MASTER

The magneto-mechanical coupling problem in a magnetic resonance imaging scanner : the co-axial circular wire loops model

Zhao, Guoying

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MASTER THESIS

The Magneto-Mechanical Coupling Problem In a Magnetic Resonance Imaging Scanner

The co-axial circular wire loops model

By
Guoying Zhao

Supervisors: Dr. Jos Maubach, Dr. Ir. Gert Mulder, Ir. Patrick Limpens
Advisors: Prof. Dr. R.M.M. Mattheij, Prof. Dr. W.H.A. Schilders, Dr. J. Boschman

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Abstract

The topic of this thesis is the modelling and simulation of the magneto-mechanical coupling problem in an MRI scanner. When an MRI scanner works, it makes unwanted noise which partly comes from the vibration of the magnet shield of the scanner due to the eddy currents and the magnetic field. The result of this research shows the relations between the vibration and the magnetic fields. The conclusions can give a help to find a method to reduce the noise. Two co-axial circular wire loops models are set up for this purpose.

The geometry models and corresponding assumptions are discussed briefly in chapter 1. In Chapter 2, we derived the mutual inductance and self inductance of the circular wire loops from Maxwell equations. In chapter 3, 4, 5, we focus on the *two coaxial wire loops model*. The formulation, simplification and numerical simulation of this model are discussed based on test data. The formulation of the *multiple coaxial wire loops model* is given in Chapter 6.

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Guoying Zhao
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Eindhoven

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Chapter 1

Introduction

In this chapter, the background of a magneto-mechanical coupling problem in a Magnetic Resonance Imaging (MRI) scanner is briefly introduced. Based on the magnetic structure of the Philips intera MRI system, two geometric models and the related physical assumptions are discussed.

1.1 The MRI scanner and its noise

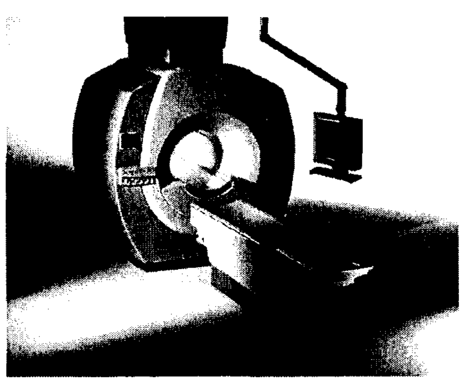

Figure 1.1: A view of the modern MRI system

Magnetic Resonance Imaging (MRI) is an imaging technique used primarily in medical settings to produce high quality images of the inside of the human body. MRI is based on the principles of Nuclear Magnetic Resonance (NMR). An MR image represents the relative response of specific nuclei to absorbed radio frequency energy. A good introduction of MRI can be found in [1].

The magnetic resonance imaging study involves:

- A static magnetic field (main magnetic field);
- Local variations of the main magnetic field (magnetic gradient field), to encode spatial information on the nuclei within a tissue sample ;
- RF pulse applied through a radio frequency pulse generation system to generate a signal;

- An RF receiver system then detects the re-emitted RF energy and transports this signal to a computer system.
- A computer system for digital processing and image display.

In a Philips Intera MRI scanner, the magnetic part is shown in Figure 1.2

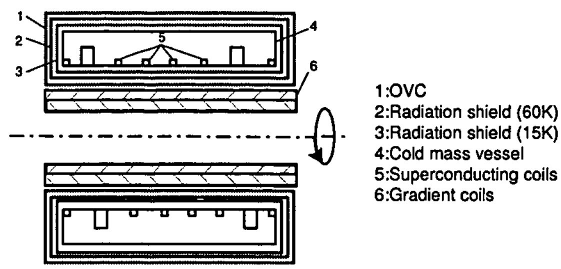

Figure 1.2: The magnetic structure of a Philips Intera MRI scanner

The cold mass vessel is located inside the Outer Vacuum Container(OVC). In the vessel, liquid helium keeps the superconducting coils at 4K temperature. These superconducting coils generate the main magnetic field. This magnetic field is steady, uniform and very strong. The field strength ranges from 0.5 to 3.0 Tesla for Philips MR systems. Between OVC and the cold mass vessel, there are two thermal shields which reduce the radiative heat transporting from the outside to the cold mass vessel. Both the shields and the OVC are made of aluminum.
The gradient coils are located inside the OVC cylinder. The gradient coils generate the gradient magnetic field in x, y and z directions. Compared to the main magnetic field, the gradient field is much smaller, it ranges from 1 to 10 mT.

The current generation of MRI scanner makes unwanted noise when it works. The main noise sources are from the vibrations of the gradient coils and the vessel. Besides the noise, the vibrations also distort the magnetic field. This could influence the imaging quality.

The vibrations of the gradient coils are a result of the magnetic forces on the gradient coils. The reason of the vibrations of the vessel is more complicated. The time-varying gradient currents generate a time-varying gradient magnetic field. This field produces the eddy currents in the OVC and the shields. The eddy currents lead to the Lorenz forces. These forces make the shields to move. Due to the movement, other eddy currents are induced. These eddy currents lead to damping. Figure 1.3 shows this mechanism.

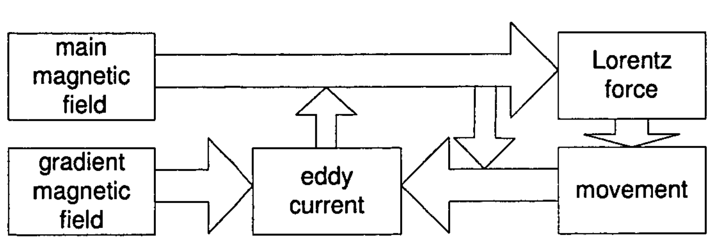

Figure 1.3: The vibration mechanism

1.2 Two geometric models for the vibrations

Since the vibrations of the main magnetic part of the MRI scanner are going to be investigated, we simplify the MRI magnetic system as two parts, the source cylinder and the shield cylinder. The source cylinder carries the source current to generate the magnetic field. The shield cylinder, which could be the OVC, vibrates due to the magnetic force.
Figure 1.4 shows the simplified geometry.

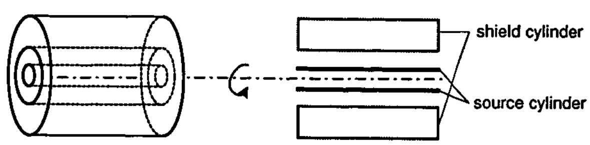

Figure 1.4: The simplified geometry model

Based on this simplification, we propose to investigate:

- A two co-axial circular wire loops model:
  Figure 1.5 shows the model in two dimensions. The fixed source loop carries a current, which generates a time-varying magnetic field. The eddy current and the movement of the shield loop will be investigated.

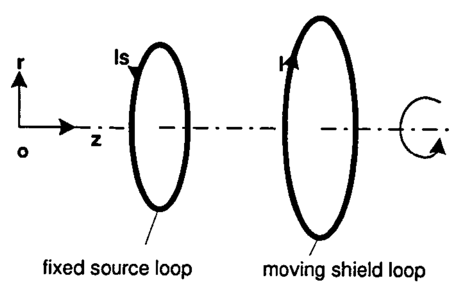

Figure 1.5: A two co-axial circular wire loops model

- A multiple co-axial circular wire loops model:
  If the two co-axial circular wire loops model works well, the same method can be extended to a multiple co-axial circular wire loops model shown in Figure 1.6. It is a discretization of the source cylinder and the shied cylinder. As for the practical reason, the number of the loops should be restricted to a small finite number. The eddy currents and the movements of the shield loops will be investigated.

The assumptions underlying these two models are:

- The changing gradient magnetic field and the main magnetic field are prescribed;
- The cylinders are very thin;
- The system is rotationally symmetric;
- The material of the shield is linear elastic, non electric polarization and non magnetic polarization;

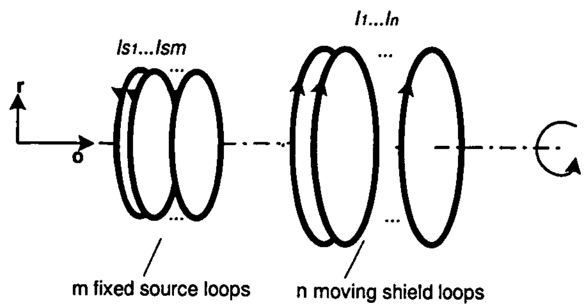

Figure 1.6: A multiple co-axial circular wire loops model

- There is no direct mechanical coupling between the gradient coils and the shield;
- The entire domain is surrounded by air;
- The acoustic aspect of the problem is neglected.

1.3 Work Plan

1. Derive the self-inductance and mutual inductance formulas from Maxwell equations.
2. Discuss on the two coaxial circular wire loops model: formulation, simplification and simulation.
3. discuss on the multiple coaxial circular wire loops model.

Chapter 2

Fundamental Equations

This chapter shows the relation between the inductance functions and the electro-magnetic field. The derivations of the formulas of the mutual inductance and the self-inductance are given,
A more elaborate introduction on the basic electromagnetic theory can be found in [4].

2.1 The Maxwell Equations

Table 2.1 shows the basic electro-magnetic field variables.

Table 2.1: The basic electro-magnetic variables

Symbol	physics meaning	unit
E	electric field strength	$V/m = kg \cdot m/C \cdot s^2$
D	dielectric displacement	C/m^2
H	magnetic field strength	$A/m = C/m \cdot s$
B	magnetic induction	$T = Wb/m^2 = kg/C \cdot s$
j	electric current density	$A/m^2 = C/m^2 \cdot s$
q	eletric charge density	C/m^3

The Maxwell equations, in the differential form and the integral form, are:
Conservation of charge:

$$\nabla \cdot \mathbf{j} + \frac{\partial q}{\partial t} = 0 \quad and \quad \oint_S \mathbf{j} \cdot ds = -\frac{\partial}{\partial t}\int_V qdv; \tag{2.1}$$

Conservation of flux:

$$\nabla \cdot \mathbf{B} = 0 \quad and \quad \oint_S \mathbf{B} \cdot ds = 0; \tag{2.2}$$

Gauss's law:

$$\nabla \cdot \mathbf{D} = q \quad and \quad \oint_S \mathbf{D} \cdot ds = \int_V qdv; \tag{2.3}$$

Maxwell's generalization of Ampere's law:

$$\nabla \times \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \quad and \quad \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_A \mathbf{j} \cdot da + \frac{\partial}{\partial t}\int_A \mathbf{D} \cdot da; \tag{2.4}$$

Faraday's law of induction:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \textit{and} \quad \oint_C \mathbf{E} \cdot dl = -\frac{\partial}{\partial t} \int_A \mathbf{B} \cdot da. \tag{2.5}$$

The electromagnetic constitutive relations for non-electric polarization and non-magnetic polarization material are:

$$\mathbf{D} = \epsilon_0 \mathbf{E}, \tag{2.6}$$
$$\mathbf{B} = \mu_0 \mathbf{H}, \tag{2.7}$$
$$\mathbf{j} = \sigma \mathbf{E}. \tag{2.8}$$

Here, ϵ_0 is the permittivity in vacuum, μ_0 is the permeability in vacuum, σ is the electric conduction coefficient.
After substituting (2.6), (2.8) into (2.3) and some simplification, we obtain

$$\nabla \cdot \mathbf{j} = \nabla \cdot (\sigma \mathbf{E}) = \frac{\sigma}{\epsilon_0} \nabla \cdot \mathbf{D} = \frac{\sigma}{\epsilon_0} q.$$

Substituting the above equation into the differential form of (2.1), we obtain:

$$\frac{\partial q}{\partial t} + \frac{\sigma}{\epsilon_0} q = 0. \tag{2.9}$$

The solution of (2.9) is proportional to $e^{-\frac{\sigma}{\epsilon_0} t}$, so that any net charge will decay exponentially in a characteristic time $\tau = \sigma/\epsilon_0$. In a good conductor, for instance, aluminum $\tau < 10^{-18}$, the almost-instantaneous decay of charge leads us to write

$$\nabla \cdot \mathbf{j} = 0, \tag{2.10}$$
$$\mathbf{j} = \nabla \times \mathbf{H}. \tag{2.11}$$

The equations (2.10) and (2.11) show that $\frac{\partial \mathbf{D}}{\partial t}$ in the equation (2.4) can be neglected in a good conductor.

2.2 The vector potential as a function of the current

From (2.11) and (2.7), it follows that

$$\mu_0 \mathbf{j} = \nabla \times \mathbf{B}. \tag{2.12}$$

The vector potential is defined by $\mathbf{B} = \nabla \times \mathbf{A}$, and we chose the gauge $\nabla \cdot \mathbf{A} = 0$. Now, because $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, the relation between the current density $\mathbf{j}$ and the vector potential $\mathbf{A}$ is:

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{j}. \tag{2.13}$$

The solution of (2.13) is:

$$\mathbf{A}_1 = \frac{\mu_0}{4\pi} \int_{V_2} \frac{\mathbf{j}_2 dv_2}{|r_{21}|}, \tag{2.14}$$

Where $|r_{21}|$ is the distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. See Figure 2.1.

$$|r_{21}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

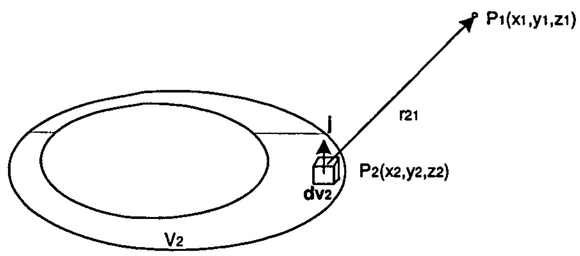

Figure 2.1: The vector potential **A** at the point 1 is given by an integral over the current element $\mathbf{j}dv$ at all points 2.

If V_2 is *thin wire loop* with current I_2, we assume that the $\mathbf{j}$ is uniform across the cross section and the current flows along the tangential direction of the loop, then

$$\mathbf{A}_1 = \frac{\mu_0}{4\pi}\int_{C_2} \frac{I_2 d\mathbf{l}_2}{|r_{21}|}, \tag{2.15}$$

$$\mathbf{B}_1 = \nabla \times \mathbf{A}_1 = \frac{\mu_0}{4\pi}\int_{V_2} \frac{\mathbf{j}_2 \times \mathbf{e}_{21}}{|r_{21}|^2} dv_2, \tag{2.16}$$

where $\mathbf{e}_{21} = \mathbf{r}_{21}/|r_{21}|$. (2.16) is the famous Boit-Savart law.
For a thin wire loop,

$$\mathbf{B}_1 = \nabla \times \mathbf{A}_1 = \frac{\mu_0}{4\pi}\int_{C_2} \frac{I_2 d\mathbf{l}_2 \times \mathbf{e}_{21}}{|r_{21}|^2}. \tag{2.17}$$

2.3 Inductance

Lenz's law: An induced current has a direction such that the magnetic field due to this current opposes the change in the magnetic field that induces the current.
Faraday's law: The induced electro-magnetic force (emf) $\varepsilon = -\frac{d\Phi}{dt}$, where the magnetic flux $\Phi = \int_S \mathbf{B} \cdot ds$. The minus sign for ϵ indicates Lenz's law.

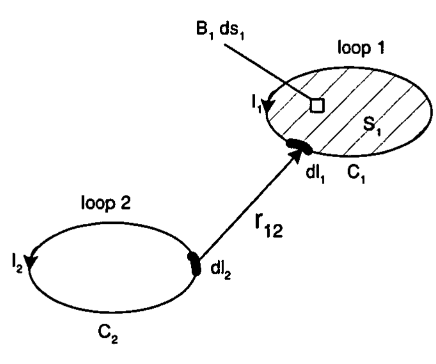

Figure 2.2: The mutual inductance for two wire loops

In Figure 2.2, loop 2 carries a time-varying current I_2. It generates the time-varying magnetic field $\mathbf{B}_1$ at the point 1. By Faraday's law, the induced emf ϵ_1 in loop 1 equals the rate of change of the magnetic flux through the loop 1. We would like to forcus on the thin wire loops from now on.

$$\begin{aligned}
\varepsilon_1 &= -\frac{\partial}{\partial t}\int_{S_1} \mathbf{B}_1 \cdot d\mathbf{s}_1 \\
&= -\frac{\partial}{\partial t}\oint_{C_1} \mathbf{A}_1 \cdot d\mathbf{l}_1 \\
&= -\frac{\partial}{\partial t}\frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2} \frac{I_2 d\mathbf{l}_2 \cdot d\mathbf{l}_1}{|r_{21}|} \\
&= -\frac{dI_2}{dt}\cdot\frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{|r_{21}|}.
\end{aligned}$$

By the definition of the mutual inductance,

$$\varepsilon_1 = -M_{12}\frac{dI_2}{dt},$$

then general form of the mutual inductance for two wire loops is:

$$M_{12} = \frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{|r_{12}|}. \tag{2.18}$$

2.3.1 The mutual inductance of two co-axial circular wire loops

The derivation of the mutual inductance of two-axial circular wire loop can be found in [3]. Figure 2.3 shows two co-axial circular current loops with the radius r_1 and r_2, position z_1 and z_2. The mutual inductance can be calculated by (2.18).

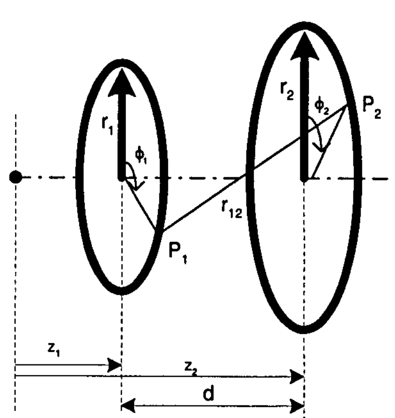

Figure 2.3: The mutual inductance between two co-axial circular loops

In the cylinder coordination system,

$$P_1 : (r_1\cos\phi_1, r_1\sin\phi_1, z_1),$$

$$P_2 : (r_2\cos\phi_2, r_2\sin\phi_2, z_2),$$

then

$$d\mathbf{l}_1 = (-r_1\sin\phi_1, r_1\cos\phi_1, 0)d\phi_1,$$

$$d\mathbf{l}_2 = (-r_2\sin\phi_2, r_2\cos\phi_2, 0)d\phi_2,$$

$$r_{12} = \sqrt{r_1^2 + r_2^2 + (z_2 - z_1)^2 - 2r_1 r_2 \cos(\phi_1 - \phi_2)}.$$

Substitute dl_1,dl_2 and r_{12} into (2.18), we obtain:

$$\begin{aligned} M_{12} &= \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{dl_1 \cdot dl_2}{r_{12}} \\ &= \frac{\mu_0}{4\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{r_1 r_2 \cos(\phi_1 - \phi_2) d\phi_1 d\phi_2}{\sqrt{r_1^2 + r_2^2 + (z_2 - z_1)^2 - 2r_1 r_2 \cos(\phi_1 - \phi_2)}}. \end{aligned}$$

Let $\phi = \phi_1 - \phi_2$, then

$$M_{12} = \mu_0 r_1 r_2 \int_0^{\pi} \frac{\cos\phi}{\sqrt{r_1^2 + r_2^2 + (z_2 - z_1)^2 - 2r_1 r_2 \cos\phi}} d\phi.$$

Let $\phi = 2\alpha$, then

$$\begin{aligned} M_{12} &= \mu_0 \sqrt{r_1 r_2} k \int_0^{\pi/2} \frac{(2\sin^2\alpha - 1) d\alpha}{\sqrt{1 - k^2 \sin^2\alpha}}, \\ &= \mu_0 \sqrt{r_1 r_2} \left[(\frac{2}{k} - k) K(k^2) - \frac{2}{k} E(k^2) \right] \end{aligned} \quad (2.19)$$

where

$$k^2 = \frac{4 r_1 r_2}{(r_1 + r_2)^2 + (z_2 - z_1)^2}, \quad (2.20)$$

$$K(k^2) = \int_0^{\pi/2} \frac{d\alpha}{(1 - k^2 \sin^2\alpha)^{1/2}}, \quad (2.21)$$

$$E(k^2) = \int_0^{\pi/2} (1 - k^2 \sin^2\alpha)^{1/2} d\alpha. \quad (2.22)$$

Here, $K(k^2)$ and $E(k^2)$ are the complete elliptic integral of the first and second kind respectively. (For more information, see [2].

2.3.2 The self-inductance of a circular wire loop

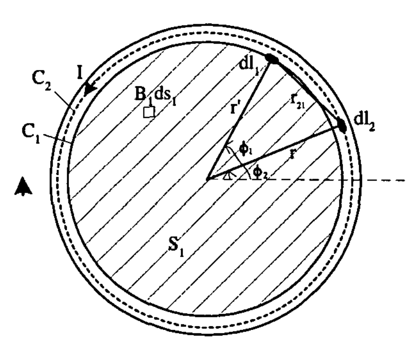

Figure 2.4: The self inductance of a wire loop

The self-inductance of a circular wire loop can be obtained by the same method. Figure 2.4 shows a circular wire loop with the radius r. We assume that the current I flows along C_2, and neglect the magnetic field inside the wire.

The emf ε generated by the current I is

$$\begin{aligned}
\varepsilon &= -\frac{\partial}{\partial t}\int_{S_1} \mathbf{B}_1 \cdot d\mathbf{s}_1 \\
&= -\frac{\partial}{\partial t}\oint_{C_1} \mathbf{A}_1 \cdot d\mathbf{l}_1 \\
&= -\frac{\partial}{\partial t}\frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2} \frac{I d\mathbf{l}_2 \cdot d\mathbf{l}_1}{|r_{21}|} \\
&= -\frac{dI}{dt}\cdot\frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{|r_{21}|}.
\end{aligned}$$

By the definition of the self-inductance,

$$\varepsilon = -L\frac{dI}{dt},$$

we obtain

$$L = \frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{|r_{21}|}.$$

Using the same method in section 2.3.1, we write the self-inductance as:

$$L = \frac{\mu_0}{4\pi}\sqrt{rr'}\left[(\frac{2}{k} - k)K(k^2) - \frac{2}{k}E(k^2)\right],$$

where

$$k^2 = \frac{4rr'}{(r+r')^2},$$

$K(k^2)$ and $E(k^2)$ are the complete elliptic integral of the first and second kind.

Since the loop is a thin wire loop, namely, $r'/r \approx 1$, this means $k \approx 1$.

When $k \approx 1$, $E(k^2) \approx 1$,

$$K(k^2) \approx \ln\frac{4}{\sqrt{1-k^2}} = \ln\frac{4(r+r')}{r-r'}.$$

Let a be the radius of the cross section of the wire loop, then $r - r' = a$. Then the formula to calculate the self inductance for a thin circular wire loop is:

$$L = \mu_0 r\left[\ln\frac{8r}{a} - 2\right]. \tag{2.23}$$

Chapter 3

The two co-axial circular wire loops model: Formulation

In this chapter, the two-axial circular wire loops model is formulated. The position, the radius and the induced current of the shield loop are three unknown variables. By Newton's law, Hooke's law and Kirchhoff's law, two dynamics equations and one circuit equation are obtained.

3.1 A description of the model

The two co-axial circular loops model for this magneto-mechanical problem is shown in Figure 3.1. Table 3.1 presents the relevant physical qualities.

Table 3.1: The variables in the two loops model

	Symbol	Physical meaning
	t	time
	$M(r(t), z(t))$	mutual inductance
shield loop	$I(t)$	induced current
	$r(t), r_0$	radius and initial radius
	$z(t), z_0$	position and initial position
	$A(r(t)), A_0$	cross section area and initial cross section area
	b_z	damping in z-direction
	b_r	dampling in r-direction
	k_z	stiffness in z-direction
	$k_r(r(t))$	stiffness in r-direction
	Q_z	quality factor in z-direction
	Q_r	quality factor in r-direction
	$R(r(t))$	resistance
	m	mass
	$L(r(t))$	self-inductance
source loop	$I_s(t)$	source current
	r_s	radius
	z_s	position
	L_s	self-inductance

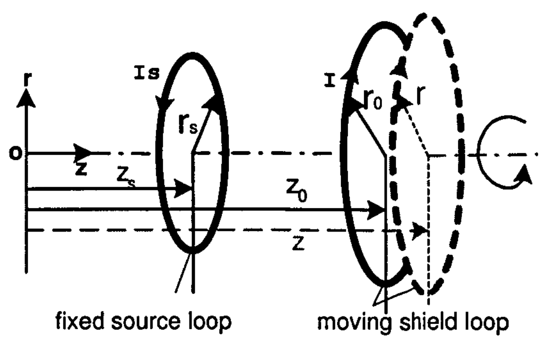

Figure 3.1: The physical model for the two co-axial circular wire loops

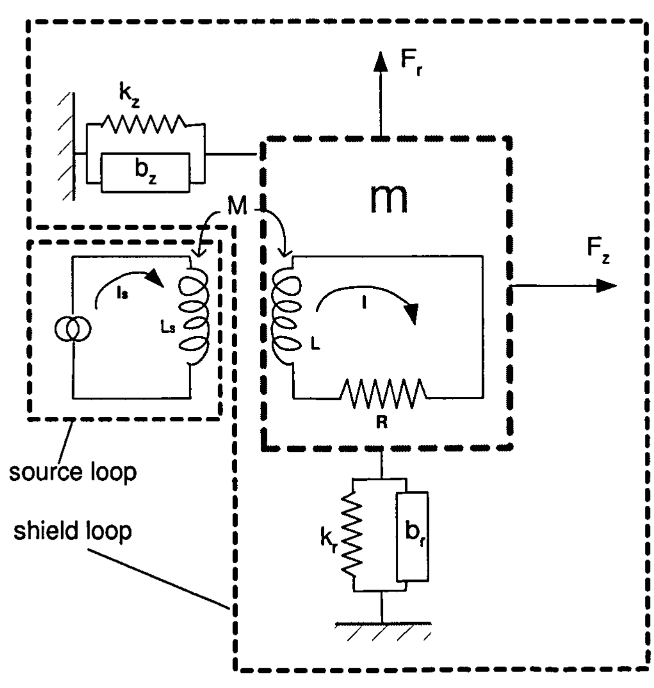

Figure 3.2: The mechanics and circuit model for the two coaxial circular wire loops

Figure 3.1 shows the physical model for the two co-axial circular wire loops. Figure 3.2 shows the mechanics and circuit model for the two co-axial circular wire loops.

- **The fixed source loop**
  The source loop with a position z_s and a radius r_s is fixed. It carries a source current I_s. The self-inductance of the source loop is L_s. In reality, the magnetic field is generated by the super-conducting coils and the gradient coils. The super-conducting coils with a constant current I_m generate the main magnetic field which is strong and steady. The gradient coils with a current $I_g \sin \omega t$ generate the gradient field which is weak and time-varying. Here, we only consider one frequency component of the gradient current signal. In the two loops model, the source loop with the current $I_s = I_m + I_g \sin \omega t$ is used to simulate both the super-conducting coils and the gradient coils with $I_m \gg I_g$ ($I_g/I_m \sim 10^{-3}$). In the multiple loops model, two groups of the source loops can be considered, in which one group carries the current I_m and the other group carries the current $I_g \sin \omega t$.

- **The moving shield loop**
  The shield loop with an initial position z_0 and a initial radius r_0 moves to a new position z and expands to a new radius r because of an induced current I in the magnetic field. So the radius r, the position z and the induced current I are the function of time t. The shield loop has a mass m and a self-inductance L. It can be a slice of the shield cylinder. It is co-axial with the source loop. We can also describe its movements by the displacements u and v, where $u = z - z_0$ and $v = r - r_0$. Because of the movement, the self-inductance L is a function of r, the mutual inductance between the shield loop and the source loop is a function of r and z. In Figure 3.1, F_z and F_r represent the magnetic forces in $z-$ and $r-$direction. k_z and k_r are the stiffness, where k_r is a function of r and $k_z = 0$ for a free loop. b_z and b_r are the damping. They are estimated by the quality factor Q_z and Q_r. The resistance R of the shield loop is a function of its radius r.

- **The unknown variables**
  In this model what we want to know the movements and the induced current of the shield loop due to the time-varying magnetic field generated by the source loop. So the position z, the radius r and the induced current I are the unknown variables.

3.2 The dynamics equations

The general mechanics theory can be found in [6] and [9].
Newton's Second Law of Motion: $\mathbf{F} = m\mathbf{a}$
The acceleration $\mathbf{a}$ of an object is directly proportional to the net force $\mathbf{F}$ acting on it and is inversely proportional to its mass m. The direction of the acceleration is in the direction of the applied net force.

