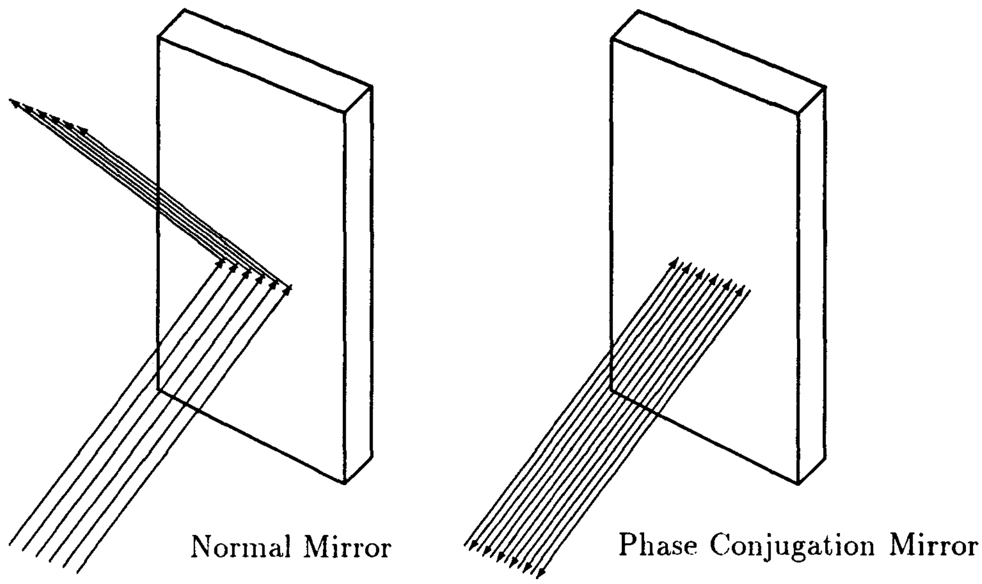

Figure 2.1: *Difference between a conventional mirror and a phase conjugation mirror.*

It is seen that, if the slowly varying amplitude $\mathcal{E}$ is a constant, the phase conjugated wave $E_c(x,t)$ can be written as $E(x,-t)$. This justifies the referring to phase conjugation as time inversion. Furthermore, it provides a mathematical equivalent to the phrase "retraces its original path".

Optical phase conjugation can be achieved by two different methods, elastic and inelastic photon scattering processes. Elastic photon scattering processes like three- and four-wave mixing leave the nonlinear medium in the *same* quantum state. Inelastic photon scattering processes, like Raman- or stimulated Brillouin scattering leave the medium in a *different* quantum state.

In the treatment of phase conjugation to be given here, only four-wave mixing (FWM) will be considered, since FWM will be used in analyzing a laser diode with phase conjugated feedback (Chapter 4). An extensive study of optical phase conjugation processes is given by Fisher [16].

Four-wave mixing is a process where three waves, i.e. two pump waves and one signal wave, produce a fourth, phase conjugated wave (see Figure 2.2). The FWM scheme can be used in different experimental setups. One of the ways FWM can be used is self-pumping. In a self-pumped situation, the laser-output is used to generate the pump waves, thus providing pump waves with exactly the same frequency as the signal wave. This is referred to as degenerate FWM [12,18,19,7,8,20,9].

In the analysis of FWM to be given here, the nearly degenerate FWM (NDFWM) method [17,18,19] will be examined. In NDFWM the frequency of the signal wave E_s is nearly equal to that of the pump waves E_1 and E_2. This makes degenerate FWM a special case of nearly degenerate FWM. It therefore suffices to analyze the nearly degenerate situation. Furthermore, only a certain type of materials, called Kerr-like media (e.g. CS_2 [37]), will be considered. A Kerr-like medium is a material that is characterized by three elements. In the first place it is nondispersive, that is, the susceptibility is not a function of the position in the material. Secondly, the material is considered to be unsaturable, which means that the polarization of the material, induced by the pump waves and the signal wave, does not have a maximum. Finally, the frequencies of the interacting optical fields are far away from atomic resonance. Therefore there is no stimulated absorption or emission in the material, which could influence the various intensities.

2.2 Phase Conjugation by Four Wave Mixing

In generating a phase conjugated wave by FWM, four optical fields interact in a Kerr-like medium via given nonlinearities, as in Figure 2.2. The optical fields induce a dipole moment density pattern P in the medium, also called the polarization. This polarization appears in the Maxwell-equations as a source-term. Since P depends in general nonlinearly on E, the optical fields will interact through the

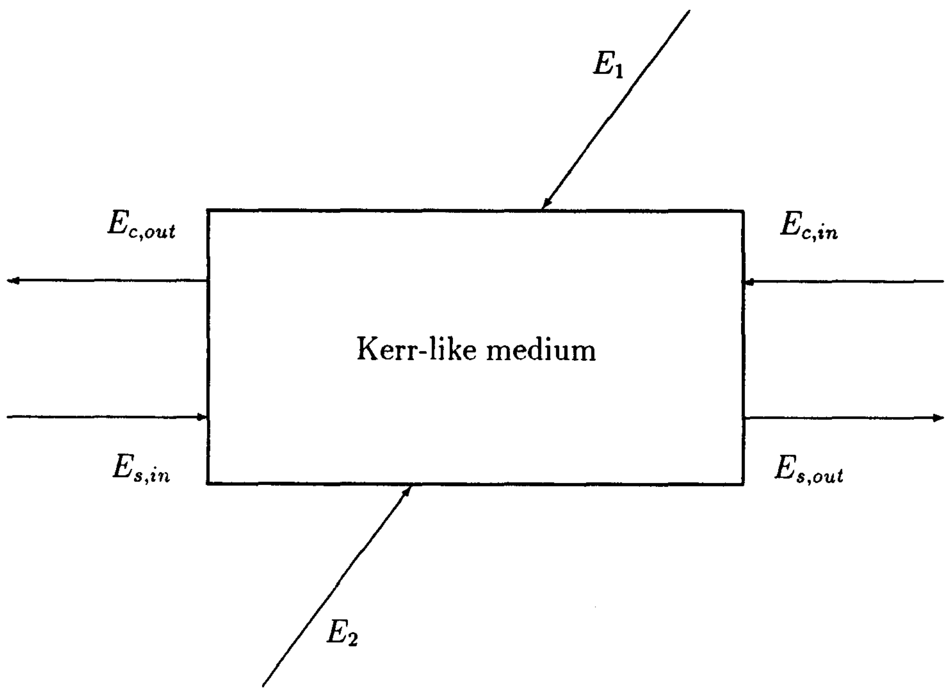

Figure 2.2: *Illustration of the creation of a phase conjugated wave. E_1 and E_2 are the non-depleted pump waves, E_s the signal wave and E_c the generated phase conjugated wave.*

polarization.

The Maxwell equations can, under certain conditions (see Appendix A), be written as a wave equation. This wave equation can be reduced to a first order differential equation by using the slowly varying envelope approximation (SVEA), as is shown in Appendix A. The relation between the slowly varying envelope amplitude of an optical electric field traveling in the positive x-direction and the slowly varying envelope of the nonlinear part of the induced polarization is then given by:

$$\left(\frac{\partial}{\partial x} + \sqrt{\mu_0 \epsilon}\,\frac{\partial}{\partial t}\right) \mathcal{E}(x,t) = -i\frac{\omega}{2}\sqrt{\frac{\mu_0}{\epsilon}}\,\mathcal{P} \tag{2.3}$$

where ϵ is the dielectric constant of the medium and μ_0 the magnetic susceptibility in vacuo. The vector nature of the fields in (2.3) has been disregarded for the sake of simplicity.

In the configuration that will be used for the description of phase conjugation, the fields are assumed to be linearly polarized and are all propagating in the same plane. In that situation the vector nature of the fields can be disregarded.

In order to be able to describe a situation when more than one traveling wave is present in the medium, the total field is decomposed in a finite number of traveling waves.

$$E(r,t) = \frac{1}{2}\sum_{\alpha}\mathcal{E}_{\alpha}(\xi_{\alpha},t)e^{i(k_{\alpha}\xi_{\alpha}-\omega_{\alpha}t)} + c.c. \tag{2.4}$$

$$P(r,t) = \frac{1}{2}\sum_{\alpha}\mathcal{P}_{\alpha}(\xi_{\alpha},t)e^{i(k_{\alpha}\xi_{\alpha}-\omega_{\alpha}t)} + c.c. \tag{2.5}$$

where ξ_{α} denotes the coordinate along the axis of propagation of the electric field and of the polarization. The associated wavenumbers and frequencies of the fields are given by k_{α} and ω_{α} and $\mathcal{E}_{\alpha}$ and $\mathcal{P}_{\alpha}$ are the slowly varying envelopes of E_{α} and P_{α}.

In (2.5) only terms with the same frequency and wavenumbers as in (2.4) are maintained. This is correct because the field power density associated with P is given by EP and for harmonically varying E and P this density is therefore significant only when E and P have the same frequency and wavenumber. This is called the *phase-matching condition.* The terms maintained in the right hand side of (2.5) are now precisely those which satisfy the phase-matching condition.

The phase-matching condition implies that (2.3) must independently apply for each index α in (2.4) and (2.5). Furthermore, it implies that the response of a wave with a particular frequency and wavenumber is determined only by the part of the polarization that has the *same* frequency and wavenumber.

2.2.1 Interaction of optical fields in a nonlinear medium

Equation (2.3) describes the way in which polarization has its effect on an electric field, but does not tell us how this polarization is created. For the latter one needs the constitutive relations, which can be expressed with a susceptibility χ as:

$$P = \chi(E)E \tag{2.6}$$

In a linearly responding medium, χ is independent of the driving field itself, but in a nonlinear medium χ will be a function of E. Assuming that χ depends analytically on E, it can be expanded in a Taylor series, yielding:

$$\chi(E) = \chi^{(1)} + \chi^{(2)}E + \chi^{(3)}E^2 + ... \tag{2.7}$$

where E is the total electric field in the medium.

In four-wave mixing the total electric field E is decomposed into four fields: a signal wave E_s, a phase conjugated wave E_c and two pump waves E_1 and E_2. The total electric field is then given by:

$$E(r,t) = \frac{1}{2} \sum_{\alpha=1,2,s,c} \mathcal{E}_\alpha(\xi_\alpha, t)e^{i(k_\alpha \xi_\alpha - \omega_\alpha t)} + c.c. \tag{2.8}$$

Substituting (2.7) and (2.8) in (2.6), an expression for the nonlinear part of the polarization is obtained:

$$\begin{aligned} P &= \chi^{(2)} \left(\frac{1}{2} \sum_{\alpha=1,2,s,c} \mathcal{E}_\alpha(\xi_\alpha, t)e^{i(k_\alpha \xi_\alpha - \omega_\alpha t)} + c.c. \right)^2 + \\ &\quad \chi^{(3)} \left(\frac{1}{2} \sum_{\alpha=1,2,s,c} \mathcal{E}_\alpha(\xi_\alpha, t)e^{i(k_\alpha \xi_\alpha - \omega_\alpha t)} + c.c. \right)^3 \end{aligned} \tag{2.9}$$

The second order nonlinearities in P, given by the first term on the r.h.s. of (2.9), do not satisfy the phase-matching condition in the configuration considered here, and are therefore omitted. However, there are terms that *do* satisfy the phase-matching condition in the second term. After these terms have been singled out of the 512 terms[1] one gets if the product is calculated, P can be written as:

$$P(r,t) = \frac{1}{2} \sum_{\alpha=1,2,s,c} \mathcal{P}_\alpha(\xi_\alpha, t)e^{i(k_\alpha \xi_\alpha - \omega_\alpha t)} + c.c. \tag{2.10}$$

so that,

$$P_s = \chi^{(3)} \left\{ \left(|\mathcal{E}_1|^2 + |\mathcal{E}_2|^2 + |\mathcal{E}_c|^2 + \frac{1}{2}|\mathcal{E}_p|^2 \right) + \mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_c^* e^{i\Delta kx} \right\} \tag{2.11}$$

$$P_c = \chi^{(3)} \left\{ \left(|\mathcal{E}_1|^2 + |\mathcal{E}_2|^2 + |\mathcal{E}_s|^2 + \frac{1}{2}|\mathcal{E}_c|^2 \right) + \mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_p^* e^{i\Delta kx} \right\} \tag{2.12}$$

[1]On page 31, right after eq. (15), Pepper and Yariv [18] mention only 156 terms instead of 512.

2.2.2 Optical phase conjugation by NDFWM

Expressions (2.8) and (2.10) for E and P are the starting point for the description of nearly degenerate four-wave mixing (NDFWM). In NDFWM there is a small frequency mismatch, $\Delta\omega$, between the frequency of the signal wave ω_s and that of the pump waves (ω), so that $\omega_s = \omega + \Delta\omega$. The phase matching condition implies that the frequency of the phase conjugated wave is then given as: $\omega_c = 2\omega - \omega_s = \omega - \Delta\omega$. This frequency mismatch corresponds to a mismatch in wavenumber Δk via the dispersion relation: $k = \epsilon^2\omega/c$.

The wave equations[2] for the slowly varying envelopes $\mathcal{E}_s$ and $\mathcal{E}_c$ can now be written as:

$$\begin{aligned}\frac{\partial}{\partial x}\mathcal{E}_s(x,t) &= i\frac{\omega_s}{2}\sqrt{\frac{\mu_0}{\epsilon}}\chi^{(3)}\{(|\mathcal{E}_1|^2 + |\mathcal{E}_2|^2 + |\mathcal{E}_c|^2 + \frac{1}{2}|\mathcal{E}_s|^2)\mathcal{E}_s + \\ &\quad \mathcal{E}_1\mathcal{E}_2\mathcal{E}_c^* e^{i\Delta kx}\} \qquad (2.13)\\ \frac{\partial}{\partial x}\mathcal{E}_c(x,t) &= -i\frac{\omega_c}{2}\sqrt{\frac{\mu_0}{\epsilon}}\chi^{(3)}\{(|\mathcal{E}_1|^2 + |\mathcal{E}_2|^2 + |\mathcal{E}_s|^2 + \frac{1}{2}|\mathcal{E}_c|^2)\mathcal{E}_c + \\ &\quad \mathcal{E}_1\mathcal{E}_2\mathcal{E}_s^* e^{i\Delta kx}\} \qquad (2.14)\end{aligned}$$

where the coordinate ξ_s has been identified with x, so that E_c propagates in the negative x-direction. The assumption that $\chi^{(3)}$ is constant, implies that the medium responds instantaneously to the presence of the interacting fields. Since the pump waves E_1 and E_2 are assumed to be non-depleted, the derivatives $\frac{\partial}{\partial\zeta}\mathcal{E}_1$ and $\frac{\partial}{\partial\zeta}\mathcal{E}_2$, where ζ is the axis of propagation of the waves, are equal to zero. Equations (2.13) and (2.14), first derived by Yariv and Pepper [19], form the basis for the description of phase conjugation by NDFWM.

It is possible to simplify (2.13) and (2.14) when $|\mathcal{E}_1|^2 + |\mathcal{E}_2|^2 \gg |\mathcal{E}_{s,c}|^2 + \frac{1}{2}|\mathcal{E}_{c,s}|^2$. The phase-dependence of $\mathcal{E}_s$ and $\mathcal{E}_c$, given by the first term on the r.h.s. of (2.13) and (2.14) can be factored out by defining:

$$\mathcal{E}_s(x,t) = \mathcal{E}_s'(x,t)e^{i\left\{\omega_s\sqrt{\mu_0/\epsilon}\,\chi^{(3)}(|\mathcal{E}_1|^2+|\mathcal{E}_2|^2)x/2\right\}} \qquad (2.15)$$

$$\mathcal{E}_c(x,t) = \mathcal{E}_c'(x,t)e^{-i\left\{\omega_c\sqrt{\mu_0/\epsilon}\,\chi^{(3)}(|\mathcal{E}_1|^2+|\mathcal{E}_2|^2)x/2\right\}} \qquad (2.16)$$

[2] In the wave equations for the slowly varying envelopes given in Pepper and Yariv [18], an error was found: the factor $\frac{1}{2}$ in front of $|\mathcal{E}_{s,c}|^2$ was written as a factor unity.

Inserting these expressions in (2.13) and (2.14), yields:

$$\frac{\partial}{\partial x}\mathcal{E}_s(x,t) = i\kappa_s\mathcal{E}_s^* e^{i\Delta kx} \quad (2.17)$$

$$\frac{\partial}{\partial x}\mathcal{E}_c(x,t) = -i\kappa_c\mathcal{E}_c^* e^{i\Delta kx} \quad (2.18)$$

where $\kappa_{s,c} = (\omega_{s,c}/2)\sqrt{\mu_0/\epsilon}\,\chi^{(3)}\mathcal{E}_1\mathcal{E}_2$ is called the coupling coefficient. The primed notation of (2.15) and (2.16) has been dropped, since the difference between, say, $\mathcal{E}_c$ and $\mathcal{E}_c'$ is an inconsequential phase-shift, and is zero at $x = 0$.

Equations (2.17) and (2.18) can be solved by Laplace-transformation which has been done in Appendix B. The incoming field $\mathcal{E}_s(0)$ is assumed to be a known quantity. Furthermore, the phase conjugated wave is generated in the medium, so that $\mathcal{E}_c(L_m) = 0$. Using these boundary conditions, the solution for $\mathcal{E}_c(x)$ can be written as:

$$\mathcal{E}_c(x) = \frac{i\kappa_c\mathcal{E}_s^*(0)e^{i\Delta kx/2}}{\beta\cos(\beta L_m) - i\frac{\Delta k}{2}\sin(\beta L_m)}\sin[\beta(L_m - x)] \quad (2.19)$$

where $\beta = \sqrt{\kappa_s\kappa_c + (\Delta k/2)^2}$. Examining (2.19) at $x = 0$ yields an expression for the "output" of the PCM:

$$\mathcal{E}_c(0) = \frac{i\kappa_c\mathcal{E}_s^*(0)}{\beta - i\frac{\Delta k}{2}\tan(\beta L_m)}\tan(\beta L_m) \quad (2.20)$$

This expression shows the characteristics of the four-wave interaction. The conjugated wave $\mathcal{E}_c(0)$ can be greater in amplitude than the input field, i.e. amplification, for certain values of κ_c, κ_s and Δk. Furthermore, it follows from the phase-matching condition that $\omega_c = \omega - \Delta\omega$ so that the frequency of the phase conjugated wave is down(up)-shifted the same amount as the frequency of the input wave is up(down)-shifted with respect to the frequency of the pump waves. Besides illustrating the characteristics of NDFWM, (2.20) can be used to find an expression for the power-reflectivity of the phase conjugation mirror, which is defined as:

$$R = |\mathcal{E}_c(0)/\mathcal{E}_s(0)|^2 \quad (2.21)$$

and becomes:

$$R = \frac{\kappa_c^2\sin^2(\beta L_m)}{\kappa_s\kappa_c(\Delta k/2)\cos^2(\beta L_m)} \quad (2.22)$$

In the case of weak coupling, when $\Delta k L_m \gg 2\kappa_{s,c}$, (2.22) reduces to:

$$R = (\kappa_c L_m)^2 \frac{\sin^2(\Delta k L_m/2)}{(\Delta k L_m/2)^2} \tag{2.23}$$

Another quantity that can be derived from (2.20) is the phase-shift of the phase conjugated wave with respect to the phase of the input wave, induced by the PCM, and is given by:

$$\Phi_{PCM} = -\arctan\left[\frac{2\beta}{\Delta k \tan(\beta L_m)}\right] \tag{2.24}$$

This is the only contribution to the phase-shift since (2.15) and (2.16) imply that the phase-shift induced by the pump wave intensity is zero at $x = 0$.

