The Stokes boundary value problem

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THE STOKES BOUNDARY VALUE PROBLEM

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1 Introduction

The Stokes equations are the linearized stationary form of the full Navier-Stokes equations. The corresponding boundary value problem is called the Stokes problem and is defined below.
Consider an open domain $\Omega \subset \mathbb{R}^n$, with boundary $\partial\Omega$ and an almost every where defined outward normal vector $\mathbf{n}$.

Definition 1.1 *The Stokes problem:*
Let $\Omega \subset \mathbb{R}^n$ be open. Suppose $f : \Omega \mapsto \mathbb{R}^n$, $h : \Omega \mapsto \mathbb{R}$ and $a : \partial\Omega \mapsto \mathbb{R}^n$.

$$P(f,h,a) : \begin{cases} \Delta v = \nabla p - f & \text{in } \Omega \\ \nabla \cdot v = -h & \text{in } \Omega \\ v = a & \text{at } \partial\Omega \end{cases} \tag{1.1}$$

Here p denotes the pressure field of the fluid and is determined up to a constant . Further $\mathbf{v} : \Omega \mapsto \mathbb{R}^n$ denotes the vector field of the fluid. Since we describe $\mathbf{v}$ in cartesian coordinates, the vector field $\mathbf{x} \mapsto \mathbf{v}(\mathbf{x})$, $\mathbf{x} \in \Omega$, may be considered as a vector *function* as well. The velocity is prescribed at the boundary. The external force field $\mathbf{f} : \Omega \to \mathbb{R}^n$ and the source function $h : \Omega \to \mathbb{R}$ are supposed to be given.

Notice that $\mathbf{a}$ and h depend on each other, since by Gauss' law we must have

$$-\int_\Omega h \, d\mathbf{x} = \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n} \, d\sigma_{\mathbf{x}}. \tag{1.2}$$

In the sequel we will always have to assume that h and $\mathbf{a}$ satisfy this relation. If $\mathbf{f}$ is a conservative force field with potential V, then $\mathbf{f}$ can be removed if we put $p' = p - V$. A direct consequence of the two equations in Ω is that:

$$\Delta p = -\Delta h + \nabla \mathbf{f} \tag{1.3}$$

An inhomogeneous Stokes problem $P(\mathbf{f}, h, \mathbf{a})$ can always be split in a homogeneous Stokes problem $P(\mathbf{f}, 0, \mathbf{0})$ and the inhomogeneous Stokes problem $P(0, h, \mathbf{a})$. Because of the hidden connection between $\mathbf{a}$ and h given by (1.2) we must be careful in splitting $P(\mathbf{f}, h, \mathbf{a}) = P(\mathbf{f}, h, 0) + P(\mathbf{0}, 0, \mathbf{a})$. In Appendix F we deal with this and suggest a more useful splitting. In this report we will barely make use of any of these splittings, since the functions $\mathbf{f}$ and h will not essentially affect the final operator equations.

Chapter 2 introduces an unusual approach to Stokes problems, which is first introduced by J. de Graaf and D.Chandra. See [14]. For a common approach the reader is referred to Appendix I. In Chapter 2 the set of solutions of the Stokes equations is parameterized by tangent vector fields at the boundary α and an operator equation at the boundary $\partial\Omega$ is derived from which α can be obtained. Using harmonic extensions $\alpha_{\mathcal{H}}$ of α to the whole region Ω, another (more useful) operator equation (see (2.16)), with $\alpha_{\mathcal{H}}$ as fundamental unknown, is obtained:

$$\alpha_{\mathcal{H}} + \mathcal{B}\alpha_{\mathcal{H}} = \beta \qquad \text{, with } \beta \text{ known and } \mathcal{B} = (\text{grad}\mathcal{N} - \mathcal{D}\text{grad})\text{div} .$$

From this we derive an operator equation (see (2.17)) for the pressure p:

$$\text{div}\mathcal{D}\text{grad}\, p = q \qquad \text{, with } q \text{ known.}$$

Some methods of solving operator equations are discussed. One of these methods, which is focused on regions in $\mathbb{R}^2$ makes use of complex function theory and in some nice regions in $\mathbb{R}^2$ it appears to be very useful.

One of the other methods is applied to the case of the n-dimensional ball in Chapter 4. In this case it works out quite nice, but in other regions it doesn't seem to be applicable.
In Chapter 4 and in the Appendix Stokes Problems on some other regions in $\mathbb{R}^2$ are discussed.

In Chapter 5 the operator equations are examined from a functional analytic point of view. In this chapter we mainly regard the three dimensional case. In the last paragraph some generalizations to the two dimensional case are made. In Section 5.1, the Hilbert space $\mathbf{V}^{(3)}$ equipped with inner product $(\cdot,\cdot)_{\mathbf{V}^{(3)}}$ given by

$$(\Phi,\Psi)_{\mathbf{V}^{(3)}} = (\operatorname{div}\Phi,\operatorname{div}\Psi)_{\mathbf{L}_2(\Omega)} + (\operatorname{rot}\Phi,\operatorname{rot}\Psi)_{\mathbf{L}_2(\Omega)}, \tag{1.4}$$

is introduced. This introduction of $\mathbf{V}^{(3)}$ is not trivial, since a trace theorem is needed.
The completion of sufficiently differentiable tangent harmonic vector functions $\widetilde{\mathbf{V}}^{(3)}$ equals $\mathbf{V}^{(3)}_{\circ}$ as is proven in Section 5.1 . Equipping the space $V^{(3)}$, which is defined as $\{\Phi \in \overset{\circ}{\mathbf{H}}_1(\Omega) \mid \mathbf{n}\cdot T_0\Phi = 0\}$, with the inner product $(\cdot,\cdot)_{V^{(3)}}$ given by (1.4) seems unusual. In literature (see for instance Duvaut & Lions[5] or Temam [7]) sort a like spaces are equipped with the inner product $(\cdot,\cdot)$ given by

$$(\Phi,\Psi) = (\Phi,\Psi)_{\mathbf{L}_2(\Omega)} + (\operatorname{div}\Phi,\operatorname{div}\Psi)_{\mathbb{L}_2(\Omega)} + (\operatorname{rot}\Phi,\operatorname{rot}\Psi)_{\mathbf{L}_2(\Omega)} \,.$$

Theorem 5.3, which is based on the second Poincaré inequality (see theorem 6.20), shows that this leads to an equivalent norm.
One of the advances of using the inner product $(\cdot,\cdot)_{\mathbf{V}^{(3)}}$ in stead of $(\cdot,\cdot)$ is that operator $\mathcal{B}$ is self adjoint. This is proved in theorem 5.23. Another advance is the natural orthogonal splitting that can be made in $\mathbf{V}^{(3)}$ (see theorem 5.11) and the geometrical aspects that go along with this. Inspired by the case of the three dimensional ball, we parameterize the subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$ by means of harmonic functions. They form an important orthogonal splitting in the harmonic subspace $\underline{\mathrm{Harm}}^{(3)}(\Omega)$ of the space $\mathbf{V}^{(3)}$.

In Chapter 5 we also prove that the operator $\operatorname{div}\mathcal{D}\operatorname{grad} : \mathbb{L}_2(\Omega)/\mathbb{R} \to \mathbb{L}_2(\Omega)/\mathbb{R}$ is self adjoint and negative definite. One of the main results in this report is Theorem 5.16 in which we show that $\operatorname{div}\mathcal{D}\operatorname{grad}$ has a bounded inverse. As a result, the operator equation $\operatorname{div}\mathcal{D}\nabla p = q$ has a bounded solution p in $\mathbb{L}_2(\Omega)/\mathbb{R}$.
The spectrum of operator $\operatorname{div}\mathcal{D}\operatorname{grad}$ is a subset of $(-1,-c^2)$, where $c > 0$ is a constant such that $\|\operatorname{grad} f\|_{\mathbf{H}_{-1}(\Omega)} \geq c\,\|f\|_{\mathbb{L}_2(\Omega)}$. In the case of the unit ball, $-c^2 = -\frac{1}{2}$.
The orthogonal complement of $\mathbb{L}_2^{\mathrm{harm}}(\Omega)/\mathbb{R}$ in $\mathbb{L}_2(\Omega)/\mathbb{R}$ appears to be the entire eigen space of eigen value -1 of operator $\operatorname{div}\mathcal{D}\operatorname{grad}$.

In the last section the problems that we have solved are reconsidered in the two dimensional case. The inner product $(\cdot,\cdot)_{\mathbf{V}^{(3)}}$ now becomes $(\cdot,\cdot)_{\mathbf{V}^{(2)}}$ given by

$$(\Phi,\Psi)_{\mathbf{V}^{(2)}} = (\operatorname{div}\Phi,\operatorname{div}\Psi)_{\mathbb{L}_2(\Omega)} + (\operatorname{div}\sigma\Phi,\operatorname{div}\sigma\Psi)_{\mathbb{L}_2(\Omega)}, \text{ with } \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This norm has a remarkable property with respect to conformal mappings, see theorem 5.58.

Chapter 6 gives an introduction to Sobolev spaces. If the reader is not familiar to Sobolev spaces then it is advisable to read this chapter first. Chapter 7 handles some fundamental theorems about harmonic functions, which are used in this report.

The Appendix consists of theoretical background and some concrete examples. Appendix A and appendix B are such examples, they show a useful and short illustration of the theory of Chapter 3 and 5.

Appendix C gives an alternative approach to the space $\mathbf{V}^{(3)}$. The operator $T_{\mathbf{n}}$ is constructed differently. Although that Appendix C is not included in part main part of this report, it might be interesting from a theoretical point of view. Appendix D and E handle specific examples which show the practical difficulties that arise in solving the operator equation div$\mathcal{D}$grad in angular regions. Appendix F examines the possibility and use of splitting Stokes problems. Appendix G deals with some mathematical subjects that arise in this report. In Appendix I the Stokes problem is approached in a common way and is intended to make a connection to the general literature. We will introduce a variational and a minimization formulation of the Stokes problem and prove existence and uniqueness of solutions of Stokes problems.

1.1 Notation

During this report the following notation will be used:

Use of Boldface style symbols:
$\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$

Let V be a normed space, then **V**denotes the cartesian product space V^n equipped with the cartesian product norm.

Operators
For any operator A with domain Dom(A) we write:
$\mathrm{im}(A) = \mathcal{R}(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathrm{Dom}(A)\}$
$\mathcal{N}(A) = \{\mathbf{x} \in \mathrm{Dom}(A) \mid A\mathbf{x} = \mathbf{0}\}$
The Dirichlet operator $\mathcal{D}$ is defined by

$$\mathbf{f} = \mathcal{D}(\mathbf{g}; \mathbf{h}) \Leftrightarrow \begin{cases} \Delta\mathbf{f} = -\mathbf{g} & \text{in } \Omega \\ \mathbf{f} = \mathbf{h} & \text{on } \partial\Omega \end{cases}$$

If either $\mathbf{g} = \mathbf{0}$ or $\mathbf{h} = \mathbf{0}$ we write shortly $\mathcal{D}(\mathbf{g}, \mathbf{0}) = \mathcal{D}(\mathbf{g})$ and $\mathcal{D}(\mathbf{0}, \mathbf{h}) = \mathbf{h}_{\mathcal{H}}$.

The Neumann operator $\mathcal{N}$ is defined by

$$\mathbf{f} = \mathcal{N}(\mathbf{g}; \mathbf{h}) \Leftrightarrow \begin{cases} \Delta\mathbf{f} = -\mathbf{g} & \text{in } \Omega \\ \frac{\partial f_i}{\partial \mathbf{n}} = h_i \text{ for } i = 1 \ldots n & \text{on } \partial\Omega \end{cases}$$

If $\mathbf{h} = \mathbf{0}$ we write shortly $\mathcal{N}(\mathbf{g}, \mathbf{0}) = \mathcal{N}(\mathbf{g})$.

During this report we use the following ∇ notation extensively:
$\mathrm{div}\,\mathbf{f} = \nabla \cdot \mathbf{f}$
$\mathrm{rot}\,\mathbf{f} = \nabla \times \mathbf{f}$
$\mathrm{grad}\,\phi = \nabla\phi$
The operators $\nabla\mathcal{N}\mathrm{div}, \mathcal{D}\nabla\mathrm{div}$ and $(\nabla\mathcal{N} - \mathcal{D}\nabla)\mathrm{div}$ will occur frequently , for convenience we write shortly:
$\mathcal{F} = \nabla\mathcal{N}\mathrm{div}$,
$\mathcal{C} = \mathcal{D}\nabla\mathrm{div}$ and
$\mathcal{B} = \mathcal{F} - \mathcal{C} = (\nabla\mathcal{N} - \mathcal{D}\nabla)\mathrm{div}$.

The Euler operator $\mathbf{x} \cdot \nabla$ will be denoted by $\mathcal{E}$.