One-dimensional Hooke's Law: $\mathbf{F} = k\mathbf{u}$
The applied force $\mathbf{F}$ of any elastic spring is proportional to its extension $\mathbf{u}$. The spring constant k describes the stiffness of the spring. Hooke's law holds when the extension of the spring is not beyond the elastic limit.

General dynamics equation for a elastic object in motion
For an elastic object in motion, by Newton's law and Hooke's law, the dynamics

equation in the equilibrium state is:

$$m\ddot{\mathbf{u}} + b\dot{\mathbf{u}} + k\mathbf{u} = \mathbf{F}, \tag{3.1}$$

where $\mathbf{F}$ is the applied force, $\mathbf{u}$ is the displacement, m is the mass, b is the damping, k is the stiffness.

Next, we will discuss the magnetic force, the stiffness and the damping, which are shown in Figure 3.2.

- **The magnetic force**

  We use the virtual work method or energy method to formulate the magneto-mechanical system. A good discussion of this method can be found in [8].

  Let W_m be the magnetic energy stored in the system of the two loops model. It can be written in terms of the mutual inductance and the self inductance:

$$W_m = \frac{1}{2}I_s^2 L_s + I_s I M + \frac{1}{2}I^2 L, \tag{3.2}$$

  where I_s and L_s are the current and self-inductance of the source loop; I and L are the current and self-inductance of the shield loop; M is the mutual inductance between the source loop and the shield loop.

  In general, the magnetic force $\mathbf{F}_m$ which leads to the displacement $\partial \mathbf{s}$ is

$$\mathbf{F}_m = \frac{\partial W_m(I, \mathbf{s})}{\partial \mathbf{s}}. \tag{3.3}$$

  In our model, $\mathbf{F}_m = (F_r, F_z)$ is the magnetic force exerted on the shield loop, and $\mathrm{s} = (r, z)$. In two dimensional coordination system, (3.3) is written as:

$$F_r = \frac{\partial W_m}{\partial r}, \quad F_z = \frac{\partial W_m}{\partial z}. \tag{3.4}$$

  Now the virtual work method or the energy method is applied. We equate the work done by the radial force per unit length f_r under a virtual displacement δr to the change in magnetic energy W_m, then,

$$2\pi r f_r \delta r = \frac{\partial W_m}{\partial r}\delta r,$$

  As in the same manner,

$$2\pi r f_z \delta z = \frac{\partial W_m}{\partial z}\delta z.$$

  When the currents don't depend on r and z, the magnetic forces per unit length on the shield loop are:

$$\begin{cases} f_z = \frac{1}{2\pi r}\frac{\partial W_m}{\partial z} = \frac{1}{2\pi r}\left[I_s I \frac{\partial M}{\partial z}\right] \\ f_r = \frac{1}{2\pi r}\frac{\partial W_m}{\partial r} = \frac{1}{2\pi r}\left[I_s I \frac{\partial M}{\partial r} + \frac{1}{2}I^2 \frac{\partial L}{\partial r}\right]. \end{cases} \tag{3.5}$$

- **The stiffness**

  Figure 3.3 shows the expansion of a wire loop from an initial radius r_0 to a radius r. We consider a small segment of the loop.

  For the stress S and the stress per unit length S', we define

$$S_\phi = S\hat{e}_\phi,$$

$$S'_r = S'\hat{e}_r.$$

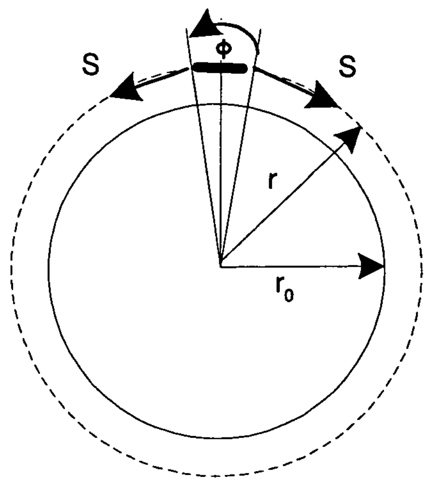

Figure 3.3: The stiffness analysis

The relation between the total stress and the stress per unit length in r direction is:

$$S_r = 2\pi r S'_r.$$

when $d\phi \ll 1$, then

$$2S_\phi \sin\frac{d\phi}{2} \approx S_\phi d\phi = S'_r r d\phi,$$

$$S'_r = \frac{S_\phi}{r}. \tag{3.6}$$

By definition, the total stiffness in $\phi-$direction k_ϕ is:

$$k_\phi = \frac{S_\phi}{\Delta l} = \frac{EA}{l}.$$

where E is the elastic modulus (the Young's modulus), A is the area of the cross section, $l = 2\pi r$ is the total length of the wire loop, $\Delta l = 2\pi\Delta r = 2\pi(r - r_0)$ is the total length extension.

Substituting Δl into the above relation, we obtain:

$$\frac{S_\phi}{\Delta r} = \frac{EA}{r}.$$

By definition, the stiffness in $r-$direction k_r is $k_r = \frac{S_r}{\Delta r}$, then k_r can be calculated by:

$$k_r = \frac{S_r}{\Delta r} = \frac{2\pi r S_r'}{\Delta r} = \frac{2\pi S_\phi}{\Delta r} = \frac{2\pi EA}{r}. \tag{3.7}$$

If we assume the stress is plain stress, then

$$A = A_0(\frac{r_0}{r})^{2\nu}, \tag{3.8}$$

where A_0 is the cross section area before the extension and ν is the Poison ratio. Substituting (3.8) into (3.7), we obtain

$$k_r = \frac{2\pi E A_0}{r}(\frac{r_0}{r})^{2\nu} = \frac{2\pi E A_0}{r_0}(\frac{r}{r_0})^{2\nu+1}. \tag{3.9}$$

- **The mechanical damping**

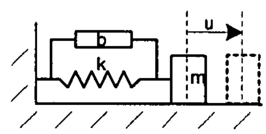

Figure 3.4: A free spring model with damping

Figure 3.4 shows a free spring system with a mass m, a damping b and a stiffness k. u denotes the displacement.

The dynamics equation for this system is:

$$m\ddot{u} + b\dot{u} + ku = 0.$$

The eigenvalues for this equation are:

$$\lambda = -\frac{b}{2m} \pm \sqrt{(\frac{b}{2m})^2 - \frac{k}{m}}.$$

We define

$$\alpha := \frac{b}{2m}, \quad \omega_0 := \sqrt{\frac{k}{m}}, \tag{3.10}$$

then the eigenvalues can be rewritten as:

$$\lambda = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}.$$

When $\alpha \leq \omega_0$, the system starts to oscillate with the frequency $\omega_1 = \sqrt{\omega_0^2 - \alpha^2}$. By the definition, the relaxation time τ is:

$$\tau := \frac{1}{2\alpha},$$

and the quality factor Q is:

$$Q := \omega_1 \tau.$$

By substituting ω_1 and τ into Q, after calculation, we obtain the following relation for b and Q

$$b = \frac{\sqrt{mk}}{\sqrt{Q^2 + 1/4}} \tag{3.11}$$

- **The complete dynamics equations**
Suppose the media of the loop is linear elastic, the stiffness and the damping of the shield loop are k_z and b_z in the z direction and k_r and b_r in the r direction. By Newton's law and Hooke's law, the dynamics equations for a

small segment of the shield loop are:

$$\begin{cases} \frac{m}{2\pi r}(z-z_0)'' + \frac{b_z}{2\pi r}(z-z_0)' + \frac{k_z}{2\pi r}(z-z_0) = f_z, \\ \frac{m}{2\pi r}(r-r_0)'' + \frac{b_r}{2\pi r}(r-r_0)' + \frac{k_r}{2\pi r}(r-r_0) = f_r. \end{cases}$$

Substitute (3.5) into above equations, we obtain:

$$\begin{cases} \frac{m}{2\pi r}(z-z_0)'' + \frac{b_z}{2\pi r}(z-z_0)' + \frac{k_z}{2\pi r}(z-z_0) = \frac{1}{2\pi r}\frac{\partial W_m}{\partial z}, \\ \frac{m}{2\pi r}(r-r_0)'' + \frac{b_r}{2\pi r}(r-r_0)' + \frac{k_r}{2\pi r}(r-r_0) = \frac{1}{2\pi r}\frac{\partial W_m}{\partial r}. \end{cases}$$

Multiply both sides of the above equations by $2\pi r$,

$$m\ddot{z} + b_z\dot{z} + k_z(z-z_0) = \frac{\partial W_m}{\partial z}, \tag{3.12}$$

$$m\ddot{r} + b_r\dot{r} + k_r(r-r_0) = \frac{\partial W_m}{\partial r}. \tag{3.13}$$

Next, substituting (3.2) into (3.12) and (3.13), we obtain the complete dynamics equations:

$$\begin{cases} m\ddot{z} + b_z\dot{z} + k_z(z-z_0) = I_s I\frac{\partial M}{\partial z}, \\ m\ddot{r} + b_r\dot{r} + k_r(r-r_0) = I_s I\frac{\partial M}{\partial r} + \frac{1}{2}I^2\frac{\partial L}{\partial r}. \end{cases} \tag{3.14}$$

3.3 The circuit equations

In the mechanical and circuit model shown in Figure 3.2, by the basic circuit theory, the current I, the resistance R and the magnetic flux Φ in the shield loop satisfy Kirchhoff's voltage law:

$$IR + \frac{d\Phi}{dt} = 0. \tag{3.15}$$

- **The magnetic flux**

The magnetic flux Φ can be written in terms of the current and the inductances:

$$\Phi = IL + I_s M. \tag{3.16}$$

The derivative of Φ is:

$$\frac{d\Phi}{dt} = I\frac{\partial L}{\partial r}\dot{r} + L\dot{I} + I_s(\frac{\partial M}{\partial r}\dot{r} + \frac{\partial M}{\partial z}\dot{z}) + M\dot{I}_s. \tag{3.17}$$

- **The Resistance**

The resistance of the shield loop is given by

$$R = \frac{\rho l}{A}, \tag{3.18}$$

where ρ is the resistivity, l is the length and A is the cross section area.

For the plain stress in the extension, the relation between the cross section area A and the initial cross section A_0 is:

$$A = A_0(\frac{r_0}{r})^{2\nu}. \tag{3.19}$$

Substituting (3.19) into (3.18), we obtain:

$$R = \frac{\rho \cdot 2\pi r}{A} = \frac{\rho 2\pi r_0}{A_0}(\frac{r}{r_0})^{2\nu+1} = R_0(\frac{r}{r_0})^{2\nu+1}, \tag{3.20}$$

where the initial resistance R_0 is

$$R_0 = \frac{\rho 2\pi r_0}{A_0}. \tag{3.21}$$

- **The complete circuit equation**

Substituting (3.20) and (3.17) into (3.15), we obtain:

$$M\dot{I}_s + I_s(\frac{\partial M}{\partial r}\dot{r} + \frac{\partial M}{\partial z}\dot{z}) + L\dot{I} + I\frac{\partial L}{\partial r}\dot{r} + IR_0(\frac{r}{r_0})^{2\nu+1} = 0. \tag{3.22}$$

We rewrite the equation in the following form:

$$L\dot{I} + (\frac{\partial L}{\partial r}\dot{r} + R_0(\frac{r}{r_0})^{2\nu+1})I = -M\dot{I}_s - I_s(\frac{\partial M}{\partial r}\dot{r} + \frac{\partial M}{\partial z}\dot{z}). \tag{3.23}$$

3.4 A more general discussion on modelling

By the previous analysis, the general form of the system of the equations are:

$$\begin{cases} m\ddot{z} + b_z\dot{z} + k_z(z - z_0) = F_z, \\ m\ddot{r} + b_r\dot{r} + k_r(r - r_0) = F_r, \\ RI = -\frac{d\Phi}{dt}. \end{cases} \tag{3.24}$$

The right hands of the first two equations of (3.24) are the forces. The right hand of the third equation of (3.24) is the changing rate of the magnetic flux. In (3.14), the forces $\mathbf{F} = \mathbf{F}_{mag} = \frac{\partial W_m}{\partial s}$ are the magnetic forces only and they are described by the currents and the inductance. In general, there are more options to describe the force and the magnetic flux:

1. If there is an extra mechanical force $\mathbf{F}_{mech}$, then

$$\mathbf{F} = \mathbf{F}_{mag} + \mathbf{F}_{mech},$$

where $\mathbf{F} = (F_z, F_r)$.

2. If the magnetic forces and flux are described by the magnetic induction $\mathbf{B} = (B_z, B_r)$, then by the formula $\mathbf{F} = I \oint_C d\mathbf{l} \times \mathbf{B}$, the magnetic forces are:

$$F_z = 2\pi r I B_r(r, z, t), \quad F_r = -2\pi r I B_z(r, z, t). \tag{3.25}$$

By the formula $\Phi = \oint_A \mathbf{B} \cdot ds$, the magnetic flux is:

$$\Phi(r, z, t) = 2\pi \int_0^r B_z(r', z, t) r' dr'.$$

3. If the magnetic forces are described by the magnetic flux Φ:
Assume the magnetic flux of the source loop is denoted by Φ_s and this magnetic flux doesn't change by the vibration of the shield loop, then the magnetic energy is

$$W_m = \frac{1}{2}(I_s \Phi_s + I\Phi).$$

The related magnetic forces are:

$$F_z = \frac{1}{2} I \frac{\partial \Phi}{\partial z}, \quad F_r = \frac{1}{2} I \frac{\partial \Phi}{\partial r}. \tag{3.26}$$

Except the forces, the other two physical quantities, B and Φ, are very difficult to be prescribed in reality because of the movements of the shield loop.

3.5 Summary

For the two-coaxial wire loops model, the mathematical equations are:
Find $t \to z(t)$, $t \to r(t)$, $t \to I(t)$, such that

$$\begin{cases} m\ddot{z} + b_z \dot{z} + k_z(z - z_0) = I_s I \frac{\partial M}{\partial z}, \\ m\ddot{r} + b_r \dot{r} + k_r(r - r_0) = I_s I \frac{\partial M}{\partial r} + \frac{1}{2} I^2 \frac{\partial L}{\partial r}, \\ L\dot{I} + (\frac{\partial L}{\partial r}\dot{r} + R)I = -M\dot{I}_s - I_s(\frac{\partial M}{\partial r}\dot{r} + \frac{\partial M}{\partial z}\dot{z}). \end{cases} \tag{3.27}$$

Here, b_z, b_r, k_r and R can be calculated by

$$\begin{cases} b_z = \frac{\sqrt{mk_z}}{\sqrt{Q_z^2 + \frac{1}{4}}}, \\ b_r = \frac{\sqrt{mk_r}}{\sqrt{Q_r^2 + \frac{1}{4}}}, \\ k_r = \frac{2\pi E A_0}{r_0} (\frac{r_0}{r})^{2\nu+1}, \\ R = R_0 (\frac{r}{r_0})^{2\nu+1} \text{ and } R_0 = \rho \frac{2\pi r_0}{A_0}, \end{cases} \tag{3.28}$$

where E is the young's modulus, μ is the Poison ratio, ρ is the resistivity.

The mutual inductance can be calculate by

$$M = \mu_0\sqrt{r_s r}\left[(\frac{2}{k} - k)K(k^2) - \frac{2}{k}E(k^2)\right],$$

where

$$k^2 = \frac{4r_s r}{(r_s + r)^2 + (z - z_s)^2},$$

and $K(k^2)$ and $E(k^2)$ are the complete elliptic integrals of the first and second kind respectively.

The self-inductance can be calculate by

$$L = \mu_0 r\left[\ln\frac{8r}{a} - 2\right],$$

where a is the radius of the cross section for a loop with circular cross section, or half of the thickness for a loop with rectangle cross section.
The source current is given by

$$I_s = I_m + I_g \sin\omega t.$$

The initial conditions are:

$$z(0) = z_0, \quad \dot{z}(0) = 0, \quad r(0) = r_0, \quad \dot{r}(0) = 0, \quad I(0) = 0. \tag{3.29}$$

In (3.27), the stiffness k_z is a sensitive parameter. For a free shield loop, $k_z = 0$. But, because we take the shield loop to be a slice of the shield of an MRI scanner, it can't be a free loop. Then, how to estimate this parameter? In the reality, we use the following relation to do the estimation.

$$\frac{k_z}{k_r} = (\frac{\omega_{0z}}{\omega_{0r}})^2,$$

where ω_{0z} and ω_{0r} are the mechanical resonant frequencies in $z-$direction and $r-$direction. The parameter ω_{0r} can be calculated by $\omega_{0r} = \sqrt{k_r/m}$. For example, for an aluminum wire loop with the radius $r = 0.45$, if the mechanical resonant frequency is $f_{0z} = 5Hz$, then $k_z/k_r = 7.7 \times 10^{-6}$.

Some explanation of the equations:
The right parts of the dynamics equations are the magnetic forces.
In the circuit equation, the induced current is produced by the three kinds of the flux change:

- $L\dot{I}$: the flux change due to the induced current change;
- $M\dot{I}_s$: the flux change due to the source current change. If the source current is a constant, this item will be zero;
- $\frac{\partial L}{\partial r}\dot{r}I$ and $(\frac{\partial M}{\partial r}\dot{r} + \frac{\partial M}{\partial z}\dot{z})I_s$: the flux change due to the movements of the loop.

Chapter 4

The two co-axial circular wire loops model: Simplification

The simplification procedure in this chapter starts from the *complete nonlinear system* (4.1) of ordinary differential equations obtained in the last chapter (see (3.27)). By the dimensionless analysis and scaling the system, the *dimensionless form of the complete nonlinear system* is obtained. Then, after linearizing the mutual inductance and self inductance function, a *simplified nonlinear system* is obtained. By neglecting the small terms in the simplified nonlinear system, finally, a *linear system* is obtained, which can be solved analytically.

4.1 The complete nonlinear system

For our convenience, we repeat the complete nonlinear system derived in the last chapter here. Table 4.1 describes the variables which are used in the system.

Find $t \to z(t)$, $t \to r(t)$, $t \to I(t)$, such that:

$$\begin{cases} m\ddot{z} + b_z\dot{z} + k_z(z - z_0) = I_s I \frac{\partial M}{\partial z}, \\ m\ddot{r} + b_r\dot{r} + k_r(r - r_0) = I_s I \frac{\partial M}{\partial r} + \frac{1}{2}I^2\frac{\partial L}{\partial r}, \\ L\dot{I} + (\frac{\partial L}{\partial r}\dot{r} + R)I = -M\dot{I}_s - (\frac{\partial M}{\partial r}\dot{r} + \frac{\partial M}{\partial z}\dot{z})I_s. \end{cases} \tag{4.1}$$

The initial conditions are:

$$z(0) = z_0, \quad \dot{z}(0) = 0, \quad r(0) = r_0, \quad \dot{r}(0) = 0, \quad I(0) = 0. \tag{4.2}$$

The system of equations (4.1) is a system of non-linear ordinary differential equations. It is not trivial, especially because the mutual inductance includes an elliptic integral, and the self-inductance includes a logarithm function. The first step of the our simplification procedure is a dimensionless analysis.

Table 4.1: The description of the variables used in (4.1)

Symbol	Physics meaning	Relation
t	time	
z	position of the shield loop	
r	radius of the shield loop	
I	induced current in the shield loop	
r_0	initial radius of the shield loop	
z_0	initial position of the shield loop	
A_0	initial cross section area of the shield loop	
A	cross section area of the shield loop	$A = A_0(\frac{r_0}{r})^{2\nu}$
a_0	initial radius of the cross section of the shield loop	
a	radius of the cross section of the shield loop	$a = a_0(\frac{r_0}{r})^{\nu}$
r_s	radius of the source loop	
z_s	position of the source loop	
ω	the source frequency	
I_m	constant current in the source loop	
I_g	amplitude of the time-varying current in the source loop	
I_s	source current in the source loop	$I_s = I_m + I_g \sin \omega t$
m	mass of the shield loop	$m = 2\pi r_0 A_0 \rho_m$
ρ_m	mass density	
E	Young's modulus	
ν	Poisson ratio for stiffness	
ρ	resistance permeability	
μ_0	permeability in vacuum	
Q_z	quality factor in z-direction	
Q_r	quality factor in r-direction	
k_z	stiffness in z-direction	
k_{r0}	stiffness in r-direction at $r = r_0$	$k_{r0} = 2\pi E A_0 / r_0$
k_r	stiffness in r-direction	$k_r = k_{r0}(\frac{r_0}{r})^{2\nu+1}$
b_z	damping in z-direction	$b_z = \sqrt{mk_z/(Q_z^2 + 1/4)}$
b_r	damping in r-direction	$b_z = \sqrt{mk_{r0}/(Q_r^2 + 1/4)}$
R_0	initial resistance of the shield loop	$R_0 = 2\pi r_0 \rho / A_0$
R	resistance of shield loop	$R = R_0(r/r_0)^{2\nu+1}$
M	mutual inductance	$M = \mu_0\sqrt{r_s r}\left[(\frac{2}{k} - k)K(k^2) - \frac{2}{k}E(k^2)\right]$ where $k^2 = \frac{4r_s r}{(r_s+r)^2+(z-z_s)^2}$
$K(k^2)$	the complete integrals of first kind	
$E(k^2)$	the complete integrals of second kind	
L	self inductance	$L = \mu_0 r\left[\ln\frac{8r}{a} - 2\right]$

4.2 The dimensionless form of the complete nonlinear system

The purpose of the dimensionless analysis is to properly scale the system. By this method, we can find the most important and dominant quantities.
In the complete nonlinear system (4.1), the basic physical qualities are the length and the current. A proper reference system of the length and the current is very important for scaling the system. There are two proper possibilities to chose the length reference system: the radius of the source loop r_s or the initial radius of the shield loop r_0. There are also two proper possibilities to chose the current reference system: the steady part of the source current I_m and the amplitude of the time varying current I_g. For a general purpose, we first use the arbitrary scaling parameters l, l', I_0, t_0 in the following dimensionless analysis. Later on, we will make a decision depending on our convenience.

4.2.1 The dimensionless form of the mutual inductance and its derivative

The mutual inductance derived in Chapter 2 (see (2.19)) is:

$$\begin{cases} M = \mu_0\sqrt{r_s r}\left[(\frac{2}{k}-k)K(k^2)-\frac{2}{k}E(k^2)\right], \\ k^2 = \frac{4r_s r}{(r_s+r)^2+(z-z_s)^2}, \end{cases} \tag{4.3}$$

where $K(k^2)$ and $E(k^2)$ are the complete elliptic integral of the first and second kind respectively.

Let us define the dimensionless radius of the source loop α_s, the dimensionless radius of the shield loop α and the dimensionless position of the shield loop β by the arbitrary length scaling parameter l':

$$\alpha_s := \frac{r_s}{l'}, \quad \alpha := \frac{r}{l'}, \quad \beta := \frac{z-z_s}{l'}, \tag{4.4}$$

The mutual inductance function can be written in terms of the dimensionless mutual inductance $\tilde{M}$:

$$\begin{cases} M = \mu_0 l'\tilde{M}(\alpha_s,\alpha,\beta), \\ \tilde{M}(\alpha_s,\alpha,\beta) = \sqrt{\alpha_s\alpha}\left[(\frac{2}{k}-k)K(k^2)-\frac{2}{k}E(k^2)\right], \\ k^2 = \frac{4\alpha_s\alpha}{(\alpha_s+\alpha)^2+\beta^2}. \end{cases} \tag{4.5}$$

The derivatives of the complete elliptic integrals are given by

$$\begin{cases} k\frac{dK}{dk} = \frac{E(k^2)}{1-k^2} - K(k^2), \\ k\frac{dE}{dk} = E(k^2) - K(k^2). \end{cases}$$

The derivatives of the mutual inductance are:

$$\begin{cases} \frac{\partial M}{\partial z} = \mu_0\frac{\partial\tilde{M}}{\partial\beta}, \quad \frac{\partial\tilde{M}}{\partial\beta} = \frac{\beta}{\sqrt{(\alpha_s+\alpha)^2+\beta^2}}\left[K(k^2) - \frac{\alpha_s^2+\alpha^2+\beta^2}{(\alpha_s-\alpha)^2+\beta^2}E(k^2)\right], \\ \frac{\partial M}{\partial r} = \mu_0\frac{\partial\tilde{M}}{\partial\alpha}, \quad \frac{\partial\tilde{M}}{\partial\alpha} = \frac{\alpha}{\sqrt{(\alpha_s+\alpha)^2+\beta^2}}\left[K(k^2) + \frac{\alpha_s^2-\alpha^2-\beta^2}{(1-\alpha)^2+\beta^2}E(k^2)\right]. \end{cases} \tag{4.6}$$

Since the radius of the source loop is fixed, we may chose $l' = r_s$, then

$$\alpha_s = 1, \quad \alpha = \frac{r}{r_s}, \quad \beta = \frac{z - z_s}{r_s}.$$

and the parameter k becomes a function of α and β:

$$k^2 = \frac{4\alpha}{(1+\alpha)^2 + \beta^2}.$$

The mutual inductance and its derivatives turn into:

$$\begin{cases} M = \mu_0 r_s \tilde{M}(\alpha,\beta), & \tilde{M}(\alpha,\beta) = \sqrt{\alpha}\left[(\frac{2}{k} - k)K(k^2) - \frac{2}{k}E(k^2)\right]; \\ \frac{\partial M}{\partial z} = \mu_0 \frac{\partial \tilde{M}}{\partial \beta}, & \frac{\partial \tilde{M}}{\partial \beta} = \frac{\beta}{\sqrt{(1+\alpha)^2+\beta^2}}\left[K(k^2) - \frac{1+\alpha^2+\beta^2}{(1-\alpha)^2+\beta^2}E(k^2)\right]; \\ \frac{\partial M}{\partial r} = \mu_0 \frac{\partial \tilde{M}}{\partial \alpha}, & \frac{\partial \tilde{M}}{\partial \alpha} = \frac{\alpha}{\sqrt{(1+\alpha)^2+\beta^2}}\left[K(k^2) + \frac{1-\alpha^2-\beta^2}{(1-\alpha)^2+\beta^2}E(k^2)\right]. \end{cases} \tag{4.7}$$

Figure 4.1 shows the dimensionless mutual inductance $\tilde{M}(\alpha,\beta)$ and its derivatives $\frac{\partial \tilde{M}}{\partial \alpha}$ and $\frac{\partial \tilde{M}}{\partial \beta}$. The length scaling parameter is r_s.

Figure 4.2 shows the distribution of the dimensionless mutual inductance $\tilde{M}(\alpha,\beta)$ and its derivatives $\frac{\partial \tilde{M}}{\partial \alpha}$ and $\frac{\partial \tilde{M}}{\partial \beta}$ by a changing dimensionless radius α for the fixed β. The length scaling parameter is r_s.

Figure 4.3 shows the distribution of the dimensionless mutual inductance $\tilde{M}(\alpha,\beta)$ and its derivatives $\frac{\partial \tilde{M}}{\partial \alpha}$ and $\frac{\partial \tilde{M}}{\partial \beta}$ by a changing dimensionless position β for the fixed α. The length scaling parameter is r_s.