Chapter 3

The diode laser with weak optical feedback

3.1 Introduction

It is well known that dynamical systems can often be stabilized by the introduction of a feedback circuit. It is also known that, if the feedback involves a time-delay, one must generally expect the possibility of instabilities [21]. Lasers are no exception to this rule and especially in semiconductor lasers the stabilizing and destabilizing effects of feedback are observed under quite common operating conditions.

Semiconductor lasers are much more sensitive to feedback than other lasers. This is due to the combination of the low Q-factor of the cavity and the high-gain of the active medium. The cleaved end facets of the diode have a power reflectivity of about 30%. This causes the low Q-factor. It also implies high transmission, not only from the laser to the outside world, but the other way around as well.

The effect of feedback manifests itself in the coherence properties of the light emitted by the laser. Weak feedback can have a coherence-improving effect (line-width reduction). Higher levels of feedback can induce instabilities, even reducing the coherence length to $\approx$1cm - compared to about 5m without feedback, thus enhancing the linewidth up to factors 500 or more (coherence collapse) [22].

The effects of external feedback on a diode laser have been studied extensively,

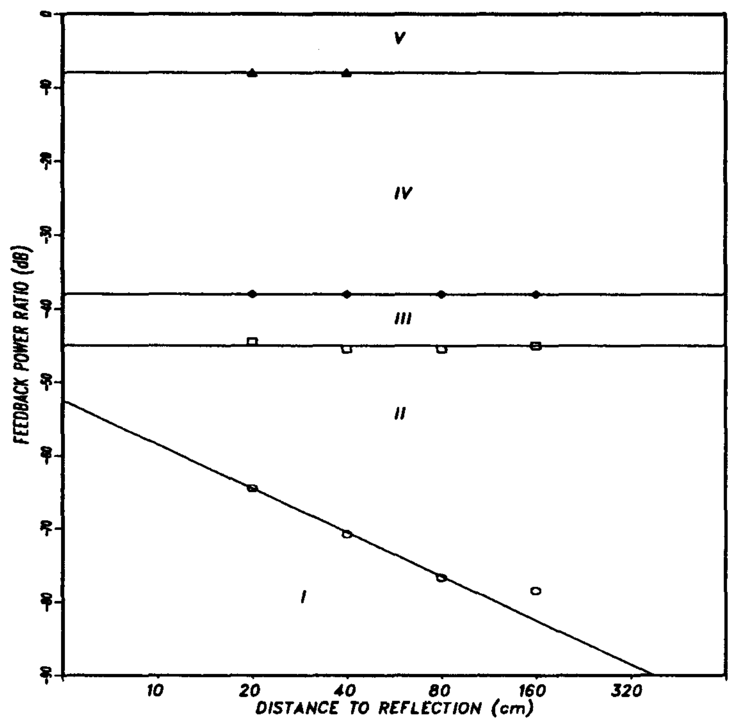

Figure 3.1: *Regimes of operation*

both experimentally [21] and theoretically [23,24]. The feedback effects can be phenomenologically classified in several regimes. Tkach & Charplyvy [25] give an overview of these regimes (Figure 3.1). In the first regime the laser exhibits single mode operation, where either line-narrowing or -broadening is observed, depending on the phase of the feedback [26]. The second regime is characterized by rapid mode hopping. In regime three, as the feedback is increased further, the mode hopping is repressed. Regime four is characterized by the development of satellite modes, separated from the main mode by the relaxation oscillation (and higher harmonics). In this regime the relaxation oscillations become undamped. As the level of feedback is increased further the laser line broadens to as much as 50 GHz. This has been termed the "coherence collapse" [22]. In regime five, the

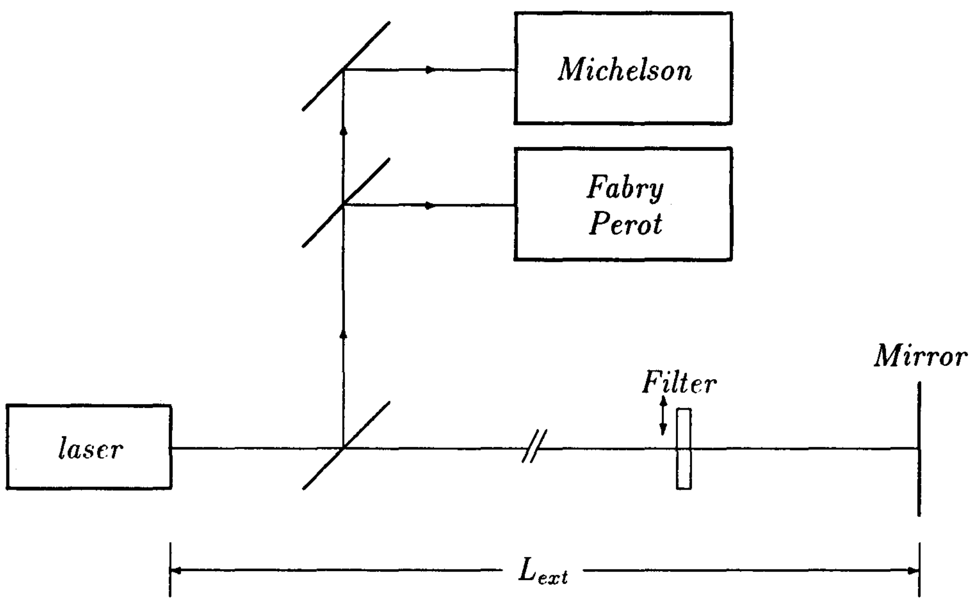

Figure 3.2: *Laser with optical feedback*

laser displays extended cavity operation with a narrow linewidth. Here, the laser operates as a long cavity laser with a short active region and is relatively insensitive to additional external optical perturbations. Regime three is only observed in distributed feedback lasers (DFB lasers). In other types of diode lasers this regime is characterized by the undamping of relaxation oscillations (regime four in Tkach & Chraplyvy). Then, regime four is solely determined by the observation of the coherence collaps.

The first complete and consistent set of equations for the laser with feedback (Figure 3.2) were given by Lang & Kobayashi in 1980 [5]. They form the basis for the theoretical analysis of the laser with optical feedback. As a convenient exercise for the analysis of the laser with phase conjugated feedback, some of the results for ordinary weak optical feedback will be rederived. The stationary state solutions of the Lang & Kobayashi equations are derived in section 3.2. Their stability, including the undamping of relaxation oscillations, will be analyzed in section 3.3.

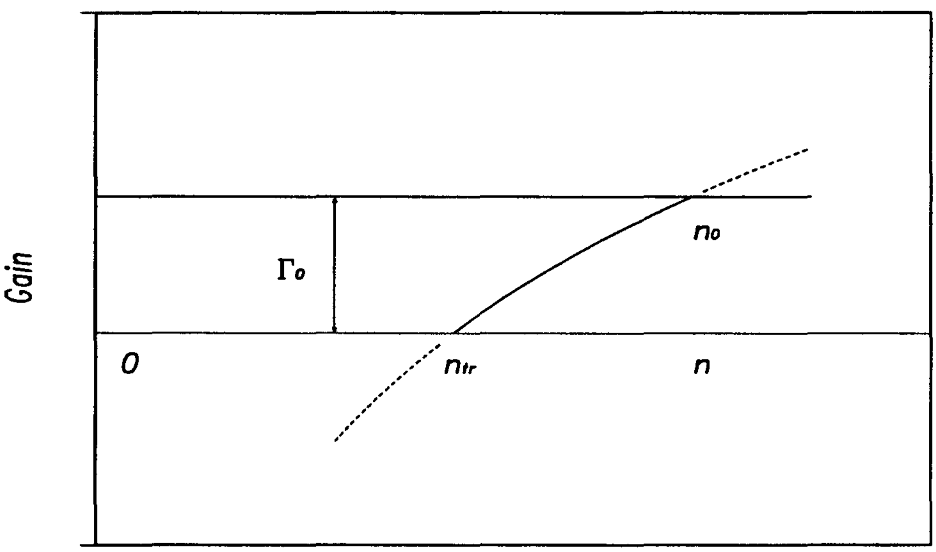

Figure 3.3: *Real part of the complex gain versus the number of electron-hole pairs. n_{tr} is the number of electron-hole pairs at transparancy and n_0 the number of electron-hole pairs at threshold.*

3.2 The Lang and Kobayashi equations

Following Lang & Kobayashi [5], the optical electric field in the laser cavity is represented as $Re\{E(t)e^{i\Omega t}\}$. Here $E(t)$ is a slowly varying envelope of the complex electric field in the active layer and Ω an angular optical frequency. As long as Ω is not specified, this representation is ambiguous since it contains two degrees of freedom for the frequency. One is hidden in $E(t)$ and the other one is Ω itself. However, this ambiguity can be an advantage, because it is now possible to make a choice for Ω later. For instance, Lenstra *et al.* [24] choose Ω to be equal to the laser frequency in the absence of feedback, while Lang & Kobayashi [5] take Ω equal to the cavity resonance frequency, i.e. the eigenfrequency at transparency, see Figure 3.3. It should be emphasized that all choices for Ω are equivalent, in the sense that in any case the total field is independent of this choice (as it should be). In this derivation Ω will be chosen as Ω_{tr}, i.e. Ω at transparancy, given by:

$$\Omega_{tr} = N\pi\frac{c}{\eta L_1} \tag{3.1}$$

where the integer N is the mode resonant frequency number, c the light velocity, η the refractive index in the active layer and L_1 the length of the diode cavity. Lang & Kobayashi describe the influence of an external mirror by adding an extra term to the standard, solitary, laser equation for the complex electric field. Thus the Lang & Kobayashi equations for the complex amplitude read:

$$\frac{d}{dt}E(t) = \frac{1}{2}\left\{G_A[n(t)] + iG_D[n(t)] - \Gamma_0\right\} E(t) + \gamma E(t-\tau)e^{-i\Omega\tau} \quad (3.2)$$

$$\frac{d}{dt}n(t) = -\frac{n(t) - n_{eq}}{T_1} - G_A[n(t)]|E(t)|^2 \quad (3.3)$$

$n(t)$ is the number of electron-hole pairs in the active layer, which are created at a rate $n_{eq}/T_1 = J/ed$, where J is the applied pump-current, e the electron charge and d the thickness of the active layer. Furthermore, $n(t)$ is annihilated by spontaneous emission, with lifetime T_1, and by stimulated emission (second term on r.h.s. in (3.3)). The complex gain function $G[n(t)]$ [21,27,28], is separated in a real, absorptive, part $G_A[n(t)]$ and an imaginary, dispersive, part $G_D[n(t)]$. The gain is normalized so that it is exclusively due to the active material and equals zero at transparancy (Figure 3.3). The real part of the gain is responsible for the intensity gain in the active layer [29]. The imaginary part describes a frequency shift, caused by changes in the real part of the index of refraction, due to variations in n. This effect cannot be neglected with respect to the feedback induced frequency shift. The quantity Γ_0 is the diode cavity intensity loss rate and accounts for the diffraction of photons and the outcoupling of the cavity, see Figure 3.3. The electric field amplitude $E(t)$ is normalized so that the intensity $P(t) = |E(t)|^2$ equals the number of photons in the cavity mode. This gives a direct connection between the number of photons and the number of electron-hole pairs in the laser cavity.

The external mirror is at a distance L from the laser, so the feedback delay time is $\tau = 2L/c$. The amount of feedback is given by the feedback rate γ as:

$$\gamma = \frac{1}{\tau_{in}}(1-R)\left(\frac{r}{R}\right)^{1/2} \quad (3.4)$$

where τ_{in} is the roundtrip time of the light in the diode. The power reflectivity of the laser facets is denoted as R and that of the external mirror as r.

The rate-equations (3.2) and (3.3) are limited in their description of the laser system to the following situations. First, the laser is operating in one single longitudinal mode. Second, the external delay time τ is much longer than the internal roundtrip time. The amount of feedback is considered to be small and multiple reflections in the external cavity are neglected. A description of a laser with feedback where multiple reflections are not neglected is given, for instance, by Mørk [30]. Lenstra and Cohen [24] have studied the laser system with moderate amounts of feedback, but in their case multiple external reflections could still be neglected. The stationary state solutions and the stability of (3.2) and (3.3) have been extensively studied [5,21,22,23,24].

Since the solitary laser can be studied separately, it will prove advantageous to write (3.2) and (3.3) in terms of parameters solely determined by the solitary laser. Here, the solitary laser is the laser without feedback ($\gamma = 0$), but under precisely the same stationary state conditions, that is, the same pump current, i.e. the same value of n_{eq}. Furthermore, the laser is assumed to operate far above threshold. Let the feedback-free stationary state be characterized by an intensity P_0, an electron-hole pair number n_0 and a frequency ω_0. It follows from (3.2) and (3.3) that:

$$G_A(n_0) = \Gamma_0; \quad G_D(n_0) = 2(\omega_0 - \Omega_{tr}); \quad \frac{n_0}{T_1} + \Gamma_0 P_0 = \frac{n_{eq}}{T_1} \tag{3.5}$$

Linearizing G_A and G_D around n_0 and using conditions (3.5) yields:

$$G_A[n(t)] = \Gamma_0 + \xi[n(t) - n_o]; \quad G_D[n(t)] = 2(\omega_0 - \Omega_{tr}) + \eta[n(t) - n_0] \tag{3.6}$$

where $\xi = dG_A/dn|_{n=n_0}$ and $\eta = dG_D/dn|_{n=n_0}$. By linearizing solely round n_0, it has been assumed that the complex gain does not depend on the frequency or the temperature [5]. All quantities, except the frequency mismatch, which is the difference between the operating frequency with feedback and Ω_{tr}, are now given at threshold for the solitary laser. Solutions obtained for feedback, will refer to the situation without feedback, except the frequency. Therefore, the frequency mismatch due to feedback will also be given with respect to the solitary laser solution (ω_0). Furthermore, α is introduced as $\alpha = \eta/\xi$ [21]. Now (3.2) and (3.3) can be completely written in quantities referring to the solitary laser as:

$$\frac{d}{dt}E(t) = \frac{1}{2}\xi(1+i\alpha)[n(t)-n_0]E(t) + \gamma E(t-\tau)e^{i\omega_0\tau} \tag{3.7}$$

$$\frac{d}{dt}[n(t)-n_0] = -\left\{\frac{1}{T_1} + \xi P(t)\right\}[n(t)-n_0] - \Gamma_0\left[P(t)-P_0\right] \tag{3.8}$$

The α-parameter, also called linewidth enhancement- [24] or anti-guiding [31] parameter, in (3.7), is an extremely important concept in the diode laser culture. Its counterpart in a laser operating on an atomic transition is the dispersive gain parameter, or effective index of refraction, which may become sizeable when the laser is slightly tuned away from line-center. In a diode laser α cannot be tuned; it is a material constant and its value may range from 2 (new generation visible quantum-well laser) to 7 (classical infrared diode laser). The typically large values of α are considered a "bad" property. It makes the laser much more sensitive to feedback than it would be if α could be made to vanish. A lot of present day research on diode lasers aims at reducing α.

Whether or not (3.7) and (3.8) allow a single frequency stationary state solution, can be examined. Stationary state solutions are time-independent, but do not necessarily have the same frequency as the solitary laser. Therefore, the possible solutions are written as:

$$E(t) = \sqrt{P_s}e^{i\Delta\omega t + i\phi_0}; \qquad n(t) = n_s \tag{3.9}$$

where P_s, $\Delta\omega$ and n_s are time-independent. Application of (3.9) to (3.7) and (3.8) shows that such stationary state solutions indeed exist and are characterized by:

$$\Delta\omega\tau = C\sin[\arctan\alpha - (\omega_0+\Delta\omega)\tau] \tag{3.10}$$

$$P_s = P_0 + 2\gamma\frac{\left(P_0 + \frac{1}{\xi T1}\right)\cos[(\omega_0+\Delta\omega)\tau]}{\Gamma_0 - 2\gamma\cos[(\omega_0+\Delta\omega)\tau]} \tag{3.11}$$

$$n_s = n_0 - 2\frac{\gamma}{\xi}\cos[(\omega_0+\Delta\omega)\tau] \tag{3.12}$$

where the dimensionless parameter C is the "dressed" feedback parameter, defined as:

$$C = \gamma\tau\sqrt{1+\alpha^2} \tag{3.13}$$

From (3.10) can be seen that the system parameter C separates the operation of the laser system in two regimes. The first is $C < 1$, where there is only one solution. This can be seen as a perturbation on the solitary laser mode due to external feedback. The other one is given by $C > 1$, where more than one stable solution can exist. Each one of these modes can be seen as a perturbed external mode [24].

3.3 Stability of the stationary state solution

In order to investigate whether a particular solution represents a *dynamically* stable state, small perturbations are added to the stationary state solution (3.9), yielding:

$$E(t) = \sqrt{P_s + \delta P(t)}e^{i[\Delta\omega t + \phi(t)]}; \quad n(t) = n_s + \delta n(t) \tag{3.14}$$

Substituting these expressions in (3.7) and (3.8) and assuming that the fluctuating quantities $\delta P(t)$, $\delta n(t)$ and $\phi(t)$ remain small, linearization leads to:

$$\frac{d}{dt}\delta P(t) = \xi P_s \delta n(t) - 2\gamma P_s \Delta\phi(t)\sin(\omega_s\tau) - \gamma\Delta P(t)\cos(\omega_s\tau) \tag{3.15}$$

$$\frac{d}{dt}\phi(t) = \frac{1}{2}\xi\alpha\delta n(t) - \gamma\Delta\phi(t)\cos(\omega_s\tau) + \frac{1}{2}\gamma\frac{\Delta P(t)}{P_s}\cos(\omega_s\tau) \tag{3.16}$$

$$\frac{d}{dt}\delta n(t) = -\left\{\frac{1}{T_1} + \xi P_s\right\}\delta n(t) - \{\Gamma_0 - 2\gamma\cos(\omega_s\tau)\}\,\delta P(t) \tag{3.17}$$

where $\Delta P(t) = \delta P(t) - \delta P(t-\tau)$, $\Delta\phi(t) = \phi(t) - \phi(t-\tau)$ and $\omega_s = \omega_0 + \Delta\omega$. The stability of a solution P_s, n_s and $\Delta\omega$ can be analyzed by seeking solutions for $\delta P(t)$, $\delta n(t)$ and $\phi(t)$ of the form e^{st}. Applying this to (3.15), (3.16) and (3.17) gives an eigenvalue problem. The solutions of this problem, the eigenvalues s, are determined by the secular determinant $D(s)$, i.e. the coefficient determinant, of these equations [32]. Only if all zeroes of $D(s)$ have a *negative* real part (or are zero), the stationary state solution is considered to be stable [32]. This determinant can be written as $D(s) = sN(s)$, where $N(s)$ is given as:

$$N(s) = \left\{s + \frac{1}{T_1} + \xi P_s\right\}\left\{s + 2\gamma\cos(\omega_s\tau)(1 - e^{-s\tau}) + \gamma^2\frac{(1-e^{-s\tau})^2}{s}\right\} +$$

$$\xi P_s \{\Gamma_0 - 2\gamma\cos(\omega_s\tau)\}\left\{1 + C\cos(\arctan\alpha - \omega_s\tau)\frac{(1-e^{-s\tau})}{s\tau}\right\} \quad (3.18)$$

From this expression a necessary condition for stability can be found. This condition is obtained by the consideration that, if s is real and $s \to \infty$, then $N(s) \to s^2$. If $N(0) < 0$, then $N(s)$ will have at least one zero on the positive real axis, thus giving rise to an instable solution. Since $N(0) = \xi P_s\{\Gamma_0 - 2\gamma\cos(\omega_s\tau)\}\{1 + C\cos(\arctan\alpha - \omega_s\tau)\}$ and $\gamma \ll \Gamma_0/2$ the condition becomes:

$$1 + C\cos(\arctan\alpha - \omega_s\tau) > 0 \quad (3.19)$$

However, this condition need not be sufficient for stability [24]. An illustration of that statement can be found by examining the root of $N(s) = 0$, which corresponds to a perturbed relaxation oscillation, with frequency ω'_R and damping constant λ'_R. Now, if $\omega_R \gg \gamma$, where ω_R is the relaxation oscillation in the absence of feedback, and $\gamma \ll \Gamma_0$, which implies $P_s \cong P_0$, $N(s)$ can be approximated in the neighborhood of its root as:

$$N(s) \cong \left(s + \frac{1}{T_1} + \xi P_0\right)\left\{s + 2\gamma\cos(\omega_s\tau)(1 - e^{-s\tau})\right\} + \xi\Gamma_0 P_0 \quad (3.20)$$

For $\gamma = 0$, i.e. without feedback, the zero's of (3.20) are given by:

$$s = -\frac{1}{2}\left(\frac{1}{T_1} + \xi P_0\right) \pm i\left[\xi\Gamma_0 P_0 - \frac{1}{4}\left(\frac{1}{T_1} + \xi P_0\right)^2\right]^{1/2} = -\lambda_R \pm i\omega_R \quad (3.21)$$

where λ_R is the damping constant in the absence of feedback given by:

$$\lambda_R = \frac{1}{2}\left(\frac{1}{T_1} + \xi P_0\right) \quad (3.22)$$

The same expressions for λ_R and ω_R can also be found in Yariv or Agrawal & Dutta [28,31], where they have been derived by a different method. In the presence of feedback and making the reasonable assumption $\omega'_R = \omega_R$, the zero's of (3.20) are given by:

$$s = -\lambda'_R \pm i\omega_R \quad (3.23)$$

where the damping constant λ'_R:

$$\lambda'_R \cong \lambda_R + \gamma\cos(\omega_s\tau)\{1 - \cos(\omega_R\tau)\} \quad (3.24)$$

if $\lambda'_R\tau \ll 1$. The relaxation oscillations will undamp if the real part of (3.23) becomes greater than zero, so when:

$$\gamma \cos(\omega_s \tau) < \frac{-\lambda_R}{1-\cos(\omega_R \tau)} \tag{3.25}$$

This illustrates that (3.19) is a *necessary* condition for stability, but not always *sufficient*. In the derivation of a similar condition for the undamping of relaxation oscillations Cohen *et al.* find an expression where γ has been replaced by $\frac{1}{2}\gamma\sqrt{1+\alpha^2}$.