Derivatives will be denoted by ∇f.
Derivatives of a vector function $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^m$
is sometimes denoted by $\nabla\mathbf{f}$, but mostly by
$D\mathbf{f}$ in order to stress that the matrix representation of this linear mapping is given by:

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

The second derivative or Hessian of a function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is denoted by:

$$\mathrm{Hessian}(f) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}.$$

Operator spaces
$\mathcal{L}(V,W)$ denotes the set of all linear operators from V to$W$.
If $V = W$ we write $\mathcal{L}(V)$ in stead of $\mathcal{L}(V,V)$.

$\mathcal{B}(V) = \{A \in \mathcal{L}(V) \mid A \text{ is bounded }\}$
$C^\infty(\Omega) =$ the set of all infinitely continuous differentiable functions on Ω.

$\mathcal{D}(\Omega) = \{f \in C^\infty(\Omega) \mid \text{supp} f \text{ is compact}\}$

The domain
The domain of a Stokes problem is usually denoted by Ω,
unless the domain is a unit ball;
$B_n = \{\mathbf{x} \in \mathbb{R}^n \mid ||\mathbf{x}|| \leq 1\}$.

We will always assume that Ω is an open simply connected subset of $\mathbb{R}^n$,
with an almost everywhere defined outward normal $\mathbf{n}$

$\partial\Omega =$ The boundary of Ω
$\overline{\Omega} =$ The closure of Ω

A Lipschitz domain is a domain Ω such that the boundary is locally Lipschitz. i.e.
in a neighborhood of any point $\mathbf{x} \in \partial\Omega$,
$\partial\Omega$ admits a representation as a hypersurface $y_n = \theta(y_1, \ldots, y_{n-1})$ where
θ is a Lipschitz function, and $(y_1, \ldots, y_n)$
are rectangular coordinates in $\mathbb{R}^n$ in a basis that may be different
from the canonical basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$.

2 A parameterized class of solutions

In Appendix I we mention how the Stokes system is usually treated. In this chapter we deal with the Stokes problem in a different and unusual way.

By solving a system of Poison equations we will parameterize the solutions of the Stokes equations by harmonic extension of tangent vector functions $\boldsymbol{\alpha}$ at the boundary, which can be obtained by solving an operator equation. After some manipulation this operator equation leads to a relative simple operator equation from which p can be obtained directly. In chapter 5 both operator equations will be further examined.

Theorem 2.1 *The pair $(\boldsymbol{v}, p)$ satisfies the inhomogeneous Stokes equations if and only if it can be written:*

$$\begin{cases} \boldsymbol{v} = \boldsymbol{\Phi} + \nabla\psi \\ p = -h - \nabla \cdot \boldsymbol{\Phi} \end{cases} \tag{2.1}$$

with (Φ, ψ) a solution of the system

$$\begin{cases} \Delta\boldsymbol{\Phi} = -\boldsymbol{f} \\ \Delta\psi = -h - \nabla \cdot \Phi \end{cases} \tag{2.2}$$

Note that the pair (Φ, ψ) is not unique. The pair $(\Phi + \nabla\chi, \psi - \chi)$, with $\Delta\chi = 0$ leads to the same pair $(\boldsymbol{v}, p)$.

Proof
The proof follows by straightforward computation.
"if" We assume that the pair (Φ, ψ) satisfies (2.2) and show by computation that the pair (v, p) satisfies the inhomogeneous Stokes equations

$$\begin{aligned} \Delta\mathrm{v} &= \Delta\Phi + \nabla\Delta\psi \\ &= -\mathbf{f} - \nabla h - \nabla\nabla \cdot \Phi \\ &= -\mathbf{f} + \nabla p \end{aligned}$$

$$\begin{aligned} \operatorname{div} \mathrm{v} &= \operatorname{div} \Phi + \operatorname{div} \nabla\psi \\ &= \operatorname{div} \Phi + \Delta\psi = -h \end{aligned}$$

"only if" We assume that the pair (v, p) satisfies the inhomogeneous Stokes equations and show by computation that the pair (Φ, ψ) satisfies (2.2).

$$\operatorname{div} \mathbf{v} = \operatorname{div}\Phi + \Delta\psi = -h \tag{2.3}$$

Substitution of (2.1) in $\Delta\mathbf{v} = \nabla p - \mathbf{f}$ and using (2.3) leads directly to the second equation of (2.2). □

First we construct $\boldsymbol{\Phi}$ by solving the Dirichlet problem:

$$\begin{cases} \Delta\boldsymbol{\Phi} = -\mathbf{f} & \text{in } \Omega \\ \boldsymbol{\Phi} = \boldsymbol{\alpha} & \text{at } \partial\Omega\ , \end{cases} \tag{2.4}$$

with $\boldsymbol{\alpha}$ a given vector field at the boundary.
The solution is given by

$$\boldsymbol{\Phi} = \mathcal{D}\mathbf{f} + \boldsymbol{\alpha}_{\mathcal{H}}$$

Next solve the Neumann problem:

$$\begin{cases} \Delta\psi = -h - \nabla\cdot\Phi & \text{in } \Omega \\ \frac{\partial\psi}{\partial n} = \gamma & \text{at } \partial\Omega\,. \end{cases} \tag{2.5}$$

The solution of the above problem is

$$\psi = \mathcal{N}(\nabla\cdot\boldsymbol{\alpha}_{\mathcal{H}}) + \mathcal{N}(\nabla\cdot\mathcal{D}\mathbf{f}) + \mathcal{N}(h;\gamma),$$

where $\mathcal{N}(h;\gamma)$ satisfies

$$\begin{cases} \Delta\mathcal{N}(h;\gamma) = -h & \text{in } \Omega \\ \frac{\partial\mathcal{N}(h;\gamma)}{\partial n} = \gamma & \text{at } \partial\Omega\,. \end{cases} \tag{2.6}$$

Note that:

- At the boundary we have $\boldsymbol{\alpha}\cdot\mathbf{n} = 0$ and $\mathcal{D}\mathbf{f}|_{\partial\Omega} = \mathbf{0}$, therefore (use Gauss' theorem) both $\mathcal{N}(\nabla\cdot\boldsymbol{\alpha}_{\mathcal{H}})$ and $\mathcal{N}(\nabla\cdot\mathcal{D}\mathbf{f})$ are compatible Neumann problems.
- The above introduced boundary function γ is given by the inner product of $\mathbf{a}$ and $\mathbf{n}$:

$$\gamma = \left.\frac{\partial\mathcal{N}(h;\gamma)}{\partial n}\right|_{\partial\Omega} = \frac{\partial\psi}{\partial n} = \mathbf{v}\cdot\mathbf{n}\Big|_{\partial\Omega} = \mathbf{a}\cdot\mathbf{n} \tag{2.7}$$

- If $\int_\Omega h\,d\mathbf{x} = \int_{\partial\Omega}\gamma\,d\sigma_{\mathbf{x}} = 0$ it makes sense to split: $\mathcal{N}(h;\gamma) = \mathcal{N}h + \mathcal{N}(0;\gamma)$, with $\mathcal{N}h = \mathcal{N}(h;0)$.

So $\mathbf{v}$ and p can be parameterized as follows:

$$\begin{cases} \mathbf{v} = \boldsymbol{\alpha}_{\mathcal{H}} + \nabla\{\mathcal{N}(\nabla\cdot\boldsymbol{\alpha}_{\mathcal{H}})\} + \mathcal{D}\mathbf{f} + \nabla\mathcal{N}(\nabla\cdot\mathcal{D}\mathbf{f}) + \nabla\mathcal{N}(h;\mathbf{a}\cdot\mathbf{n}) \\ p = -h - \nabla\cdot\boldsymbol{\alpha}_{\mathcal{H}} - \nabla\cdot\mathcal{D}\mathbf{f} \end{cases} \tag{2.8}$$

At the boundary $\partial\Omega$ we have $\mathbf{v}|_{\partial\Omega} = \mathbf{a}$, which leads to *the first version of the operator equation at the boundary:*

$$\boldsymbol{\alpha} + (\nabla\{\mathcal{N}(\nabla\cdot\boldsymbol{\alpha})\} + \nabla\mathcal{N}(h;\mathbf{a}\cdot\mathbf{n}) + \nabla\{\mathcal{N}(\nabla\cdot\mathcal{D}\mathbf{f})\})_{\partial\Omega} = \mathbf{a}. \tag{2.9}$$

The left-hand side of the above equation can be modified. Since, $\Delta\left(-\frac{1}{2}\mathbf{x}\cdot\boldsymbol{\alpha}_{\mathcal{H}}\right) = -\nabla\cdot\boldsymbol{\alpha}_{\mathcal{H}}$, we have that

$$\mathcal{N}(\nabla\cdot\boldsymbol{\alpha}_{\mathcal{H}}) = -\frac{1}{2}\mathbf{x}\cdot\boldsymbol{\alpha}_{\mathcal{H}} + \mathcal{N}(0;\frac{1}{2}\frac{\partial}{\partial n}\mathbf{x}\cdot\boldsymbol{\alpha}_{\mathcal{H}}), \text{ so that (2.9) can be rewritten into} \tag{2.10}$$

the second version of the operator equation at the boundary:

$$\boldsymbol{\alpha} + \nabla\left\{\mathcal{N}(0;\frac{1}{2}\frac{\partial}{\partial n}\mathbf{x}\cdot\boldsymbol{\alpha}_{\mathcal{H}}) - \frac{1}{2}\mathbf{x}\cdot\boldsymbol{\alpha}_{\mathcal{H}}\right\} = \mathbf{a} - \nabla\{\mathcal{N}(\nabla\cdot\mathcal{D}\mathbf{f}) - \mathcal{N}(h;\mathbf{a}\cdot\mathbf{n})\} \tag{2.11}$$

Using the identity $\nabla(\mathbf{x}\cdot\Phi) = \Phi + [D\Phi]^T\mathbf{x}$, with Φ replaced by $\boldsymbol{\alpha}_{\mathcal{H}}$, we arrive at *the third version of the operator equation at the boundary:*

$$\frac{1}{2}\boldsymbol{\alpha} - \frac{1}{2}\left[(D\boldsymbol{\alpha}_{\mathcal{H}})^T\mathbf{x}\right]_{\text{tg}} + \left[\nabla\left\{\mathcal{N}(0;\frac{1}{2}\frac{\partial}{\partial n}\mathbf{x}\cdot\boldsymbol{\alpha}_{\mathcal{H}})\right\}\right]_{\text{tg}} = \mathbf{a} - \nabla\left\{\mathcal{N}(\nabla\cdot\mathcal{D}\mathbf{f}) - \mathcal{N}(h;\mathbf{a}\cdot\mathbf{n})\right\} \tag{2.12}$$

Here the subscript tg denotes the tangential component of the vector field within [·]. Note that the second term of the lefthand-side in (2.12) only depends on α *along the boundary* $\partial\Omega$.

Apply harmonic extensions to the first version of the operator equation at the boundary. This leads to the following equation on Ω:

$$\alpha_{\mathcal{H}}+\nabla\{\mathcal{N}(\nabla\cdot\alpha_{\mathcal{H}})\}_{\mathcal{H}}+(\nabla\mathcal{N}(h;\mathbf{a}\cdot\mathbf{n}))_{\mathcal{H}}+(\mathcal{D}\mathbf{f})_{\mathcal{H}}+\nabla\{\mathcal{N}(\nabla\cdot\mathcal{D}\mathbf{f})\}_{\mathcal{H}} = \mathbf{a}_{\mathcal{H}} \,. \tag{2.13}$$

One can easily verify, by using $\Delta\mathcal{D} = \Delta\mathcal{N} = -\mathcal{I}$, that:

$$\begin{cases} \nabla\{\mathcal{N}(\nabla\cdot\alpha_{\mathcal{H}})\}_{\mathcal{H}} & = \nabla\mathcal{N}(\nabla\cdot\alpha_{\mathcal{H}}) - \mathcal{D}\nabla\nabla\cdot\alpha_{\mathcal{H}} \\ \nabla\{\mathcal{N}(\nabla\cdot\mathcal{D}\mathbf{f}+h)\}_{\mathcal{H}} & = \nabla\mathcal{N}(\nabla\cdot\mathcal{D}\mathbf{f}+h) - \mathcal{D}(\nabla\nabla\cdot\mathcal{D}\mathbf{f}+\nabla h) \\ (\mathcal{D}\mathbf{f})_{\mathcal{H}} = 0 & \\ (\nabla\mathcal{N}(h;\mathbf{a}\cdot\mathbf{n}))_{\mathcal{H}} & = \nabla\mathcal{N}(h;\mathbf{a}\cdot\mathbf{n}) - \mathcal{D}(\nabla h) \,. \end{cases} \tag{2.14}$$