4.2.2 The dimensionless form of the self-inductance and its derivative

The self-inductance derived in Chapter 2 is given by

$$L(r) = \mu_0 r \left[\ln\frac{8r}{a} - 2\right], \tag{4.8}$$

and for a plain stress,

$$a = a_0(\frac{r_0}{r})^{\nu}, \tag{4.9}$$

Here, r_0 and r are the radii; a_0 and a are the cross section radii or half of the thickness for a rectangular cross section loop; ν is the Poisson ratio. The suffix 0 denotes the corresponding initial state.
Then the self-inductance and its derivative can be written as:

$$\begin{cases} L = \mu_0 r \left[\ln\frac{8r^{\nu+1}}{a_0 r_0^{\nu}} - 2\right], \\ \frac{\partial L}{\partial r} = \mu_0 \left[\ln\frac{8r^{\nu+1}}{a_0 r_0^{\nu}} - 1 + \nu\right]. \end{cases} \tag{4.10}$$

Figure 4.4 shows the self-inductance and its derivative.
Let us define the ratio:

$$\gamma = \frac{a}{l'}. \tag{4.11}$$

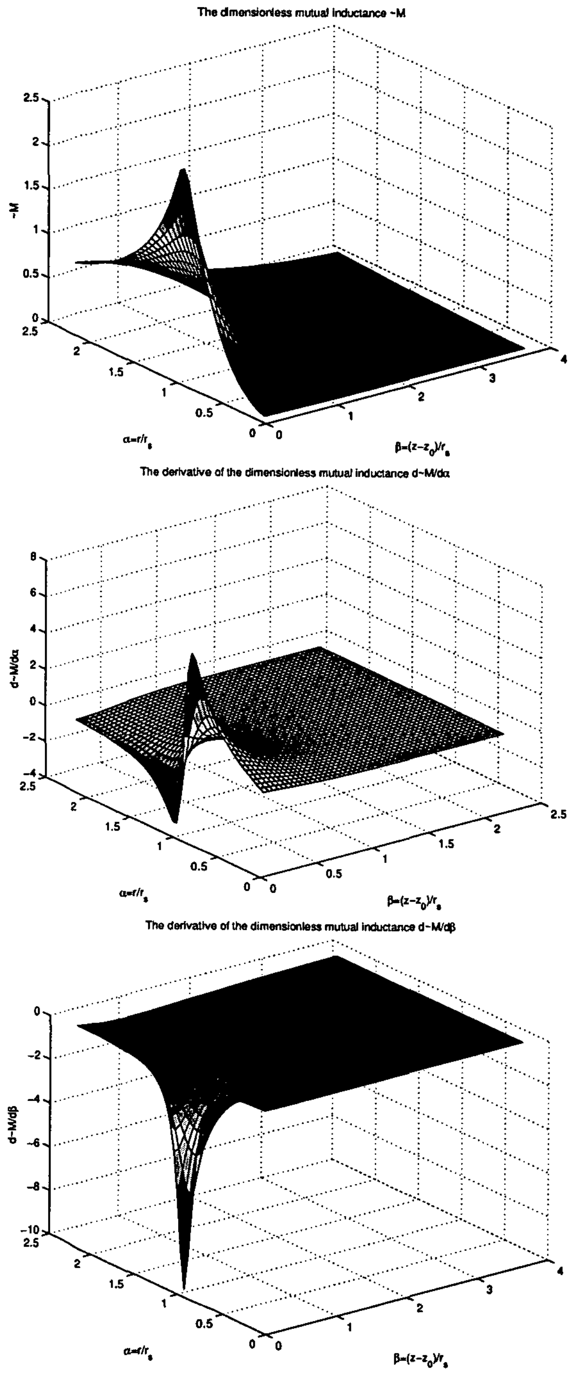

Figure 4.1: The dimensionless mutual inductance and its derivatives

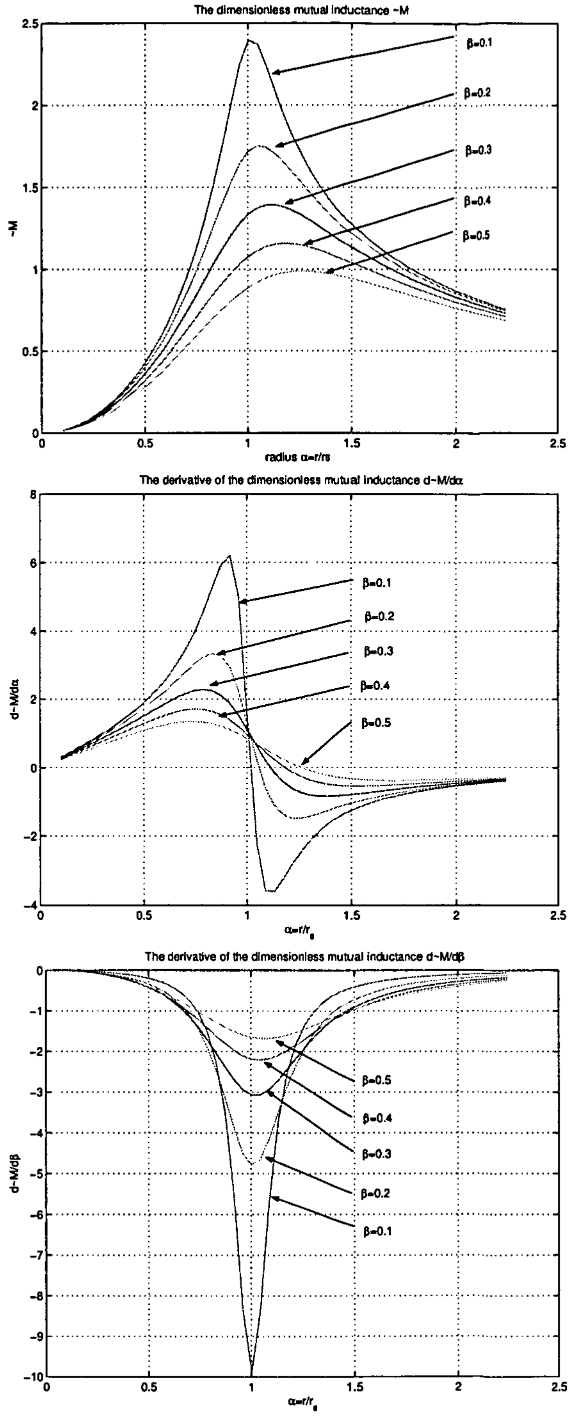

Figure 4.2: The dimensionless mutual inductance and its derivatives, depending on α for some β

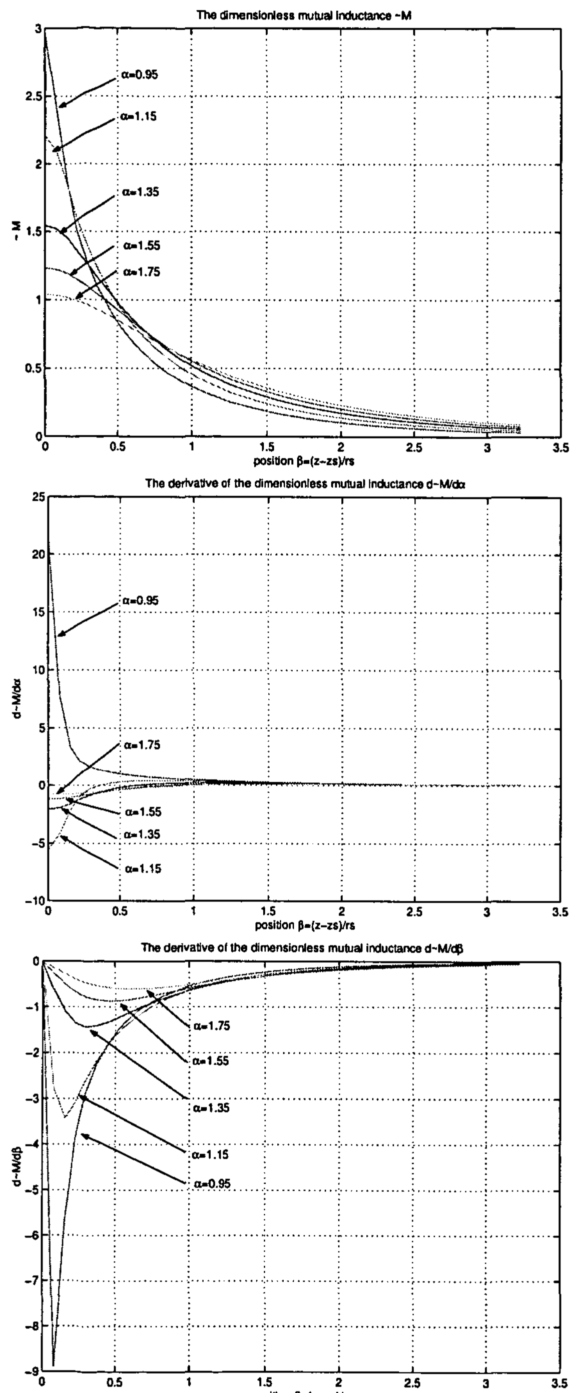

Figure 4.3: The dimensionless mutual inductance and its derivatives, depending on β for some α

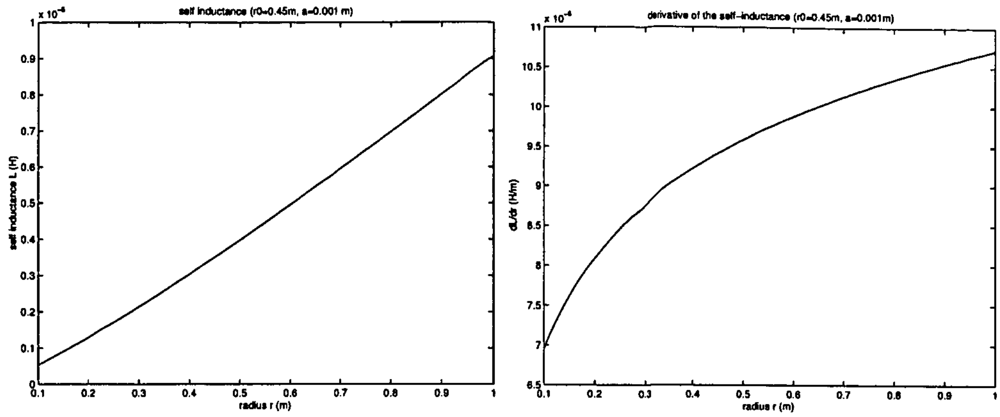

Figure 4.4: The self inductance function and its derivative

The self-inductance can be written as

$$\begin{cases} L = \mu_0 l' \tilde{L}, \\ \tilde{L} = \alpha \left[\ln \frac{8\alpha}{\gamma} - 2 \right]. \end{cases} \tag{4.12}$$

The derivative of the self-inductance is:

$$\begin{cases} \frac{\partial L}{\partial r} = \mu_0 \frac{\partial \tilde{L}}{\partial \alpha}, \\ \frac{\partial \tilde{L}}{\partial \alpha} = \ln \frac{8\alpha^{\nu+1}}{\gamma_0 \alpha_0^{\nu}} - 1 + \nu. \end{cases} \tag{4.13}$$

Here

$$\alpha = \frac{r}{l'} \text{ and } \alpha_0 = \frac{r_0}{l'}, \gamma_0 = \frac{a_0}{l'}.$$

4.2.3 The dimensionless form of the complete nonlinear system

We introduce two new unknown variables, the displacement U and V instead of the unknown variables r and z in (4.1), because the displacement U and V are really what we are interested of. The reason to use the radius r and the position z to be the unknown variables is that they are more intuitive for deriving the inductance function and the magnetic forces.
U denotes the displacement in $z-$ direction and V denotes the displacement in $r-$ direction,

$$U = z - z_0 \quad \text{and} \quad V = r - r_0 \tag{4.14}$$

Then we define the dimensionless parameters by

$$\left.\begin{array}{ll} \text{the dimensionless displacement in } z- \text{ direction:} & \tilde{U} := \frac{U}{l} = \frac{z-z_0}{l}, \\ \text{the dimensionless displacement in } r- \text{ direction:} & \tilde{V} := \frac{V}{l} = \frac{r-r_0}{l}, \\ \text{the dimensionless induced current:} & \tilde{I} := \frac{I}{I_0}, \\ \text{the dimensionless time:} & \tau := \frac{t}{t_0}, \\ \text{the dimensionless source current:} & \tilde{I}_s = \frac{I_s}{I_0}. \end{array}\right\} \tag{4.15}$$

Here, $\tilde{U}$, $\tilde{V}$, $\tilde{I}$ are the new unknown variables. l, I_0 and t_0 are the general scaling parameters for the length, current and time, respectively.

Let $\frac{d\tilde{U}}{d\tau} = \dot{\tilde{U}}$, $\frac{d\tilde{V}}{d\tau} = \dot{\tilde{V}}$, $\frac{d\tilde{I}}{d\tau} = \dot{\tilde{I}}$, then by using the transformation rules for the derivatives, we obtain:

$$\begin{cases} z = z_0 + l\tilde{U}, & \dot{z} = \frac{l}{t_0}\dot{\tilde{U}}, \quad \ddot{z} = \frac{l}{t_0^2}\ddot{\tilde{U}}; \\ r = r_0 + l\tilde{V}, & \dot{r} = \frac{l}{t_0}\dot{\tilde{V}}, \quad \ddot{r} = \frac{l}{t_0^2}\ddot{\tilde{V}}; \\ I = I_0\tilde{I}, & \dot{I} = \frac{I_0}{t_0}\dot{\tilde{I}}, \end{cases} \tag{4.16}$$

Substituting (4.15) and (4.16) into (4.1), we obtain the dimensionless form of the complete nonlinear system:

$$\begin{cases} \left[\frac{ml}{t_0^2}\right]\ddot{\tilde{U}} + \left[\frac{b_z l}{t_0}\right]\dot{\tilde{U}} + [k_z]\tilde{U} = \left[I_0^2\mu_0\right]\tilde{I}_s\tilde{I}\frac{\partial\tilde{M}}{\partial\beta}, \\ \left[\frac{ml}{t_0^2}\right]\ddot{\tilde{V}} + \left[\frac{b_r l}{t_0}\right]\dot{\tilde{V}} + \left[k_{r0}\tilde{k}_r\right]\tilde{V} = \left[I_0^2\mu_0\right]\tilde{I}_s\tilde{I}\frac{\partial\tilde{M}}{\partial\alpha} + \frac{1}{2}\left[I_0^2\mu_0\right]\tilde{I}^2\frac{\partial\tilde{L}}{\partial\alpha}, \\ \left[\frac{I_0\mu_0 l'}{t_0}\tilde{L}\right]\dot{\tilde{I}} + \left[\frac{I_0\mu_0}{t_0}\frac{\partial\tilde{L}}{\partial\alpha}l\dot{\tilde{V}} + I_0R_0\tilde{R}\right]\tilde{I} = -\left[\frac{I_0\mu_0 l'}{t_0}\tilde{M}\right]\dot{\tilde{I}}_s - \left[\frac{I_0\mu_0}{t_0}l\left(\frac{\partial\tilde{M}}{\partial\alpha}\dot{\tilde{V}} + \frac{\partial\tilde{M}}{\partial\beta}\dot{\tilde{U}}\right)\right]\tilde{I}_s. \end{cases} \tag{4.17}$$

namely,

$$\begin{cases} \ddot{\tilde{U}} + \left[\frac{b_z t_0}{m}\right]\dot{\tilde{U}} + \left[\frac{k_z t_0^2}{m}\right]\tilde{U} = \left[\frac{\mu_0 I_0^2 t_0^2}{ml}\right]\tilde{I}_s\tilde{I}\frac{\partial\tilde{M}}{\partial\beta}, \\ \ddot{\tilde{V}} + \left[\frac{b_r t_0}{m}\right]\dot{\tilde{V}} + \left[\frac{k_{r0}t_0^2}{m}\tilde{k}_r\right]\tilde{V} = \left[\frac{\mu_0 I_0^2 t_0^2}{ml}\right]\tilde{I}_s\tilde{I}\frac{\partial\tilde{M}}{\partial\alpha} + \frac{1}{2}\left[\frac{\mu_0 I_0^2 t_0^2}{ml}\right]\tilde{I}^2\frac{\partial\tilde{L}}{\partial\alpha}, \\ \tilde{L}\dot{\tilde{I}} + \left[\frac{l}{l'}\frac{\partial\tilde{L}}{\partial\alpha}\dot{\tilde{V}} + \frac{R_0 t_0}{\mu_0 r_s}\tilde{R}\right]\tilde{I} = -\tilde{M}\dot{\tilde{I}}_s - \left[\frac{l}{l'}\left(\frac{\partial\tilde{M}}{\partial\alpha}\dot{\tilde{V}} + \frac{\partial\tilde{M}}{\partial\beta}\dot{\tilde{U}}\right)\right]\tilde{I}_s, \end{cases} \tag{4.18}$$

where

$$\begin{cases} \tilde{k}_r = \left(\frac{r_0}{r}\right)^{2\nu+1} = \left(\frac{\alpha_0}{\alpha_0 + \frac{l}{l'}\tilde{V}}\right)^{2\nu+1}, \\ \tilde{R} = \left(\frac{r}{r_0}\right)^{2\nu+1} = \left(\frac{\alpha_0 + \frac{l}{l'}\tilde{V}}{\alpha_0}\right)^{2\nu+1}, \\ \alpha = \frac{r}{l'} = \alpha_0 + \frac{l}{l'}\tilde{V}, \\ \beta = \frac{z - z_s}{l'} = \beta_0 + \frac{l}{l'}\tilde{U}, \end{cases} \tag{4.19}$$

and

$$\alpha_0 = \frac{r_0}{l'} \quad \beta_0 = \frac{z_0 - z_s}{l'}. \tag{4.20}$$

4.2.4 Scaling the system

In the last section, the scaling parameters l, l', I_0 and t_0 are very general parameters. In theory, changing the scaling parameter doesn't change the solution. In practice, because of the numerical computing accuracy, we always chose the proper scaling parameters such that the solutions are close to order 1.

We choose the scaling parameters as the following:

$$\left.\begin{array}{lll} \text{the length scaling:} & l' := r_0 \\ \text{the displacement scaling:} & l = \epsilon r_0 = \epsilon l', \\ \text{the current scaling:} & I_0 = \epsilon I_m, \\ \text{the time scaling:} & t_0 = \frac{1}{\omega_{0r}} = \sqrt{\frac{m}{k_{r0}}}. \end{array}\right\} \tag{4.21}$$

Here are some explanations for the scaling parameters:

1. the small parameter ϵ
   In (4.21), ϵ is a general small parameter. It shows that $r - r_0$ and $z - z_0$ are much smaller than r_0 and the induced current I is much smaller than I_m which is the steady part of the source current. In the system, there is another small parameter $\epsilon' = \frac{I_a}{I_m}$. We may take $\epsilon = \epsilon'$ for our convenience.

2. the time scaling $t_0 = 1/\omega_{0r}$
   $\omega_{0r} = \sqrt{\frac{k_{r0}}{m}}$ is the mechanical resonance frequency in $r-$ direction. By this scaling, the dimensionless resonant frequency in $r-$ direction is about 1. The dimensionless source frequency $\tilde{\omega} := \frac{\omega}{\omega_{0r}}$.

The dimensionless source current can be written in terms of the dimensionless source frequency $\tilde{\omega}$ as

$$\tilde{I}_s = \frac{1}{\epsilon} + \frac{\epsilon'}{\epsilon} \sin \tilde{\omega}\tau. \tag{4.22}$$

By (3.11), the damping b_z and b_r are

$$b_z = \frac{mk_z}{\sqrt{Q1^2+1/4}} \quad \text{and} \quad b_r = \frac{mk_{r0}}{\sqrt{Q2^2+1/4}}. \tag{4.23}$$

Substituting (4.21), (4.22) and (4.23) into (4.18), we obtain:

$$\left\{\begin{array}{l} \ddot{\tilde{U}} + \left[\frac{\sqrt{\frac{k_z}{k_{r0}}}}{\sqrt{Q_z^2+1/4}}\right]\dot{\tilde{U}} + \left[\frac{k_z}{k_{r0}}\right]\tilde{U} = \left[\frac{\mu_0 I_m^2}{k_{r0} r_0}\right](1+\epsilon' \sin\tilde{\omega}\tau)\tilde{I}\frac{\partial \tilde{M}}{\partial \beta}, \\ \ddot{\tilde{V}} + \left[\frac{1}{\sqrt{Q_r^2+1/4}}\right]\dot{\tilde{V}} + \left[\tilde{k}_r\right]\tilde{V} = \left[\frac{\mu_0 I_m^2}{k_{r0} r_0}\right](1+\epsilon' \sin\tilde{\omega}\tau)\tilde{I}\frac{\partial \tilde{M}}{\partial \alpha} + \frac{\epsilon}{2}\left[\frac{\mu_0 I_m^2}{k_{r0} r_0}\right]\tilde{I}^2\frac{\partial \tilde{L}}{\partial \alpha}, \\ \tilde{L}\dot{\tilde{I}} + \left[\epsilon\frac{\partial \tilde{L}}{\partial \alpha}\dot{\tilde{V}} + \frac{R_0 t_0}{\mu_0 r_0}\tilde{R}\right]\tilde{I} = -\tilde{M}\tilde{\omega}\frac{\epsilon'}{\epsilon}\cos\tilde{\omega}\tau - \left[\frac{\partial \tilde{M}}{\partial \alpha}\dot{\tilde{V}} + \frac{\partial \tilde{M}}{\partial \beta}\dot{\tilde{U}}\right](1+\epsilon' \sin\tilde{\omega}\tau), \end{array}\right. \tag{4.24}$$

where

$$\left\{\begin{array}{l} \tilde{k}_r = \left(\frac{\alpha_0}{\alpha_0+\epsilon\tilde{V}}\right)^{2\nu+1} = \left(\frac{1}{1+\epsilon\tilde{V}}\right)^{2\nu+1}, \\ \tilde{R} = \left(\frac{r}{r_0}\right)^{2\nu+1} = \left(\frac{\alpha_0+\epsilon\tilde{V}}{\alpha_0}\right)^{2\nu+1} = (1+\epsilon\tilde{V})^{2\nu+1}. \end{array}\right. \tag{4.25}$$

After scaling, the mutual inductance and self inductance and their derivatives are:

$$\begin{cases} \tilde{M} = \sqrt{\alpha_s\alpha}\left[(\frac{2}{k}-k)K(k^2)-\frac{2}{k}E(k^2)\right], \\ \frac{\partial\tilde{M}}{\partial\beta} = \frac{\beta}{\sqrt{(\alpha_s+\alpha)^2+\beta^2}}\left[K(k^2)-\frac{\alpha_s^2+\alpha^2+\beta^2}{(\alpha_s-\alpha)^2+\beta^2}E(k^2)\right], \\ \frac{\partial\tilde{M}}{\partial\alpha} = \frac{\alpha}{\sqrt{(1+\alpha)^2+\beta^2}}\left[K(k^2)+\frac{1-\alpha^2-\beta^2}{(1-\alpha)^2+\beta^2}E(k^2)\right], \\ \\ \tilde{L} = \alpha\left[\ln\frac{8\alpha}{\gamma}-2\right], \\ \frac{\partial\tilde{L}}{\partial\alpha} = \ln\frac{8\alpha^{\nu+1}}{\gamma_0\alpha_0^{\nu}}-1+\nu. \end{cases} \tag{4.26}$$

where

$$k^2 = \frac{4\alpha_s\alpha}{(\alpha_s+\alpha)^2+\beta^2},$$

and

$$\alpha = 1+\epsilon\tilde{V},\quad \beta=\beta_0+\epsilon\tilde{U},\quad \alpha_s=\frac{r_s}{r_0},\quad \beta_0=\frac{z_0-z_s}{r_0},\quad \gamma_0=\frac{a_0}{r_0}. \tag{4.27}$$

4.3 A simplified nonlinear system

As we have seen in the last section, the mutual inductance and self inductance function are nonlinear. The elliptic integral and the logarithm function are the functions of the unknown variables $\alpha = 1+\epsilon\tilde{V}$ and $\beta = \beta_0+\epsilon\tilde{U}$. To make the system simpler, we linearize the inductance functions by the Taylor expansion, and obtain "*a simplified nonlinear system*" by using the linearized inductance functions instead of the original inductance functions.

4.3.1 Simplification of the inductance functions

By (4.27), $\alpha = \alpha_0+\epsilon\tilde{V}$, $\beta=\beta_0+\epsilon\tilde{U}$. Thus, the Taylor expansion of the inductance functions and their derivatives at (α_0,β_0) are

$$\begin{cases} \tilde{M}(\alpha,\beta) = \tilde{M}_0+\tilde{M}_\alpha(\alpha-\alpha_0)+\tilde{M}_\beta(\beta-\beta_0) = \tilde{M}_0+\epsilon\tilde{M}_\alpha\tilde{V}+\epsilon\tilde{M}_\beta\tilde{U}, \\ \frac{\partial\tilde{M}}{\partial\alpha}(\alpha,\beta) = \tilde{M}_\alpha+\tilde{M}_{\alpha\alpha}(\alpha-\alpha_0)+\tilde{M}_{\alpha\beta}(\beta-\beta_0) = \tilde{M}_\alpha+\epsilon\tilde{M}_{\alpha\alpha}\tilde{V}+\epsilon\tilde{M}_{\alpha\beta}\tilde{U}, \\ \frac{\partial\tilde{M}}{\partial\beta}(\alpha,\beta) = \tilde{M}_\beta+\tilde{M}_{\alpha\beta}(\alpha-\alpha_0)+\tilde{M}_{\beta\beta}(\beta-\beta_0) = \tilde{M}_\beta+\epsilon\tilde{M}_{\alpha\beta}\tilde{V}+\epsilon\tilde{M}_{\beta\beta}\tilde{U}, \\ \\ \tilde{L}(\alpha) = \tilde{L}_0+\tilde{L}_\alpha(\alpha-\alpha_0) = \tilde{L}_0+\epsilon\tilde{L}_\alpha\tilde{V}, \\ \frac{\partial\tilde{L}}{\partial\alpha}(\alpha) = \tilde{L}_\alpha+\tilde{L}_{\alpha\alpha}(\alpha-\alpha0) = \tilde{L}_\alpha+\epsilon\tilde{L}_{\alpha\alpha}\tilde{V}, \end{cases} \tag{4.28}$$

where

$$\begin{cases} \tilde{M}_0 = \tilde{M}(\alpha_0, \beta_0), & \tilde{M}_\alpha = \frac{\partial \tilde{M}}{\partial \alpha}(\alpha_0, \beta_0), & \tilde{M}_\beta = \frac{\partial \tilde{M}}{\partial \beta}(\alpha_0, \beta_0), \\ \tilde{M}_{\alpha\alpha} = \frac{\partial^2 \tilde{M}}{\partial \alpha^2}(\alpha_0, \beta_0), & \tilde{M}_{\beta\beta} = \frac{\partial^2 \tilde{M}}{\partial \beta^2}(\alpha_0, \beta_0), & \tilde{M}_{\alpha\beta} = \frac{\partial^2 \tilde{M}}{\partial \alpha \partial \beta}(\alpha_0, \beta_0), \\ \tilde{L}_0 = \tilde{L}(\alpha_0), & \tilde{L}_\alpha = \frac{\partial \tilde{L}}{\partial \alpha}(\alpha_0), & \tilde{L}_{\alpha\alpha} = \frac{\partial^2 \tilde{L}}{\partial \alpha^2}(\alpha_0). \end{cases} \tag{4.29}$$

Figure 4.5 shows the relative errors due to the linearization for the mutual inductance and its derivatives:

$$\frac{\|\tilde{M}(\alpha,\beta) - \tilde{M}_{linear}(\alpha,\beta)\|}{\|\tilde{M}(\alpha,\beta)\|}, \quad \frac{\|\frac{\partial \tilde{M}}{\partial \alpha}(\alpha,\beta) - \frac{\partial \tilde{M}}{\partial \alpha}_{linear}(\alpha,\beta)\|}{\|\frac{\partial \tilde{M}}{\partial \alpha}(\alpha,\beta)\|}, \quad \frac{\|\frac{\partial \tilde{M}}{\partial \beta}(\alpha,\beta) - \frac{\partial \tilde{M}}{\partial \beta}_{linear}(\alpha,\beta)\|}{\|\frac{\partial \tilde{M}}{\partial \beta}(\alpha,\beta)\|},$$

where $\alpha \in [\alpha_0 - \delta, \alpha_0 + \delta]$, $\beta \in [\beta_0 - \delta, \beta_0 + \delta]$ with $\delta \in [0.0001, 0.05]$. The original dimensionless mutual inductance and its derivatives are calculated by (4.5) and (4.6). The linearized mutual inductance and its derivatives are calculated by (4.28).
Figure 4.6 shows the error estimation of the linearization for the self inductance and its derivative:

$$\frac{\|\tilde{L}(\alpha) - \tilde{L}_{linear}(\alpha)\|}{\|\tilde{L}(\alpha)\|}, \quad \frac{\|\frac{\partial \tilde{L}}{\partial \alpha}(\alpha) - \frac{\partial \tilde{L}}{\partial \alpha}_{linear}(\alpha)\|}{\|\frac{\partial \tilde{L}}{\partial \alpha}(\alpha)\|},$$

where $\alpha \in [\alpha_0 - \delta, \alpha_0 + \delta]$, $\delta \in [0.0001, 0.05]$. The original dimensionless self inductance and its derivatives are calculated by (4.12) and (4.13). The linearized self inductance and its derivatives are calculated by (4.28).

The figures show that the linearization errors increase fast when $\delta > 10^{-2}$.