In the theory presented so far, the effects of noise, which, in semiconductor lasers, is dominated by spontaneous emission, have been neglected. Even though spontaneous emission is the fundamental mechanism that makes laser operation possible, it can be neglected when the laser operates far above threshold. Since then the intensity is mainly due to stimulated emission. If processes based on random fluctuating forces, like linewidth broadening, are considered, spontaneous emission must be taken into account. A detailed study of the effects of noise can be found in [22,23,24,33,34].

Chapter 4

The diode laser with weak phase conjugated feedback

4.1 Introduction

In the previous chapter, the diode laser with normal feedback has been studied. Some results concerning the operating characteristics of such a system were derived. Recently a lot of research has been done on diode laser systems with some form of phase conjugated feedback [7,8,9,10,11,12,20]. This research has mainly been restricted to phase conjugated feedback that is generated by self-pumping, i.e. through degenerate four-wave mixing (DFWM). Much of this research is experimental [8,9,10,11,20]. A theoretical model, based on the modified Lang & Kobayashi equations, has been presented by Agrawal & Klaus [12]. In their paper Agrawal & Klaus derive some very important concepts which are characteristic for the phase conjugated nature of the optical feedback. One of the most striking results of their analysis is that the overall phase of the optical electric field is determined by the phase conjugation mirror (PCM). This situation is quite different from normal feedback, since then the overall phase remains arbitrary. Another important difference between normal feedback and phase conjugated feedback is related to the laser frequency. In normal feedback, there are multiple solutions which correspond to the longitudinal modes of the external cavity. However, the physics of phase conjugated feedback implies that such external cavity mode solutions can no longer occur. This is due to the fact that the phase shift obtained on the way back to the mirror is compensated by the phase shift on the way to the laser (time inversion). Thus the stationary state solution is *independent* of the position of the PCM. As will be shown, the only effect of the phase conjugated feedback is to *shift* the stationary state frequency of the laser.

DFWM is a special case of nearly degenerate FWM (NDFWM), since in DFWM the reflected phase conjugated wave has exactly the same frequency as the input wave. In NDFWM the reflected wave does *not* have the same frequency as the input wave, there is a, small, frequency mismatch. This is due to a frequency difference between the input and the pump waves. Therefore, a study of the laser with phase conjugated feedback generated by NDFWM is more general than the analysis given by Agrawal & Klaus [12].

In this chapter the rate-equations will be derived for the laser with phase conjugated feedback generated by NDFWM. Furthermore, it will be shown that these rate-equations reduce to those used by Agrawal & Klaus if DFWM is assumed. However, it will also be shown that the solutions for NDFWM phase conjugated feedback are quite different from those obtained in the DFWM scheme.

4.2 Derivation of the rate-equations

When the external mirror in the laser system with feedback is replaced by a phase conjugation mirror, the rate equations will change. The method used to derive these modified equations is very similar to the method used for normal feedback. Since the number of electron-hole pairs is not directly affected by the feedback, but only through the rate-equation for the electric field, the equation governing the number of electron-hole pairs does not change. It therefore suffices to derive the equation for the electric field. Figure 4.1 is a schematic representation of the

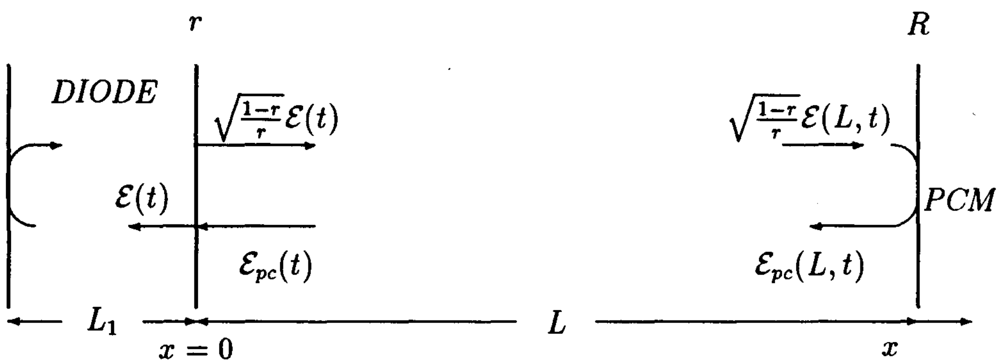

Figure 4.1: *Schematic representation of a laser with external feedback.* $\tau_1 = 2nL_1/c = 2l_d/c$ *is the internal roundtrip time,* n *the refractive index and* l_d *the optical length.* $\tau = 2L/c$ *is the external roundtrip time and* r *and* R *are the power reflectivities of the laser facet and the PCM.*

laser system. The electric field is represented as:

$$Field = E(t)e^{i\omega t} + c.c. = \mathcal{E}(t) + c.c. \tag{4.1}$$

The changes in time of the electric field can be expressed with a differential quotient as:

$$\frac{d}{dt}\mathcal{E}(t) = \frac{\mathcal{E}(t) - \mathcal{E}(t-\tau_1)}{\tau_1} \tag{4.2}$$

where $\mathcal{E}(t)$ is the complex electric field in the laser cavity and τ_1 is the internal roundtrip time $\tau_1 = 2l_d/c$, where l_d is the optical length of the laser cavity and c the velocity of light. The expression (4.2) is valid as long as the changes in the field are small during one roundtrip in the laser cavity. From Figure 4.1 it can be seen that the field $\mathcal{E}(t)$ at $x = 0$ has two contributions: one from the roundtrip in the laser cavity and one due to the external feedback, yielding:

$$\mathcal{E}(t) = G_r\mathcal{E}(t-\tau_1) + \sqrt{(1-r)}\mathcal{E}_{pc}(t) \tag{4.3}$$

so that

$$\mathcal{E}(t-\tau_1) = \frac{1}{G_r}\left\{\mathcal{E}(t) - \sqrt{(1-r)}\mathcal{E}_{pc}(t)\right\} \tag{4.4}$$

where G_r is the gain of the field in the medium in one roundtrip and r is the power reflectivity of the laser facet. Applying (4.4) to (4.2) gives:

$$\frac{d}{dt}\mathcal{E}(t) \cong \frac{G_r - 1}{\tau_1}\mathcal{E}(t) + \frac{1}{\tau_1}\sqrt{1-r}\mathcal{E}_{pc}(t) \tag{4.5}$$

where is used that $G_r \cong 1$, so that $1 - 1/G_r$ can be replaced by $G_r - 1$.

In order to solve (4.5), $\mathcal{E}_{pc}(t)$ must be written in terms of $\mathcal{E}(t)$. The electric field that exits the laser can be written as:

$$\mathcal{E}'(x,t) \cong \sqrt{\frac{1-r}{r}}\mathcal{E}(x,t) \tag{4.6}$$

The electric field is expressed as:

$$\mathcal{E}(x,t) = \int A_\omega e^{i(\omega t - kx)} d\omega \tag{4.7}$$

So that a component of the field incident on the PCM is given by $\sqrt{(1-r)/r}A_\omega e^{i\omega t}$ (at $x = 0$). At $x = L$, the PCM "sees" this component as $\sqrt{(1-r)/r}A_\omega e^{i(\omega t - kL)}$. The PCM generates a phase conjugated wave that satisfies the phase-matching condition. The frequency mismatch between the input wave and the pump waves yields a frequency $2\omega_p - \omega$ for the reflectd wave. This must be associated with the corresponding wavenumber $2k_p - k$. Furthermore, the phase of the component incident on the PCM will be conjugated so that a component of the field, reflected by the PCM, is given by:

$$\sqrt{\frac{(1-r)R}{r}}A_\omega^* e^{i[(2\omega_p-\omega)t + kL + (2k_p - k)(x-L)]}e^{i\Phi_{PCM}} \tag{4.8}$$

where Φ_{PCM} is a phase shift, induced by the PCM, given by (2.24), and $\sqrt{R}$ the effective reflectivity given by (2.22). This expression can be written in terms of the external roundtrip time τ so that the component of the phase conjugated field at $x = 0$ is given by:

$$\sqrt{\frac{(1-r)R}{r}}A_\omega^* e^{i[(2\omega_p-\omega)t - (\omega_p - \omega)\tau]}e^{i\Phi_{PCM}}$$

Assume that, if $\mathcal{E}(t)$ is sharply peaked around ω, the PCM acts the same for *every* frequency component of $\mathcal{E}(t)$. Then the total phase conjugated field at $x = 0$ can be written as:

$$\mathcal{E}_{pc}(t) = \sqrt{\frac{(1-r)R}{r}} e^{2i\omega_p(t-1/2\tau)} e^{i\Phi_{PCM}} \int A^*_\omega e^{-i\omega(t-\tau)} d\omega \tag{4.9}$$

Comparing (4.9) with (4.7) yields:

$$\mathcal{E}_{pc}(t) = \sqrt{\frac{(1-r)R}{r}} e^{2i\omega_p(t-1/2\tau)} e^{i\Phi_{PCM}} \mathcal{E}^*(t-\tau) \tag{4.10}$$

If (4.10) is applied to (4.5) then the derivative with respect to time for the electric field is given by:

$$\frac{d}{dt}\mathcal{E}(t) = \left(\frac{G_r - 1}{\tau_1}\right) \mathcal{E}(t) + \frac{1}{\tau_1}(1-r)\sqrt{\frac{R}{r}} e^{2i\omega_p t} e^{i\Phi_{PCM}} \mathcal{E}^*(t-\tau) e^{-i\omega_p \tau} \tag{4.11}$$

This yields the following expression for the complex amplitude:

$$\frac{d}{dt}E(t) = \left(\frac{G_r - 1}{\tau_1} - i\omega\right) E(t) + \gamma E^*(t-\tau) e^{2i(\omega_p - \omega)(t-\tau/2)} e^{i\Phi_{PCM}} \tag{4.12}$$

where γ is the feedback rate coefficient given by:

$$\gamma = \frac{1}{\tau_1}(1-r)\sqrt{\frac{R}{r}} \tag{4.13}$$

The complex function $(G_r - 1)/\tau_1$ can be written as $\frac{1}{2}\left(G_A[n(t)] + iG_D[n(t)] - \Gamma_0\right) +$ $i\omega$. The imaginary part of the gain G_D is chosen so that it equals zero if $\omega = \omega_0$. So (4.12) can be written as:

$$\frac{d}{dt}E(t) = \frac{1}{2}\left\{G_A(n) + iG_D(n) - \Gamma_0\right\} E(t) + \gamma E^*(t-\tau) e^{2i(\omega_p - \omega_0)(t-\tau/2)} e^{i\Phi_{PCM}} \tag{4.14}$$

The rate-equation for the number of electron-hole pairs is given by:

$$\frac{d}{dt}n(t) = -\frac{n(t) - n_{eq}}{T_1} - G_A[n(t)]|E(t)|^2 \tag{4.15}$$

Equations (4.14) and (4.15) describe the laser subject to weak phase conjugated feedback generated by NDFWM.

If the DFWM scheme is assumed, the wave reflected by the PCM is given by:

$$\sqrt{\frac{(1-r)R}{r}} A^*_\omega e^{i[\omega t + kL + k(x-L)]} e^{i\Phi_{PCM}}$$

so that the time dependent factor with $(\omega_p - \omega_0)$ in (4.14) vanishes and the rate-equations as given by Agrawal & Klaus [12] are obtained.

Reflection from a normal mirror yields a reflected wave given by:

$$\sqrt{\frac{(1-r)R}{r}}A_\omega e^{i[\omega t - kL + k(x-L)]}$$

so that (4.14) becomes the rate-equation for normal feedback, as, e.g., given by Lenstra *et al.* [24].

4.3 Stationary state solutions

The rate-equations for the laser with normal feedback were written in quantities referring to the solitary laser. If this is done for the laser with phase conjugated feedback (4.14) and (4.15) become:

$$\frac{d}{dt}E(t) = \frac{1}{2}\xi(1+i\alpha)n_s(t)E(t) + \gamma E^*(t-\tau)e^{2i(\omega_p-\omega_0)(t-\tau/2)}e^{i\Phi_{PCM}} \quad (4.16)$$

$$\frac{d}{dt}n_s(t) = -\left\{\frac{1}{T_1} + \xi|E(t)|^2\right\}n_s(t) - \Gamma_0\{P(t) - P_0\} \quad (4.17)$$

where $n_s(t) = n(t) - n_0$. Furthermore, n_0 is the number of electron-hole pairs, ω_0 the operating frequency and P_0 the number of photons in the absence of feedback. The alpha-parameter α is defined as η/ξ, with $\eta = dG_D/dn$ and $\xi = dG_A/dn$, where both derivatives are taken at $n = n_0$.

As in normal feedback, it is now possible to search for stationary state solutions. Since this corresponds to a harmonic field, it can be written as:

$$E(t) = \sqrt{P_s}e^{i\Delta\omega t + i\phi_0}; \qquad n_s(t) = n_s. \quad (4.18)$$

Applying these expressions to (4.16) and (4.17) yields:

$$\Delta\omega = \omega_p - \omega_0 \quad (4.19)$$

which implies that the harmonic solution can only exist if the laser frequency locks to the pump frequency ω_p. The possible solutions are characterized by:

$$\Delta\omega\tau = -C\sin(\arctan\alpha + \theta) \tag{4.20}$$

$$n_s = -2\frac{\gamma}{\xi}\cos\theta \tag{4.21}$$

$$P_s = P_0 + 2\gamma\cos\theta\frac{P_0 + \frac{1}{\xi T_1}}{\Gamma_0 - 2\gamma\cos\theta} \tag{4.22}$$

where $\theta = 2\phi_0 - \Phi_{PCM}$ and $C = \gamma\tau\sqrt{1+\alpha^2}$ the dressed feedback parameter. According to (4.20) the overall phase ϕ_0 is known, i.e. it locks. Another requirement that has to be satisfied follows from (4.20). Since $\sin(\arctan\alpha + \theta) \leq 1$, so:

$$|\Delta\omega\tau| \leq C \tag{4.23}$$

must be met if a harmonic solution exists. From (4.23) it can be seen that, if $\omega_0 \neq \omega_p$, C can always be chosen so that (4.23) is not satisfied. This means that, if $\omega_0 \neq \omega_p$, *the stationary state solution does* not *simply reduce to the solitary laser solution* $\gamma \longrightarrow 0$. So, if the amount of feedback is continuously reduced to zero, the laser *must* become *unstable* first before it can reach the *stable* solution at $\gamma = 0$.

4.4 Frequency difference solution

If the harmonic solution satisfies the rate-equations (4.16) and (4.17) then (4.19) says that the laser operates at frequency $\omega = \omega_p$, i.e. the frequency locks. According to (4.23) the state where the laser frequency locks to ω_p cannot be reached for very weak feedback if $\omega_p \neq \omega_0$. Apparently (4.23) must be interpreted as a locking condition. This condition implies that a too small amount of feedback will not be enough to lock the frequency to that of the pump. Or, the farther the pump frequency is away from the operating frequency of the solitary laser, the more feedback is required to keep the frequency locked to that of the pump.