Substituting (2.14) into (2.13) leads to:

$$\alpha_{\mathcal{H}}+\nabla\mathcal{N}(\nabla\cdot\alpha_{\mathcal{H}})-\mathcal{D}\nabla\nabla\cdot\alpha_{\mathcal{H}}+\nabla\mathcal{N}(\nabla\cdot\mathcal{D}\mathbf{f})-\mathcal{D}(\nabla\nabla\cdot\mathcal{D}\mathbf{f}+\nabla h)+\nabla\mathcal{N}(h;\mathbf{a}\cdot\mathbf{n}) = \mathbf{a}_{\mathcal{H}}. \tag{2.15}$$

Introduce the operator $\mathcal{B} = (\nabla\mathcal{N}-\mathcal{D}\nabla)\nabla\cdot$ and denote the set of tangent harmonic n-component vector functions on Ω by $\underline{\mathrm{Harm}}^{(n)}(\Omega)$, then it is easy to see that the linear operator $\mathcal{B}$ maps $\underline{\mathrm{Harm}}^{(n)}(\Omega)$ into itself and that equation (2.15) can be written as:

$$\alpha_{\mathcal{H}}+\mathcal{B}\alpha_{\mathcal{H}} = \mathbf{a}_{\mathcal{H}}-\nabla\mathcal{N}(\nabla\cdot\mathcal{D}\mathbf{f})+\mathcal{D}(\nabla\nabla\cdot\mathcal{D}\mathbf{f}+\nabla h)-\nabla\mathcal{N}(h;\mathbf{a}\cdot\mathbf{n}) \tag{2.16}$$

Taking the divergence on both sides of (2.15) and again using $\Delta\mathcal{D} = \Delta\mathcal{N} = -\mathcal{I}$, (2.8), $\Delta\mathcal{N}(h;\mathbf{a}\cdot\mathbf{n}) = -h$ we obtain:

$$\mathrm{div}\mathcal{D}\nabla p = \mathrm{div}\,\mathbf{a}_{\mathcal{H}}+(\nabla\cdot\mathcal{D}\mathbf{f}+h) \tag{2.17}$$

This fundamental equation can be obtained in a more direct manner:
Put $\mathbf{v} = \mathbf{a}_{\mathcal{H}}+\mathbf{w}$, then by using the Stokes equations we have that $\mathbf{w} = \mathcal{D}(-\nabla p+\mathbf{f})$ and $\mathrm{div}\,\mathbf{w} = -h - \mathrm{div}\,\mathbf{a}_{\mathcal{H}}$, from which equation (2.17) follows.

In order to solve operator equation (2.17) we must invert $\mathrm{div}\mathcal{D}\nabla$. First, we investigate $\mathcal{D}\nabla p$. We will discuss three methods that may lead to a solution of operator equation (2.17):

1. Using the formula

   $$\Delta\{f\ \mathbf{v}\} = f\Delta\mathbf{v}+\Delta f\mathbf{v}+2\nabla f[D\mathbf{v}]^T$$

   and the fact that both p and $\mathbf{x}$ are harmonic it follows that $\mathcal{D}\nabla p$ can be split:

   $$\mathcal{D}\nabla p = -\frac{1}{2}p\mathbf{x}+(\frac{1}{2}\mathbf{x}p)_{\mathcal{H}}. \tag{2.18}$$

   As a result, we have

   $$\mathrm{div}\mathcal{D}\nabla p = -\frac{n}{2}p-\frac{1}{2}\mathcal{E}p+\mathrm{div}\{(\frac{1}{2}\mathbf{x}p)_{\mathcal{H}}\}. \tag{2.19}$$

Although this method seems most obvious, it has an unpleasant feature, namely the term $\text{div}\{(\frac{1}{2}\mathbf{x}p)_{\mathcal{H}}\}$, which can be tedious in inverting $\text{div}\mathcal{D}\nabla$. For instance on a square, see Appendix D. Remark: In the examples we often write $\mathbf{W} = (\frac{1}{2}\mathbf{x}p)_{\mathcal{H}}$.

2. Write $\mathcal{D}\nabla p = \theta\,\mathbf{H}$, with

   θ, $\mathbf{H}$ cleverly chosen, such that:
   $\Delta\mathbf{H} = \mathbf{0}$ in Ω,
   $(\theta\mathbf{H})|_{\partial\Omega} = \mathbf{0}$ and
   $\Delta\theta\,\mathbf{H} = (\Delta\theta)\,\mathbf{H} + 2\nabla\theta[D\mathbf{H}]^T = -\nabla p$.

   Although this is quite an ad hoc method, in the ball case (see chapter 3) it works out very nice (it leads to the inverse of $\text{div}\mathcal{D}\nabla$, see 3.29) if we choose theta as the solution of

$$\left\{\begin{array}{ll} \Delta\theta = -1 & \text{in } B_n \\ \theta = 0 & \text{on } \partial B_n \end{array}\right., \text{ i.e. } \theta = -\frac{1}{2}(\|\mathbf{x}\|^2 - 1)\,. \tag{2.20}$$

   Unfortunately, this method didn't work out on other regions. See, for instance Appendix D, where many attempts (choices for θ) failed. Note that a strong and necessary condition to θ and ∇ is given by:

$$\begin{array}{ll} \Delta^2\theta\, H_i & = \Delta^2\theta H_i + 4\,\nabla H_i\cdot\nabla\Delta\theta + 4\text{trace}\left\{\text{Hessian}(\theta)\text{Hessian}(H^{(i)})\right\} \\ & = \Delta\nabla p = \nabla\Delta p = 0 \end{array} . \tag{2.21}$$

3. First some preparations:
   Put $z = x + iy$ and Let $f : \mathbb{C} \mapsto \mathbb{C}$ be analytic and put $z = x + iy$, $f(x+iy) = p(x,y) + i\,q(x,y)$, then we have

$$f'(z) = p_x + i\,q_x = q_y - i\,p_y\,. \tag{2.22}$$

   From this we conclude that both the real and imaginary part of f are harmonic and that $\frac{\partial}{\partial x}\text{Re}\,f = \text{Re}\,\frac{\partial f}{\partial x}$ and $\frac{\partial}{\partial y}\text{Re}\,f = \text{Re}\,\frac{\partial f}{\partial y}$. :

   $\text{Re}\,f'(z) = p_x = q_y$ and $\text{Im}\,f'(z) = -p_y = q_x$
   (Cauchy-Riemann equations) $\Rightarrow$

   $\Delta p = p_{xx} + p_{yy} = 0$, $\Delta q = q_{xx} + q_{yy} = 0$ and
   $\frac{\partial}{\partial x}\text{Re}\,f(z) = \text{Re}\,f'(z)$ and $\frac{\partial}{\partial y}\text{Re}\,f(z) = -\text{Im}\,f'(z) \Rightarrow$

   $\frac{\partial}{\partial x}\text{Re}\,f = \text{Re}\,\frac{\partial f}{\partial x}$ and $\frac{\partial}{\partial y}\text{Re}\,f = \text{Re}\,\frac{\partial f}{\partial y}$

   From $\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}$ and $\frac{\partial}{\partial y} = i\,\frac{\partial}{\partial z} - i\,\frac{\partial}{\partial \overline{z}}$ it follows that

$$\begin{array}{l} \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right), \\ \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right), \\ \Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}. \end{array}$$

   Notice that f is analytic $\Rightarrow \frac{\partial}{\partial x}f = \frac{\partial}{\partial z}f \Leftrightarrow \frac{\partial}{\partial \overline{z}}f(z) = 0 \Leftrightarrow \frac{\partial}{\partial z}f(\overline{z}) = 0$.
   We will use the above to determine $\mathbf{u} = (u,v) := \mathcal{D}\nabla p$. From (2.22) and

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}\{-\frac{1}{4}\overline{z}f(z)\} = -\frac{\partial}{\partial z}\left(f(z) + \overline{z}\frac{\partial}{\partial \overline{z}}f(z)\right) = -f'(z)\,,$$

it follows that

$$\begin{array}{l} u(x,y) = \text{Re}\left\{-\frac{1}{4}\bar{z}f(z)\right\} + H_1(x,y) \\ v(x,y) = \text{Im}\left\{\frac{1}{4}\bar{z}f(z)\right\} + H_2(x,y), \end{array} \tag{2.23}$$

where $\mathbf{H}(x,y) = (H_1(x,y), H_2(x,y))$ must be harmonic and compensate at the boundary such that $u(x,y)|_{\partial\Omega} = v(x,y)|_{\partial\Omega} = 0$. i.e.

$$\begin{array}{l} H_1 = \left[\text{Re}\ \left\{\frac{1}{4}\bar{z}f(z)\right\}|_{\partial\Omega}\right]_{\mathcal{H}} \\ H_2 = \left[\text{Re}\ \left\{\frac{1}{4}i\bar{z}f(z)\right\}|_{\partial\Omega}\right]_{\mathcal{H}} \end{array}$$

Now, $\text{div}\mathcal{D}\nabla p$ can be determined :

$$\begin{aligned} \text{div}\mathcal{D}\nabla p &= \tfrac{\partial u}{\partial x} + \tfrac{\partial v}{\partial y} = \text{Re}\left\{\tfrac{\partial}{\partial x} - \tfrac{1}{4}\bar{z}f(z)\right\} + \text{Im}\left\{\tfrac{\partial}{\partial y}\tfrac{1}{4}\bar{z}f(z)\right\} + \text{div}\mathbf{H} \\ &= \text{Re}\left\{-\tfrac{1}{4}f(z) - \tfrac{1}{4}\bar{z}f'(z)\right\} + \text{Im}\left\{-\tfrac{1}{4}i\,f(z) + \tfrac{1}{4}i\,\bar{z}\,f'(z)\right\} + \text{div}\mathbf{H} \\ &= -\tfrac{1}{2}\text{Re}\,(f(z)) + \text{div}\mathbf{H} = -\tfrac{1}{2}p + \text{div}\mathbf{H} \end{aligned} \tag{2.24}$$

A nice property is that the factor $-\frac{1}{2}$, which plays a dominant role in the spectra of $\text{div}\mathcal{D}\nabla$, appears in a natural way. In some regions it is quite easy to derive $\mathbf{H}$. For instance, in the unit disc case we have $H_1(x,y) = \text{Re}\ \left\{\frac{1}{4}\frac{1}{z}f(z)\right\}$ and $H_2(x,y) = \text{Re}\ \left\{\frac{1}{4}\frac{i}{z}f(z)\right\}$.
Another example where $\mathbf{H}$ can be determined easily is the half space case, see paragraph 4.1.

It is of course, not always that easy to determine $\mathbf{H}$, in general it is not only dependent on p but also on q, the harmonic conjugate of p. From the Cauchy-Riemann equations it follows that along a path K with normal $\mathbf{n}$ and tangent vector $\mathbf{t}$ we have that $\frac{\partial p}{\partial n} = \frac{\partial q}{\partial s}$[1] From this we obtain:

$$q(\mathbf{x}) = p(\mathbf{x}_0) + \int_K \frac{\partial p}{\partial n}\, ds\ , \tag{2.25}$$

, where K is some path from a certain $\mathbf{x}_0 \in \Omega$ to $\mathbf{x}$. Note that q is determined up to a constant and that equation (2.25) is independent of the choice of K, since $\Delta p = 0$.

In chapter (5) the spectral properties of operator $\nabla\cdot\mathcal{D}\nabla$ and operator $\mathcal{B}$ are further examined. Note that the direct relation between them *in the special case $f = 0$ and* $h = 0$ is given by:

$$-\nabla\cdot\mathcal{D}\nabla\nabla\cdot\alpha_{\mathcal{H}} = \nabla\cdot(\mathcal{B}+I)\alpha_{\mathcal{H}} \tag{2.26}$$

[1] Actually, these are the Cauchy-Riemann equations in a more generalized form.

3 Stokes equations on a n-dimensional unit ball

In this section the special case $\Omega = B^n_{0,1}$ will be examined. In the sequel the n-dimensional unit ball will be denoted by B_n and its boundary will be denoted by ∂B_n. According to the next theorem an explicit splitting for harmonic extensions on a 3 dimensional unit ball of functions defined at the boundary can be given. As a preparation some lemmas are stated first.

Theorem 3.1 *If a continuous vector function $\Phi : \overline{B} \mapsto \mathbb{R}^3$ satisfies*

$$\begin{cases} \Delta\Phi(x) = 0 & x \in B \\ div\,\Phi(x) = 0 & x \in B \\ x \cdot \Phi = 0 & x \in \partial B \end{cases}$$

then $\Phi = x \times \nabla\phi$, with $\Delta\phi = 0$ and ϕ is unique if we require $\phi(0) = 0$.