As we have mentioned before, $\delta = \epsilon \tilde{U}$ and $\delta = \epsilon \tilde{V}$. If $\tilde{U}$ and $\tilde{V}$ are $O(1)$, the simplified nonlinear system will not be a good approximation of the complete nonlinear system when $\epsilon > 10^{-2}$. If we take the $\epsilon = \epsilon' = \frac{I_a}{I_m}$, it means that the simplified nonlinear system loses the accuracy when $\epsilon' > 10^{-2}$. If $\tilde{U}, \tilde{V} \sim O(10)$, then $\epsilon (= \epsilon')$ should be kept less than 10^{-3} such that the errors of the linearization errors of the inductance functions stay in a low level.

4.3.2 Order reduction of the simplified nonlinear system

Since most of the ODE solvers solve the first order systems, we reduce the order of our system of second order system to the first order here.
At first, we define some dimensionless constants to make the equations easier to read.

$$c_1 := \frac{k_z}{k_{r0}}, \quad c_2 := \frac{\mu_0 I_m^2}{k_{r0} r_0}, \quad c_3 := \frac{\sqrt{\frac{k_z}{k_{r0}}}}{\sqrt{Q_z^2 + 1/4}}, \quad c_4 := \frac{1}{\sqrt{Q_r^2 + 1/4}}, \quad c_5 := \frac{R_0 t_0}{\mu_0 r_0}, \tag{4.30}$$

where

$$m = 2\pi r_0 \cdot A_0 \cdot \rho_m, \quad R_0 = 2\pi r_0 \rho / A_0, \quad k_{r0} = 2\pi E A_0 / r_0. \tag{4.31}$$

Using these definitions, the nonlinear system (4.24) in terms of c parameters can

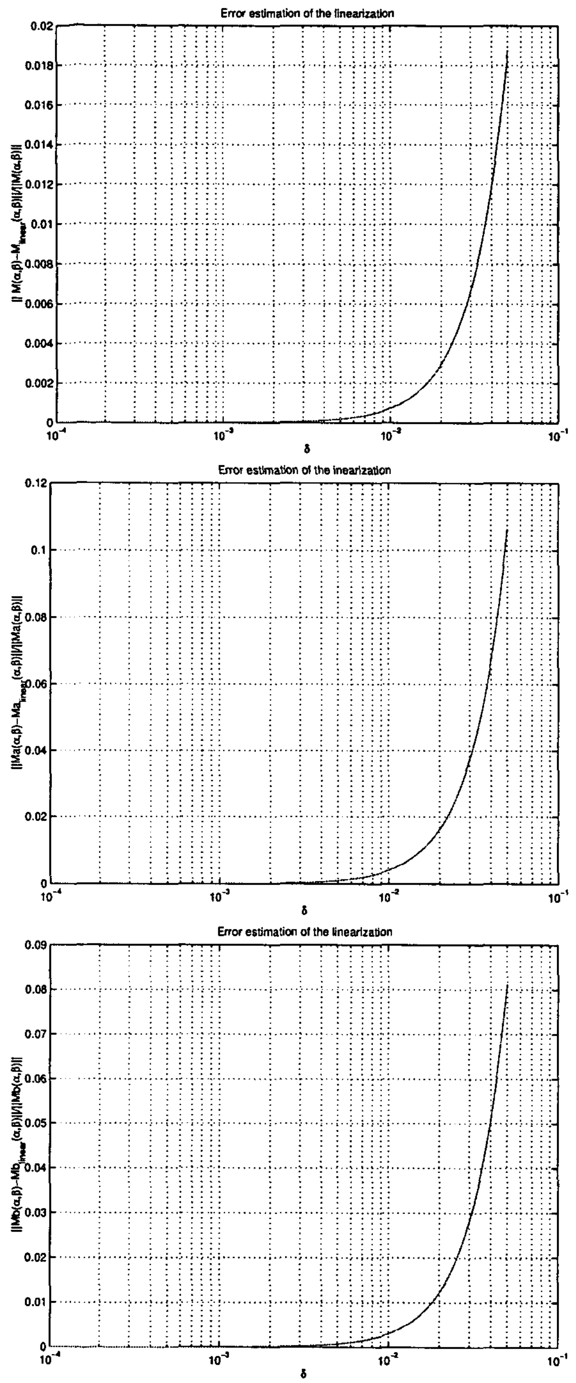

Figure 4.5: Linearization of the mutual inductance and its derivatives at $(\alpha_0, \beta_0) = (1.0000, 0.1111)$, $\alpha_s = 0.8889$, $\alpha \in [\alpha_0 - \delta, \alpha_0 + \delta], \beta \in [\beta_0 - \delta, \beta_0 + \delta]$

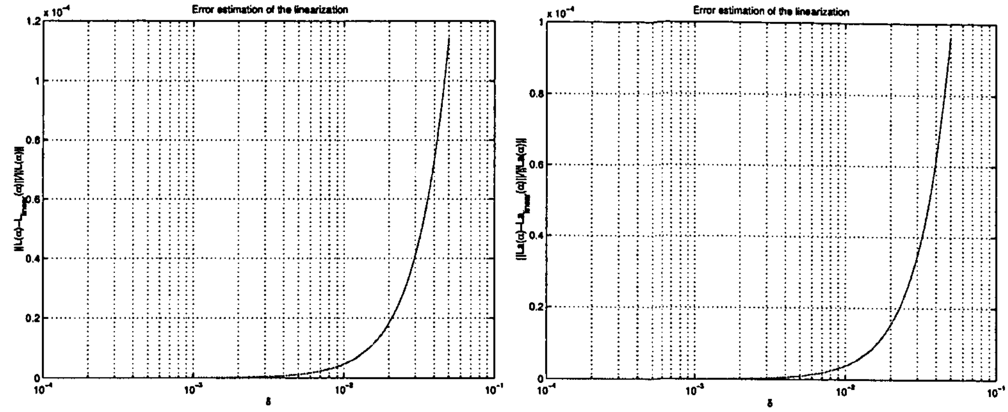

Figure 4.6: Linearization of the self inductance at $\alpha_0 = 1$, $\alpha \in [\alpha_0 - \delta, \alpha_0 + \delta]$

be written as

$$\begin{cases} \ddot{\tilde{U}} + c_3\dot{\tilde{U}} + c_1\tilde{U} = c_2(1+\epsilon' \sin\tilde{\omega}\tau)\tilde{I}\frac{\partial \tilde{M}}{\partial \beta}, \\ \ddot{\tilde{V}} + c_4\dot{\tilde{V}} + \tilde{k}_r\tilde{V} = c_2\left[(1+\epsilon' \sin\tilde{\omega}\tau)\tilde{I}\frac{\partial \tilde{M}}{\partial \alpha} + \frac{\epsilon}{2}\tilde{I}^2\frac{\partial \tilde{L}}{\partial \alpha}\right], \\ \tilde{L}\dot{\tilde{I}} + \left[\epsilon\frac{\partial \tilde{L}}{\partial \alpha}\dot{\tilde{V}} + c_5\tilde{R}\right]\tilde{I} = -\tilde{M}\tilde{\omega}\frac{\epsilon'}{\epsilon}\cos\tilde{\omega}\tau - [\frac{\partial \tilde{M}}{\partial \alpha}\dot{\tilde{V}} + \frac{\partial \tilde{M}}{\partial \beta}\dot{\tilde{U}}](1+\epsilon' \sin\omega\tilde{\tau}). \end{cases} \tag{4.32}$$

Now introduce $\mathbf{Y} = (y_1, y_2, y_3, y_4, y_5)^T = (\tilde{U}, \dot{\tilde{U}}, \tilde{V}, \dot{\tilde{V}}, \tilde{I})^T$,
a mapping function: $\mathbf{F}(y_1, y_2, y_3, y_4, y_5, \tau) = (f_1, f_2, f_3, f_4, f_5)^T$,
where $f_i = f_i(y_1, y_2, y_3, y_4, y_5, \tau)$ for $i = 1, 2, ..., 5$.

Then, the vector form of (4.32) is

$$\begin{cases} \dot{\mathbf{Y}} = \mathbf{F}(\mathbf{Y}, \tau), \\ \mathbf{Y}(0) = \mathbf{0}, \end{cases} \tag{4.33}$$

where

$$\begin{cases} \dot{y_1} = & y_2, \\ \dot{y_2} = & -c_3y_2 - c_1y_1 + c_2\left[\frac{\partial \tilde{M}}{\partial \beta}(1+\epsilon' \sin\tilde{\omega}\tau)\right]y_5, \\ \dot{y_3} = & y_4, \\ \dot{y_4} = & -c_4y_4 - \tilde{k}_r y_3 + c_2\left[\frac{\partial \tilde{M}}{\partial \alpha}(1+\epsilon' \sin\tilde{\omega}\tau)y_5 + \frac{\epsilon}{2}y_5^2\frac{\partial \tilde{L}}{\partial \alpha}\right], \\ \dot{y_5} = & -\left[(\epsilon\frac{\partial \tilde{L}}{\partial \alpha}y_4 + c_5\tilde{R})y_5 + \tilde{M}\tilde{\omega}\frac{\epsilon'}{\epsilon}\cos\tilde{\omega}\tau + (\frac{\partial \tilde{M}}{\partial \alpha}y_4 + \frac{\partial \tilde{M}}{\partial \beta}y_2)(1+\epsilon' \sin\omega\tilde{\tau})\right]/\tilde{L}, \end{cases} \tag{4.34}$$

and

$$\begin{cases} \tilde{k}_r = (\frac{1}{1+\epsilon y_3})^{2\nu+1} \approx 1 - (2\nu+1)\epsilon y_3, \\ \tilde{R} = (\frac{r}{r_0})^{2\nu+1} = (1+\epsilon y_3)^{2\nu+1} \approx 1 + (2\nu+1)\epsilon y_3. \end{cases} \tag{4.35}$$

For the simplified nonlinear system, the linearized inductance functions are

$$
\begin{cases}
\tilde{M}(\alpha,\beta) = \tilde{M}_0 + \epsilon\tilde{M}_\alpha y_3 + \epsilon\tilde{M}_\beta y_1, \\
\frac{\partial\tilde{M}}{\partial\alpha} = \tilde{M}_{\alpha_0} + \epsilon\tilde{M}_{\alpha\alpha} y_3 + \epsilon\tilde{M}_{\alpha\beta} y_1, \\
\frac{\partial\tilde{M}}{\partial\beta} = \tilde{M}_\beta + \epsilon\tilde{M}_{\alpha\beta} y_3 + \epsilon\tilde{M}_{\beta\beta} y_1, \\
\\
\tilde{L}(\alpha) = \tilde{L}_0 + \epsilon\tilde{L}_\alpha y_3, \\
\frac{\partial\tilde{L}}{\partial\alpha} = \tilde{L}_\alpha + \epsilon\tilde{L}_{\alpha\alpha} y_3.
\end{cases}
\tag{4.36}
$$

4.4 The linear system

The simplified nonlinear system (4.32) is still complicated. But, fortunately, it turns to be a simple linear system (4.37) when we neglect all the small terms of ϵ and ϵ':

$$
\begin{cases}
\ddot{\tilde{U}} + c_3\dot{\tilde{U}} + c_1\tilde{U} = c_2\tilde{M}_\beta\tilde{I}, \\
\ddot{\tilde{V}} + c_4\dot{\tilde{V}} + \tilde{V} = c_2\tilde{M}_\alpha\tilde{I}, \\
\dot{\tilde{I}} + \frac{c_5}{\tilde{L}_0}\tilde{I} = -\frac{\tilde{M}_\alpha}{\tilde{L}_0}\dot{\tilde{V}} - \frac{\tilde{M}_\beta}{\tilde{L}_0}\dot{\tilde{U}} - \frac{\tilde{M}_0}{\tilde{L}_0}\frac{\epsilon'}{\epsilon}\tilde{\omega}\cos\tilde{\omega}\tau.
\end{cases}
\tag{4.37}
$$

We reduce (4.37) into the first order:

$$
\begin{cases}
\dot{y}_1 = y_2, \\
\dot{y}_2 = -c_3 y_2 - c_1 y_1 + c_2\tilde{M}_\beta y_5, \\
\dot{y}_3 = y_4, \\
\dot{y}_4 = -c_4 y_4 - y_3 + c_2\tilde{M}_\alpha y_5, \\
\dot{y}_5 = -\left[c_5 y_5 + \tilde{M}_\alpha y_4 + \tilde{M}_\beta y_2 + \tilde{M}_0\frac{\epsilon'}{\epsilon}\tilde{\omega}\cos\tilde{\omega}\tau\right]/\tilde{L}_0.
\end{cases}
\tag{4.38}
$$

Let

$$
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
-c_1 & -c_3 & 0 & 0 & c_2\tilde{M}_\beta \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & -c_4 & c_2\tilde{M}_\alpha \\
0 & -\frac{\tilde{M}_\beta}{\tilde{L}_0} & 0 & -\frac{\tilde{M}_\alpha}{\tilde{L}_0} & -\frac{c_5}{\tilde{L}_0}
\end{bmatrix}
$$

and

$$
b = (0,0,0,0,-\frac{\tilde{M}_0}{\tilde{L}_0}\frac{\epsilon'}{\epsilon}\tilde{\omega}\cos\tilde{\omega}\tau)^T,
$$

then, the matrix form of (4.38) is

$$
\begin{cases}
\dot{\mathbf{Y}} = A\mathbf{Y} + \mathbf{b}, \\
\mathbf{Y}(0) = \mathbf{0}.
\end{cases}
\tag{4.39}
$$

4.5 Summary

In this chapter, we followed the following procedure to simplify the system obtained in the last chapter:

1. The dimensionless analysis:
   After this step, we obtained the dimensionless form of the complete nonlinear system.

2. The simplification of the inductance functions by linearization.
   After this step, we obtained a simplified nonlinear system.

3. The simplification of the simplified nonlinear system by neglecting the small terms.
   After this step, fortunately, we obtained a linear system.

In the next chapter, we will analyze the properties of these three systems separately.

Chapter 5

The two co-axial circular wire loops model: Simulation

In the previous chapter, we obtained three systems for the co-axial circular wire loops model: the complete nonlinear system, the simplified nonlinear system and the linear system. In this chapter, we will analyze these three systems by numerical simulation with some typical test data. Table 5.1 gives the test data which will be used in the simulations.

Table 5.1: Test data for the simulation

object	physical meaning	symbol	value	unit	note
source loop	main current	I_m	6.0×10^5	A	
	gradient current	I_g	6.0×10^2	A	
	radius	r_s	0.4	m	
	position	z_s	0	m	
shield loop	radius	r_0	0.45	m	
	position	z_0	0.05	m	
	area of cross section	A_0	10^{-4}	m^2	
	mass density	ρ_m	2.7×10^3	$kg \cdot /m^3$	aluminum
	Young's modulus	E	7.0×10^{10}	N/m^2	aluminum
	resistance permeability	ρ	3.0×10^{-8}	$\Omega \cdot m$	aluminum
	Possion ratio	ν	0.3	-	aluminum
	quality factor in $z-$direction	Q_z	20		
	quality factor in $r-$direction	Q_r	500		
	ratio between the stiffness	k_z/k_{r0}	10^{-4}	kg/s^2	
	permeability of vacuum	μ_0	$4\pi \times 10^{-7}$	H/m	
	scaling parameter	ϵ	0.001		

5.1 The linear system

The linear system derived in the last chapter:

$$\begin{cases} \ddot{\tilde{U}} + c_3\dot{\tilde{U}} + c_1\tilde{U} = c_2\tilde{M}_\beta\tilde{I}, \\ \ddot{\tilde{V}} + c_4\dot{\tilde{V}} + \tilde{V} = c_2\tilde{M}_\alpha\tilde{I}, \\ \dot{\tilde{I}} + \frac{c_5}{L_0}\tilde{I} = -\frac{\tilde{M}_\alpha}{L_0}\dot{\tilde{V}} - \frac{\tilde{M}_\beta}{L_0}\dot{\tilde{U}} - \frac{\tilde{M}_0}{L_0}\frac{\epsilon'}{\epsilon}\tilde{\omega}\cos\tilde{\omega}\tau. \end{cases} \tag{5.1}$$

The matrix form of the linear system in the first order:

$$\begin{cases} \dot{\mathbf{Y}} = A\mathbf{Y} + \mathbf{b} \\ \mathbf{Y}(0) = \mathbf{0} \end{cases} \tag{5.2}$$

where

$$Y = \begin{bmatrix} \tilde{U} \\ \dot{\tilde{U}} \\ \tilde{V} \\ \dot{\tilde{V}} \\ \tilde{I} \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -c_1 & -c_3 & 0 & 0 & c_2\tilde{M}_\beta \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -c_4 & c_2\tilde{M}_\alpha \\ 0 & -\frac{\tilde{M}_\beta}{L_0} & 0 & -\frac{\tilde{M}_\alpha}{L_0} & -\frac{c_5}{L_0} \end{bmatrix}, \quad b = \cos\tilde{\omega}\tau \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\frac{\tilde{M}_0}{L_0}\frac{\epsilon'}{\epsilon}\tilde{\omega} \end{bmatrix}.$$

5.1.1 The analytic solution

The system (5.2) can be solved analytically. The following method can be found from page 85 to page 88 in [7].

Suppose:

1. λ_j for $j = 1, 2, ..., 5$ are the *eigenvalues* of the matrix A; The eigenvalue matrix is $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_5)$.

2. $\mathbf{v}_j$ for j=1,2,...,5 are the corresponding eigenvectors of the matrix A. The *eigenvector matrix* $V = [\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_5]$.

3. Let $\mathbf{b}' = [b'_1, b'_2, ..., b'_5]^T$ be a constant vector related to the source item $\mathbf{b}$ and the eigenvector matrix V.
   Since the source terms can be written as $\mathbf{b}(\tau) = \mathbf{b}_c \cos\tilde{\omega}\tau$, where $\mathbf{b}_c = [0, 0, 0, 0, -\frac{\tilde{M}_0}{L_0}\frac{\epsilon'}{\epsilon}\tilde{\omega}]^T$, then the relation between $\mathbf{b}'$ and $\mathbf{b}_c$ is $\mathbf{b}' = V^{-1}\mathbf{b}_c$.
   The source term can be taken as the real part of the complex form. The complex form of the source can be written as $\mathbf{b}(\tau) = b_c e^{i\tilde{\omega}\tau}$. Now we use the complex form of the source terms to do the following analysis for the linear system.

From the linear algebra theory, the relation between the matrix A and the matrix Λ is $\Lambda = V^{-1}AV$.
Let $X = V^{-1}Y$, then (5.2) can be written as

$$\begin{cases} \dot{\mathbf{X}} = \Lambda\mathbf{X} + e^{i\tilde{\omega}\tau}\mathbf{b}', \\ \mathbf{X}(0) = \mathbf{0}, \end{cases} \tag{5.3}$$

where $\mathbf{b}' = V^{-1}\mathbf{b}_c$.

The solution of (5.3) is $\mathbf{X} = [x_1, x_2, ..., x_5]^T$, where

$$x_j = \begin{cases} \frac{b'_j}{i\tilde{\omega}-\lambda_j}\left[e^{i\tilde{\omega}\tau} - e^{\lambda_j\tau}\right] & \text{if } \lambda_j \neq i\tilde{\omega}; \\ \tau e^{i\tilde{\omega}\tau} & \text{if } \lambda_j = i\tilde{\omega}, \end{cases} \tag{5.4}$$

for $j = 1, 2, ..., 5$.
The solution of (5.2) is

$$\mathbf{Y} = V\mathbf{X}. \tag{5.5}$$

We are only interested in the real parts of Y.

5.1.2 The distribution of the relative resonant frequencies by varying the parameters

By the test data in Table 5.1, the eigenvalues of the matrix A are:

$$\lambda_1 = -0.0001, \quad \lambda_{2,3} = -0.0144 \pm 0.1928i, \quad \lambda_{4,5} = -0.0013 \pm 1.0088i$$

Since the matrix A is a 5 by 5 matrix, it has 5 eigenvalues for each choice of parameters.λ_1 is a negative real number. The others are two pairs of the conjugate complex numbers. These show there are two resonant frequencies in the linear system. One is in $z-$direction and the other is in $r-$direction. The real parts of the eigenvalues are negative. They are the damping factors. The imaginary parts of the eigenvalues are the relative resonant frequencies:

$$\tilde{\omega}_{0z} = \frac{\omega_{0z}}{\omega_0}, \quad \tilde{\omega}_{0r} = \frac{\omega_{0r}}{\omega_0}, \tag{5.6}$$

where $\omega_0 = \sqrt{m/k_{r0}}$ is the mechanical resonant frequency in $r-$direction if there is no damping.

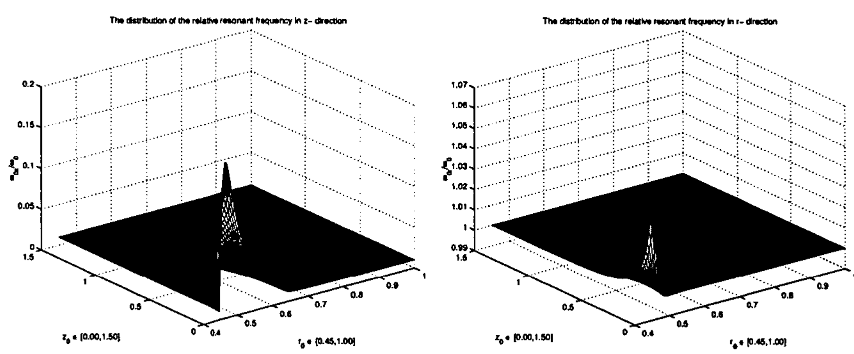

Figure 5.1: Distribution of the relative resonant frequencies by varying r_0 and z_0

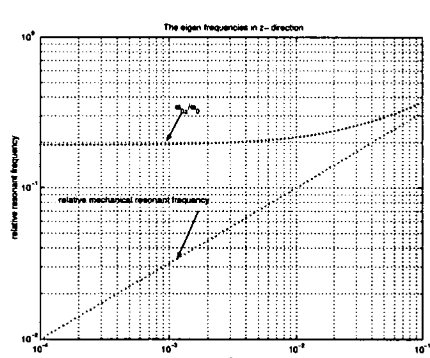

Figure 5.2: Distribution of the relative resonant frequency

By varying the parameters of the system, the entries of the matrix A vary and its eigenvalues vary. Figure 5.1 shows the distribution of the relative resonant frequencies (the eigen frequencies) by varying the radius r_0 and the position z_0. r_0 varies from 0.45m to 1.0m; z_0 varies from 0.00m to 1.50m. The other parameters are the test data in Table 5.1. The peaks show the frequency shifting of $\tilde{\omega}_{0z}$ and $\tilde{\omega}_{or}$

due to the magneto-mechanical coupling. There is magnetic stiffness super-imposed on the mechanical stiffness. In the plain area, the mechanical stiffness is dominant. Figure 5.2 shows the distribution of the relative resonant frequencies by varying the parameters c_1. c_1 is a sensitive parameter. In the linear system (5.1), $c_1 = k_z/k_{r0}$ is the relative mechanical stiffness in $z-$ direction. The figure shows both $\tilde{\omega}_{0z}$ and the relative mechanical resonant frequency which is $\sqrt{c_1}$. The difference between these two lines shows the effect of magnetic stiffness due to the magneto-mechanical coupling. When the relative mechanical stiffness c_1 is small, the magnetic stiffness in the system is dominant.

5.1.3 The steady state

The analytic solution of the system (5.4) shows that, in the steady state, all the resonant frequencies components $e^{\lambda_j \tau}$ damp out except the source frequency part since the real part of λ_j is negative. For the steady state, we use the *harmonic balance method* to do the analysis.
Let the source items to be

$$\sin \tilde{\omega}\tau = Real(-ie^{i\tilde{\omega}\tau}), \quad \cos \tilde{\omega}\tau = Real(e^{i\tilde{\omega}\tau}),$$

and we assume the steady solution are:

$$\tilde{U} = \tilde{U}_1 e^{i\tilde{\omega}\tau}, \quad \tilde{V} = \tilde{V}_1 e^{i\tilde{\omega}\tau}, \quad \tilde{I} = \tilde{I}_1 e^{i\tilde{\omega}\tau}, \tag{5.7}$$

here, $\tilde{U}_1$, $\tilde{V}_1$ and $\tilde{I}_1$ are complex numbers.
By substituting the steady solution and the source into (5.1), we balance the coefficients of $e^{i\tilde{\omega}\tau}$ and obtain:

$$\begin{cases} (-\tilde{\omega}^2 + i\tilde{\omega}c_3 + c_1)\tilde{U}_1 = c_2 \tilde{M}_\beta \tilde{I}_1, \\ (-\tilde{\omega}^2 + i\tilde{\omega}c_4 + 1)\tilde{V}_1 = c_2 \tilde{M}_\alpha \tilde{I}_1, \\ (i\tilde{\omega}\tilde{L}_0 + c_5)\tilde{I}_1 = -i\tilde{\omega}(\tilde{M}_\alpha \tilde{V}_1 + \tilde{M}_\beta \tilde{U}_1) - \tilde{M}_0 \frac{\epsilon'}{\epsilon}\tilde{\omega}. \end{cases} \tag{5.8}$$

By solving this system, we obtain:

$$\begin{cases} \tilde{U}_1 = \frac{c_2 \tilde{M}_\beta \tilde{I}_1}{-\tilde{\omega}^2 + c_1 + i\tilde{\omega}c_3} \\ \tilde{V}_1 = \frac{c_2 \tilde{M}_\alpha \tilde{I}_1}{-\tilde{\omega}^2 + 1 + i\tilde{\omega}c_4} \\ \tilde{I}_1 = -\frac{\tilde{M}_0 \tilde{\omega} \frac{\epsilon'}{\epsilon}}{c_5 + i\tilde{\omega}\left[\tilde{L}_0 + c_2\left(\frac{\tilde{M}_\alpha^2}{-\tilde{\omega}^2 + 1 + i\tilde{\omega}c_4} + \frac{M_\beta^2}{-\tilde{\omega}^2 + c_1 + i\tilde{\omega}c_3}\right)\right]} \end{cases} \tag{5.9}$$

The amplitudes are:

$$|\tilde{U}_1|, |\tilde{V}_1|, |\tilde{I}_1|$$

.
The phases are:

$$\tan^{-1}\frac{\text{imag}(\tilde{U}_1)}{\text{real}(\tilde{U}_1)}, \tan^{-1}\frac{\text{imag}(\tilde{V}_1)}{\text{real}(\tilde{V}_1)}, \tan^{-1}\frac{\text{imag}(\tilde{I}_1)}{\text{real}(\tilde{I}_1)}$$

.
Figure 5.3 shows the Bode diagrams of the linear system, which are the distribution of the amplitudes and the phases of the steady solutions in the frequency

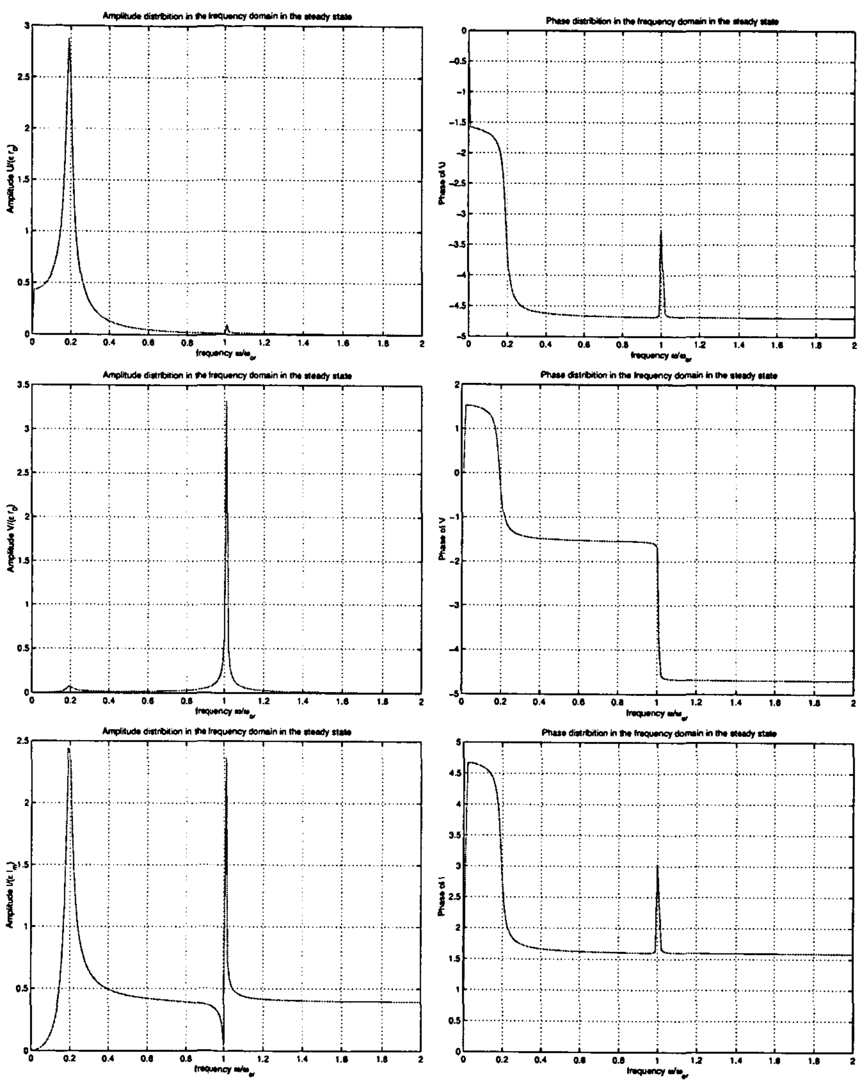

Figure 5.3: Bode-diagrams of the linear system with the test data

domain. The system parameters are the test data in Table 5.1. It shows that strong vibrations will happen only when the system resonates.