Since the locking condition is not always satisfied, we must consider what might happen when the solitary laser suddenly experiences phase conjugated feedback. Suppose there is just a small amount of feedback. The light emitted from the laser will be phase conjugated. The reflected wave will enter the laser with a frequency difference with respect to the solitary laser frequency given by $\nu = 2(\omega_p - \omega_0)$. But if the amount of feedback is too small, it may not be sufficient to lock the laser frequency to that of the pump. This suggests that the laser will *remain* operating at its solitary laser frequency, but with a small additional field component at frequency $2\omega_p - \omega_0$. The complex amplitude of the total field can then be written as:

$$E(t) = \left\{E_\gamma + \mathcal{E}_1 e^{i\nu t + i\phi_1}\right\} e^{i\phi_0} \tag{4.24}$$

where $E_\gamma = E_0 + \Delta E_0$ and $\mathcal{E}_1$ the complex amplitude of the additional field. Here, ΔE_0 is a small correction to the amplitude of the solitary laser due to feedback. Since the rate-equations are coupled, and the rate-equation for $n(t)$ contains a term $|E(t)|^2$, the laser then generates a field proportional to $|E(t)|^2$. This effect is illustrated in Figure 4.2. Using (4.24), $|E(t)|^2$ can be calculated. From that expression can be seen that the laser generates a solitary laser field and fields with frequencies proportional to $\pm\nu$. If only first order term involving $\mathcal{E}_1$ are considered, which can be done if $\mathcal{E}_1$ is small, the complex amplitude of the field generated by the laser can be written as:

$$E(t) = \left\{E_\gamma + \mathcal{E}_1 e^{i\nu t + i\phi_1} + \mathcal{E}_2 e^{-i\nu t + \phi_2}\right\} e^{i\phi_0} \tag{4.25}$$

Applying (4.25) to the rate-equations (4.14) and 4.15), which is done in Appendix C, yields:

$$\Delta E_0 = 0 \tag{4.26}$$

$$\mathcal{E}_1 = \frac{1}{2}\left|\frac{\gamma E_0}{\kappa\nu}\right| \left\{4\nu\left[\nu(1-\beta) - \alpha\beta\lambda_R\right] + \beta^2\left(1+\alpha^2\right)\left(\lambda_R^2+\nu^2\right)\right\}^{\frac{1}{2}} \tag{4.27}$$

$$\mathcal{E}_2 = \frac{1}{2}\left|\frac{\gamma E_0}{\kappa\nu}\right| \beta\left\{\left(1+\alpha^2\right)\left(\lambda_R^2+\nu^2\right)\right\}^{\frac{1}{2}} \tag{4.28}$$

$$\tan\phi_1 = -\frac{2\nu^2 - \nu\beta\left(3\nu + \alpha\beta\lambda_R\right) + \beta^2\left(\lambda_R^2+\nu^2\right)}{\alpha\beta^2\left(\lambda_R^2+\nu^2\right) - \beta\nu\left(\alpha\nu - \lambda_R\right)} \tag{4.29}$$

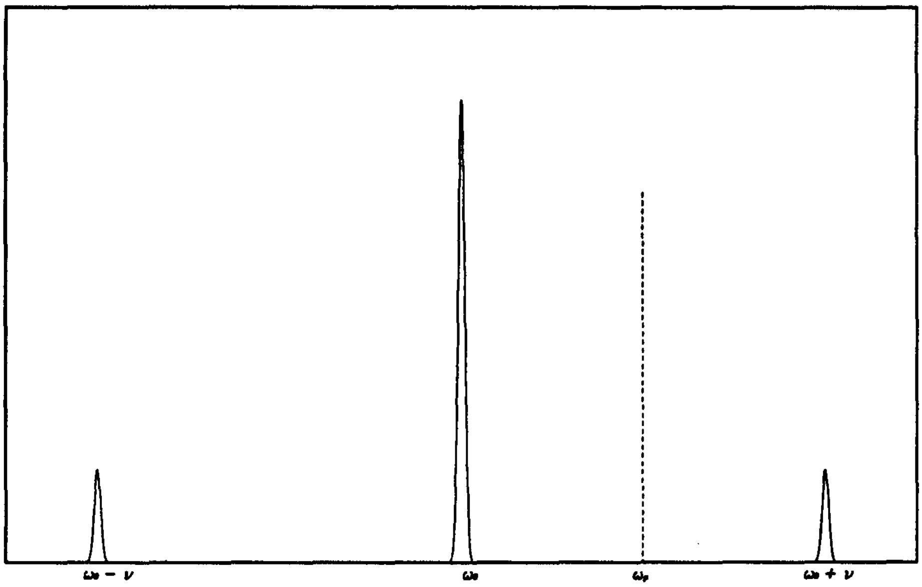

Figure 4.2: *Schematic representation of the effect of phase conjugated feedback on the frequency.* ω_0 *is the solitary laser frequency,* ω_p *the frequency of the pump waves and* $\nu = 2(\omega_p - \omega)$ *is the frequency difference of the feedback field with respect to* ω_0.

$$\tan\phi_2 = \frac{\beta^2(\lambda_R^2 + \nu^2) - \nu(\alpha\lambda_R - \nu)}{\alpha\beta^2(\lambda_R^2 + \nu^2) - \beta\nu(\alpha\nu - \lambda_R)} \tag{4.30}$$

where λ_R is the damping constant of the solitary laser and ν, κ and β are given by:

$$\nu = 2(\omega_p - \omega_0) \tag{4.31}$$

$$\kappa = \sqrt{\nu^2(1-\beta)^2 + \lambda_R^2\beta^2} \tag{4.32}$$

$$\beta = \xi\frac{E_0^2\Gamma_0}{\lambda_R^2 + \nu^2} \tag{4.33}$$

The solutions for $\mathcal{E}_1$ and $\mathcal{E}_2$, (4.27) and (4.28), are proportional to γ. This means that, for $\gamma \longrightarrow 0$, the solution for $E(t)$ reduces to the solitary laser solution, as it should. Since both the root and κ in (4.27) and (4.28) are proportional to ν, $\mathcal{E}_1$ and $\mathcal{E}_2$ are inversely proportional to ν. So this solution is expected to be valid for values of ν that are large compared to γ. Note that it is by no means proven that a solution of the kind (4.25) indeed exists. However, if it exists, it has to satisfy (4.27) - (4.30).

4.5 Stability of the stationary state solutions

As was done for normal feedback, the stability of the possible stationary state solutions can be examined. The time-dependent complex amplitude and the number of electron-hole pairs are now given as:

$$E(t) = \sqrt{P_s + \delta P(t)} e^{i\Delta\omega t + i\phi(t) + i\phi_0}; \qquad n(t) = n_s + \delta n(t) \tag{4.34}$$

If expressions (4.34) are applied to the rate-equations (4.16) and (4.17), the following set of equations is, after linearization, obtained:

$$\frac{d}{dt}\delta P(t) = \xi \delta n(t) P_s - 2\gamma P_s \Delta\phi \sin\theta - \gamma \Delta P \cos\theta \tag{4.35}$$

$$\frac{d}{dt}\phi(t) = \frac{1}{2}\alpha\xi\delta n(t) - \gamma\Delta\phi\cos\theta + \gamma\frac{\Delta P}{2P_s}\sin\theta \tag{4.36}$$

$$\frac{d}{dt}\delta n(t) = -\left(\frac{1}{T_1} + \xi P_s\right)\delta n(t) - (\Gamma_0 - 2\gamma\cos\theta)\,\delta P(t) \tag{4.37}$$

where $\theta = 2\phi_0 - \Phi_{PCM}$, $\Delta\phi = \phi(t) + \phi(t-\tau)$ and $\Delta P = \delta P(t) - \delta P(t-\tau)$. The stability of this system can be analyzed by seeking solutions for $\delta P(t)$, $\phi(t)$ and $\delta n(t)$ of the form e^{st}, as was done in section 3.3. Applying solutions of that form to (4.35), (4.36) and (4.37) yields a secular determinant $D(s)$, given by:

$$\begin{aligned} D(s) = {} & \left(s + \frac{1}{T_1} + \xi P_s\right)\left\{s^2 + 2s\gamma\cos\theta + \gamma^2\left(1 - e^{-2s\tau}\right)\right\} + \\ & \xi P_0\,(\Gamma_0 - 2\gamma\cos\theta)\left\{s + \frac{C}{\tau}\cos(\arctan\alpha + \theta)\left(1 + e^{-s\tau}\right)\right\} \end{aligned} \tag{4.38}$$

The zero's of $D(s)$ determine the stability of the system. If all zeros of $D(s)$ have a negative real part, the solutions are stable. As was the case with normal feedback, a necessary condition for stability can be found from (4.38). If s is real and $s \to \pm\infty$, then $D(s) \to s^3$. So, if $s \to -\infty$, $D(s) \to -\infty$ and if $s \to +\infty$, $D(s) \to +\infty$. This means that if $D(0) < 0$, there must be at least one zero of $D(s)$ with a positive real part of s, thus yielding an unstable solution. This condition can be written as:

$$\cos(\arctan\alpha + \theta) \geq 0 \tag{4.39}$$

In the stationary state (4.20) must apply. Here, an initial stationary state is assumed, so before the system can become unstable this relationship has to apply here as well, so that:

$$\theta = -\arcsin\left(\frac{\Delta\omega\tau}{C}\right) - \arctan\alpha \tag{4.40}$$

or, since $\theta + \arctan\alpha$ has two roots, y and $\pi - y$, where $y = -\arcsin(\Delta\omega\tau/C)$:

$$\theta = \pi + \arcsin\left(\frac{\Delta\omega\tau}{C}\right) - \arctan\alpha \tag{4.41}$$

Using (4.40), the condition (4.39) can be rewritten to:

$$\cos\left[\arcsin\left(\frac{\Delta\omega\tau}{C}\right)\right] \geq 0 \tag{4.42}$$

which is always satisfied since $-\pi/2 \leq \arcsin x \leq \pi/2$. If (4.41) is used in (4.39), that can be written as:

$$-\cos\left[\arcsin\left(\frac{\Delta\omega\tau}{C}\right)\right] \geq 0 \tag{4.43}$$

which is *never* satisfied, so the second root corresponds with an unstable solution. So, if there are stationary state solutions, the first root, i.e. (4.40), must apply.

This situation is quite different from the results found for normal feedback. In normal feedback, the stability of the laser was determined by the length of the external cavity and the dressed feedback parameter, see (3.19). In that case the laser output could become unstable for certain values of C and $\omega\tau$. Here, if the laser exhibits stationary state behaviour, no simple method such as changing the external roundtrip time suffices to obtain a different solution.

In the stability analysis for normal feedback, it was possible to obtain a condition for the undamping of relaxation oscillations. A similar analysis can be used to find such a condition for phase conjugated feedback. If the assumption is made that if the relaxation oscillations undamp the solution for s will be $\cong \omega_R$, where ω_R is the relaxation oscillation frequency of the solitary laser, the secular determinant (4.38) can be approximated by:

$$D(s) \cong (s + 2\lambda_R)\left\{s^2 + 2\gamma s \cos\theta\right\} + s\xi\Gamma_0 P_0 \tag{4.44}$$

where λ_R is the damping constant in the absence of feedback. This equation can be solved for the case when $D(s) = 0$, yielding:

$$s = -\lambda_R - \gamma\cos\theta \pm i\omega_R \tag{4.45}$$

so that the damping constant in the presence of feedback is given by:

$$\lambda'_R = \lambda_R + \gamma\cos\theta \tag{4.46}$$

where θ must satisfy condition (4.23). If this condition is applied to (4.46), the following condition for the undamping of relaxation oscillations in the presence of phase conjugated feedback is obtained:

$$\frac{1}{1+\alpha^2}\left\{\pm\left|\sqrt{\gamma^2(1+\alpha^2) - \Delta\omega^2}\right| - \alpha\Delta\omega\right\} < -\lambda_R \tag{4.47}$$

The possible undamping of the relaxation oscillation is determined by the boundary condition, $l.h.s. = -\lambda_R$. This condition can be written as:

$$\gamma^2 = \lambda_R^2 + (\alpha\lambda_R - \Delta\omega)^2 \tag{4.48}$$

This expression now suggests an asymmetry between a positive and negative frequency mismatch, which depends on α. Furthermore, condition (4.23) must be satisfied before the relaxation oscillation can undamp. The minimum amount of feedback necessary in order for the relaxation oscillations to undamp can be derived from (4.48). In that case $\Delta\omega$ must satisfy:

$$\Delta\omega = \alpha\lambda_R$$

Relationship (4.48) can be calculated, using the values from Table 4.6. If the values found for γ and $\Delta\omega$ are compared with the numerical results that are summarized in Figure 4.3, the asymmetric nature of the undamping of the relaxation oscillation for a positive or negative frequency mismatch is reproduced. However, the values calculated with (4.48) are substantially larger than those found numerically. So, even though this analytic expression for the undamping of the relaxation oscillation can be used as a condition that *must* be satisfied, the numerical calculations show that it is not strict enough. Summarizing, it can be said that the undamping of the relaxation oscillation for the laser with phase conjugated feedback has not yet been completely understood. A numerical analysis of the secular determinant may provide this understanding.

4.6 Numerical analysis

In the previous sections the rate-equations for the laser with phase conjugated feedback were derived. Furthermore, stationary state solutions and some conditions under which the laser may show stable behavior were found. As was mentioned in section 4.3, the stationary state solution does not directly reduce to the solitary laser solution, $\gamma \rightarrow 0$. There may exist another regime, where the laser oscillates at twice the frequency difference $|\omega_p - \omega_0|$. To check whether or not the analytical expressions and conditions indeed describe the dynamical behavior of the laser , the rate-equations have been solved numerically. For this analysis, the temporal evolution of the amplitude, phase and number of electron-hole pairs is expressed as differential quotient:

$$f(t + \Delta t) \cong f(t) + \Delta t \frac{d}{dt} f(t) \tag{4.49}$$

where Δt must be very small and $f(t)$ represents any of the functions $E(t)$, $n(t)$ or $\phi(t)$. The numerical scheme based on this approximation will not be very stable, and will only be valid if $f(t + \Delta t) \cong f(t)$. So, this method cannot be used to

Photon population	E_0^2	7.8 10^4	
alpha parameter	α	4	
carrier lifetime	T_1	1.2 10^{-9}	s
photon decay rate	Γ_0	6.4 10^{11}	s^{-1}
Gain derivative $(\delta G_A/\delta N)$	ξ	5.62 10^3	s^{-1}
phase	θ	0	
external roundtrip time	τ	1	ns
time interval	Δt	2 10^{-4}	τ

Table 4.1: *Values used for numerical calculations*

analyze chaotic behavior, but it can indicate whether or not chaotic behavior or quasi-chaotic behavior is to be expected. In that case the results will not be reproducible if, e.g., different integration steps Δt are taken.

Let the complex amplitude and the number of electron-hole pairs be given by:

$$E(t) = \{E_0 + \delta E(t)\}\, e^{i\phi_0 + i\phi(t)}; \qquad n(t) = n_0 + \delta n(t) \tag{4.50}$$

where E_0 and n_0 refer to the situation without feedback. The rate-equations can then be rewritten in terms of deviations from the solitary laser solution as:

$$\frac{d}{dt}\delta E(t) = \frac{1}{2}\xi\delta n(t)\,[E_0 + \delta E(t)] + \gamma\,\{E_0 + \delta E(t-\tau)\}\cos(\Delta\phi) \tag{4.51}$$

$$\frac{d}{dt}\phi(t) = \frac{1}{2}\alpha\xi\delta n(t) + \gamma\frac{E_0 + \delta E(t-\tau)}{E_0 + \delta E(t)}\sin(\Delta\phi) \tag{4.52}$$

$$\frac{d}{dt}\delta n(t) = -\left(\frac{1}{T_1} + \xi\,|E(t)|^2\right)\delta n(t) - \Gamma_0\left(|E(t)|^2 - E_0^2\right) \tag{4.53}$$

where $\Delta\phi = 2\Delta\omega(t-\tau/2) - [\phi(t) + \phi(t-\tau) + \theta]$. Using (4.49), δE, ϕ and δn can be calculated at $t + \Delta t$, if they are known at t.

A computer program to solve the rate-equations using (4.51), (4.52), (4.53) and based on the algorithm (4.49) was written. The parameters used in these calculations are taken from Agrawal & Dutta [31] and are presented in Table 4.6. Some of the typical results are presented in Figures 4.4 - 4.8. For very small amounts of feedback the laser exhibits behavior in accordance with the frequency difference solution as is shown in Figure 4.4. If the amount of feedback is increased, higher order effects, which have been neglected in the trial expression for this solution, can apparently no longer be omitted. These effects on the laser's operation are shown in Figure 4.5. For higher amounts of feedback, the predicted stationary state behavior is indeed found, as is shown in Figure 4.6. The level of feedback can be increased further, until the level at which the relaxation oscillation becomes undamped. This is shown in Figure 4.7. The calculated relaxation oscillation frequency in the presence of feedback is indicated in Table 4.2. The relaxation oscillation frequency in the presence of feedback deviates only a little from that of the solitary laser. Thus giving a numerical basis for the assumption, in the stability analysis, that the relaxation oscillation in the presence of feedback can be replaced with that of the solitary laser. More figures concerning laser operation in the different regimes can be found in Appendix D

As mentioned earlier, the numerical method for solving the rate-equations is rather crude, so the results have to be checked. This has been done by examining the reproducibility of the results obtained if the time interval Δt is changed. For $C < 8$ the results could be reproduced within 0.1% accuracy. For higher levels of feedback reproduction was no longer possible, indicating that the behavior might be chaotic. Another check that was performed was to see whether the laser behavior remained the same over a very long time interval. From this was learned that 50 external roundtrip times are usually sufficient to overcome the initial switch on effects and obtain stationary state results.

The calculations have been performed for a wide range of values for C and $\Delta\omega$. The results are summarized in one figure, Figure 4.3. According to this figure the laser operation can be classified in three regimes. In regime I the frequency of the laser is locked to that of the pump. In regime II the laser exhibits the frequency difference behavior and in regime III the relaxation oscillation is undamped. Furthermore, preliminary evidence is found that a fourth regime (IV) should be distinguished in which the laser possibly exhibits chaotic behavior. However, the

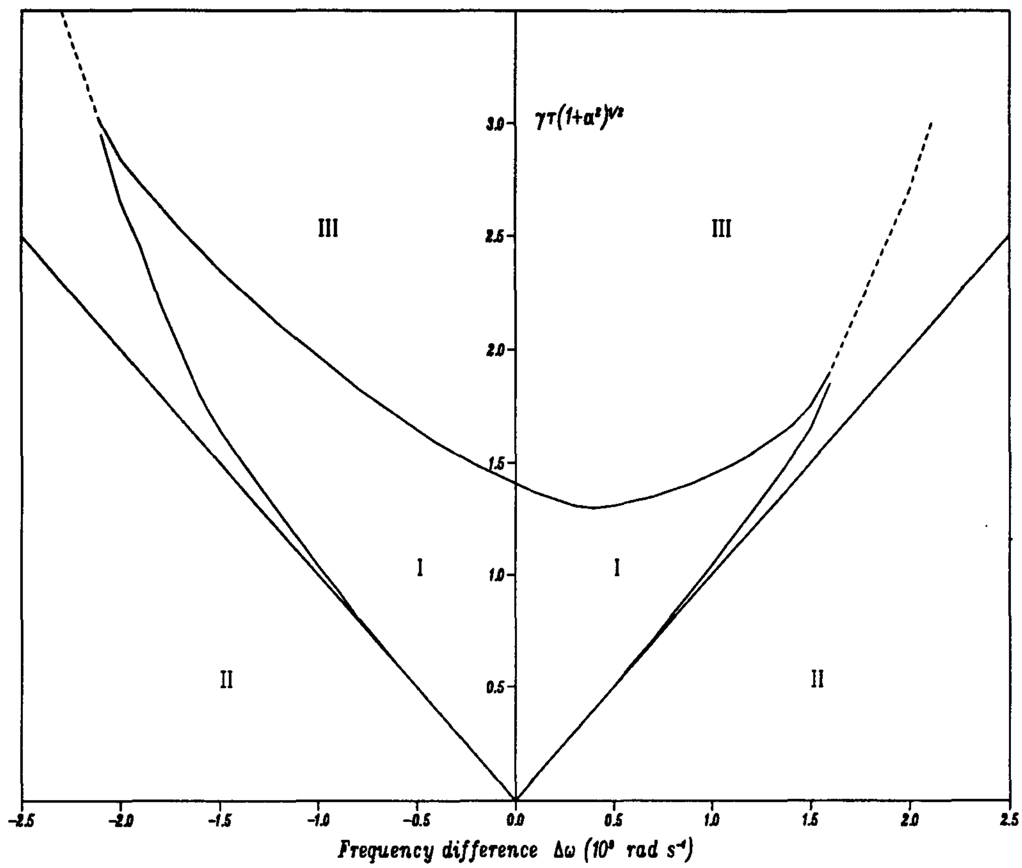

Figure 4.3: *Phase-space topology for the laser with PC-feedback*

method used for solving the equations is too crude to conclude this. Figure 4.3 can be seen as an analog of Figure 3.1 for the laser with normal feedback.