Proof First we define a necessary condition on $\nabla\phi$ by applying rotation on both sides.

$$\nabla\times\Phi = \nabla\times(\mathbf{x}\times\nabla\phi) = (\nabla\cdot\nabla\phi)\mathbf{x}+(\nabla\phi\cdot\nabla)\mathbf{x}-(\nabla\cdot\mathbf{x})\nabla\phi-(\mathbf{x}\nabla)\nabla\phi = -(\mathcal{E}+2)\nabla\phi$$

So a candidate for $\nabla\phi$ is given by $-(\mathcal{E}+2)^{-1}(\nabla\times\Phi)$. We investigate whether $-(\mathcal{E}+2)^{-1}(\nabla\times\Phi) = -\nabla\times\left((\mathcal{E}+1)^{-1}\Phi\right)$ has a potential. Calculate

$$\begin{aligned} \nabla\times\left((\mathcal{E}+2)^{-1}(\nabla\times\Phi)\right) &= \nabla\times\left(\nabla\times(\mathcal{E}+1)^{-1}\Phi\right) \\ &= \nabla\left(\nabla\cdot\left((\mathcal{E}+1)^{-1}\Phi\right)\right) - \Delta(\mathcal{E}+1)^{-1}\Phi \\ &= \nabla(\mathcal{E}+2)^{-1}\text{div}\,\Phi - (\mathcal{E}+3)^{-1}\Delta\Phi = \mathbf{0} \end{aligned} \tag{3.1}$$

So the vector field $-(\mathcal{E}+2)^{-1}(\nabla\times\Phi)$ has indeed a potential ϕ, say. We write $-(\mathcal{E}+2)^{-1}(\nabla\times\Phi) = \nabla\phi$. Further note that $\nabla\cdot\nabla\phi = -\nabla\cdot\nabla\times(\mathcal{E}+1)^{-1}\Phi = 0$. So, ϕ is harmonic.

Finally we prove that the vector field $\mathbf{u} = \Phi+\mathbf{x}\times\left(\nabla\times(\mathcal{E}+1)^{-1}\Phi\right) = \Phi-\mathbf{x}\times\nabla\phi$ is identically zero. Observe

1. $\text{rot}\,\mathbf{u} = \mathbf{0}$
2. $\text{div}\,\mathbf{u} = \nabla\cdot\Phi + (\nabla\times\mathbf{x})\cdot(\nabla\times(\mathcal{E}+1)^{-1}\Phi) - (\nabla\times(\nabla\times(\mathcal{E}+1)^{-1}\Phi)\cdot\mathbf{x}) = 0$
3. we conclude that $\mathbf{u} = \nabla\sigma$, with $\Delta\sigma = 0$. At the boundary we have $\mathbf{x}\cdot\mathbf{u} = \mathcal{E}\sigma = 0$. This equality holds also on the whole region Ω because $\Delta\mathcal{E}\sigma = 0$. Expand σ in spherical harmonics $\sigma = \sum_{n=0}^{\infty}\sigma_n$. Then $\mathcal{E}\sigma = \sum_{n=0}^{\infty} n\ \sigma_n$. We conclude that $\sigma_n = 0$ for all $n \in \mathbb{N}\setminus\{0\}$. Therefore σ must be a constant and hence $\mathbf{u} = \mathbf{0}$ □.

Theorem 3.2 *If a continuous vector function $\Phi : \overline{B} \mapsto \mathbb{R}^3$ satisfies*

$$\begin{cases} \Delta\Phi(x) = 0 & x \in B \\ div\,\Phi(x) = 0 & x \in B \end{cases}$$

then Φ can be written as $\Phi = \Phi_0 + \Phi_1$, with

$$\begin{aligned} \Phi_0 &= x \times \nabla\phi \\ \Phi_1 &= \nabla\chi - (\mathcal{E}\ \chi)x + \tfrac{1}{2}(\|x\|^2 - 1)\nabla[(\mathcal{E}+\tfrac{1}{2})^{-1}\mathcal{E}]\chi \end{aligned}$$

with both $\phi : \overline{B} \mapsto \mathbb{R}$ and $\chi : \overline{B} \mapsto \mathbb{R}$ harmonic. If we require $\phi(0) = \chi(0) = 0$, both ϕ and χ are unique. Moreover χ satisfies $[-(\mathcal{E}+\frac{3}{2})(\mathcal{E}+1)(\mathcal{E}+\frac{1}{2})^{-1}\mathcal{E}]\chi$.

Proof
First, construct χ from $\operatorname{div}\Phi$, i.e. $\chi = -(\mathcal{E}+\frac{3}{2})^{-1}(\mathcal{E}+1)^{-1}(\mathcal{E}+\frac{1}{2})\mathcal{E}^{-1}\operatorname{div}\Phi$ and define Φ_1 according to (3.2). Put $\Phi_0 = \Phi - \Phi_1$. Observe that $\operatorname{div}\Phi_0 = 0$. Next we obtain
$\mathbf{x}\cdot\Phi_1 = (1-\|\mathbf{x}\|^2)\mathcal{E}\left[I - \frac{1}{2}(\mathcal{E}+\frac{1}{2})^{-1}\mathcal{E}\right]\chi$. Therefore at the boundary, we get $\mathbf{x}\cdot\Phi_0 = 0$. Further, we also have that $\Delta\Phi_0 = \mathbf{0}$,since $\Delta\Phi_1 = \mathbf{0}$. Hence, according to the preceding lemma, we obtain $\Phi_0 = \mathbf{x}\times\nabla\phi$. Finally, note that until now ϕ and χ are fixed up to an additive constant. By taking $\phi(\mathbf{0}) = \chi(\mathbf{0}) = 0$, this arbitrariness is removed □.

Theorem 3.3 *Let $\psi : \overline{B_3} \to \mathbb{R}$ be continuous and such that $\Delta\psi = 0$ in B_3 and $\psi(0) = 0$. If the vector function $w : \overline{B_3} \to \mathbb{R}^3$ satisfies*

$$\begin{cases} \Delta w(x) = 0 & x \in B_3 \\ w(x) = x\psi(x) + cx & x \in \partial B\, c \in R. \end{cases}$$

Then w can be written $w = \nabla\mathcal{E}^{-1}\psi + cx$ in which $c = \frac{1}{4\pi}\int\limits_{\partial B_3}(a\cdot x)\,d\sigma$.

Proof
Straightforward computation yield $\Delta\mathrm{w} = \Delta(\nabla\mathcal{E}^{-1}\psi + c\mathrm{x}) = \mathbf{0}$. Next by taking the inner product $\mathbf{w}(\mathbf{x})\cdot\mathbf{x}$ at the boundary, one finds

$$\mathrm{x}\cdot\mathrm{w} = \mathcal{E}\mathcal{E}^{-1}\psi + c\mathrm{x}\cdot\mathrm{x} = \psi(\mathrm{x}) + c$$

Finally, take the inner product $w(\mathbf{x})\cdot\mathbf{x}$ at the boundary, calculate the integral over the boundary, and use the mean value theorem for harmonic functions, to obtain

$$\int\limits_{\partial B_3} \mathrm{w}\cdot\mathrm{x}\,d\sigma = \int\limits_{\partial B_3} \psi(\mathrm{x})\,d\sigma + 4\pi\,c = 4\pi\,\psi(\mathbf{0}) + 4\pi c$$

Note that $\psi(\mathbf{0}) = 0$ and the result is obtained □.

Theorem 3.4 *Consider the continuous vector field $a : \partial B_3 \mapsto \mathbb{R}^3$. Let $a_{\mathcal{H}} : \overline{B_3} \mapsto \mathbb{R}^3$ be the harmonic extension of a. Then there exist unique harmonic functions ϕ, χ, $\psi : B_3 \mapsto \mathbb{R}$, with $\phi(0) = \chi(0) = \psi(0) = 0$ such that*

$$a_{\mathcal{H}} = h_0[\phi] + h_1[\chi] + h_2[\psi] + cx$$

In which h_i for $i = 0, 1, 2$ and c are defined by

$$\begin{aligned} & h_0[\phi] : x \mapsto x \times \nabla\phi(x) \\ & h_1[\chi] : x \mapsto \nabla\chi - (\mathcal{E}\chi)x + \tfrac{1}{2}(\|x\|^2 - 1)\nabla[(\mathcal{E}+\tfrac{1}{2})^{-1}\mathcal{E}]\chi \\ & h_2[\psi] : x \mapsto \nabla\mathcal{E}^{-1}\psi \\ & c = \tfrac{1}{4\pi}\int\limits_{\partial B_3}(a\cdot x)d\sigma \end{aligned} \tag{3.2}$$

Proof
The proof is divided into several steps.

1. Split the vector field $\mathbf{a}$ into a tangential and a normal component as follows: $\mathbf{a} = \mathbf{a}_t + \mathbf{a}_n + c\mathbf{n}$, with $c = \frac{1}{4\pi}\int\limits_{\partial B_3}(\mathbf{a}\cdot\mathbf{n})d\sigma$, $\mathbf{a}_n = (\mathbf{a}\cdot\mathbf{n} - c)\mathbf{n}$ and $\mathbf{a}_t = \mathbf{a} - \mathbf{a}_n - c\mathbf{n}$.
   Notice that $\mathbf{a}_t$ equals the tangential part of $\mathbf{a}$.

2. Define ψ as the solution of the Dirichlet problem

$$\begin{cases} \Delta\psi(\mathrm{x}) = 0 & \mathrm{x} \in B_3 \\ \psi(\mathrm{x}) = \mathbf{a}\cdot\mathbf{n} - c & \mathrm{x} \in \partial B_3 \end{cases}$$

3. Note that for any $\mathbf{x} \in \partial B_3$, the vector field $\mathbf{a} - \nabla\mathcal{E}^{-1}\psi(\mathbf{x}) - c\mathbf{x}$ is tangential.

4. The harmonic extension $\mathbf{x} \mapsto \mathbf{a}_{\mathcal{H}}(\mathbf{x}) - \nabla\mathcal{E}^{-1}\psi(\mathbf{x}) - c\mathbf{x}$ satisfies the conditions of lemma 3.2. This means that it can be uniquely written as $\mathbf{h}_0[\phi] + \mathbf{h}_1[\xi]$ for suitable harmonic ϕ and χ. □

Note that in our case $\mathbf{a} = \mathbf{v}|_{\partial B_3} \ \wedge\ \nabla \cdot \mathbf{v} = 0 \Rightarrow c = 0$.
So, because $\mathbf{h}_0[\phi]$ and $\mathbf{h}_1[\chi]$ are tangent, the normal part of $\mathbf{a}_{\mathcal{H}}$ equals $\mathbf{h}_2[\psi]$. Also note that $\mathbf{h}_0[\phi]$ and $\mathbf{h}_1[\chi]$ are respectively, the divergence-less and divergence-full tangential part of $\mathbf{a}_{\mathcal{H}}$.

The question that arises is whether such explicit splitting can be found in the n dimensional case. It is not difficult to generalize $\mathbf{h}_1$ to the n dimensional case and $\mathbf{h}_2$ can be taken exactly the same as in the 3 dimensional case. $\mathbf{h}_0$ causes more problems because of the use of a cross product which, of course, only makes sense in $\mathbb{R}^3$.

The generalization of $\mathbf{h}_1$ and $\mathbf{h}_2$ to the n dimensional case is given by

$$\begin{array}{l} \mathbf{h}_1[\chi] = \nabla\chi - (\mathcal{E}\chi)\mathbf{x} + \frac{1}{2}(\|\mathbf{x}\|^2 - 1)\nabla(\mathcal{E} + \frac{n-2}{2})^{-1}\mathcal{E}\chi \\ \mathbf{h}_2[\psi] = \nabla\mathcal{E}^{-1}\psi \ . \end{array} \tag{3.3}$$

Straight calculations and the use of $\Delta(f\mathbf{v}) = f\Delta\mathbf{v} + \Delta f\mathbf{v} + 2[\nabla\mathbf{v}][\nabla f]^T$ show that indeed $\Delta\mathbf{h}_1[\chi] = 0$. One can also easily derive

$$\begin{array}{l} \nabla \cdot \mathbf{h}_1[\chi] = \\ (-n\mathcal{E} - \mathcal{E}^2 + \mathcal{E}(\mathcal{E} + \frac{n-2}{2})^{-1}\mathcal{E})\ \chi = \\ -(\frac{n-2}{2} + \mathcal{E})^{-1}((n-2) + \mathcal{E})\mathcal{E}(\frac{n}{2} + \mathcal{E})\ \chi \ . \end{array} \tag{3.4}$$

Again, χ can be chosen such that

$$\begin{array}{l} \nabla \cdot \mathbf{h}_1[\chi] = \nabla \cdot \mathbf{a}_{\mathcal{H}} \Leftrightarrow \\ \chi = -(\frac{n-2}{2} + \mathcal{E})(n-2+\mathcal{E})^{-1}\mathcal{E}^{-1}(\frac{n}{2} + \mathcal{E})^{-1}\nabla \cdot \mathbf{a}_{\mathcal{H}} \ . \end{array} \tag{3.5}$$

Remark: $\nabla \cdot \mathbf{h}_2[\psi] = \nabla \cdot \nabla\mathcal{E}^{-1}\psi = 0$.

In order to find the generalization of $\mathbf{h}_0[\phi]$ one might use theorem 3.5.