Figure 5.4 show the mode shapes $(\tilde{U}(t), \tilde{V}(t))$ of the linear system in the steady state. The left figure shows the resonant states and the right figure shows the non-resonant states. The numbers in the figures are the relative source frequencies $\tilde{\omega} = \omega/\omega_0$. We should notice that the scales for the axis $\tilde{U}$ and the axis $\tilde{V}$ in the right figure are not in the same order.

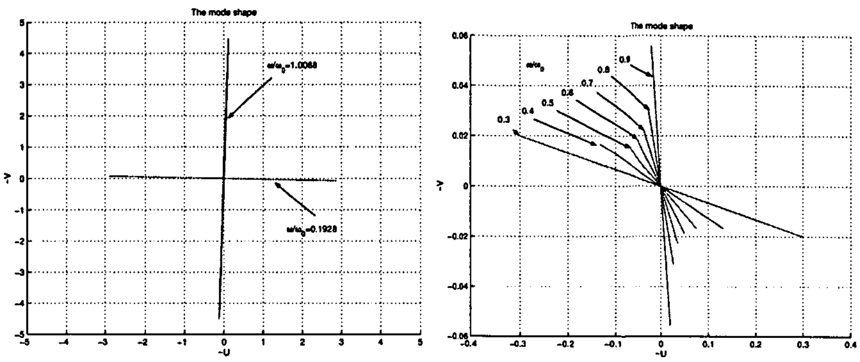

Figure 5.4: The mode shapes of the linear system

5.2 The simplified nonlinear system

The simplified nonlinear system derived in the last chapter is:

$$
\begin{cases}
\ddot{\tilde{U}} + c_3\dot{\tilde{U}} + c_1\tilde{U} = c_2(1+\epsilon'\sin\tilde{\omega}\tau)\tilde{I}\frac{\partial\tilde{M}}{\partial\beta}, \\
\ddot{\tilde{V}} + c_4\dot{\tilde{V}} + \tilde{k}_r\tilde{V} = c_2\left[(1+\epsilon'\sin\tilde{\omega}\tau)\tilde{I}\frac{\partial\tilde{M}}{\partial\alpha} + \frac{\epsilon}{2}\tilde{I}^2\frac{\partial\tilde{L}}{\partial\alpha}\right], \\
\tilde{L}\dot{\tilde{I}} + \left[\epsilon\frac{\partial\tilde{L}}{\partial\alpha}\dot{\tilde{V}} + c_5\tilde{R}\right]\tilde{I} = -\tilde{M}\tilde{\omega}\frac{\epsilon'}{\epsilon}\cos\tilde{\omega}\tau - [\frac{\partial\tilde{M}}{\partial\alpha}\dot{\tilde{V}} + \frac{\partial\tilde{M}}{\partial\beta}\dot{\tilde{U}}](1+\epsilon'\sin\tilde{\omega}\tau),
\end{cases}
\tag{5.10}
$$

where

$$
\begin{cases}
\tilde{k}_r = (\frac{\alpha_0}{\alpha_0+\epsilon\tilde{V}})^{2\nu+1} = (\frac{1}{1+\epsilon\tilde{V}})^{2\nu+1} \approx 1 - \epsilon(2\nu+1)\tilde{V}, \\
\tilde{R} = (\frac{r}{r_0})^{2\nu+1} = (\frac{\alpha_0+\epsilon\tilde{V}}{\alpha_0})^{2\nu+1} = (1+\epsilon\tilde{V})^{2\nu+1} \approx 1+\epsilon(2\nu+1)\tilde{V}, \\
\tilde{M}(\alpha,\beta) = \tilde{M}_0 + \epsilon\tilde{M}_\alpha\tilde{V} + \epsilon\tilde{M}_\beta\tilde{U}, \\
\frac{\partial\tilde{M}}{\partial\alpha} = \tilde{M}_\alpha + \epsilon\tilde{M}_{\alpha\alpha}\tilde{V} + \epsilon\tilde{M}_{\alpha\beta}\tilde{U}, \\
\frac{\partial\tilde{M}}{\partial\beta} = \tilde{M}_\beta + \epsilon\tilde{M}_{\alpha\beta}\tilde{V} + \epsilon\tilde{M}_{\beta\beta}\tilde{U}, \\
\tilde{L}(\alpha) = \tilde{L}_0 + \epsilon\tilde{L}_\alpha\tilde{V}, \\
\frac{\partial\tilde{L}}{\partial\alpha} = \tilde{L}_\alpha + \epsilon\tilde{L}_{\alpha\alpha}\tilde{V}.
\end{cases}
\tag{5.11}
$$

For this nonlinear system, the harmonic balance method in the real domain are applied to analyze the steady state instead of the harmonic balance method in the complex domain which is used in the linear system. Since the source term is a periodic signal, we use the truncated Fourier series to approximate the steady solutions. Assume the steady solutions are:

$$\begin{cases} \tilde{U} = \tilde{U}_0 + \tilde{U}_{1a} \cos\tilde{\omega}\tau + \tilde{U}_{1b} \sin\tilde{\omega}\tau + \tilde{U}_{2a} \cos 2\tilde{\omega}\tau + \tilde{U}_{2b} \sin 2\tilde{\omega}\tau, \\ \tilde{V} = \tilde{V}_0 + \tilde{V}_{1a} \cos\tilde{\omega}\tau + \tilde{V}_{1b} \sin\tilde{\omega}\tau + \tilde{V}_{2a} \cos 2\tilde{\omega}\tau + \tilde{V}_{2b} \sin 2\tilde{\omega}\tau, \\ \tilde{I} = \tilde{I}_0 + \tilde{I}_{1a} \cos\tilde{\omega}\tau + \tilde{I}_{1b} \sin\tilde{\omega}\tau + \tilde{I}_{2a} \cos 2\tilde{\omega}\tau + \tilde{I}_{2b} \sin 2\tilde{\omega}\tau. \end{cases} \tag{5.12}$$

The reason to omit the higher order harmonic terms in the assumption is that the higher order harmonic terms are invisible in the numerical steady solutions which are given in the appendix chapter A. Apparently they are negligible.
By substituting (5.12) into (5.10) and (5.11), and balancing corresponding constant items: the coefficients of $\cos\tilde{\omega}\tau$, $\sin\tilde{\omega}\tau$, $\cos 2\tilde{\omega}\tau$ and $\sin 2\tilde{\omega}\tau$, then, an algebraic system of the nonlinear polynomial equations with 15 unknown variables is obtained. It has the following format:

$$\mathbf{F} = \mathbf{0},$$

where

$$F = [f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_{15}(\mathbf{x})]^T,$$

and

$$\mathbf{x} = [\tilde{U}_0, \tilde{U}_{1a}, \tilde{U}_{1b}, \tilde{U}_{2a}, \tilde{U}_{2b}, \tilde{V}_0, \tilde{V}_{1a}, \tilde{V}_{1b}, \tilde{V}_{2a}, \tilde{V}_{2b}, \tilde{I}_0, \tilde{I}_{1a}, \tilde{I}_{1b}, \tilde{I}_{2a}, \tilde{I}_{2b}]^T.$$

In practice, this system of equations is obtained by Mathematica. As shown in Appendix B.6.3, the system is so complicated that even Mathematica can't solve it analytically. Here, the Newton iteration method is applied to solve this system numerically.
The Newton iteration method:

$$\mathbf{x}^{(1)} = \mathbf{x}^{(0)} - \left[\frac{d\mathbf{F}}{d\mathbf{x}}\right]^{-1} \cdot \mathbf{F}(\mathbf{x}^{(0)})$$

When $\tilde{\omega} = 1.0088$, the solution of the algebraic system $\mathbf{F} = \mathbf{0}$ with the test data in Table 5.1 is:

$$\mathbf{x} = \begin{bmatrix} \tilde{U}_0 \\ \tilde{U}_{1a} \\ \tilde{U}_{1b} \\ \tilde{U}_{2a} \\ \tilde{U}_{2b} \\ \tilde{V}_0 \\ \tilde{V}_{1a} \\ \tilde{V}_{1b} \\ \tilde{V}_{2a} \\ \tilde{V}_{2b} \\ \tilde{I}_0 \\ \tilde{I}_{1a} \\ \tilde{I}_{1b} \\ \tilde{I}_{2a} \\ \tilde{I}_{2b} \end{bmatrix} = \begin{bmatrix} 8.7027 \\ -0.0315 \\ -0.0579 \\ 0.0001 \\ -0.0002 \\ 0.0042 \\ -1.2932 \\ -1.8431 \\ 0.0005 \\ -0.0013 \\ -0.0044 \\ -0.7602 \\ -1.3973 \\ 0.0018 \\ -0.0028 \end{bmatrix}$$

Comparing to the linear system, besides the first order harmonic terms, the solutions include the zero-th order and second order harmonic terms. The zero-th order harmonic terms (the DC components) are $O(1)$. The second order harmonic terms are much smaller than the first order harmonic terms.

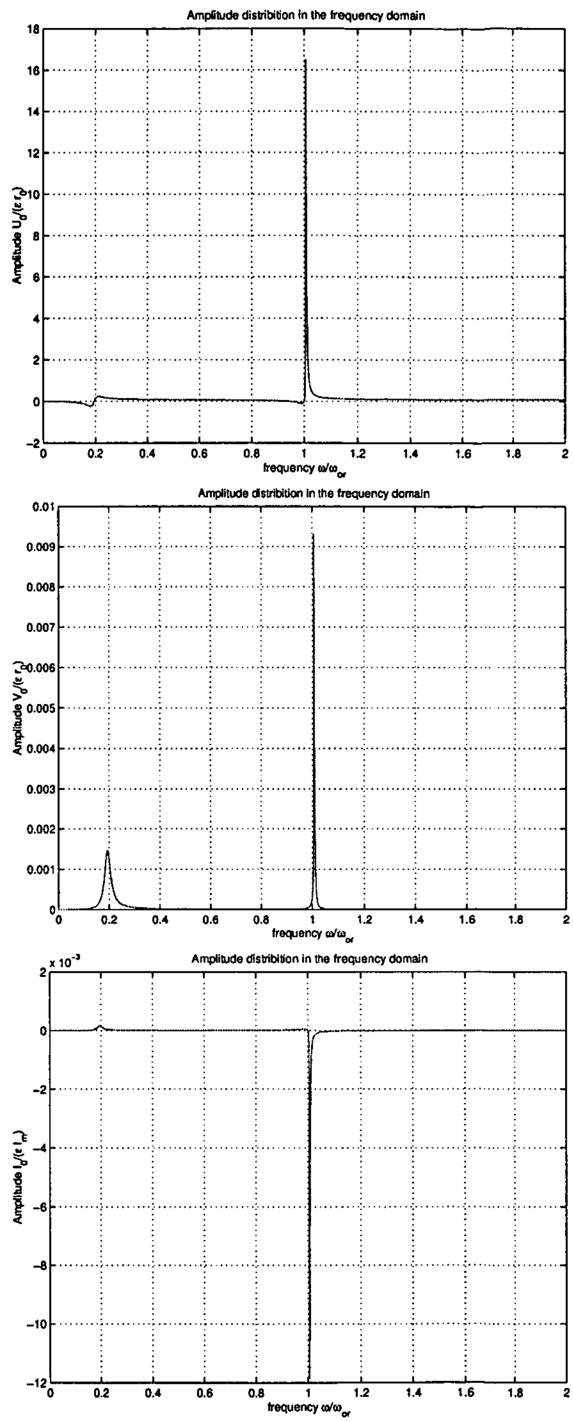

Figure 5.5: Bode diagrams of the zero-th order harmonic terms of the simplified nonlinear system

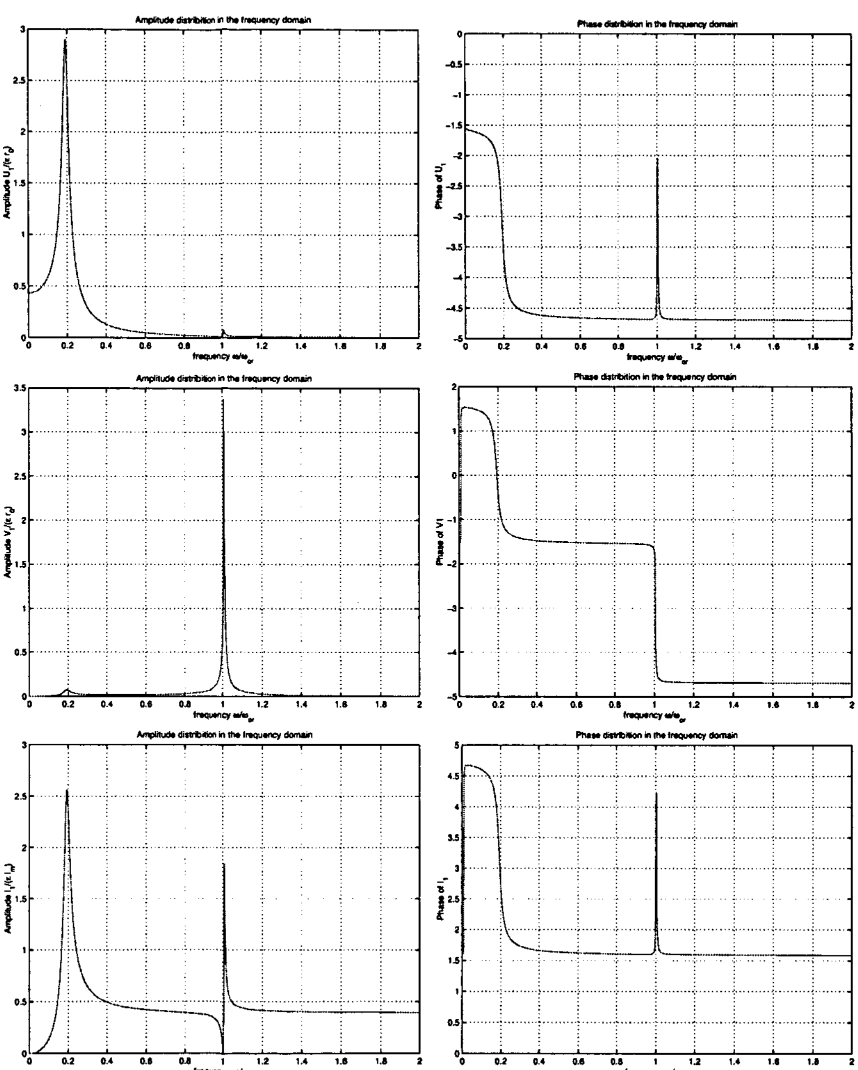

Figure 5.6: Bode diagrams of the first order harmonic terms of the simplified non-linear system

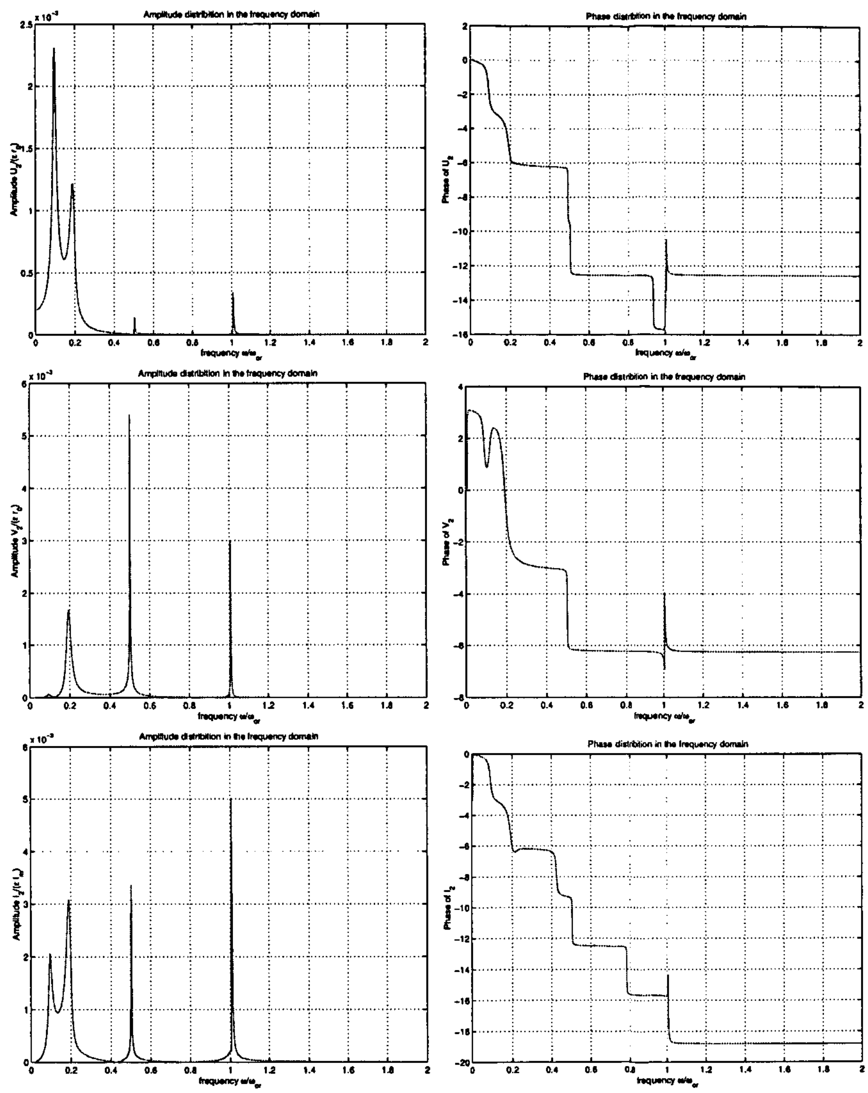

Figure 5.7: Bode diagrams of the second order harmonic terms of the simplified nonlinear system

Figure 5.5 to 5.7 show the Bode diagrams of the simplified nonlinear system based the test data. Here are some more detailed discussion on the solutions.

1. **The zero-th order harmonic terms (Figure 5.5):**

   (a) The zero-th order harmonic terms are effected by the source frequency. Figure 5.8 shows the distribution of $\tilde{U}_0$ when the source frequency is not nearby the resonant frequencies. Comparing to Figure 5.5, $\tilde{U}_0$ is much smaller in the non-resonant steady states than in the resonant steady states.

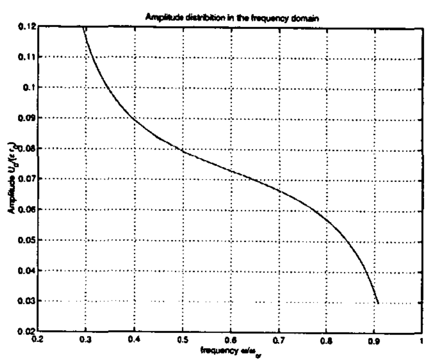

Figure 5.8: $\tilde{U}_0$ in the non resonant states

   (b) The zero-th order harmonic terms are effected by ϵ'
   Figure 5.9 shows the distribution of $\tilde{U}_0$ by varying the source amplitude ϵ'. The left figure shows that $\tilde{U}_0$ in the resonant steady state where $\tilde{\omega} = 1.0088$. It shows that $\tilde{U}_0$ is a linear function of ϵ' when $\epsilon' \leq 10^{-4}$. The right figure shows $\tilde{U}_0$ in the non-resonant steady state, where $\tilde{\omega} = 0.9028$. It shows that $\tilde{U}_0$ is a linear function of ϵ' when $\epsilon' \leq 10^{-2}$.

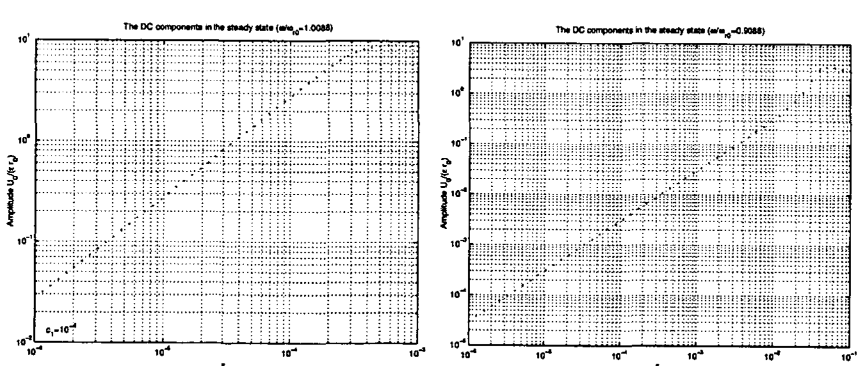

Figure 5.9: $\tilde{U}_0$ as a function of ϵ', where $c_1 = 10^{-4}$

   (c) The zero-th order harmonic terms are effected by the parameter c_1.
   Besides c_1 can effect on the eigen frequency in $z-$ direction ,which was mentioned in Section 5.1.2, Figure 5.10 shows that how c_1 effects on $\tilde{U}_0$. The difference between the left and right figure is the source frequency. Larger c_1 is, smaller $\tilde{U}_0$ is.

In fact, the zero-th order harmonic terms are the nonlinear effect of the simplified nonlinear system. Here we conclude that a smaller ϵ', a source frequency

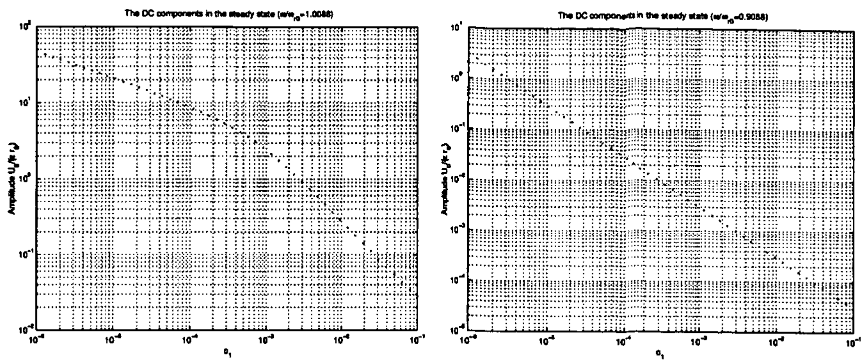

Figure 5.10: $\tilde{U}_0$ is sensitive to c_1, where $\epsilon = \epsilon' = 0.001$

which are sufficient far from the eigen frequencies and a larger c_1 can decrease this nonlinear effect.

2. **The first order harmonic terms (Figure 5.6):**
The Bode diagrams of the first order harmonic terms of the simplified nonlinear system are very similar to those of the linear system. There are two peaks nearby the resonant frequencies, but the amplitudes of the peaks in the simplified nonlinear system are smaller than those in the linear system. This is reasonable because of the zero-th order harmonic terms nearby the resonant frequencies. Figure 5.11 shows the amplitude of $\tilde{V}_1$ for both the simplified nonlinear system and the linear system in the resonant and non-resonant steady states. In the non-resonant states shown in the left figure, the solutions are the same for the linear system and simplified nonlinear system. In the resonant states shown in the right figure, the peak of the simplified nonlinear system is smaller, also there is a slight shift comparing to the linear system. Later we will show that there are multiple solutions for the simplified nonlinear system when the source frequency is nearby the resonant frequencies.

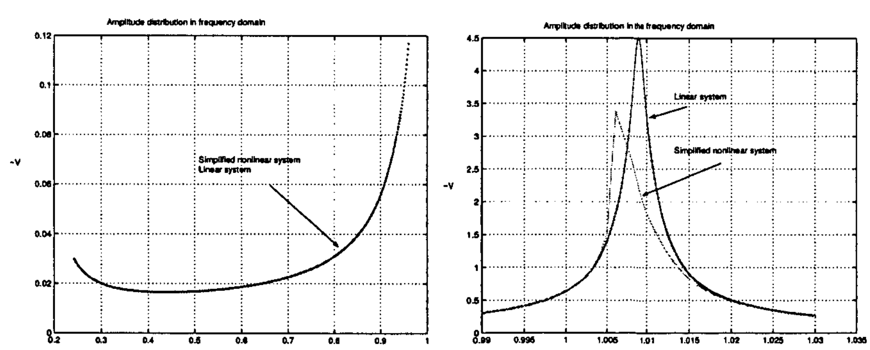

Figure 5.11: $\tilde{V}_1$ in the linear system and the simplified nonlinear system

3. **The second order harmonic terms (Figure 5.7):**
The Bode diagrams of the second order harmonic terms are more complicated. There are four peaks of the amplitudes in the frequency domain. There are the resonant frequencies and the halves of the resonant frequencies. Though they are very complicated, they can be neglected in the reality, since the

amplitudes of the peaks are much smaller than those of the zero-th order and first order harmonic terms.

5.3 The complete nonlinear system

In the complete nonlinear system, the mutual inductance and self inductance functions are the original forms which include the elliptic integrals and the logarithm function. The harmonic balance method doesn't work here. The straight method to obtain the solution is to use the proper numerical integration method.

- **Numerical integration methods**
  For solving a system of the nonlinear ordinary differential equations, the possible numerical integration methods are in Table 5.2. They are the available solvers in Matlab.

Table 5.2: The available solver in Matlab

Solver	Solver Type	Problem Type	Order of Accuracy	Algorithm
ode23	one-step	Non-stiff	Low	explicit Runge-kutta(2,3) Bagacki and Shampine pair
ode45	one-step	Non-stiff	Medium	explicit Runge-kutta(4,5) Dormand-Prince pair
odel13	multi-step	Non-stiff	Low to High	Adam-Bashforth-Moulton PECE
odel5s	multi-step	stiff	Low to Medium	Numerical difference formula Backward difference formular
ode23s	one-step	stiff	Low	Rosenbrock formula (2)
ode23t	one-step	moderately stiff	Low	trapzoidal rule
ode23tb	one-step	stiff	Low	TR(1)+BDF(2)

- **Special stiff problem**
  Table 5.2 shows that how to chose a solver depends on which type the problem belongs to. At the moment, since the complete nonlinear system is too complicated, we start to analyze the simplest system, the linear system. By the test data, The eigenvalues of the linear system are:

```
-0.0001
-0.0144 + 0.1928i
-0.0144 - 0.1928i
-0.0013 + 1.0088i
-0.0013 - 1.0088i
```

  . The eigen frequencies which are the imaginary parts of the eigenvalues are $O(1)$. One of damping factors, the real parts of the eigenvalues, are much smaller than $O(1)$. This means from the initial state to the steady state takes a very long time interval. In this sense, we think our problem is a stiff problem because there are two time scales in the system. However, this stiff problem is different from those problems which have real eigenvalues in the

Table 5.3: The accuracy and the efficiency analysis for the different solver

Solvers	$\bar{U}$	$\bar{V}$	$\bar{I}$	flops	time(s)
ode113	8.6453	-2.2502	-1.5901	427,959,116	837.2350
ode45	8.6455	-2.2502	-1.5900	681,633,092	1049.1000
ode15s	8.6329	-2.2490	-1.5893	377,196,873	673.9530

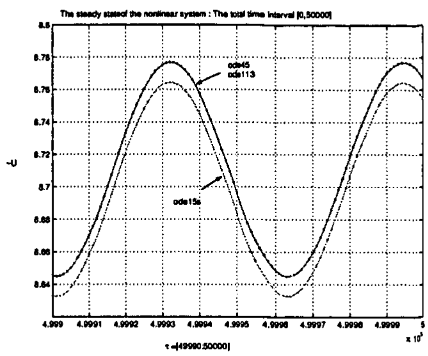

Figure 5.12: The steady state of the simplified nonlinear system by different solvers

different order. So the normal stiff solvers don't work well for this problem. Table 5.3 and Figure 5.12 show the steady solutions by the different solvers for the simplified nonlinear system with the test data and with $\epsilon = \epsilon' = 10^{-3}$, $\tilde{\omega} = 1.0088$, $RelTol = 10^{-5}$ and $AbsTol = 10^{-8}$. These show that the ode113 is the best numerical method. It gives almost the same result as the ode45, but it is more efficient. The solutions by the stiff solver ode15s damp too much.