Figure	C	$\Delta\omega$ $10^9\ rad\ s^{-1}$	Frequency $10^9\ rad\ s^{-1}$	Relaxation Oscillation $10^9\ rad\ s^{-1}$
4.1	0.05	0.5	0.9958	-
4.2	0.3	0.5	-	-
4.3	0.8	0.5	-	-
4.4	2.0	0.5	17.21	16.69
4.5	10.0	0.5	-	-

Table 4.2: *Parameters and calculated values for Figures 4.1 - 4.5*

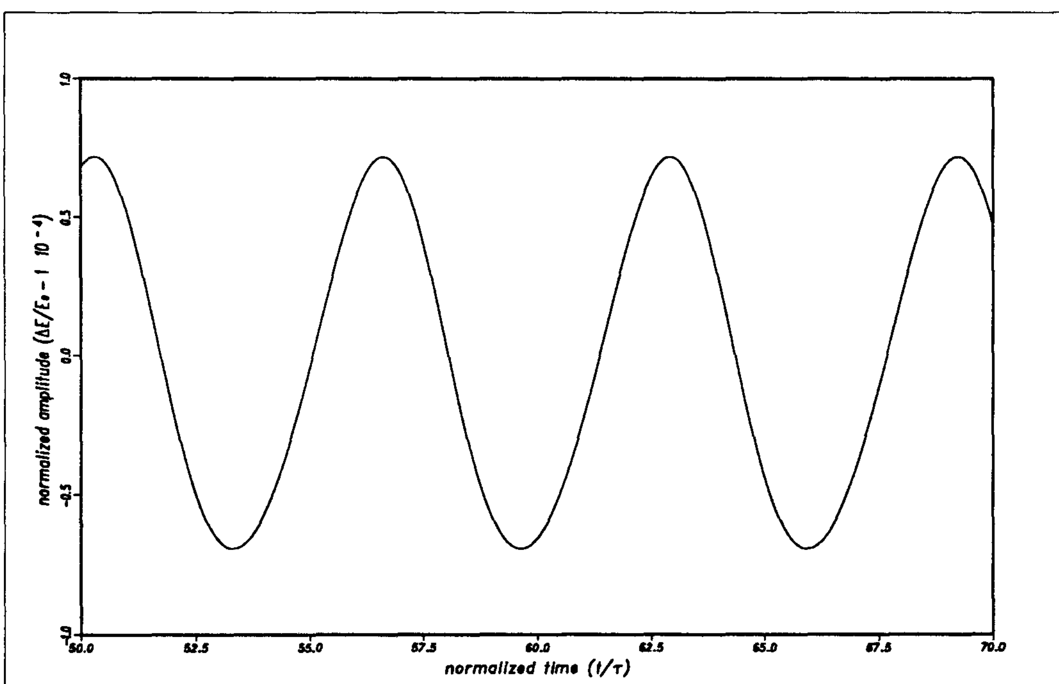

Figure 4.4: *Frequency difference behavior.*

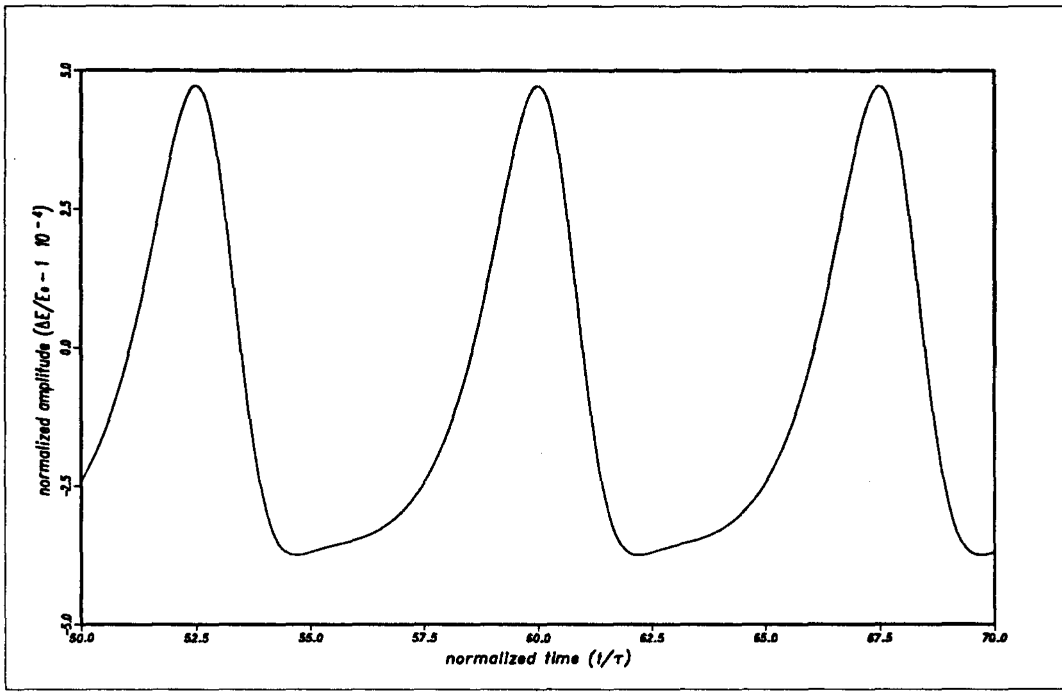

Figure 4.5: *Higher order effects can no longer be neglected.*

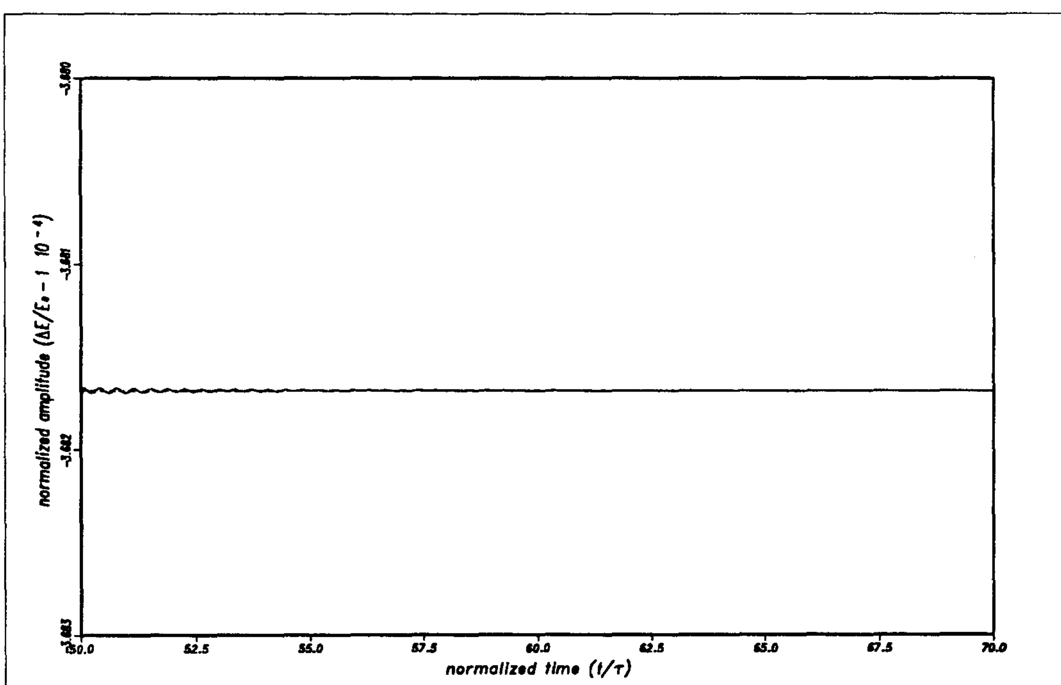

Figure 4.6: *Stationary state behavior.*

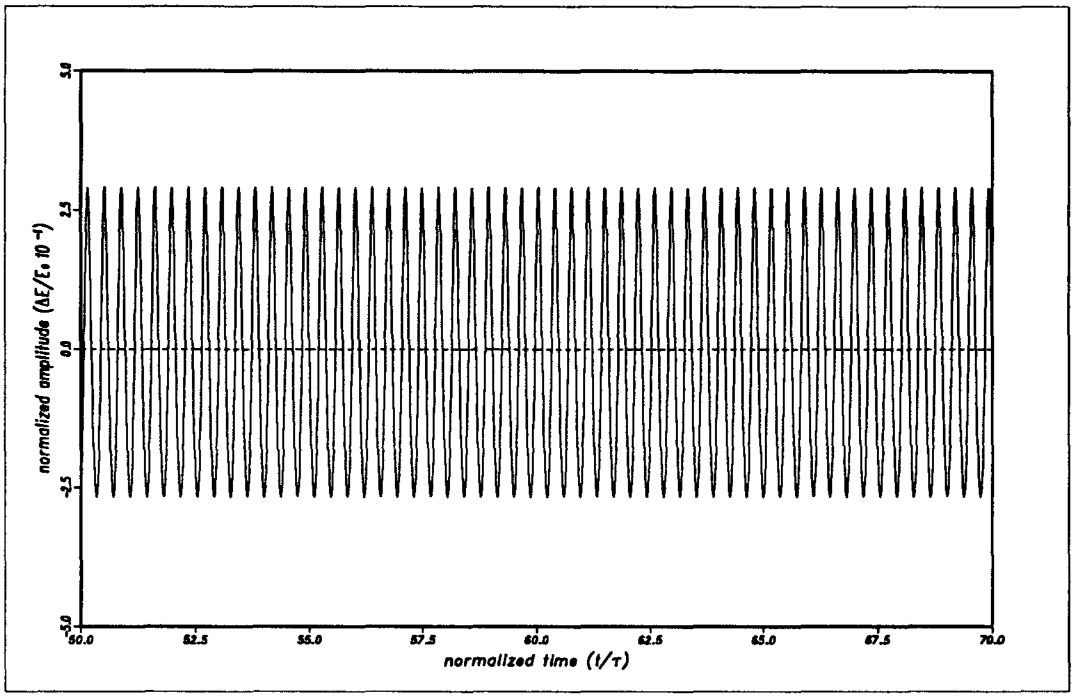

Figure 4.7: *Undamping of relaxation oscillations.*

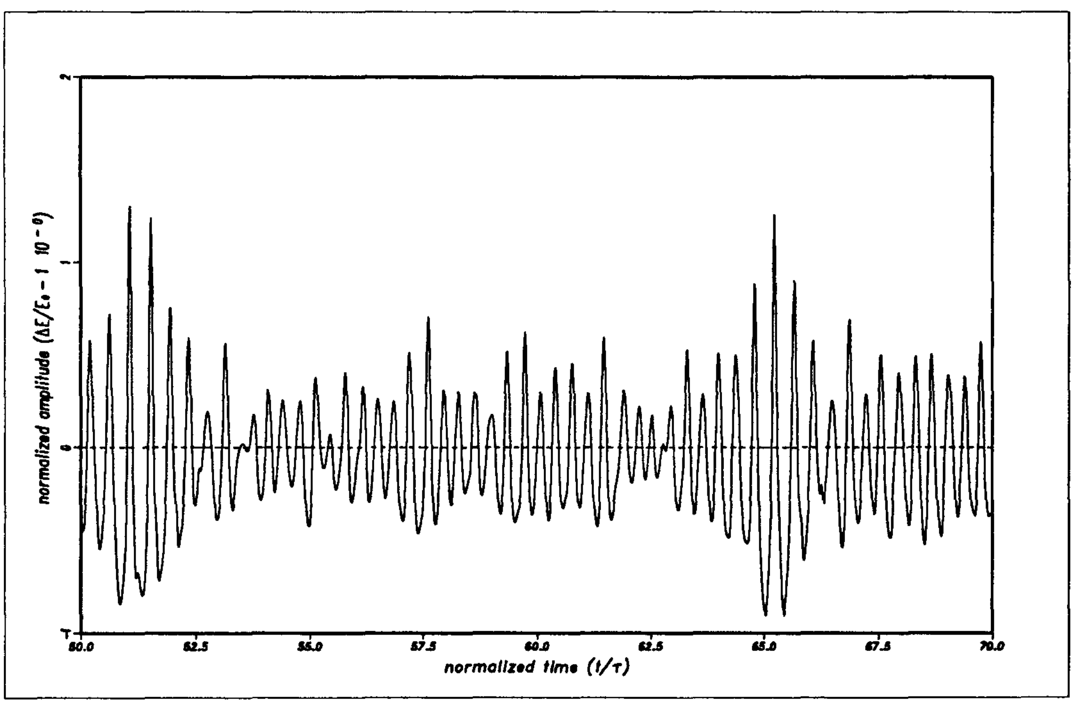

Figure 4.8: *Possible chaotic behavior.*

Chapter 5

Conclusions

In this report, the effects of phase conjugated feedback on a diode laser have been studied theoretically. This work splits naturally into two parts; theory of phase conjugation and theory of semiconductor lasers. Phase conjugation can be achieved by several methods. In this thesis, we have confined ourselves to the nearly degenerate four-wave mixing (NDFWM) scheme, which is presented in chapter 2.

Since the feedback imposed on a semiconductor laser plays a central role in this study, the results for "normal" feedback are, to a certain extent, rederived. In that case, the laser is described by the Lang & Kobayashi equations. The stationary state solutions and their stability in the case of normal feedback are analyzed in chapter 3.

A dynamical description of the laser with phase conjugated feedback is given in chapter 4. To describe this system, a set of rate-equations, similar to the Lang & Kobayashi equations, are derived, to our knowledge for the first time. As was done for the laser with normal feedback, the stationary state solutions and their stability have been analyzed. Furthermore, the rate-equations have been solved numerically to examine the dynamical behavior of the system, which confirmed the analytical results.

In the case of normal feedback, the stationary state solutions for the operating frequency depend on the distance between the laser and the mirror. Due to the time reversal nature of phase conjugation, the stationary state solutions are independent of this distance, therefore no external cavity mode structure is seen. Instead, the operating frequency is now determined by the frequency of the reference waves, i.e. the laser frequency *locks* to that of the pump. Furthermore, another quantity, the phase, locks unlike in the case of normal feedback. This is governed by a phase shift induced by the phase conjugation mirror (PCM) and the frequency of the reference waves. The equation describing the phase locking also provides a condition that must be satisfied if the laser is to show stationary state behavior. This condition gives a, theoretical, lower limit for the feedback rate γ as a function of the frequency mismatch for which the frequency- and phase locked solution may be found. An expectation with respect to the linewidth of the laser output is suggested by the phase locking. Since in a laser the linewidth is mainly governed by phase fluctuations, it is expected that if the phase locks, the linewidth will be reduced. In that case the linewidth is expected to be determined by the phase fluctuations in the pump beams. However, this has not yet been analyzed, but investigating the effect of spontaneous emission would be a very interesting future topic.

In their paper, Agrawal & Klaus [12] used a different scheme for achieving phase conjugation, i.e. degenerate four-wave mixing (DFWM). The equations they found can be obtained from our equations if the reference frequency is assumed to be the same as the operating frequency of the laser. As in NDFWM, the stationary state solutions are independent of the distance between the laser and the mirror. However, the frequency does not lock, but the phase does. In this case the phase is governed by a phase shift induced by the PCM and the operating frequency of the laser.

The differences in the stationary state solutions between the various types of feedback are: In normal feedback, *neither* frequency *nor* phase locks. Furthermore, the position of the mirror affects the laser operation, so that, e.g. mode hopping between external cavity modes may be observed. Applying phase conjugated feedback gives stationary state solutions that are independent of the position of the mirror. Phase conjugated feedback created by DFWM yields *only* a locked phase. If NDFWM is used to obtain phase conjugation, *both* the phase *and* the operating frequency of the laser may lock.

These differences can be explained by examining the effect the "mirror" has on the feedback. The mirror used for normal feedback does not affect the characteristics of the light it reflects, i.e. it is considered to be passive. Therefore, neither frequency nor phase are affected by the mirror and are still free parameters which can be determined by the laser: the laser has two degrees of freedom. However, a PCM does affect the nature of the light it "reflects", i.e. it is *active*. If a PCM "reflects" light by DFWM, only the phase of the light is affected, its frequency is unaffected. In this case the laser has one degree of freedom left, the frequency. The phase can no longer be determined by the laser: it locks. In reflecting the light by the NDFWM scheme, both phase and frequency are affected by the PCM. The laser has no degrees of freedom left: frequency and phase lock.

In both normal feedback and phase conjugated feedback by DFWM, the stationary state solutions reduce to the solitary laser solution ($\gamma = 0$) if $\gamma \rightarrow 0$. However, in the case of phase conjugated feedback by NDFWM this is *not* true. This is not really surprising, since when the laser is stationary, it has no free parameters left, due to the phase- and frequency locking. However, such an explanation is not very satisfactory. There is a "gap" between the, stationary, solitary laser solution and the stationary state solutions for the laser with feedback. A model that may describe the laser operation in this regime has been suggested: the frequency difference solution. Here, the total field is thought of as a sum of two fields: One, a component due to solitary laser operation, oscillating at the solitary laser frequency. And two, a component due to feedback, oscillating with a frequency mismatch with respect to the solitary laser frequency of twice the frequency difference between the solitary laser frequency and that of the reference waves. This solution has been tried and it was found that if the feedback is sufficiently weak and if higher order effects can be neglected, this trial solution indeed satisfies the rate-equations. It *may* bridge the gap between the locking behavior and the solitary laser solution.

Besides analyzing possible stationary state solutions, the stability of these solutions has been examined. For normal feedback, a condition is found that must be satisfied if the laser is to operate stably. A similar analysis for the laser with phase conjugated feedback yields a condition that is *automatically* satisfied, so long as the laser is initially in a stationary state. This is due to the phase- and frequency locking of the system.

In the case of normal feedback another condition exists, which indicates when the relaxation oscillation may undamp. Thus, the stability condition is shown to be a *necessary* condition, however, it doesn't have to be *sufficient.* A similar approximate condition for the undamping of the relaxation oscillation is found for the laser with phase conjugated feedback. The condition has been compared with the numerical results. The general behavior of this condition, e.g. the asymmetry with respect to the frequency mismatch, can be reproduced. However, the associated values involved for the amount of feedback and the frequency mismatch can *not* be reproduced. So, although a qualitative consideration seems satisfactory, the undamping of the relaxation oscillation has not yet been completely understood. A numerical analysis of the secular determinant may provide a way to reproduce the values for which the undamping of the relaxation oscillation is numerically observed. Such a numerical analysis was outside the scope of this thesis.

Theoretically, the operation of the laser with phase conjugated feedback can be separated in three regimes. In regime I, the laser has a stationary state solution where the phase and frequency lock. In regime II, the frequency difference solution governs the system, providing a way to reduce to the solitary laser solution ($\gamma = 0$). In regime III, the relaxation oscillation undamps, thus providing a possible route to chaos.

To check the theoretical analysis, the rate-equations have been solved numerically. These calculations, performed over a wide range of values for the frequency mismatch and the feedback rate, are summarized in Figure 4.3. From this figure it can be seen that the ideas presented in the theoretical analysis of the system are supported by the calculations. The occurrence of a frequency difference- and a stationary state solution regime as well as the undamping of the relaxation oscillation are observed numerically. Furthermore, the calculations give rise to the possible existence of a fourth regime, where the laser may exhibit chaotic behavior. To study possible chaotic behavior, another numerical method must be used, since the method used here is rather crude.

This research has been restricted to the following situations: The laser is assumed to be operating in one single longitudinal mode. The external delay time is much longer than the internal roundtrip time. The amount of feedback is considered to be small and multiple reflections in the external cavity are neglected. Furthermore, the laser is assumed to operate far above threshold and the effects of spontaneous emission are neglected. In future, theoretical, research the effects of spontaneous emission on the laser characteristics can be analyzed. Another interesting regime of laser operation is the lasers behavior around threshold. Besides further theoretical work, it would be very interesting to try and observe the predicted phenomena experimentally. A proposal to that extent has been submitted [40].

The results of this thesis will be presented at the IPR- and IQEC conferences [13,14]. Furthermore, a publication is in progress [15].