Definition 3.1 *Given an operator B and a vector function f defined on $\Omega \subset \mathbb{R}^n$ the vector function $\mathcal{L}_B f$ is defined by*

$$(\mathcal{L}_B f)(x) = Df\,Bx = \begin{pmatrix} (\nabla f_1, Bx) \\ \vdots \\ (\nabla f_n, Bx) \end{pmatrix} .$$

Operator $\mathcal{L}_B(\mathbf{f})$ has the following geometrical property:
Observe the begin value problem:

$$\begin{cases} \frac{d}{dt}\mathbf{x} = B\mathbf{x} \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases} . \tag{3.6}$$

Its solution is given by $\mathbf{x}(t) = e^{tB}\mathbf{x}_0$. Further

$$(\mathcal{L}_B\mathbf{f})(\mathbf{x}(t)) = (B\mathbf{x}(t), \nabla\mathbf{f}) = (\frac{d\mathbf{x}}{dt}, \frac{\partial\mathbf{f}}{\partial\mathbf{x}}) = \frac{d}{dt}\mathbf{f}(\mathbf{x}(\mathbf{t})) \tag{3.7}$$

In particular for $t = 0$ we have

$$(\mathcal{L}_B(f))(\mathbf{x}_0) = \frac{d}{dt}\ (f(e^{tB}\mathbf{x}_0))\big|_{t=0} \ .$$

So $(\mathcal{L}_B\mathbf{f})(\mathbf{x})$ is the time derivative evaluated at $t = 0$ of the function $\mathbf{f}$ along the solution (path) of (3.6). If B is skew-symmetric then this path lies in the sphere with radius $||\mathbf{x}_0||$ and origin $\mathbf{0}$.

Theorem 3.5 *Let B be a skew-symmetric operator on Ω. Let $\boldsymbol{f}, \boldsymbol{g} : \Omega \mapsto \mathbb{R}$ be harmonic such that $D\boldsymbol{f}B$ and $D\,\boldsymbol{g}$ are skew-symmetric, then $\mathcal{L}_B\boldsymbol{f}$ and $\mathcal{E}\ \boldsymbol{g}$ satisfy*

$$\begin{cases} \Delta\Phi = 0 \\ \nabla \cdot \boldsymbol{\Phi} = 0 \\ \boldsymbol{x} \cdot \boldsymbol{\Phi} = 0 \end{cases} \tag{3.8}$$

Proof

Let $A(t)$ be a one parameter subgroup of $O(n)$ such that $A(t) = e^{Bt}$. Because of $\text{trace}(A(t)\ [D^2 f_i]\ A^T(t)) = \text{trace}(A(t)\ [D^2 f_i]\ A^{-1}(t)) = 0$,
$\Delta f_i(A(t)\mathbf{x}) = 0$ for all t. So,

$$\frac{d}{dt}\Delta f_i(A(t)\mathbf{x}) =$$
$$\frac{d}{dt}\Delta f_i(e^{Bt}\mathbf{x}) =$$

$$\Delta(\frac{\partial f_i}{\partial \mathbf{x}} e^{-Bt} B e^{Bt}\mathbf{x}) =$$

$$\Delta(\nabla f_i B\mathbf{x}) =$$

$$\Delta\mathcal{L}_B(f_i) = 0$$

Using $\nabla \cdot A\mathbf{v} = (\nabla \cdot A, \mathbf{v}) + \text{trace}(AD\mathbf{v})$ one gets

$$\nabla \cdot (\mathcal{L}_B(\mathbf{f}))(\mathbf{x}) = \text{trace}(D\mathbf{f}B).$$

Finally note that

$$A^T = -A \Rightarrow (\mathbf{x}, A\mathbf{x}) = -(A\mathbf{x}, \mathbf{x}) = 0, \tag{3.9}$$

so we have

$$\forall_{\mathbf{x} \in \partial B_n} : \mathbf{x}^T\ D\mathbf{f}B\,\mathbf{x} = \mathbf{x}^T\ \mathcal{L}_B\mathbf{f} = 0 \text{ and}$$

we conclude: $\mathcal{L}_B\mathbf{f}$ is a solution of (3.8).
That $\mathcal{E}\ \mathbf{g}$ satisfies (3.8) as well, is due to (3.9) and

$$\Delta\mathcal{E} = 2\Delta + \mathcal{E}\Delta$$
$$\nabla \cdot \mathcal{E}g = \text{trace}(\nabla\mathbf{g}). \qquad \square$$

Essential is the skew-symmetry requirement of $\nabla f B$. Because B must be skew-symmetric, this requirement is equivalent to:

$$B\ \nabla\mathbf{f}^T = [\nabla\mathbf{f}]\ B \tag{3.10}$$

The most obvious situation in which this appears to be true is the case that ∇f is symmetric and commutes with B. This situation takes place at the *disk example*:

$$B^T = -B \Leftrightarrow B = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \tag{3.11}$$

So, if $\nabla\mathbf{f}$ is symmetric and commutes with a certain skew-symmetric operator, it commutes with them all. Now, Schur's lemma together with $[A, B] = 0 \Rightarrow [A, e^B] = 0$ imply that $\nabla\mathbf{f} = cI$. The result is that

$$(\mathcal{L}_B\mathbf{f})(\mathbf{x}) = C\begin{pmatrix} -y \\ x \end{pmatrix}$$

which is indeed the unique solution of (3.8) in the disc-case.

Let's investigate whether the solution of (3.8) in *the 3 dimensional unit ball case* is of the form $\mathcal{L}_B\mathbf{f}$, with $B^T = -B$ or $B = I$. According to theorem 3.1, the solution of (3.8) if $\Omega = B_3$ can be written as $\mathbf{\Phi} = \mathbf{x} \times \nabla\phi$, with unique ϕ such that $\Delta\phi = 0 = \phi(\mathbf{0})$.
Define In $\mathbb{R}^3$ every skew-symmetric operator B corresponds to a unique $\mathbf{b} \in \mathbb{R}^3$ such that $B\mathbf{x} = \mathbf{b} \times \mathbf{x}$. Therefore,

$$(\mathcal{L}_B\,\mathbf{f})(\mathbf{x}) = D\mathbf{f}\cdot(\mathbf{b}\times\mathbf{x}) = \begin{pmatrix} \nabla f_1 \cdot (\mathbf{b}\times\mathbf{x}) \\ \nabla f_2 \cdot (\mathbf{b}\times\mathbf{x}) \\ \nabla f_3 \cdot (\mathbf{b}\times\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \mathbf{b}\cdot(\mathbf{x}\times\nabla f_1) \\ \mathbf{b}\cdot(\mathbf{x}\times\nabla f_2) \\ \mathbf{b}\cdot(\mathbf{x}\times\nabla f_3) \end{pmatrix}$$

Taking $\mathbf{b} = \mathbf{e}_1$ gives $\mathcal{L}_B\ \mathbf{f} = \sum_{i=1}^{3} L_x f_i \mathbf{e}_i$. So, if one chooses

$$\begin{aligned} f_1 &= \phi \\ f_2 &= L_x^{-1} L_y \phi \\ f_3 &= L_x^{-1} L_z \phi \end{aligned} \tag{3.12}$$

then $(\mathcal{L}_B\mathbf{f})(\mathbf{x}) = \mathbf{L}\phi = \mathbf{x} \times \nabla\phi$.

Notice that $[\Delta, \mathbf{L}] = [\mathcal{E}, \mathbf{L}] = 0$. Therefore, these self-adjoint operators have a common orthogonal base of eigen functions which are the spherical harmonics $Q_{l,m}$. They form an orthogonal base for $\text{HarmHomPol}_k(B_3)$.
For every ϕ which is a harmonic function on B_3 there exists a unique expansion

$$\phi = \sum_{k=0}^{\infty} \phi_k \ , \text{ with } \phi_k \in \text{HarmHomPol}_k(\mathbb{R}^3) \tag{3.13}$$

which converges uniformly on compact sets in B_3.
Define $L^+ = L_x + iL_y$ and $L^- = L_x - iL_y$, then $[L_z, L^\pm] = \pm iL^\pm$ and $L_x = \frac{1}{2}(L^+ + L^-)$.

$$\begin{aligned} L_z Q_{l,m} &= im Q_{l,m} \\ L^\pm Q_{l,m} &= A^\pm_{l,m} Q_{l,m\pm1} \text{ for } m \pm 1 \in (-l, l) \text{ and some constants } A^\pm_{l,m}. \end{aligned} \tag{3.14}$$

So, L_x^{-1} in the expressions (3.12) makes sense (NB. $\Delta\phi = \phi(\mathbf{0}) = 0$ implies that $\phi, L_y\phi$ and $L_z\phi$ can not be constant) and indeed the solution of (3.8) in the 3 dimensional unit ball case can be written as $\mathcal{L}_B(\mathbf{f})$, with $\mathbf{f}$ harmonic and B skew-symmetric.

The following questions arise:

- How many independent solutions has problem (3.8) in the *n-dimensional* unit ball ?
- Can they all be parameterized by harmonic functions ?
- Can they all be parameterized as suggested in theorem 2 ?

At the moment it isn't clear in what way the solutions of (3.8) in the n-dimensional unit ball case can be parameterized, therefore they will be written as $\mathbf{h}_0$.

Theorem 3.6 *The general solution of the Stokes problem, with $f = 0$ and $h = 0$, on the n-dimensional unit ball is given by:*

$$\begin{cases} p = -2(\mathcal{E} + \frac{n-2}{2})\mathcal{E}^{-1}\nabla \cdot a_{\mathcal{H}} \\ v = a_{\mathcal{H}} + \frac{(1-\|x\|^2)}{2}\nabla\mathcal{E}^{-1}\nabla \cdot a_{\mathcal{H}} \end{cases} \tag{3.15}$$

or expressed in h_i $i = 0, 1, 2$, referring to (3.3):

$$\begin{cases} p &= -2(\mathcal{E} + \frac{n-2}{2})\mathcal{E}^{-1}\nabla \cdot h_1[\chi] \\ &= 2(\mathcal{E} + n - 2)(\frac{n}{2} + \mathcal{E})\ \chi \\ v &= h_0 + h_1[\chi] + h_2[\psi] + \frac{(1-\|x\|^2)}{2}\nabla\mathcal{E}^{-1}\nabla \cdot h_1[\chi] \\ &= h_0 + \nabla\chi - (\mathcal{E}\chi)x + \frac{1}{2}(\|x\|^2 - 1)\nabla(\mathcal{E} + \frac{n-2}{2})^{-1}\mathcal{E}\chi + \nabla\mathcal{E}^{-1}\psi + \\ &\quad -\frac{(1-\|x\|^2)}{2}\nabla(\frac{n-2}{2} + \mathcal{E})^{-1}((n-2) + \mathcal{E})(\frac{n}{2} + \mathcal{E})\ \chi \end{cases} \tag{3.16}$$

, with the harmonic functions ψ and χ such that

$$\begin{aligned} &(a)_{\mathcal{H}} = h_0 + h_1[\chi] + h_2[\psi] \\ &\chi(0) = \psi(0) = 0\ . \end{aligned} \tag{3.17}$$

Proof

First, determine p :
Taking $\mathbf{f} = \mathbf{0}$ and $h = 0$ in (2.17), we obtain

$$\nabla \cdot \mathcal{D}\nabla p = \operatorname{div}(\mathbf{a})_{\mathcal{H}} = \operatorname{div}\mathbf{h}_1[\chi]\ . \tag{3.18}$$

$\mathcal{D}\nabla p$ can be expressed in ∇p as follows:
In order to solve the Dirichlet problem set $\mathcal{D}\nabla p = (1 - \|\mathbf{x}\|^2)\mathbf{H}$, with $\mathbf{H}$ a harmonic yet to be derived function. See chapter 3, the second method.
Substitution of this expression in the Dirichlet problem yields[2]

$$\begin{aligned} &-4(\tfrac{n}{2} + \mathcal{E})H_j = -\tfrac{\partial}{\partial x_j} \text{ for } j = 1, \ldots, n \Leftrightarrow \\ &\mathbf{H} = \tfrac{1}{4}(\mathcal{E} + \tfrac{n}{2})^{-1}\nabla p \Leftrightarrow \\ &\mathcal{D}\nabla p = \tfrac{(1-\|\mathbf{x}\|^2)}{4}\nabla(\mathcal{E} + \tfrac{n-2}{2})^{-1}p \end{aligned} \tag{3.19}$$

Notice that $\nabla((\mathcal{E}+s)^{-1}f)(\mathbf{x}) = (\mathcal{E}+s+1)^{-1}\nabla f(\mathbf{x})$ is used (the special case $s = \frac{n}{2}$).
Moreover, the resolvents of $\mathcal{E}$ are given by

$$(\mathcal{E}+s)^{-1}f(\mathbf{x}) = \int_0^1 \lambda^{s-1}f(\lambda\mathbf{x})d\lambda\ . \tag{3.20}$$

Therefore it is obvious that $\mathbf{H}$ is harmonic. Straight calculations show that:

$$\nabla \cdot \mathcal{D}\nabla p = -\frac{1}{2}\mathcal{E}\left(\mathcal{E} + \frac{(n-2)}{2}\right)^{-1} p \tag{3.21}$$

[2]Note that $\mathcal{E}^{-1}p$ makes no sence if p is a constant. However, the final solution for p of the Stokes problem is determined up to a constant, so there is no need to bother about this.