- **Numerical solutions**

  Now, for the complete nonlinear system, ode113 can be used to obtain the numerica solutions. As a comparison, the solutions of the simplified nonlinear system and linear system also are given by ode113. Figure 5.13 to Figure 5.16 shows the steady solution for both the resonant and non-resonant states.

  Because of the practical reason, the parameter $c_1 = 0.001$, since it takes too long integration time to reach the steady state if $c_1 = 0.0001$. The other parameters which are not indicated in Table 5.1 are: $AbsTol = 10^{-4}$, $AbsTol = 10^{-7}$ For the resonant state $\tilde{\omega} = 1.0088$; for the non-resonant state $\tilde{\omega} = 0.9088$.

  Figure 5.13 shows, when $\epsilon' = 10^{-6}$, for both the resonant state and the non-resonant state, the numerical steady solutions off the three systems state are almost same. At least, the differences are invisible in the figure.

  Figure 5.14 shows, when $\epsilon' = 10^{-4}$, for the resonant state, the numerical steady solution of the linear system are different from those of the other two systems in the steady state; for the non-resonant state, the numerical steady solutions are the same for these three systems.

  Figure 5.15 shows, when $\epsilon' = 10^{-3}$, for the resonant state, even the numerical steady solutions of the simplified nonlinear system and the complete nonlinear system are different; but for the non-resonant state, the numerical steady solutions of the simplified nonlinear system and the complete nonlinear system are the same.

Figure 5.16 shows, when $\epsilon' = 10^{-2}$ and $\epsilon' = 0.1$, for the resonant state, the numerical solutions of these three systems are very different. The figure just gives an example of the global solution of the $\tilde{U}$ to see the differences.

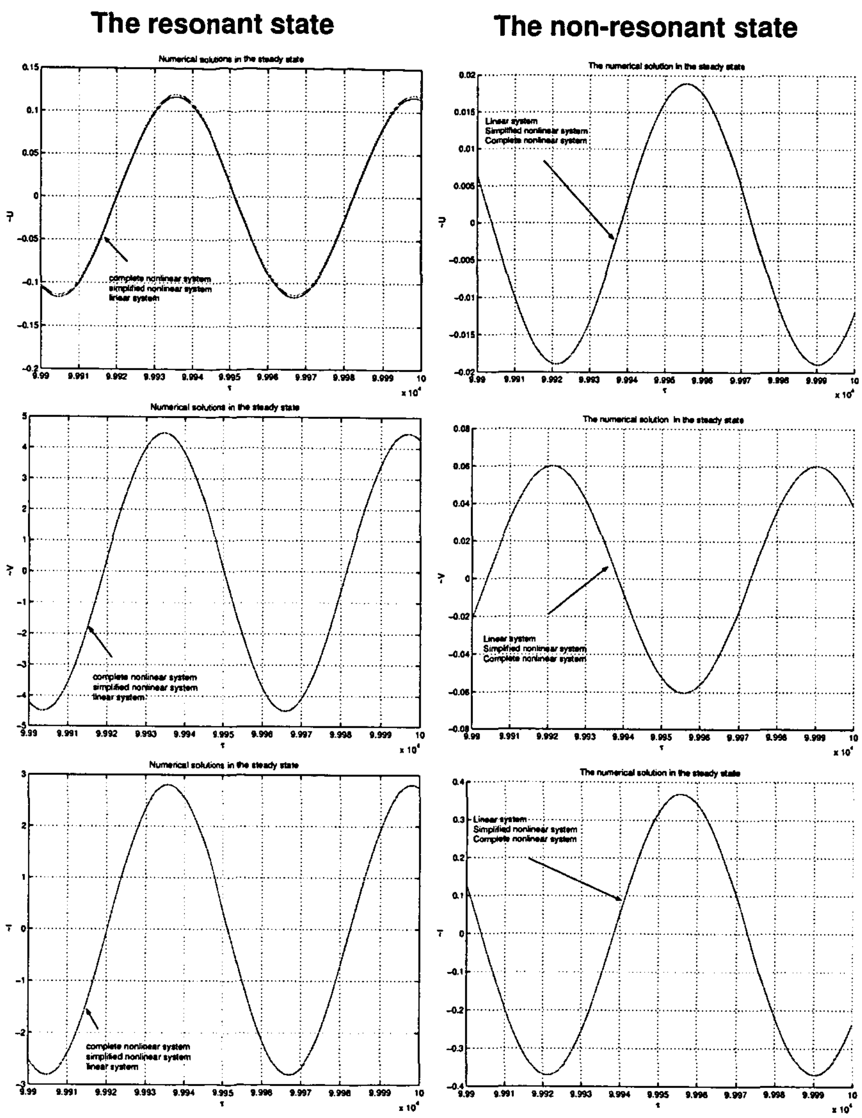

Figure 5.13: The numerical solutions in the steady states where $\epsilon' = 10^{-6}$

The resonant state **The non-resonant state**

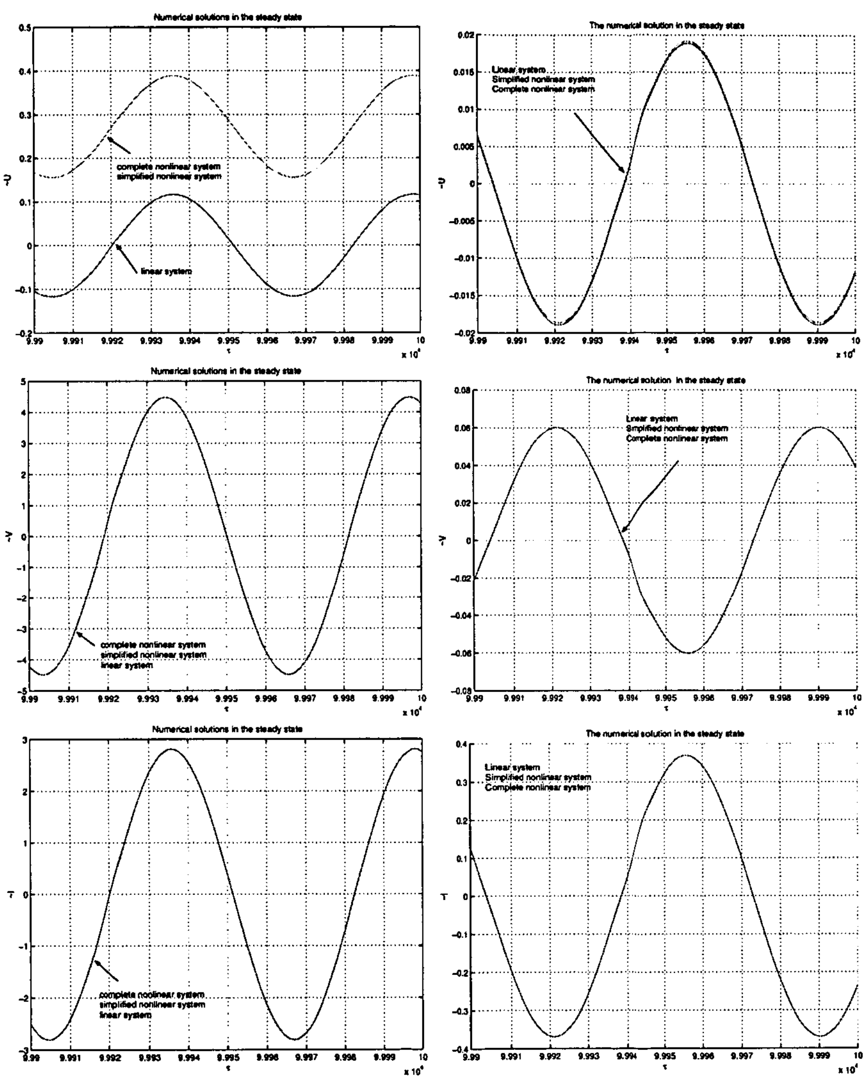

Figure 5.14: The numerical solutions in the steady states where $\epsilon' = 10^{-4}$

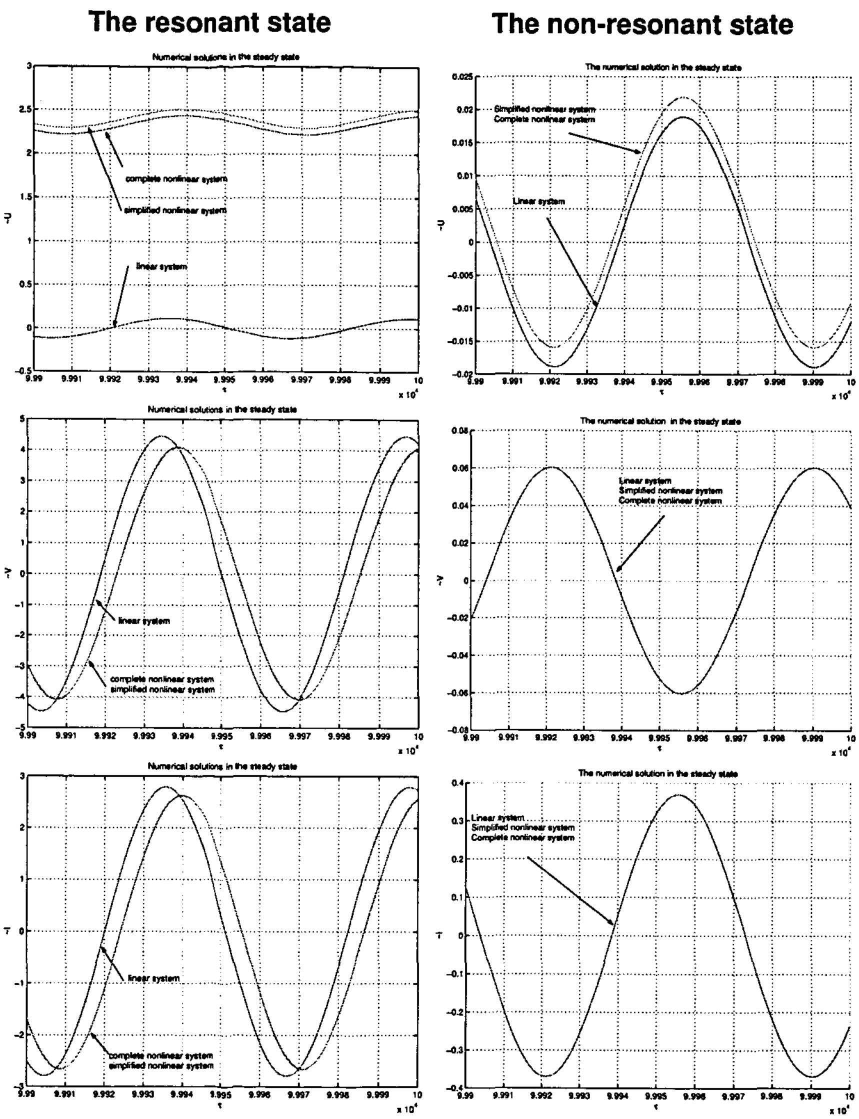

Figure 5.15: The numerical solutions in the steady states where $\epsilon' = 10^{-3}$

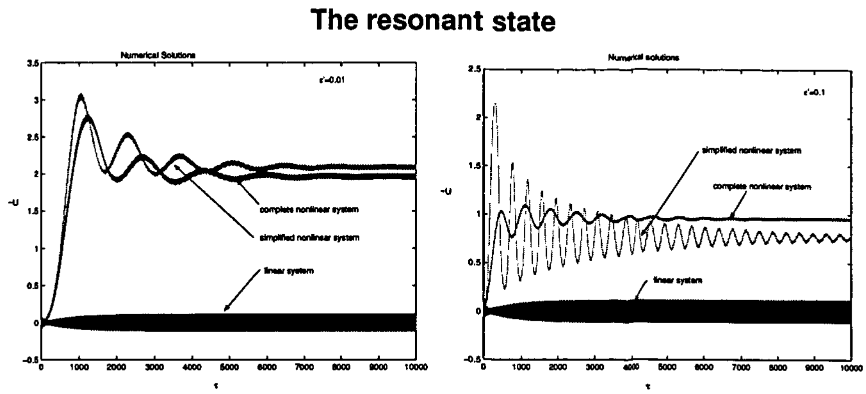

Figure 5.16: The numerical solutions in the steady states where $\epsilon' = 10^{-2}$ and $\epsilon' = 0.1$

5.4 Further discussion and conclusions

- **An efficient numerical integration method:**
  Since our problem is a special stiff problem, the normal stiff solvers don't work well here. Though the robust numerical integration methods lead to the exact solutions, we are not satisfied with the efficiency of these methods. In the left figure of Figure 5.17, the initial point is **0** for the Part I. The initial point for Part II is the last point of part I and the source frequency keeps the same. In the right figure of Figure 5.17, the initial point is 0 for Part I. The initial point for Part II is the last point of Part I but the source frequency changes slightly. Figure 5.17 shows the time interval to reach the steady state are also long even when we start from a point of the steady state but slightly change the source frequency.

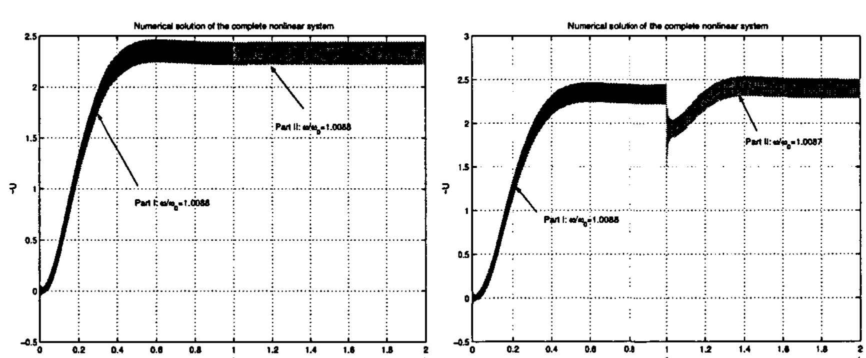

Figure 5.17: The numerical solutions obtained by ode113

- **The multiple solutions of the nonlinear systems:**
  When ϵ' is large enough, in the very narrow frequency ranges nearby the resonant frequencies, the nonlinear systems have the multiple solutions in the steady state. Figure 5.18 and Table 5.4 show the examples of the multiple solutions of the nonlinear systems, where $\epsilon' = 0.01$.

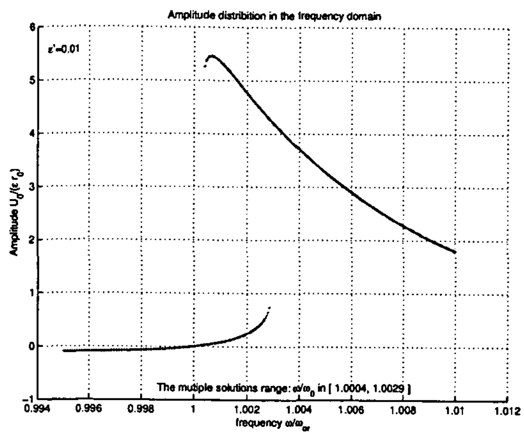

Figure 5.18: The multiple solutions of the simplified nonlinear system

Table 5.4: The multiple solutions of the complete nonlinear system

$\tilde{\omega}$	$\bar{U}_0$ when **x(0) = 0**	$\bar{U}_0$ when **x(0) = 5**
1.0020	0.205	0.205
1.0025	0.321	3.876
1.0030	0.527	4.028
1.0035	3.858	3.858
1.0040	3.647	3.647

- **Conclusions**

  In this chapter, the linear system, the simplified nonlinear system and the complete nonlinear system have been analyzed. The following conclusions are based the numerical simulations with the test data in Table 5.1.

  1. The applicability of the systems:
     Table 5.5 and Table 5.6 show the applicability of the systems. For the resonant state, $\tilde{\omega} = 1.0088$; for the non-resonant state, $\tilde{\omega} = 0.9088$. The parameter c_1 varies. L indicates that the linear system; S indicates the simplified nonlinear system; C indicates the complete nonlinear system. The tables show the system which can be a good approximation of the complete nonlinear system for the different ϵ' and c_1.

Table 5.5: The applicability analysis ($\tilde{\omega} = 1.0088$)

ϵ'	$c_1 = 0.001$	$c_1 = 0.01$	$c_1 = 0.1$
$\leq 10^{-6}$	L	L	L
$10^{-6} \sim 10^{-4}$	S	S	L
$10^{-4} \sim 10^{-3}$	C	S	S
$10^{-3} \sim 10^{-2}$	C	C	C

  2. The vibrations in the resonant states are $O(\epsilon')$:
     Only when the source frequency in the source loop is very close to the resonant frequencies, it will stimulate a relative large vibration in the shield loop. In these worst cases, the amplitude of the vibration in $z-$ direction and $r-$ direction $U/r_0 \sim O(\epsilon')$, $V/r_0 \sim O(\epsilon')$, and the induced currents in the shield loop $I/I_m \sim O(\epsilon')$, where $\epsilon' = I_g/I_m$.

Table 5.6: The applicability analysis ($\tilde{\omega} = 0.9088$)

ϵ'	$c_1 = 0.001$	$c_1 = 0.01$	$c_1 = 0.1$
$\leq 10^{-6}$	L	L	L
$10^{-6} \sim 10^{-4}$	L	L	L
$10^{-4} \sim 10^{-3}$	S	L	L
$10^{-3} \sim 10^{-2}$	S	S	L

3. The amplitude of the vibration can be reduced when the source frequency is sufficient far from the resonant frequencies. The resonant frequencies mainly depend on:

   (a) the loops: mutual inductance $M(r_s, z_s, r_0, z_0)$ and the self inductance $L(r_0)$ and the resistance $R(r_0)$;

   (b) the system: the stiffness k_z and k_r and the quality factor Q_z and Q_r;

   (c) the main magnetic field:I_m.

Chapter 6

The multiple co-axial circular wire loops model: Formulation

The geometry model for multiple co-axial circular loops is shown in Figure 1.6, where the source loops are $I_{s_1}, I_{s_2}, ..., I_{s_m}$, and the shield loops are $I_1, I_2, ..., I_n$. Table 6.1 shows the physical meaning of the symbols.

Table 6.1: **The explanation of the symbols**

Symbols	The physical meaning
i	the index number for the shield loops, $i = 1, 2, ..., n$
s_j	the index number for the source loops, $s_j = s_1, s_2, ..., s_m$
z_i	the position of the shield loop i
r_i	the radius of the shield loop i
I_i	the currents in the shield loop i
z_{0i}	the initial position of the shield loop i
r_{0i}	the initial radius of the shield loop i
a_{0i}	the initial radius of the cross section of the shield loop i
m_i	the mass of the shield loop i;
b_{z_i}	the damping of the shield loop i in $z-$ direction;
b_{r_i}	the damping of the shield loop i in $r-$ direction;
k_{z_i}	the stiffness of the shield loop i in $z-$ direction
k_{r_i}	the stiffness of the shield loop i in $r-$ direction
R_i	the resistance of the shield loop i
W_m	the total magnetic energy stored in the system
ϕ_i	the magnetic flux through the shield loop i
L_i	the self-inductance of the shield loop i
M_{ij}	the mutual inductance between the shield loops i and j
M_{is_j}	the mutual inductance between the coil i and the source coil s_j
I_{s_j}	the current in the source loop s_j
z_{s_j}	the position of the source loop s_j
r_{s_j}	the radius of the source loop s_j

We use the same energy method to analyze the magnetic forces in the system as what we have done in the two loops model. The total magnetic energy stored in the system is:

$$W_m = \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} I_{s_i} I_{s_j} M_{s_i s_j} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} I_i I_{s_j} M_{i s_j} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} I_i I_j M_{ij}, \tag{6.1}$$

where $M_{ii} = L_i$ and $M_{s_i s_i} = L_{s_i}$ for $i = 1, 2, ..., n$.
The magnetic flux in the shield loop i is:

$$\Phi_i = \sum_{j=1}^{m} M_{i s_j} I_{s_j} + \sum_{j=1}^{n} I_j M_{ij} \text{ where } i = 1, 2, ..., n \tag{6.2}$$

Using Newton's law and Hooke's law, we obtain the dynamics equations for every shield loop i. They are:

$$m_i \ddot{z}_i + b_{z_i} \dot{z}_i + k_{z_i}(z_i - z_{0i}) = \frac{\partial W_m}{\partial z_i}, \tag{6.3}$$

$$m_i \ddot{r}_i + b_{r_i} \dot{r}_i + k_{r_i}(r_i - r_{0i}) = \frac{\partial W_m}{\partial r_i}. \tag{6.4}$$

Using the Kirchhoff's law, we obtain the circuit equation for every shield loop i:

$$I_i R_i + \frac{d\phi_i}{dt} = 0. \tag{6.5}$$

- **Operator definition**

  1. Let vector $\mathbf{v} = (v_1, v_2, ..., v_n)$, then

$$diag(\mathbf{v}) = \begin{pmatrix} v_1 & 0 & 0 & ... & 0 \\ 0 & v_2 & 0 & ... & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & 0 & ... & v_n \end{pmatrix}$$

  2. Let W be a scalar, and a vector $\mathbf{u} = (u_1, u_2, ..., u_n)^T$, then

$$\frac{\partial W}{\partial \mathbf{u}} = (\frac{\partial W}{\partial u_1}, \frac{\partial W}{\partial u_2}, ..., \frac{\partial W}{\partial u_n})^T$$

  3. Let M to be a matrix by $n \times m$, a vector $\mathbf{u} = (u_1, u_2, ..., u_n)^T$, then

$$\frac{\partial M}{\partial \mathbf{u}} = \begin{pmatrix} \frac{\partial M_{11}}{\partial u_1} & \frac{\partial M_{12}}{\partial u_1} & ... & \frac{\partial M_{1m}}{\partial u_1} \\ \frac{\partial M_{21}}{\partial u_2} & \frac{\partial M_{22}}{\partial u_2} & ... & \frac{\partial M_{2m}}{\partial u_2} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial M_{n1}}{\partial u_n} & \frac{\partial M_{n2}}{\partial u_n} & ... & \frac{\partial M_{nm}}{\partial u_n} \end{pmatrix}$$

- **The system parameters**

  1. The definition of the vectors:
     (a) The induced current: $\mathbf{I} = (I_1(t), I_2(t), ..., I_n(t))^T$.

(b) The radius: $\mathbf{r} = (r_1(t), r_2(t), ..., r_n(t))^T$.
(c) The position: $\mathbf{z} = (z_1(t), z_2(t), ..., z_n(t))^T$.
(d) The source current: $\mathbf{I}_s = (I_{s_1}(t), I_{s_2}(t), ..., I_{s_m}(t))^T$.
(e) The initial radius: $\mathbf{r}_0 = (r_{01}, r_{02}, ..., r_{0n})^T$.
(f) The initial position: $\mathbf{z}_0 = (z_{01}, z_{02}, ..., z_{0n})^T$.
(g) The mass vector: $\mathbf{m} = (m_1, m_2, ..., m_n)$.
(h) The damping vector in $z-$ direction: $\mathbf{b}_z = (b_{z_1}, b_{z_2}, ..., b_{z_n})$.
(i) The damping vector in $r-$ direction: $\mathbf{b}_r = (b_{r_1}, b_{r_2}, ..., b_{r_n})$.
(j) The stiffness vector in $z-$ direction $\mathbf{k}_z = (k_{z_1}, k_{z_2}, ..., k_{z_n})$.
(k) The stiffness vector in $r-$ direction: $\mathbf{k}_r = (k_{r_1}, k_{r_2}, ..., k_{r_n})$, where $k_{r_i} = \frac{2\pi E A_{i0}}{r_i}(\frac{r_{i0}}{r_i})^{2\nu}$

(l) The resistance vector $\mathbf{R} = (R_1, R_2, ..., R_n)$, where $R_i = R_{0i}(\frac{r_i}{r_{i0}})^{2\nu+1}$.

2. The definition of the matrices:
   (a) The mutual inductance matrix M_s': its entry $M'_{s_is_j}$ is the mutual inductance between the source loops s_i and s_j. $M_{s_is_i} = L_{s_i}$.
   (b) The mutual-inductance matrix M_s: its entry M_{is_j} is the mutual inductance between the shield loop i and the source loop s_j.
   (c) The mutual inductance matrix M: its entry M_{ij} is the mutual inductance between the shield loops i and j. $M_{ii} = L_i$.

- **The equations in matrix form**

The total energy is:

$$W_m = \frac{1}{2}\mathbf{I}_s^T M_s' \mathbf{I}_s + \frac{1}{2}\mathbf{I}^T M_s \mathbf{I}_s + \frac{1}{2}\mathbf{I}^T M \mathbf{I} \tag{6.6}$$

The magnetic flux vector $\mathbf{\Phi} = (\phi_1, \phi_2, ..., \phi_n)^T$ of the shield loops is:

$$\mathbf{\Phi} = M_s \mathbf{I}_s + M\mathbf{I} \tag{6.7}$$

Then the the dynamics equation (6.3), (6.4) and the circuit equation (6.5) can be written as:

$$\begin{cases} diag(\mathbf{m})\ddot{\mathbf{z}} + diag(\mathbf{b}_z)\dot{\mathbf{z}} + diag(\mathbf{k}_z)(\mathbf{z} - \mathbf{z}_0) = \frac{\partial W_m}{\partial \mathbf{z}} \\ diag(\mathbf{m})\ddot{\mathbf{r}} + diag(\mathbf{b}_r)\dot{\mathbf{r}} + diag(\mathbf{k}_r)(\mathbf{r} - \mathbf{r}_0) = \frac{\partial W_m}{\partial \mathbf{r}} \\ \frac{d\mathbf{\Phi}}{dt} + diag(\mathbf{R})\mathbf{I} = \mathbf{0} \end{cases} \tag{6.8}$$

Substituting (6.6) and (6.7) into (6.8), we obtain the system of the equations:

Find $t \to \mathbf{z}(t), t \to \mathbf{r}(t), t \to \mathbf{I}(t)$, such that

$$\begin{cases} diag(\mathbf{m})\ddot{\mathbf{z}} + diag(\mathbf{b}_z)\dot{\mathbf{z}} + diag(\mathbf{k}_z)(\mathbf{z} - \mathbf{z}_0) = \frac{1}{2}\mathbf{I}^T\frac{\partial M_s}{\partial \mathbf{z}}\mathbf{I}_s + \frac{1}{2}\mathbf{I}^T\frac{\partial M}{\partial \mathbf{z}}\mathbf{I} \\ diag(\mathbf{m})\ddot{\mathbf{r}} + diag(\mathbf{b}_r)\dot{\mathbf{r}} + diag(\mathbf{k}_r)(\mathbf{r} - \mathbf{r}_0) = \frac{1}{2}\mathbf{I}^T\frac{\partial M_s}{\partial \mathbf{r}}\mathbf{I}_s + \frac{1}{2}\mathbf{I}^T\frac{\partial M}{\partial \mathbf{r}}\mathbf{I} \\ M\dot{\mathbf{I}} + \left(\left[diag(\dot{\mathbf{r}})\frac{\partial M}{\partial \mathbf{r}} + diag(\dot{\mathbf{z}})\frac{\partial M}{\partial \mathbf{z}}\right] + diag(\mathbf{R})\right)\mathbf{I} = -M_s\dot{\mathbf{I}}_s - \left[diag(\dot{\mathbf{r}})\frac{\partial M_S}{\partial \mathbf{r}} + diag(\dot{\mathbf{z}})\frac{\partial M_S}{\partial \mathbf{z}}\right]\mathbf{I}_s \end{cases} \tag{6.9}$$

since

$$\frac{d\mathbf{\Phi}}{dt} = M_s\dot{\mathbf{I}}_s + \left[diag(\dot{\mathbf{r}})\frac{\partial M_S}{\partial \mathbf{r}} + diag(\dot{\mathbf{z}})\frac{\partial M_S}{\partial \mathbf{z}}\right]\mathbf{I}_s + M\dot{\mathbf{I}} + \left[diag(\dot{\mathbf{r}})\frac{\partial M}{\partial \mathbf{r}} + diag(\dot{\mathbf{z}})\frac{\partial M}{\partial \mathbf{z}}\right]\mathbf{I}$$

- **Conclusions and suggestions for future study**

  This multiple loops model is a discretization of the shield cylinder. Because we neglect the mechanical interaction between the loops, $diag(\mathbf{b}_z)$, $diag(\mathbf{b}_r)$, $diag(\mathbf{k}_z)$, $diag(\mathbf{k}_r)$ are the diagonal matrices here. In reality, these matrices should be band matrices. The method of model reduction and numerical simulation of the two loops model can be extended to the multiple loops model.