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Appendix A

Derivation of the SVEA form

The Maxwell-equations are given by:

$$\begin{aligned} \underline{\nabla}\cdot\underline{D} &= \rho & \underline{\nabla}\times\underline{E} &= -\frac{\partial \underline{B}}{\partial t} \\ \underline{\nabla}\cdot\underline{B} &= 0 & \underline{\nabla}\times\underline{H} &= \underline{J}+\frac{\partial \underline{D}}{\partial t} \end{aligned} \tag{A.1}$$

Assume that the material is homogeneous (χ is independent of the position), nonmagnetic ($\underline{B} = \mu_0 \underline{H}$), nonconducting ($\underline{J} = 0$) and that there are no free charges ($\rho = 0$). Write the dielectric displacement as $\underline{D} = \epsilon_0 \underline{E} + \underline{P}$, where $\underline{P} = \underline{P}_L + \underline{P}_{NL}$ can be separated in a linear and a nonlinear term. Furthermore, assume that all electric fields are perpendicular to their corresponding $\underline{k}$-vectors. The wave equation then becomes:

$$\nabla^2 \underline{E} - \mu_0 \epsilon \frac{\partial^2 \underline{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \underline{P}_{NL}}{\partial t^2} \tag{A.2}$$

Here the linear part $\underline{P}_L$ has been accounted for in the dielectric constant ϵ. Assume that the fields are scalar fields, then the electric field and the nonlinear part of the polarization are given by:

$$E(x,t) = \frac{1}{2}\mathcal{E}(x,t)e^{i(kx-\omega t)} + c.c. \tag{A.3}$$

$$P(x,t) = \frac{1}{2}\mathcal{P}(x,t)e^{i(kx-\omega t)} + c.c. + other\ terms \tag{A.4}$$

where the other terms correspond to other frequencies or wave-vectors as those of the electric field and do not couple to the particular electric field (phase-matching condition). Substitution of (A.3) and (A.4) in (A.2) and multiplying by $ike^{-i(kx-\omega t)}$ yields:

$$\{-\frac{i}{2k}\mathcal{E}_{xx}(x,t) + \mathcal{E}_x(x,t) + i\frac{\epsilon\mu_0}{2k}\mathcal{E}_{tt}(x,t) + \sqrt{\mu_0\epsilon}\,\mathcal{E}_t(x,t)\} +$$
$$\{-\frac{i}{2k}\mathcal{E}^*_{xx}(x,t) + \mathcal{E}^*_x(x,t) + i\frac{\epsilon\mu_0}{2k}\mathcal{E}^*_{tt}(x,t) + \sqrt{\mu_0\epsilon}\,\mathcal{E}^*_t(x,t)\}e^{-2i(kx-\omega t)} =$$
$$-i\frac{\mu_0}{2k}\{\mathcal{P}_{tt}(x,t) + 2i\omega\mathcal{P}_t(x,t) - \omega^2\mathcal{P}(x,t)\}$$
$$-i\frac{\mu_0}{2k}\{\mathcal{P}^*_{tt}(x,t) + 2i\omega\mathcal{P}^*_t(x,t) - \omega^2\mathcal{P}^*(x,t)\}e^{-2i(kx-\omega t)} \quad (A.5)$$

where the dispersion relation $k = \omega\sqrt{\mu_0\epsilon}$ has been used. The subscripts x and t denote the partial derivatives with respect to x and t. Applying the slowly varying conditions,

$$|k^2\mathcal{E}| \gg |k\frac{\partial\mathcal{E}}{\partial x}| \gg |\frac{\partial^2\mathcal{E}}{\partial x^2}| \quad (A.6)$$

which can, equivalently, be rewritten by substituting t for x and ω for k, $\mathcal{P}$ for $\mathcal{E}$, to (A.5) gives:

$$\mathcal{E}_x(x,t) + \sqrt{\mu_0\epsilon}\,\mathcal{E}_t(x,t) + \{\mathcal{E}^*_t(x,t) + \sqrt{\mu_0\epsilon}\,\mathcal{E}^*_t(x,t)\}e^{-2i(kx-\omega t)} =$$
$$i\frac{\omega}{2}\{\mathcal{P}(x,t) + \mathcal{P}^*(x,t)e^{-2i(kx-\omega t)}\} \quad (A.7)$$

To eliminate the rapidly varying term in (A.7), the equation is averaged over one optical period at a position x. $\mathcal{E}_x(x,t)$, $\mathcal{E}_t(x,t)$, $\mathcal{P}(x,t)$ and their complex conjugates are considered to be constants since they vary slowly with respect to x and t. Carrying out the integration and using:

$$\frac{1}{\tau}\int_{t_0}^{t_0+\tau} e^{-2i\omega t'}dt' \quad (A.8)$$

where $\tau = \pi/2\omega$. Equation (A.7) now reduces to the SVEA form:

$$\frac{\partial}{\partial x}\mathcal{E}(x,t) + \sqrt{\mu_0\epsilon}\,\frac{\partial}{\partial t}\mathcal{E}(x,t) = i\frac{\omega}{2}\sqrt{\frac{\mu_0}{\epsilon}}\,\mathcal{P}(x,t) \quad (A.9)$$

Appendix B

Derivation of $\mathcal{E}_{s,c}$ by Laplace transformation

The slowly varying wave equations for the input wave and the phase conjugated wave in NDFWM are given by:

$$\frac{\partial}{\partial x}\mathcal{E}_s = i\kappa_s \mathcal{E}_c^* e^{i\Delta kx} \tag{B.1}$$

$$\frac{\partial}{\partial x}\mathcal{E}_c = i\kappa_c \mathcal{E}_s^* e^{i\Delta kx} \tag{B.2}$$

These equations can be written as one second order differential equation as:

$$\frac{\partial^2}{\partial x^2}\mathcal{E} + i\Delta k\frac{\partial}{\partial x}\mathcal{E} + \kappa_s\kappa_c\mathcal{E} = 0 \tag{B.3}$$

where

$$\mathcal{E} = \mathcal{E}_s e^{-i\Delta kx} \tag{B.4}$$

Laplace transformation of (B.3) yields:

$$(s^2 + i\Delta ks + \kappa_s\kappa_c)\tilde{\mathcal{E}} = \mathcal{E}'(0) + (s + i\Delta k)\mathcal{E}(0) \tag{B.5}$$

where $\tilde{\mathcal{E}}$ denotes the Laplace-transformed of $\mathcal{E}$ and the prime denotes te derivative with respect to x. Rewriting (B.5) gives:

$$\tilde{\mathcal{E}} = \frac{\alpha_1}{s + \gamma_1} + \frac{\alpha_2}{s + \gamma_2} \tag{B.6}$$

where $\gamma_1 = i[\beta + \Delta k/2]$, $\gamma_2 = -i[\beta - \Delta k/2]$ and $\beta = \sqrt{\kappa_s\kappa_c + (\Delta k/2)^2}$ and α_1 and α_2 must satisfy the conditions:

$$\alpha_1 + \alpha_2 = \mathcal{E}(0) \tag{B.7}$$

$$\alpha_1\gamma_2 + \alpha_2\gamma_1 = \mathcal{E}'(0) + i\Delta k\mathcal{E}(0) \tag{B.8}$$

so that:

$$\alpha_1 = \frac{1}{2}\mathcal{E}(0)\left[1 - \frac{\Delta k}{2\beta}\right] + i\frac{\mathcal{E}'(0)}{2\beta} \tag{B.9}$$

$$\alpha_2 = \frac{1}{2}\mathcal{E}(0)\left[1 + \frac{\Delta k}{2\beta}\right] - i\frac{\mathcal{E}'(0)}{2\beta} \tag{B.10}$$

Inverse Laplace transformation of (B.6) yields:

$$\mathcal{E}(x) = \alpha_1 e^{-\gamma_1 x} + \alpha_2 e^{-\gamma_2 x} \tag{B.11}$$

Inserting (B.11) in (B.2) and integrating over x gives:

$$\mathcal{E}_c(x) = -\kappa_c\left\{\frac{\alpha_1^*}{|\gamma_1|}e^{i|\gamma_1|x} + \frac{\alpha_2^*}{|\gamma_2|}e^{i|\gamma_2|x}\right\} + constant \tag{B.12}$$

The constant must equal zero because, if $x = 0$ and $\Delta k = 0$, (B.12) reduces to:

$$\mathcal{E}_c(0) = i\kappa_c\frac{\mathcal{E}'(0)^*}{\beta^2} + constant \tag{B.13}$$

Together with the expression found for $\mathcal{E}'(0)$ by substituting (B.4) in (B.1) and setting Δk and x equal to zero, which yields:

$$\mathcal{E}'(0) = i\kappa_s\mathcal{E}_c^*(0) \tag{B.14}$$

It now follows from (B.13) and (B.14) that the integration constant must be zero. $\mathcal{E}'(0)$ can be expressed in known quantities by using the condition $\mathcal{E}_c(L) = 0$ in (B.12), yielding:

$$\mathcal{E}'(0) = \beta\mathcal{E}(0)\frac{\left(2\beta + \frac{(\Delta k)^2}{2\beta}\right)\sin(\beta L) - 2i\Delta k\cos(\beta L)}{2\beta\cos(\beta L) + i\Delta k\sin(\beta L)} \tag{B.15}$$

With (B.15), $\mathcal{E}_s$ and $\mathcal{E}_c$ can be expressed in known quantities, yielding:

$$\mathcal{E}_s(x) = \frac{\mathcal{E}_s(0)e^{i\Delta kx/2}}{\beta\cos(\beta L) + i(\Delta k/2)\sin(\beta L)}\{\beta\cos[\beta(L-x)] + i\frac{\Delta k}{2}\sin[\beta(L-x)]\} \quad \text{(B.16)}$$

$$\mathcal{E}_c(x) = \frac{\mathcal{E}_s^*(0)e^{i\Delta kx/2}}{\beta\cos(\beta L) - i(\Delta k/2)\sin(\beta L)}\sin[\beta(L-x)] \quad \text{(B.17)}$$

Appendix C

Derivation of $\mathcal{E}_{1,2}$ and $\phi_{1,2}$

For very small amounts of feedback, in the region where the stability condition (4.23) is not satisfied, the electric field and the number of electron-hole pairs can be represented, see section 4.4, as:

$$E(t) = \left\{E_\gamma + \mathcal{E}_1 e^{i\nu t + i\phi_1} + \mathcal{E}_2 e^{-i\nu + \phi_2}\right\} e^{i\phi_0}; \quad n(t) = n_0 + n_{qs}(t) \tag{C.1}$$

where n_{qs} is the perturbation of the number of electron-hole pairs in the quasi-steady state solution and $E_\gamma = E_0 + \Delta E_0$, where E_0 is the amplitude of the field without feedback and ΔE_0 a small, constant, amplitude change of the solitary laser amplitude, due to feedback. Applying (C.1) to the rate-equation for the number of electron-hole pairs (4.15) yields:

$$\frac{d}{dt} n_{qs}(t) = -\lambda_R n_{qs}(t) - 2\Gamma_0 E_0 \Delta E_0 - 2\Gamma_0 E_0 \{\kappa_c \cos(\nu t) - \kappa_s \sin(\nu t)\} \tag{C.2}$$

where λ_R the damping constant in the absence of feedback and the constants $\kappa_{s,c}$ are given by:

$$\kappa_s = \mathcal{E}_1 \sin\phi_1 - \mathcal{E}_2 \sin\phi_2 \tag{C.3}$$

$$\kappa_c = \mathcal{E}_1 \cos\phi_1 + \mathcal{E}_2 \cos\phi_2 \tag{C.4}$$

Equation (C.2) is a first order differential equation and its derivative with respect to the time is itself and a term containing sines and cosines, the solution for this sort of differential equation can be written as:

$$n_{qs}(t) = a\sin(\nu t) + b\cos(\nu t) + c \tag{C.5}$$

This expression must satisfy (C.2). Furthermore, the ac- and dc parts of the expression found if (C.5) is applied to (C.2) must *independently* be satisfied. Using this condition, a, b and c are found to be:

$$a = 2\frac{\Gamma_0 E_0}{\lambda_R^2 + \nu^2}(\lambda_R \kappa_s - \nu \kappa_c) \tag{C.6}$$

$$b = -2\frac{\Gamma_0 E_0}{\lambda_R^2 + \nu^2}(\nu \kappa_s + \lambda_R \kappa_c) \tag{C.7}$$

$$c = -2\frac{\Gamma_0 E_0}{\lambda_R}\Delta E_0 \tag{C.8}$$

If (C.1) and (C.5) are applied to the rate-equation for the electric field (4.14), the following set of equations is found

$$\Delta E_0 = 0 \tag{C.9}$$

$$\begin{pmatrix} \nu(1-\beta) & -\lambda_R\beta & 0 & 0 \\ \lambda_R\beta & \nu(1-\beta) & 0 & 0 \\ \alpha\beta\lambda_R & -\alpha\beta\nu & \nu & 0 \\ \alpha\beta\nu & \alpha\beta\lambda_R & 0 & \nu \end{pmatrix} \begin{pmatrix} \kappa_s \\ \kappa_c \\ \eta_s \\ \eta_c \end{pmatrix} = -\gamma E_0 \begin{pmatrix} \cos\theta \\ \sin\theta \\ \cos\theta \\ \sin\theta \end{pmatrix} \tag{C.10}$$

where

$$\eta_s = \mathcal{E}_1 \sin\phi_1 + \mathcal{E}_2 \sin\phi_2 \tag{C.11}$$

$$\eta_c = \mathcal{E}_1 \cos\phi_1 - \mathcal{E}_2 \cos\phi_2 \tag{C.12}$$

The set of equations (C.10) can be solved. In order to solve (C.10), this equation is written as:

$$\begin{pmatrix} D_A \underline{\underline{U}}(\Psi) & \underline{\underline{0}} \\ D_B \underline{\underline{U}}(\Theta) & \nu \underline{\underline{I}} \end{pmatrix} \begin{pmatrix} \underline{\kappa} \\ \underline{\eta} \end{pmatrix} = -\gamma E_0 \begin{pmatrix} \underline{e}_x \\ \underline{e}_x \end{pmatrix} \tag{C.13}$$

where the matrices $\underline{\underline{U}}$ are unitary rotation matrices that rotate a vector over an angle Ψ or Θ and $\underline{\underline{I}}$ is the unity matrix.

The angle θ is chosen equal to zero, which can be done because choosing ϕ_0 is just choosing a reference time. The quantities D_A, D_B, $\underline{\kappa}$, $\underline{\eta}$ and $\underline{e}_x$ are given by:

$$\underline{\kappa} = \begin{pmatrix} \kappa_s \\ \kappa_c \end{pmatrix}; \quad \underline{\eta} = \begin{pmatrix} \eta_s \\ \eta_c \end{pmatrix}; \quad \underline{e}_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \tag{C.14}$$

$$D_A = \sqrt{\nu^2(1-\beta)^2 + \lambda_R^2\beta^2} \quad (C.15)$$

$$D_B = \sqrt{\alpha^2\beta^2(\lambda_R^2 + \nu^2)} \quad (C.16)$$

From (C.3), (C.4), (C.11) and (C.12) follows that:

$$\mathcal{E}_1 = \frac{1}{2}\left\{(\kappa_s + \eta_s)^2 + (\kappa_c + \eta_c)^2\right\}^{\frac{1}{2}} \quad (C.17)$$

$$\mathcal{E}_2 = \frac{1}{2}\left\{(\kappa_s - \eta_s)^2 + (\kappa_c - \eta_c)^2\right\}^{\frac{1}{2}} \quad (C.18)$$

$$\tan\phi_1 = \frac{\kappa_s + \eta_s}{\kappa_c + \eta_c} \quad (C.19)$$

$$\tan\phi_2 = -\frac{\kappa_s - \eta_s}{\kappa_c - \eta_c} \quad (C.20)$$

The quantities $\kappa_s \pm \eta_s$, $\kappa_c \pm \eta_c$ can be solved from (C.13), yielding:

$$\kappa_s + \eta_s = -\frac{\gamma E_0}{D_A\nu}\{\nu\cos\Psi - D_B\cos(\Theta - \Psi) + D_A\} \quad (C.21)$$

$$\kappa_c + \eta_c = \frac{\gamma E_0}{D_A\nu}\{\nu\sin\Psi + D_B\sin(\Theta - \Psi)\} \quad (C.22)$$

$$\kappa_s - \eta_s = -\frac{\gamma E_0}{D_A\nu}\{\nu\cos\Psi + D_B\cos(\Theta - \Psi) - D_A\} \quad (C.23)$$

$$\kappa_c - \eta_c = \frac{\gamma E_0}{D_A\nu}\{\nu\sin\Psi - D_B\sin(\Theta - \Psi)\} \quad (C.24)$$

with these expressions (C.17), (C.18), (C.19) and (C.20) can be solved, yielding:

$$\mathcal{E}_1 = \frac{1}{2}\left|\frac{\gamma E_0}{D_A\nu}\right|\left\{4\nu\left[\nu(1-\beta) - \alpha\beta\lambda_R\right] + \beta^2(1+\alpha^2)(\lambda_R^2 + \nu^2)\right\}^{\frac{1}{2}} \quad (C.25)$$

$$\mathcal{E}_2 = \frac{1}{2}\left|\frac{\gamma E_0}{D_A\nu}\right|\beta(1+\alpha^2)^{\frac{1}{2}}(\lambda_R^2 + \nu^2)^{\frac{1}{2}} \quad (C.26)$$

$$\tan\phi_1 = -\frac{2\nu^2 - \nu\beta(3\nu + \alpha\beta\lambda_R) + \beta^2(\lambda_R^2 + \nu^2)}{\alpha\beta^2(\lambda_R^2 + \nu^2) - \beta\nu(\alpha\nu - \lambda_R)} \quad (C.27)$$

$$\tan\phi_2 = \frac{\beta^2(\lambda_R^2 + \nu^2) - \nu(\alpha\lambda_R - \nu)}{\alpha\beta(\lambda_R^2 + \nu^2) - \nu(\alpha\nu + \lambda_R)} \quad (C.28)$$

$$(C.29)$$

Appendix D

Figures

Figure	C	$\Delta\omega$ 10^9 $rad\ s^{-1}$	Frequency 10^9 $rad\ s^{-1}$	Rel. Osc. 10^9 $rad\ s^{-1}$
D.1	0.05	+0.5	0.9958	–
D.2	0.05	−0.5	0.9963	–
D.3	0.3	+0.5	–	–
D.4	0.3	−0.5	–	–
D.5	0.49	+0.5	–	–
D.6	0.49	−0.5	–	–
D.7	0.8	+0.5	–	–
D.8	0.8	−0.5	–	–
D.9	1.30	+0.5	17.28	16.69
D.10	1.30	−0.5	17.38	16.69
D.11	1.60	+0.5	17.22	16.69
D.12	1.60	−0.5	17.44	16.69
D.13	5.0	+0.5	17.06	16.69
D.14	5.0	−0.5	17.21	16.69
D.15	10.0	+0.5	–	–
D.16	10.0	−0.5	–	–