So,

$$
\begin{aligned}
p &= -2(\mathcal{E} + \tfrac{(n-2)}{2})\mathcal{E}^{-1}\mathrm{div}(\mathbf{a})_{\mathcal{H}} \\
&= -2(\mathcal{E} + \tfrac{(n-2)}{2})\mathcal{E}^{-1}\mathrm{div}\mathbf{h}_1[\chi] \\
&= -2(\mathcal{E} + \tfrac{(n-2)}{2})\mathcal{E}^{-1}(-n\mathcal{E} - \mathcal{E}^2 + \mathcal{E}(\mathcal{E} + \tfrac{n-2}{2})^{-1}\mathcal{E})\chi \\
&= 2(\mathcal{E} + \tfrac{(n-2)}{2})\mathcal{E}^{-1}(\tfrac{n-2}{2} + \mathcal{E})^{-1}\mathcal{E}((n-2) + \mathcal{E})(\tfrac{n}{2} + \mathcal{E})\chi \\
&= 2((n-2) + \mathcal{E})(\tfrac{n}{2} + \mathcal{E})\chi
\end{aligned}
$$

Note that $\nabla \cdot (\mathbf{a})_{\mathcal{H}}$ can *not* be constant, since

$$\int_{\partial\Omega} \mathrm{div}\, \mathbf{a}_{\mathcal{H}} d\tau = \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n}\, d\sigma = \int_{\partial\Omega} \mathrm{div}\, \mathbf{v}\ d\tau = 0\,.$$

Next, $\mathbf{v}$ is determined. Substituting of (2.15) in the first equation of (2.8) and using $\mathbf{f} = \mathbf{0}$, $h = 0$ and finally the second equation of (2.8) gives

$$\mathbf{v} = \mathbf{a}_{\mathcal{H}} - \mathcal{D}\nabla p\,. \tag{3.22}$$

So, using (2.17) one gets:

$$
\begin{aligned}
&\mathbf{v} = \mathbf{a}_{\mathcal{H}} + \tfrac{(1-\|\mathbf{x}\,\|^2)}{2}\nabla\mathcal{E}^{-1}\nabla \cdot \mathbf{a}_{\mathcal{H}} \Leftrightarrow \\
&\mathbf{v} = \mathbf{h}_0 + \mathbf{h}_1[\chi] + \nabla\mathcal{E}^{-1}\nabla \cdot \mathbf{h}_1[\chi]
\end{aligned}
\tag{3.23}
$$

For explicit formulas for $\mathbf{h}_i$ and $\nabla\mathbf{h}_i$ see (3.4) and (3.3). □

Theorem 3.7 *The general solution of the Stokes problem, with f and h arbitrary but given, on the n-dimensional unit ball is given by:*

$$
\begin{cases}
p = -2(\mathcal{E} + \tfrac{n-2}{2})\mathcal{E}^{-1}(\nabla \cdot (a_{\mathcal{H}} + \mathcal{D}f\,) + h) \\
v = a_{\mathcal{H}} + \mathcal{D}f\ + \tfrac{(1-\|x\,\|^2)}{2}\nabla\mathcal{E}^{-1}(\nabla \cdot (a_{\mathcal{H}} + \mathcal{D}f\,) + h)
\end{cases}
\tag{3.24}
$$

Proof
First note that equation (3.21) is valid for p in general. So, substitute (3.21) in (2.17). The result is:

$$
\begin{aligned}
&-\tfrac{1}{2}\mathcal{E}(\mathcal{E} + \tfrac{(n-2)}{2})^{-1}p = \nabla \cdot (\mathbf{a}_{\mathcal{H}} + \mathcal{D}\mathbf{f}\,) + h \Leftrightarrow \\
&p = -2(\mathcal{E} + \tfrac{(n-2)}{2})\mathcal{E}^{-1}(\nabla \cdot (\mathbf{a}_{\mathcal{H}} + \mathcal{D}\mathbf{f}\,) + h)
\end{aligned}
\tag{3.25}
$$

Now $\mathbf{v}$ can be derived easily. Substituting of (2.15) in the first equation of (2.8) and using the second equation of (2.8) gives

$$\mathbf{v} = \mathbf{a}_{\mathcal{H}} - \mathcal{D}\nabla p + \mathcal{D}\mathbf{f} \tag{3.26}$$

Substitution of (3.25) in (3.21) leads to:

$$\mathcal{D}\nabla p = \frac{(1 - \|\mathbf{x}\,\|^2)}{2}\nabla\mathcal{E}^{-1}(\nabla \cdot (\mathbf{a}_{\mathcal{H}} + \mathcal{D}\mathbf{f}\,) + h) \tag{3.27}$$

Substitute (3.27) in (3.26) and the final result is obtained. □

Note that :

- on the unit ball[3]:

$$\nabla\cdot\mathcal{D}\nabla = -\frac{1}{2}\mathcal{E}\left(\mathcal{E} + \frac{(n-2)}{2}\right)^{-1} \tag{3.28}$$

  This equation was of fundamental importance in finding the general solutions of the Stokes problem, since the righthand side can be inverted easily:

$$[\nabla\cdot\mathcal{D}\nabla]^{-1} = -2\left(\mathcal{E} + \frac{(n-2)}{2}\right)\mathcal{E}^{-1} \tag{3.29}$$

- in stead of the proof above one could also use the splitting that is introduced in appendix F (F.2) in combination with theorem (3.6). If (3.28) is kept in mind, it is not difficult to see that this leads to the same result.

In Appendix A and B the solutions in the case $n = 2$ (Disk) and $n = 3$ (3D Unit ball) are derived following the theory of this chapter.

[3]see (3.21) in proof theorem 3.6

4 Stokes problems on some domains in $\mathbb{R}^2$

Just like in the unit ball case, we will examine operator div$\mathcal{D}$grad. in case Ω is a half space respectively infinite strip. In both examples we only regard $\mathbf{f} = \mathbf{0}$ and $h = 0$.

4.1 Example: Operator div$\mathcal{D}$grad on a half plane

In this section we will solve operator equation (2.17) in case
$\Omega = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$. For that purpose we first follow the first method as discussed in chapter 3. So, we have to determine $\mathbf{W} = \{-\frac{1}{2}p\,\mathbf{x}\}_{\mathcal{H}}$ which satisfies,

$$\begin{cases} \Delta \mathbf{W} = 0 & \text{in } \Omega \\ \mathbf{W}(x,0) = -\frac{1}{2}p(x,0)\begin{pmatrix} x \\ 0 \end{pmatrix} & \text{by definition.} \end{cases}$$

Obviously we have $W_2 = 0$. In order to derive W_1 we use
Fourier transform with respect to x, $x \mapsto \xi$:

$$\begin{aligned} &\mathcal{F}(W_1) = \mathcal{F}(X)Y(y) \\ &\mathcal{F}(\Delta W_1) = 0 \ \wedge\ Y(0) = 1 \Rightarrow Y(y) = e^{-|\xi|y} \\ &\mathcal{F}(W_1) = -\tfrac{1}{2}e^{-|\xi|y}\mathcal{F}(p(x,0)x) \Rightarrow \\ &W_1(x,y) = -\tfrac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}[\mathcal{F}(p(x,0)x)](\xi)\ e^{-|\xi|y}\ e^{i\xi x}\ d\xi \end{aligned} \tag{4.1}$$

So we have,

$$\nabla\cdot\mathcal{D}\nabla p = -p - \frac{1}{2}\mathcal{E}p + \frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}[\mathcal{F}(p(x,0)x)](\xi)\ e^{-|\xi|y}\ e^{i\xi x}(i\xi)\ d\xi \tag{4.2}$$

Next, we will show :

$$\begin{aligned} &-\tfrac{1}{2}\mathcal{E}p = \tfrac{1}{2}p - \tfrac{1}{2\sqrt{2\pi}}\int_{-\infty}^{\infty}[\mathcal{F}(p(x,0)x)](\xi)\ e^{-|\xi|y}\ e^{i\xi x}(i\xi)\ d\xi \Leftrightarrow \\ &\mathcal{F}(-\tfrac{1}{2}\mathcal{E}p - \tfrac{1}{2}p)(\xi) = \mathcal{F}(\tfrac{\partial W_1}{\partial x}) \end{aligned}$$

This is indeed the case because of

$$\frac{\partial W_1}{\partial x}(x,0) = -\frac{1}{2}\frac{\partial}{\partial x}[p(x,0)x] = -\frac{1}{2}\mathcal{E}p\Big|_{y=0} - \frac{1}{2}p(x,0)\,. \tag{4.3}$$

NB. Note that because div $\mathbf{W} + \frac{1}{2}(\mathcal{E} + I)p$ is harmonic it's sufficient to show that this equality holds at the boundary.

As a result,

$$\text{div}\mathcal{D}\text{grad}\, p = -\frac{1}{2}p \tag{4.4}$$

, which is exactly the same as in the unit disk case. This result can be obtained directly, if we follow the third method of determination of div$\mathcal{D}\nabla$ as described in chapter 3. On the boundary we have that $z = \overline{z}$, therefore

$$\begin{aligned} H_1 &= \left[\text{Re}\ \{\tfrac{1}{4}\overline{z}f(z)\}\big|_{\partial\Omega}\right]_{\mathcal{H}} = \text{Re}\ \{\tfrac{1}{4}z\,f(z)\} \\ H_2 &= \left[\text{Re}\ \{\tfrac{1}{4}\,i\overline{z}f(z)\}\big|_{\partial\Omega}\right]_{\mathcal{H}} = \text{Re}\{\tfrac{1}{4}i\,z\,f(z)\} \end{aligned} \tag{4.5}$$

and thus div$\mathbf{H} = \frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0$. From equation (2.24) equation (4.4) now follows.

Corollary :

$$\begin{aligned} \nabla \cdot \mathcal{D}\nabla p &= -\tfrac{1}{2}p \Rightarrow \\ p &= -2\nabla \cdot \mathbf{a}_{\mathcal{H}} \\ &= -2\nabla \cdot \mathcal{F}^{-1}(e^{-|\xi|y}[\mathcal{F}(\mathbf{a})](\xi)) \\ &= \frac{-2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\xi(\hat{a}(\xi) + i\hat{b}(\xi)\mathrm{sgn}(\xi))\ e^{-|\xi|y}\ e^{i\xi x}\ d\xi \end{aligned} \tag{4.6}$$

4.2 Example: Operator div$\mathcal{D}$grad on a infinite strip

In this chapter we examine div$\mathcal{D}$grad in the case $\Omega = \{(x,y) \in \mathbb{R}^2 \,|\, y \in [0,a]\}$. Although that a simple formula is found for operator div$\mathcal{D}$grad, see (4.2) it should be noticed that a simple explicit solution of the operator equation div$\mathcal{D}\nabla = \text{div}\mathbf{a}_{\mathcal{H}}$ will not be presented.