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Appendix A

Numerical solutions with the test data

In this chapter, the numerical solutions are obtained by using the numerical integration method ode113. All the pictures are generated by Matlab.
The source frequency $\tilde{\omega} = 1.0088$. It is the eigen frequency in $r-$ direction. The source amplitude $\epsilon' = 0.001$. For the practical reason, $c_1 = 0.001$. The scalar relative error tolerance is RelTol 1e-4 and the vector of absolute error tolerances is AbsTol 1e-7 (all components).
The Figures show the global solutions in the first line, the zoom-in initial part of the global solutions in the middle line and the zoom-in final part of the global solutions.

A.1 Numerical solutions for the linear system

A.2 Numerical solutions for the simplified nonlinear system

A.3 Numerical solutions for the complete nonlinear system

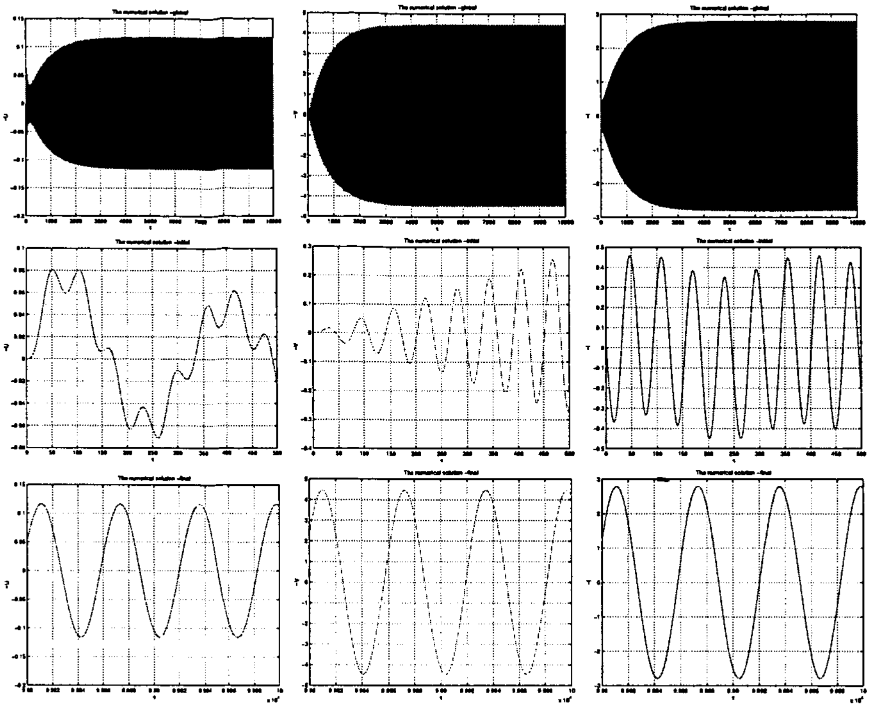

Figure A.1: Numerical solutions of the linear system

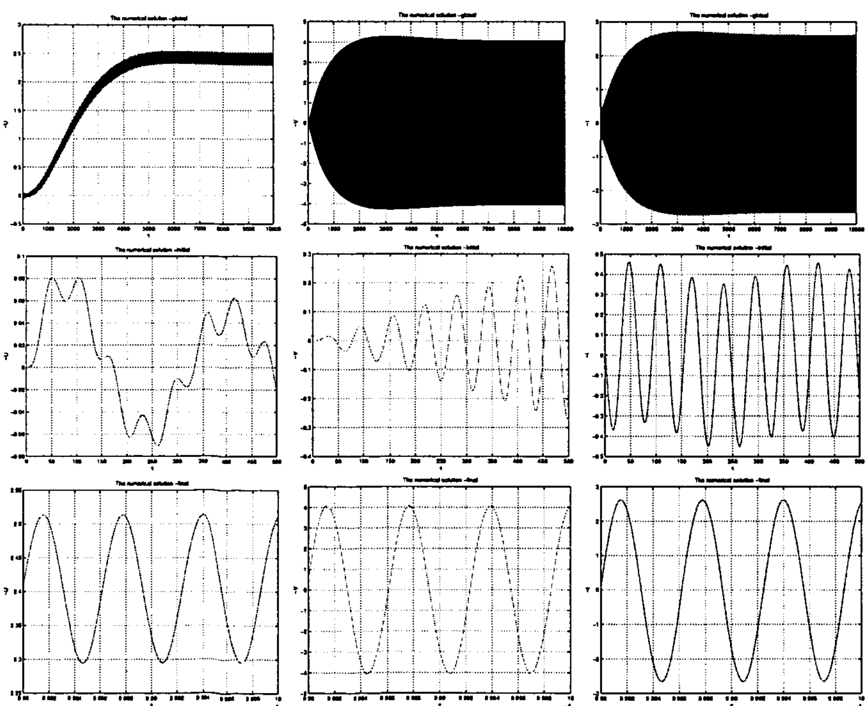

Figure A.2: Numerical solutions of the simplified nonlinear system

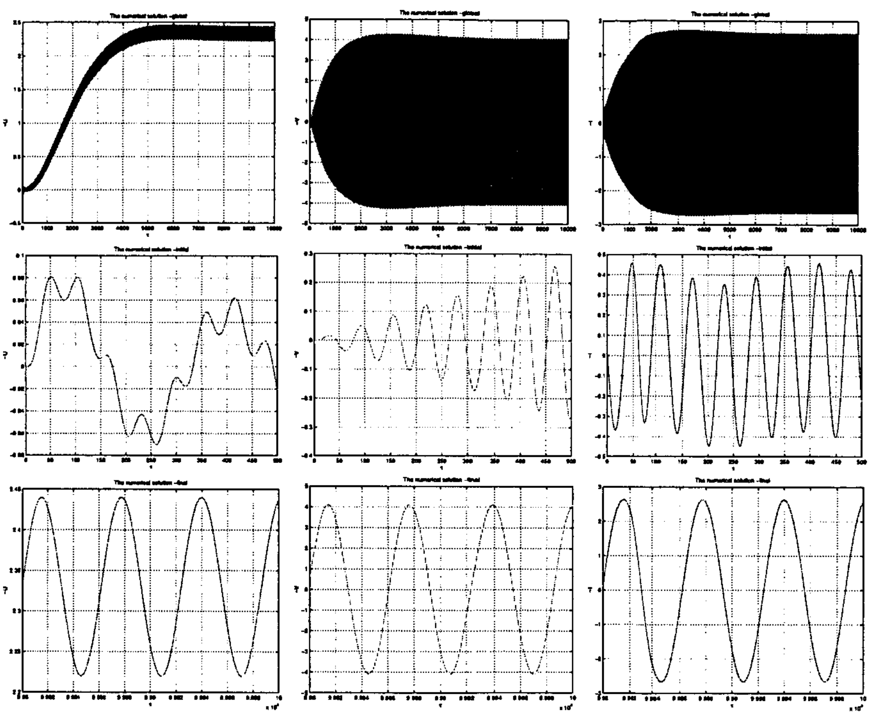

Figure A.3: Numerical solutions of the complete nonlinear system

Appendix B

Matlab scripts

B.1 The mutual inductance

```
% Function to calculate the mutual inductance of two co-axial circular loop and its derivatives;
% R1 and R2 are the radius of the loops;
% d is the seperation between the two loops;
% M is the mutual inductance
% Mr=dM/dR2; Mz=dM/dz2; Mrr=d^2M/dR2^2; Mrz=d^2M/dR2dz2; Mzz=d^2M/dz2^2;
%function [M,Mr,Mz,Mrr,Mrz,Mzz]=f_mi(R1,R2,d)
function [M0,Mr,Mz,Mrr,Mrz,Mzz]=t_f_mi(as,a,b)

R1=as; R2=a; d=b;

RA=R1+R2; RA2=RA*RA;

RJ=R1-R2; RJ2=RJ*RJ;

RM=R1*R2; R12=R1*R1; R22=R2*R2; d2=d*d;

k2=4*RM/(RA2+d2); k=sqrt(k2);

[Ke,Ee]=ellipke(k2);

F=((2-k2)*Ke-2*Ee)/k;

M0=sqrt(RM)*F;

F2=Ke+Ee*(R12-R22-d2)/(RJ2+d2); F3=Ke-Ee*(R12+R22+d2)/(RJ2+d2);

Mr=R2*F2/sqrt(RA2+d2); Mz=d*F3/sqrt(RA2+d2);

%F4k=R1^4-2*R1^3*R2+2*R1^2*(R2^2+d^2)^2)-2*r1*r2(R2^2+d^2)+(R2^2+d^2)(d^2)
F4k=(R12*R12-2*R12*RM+2*R12*(R22+d2)-2*RM*(R22+d2)+(R22+d2)*(R22+d2))*d2;
%F4e=2*R1^6+2*R1^2*R2^2*(R2^2-5*d^2)+R1^4*(-4*R2^2+3*d^2-(R2^2+d^2)^2*d^2
F4e=2*R12*R12*R12+2*R12*R22*(R22-5*d2)+R12*R12*(3*d2-4*R22)-(R22+d2)*(R22+d2)*d2;
F4m=(RJ2+d2)*(RJ2+d2)*(RA2+d2)*(RA2+d2)*k/2;

Mrr=sqrt(RM)*(F4e*Ee+F4k*Ke)/F4m;
```

```
F5e=-7*R12*R12 + 6*R12*(R22-d2) + (R22+d2)*(R22+d2); F5k=R12*R12 -
2*R12*RM + 2*RM*(R22+d2) - (R22+d2)*(R22+d2);

Mrz=R2*sqrt(RM)*d*(F5e*Ee+F5k*Ke)/F4m;

F6e=-(R12*R12*R12 - R12*R12*(R22-2*d2) + R22*(R22+d2)*(R22+d2) +
R12*(-R22*R22-12*R22*d2+d2*d2)); F6k=R12*R12*R12 - 2*R12*R12*RM -
R12*R12*(R22-2*d2) + 2*R12*RM*(2*R22-d2) - 2*RM*R22*(R22+d2)
+R22*(R22+d2)*(R22+d2)+R12*(-R22*R22+d2*d2);

Mzz=sqrt(RM)*(F6e*Ee+F6k*Ke)/F4m; return
```

B.2 The self inductance

```
% function to calculate the self inductance of a wire circular loop
% r is the radius of the loop
% a0 is the raduts of the cross section(circular), or half of the width of the cross section (re

function [L0,Lr,Lrr]=t_f_si(a,a0,gamma0) nu=0.3;
L0=a*(log(8*a^(1+nu)/(gamma0*a0^nu))-2);
Lr=log(8*a^(nu+1)/(gamma0*a0^nu))-1+nu; Lrr=(1+nu)/a; return
```

B.3 The linear system

```
% ode function to define the linear system
% with the mutual inductance and self inductance are constant.
function dy=equ_nonlinear(t,y)

 global c
 global epsilon
 global t_omega
 global nu
 global a0
 global b0
 global as
 global gamma0
 global e
 global ep

dy=zeros(5,1);


[M0,Ma,Mb,Maa,Mab,Mbb]=t_f_mi(as,a0,b0);
[L0,La,Laa]=t_f_si(a0,a0,gamma0);


dy(1)=y(2); dy(2)=-c(1)*y(1)-c(3)*y(2)+c(2)*Mb*y(5); dy(3)=y(4);
dy(4)=-y(3)-c(4)*y(4)+c(2)*Ma*y(5);
dy(5)=-c(5)/L0*y(5)-Ma/L0*y(4)-Mb/L0*y(2)-M0/L0*t_omega*cos(t_omega*t)*ep/e;

return
```

B.4 The simplified nonlinear system

```
% ode function to define the simplified nonlinear system,
% with the linearized mutual inductance and the self inductance

function dy=equ_nonlinear_new(t,y)
  global c
  global epsilon
  global t_omega
  global nu
  global a0
  global b0
  global as
  global gamma0
  global e
  global ep

nu2=1.6; dy=zeros(5,1);

[M0,Ma,Mb,Maa,Mab,Mbb]=t_f_mi(as,a0,b0);
[L0,La,Laa]=t_f_si(a0,a0,gamma0);


tM=M0+e*Ma*y(3)+e*Mb*y(1); tMa=Ma+e*Maa*y(3)+e*Mab*y(1);
tMb=Mb+e*Mab*y(3)+e*Mbb*y(1); tL=L0+e*La*y(3); tLa=La+e*Laa*y(3);

tk2=(a0/(a0+e*y(3)))^nu2; tR=1/tk2;

ss=1+ep*sin(t_omega*t); dss=t_omega*cos(t_omega*t)*ep/e;


dy(1)=y(2); dy(2)=-c(1)*y(1)-c(3)*y(2)+c(2)*tMb*ss*y(5);
dy(3)=y(4);
dy(4)=-tk2*y(3)-c(4)*y(4)+c(2)*(tMa*ss*y(5)+0.5*e*tLa*y(5)*y(5));
dy(5)=-((e*tLa*y(4)+c(5)*tR)*y(5)+tM*dss+(tMa*y(4)+tMb*y(2))*ss)/tL;

return
```

B.5 The complete nonlinear system

```
% ode function to define the complete nonlinear system
% ep=epsilon'=Ig/Im; e is the scaling parameter. mostly we chose e=ep


function dy=equ_nonlinear_c(t,y) global c global epsilon global
t_omega global nu global a0 global b0 global as global gamma0
global e global ep

nu2=1.6; dy=zeros(5,1);
```

```
a=a0+e*y(3); b=b0+e*y(1);
[M,Ma,Mb,Maa,Mab,Mbb]=t_f_mi(as,a,b);
[L,La,Laa]=t_f_si(a,a0,gamma0);

tk2=(a0/(a0+e*y(3)))^nu2; tR=1/tk2;

ss=1+ep*sin(t_omega*t); dss=t_omega*cos(t_omega*t)*ep/e;

dy(1)=y(2); dy(2)=-c(1)*y(1)-c(3)*y(2)+c(2)*Mb*ss*y(5);
dy(3)=y(4);
dy(4)=-tk2*y(3)-c(4)*y(4)+c(2)*(Ma*ss*y(5)+0.5*e*La*y(5)*y(5));
dy(5)=-((e*La*y(4)+c(5)*tR)*y(5)+M*dss+(Ma*y(4)+Mb*y(2))*ss)/L;

return
```

B.6 The harmonic balance method for the simplified nonlinear system

B.6.1 The main program

```
%main program
clear
global epsilon
global c %the constant coefficent
global a0
global b0
global as
global gamma0
global t_omega
global X

%The constant
pi=3.14159265; mu0=4*pi*1.0e-7; rho_m=2.7e3; rho=3.0e-8;
E=7.0e10;nu=0.3;
%Input the parameters
rs=0.4; zs=0; r0=0.45; z0=0.05; Im=6.0e5; Ig=6.0e2; aa0=1.0e-2/2;
A0=4*aa0*aa0; Q1=20; Q2=500; stiff_r=1e-4; e=0.0001;ep=0.0001;
%Calculate the mass, the resistance, and stiffness k_{20}
m=2*pi*r0*A0*rho_m; R0=rho*2*pi*r0/A0; k20=2*pi*E*A0/r0;
k1=stiff_r*k20; t0=sqrt(m/k20); omega0=1/t0; epsilon=Ig/Im;

as=rs/r0; a0=r0/r0; b0=z0/r0; gamma0=aa0/r0;

c=zeros(5,1);
%=====================
c(1)=k1/k20; c(2)=Im*Im*mu0/(k20*r0);
c(3)=sqrt(k1/k20)/sqrt(Q1*Q1+0.25); c(4)=1/sqrt(Q2*Q2+0.25);
c(5)=R0*t0/(mu0*r0);
%=====================
```

```
[M0,Ma,Mb,Maa,Mab,Mbb]=t_f_mi(as,a0,b0);
[L0,La,Laa]=t_f_si(a0,a0,gamma0);

%===================================
A=[0,1,0,0,0;
   -c(1),-c(3),0,0,c(2)*Mb;
   0,0,0,1,0;
   0,0,-1,-c(4),c(2)*Ma;
   0,-Mb/L0,0,-Ma/L0,-c(5)/L0];
b=[0,0,0,0,1]';
%=====================================

t_omega=0:0.001:2; [mm,nn]=size(t_omega);
Au1=zeros(1,nn);Av1=zeros(1,nn);Ai1=zeros(1,nn);
Pu1=zeros(1,nn);Pv1=zeros(1,nn);Pi1=zeros(1,nn);
Au2=zeros(1,nn);Av2=zeros(1,nn);Ai2=zeros(1,nn);
Pu2=zeros(1,nn);Pv2=zeros(1,nn);Pi2=zeros(1,nn);

U0=zeros(1,nn);V0=zeros(1,nn);I0=zeros(1,nn);


x=zeros(15,1);    x0=zeros(15,1); for i=1:nn
   [x,flag]=nonlinear_steady_f(c,e,ep,t_omega(i),M0,Ma,Mb,Maa,Mab,Mbb,L0,La,Laa,x0);
   Au1(i)=sqrt(x(2)*x(2)+x(3)*x(3));
   Pu1(i)=atan2(-x(3),x(2));
   Av1(i)=sqrt(x(7)*x(7)+x(8)*x(8));
   Pv1(i)=atan2(-x(8),x(7));
   Ai1(i)=sqrt(x(12)*x(12)+x(13)*x(13));
   Pi1(i)=atan2(-x(13),x(12));

   Au2(i)=sqrt(x(4)*x(4)+x(5)*x(5));
   Pu2(i)=atan2(-x(5),x(4));
   Av2(i)=sqrt(x(9)*x(9)+x(10)*x(10));
   Pv2(i)=atan2(-x(10),x(9));
   Ai2(i)=sqrt(x(14)*x(14)+x(15)*x(15));
   Pi2(i)=atan2(-x(15),x(14));
   U0(i)=x(1);V0(i)=x(6);I0(i)=x(11);
   x0=x;
end
   Pu1=unwrap(Pu1);
   Pv1=unwrap(Pv1);
   Pi1=unwrap(Pi1);

   Pu2=unwrap(Pu2);
   Pv2=unwrap(Pv2);
   Pi2=unwrap(Pi2);

%=========================================
figure(01) plot(t_omega,U0); grid xlabel('frequency
\omega/\omega_{or}'); ylabel('Amplitude U_0/(\epsilon r_0)');
title('Amplitude distribition in the frequency domain');
figure(02) plot(t_omega,V0); grid xlabel('frequency
\omega/\omega_{or}'); ylabel('Amplitude V_0/(\epsilon r_0)');
title('Amplitude distribition in the frequency domain');
```

```
figure(03) plot(t_omega,I0); grid xlabel('frequency
\omega/\omega_{or}'); ylabel('Amplitude I_0/(\epsilon I_m)');
title('Amplitude distribition in the frequency domain');
%======================================
figure(11) plot(t_omega,Au1); grid xlabel('frequency
\omega/\omega_{or}'); ylabel('Amplitude U_1/(\epsilon r_0)');
title('Amplitude distribition in the frequency domain');

figure(12) plot(t_omega,Pu1) grid xlabel('frequency
\omega/\omega_{or}'); ylabel('Phase of U_1'); title('Phase
distribition in the frequency domain');


figure(13) plot(t_omega,Av1); grid xlabel('frequency
\omega/\omega_{or}'); ylabel('Amplitude V_1/(\epsilon r_0)');
title('Amplitude distribition in the frequency domain');

figure(14) plot(t_omega,Pv1) grid xlabel('frequency
\omega/\omega_{or}'); ylabel('Phase of V1'); title('Phase
distribition in the frequency domain');


figure(15) plot(t_omega,Ai1); grid xlabel('frequency
\omega/\omega_{or}'); ylabel('Amplitude I_1/(\epsilon I_m)');
title('Amplitude distribition in the frequency domain');

figure(16) plot(t_omega,Pi1) grid xlabel('frequency
\omega/\omega_{or}'); ylabel('Phase of I_1'); title('Phase
distribition in the frequency domain');

%==========================================
figure(21) plot(t_omega,Au2); grid xlabel('frequency
\omega/\omega_{or}'); ylabel('Amplitude U_2/(\epsilon r_0)');
title('Amplitude distribition in the frequency domain');
figure(22) plot(t_omega,Pu2) grid xlabel('frequency
\omega/\omega_{or}'); ylabel('Phase of U_2'); title('Phase
distribition in the frequency domain');


figure(23) plot(t_omega,Av2); grid xlabel('frequency
\omega/\omega_{or}'); ylabel('Amplitude V_2/(\epsilon r_0)');
title('Amplitude distribition in the frequency domain ');
figure(24) plot(t_omega,Pv2) grid xlabel('frequency
\omega/\omega_{or}'); ylabel('Phase of V_2'); title('Phase
distribition in the frequency domain');


figure(25) plot(t_omega,Ai2); grid xlabel('frequency
\omega/\omega_{or}'); ylabel('Amplitude I_2/(\epsilon I_m)');
title('Amplitude distribition in the frequency domain ');
figure(26) plot(t_omega,Pi2) grid xlabel('frequency
\omega/\omega_{or}'); ylabel('Phase of I_2'); title('Phase
distribition in the frequency domain');
```

B.6.2 The Newton iteration method

```
function
[x1,flag]=nonlinear_steady_f(c,e,ep,w,M0,Ma,Mb,Maa,Mab,Mbb,L0,La,Laa,x0);

x1=zeros(15,1); dF=zeros(15,15); F=zeros(15,1);

Tol=1e-6; maxstep=10000; flag=0;j=0; while flag==0
   j=j+1;
   F=new_newton_F(c,e,ep,w,M0,Ma,Mb,Maa,Mab,Mbb,L0,La,Laa,x0);
   dF=new_newton_DF(c,e,ep,w,M0,Ma,Mb,Maa,Mab,Mbb,L0,La,Laa,x0);
   x1=x0-inv(dF)*F';
   if norm(x1-x0) < Tol
      flag=1;
   else if j > maxstep
         flag=2;
   else
      x0=x1;
   end
   end
end
```