Table D.1: *Parameters and calculated values for Figures D.1 - D.16*

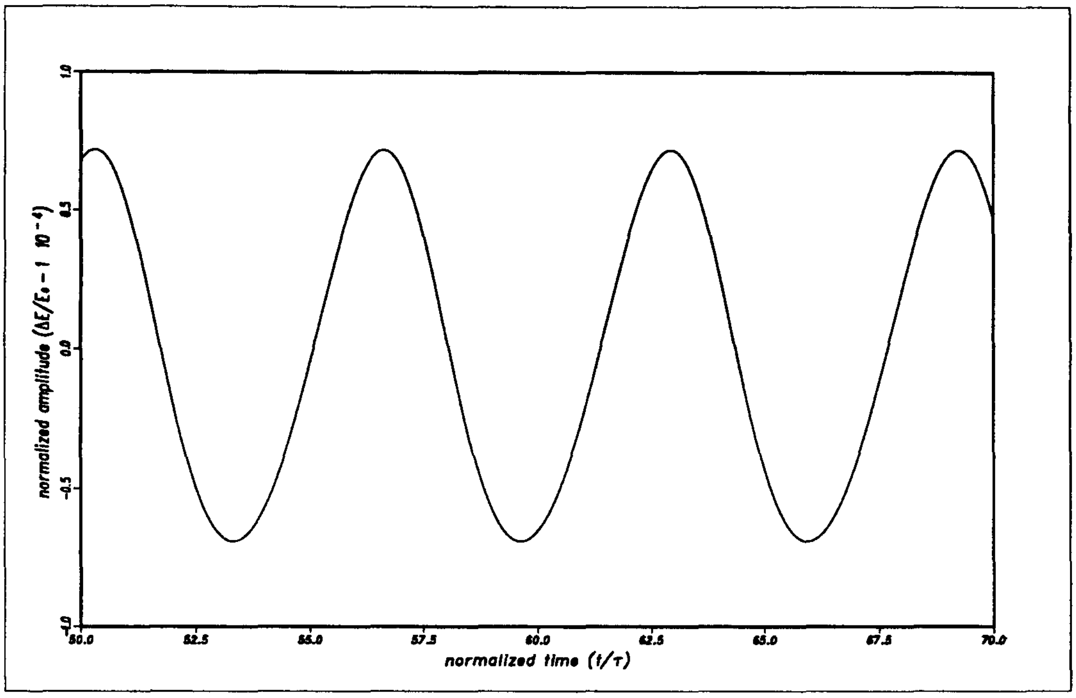

Figure D.1: *Frequency difference solution for positive* $\Delta\omega$

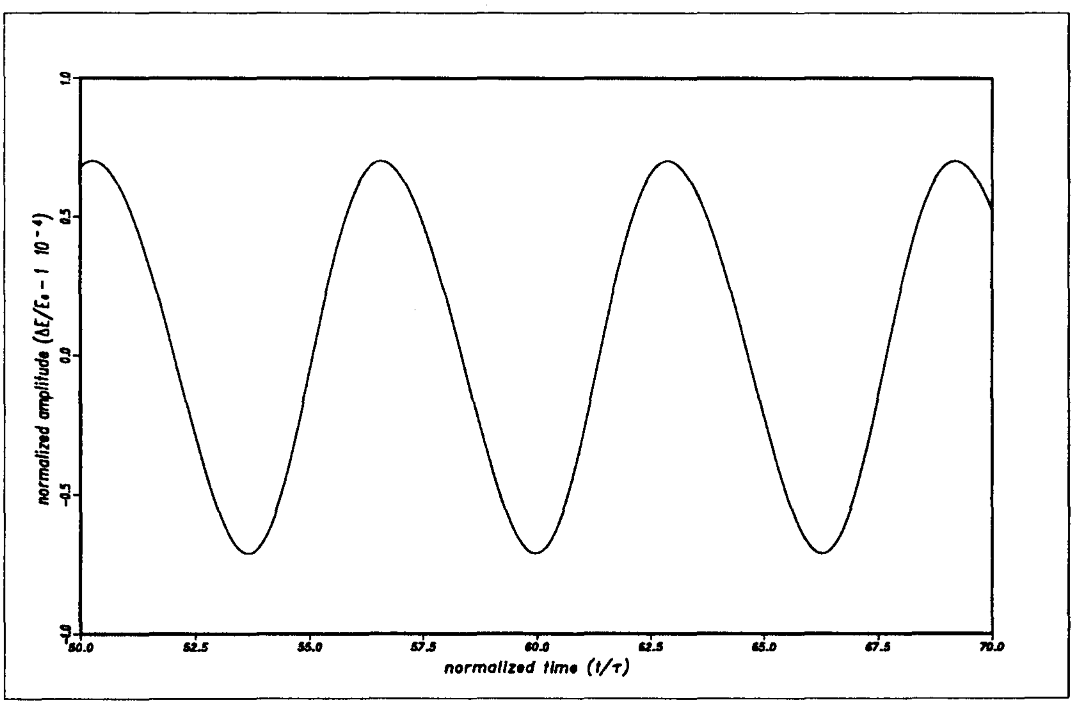

Figure D.2: *Frequency difference solution for negative* $\Delta\omega$

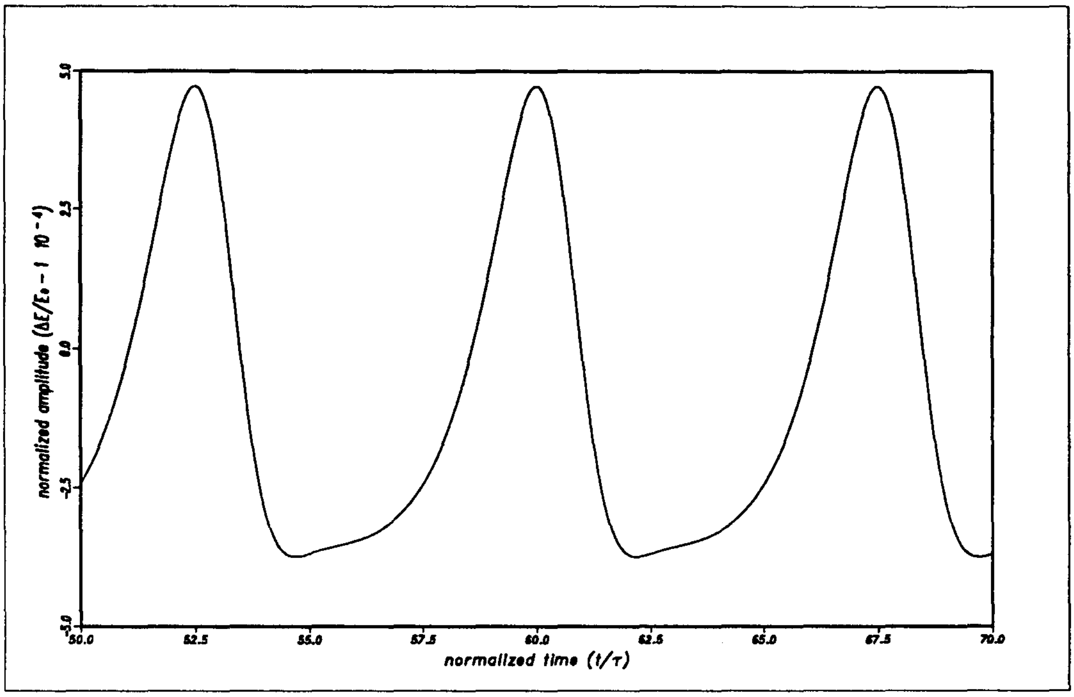

Figure D.3: *Onset higher order effects for positive* $\Delta\omega$

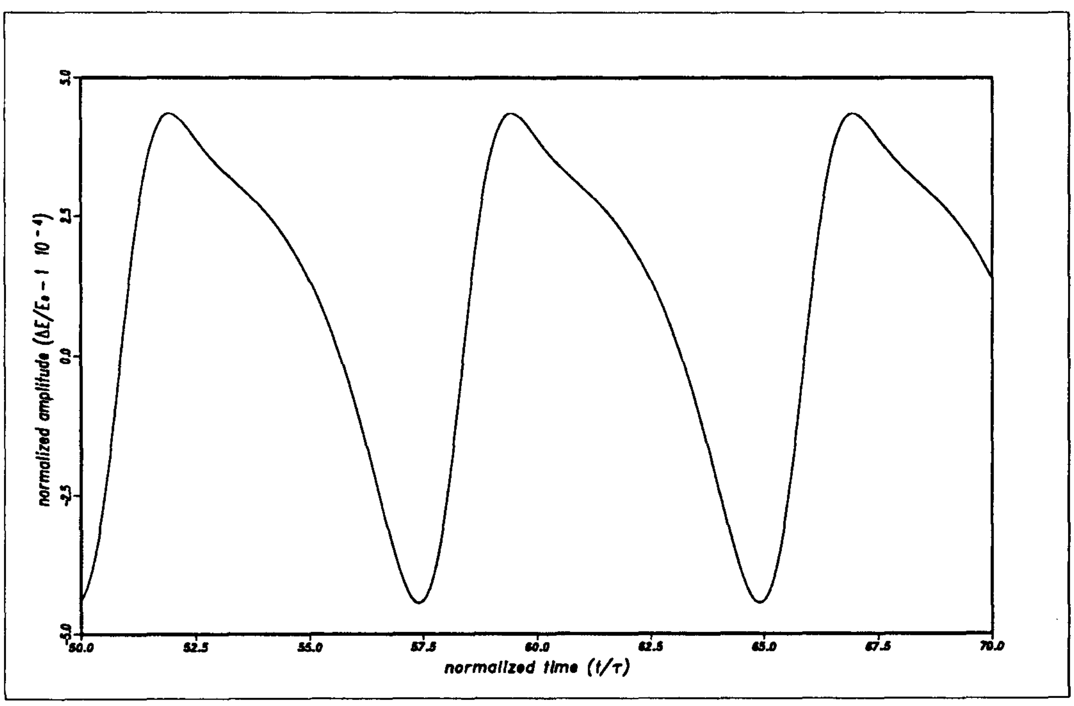

Figure D.4: *Onset higher order effects for negative* $\Delta\omega$

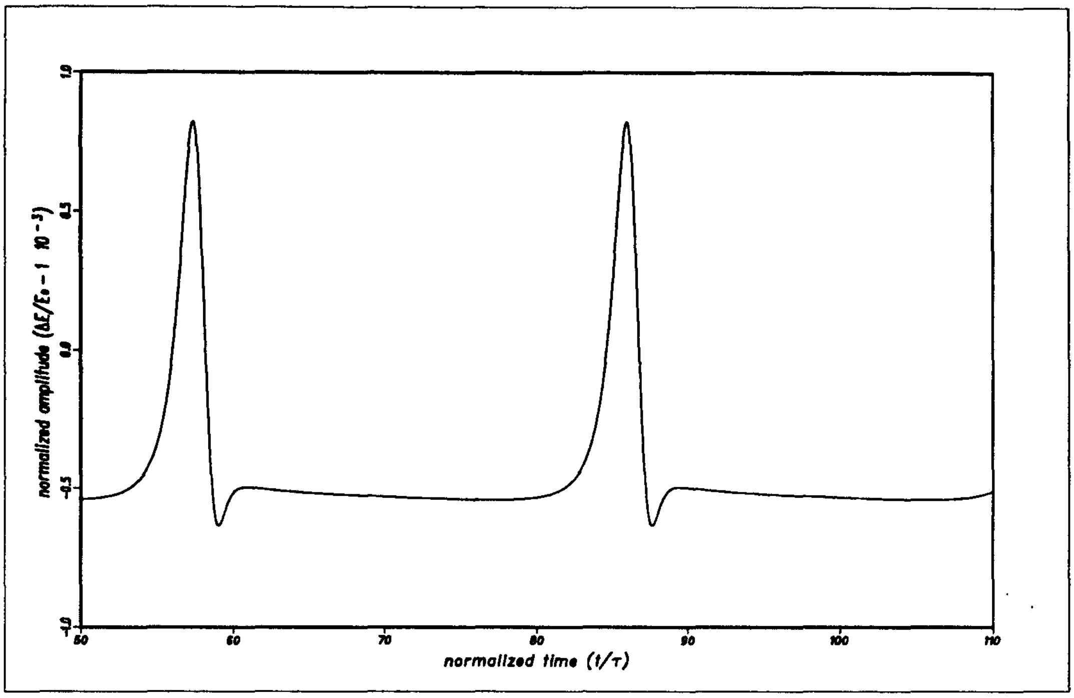

Figure D.5: *Near locking for positive* $\Delta\omega$

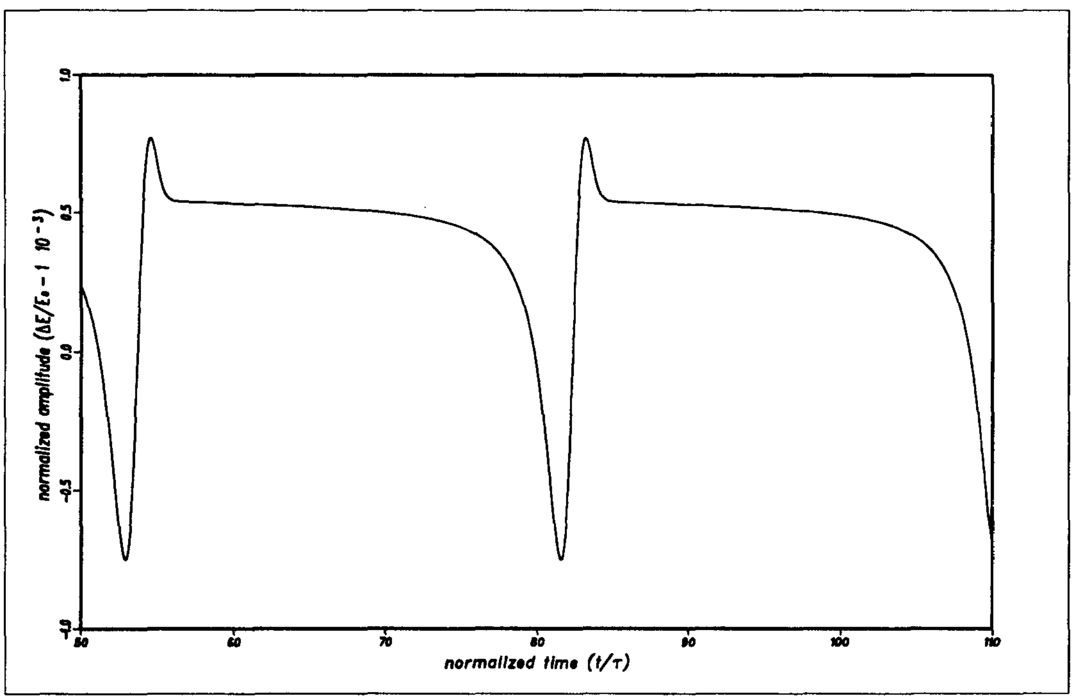

Figure D.6: *Near locking for negative* $\Delta\omega$

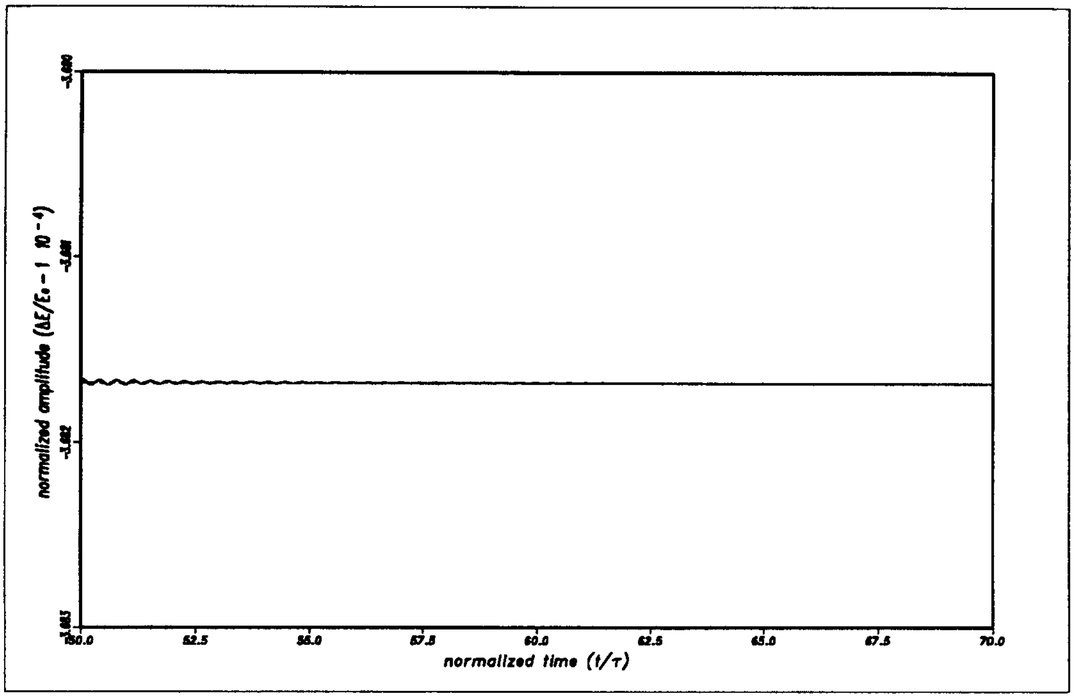

Figure D.7: *Locking for positive* $\Delta\omega$

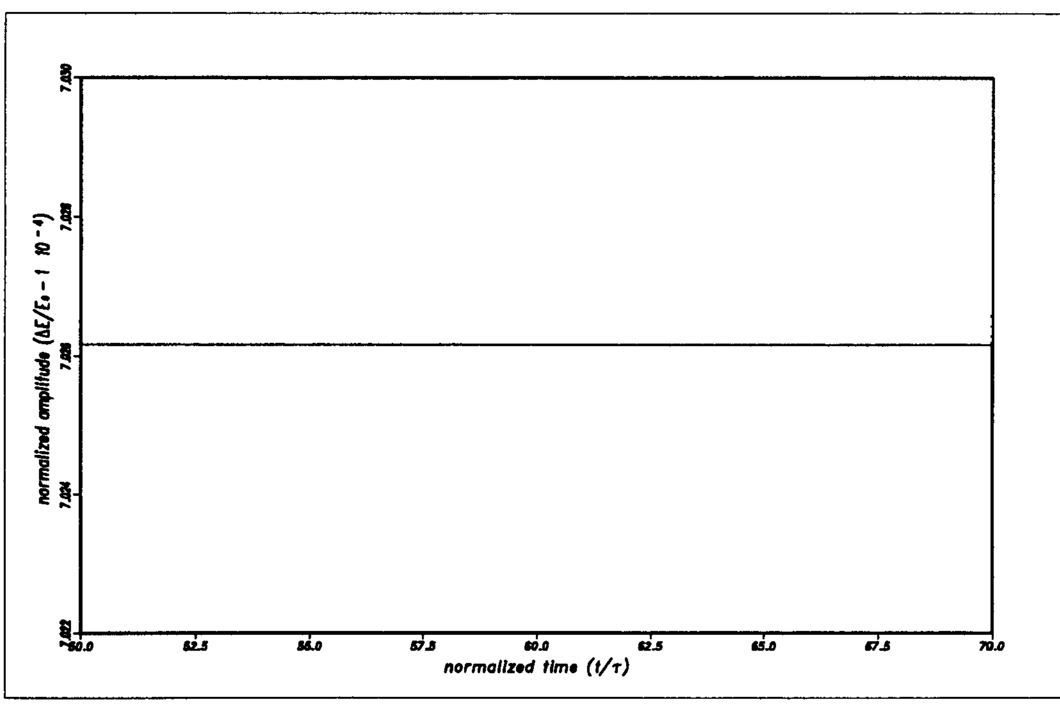

Figure D.8: *Locking for negative* $\Delta\omega$

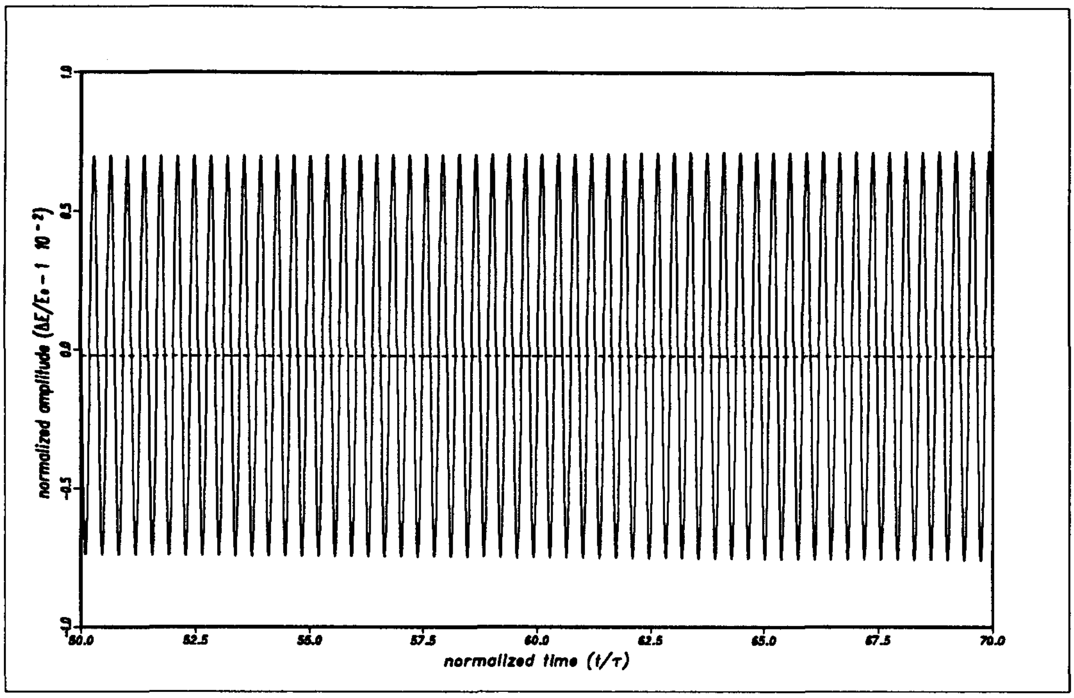