To examine how operator div$\mathcal{D}\nabla$ deals with harmonic functions, we first determine $\mathcal{D}\nabla p$. We follow the third method of determination of div$\mathcal{D}\nabla$ as described in chapter 3. Remind that we must find $\mathbf{H} = (H_1, H_2)$.
We will see that in comparison to the disk and half plane case "the" harmonic extension q of p appears. Note that

$$\begin{cases} u(x,y) = \text{Re}\left\{\frac{1}{4}\, z\, f(z)\right\} + H_1(x,y) & , z = x+iy \\ v(x,y) = \text{Re}\left\{-\frac{1}{4}\, i\, z\, f(z)\right\} + H_2(x,y) & , z = x+iy \end{cases}$$

with $f(z) = p(x,y) + i\,q(x,y) \qquad z = x + iy$ and therefore we write

$$\begin{array}{ll} H_1(x,y) = \text{Re}\left\{\frac{1}{4} z\, f(z)\right\} + Q(x,y) & z = x+iy. \\ H_2(x,y) = \text{Re}\left\{\frac{1}{4} z\, f(z)\right\} + R(x,y) & z = x+iy. \end{array}$$

Where Q and R are the solutions of respectively (4.7) and (4.8):

$$\begin{cases} \Delta Q = 0 & \text{in } \Omega \\ Q(x,0) = 0 & x \in \mathbb{R} \\ Q(x,a) = \frac{a}{2} q(x,a) & x \in \mathbb{R}, \end{cases} \tag{4.7}$$

$$\begin{cases} \Delta R = 0 & \text{in } \Omega \\ R(x,0) = 0 & x \in \mathbb{R} \\ R(x,a) = \frac{a}{2} q(x,a) & x \in \mathbb{R}. \end{cases} \tag{4.8}$$

Using Fourier transformation $\mathcal{F}$ with respect to x, $(x \mapsto \omega)$ we obtain the solutions of these Dirichlet problems:

$$\begin{array}{l} Q(x,y) = \frac{a}{2}\mathcal{F}^{-1}\left(\hat{q}(\omega,a)\frac{\sinh(y\omega)}{\sinh(a\omega)}\right) \\ R(x,y) = \frac{a}{2}\mathcal{F}^{-1}\left(\hat{p}(\omega,a)\frac{\sinh(y\omega)}{\sinh(a\omega)}\right) \end{array} \tag{4.9}$$

So we have

$$\begin{aligned} \text{div}\mathcal{D}\text{grad}p &= -\tfrac{1}{2}p + \tfrac{\partial Q}{\partial x} + \tfrac{\partial R}{\partial y} \\ &= -\tfrac{1}{2}p + \tfrac{a}{2}\mathcal{F}^{-1}\left(i\omega\,\hat{q}(\omega,a)\tfrac{\sinh(y\omega)}{\sinh(a\omega)}\right) + \tfrac{a}{2}\mathcal{F}^{-1}\left(\omega\,\hat{p}(\omega,a)\tfrac{\cosh(y\omega)}{\sinh(a\omega)}\right). \end{aligned}$$

Note that by Plancherel's theorem we have that Fourier transform is a unitary operator on $\mathbb{L}_2(\mathbb{R})$. Therefore, $\|p(x,a)\|_{\mathbb{L}_2(\Omega)} = \|\hat{p}(\omega,a)\|_{\mathbb{L}_2(\Omega)}$. Further note that applying Fourier transform to the Cauchy-Riemann equations gives

$$\begin{cases} \frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} \\ \frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x} \end{cases} \Rightarrow \begin{array}{l} i\,\omega\,\hat{p}(\omega,y) = \frac{\partial}{\partial y}\hat{q}(\omega,y) \\ \frac{\partial}{\partial y}\hat{p}(\omega,y) = -i\,\omega\,\hat{q}(\omega,y)\ . \end{array}$$

So, in order to obtain estimates for $(p, \text{div}\mathcal{D}\text{grad}\ p)$ it doesn't seem necessary, to derive "the" harmonic conjugate q of p by formula (2.25).

5 Spectral analysis of the operators $\mathcal{B}$ and div$\mathcal{D}$grad

In order to solve the Stokes problem one only needs to solve either operator equation (2.17) or operator equation (2.16). Solving these equations means inverting the operators $\mathcal{B} + I$, div$\mathcal{D}$grad. Therefore it might be useful to investigate whether $-1 \in \sigma(B + I)$ or $0 \in \sigma(\text{div}\mathcal{D}\text{grad})$. Note that if $||B|| < 1$ then the solution of $(I + \mathcal{B})\alpha_{\mathcal{H}} = \beta$ can be represented by the van Neumann series:

$$\alpha_{\mathcal{H}} = \sum_{k=0}^{\infty}(-1)^k B^k \, \beta \tag{5.1}$$

Reminding (2.26) in chapter (2) it is obvious that spectral properties of operator $\mathcal{B}$ and operator div$\mathcal{D}$grad are of course strongly related to each other. In order to examine the operators $\mathcal{B}$ and $\nabla \cdot \mathcal{D}\nabla$ we must have a closer look at the original spaces on which they work: $\mathbf{V}^{(n)}$ and $\mathbb{L}_2(\Omega)/_{\mathbb{R}}$. First, some notations / definitions that will be used during this chapter:

Spaces:
$\mathbf{L}_2(\Omega) = \{\mathbb{L}_2(\Omega)\}^3$
$\mathbf{H}_m(\Omega) = \{\mathbb{H}_m(\Omega)\}^3$
$\mathbf{C}^1(\Omega) = \{C^1(\Omega)\}^3$
equipped with the usual cartesian product norms, unless another norm is stated explicitly. (5.2)

Operators:
$\mathcal{C} = \mathcal{D}\nabla\nabla\cdot = \mathcal{D}\text{grad}\,\text{div}$
$\mathcal{F} = \nabla\mathcal{N}\nabla\cdot = \text{grad}\mathcal{N}\text{div}$
Note that
$\mathcal{B} = (\nabla\mathcal{N} - \mathcal{D}\nabla)\text{div} = \mathcal{F} - \mathcal{C}$

Notice that the fundamental relation (2.26) is equivalent to:

$$-\text{div}\mathcal{D}\text{grad}\,[\text{div}\,\alpha_{\mathcal{H}}] = -\text{div}\,\mathcal{C}\alpha_{\mathcal{H}} = \text{div}\,(I+\mathcal{B})\alpha_{\mathcal{H}} \tag{5.3}$$

Remark 5.1
During this chapter we shall denote the Sobolev space $W_{2,1}(\Omega) = \{\phi \in \mathbb{L}_2(\Omega) \mid \forall_{|s|\leq 1} : D^s\phi \in \mathbb{L}_2(\Omega)\}$ by $\mathbb{H}_1(\Omega)$. Although this is common use we remark that (in strict sense) this is not correct. In chapter 6 we define the space $\mathbb{H}_1(\Omega)$ differently, see definition 6.5. In Theorem 6.14 sufficient conditions are mentioned such that the spaces $\mathbb{H}_1(\Omega)$ and $W_{2,1}(\Omega)$ *are* the same.

5.1 The space $\mathbf{V}^{(3)}$

In this section we define and examine the space $\mathbf{V}^{(3)}$, the space on which $\mathcal{B} = (\nabla\mathcal{N} - \mathcal{D}\nabla)\nabla\cdot$ acts. It will turn out that the definition of the space $\mathbf{V}^{(3)}$ is not a triviality, since a Trace-operator $T_{\mathbf{n}}$ will be needed. In praxis we are only interested in how operator $\mathcal{B}$ acts on *tangent* (harmonic) vector fields. (NB. See operator equation 2.16 in which operator $\mathcal{B}$ acts on $\alpha_{\mathcal{H}}$)
See Appendix C for a different and more theoretical approach to the space $\mathbf{V}^{(3)}$!

First we define $\widetilde{\mathbf{V}}^{(n)}$

$$\widetilde{\mathbf{V}}^{(n)} = \{\Phi \in \mathbf{C}^1(\Omega) \mid \Phi \cdot \mathbf{n}|_{\partial\Omega} = 0\} \tag{5.4}$$

For the present we will only look at the special case in which $n = 3$. Then we obtain $\widetilde{\mathbf{V}}^{(3)}$ which will be equipped with the following norm $\|\cdot\|_{\widetilde{\mathbf{V}}^{(3)}}$ defined by:

$$\|\Phi\|^2_{\widetilde{\mathbf{V}}^{(3)}} = \|\mathrm{rot}\,\Phi\|^2_{\mathbf{L}_2(\Omega)} + \|\mathrm{div}\,\Phi\|^2_{\mathbf{L}_2(\Omega)} = \int_\Omega (\mathrm{div}\,\Phi)^2 + \|\mathrm{rot}\,\Phi\|^2 d\mathbf{x}\,. \tag{5.5}$$

Notice that $\|\Phi\|_{\widetilde{\mathbf{V}}^{(3)}} = 0$ implies that $\Phi = 0$:
On the one hand $\|\Phi\|_{\widetilde{\mathbf{V}}^{(3)}} = 0$ implies $\mathrm{rot}\Phi = \mathbf{0}$, so in a simply connected region with sufficiently smooth boundary we have $\Phi = \nabla f$ for a certain scalar function f. On the other hand $\|\Phi\|_{\widetilde{\mathbf{V}}^{(3)}} = 0$ implies $\mathrm{div}\Phi = 0$, so f must be harmonic. Since $\Phi \cdot \mathbf{n} = 0$ it follows that f satisfies

$$\begin{cases} \Delta f = 0 & \text{in } \Omega \\ \frac{\partial f}{\partial n} = 0 & \text{at } \partial\Omega \end{cases} . \tag{5.6}$$

Therefore f must be constant and as a result Φ must be zero.

The space $\widetilde{\mathbf{V}}^{(3)}$ is *not* complete. Every metric space has a completion, see Appendix G.1. So has $\widetilde{\mathbf{V}}^{(3)}$. In this subsection we will construct a space $\mathbf{V}^{(3)}$, which will turn out to be the completion of $\widetilde{\mathbf{V}}^{(3)}$.

If $\mathbf{f}$ is a continuous function on $\overline{\Omega}$ then the point wise restriction of $\mathbf{f}$ to the boundary times the normal on the boundary, $\mathbf{f}|_{\partial\Omega} \cdot \mathbf{n}$, makes sense. With elements in $\mathbf{L}_2(\Omega)$ such a restriction to boundary doesn't make sense because the boundary has zero three dimensional measure. $\mathbf{f}$ represents a class of functions and any assertion on $\mathbf{f}$ should be representative independent. The next theorem shows that restriction of elements in $\mathbb{H}_1(\Omega)$ to $\mathbb{L}_2(\Omega)$ makes sense.

Theorem 5.1 *(Trace Theorem) Let Ω be an open bounded Lipschitz domain in $\mathbb{R}^n$. Then there exists a continuous operator $T_0 : \mathbb{H}_1(\Omega) \to \mathbb{L}_2(\Omega)$ such that the image of all $u \in \mathbb{H}_1(\Omega) \cap C^1(\overline{\Omega})$ under T_0 equals the point wise restriction of u to the boundary.*

Proof
See for instance Wloka[2] Theorem 8.7 p.127-129 in which Wloka proofs that there exists a continuous trace operator that maps $\mathbb{H}_1(\Omega)$ into $\mathbb{H}^{1/2}(\partial\Omega)$, a dense subset of $\mathbb{L}_2(\partial\Omega)$.
For a special case of the above proof see appendix G.3.

Next we define $\mathbf{V}^{(3)}$.

Definition 5.1 *By theorem 5.1 there exists a trace operator $T_0 : \mathbf{H}_1(\Omega) \to \mathbf{L}_2(\partial\Omega)$. From this it follows that $T_n = \boldsymbol{n} \cdot T_0$ is a continuous linear operator from $\mathbf{H}_1(\Omega)$ into $\mathbb{L}_2(\partial\Omega)$, since we have by Cauchy-Schwarz for all $\Phi \in \mathbf{H}_1(\Omega)$*

$$\int_{\partial\Omega} |\boldsymbol{n}\cdot(T_0\,\Phi)|^2\, d\sigma \le \int_{\partial\Omega} |T_0\,\Phi|^2\, d\sigma \le \|T_0\|\,\|\Phi\|_{\mathbf{H}_1(\Omega)}$$

The space $\mathbf{V}^{(3)}$ is defined as the nil-space of this operator, i.e.

$$\mathbf{V}^{(3)} = \mathcal{N}(\boldsymbol{n}\cdot T_0) \tag{5.7}$$

For the present we will equip $\mathbf{V}^{(3)}$ with the inner product/norm carried from $\mathbf{H}_1(\Omega)$, i.e. $(\Phi, \Psi)_{\mathbf{H}_1(\Omega)} = (\Phi, \Psi)_{\mathbf{L}_2(\Omega)} + \sum_{k=1}^{3} (\nabla\phi_k, \nabla\psi_k)_{\mathbf{L}_2(\Omega)} \qquad (\Phi, \Psi \in \mathbf{V}^{(3)}),$

with $\Phi = (\phi_1, \phi_2, \phi_3)$ and $\Psi = (\psi_1, \psi_2, \psi_3)$.

The nill-space of a continuous linear operator is closed, therefore $\mathbf{V}^{(3)}$ is a closed subspace of the Hilbert space $\mathbf{H}_1(\Omega)$ and therefore $\mathbf{V}^{(3)}$ is again a Hilbert space. Next, we define the normed linear space $\mathbf{W}^{(3)}$. Afterwards we will show that $\mathbf{W}^{(3)} = \mathbf{V}^{(3)}$ both set theoretically as topologically.

Definition 5.2 *Define the space* $\mathbf{W}^{(3)}$ *by*

$$\mathbf{W}^{(3)} = \{\Phi \in \mathbf{L}_2(\Omega) \mid div\,\Phi \in \mathbb{L}_2(\Omega) \text{ and } rot\,\Phi \in \mathbf{L}_2(\Omega), T_n\Phi = 0\}.$$

The space $\mathbf{W}^{(3)}$ *can be equipped with the inner product* $(\cdot,\cdot)_{\mathbf{W}^{(3)}}$ *defined by*

$$(\Phi, \Psi)_{\mathbf{W}^{(3)}} = (\Phi, \Psi)_{\mathbf{L}_2(\Omega)} + (div\,\Phi, div\,\Psi)_{\mathbb{L}_2(\Omega)} + (rot\,\Phi, rot\,\Psi)_{\mathbf{L}_2(\Omega)} \,. \tag{5.8}$$

The according norm on $\mathbf{W}^{(3)}$ *will be denoted by*

$$||\Phi||_{\mathbf{W}^{(3)}} = (\Phi, \Phi)^{1/2} \,. \tag{5.9}$$

Theorem 5.2 *The following equality holds both set theoretically and topologically*

$$\mathbf{W}^{(3)} = \mathbf{V}^{(3)} \,. \tag{5.10}$$

Proof
Duvaut-Lions[5](Theorem 6.1 Chapter 7)

From the above theorem it follows that we can equip the space $\mathbf{V}^{(3)}$ with the inner product respectively norm given by respectively (5.8) and (5.9). If the norm given by (5.9) is applied to $\mathbf{V}^{(3)}$ we simply write $||\Phi||$ in stead of $||\Phi||_{\mathbf{W}^{(3)}}$!