B.6.3 The nonlinear algebraic system in harmonic balance method

```
function F=newton_F(c,e,ep,w,M0,Ma,Mb,Maa,Mab,Mbb,L0,La,Laa,x0)

nu=0.3; c1=c(1);c2=c(2);c3=c(3);c4=c(4);c5=c(5);
U0=x0(1);U1a=x0(2);U1b=x0(3);U2a=x0(4);U2b=x0(5);
V0=x0(6);V1a=x0(7);V1b=x0(8);V2a=x0(9);V2b=x0(10);
J0=x0(11);J1a=x0(12);J1b=x0(13);J2a=x0(14);J2b=x0(15);



%================================================================
% Calculate F(x)
%================================================================
eq10 =    1/4*((-4)*c2*J0*Mb - 2*c2*ep*J1b*Mb + 4*c1*U0 - ...
           4*c2*e*J0*Mbb*U0 - 2*c2*e*ep*J1b*Mbb*U0 - ...
           2*c2*e*J1a*Mbb*U1a - c2*e*ep*J2b*Mbb*U1a - ...
           2*c2*e*ep*J0*Mbb*U1b - 2*c2*e*J1b*Mbb*U1b + ...
           c2*e*ep*J2a*Mbb*U1b + c2*e*ep*J1b*Mbb*U2a - ...
           2*c2*e*J2a*Mbb*U2a - c2*e*ep*J1a*Mbb*U2b - ...
           2*c2*e*J2b*Mbb*U2b - 4*c2*e*J0*Mab*V0 - ...
           2*c2*e*ep*J1b*Mab*V0 - 2*c2*e*J1a*Mab*V1a -...
           c2*e*ep*J2b*Mab*V1a - 2*c2*e*ep*J0*Mab*V1b - ...
           2*c2*e*J1b*Mab*V1b + c2*e*ep*J2a*Mab*V1b + ...
           c2*e*ep*J1b*Mab*V2a - 2*c2*e*J2a*Mab*V2a - ...
           c2*e*ep*J1a*Mab*V2b - 2*c2*e*J2b*Mab*V2b);

eq20 =    1/8*((-4)*c2*e*J0^2*La - 2*c2*e*J1a^2*La - ...
           2*c2*e*J1b^2*La - 2*c2*e*J2a^2*La - ...
```

```
        2*c2*e*J2b^2*La - 8*c2*J0*Ma - 4*c2*ep*J1b*Ma -...
        8*c2*e*J0*Mab*U0 - 4*c2*e*ep*J1b*Mab*U0 - ...
        4*c2*e*J1a*Mab*U1a - 2*c2*e*ep*J2b*Mab*U1a - ...
        4*c2*e*ep*J0*Mab*U1b - 4*c2*e*J1b*Mab*U1b + ...
        2*c2*e*ep*J2a*Mab*U1b + 2*c2*e*ep*J1b*Mab*U2a -...
        4*c2*e*J2a*Mab*U2a - 2*c2*e*ep*J1a*Mab*U2b - ...
        4*c2*e*J2b*Mab*U2b + 8*V0 - 4*c2*e^2*J0^2*Laa*V0 -...
        2*c2*e^2*J1a^2*Laa*V0 - 2*c2*e^2*J1b^2*Laa*V0 - ...
        2*c2*e^2*J2a^2*Laa*V0 - 2*c2*e^2*J2b^2*Laa*V0 - ...
        8*c2*e*J0*Maa*V0 - 4*c2*e*ep*J1b*Maa*V0 - 8*e*V0^2 -...
        16*e*nu*V0^2 - 4*c2*e^2*J0*J1a*Laa*V1a - ...
        2*c2*e^2*J1a*J2a*Laa*V1a - 2*c2*e^2*J1b*J2b*Laa*V1a - ...
        4*c2*e*J1a*Maa*V1a - 2*c2*e*ep*J2b*Maa*V1a - ...
        4*e*V1a^2 - 8*e*nu*V1a^2 - 4*c2*e^2*J0*J1b*Laa*V1b +...
        2*c2*e^2*J1b*J2a*Laa*V1b - 2*c2*e^2*J1a*J2b*Laa*V1b - ...
        4*c2*e*ep*J0*Maa*V1b - 4*c2*e*J1b*Maa*V1b + ...
        2*c2*e*ep*J2a*Maa*V1b - 4*e*V1b^2 - 8*e*nu*V1b^2 -...
        c2*e^2*J1a^2*Laa*V2a + c2*e^2*J1b^2*Laa*V2a - ...
        4*c2*e^2*J0*J2a*Laa*V2a + 2*c2*e*ep*J1b*Maa*V2a -...
        4*c2*e*J2a*Maa*V2a - 4*e*V2a^2 - 8*e*nu*V2a^2 - ...
        2*c2*e^2*J1a*J1b*Laa*V2b - 4*c2*e^2*J0*J2b*Laa*V2b -...
        2*c2*e*ep*J1a*Maa*V2b - 4*c2*e*J2b*Maa*V2b - ...
        4*e*V2b^2 - 8*e*nu*V2b^2);

eq30 = (1/(4*e))*(4*c5*e*J0 + 4*c5*e^2*J0*V0 +...
        8*c5*e^2*J0*nu*V0 + 2*c5*e^2*J1a*V1a + ...
        4*c5*e^2*J1a*nu*V1a + 2*c5*e^2*J1b*V1b + ...
        4*c5*e^2*J1b*nu*V1b + 2*c5*e^2*J2a*V2a + ...
        4*c5*e^2*J2a*nu*V2a + 2*c5*e^2*J2b*V2b + ...
        4*c5*e^2*J2b*nu*V2b - 2*e^2*ep*Mbb*U0*U1a*w -...
        e^2*ep*Mbb*U1a*U2a*w - e^2*ep*Mbb*U1b*U2b*w - ...
        2*e^2*ep*Mab*U1a*V0*w - 2*e^2*ep*Mab*U0*V1a*w - ...
        e^2*ep*Mab*U2a*V1a*w - 2*e^3*J1b*Laa*V0*V1a*w - ...
        2*e^2*ep*Maa*V0*V1a*w - e^3*J2b*Laa*V1a^2*w - ...
        e^2*ep*Mab*U2b*V1b*w + 2*e^3*J1a*Laa*V0*V1b*w + ...
        2*e^3*J2a*Laa*V1a*V1b*w + e^3*J2b*Laa*V1b^2*w - ...
        e^2*ep*Mab*U1a*V2a*w - 4*e^3*J2b*Laa*V0*V2a*w - ...
        e^3*J1b*Laa*V1a*V2a*w - e^2*ep*Maa*V1a*V2a*w - ...
        e^3*J1a*Laa*V1b*V2a*w - e^2*ep*Mab*U1b*V2b*w + ...
        4*e^3*J2a*Laa*V0*V2b*w + e^3*J1a*Laa*V1a*V2b*w - ...
        e^3*J1b*Laa*V1b*V2b*w - e^2*ep*Maa*V1b*V2b*w);

eq11 =  1/4*((-4)*c2*J1a*Mb - 2*c2*ep*J2b*Mb - ...
        4*c2*e*J1a*Mbb*U0 - 2*c2*e*ep*J2b*Mbb*U0 + 4*c1*U1a -...
        4*c2*e*J0*Mbb*U1a - c2*e*ep*J1b*Mbb*U1a - ...
        2*c2*e*J2a*Mbb*U1a - c2*e*ep*J1a*Mbb*U1b - ...
        2*c2*e*J2b*Mbb*U1b - 2*c2*e*J1a*Mbb*U2a - ...
        2*c2*e*ep*J0*Mbb*U2b - 2*c2*e*J1b*Mbb*U2b - ...
        4*c2*e*J1a*Mab*V0 - 2*c2*e*ep*J2b*Mab*V0 - ...
        4*c2*e*J0*Mab*V1a - c2*e*ep*J1b*Mab*V1a - ...
        2*c2*e*J2a*Mab*V1a - c2*e*ep*J1a*Mab*V1b - ...
        2*c2*e*J2b*Mab*V1b - 2*c2*e*J1a*Mab*V2a - ...
        2*c2*e*ep*J0*Mab*V2b - 2*c2*e*J1b*Mab*V2b + ...
        4*c3*U1b*w - 4*U1a*w^2);
```

```
eq12 =   1/4*((-4)*c2*ep*J0*Mb - 4*c2*J1b*Mb + 2*c2*ep*J2a*Mb -...
         4*c2*e*ep*J0*Mbb*U0 - 4*c2*e*J1b*Mbb*U0 + ...
         2*c2*e*ep*J2a*Mbb*U0 - c2*e*ep*J1a*Mbb*U1a - ...
         2*c2*e*J2b*Mbb*U1a + 4*c1*U1b - 4*c2*e*J0*Mbb*U1b -...
         3*c2*e*ep*J1b*Mbb*U1b + 2*c2*e*J2a*Mbb*U1b + ...
         2*c2*e*ep*J0*Mbb*U2a + 2*c2*e*J1b*Mbb*U2a - ...
         2*c2*e*ep*J2a*Mbb*U2a - 2*c2*e*J1a*Mbb*U2b - ...
         2*c2*e*ep*J2b*Mbb*U2b - 4*c2*e*ep*J0*Mab*V0 - ...
         4*c2*e*J1b*Mab*V0 + 2*c2*e*ep*J2a*Mab*V0 - ...
         c2*e*ep*J1a*Mab*V1a - 2*c2*e*J2b*Mab*V1a - ...
         4*c2*e*J0*Mab*V1b - 3*c2*e*ep*J1b*Mab*V1b + ...
         2*c2*e*J2a*Mab*V1b + 2*c2*e*ep*J0*Mab*V2a + ...
         2*c2*e*J1b*Mab*V2a - 2*c2*e*ep*J2a*Mab*V2a - ...
         2*c2*e*J1a*Mab*V2b - 2*c2*e*ep*J2b*Mab*V2b - ...
         4*c3*U1a*w - 4*U1b*w^2);


eq13 =   1/4*(2*c2*ep*J1b*Mb - 4*c2*J2a*Mb +...
         2*c2*e*ep*J1b*Mbb*U0 - 4*c2*e*J2a*Mbb*U0 -...
         2*c2*e*J1a*Mbb*U1a + 2*c2*e*ep*J0*Mbb*U1b + ...
         2*c2*e*J1b*Mbb*U1b - 2*c2*e*ep*J2a*Mbb*U1b + ...
         4*c1*U2a - 4*c2*e*J0*Mbb*U2a - 2*c2*e*ep*J1b*Mbb*U2a +...
         2*c2*e*ep*J1b*Mab*V0 - 4*c2*e*J2a*Mab*V0 - ...
         2*c2*e*J1a*Mab*V1a + 2*c2*e*ep*J0*Mab*V1b + ...
         2*c2*e*J1b*Mab*V1b - 2*c2*e*ep*J2a*Mab*V1b - ...
         4*c2*e*J0*Mab*V2a - 2*c2*e*ep*J1b*Mab*V2a + ...
         8*c3*U2b*w - 16*U2a*w^2);


eq14 =   1/4*((-2)*c2*ep*J1a*Mb - 4*c2*J2b*Mb -...
         2*c2*e*ep*J1a*Mbb*U0 - 4*c2*e*J2b*Mbb*U0 -...
         2*c2*e*ep*J0*Mbb*U1a - 2*c2*e*J1b*Mbb*U1a - ...
         2*c2*e*J1a*Mbb*U1b - 2*c2*e*ep*J2b*Mbb*U1b + ...
         4*c1*U2b - 4*c2*e*J0*Mbb*U2b - 2*c2*e*ep*J1b*Mbb*U2b -...
         2*c2*e*ep*J1a*Mab*V0 - 4*c2*e*J2b*Mab*V0 - ...
         2*c2*e*ep*J0*Mab*V1a - 2*c2*e*J1b*Mab*V1a - ...
         2*c2*e*J1a*Mab*V1b - 2*c2*e*ep*J2b*Mab*V1b - ...
         4*c2*e*J0*Mab*V2b - 2*c2*e*ep*J1b*Mab*V2b - ...
         8*c3*U2a*w - 16*U2b*w^2);

eq21 =   1/8*((-8)*c2*e*J0*J1a*La - 4*c2*e*J1a*J2a*La -...
         4*c2*e*J1b*J2b*La - 8*c2*J1a*Ma - 4*c2*ep*J2b*Ma -...
         8*c2*e*J1a*Mab*U0 - 4*c2*e*ep*J2b*Mab*U0 - ...
         8*c2*e*J0*Mab*U1a - 2*c2*e*ep*J1b*Mab*U1a - ...
         4*c2*e*J2a*Mab*U1a - 2*c2*e*ep*J1a*Mab*U1b - ...
         4*c2*e*J2b*Mab*U1b - 4*c2*e*J1a*Mab*U2a - ...
         4*c2*e*ep*J0*Mab*U2b - 4*c2*e*J1b*Mab*U2b - ...
         8*c2*e^2*J0*J1a*Laa*V0 - 4*c2*e^2*J1a*J2a*Laa*V0 -...
         4*c2*e^2*J1b*J2b*Laa*V0 - 8*c2*e*J1a*Maa*V0 - ...
         4*c2*e*ep*J2b*Maa*V0 + 8*V1a - ...
         4*c2*e^2*J0^2*Laa*V1a - 3*c2*e^2*J1a^2*Laa*V1a - ...
         c2*e^2*J1b^2*Laa*V1a - 4*c2*e^2*J0*J2a*Laa*V1a - ...
         2*c2*e^2*J2a^2*Laa*V1a - 2*c2*e^2*J2b^2*Laa*V1a - ...
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        8*c2*e*J0*Maa*V1a - 2*c2*e*ep*J1b*Maa*V1a - ...
        4*c2*e*J2a*Maa*V1a - 16*e*V0*V1a - 32*e*nu*V0*V1a -...
        2*c2*e^2*J1a*J1b*Laa*V1b - 4*c2*e^2*J0*J2b*Laa*V1b - ...
        2*c2*e*ep*J1a*Maa*V1b - 4*c2*e*J2b*Maa*V1b - ...
        4*c2*e^2*J0*J1a*Laa*V2a - 4*c2*e^2*J1a*J2a*Laa*V2a -...
        4*c2*e*J1a*Maa*V2a - 8*e*V1a*V2a - 16*e*nu*V1a*V2a - ...
        4*c2*e^2*J0*J1b*Laa*V2b - 4*c2*e^2*J1a*J2b*Laa*V2b - ...
        4*c2*e*ep*J0*Maa*V2b - 4*c2*e*J1b*Maa*V2b - ...
        8*e*V1b*V2b - 16*e*nu*V1b*V2b + 8*c4*V1b*w - ...
        8*V1a*w^2);

eq22 =  1/8*((-8)*c2*e*J0*J1b*La + 4*c2*e*J1b*J2a*La -...
        4*c2*e*J1a*J2b*La - 8*c2*ep*J0*Ma - 8*c2*J1b*Ma + ...
        4*c2*ep*J2a*Ma - 8*c2*e*ep*J0*Mab*U0 - ...
        8*c2*e*J1b*Mab*U0 + 4*c2*e*ep*J2a*Mab*U0 -...
        2*c2*e*ep*J1a*Mab*U1a - 4*c2*e*J2b*Mab*U1a - ...
        8*c2*e*J0*Mab*U1b - 6*c2*e*ep*J1b*Mab*U1b + ...
        4*c2*e*J2a*Mab*U1b + 4*c2*e*ep*J0*Mab*U2a + ...
        4*c2*e*J1b*Mab*U2a - 4*c2*e*ep*J2a*Mab*U2a - ...
        4*c2*e*J1a*Mab*U2b - 4*c2*e*ep*J2b*Mab*U2b - ...
        8*c2*e^2*J0*J1b*Laa*V0 + 4*c2*e^2*J1b*J2a*Laa*V0 -...
        4*c2*e^2*J1a*J2b*Laa*V0 - 8*c2*e*ep*J0*Maa*V0 - ...
        8*c2*e*J1b*Maa*V0 + 4*c2*e*ep*J2a*Maa*V0 - ...
        2*c2*e^2*J1a*J1b*Laa*V1a - 4*c2*e^2*J0*J2b*Laa*V1a -...
        2*c2*e*ep*J1a*Maa*V1a - 4*c2*e*J2b*Maa*V1a + 8*V1b - ...
        4*c2*e^2*J0^2*Laa*V1b - c2*e^2*J1a^2*Laa*V1b - ...
        3*c2*e^2*J1b^2*Laa*V1b + 4*c2*e^2*J0*J2a*Laa*V1b -...
        2*c2*e^2*J2a^2*Laa*V1b - 2*c2*e^2*J2b^2*Laa*V1b - ...
        8*c2*e*J0*Maa*V1b - 6*c2*e*ep*J1b*Maa*V1b + ...
        4*c2*e*J2a*Maa*V1b - 16*e*V0*V1b - 32*e*nu*V0*V1b +...
        4*c2*e^2*J0*J1b*Laa*V2a - 4*c2*e^2*J1b*J2a*Laa*V2a + ...
        4*c2*e*ep*J0*Maa*V2a + 4*c2*e*J1b*Maa*V2a - ...
        4*c2*e*ep*J2a*Maa*V2a + 8*e*V1b*V2a + ...
        16*e*nu*V1b*V2a - 4*c2*e^2*J0*J1a*Laa*V2b -...
        4*c2*e^2*J1b*J2b*Laa*V2b - 4*c2*e*J1a*Maa*V2b -...
        4*c2*e*ep*J2b*Maa*V2b - 8*e*V1a*V2b - ...
        16*e*nu*V1a*V2b - 8*c4*V1a*w - 8*V1b*w^2);


eq23 =  1/8*((-2)*c2*e*J1a^2*La + 2*c2*e*J1b^2*La -...
        8*c2*e*J0*J2a*La + 4*c2*ep*J1b*Ma - 8*c2*J2a*Ma +...
        4*c2*e*ep*J1b*Mab*U0 - 8*c2*e*J2a*Mab*U0 - ...
        4*c2*e*J1a*Mab*U1a + 4*c2*e*ep*J0*Mab*U1b + ...
        4*c2*e*J1b*Mab*U1b - 4*c2*e*ep*J2a*Mab*U1b - ...
        8*c2*e*J0*Mab*U2a - 4*c2*e*ep*J1b*Mab*U2a - ...
        2*c2*e^2*J1a^2*Laa*V0 + 2*c2*e^2*J1b^2*Laa*V0 -...
        8*c2*e^2*J0*J2a*Laa*V0 + 4*c2*e*ep*J1b*Maa*V0 - ...
        8*c2*e*J2a*Maa*V0 - 4*c2*e^2*J0*J1a*Laa*V1a - ...
        4*c2*e^2*J1a*J2a*Laa*V1a - 4*c2*e*J1a*Maa*V1a - ...
        4*e*V1a^2 - 8*e*nu*V1a^2 + 4*c2*e^2*J0*J1b*Laa*V1b -...
        4*c2*e^2*J1b*J2a*Laa*V1b + 4*c2*e*ep*J0*Maa*V1b + ...
        4*c2*e*J1b*Maa*V1b - 4*c2*e*ep*J2a*Maa*V1b + ...
        4*e*V1b^2 + 8*e*nu*V1b^2 + 8*V2a - ...
        4*c2*e^2*J0^2*Laa*V2a - 2*c2*e^2*J1a^2*Laa*V2a -...
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        2*c2*e^2*J1b^2*Laa*V2a - 3*c2*e^2*J2a^2*Laa*V2a - ...
        c2*e^2*J2b^2*Laa*V2a - 8*c2*e*J0*Maa*V2a - ...
        4*c2*e*ep*J1b*Maa*V2a - 16*e*V0*V2a - 32*e*nu*V0*V2a - ...
        2*c2*e^2*J2a*J2b*Laa*V2b + 16*c4*V2b*w - ...
        32*V2a*w^2);

eq24 =  1/8*((-4)*c2*e*J1a*J1b*La - 8*c2*e*J0*J2b*La -...
        4*c2*ep*J1a*Ma - 8*c2*J2b*Ma - 4*c2*e*ep*J1a*Mab*U0 -...
        8*c2*e*J2b*Mab*U0 - 4*c2*e*ep*J0*Mab*U1a - ...
        4*c2*e*J1b*Mab*U1a - 4*c2*e*J1a*Mab*U1b - ...
        4*c2*e*ep*J2b*Mab*U1b - 8*c2*e*J0*Mab*U2b - ...
        4*c2*e*ep*J1b*Mab*U2b - 4*c2*e^2*J1a*J1b*Laa*V0 - ...
        8*c2*e^2*J0*J2b*Laa*V0 - 4*c2*e*ep*J1a*Maa*V0 - ...
        8*c2*e*J2b*Maa*V0 - 4*c2*e^2*J0*J1b*Laa*V1a - ...
        4*c2*e^2*J1a*J2b*Laa*V1a - 4*c2*e*ep*J0*Maa*V1a - ...
        4*c2*e*J1b*Maa*V1a - 4*c2*e^2*J0*J1a*Laa*V1b - ...
        4*c2*e^2*J1b*J2b*Laa*V1b - 4*c2*e*J1a*Maa*V1b - ...
        4*c2*e*ep*J2b*Maa*V1b - 8*e*V1a*V1b - ...
        16*e*nu*V1a*V1b - 2*c2*e^2*J2a*J2b*Laa*V2a + 8*V2b -...
        4*c2*e^2*J0^2*Laa*V2b - 2*c2*e^2*J1a^2*Laa*V2b - ...
        2*c2*e^2*J1b^2*Laa*V2b - c2*e^2*J2a^2*Laa*V2b - ...
        3*c2*e^2*J2b^2*Laa*V2b - 8*c2*e*J0*Maa*V2b - ...
        4*c2*e*ep*J1b*Maa*V2b - 16*e*V0*V2b - 32*e*nu*V0*V2b -...
        16*c4*V2a*w - 32*V2b*w^2);

eq31 = (1/(4*e))*(4*c5*e*J1a + 4*c5*e^2*J1a*V0 +...
        8*c5*e^2*J1a*nu*V0 + 4*c5*e^2*J0*V1a + ...
        2*c5*e^2*J2a*V1a + 8*c5*e^2*J0*nu*V1a + ...
        4*c5*e^2*J2a*nu*V1a + 2*c5*e^2*J2b*V1b + ...
        4*c5*e^2*J2b*nu*V1b + 2*c5*e^2*J1a*V2a + ...
        4*c5*e^2*J1a*nu*V2a + 2*c5*e^2*J1b*V2b + ...
        4*c5*e^2*J1b*nu*V2b + 4*e*J1b*L0*w + 4*ep*M0*w +...
        4*e*ep*Mb*U0*w - e^2*ep*Mbb*U1a^2*w + 4*e*Mb*U1b*w +...
        4*e^2*Mbb*U0*U1b*w + e^2*ep*Mbb*U1b^2*w - ...
        2*e*ep*Mb*U2a*w - 4*e^2*ep*Mbb*U0*U2a*w - ...
        2*e^2*Mbb*U1b*U2a*w + 2*e^2*Mbb*U1a*U2b*w + ...
        4*e^2*J1b*La*V0*w + 4*e*ep*Ma*V0*w + ...
        4*e^2*Mab*U1b*V0*w - 4*e^2*ep*Mab*U2a*V0*w +...
        2*e^2*J2b*La*V1a*w - 2*e^2*ep*Mab*U1a*V1a*w + ...
        2*e^2*Mab*U2b*V1a*w - 2*e^3*J2b*Laa*V0*V1a*w - ...
        e^3*J1b*Laa*V1a^2*w - e^2*ep*Maa*V1a^2*w + ...
        4*e^2*J0*La*V1b*w - 2*e^2*J2a*La*V1b*w + ...
        4*e*Ma*V1b*w + 4*e^2*Mab*U0*V1b*w + ...
        2*e^2*ep*Mab*U1b*V1b*w - 2*e^2*Mab*U2a*V1b*w +...
        4*e^3*J0*Laa*V0*V1b*w + 2*e^3*J2a*Laa*V0*V1b*w + ...
        4*e^2*Maa*V0*V1b*w + 2*e^3*J1a*Laa*V1a*V1b*w + ...
        e^3*J1b*Laa*V1b^2*w + e^2*ep*Maa*V1b^2*w - ...
        2*e^2*J1b*La*V2a*w - 2*e*ep*Ma*V2a*w - ...
        4*e^2*ep*Mab*U0*V2a*w - 2*e^2*Mab*U1b*V2a*w -...
        4*e^3*J1b*Laa*V0*V2a*w - 4*e^2*ep*Maa*V0*V2a*w -...
        4*e^3*J2b*Laa*V1a*V2a*w - 2*e^3*J0*Laa*V1b*V2a*w + ...
        2*e^3*J2a*Laa*V1b*V2a*w - 2*e^2*Maa*V1b*V2a*w + ...
        2*e^2*J1a*La*V2b*w + 2*e^2*Mab*U1a*V2b*w + ...
        4*e^3*J1a*Laa*V0*V2b*w + 2*e^3*J0*Laa*V1a*V2b*w +...
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         4*e^3*J2a*Laa*V1a*V2b*w + 2*e^2*Maa*V1a*V2b*w + ...
         2*e^3*J2b*Laa*V1b*V2b*w);

eq32 = (1/(4*e))*(4*c5*e*J1b + 4*c5*e^2*J1b*V0 +...
         8*c5*e^2*J1b*nu*V0 + 2*c5*e^2*J2b*V1a + ...
         4*c5*e^2*J2b*nu*V1a + 4*c5*e^2*J0*V1b - ...
         2*c5*e^2*J2a*V1b + 8*c5*e^2*J0*nu*V1b - ...
         4*c5*e^2*J2a*nu*V1b - 2*c5*e^2*J1b*V2a - ...
         4*c5*e^2*J1b*nu*V2a + 2*c5*e^2*J1a*V2b + ...
         4*c5*e^2*J1a*nu*V2b - 4*e*J1a*L0*w - 4*e*Mb*U1a*w -...
         4*e^2*Mbb*U0*U1a*w - 2*e^2*ep*Mbb*U1a*U1b*w - ...
         2*e^2*Mbb*U1a*U2a*w - 2*e*ep*Mb*U2b*w - ...
         4*e^2*ep*Mbb*U0*U2b*w - 2*e^2*Mbb*U1b*U2b*w -...
         4*e^2*J1a*La*V0*w - 4*e^2*Mab*U1a*V0*w - ...
         4*e^2*ep*Mab*U2b*V0*w - 4*e^2*J0*La*V1a*w - ...
         2*e^2*J2a*La*V1a*w - 4*e*Ma*V1a*w - ...
         4*e^2*Mab*U0*V1a*w - 2*e^2*ep*Mab*U1b*V1a*w -...
         2*e^2*Mab*U2a*V1a*w - 4*e^3*J0*Laa*V0*V1a*w + ...
         2*e^3*J2a*Laa*V0*V1a*w - 4*e^2*Maa*V0*V1a*w - ...
         e^3*J1a*Laa*V1a^2*w - 2*e^2*J2b*La*V1b*w - ...
         2*e^2*ep*Mab*U1a*V1b*w - 2*e^2*Mab*U2b*V1b*w +...
         2*e^3*J2b*Laa*V0*V1b*w - 2*e^3*J1b*Laa*V1a*V1b*w -...
         2*e^2*ep*Maa*V1a*V1b*w + e^3*J1a*Laa*V1b^2*w - ...
         2*e^2*J1a*La*V2a*w - 2*e^2*Mab*U1a*V2a*w - ...
         4*e^3*J1a*Laa*V0*V2a*w - 2*e^3*J0*Laa*V1a*V2a*w -...
         2*e^3*J2a*Laa*V1a*V2a*w - 2*e^2*Maa*V1a*V2a*w - ...
         4*e^3*J2b*Laa*V1b*V2a*w - 2*e^2*J1b*La*V2b*w - ...
         2*e*ep*Ma*V2b*w - 4*e^2*ep*Mab*U0*V2b*w - ...
         2*e^2*Mab*U1b*V2b*w - 4*e^3*J1b*Laa*V0*V2b*w -...
         4*e^2*ep*Maa*V0*V2b*w - 2*e^3*J2b*Laa*V1a*V2b*w -...
         2*e^3*J0*Laa*V1b*V2b*w + 4*e^3*J2a*Laa*V1b*V2b*w - ...
         2*e^2*Maa*V1b*V2b*w);

eq33 = (1/(4*e))*(4*c5*e*J2a + 4*c5*e^2*J2a*V0 +...
         8*c5*e^2*J2a*nu*V0 + 2*c5*e^2*J1a*V1a + ...
         4*c5*e^2*J1a*nu*V1a - 2*c5*e^2*J1b*V1b - ...
         4*c5*e^2*J1b*nu*V1b + 4*c5*e^2*J0*V2a + ...
         8*c5*e^2*J0*nu*V2a + 8*e*J2b*L0*w + ...
         4*e*ep*Mb*U1a*w + 2*e^2*ep*Mbb*U0*U1a*w +...
         4*e^2*Mbb*U1a*U1b*w - 2*e^2*ep*Mbb*U1a*U2a*w +...
         8*e*Mb*U2b*w + 8*e^2*Mbb*U0*U2b*w + ...
         4*e^2*ep*Mbb*U1b*U2b*w + 8*e^2*J2b*La*V0*w +...
         2*e^2*ep*Mab*U1a*V0*w + 8*e^2*Mab*U2b*V0*w + ...
         4*e^2*J1b*La*V1a*w + 4*e*ep*Ma*V1a*w + ...
         2*e^2*ep*Mab*U0*V1a*w + 4*e^2*Mab*U1b*V1a*w -...
         2*e^2*ep*Mab*U2a*V1a*w + 2*e^3*J1b*Laa*V0*V1a*w +...
         2*e^2*ep*Maa*V0*V1a*w + 4*e^2*J1a*La*V1b*w + ...
         4*e^2*Mab*U1a*V1b*w + 4*e^2*ep*Mab*U2b*V1b*w + ...
         2*e^3*J1a*Laa*V0*V1b*w + 4*e^3*J0*Laa*V1a*V1b*w + ...
         4*e^2*Maa*V1a*V1b*w - 2*e^2*ep*Mab*U1a*V2a*w - ...
         2*e^3*J1b*Laa*V1a*V2a*w - 2*e^2*ep*Maa*V1a*V2a*w +...
         2*e^3*J1a*Laa*V1b*V2a*w - 2*e^3*J2b*Laa*V2a^2*w + ...
         8*e^2*J0*La*V2b*w + 8*e*Ma*V2b*w + ...
         8*e^2*Mab*U0*V2b*w + 4*e^2*ep*Mab*U1b*V2b*w +...
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```
          8*e^3*J0*Laa*V0*V2b*w + 8*e^2*Maa*V0*V2b*w + ...
          4*e^3*J1a*Laa*V1a*V2b*w + 4*e^3*J1b*Laa*V1b*V2b*w + ...
          4*e^2*ep*Maa*V1b*V2b*w + 4*e^3*J2a*Laa*V2a*V2b*w + ...
          2*e^3*J2b*Laa*V2b^2*w);

eq34 = (1/(4*e)) *(4*c5*e*J2b + 4*c5*e^2*J2b*V0 +...
          8*c5*e^2*J2b*nu*V0 + 2*c5*e^2*J1b*V1a + ...
          4*c5*e^2*J1b*nu*V1a + 2*c5*e^2*J1a*V1b + ...
          4*c5*e^2*J1a*nu*V1b + 4*c5*e^2*J0*V2b + ...
          8*c5*e^2*J0*nu*V2b - 8*e*J2a*L0*w - ...
          2*e^2*Mbb*U1a^2*w + 4*e*ep*Mb*U1b*w + ...
          2*e^2*ep*Mbb*U0*U1b*w + 2*e^2*Mbb*U1b^2*w -...
          8*e*Mb*U2a*w - 8*e^2*Mbb*U0*U2a*w - ...
          4*e^2*ep*Mbb*U1b*U2a*w - 2*e^2*ep*Mbb*U1a*U2b*w -...
          8*e^2*J2a*La*V0*w + 2*e^2*ep*Mab*U1b*V0*w - ...
          8*e^2*Mab*U2a*V0*w - 4*e^2*J1a*La*V1a*w - ...
          4*e^2*Mab*U1a*V1a*w - 2*e^2*ep*Mab*U2b*V1a*w -...
          2*e^3*J1a*Laa*V0*V1a*w - 2*e^3*J0*Laa*V1a^2*w - ...
          2*e^2*Maa*V1a^2*w + 4*e^2*J1b*La*V1b*w + ...
          4*e*ep*Ma*V1b*w + 2*e^2*ep*Mab*U0*V1b*w + ...
          4*e^2*Mab*U1b*V1b*w - 4*e^2*ep*Mab*U2a*V1b*w +...
          2*e^3*J1b*Laa*V0*V1b*w + 2*e^2*ep*Maa*V0*V1b*w + ...
          2*e^3*J0*Laa*V1b^2*w + 2*e^2*Maa*V1b^2*w - ...
          8*e^2*J0*La*V2a*w - 8*e*Ma*V2a*w - ...
          8*e^2*Mab*U0*V2a*w - 4*e^2*ep*Mab*U1b*V2a*w -...
          8*e^3*J0*Laa*V0*V2a*w - 8*e^2*Maa*V0*V2a*w - ...
          4*e^3*J1a*Laa*V1a*V2a*w - 4*e^3*J1b*Laa*V1b*V2a*w -...
          4*e^2*ep*Maa*V1b*V2a*w - 2*e^3*J2a*Laa*V2a^2*w - ...
          2*e^2*ep*Mab*U1a*V2b*w - 2*e^3*J1b*Laa*V1a*V2b*w - ...
          2*e^2*ep*Maa*V1a*V2b*w + 2*e^3*J1a*Laa*V1b*V2b*w - ...
          4*e^3*J2b*Laa*V2a*V2b*w + 2*e^3*J2a*Laa*V2b^2*w);


      F(1)=eq10;F(2)=eq11;F(3)=eq12;F(4)=eq13;F(5)=eq14;
      F(6)=eq20;F(7)=eq21;F(8)=eq22;F(9)=eq23;F(10)=eq24;
      F(11)=eq30;F(12)=eq31;F(13)=eq32;F(14)=eq33;F(15)=eq34;
```