Figure D.9: *Undamping relaxation oscillation for positive* $\Delta\omega$

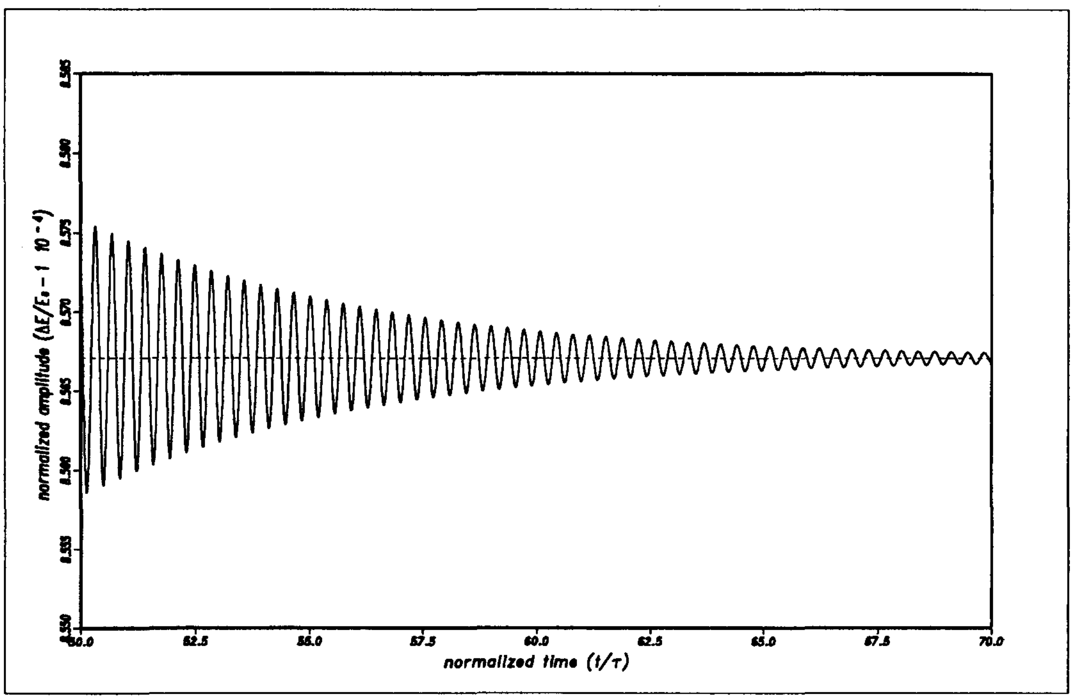

Figure D.10: *Relaxation oscillation not yet undamped for negative* $\Delta\omega$

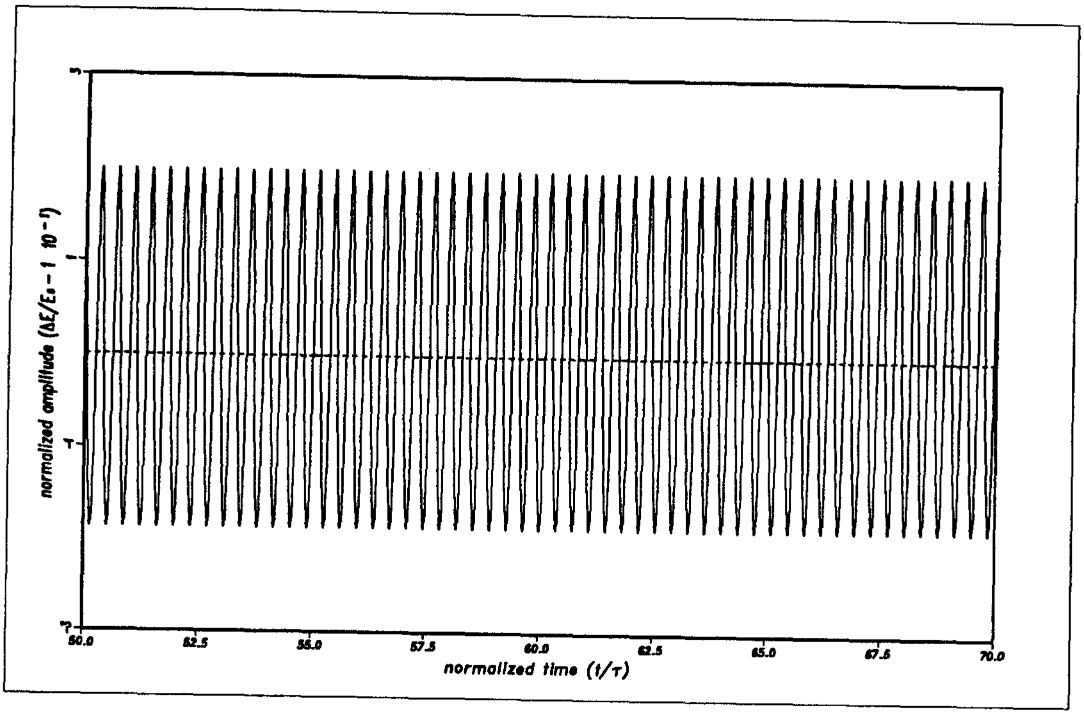

Figure D.11: *Undamped relaxation oscillation for positive* $\Delta\omega$

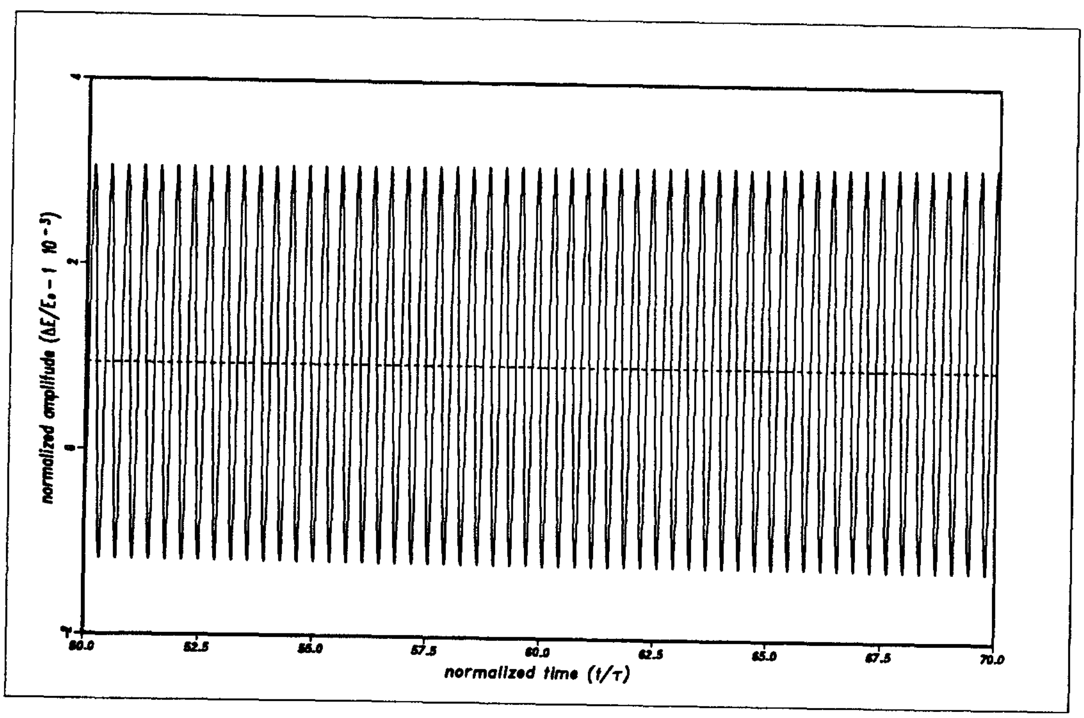

Figure D.12: *Undamping relaxation oscillation for negative* $\Delta\omega$

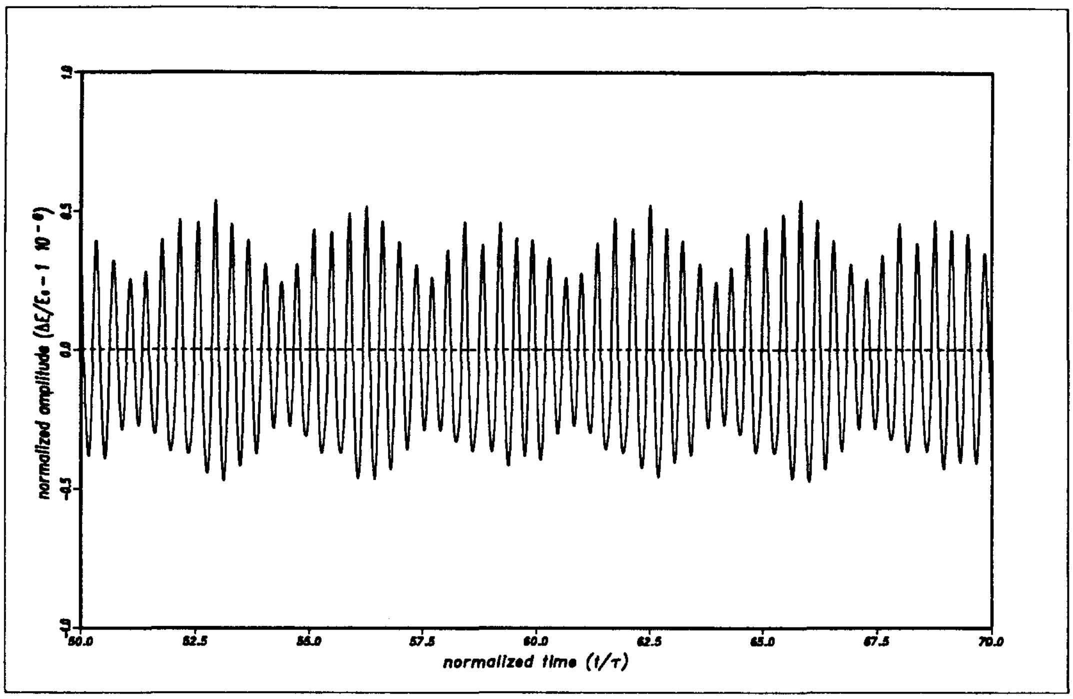

Figure D.13: *Undamped relaxation oscillation and higher orders for positive* $\Delta\omega$

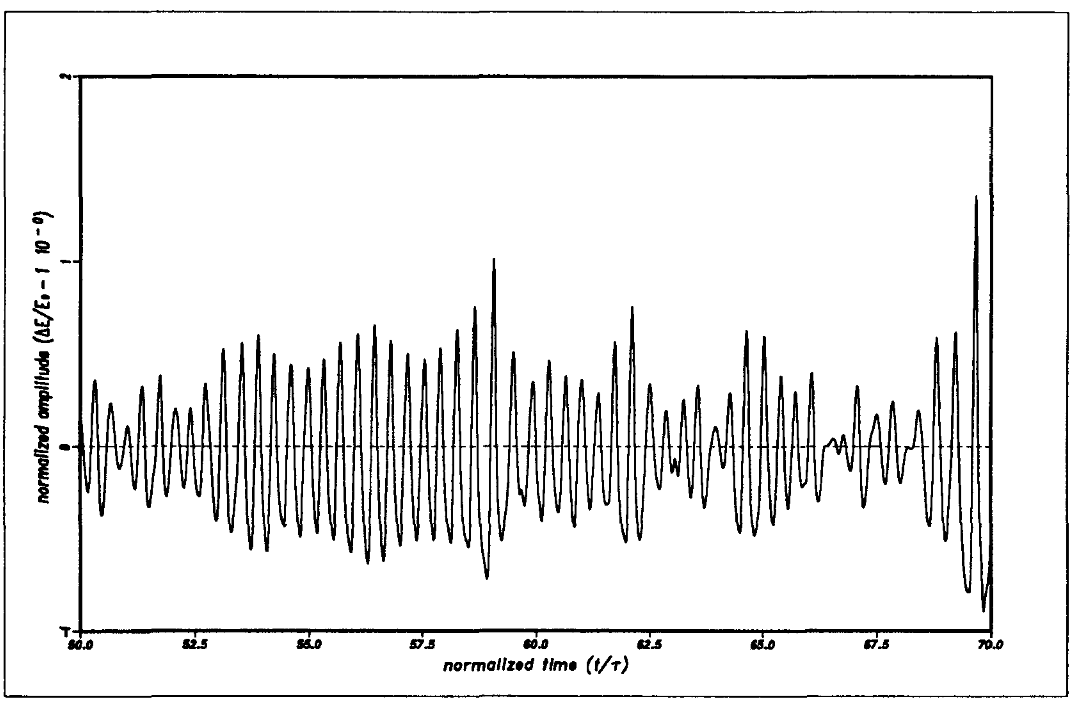

Figure D.14: *Undamped relaxation oscillation and higher orders for negative* $\Delta\omega$

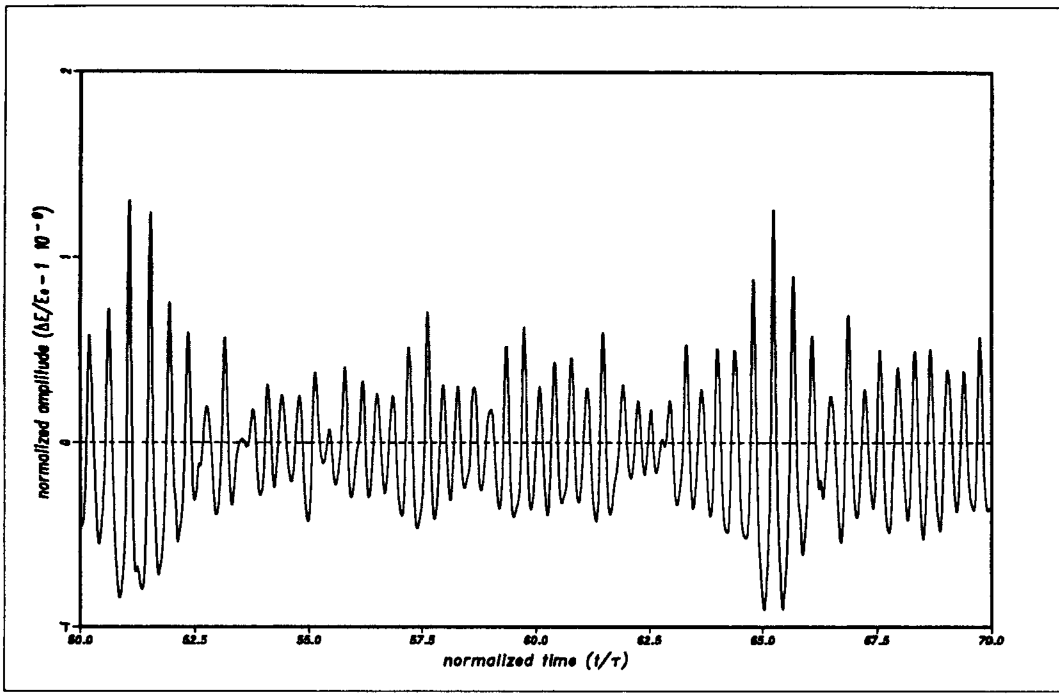

Figure D.15: *Possible chaotic behavior for positive* $\Delta\omega$

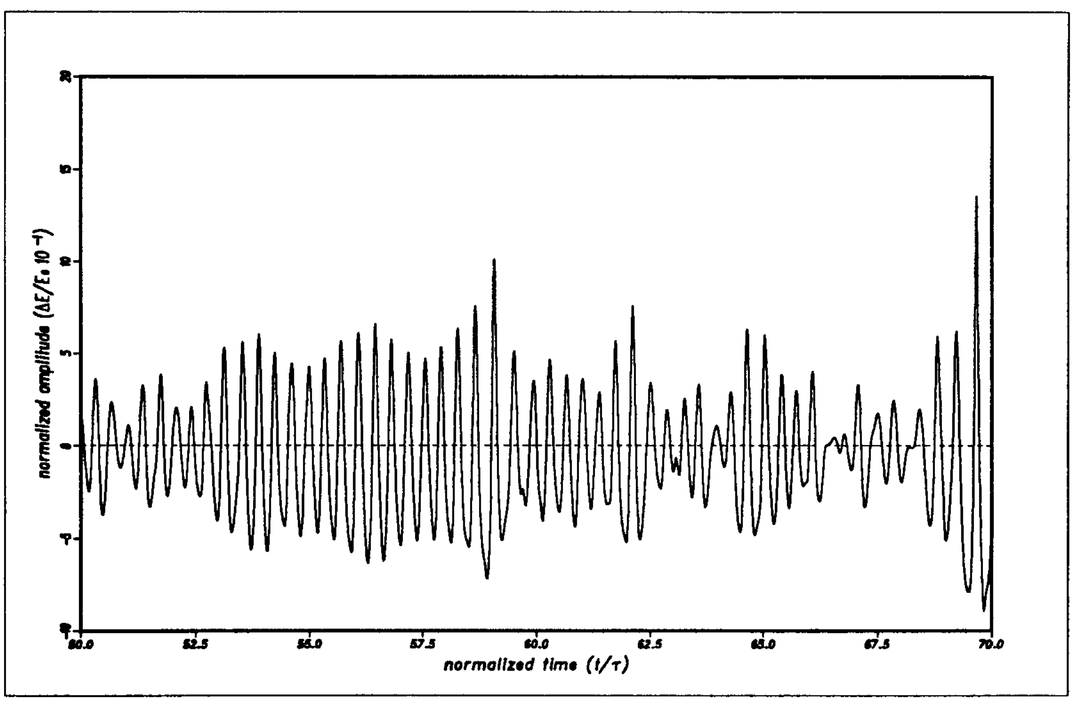

Figure D.16: *Possible chaotic behavior for negative* $\Delta\omega$