For every element $\Phi \in \mathbf{V}^{(3)}$ we have

$$\begin{cases} \operatorname{div}\Phi = 0 \\ \operatorname{rot}\Phi = \mathbf{0} \end{cases} \Rightarrow \Phi = 0,$$

with similar arguments as between (5.5) and (5.6). Therefore a norm on $||\cdot||_{\mathbf{V}^{(3)}}$ on $\mathbf{V}^{(3)}$ is given by

$$||\Psi||^2_{\mathbf{V}^{(3)}} = ||\operatorname{div}\Psi||^2_{\mathbb{L}_2(\Omega)} + ||\operatorname{rot}\Psi||^2_{\mathbf{L}_2(\Omega)}. \tag{5.11}$$

The next theorem shows that this norm is equivalent to the norm given by (5.9).

Theorem 5.3 *Let* $\mathbf{V}^{(3)}$ *be equipped with the inner product given by 5.9. Then*

$$\exists_{C>0}\forall_{\Phi\in\mathbf{V}^{(3)}} : ||\Phi||^2 \le C\left(||rot\,\Phi||^2_{\mathbf{L}_2(\Omega)} + ||div\,\Phi||^2_{\mathbb{L}_2(\Omega)}\right) .$$

As a result $||\cdot||$ *and* $||\cdot||_{\mathbf{V}^{(3)}}$ *are equivalent.*

Proof
Suppose

$$\forall_{C>0}\exists_{\Phi\in\mathbf{V}^{(3)}} : ||\Phi||^2 \ge C\left(||\operatorname{rot}\Phi||^2_{\mathbf{L}_2(\Omega)} + ||\operatorname{div}\Phi||^2_{\mathbb{L}_2(\Omega)}\right)$$

By taking $c = 1, 2, \ldots$ a sequence $\{\Phi_n\}_{n\in\mathbb{N}}$ can be obtained such that:

$$1 = ||\Phi_n||^2 \ge n\left(||\operatorname{rot}\Phi_n||^2_{\mathbf{L}_2(\Omega)} + ||\operatorname{div}\Phi_n||^2_{\mathbb{L}_2(\Omega)}\right)$$

Therefore

$$\begin{array}{ll} \operatorname{div} \Phi_n \to 0 & \text{in } \mathbb{L}_2(\Omega) \\ \operatorname{rot} \Phi_n \to \mathbf{0} & \text{in } \mathbf{L}_2(\Omega) \end{array} \tag{5.12}$$

Therefore $\Phi = 0$.
The embedding $\mathbb{H}_1(\Omega) \hookrightarrow \mathbb{L}_2(\Omega)$ is compact. So is the embedding $\mathbf{H}_1(\Omega) \hookrightarrow \mathbf{L}_2(\Omega)$. As a result $\{\Phi_n\}$ contains a subsequence $\{\Phi_{n_l}\}_{l\in\mathbb{N}}$ which converges to a certain Φ in $\mathbf{L}_2(\Omega)$. From (5.12) follows that $\{\Phi_{n_l}\}$ is a Cauchy sequence in $\mathbf{H}_1(\Omega)$ which is complete and therefore $\Phi_{n_l} \to \Phi$ for $l \to \infty$.
$\operatorname{div} \Phi = 0$ and $\operatorname{rot} \Phi = \mathbf{0}$. This means in our simply connected region Ω that $\Phi = \mathbf{0}$.
But

$$\lim_{n\to\infty} \|\Phi_n\| = \lim_{n\to\infty} 1 = 1 \neq 0$$

As a consequence we have a contradiction. $\square$

Note that $\|\cdot\|_{\mathbb{H}_1(\Omega)} \asymp \|\cdot\|$ and $\|\cdot\| \asymp \|\cdot\|_{\mathbb{H}_1(\Omega)}$ implies that $\|\cdot\|_{\mathbb{H}_1(\Omega)} \asymp \|\cdot\|_{\mathbf{V}^{(3)}}$. So, we have that $\mathbf{V}^{(3)}$ equipped with $\|\cdot\|_{\mathbf{V}^{(3)}}$ is complete. [4] So $\mathbf{V}^{(3)}$ is a *Hilbert space* according to the inner product $(\cdot,\cdot)_{\mathbf{V}^{(3)}}$.

Theorem 5.4 *The space* $\mathbf{V}^{(3)}$ *is the completion of* $\widetilde{\mathbf{V}}^{(3)}$ *according to norm* $\|\cdot\|_{\mathbf{V}^{(3)}}$.

Proof
As was seen above, $\mathbf{V}^{(3)}$ is complete. $\widetilde{\mathbf{V}}^{(3)}$ is dense in $\mathbf{V}^{(3)}$ according to the norm $\|\cdot\|$ given by (5.9). See, [5] p.354 lemme 6.1.
Using the equivalence of both norms it follows that $\widetilde{\mathbf{V}}^{(3)}$ is dense in $\mathbf{V}^{(3)}$ according to both norms. $\square$

Because of the mean value theorem for harmonic functions $\underline{\mathrm{Harm}}^{(3)}(\Omega)$ is a closed subspace of $\mathbf{V}^{(3)}$. (see theorem 7.3 in chapter 7.)
In this subspace one can construct an orthogonal splitting, as follows from the next theorem:

Theorem 5.5 *Defining,*

$$\begin{array}{l} \mathcal{H}_0 = \{\Phi \in \underline{Harm}^{(3)}(\Omega) \mid div\,\Phi = 0\} \\ \mathcal{H}_1 = \mathcal{H}_0^{\perp} \end{array}$$

the following relations hold

1. $\mathcal{H}_0(\Omega) = \overline{\{h_0[\phi] := \boldsymbol{x} \times \nabla\phi - \nabla\mathcal{N}(0, (\boldsymbol{n} \times \boldsymbol{x}) \cdot \nabla\phi) \mid \Delta\phi = 0\}}$
2. $\mathcal{H}_1(\Omega) = \overline{\{\nabla\mathcal{N}(0; f) - [f\boldsymbol{n}]_{\mathcal{H}} \mid f \in \mathbb{L}_2(\partial\Omega)\}}$
3. $\mathcal{H}_1(\Omega) = \overline{\{h_1[\chi] := \nabla\chi - F[\chi]\boldsymbol{n}_{\mathcal{H}} - 2\,\mathcal{D}(\nabla \boldsymbol{n}_{\mathcal{H}}[\nabla F[\chi]]^T) \mid \Delta\chi = 0\}}$

with

$$\begin{array}{ll} \chi & = \mathcal{N}(0; f) \\ F[\chi] & = (f_{\mathcal{H}} =)\ \boldsymbol{n}_{\mathcal{H}} \cdot \nabla\chi + 2\,\mathcal{D}(trace\{[\nabla \boldsymbol{n}_{\mathcal{H}} Hessian(\chi)]\}) \Rightarrow \\ \nabla F[\chi] & = [I + 2\,\nabla\mathcal{D} div](\nabla\chi[\nabla \boldsymbol{n}_{\mathcal{H}}]^T) + \boldsymbol{n}_{\mathcal{H}}^T Hessian(\chi) \end{array} \tag{5.13}$$

[4] NB. There are *metric* spaces (M, d_1) and (M, d_2) in which d_1 and d_2 induce the same topology and (M, d_1) is complete and (M, d_2) is not. But in these cases the equivalence constant C depends on $x \in M$.

Proof

1. $\subset$

$\operatorname{div}(\mathbf{x}\times\nabla\phi)=0$, as follows directly from $\operatorname{div}(\mathbf{a}\times\mathbf{b})=\mathbf{a}\cdot\operatorname{rot}\mathbf{b}-\mathbf{b}\cdot\operatorname{rot}\mathbf{a}$. The extra term $\{[(\mathbf{n}\times\mathbf{x})\cdot\nabla\phi]\mathbf{n}\}_{\mathcal{H}}=\nabla\mathcal{N}(\mathbf{0},(\mathbf{n}\times\mathbf{x})\cdot\nabla\phi)$ compensates $(\mathbf{x}\times\nabla\phi)\cdot\mathbf{n}|_{\partial\Omega}$ at the boundary. Note that $\mathcal{N}(\mathbf{0},(\mathbf{n}\times\mathbf{x})\cdot\nabla\phi)$ is well defined:

$$\begin{aligned}\int_{\partial\Omega}\nabla\phi\cdot(\mathbf{n}\times\mathbf{x})d\sigma &= \int_{\partial\Omega}\mathbf{n}\cdot(\mathbf{x}\times\nabla\phi)d\sigma\\ &= \int_{\Omega}\operatorname{div}\ (\mathbf{x}\times\nabla\phi)d\sigma_{\mathbf{x}}=0\end{aligned}$$

1. $\supset$

Follow the proof from Theorem 3.1 ! Note that

$$\operatorname{rot}\nabla\mathcal{N}(\mathbf{0},(\mathbf{n}\times\mathbf{x})\cdot\nabla\phi)=\mathbf{0}$$
$$\operatorname{div}\nabla\mathcal{N}(\mathbf{0},(\mathbf{n}\times\mathbf{x})\cdot\nabla\phi)=0$$

So, again we have $\operatorname{rot}\mathbf{u}=\mathbf{0}$ and $\operatorname{div}\mathbf{u}=0$. Only Point 3. must now be modified into: $T_{\mathbf{n}}\mathbf{u}=0$, so we have $\mathbf{u}=\mathbf{0}$. (NB. Remind that $\|\cdot\|_{\mathbf{V}^{(3)}}$ is a norm on $\mathbf{V}^{(3)}$).

2. $\subset$

Let

$$\begin{aligned}\mathbf{v}&=\mathbf{x}\times\nabla\phi-\nabla\mathcal{N}(\mathbf{0},(\mathbf{n}\times\mathbf{x})\cdot\nabla\phi) &&\text{with } \Delta\phi=0\\ \mathbf{w}&=\nabla\mathcal{N}(\mathbf{0};\mathbf{f})-[f\mathbf{n}]_{\mathcal{H}} &&\text{with } f:\partial\Omega\mapsto\mathbb{R}\end{aligned}$$

Then

$$\begin{aligned}(\mathbf{v},\mathbf{w})_{V^{(3)}} &= \int_{\Omega}-\operatorname{rot}(\mathbf{x}\times\nabla\phi)\cdot\operatorname{rot}[f\mathbf{n}]_{\mathcal{H}}\,d\mathbf{x}\\ &= \int_{\Omega}\operatorname{div}(-\nabla(\mathcal{E}+I)\phi\times[f\mathbf{n}]_{\mathcal{H}})\,d\mathbf{x}\\ &= \int_{\partial\Omega}(-\nabla(\mathcal{E}+I)\phi\times[f\mathbf{n}]_{\mathcal{H}})\cdot\mathbf{n}\,d\mathbf{x}=0\end{aligned}$$

2. $\supset$

We will show that any element that is orthogonal to $\{\nabla\mathcal{N}(\mathbf{0};\mathbf{f})-[f\mathbf{n}]_{\mathcal{H}}\mid f\in\mathbb{L}_2(\partial\Omega)\}$ lies in $\mathcal{H}_0$.

First note that $\mathcal{H}_0$ and $\mathcal{H}_1$ are closed subspaces of a Hilbert space and therefore complete.

Let $\Phi\in\underline{\text{Harm}}^{(3)}(\Omega)$ such that it is orthogonal to $\{\nabla\mathcal{N}(\mathbf{0};\mathbf{f})-[f\mathbf{n}]_{\mathcal{H}}\mid f\in\mathbb{L}_2(\partial\Omega)\}$
Then for any $f\in\mathbb{L}_2(\partial\Omega)$ we have

$$\begin{aligned}&\int_{\Omega}\{\operatorname{div}\Phi\operatorname{div}[f\mathbf{n}]_{\mathcal{H}}+\operatorname{rot}\Phi\cdot\operatorname{rot}[f\mathbf{n}]_{\mathcal{H}}\}\,d\mathbf{x}=\\ &\int_{\Omega}\{-\nabla\operatorname{div}\Phi\operatorname{div}[f\mathbf{n}]_{\mathcal{H}}+\operatorname{div}\{\operatorname{div}\Phi[f\mathbf{n}]_{\mathcal{H}}\}\}\,d\mathbf{x}+\\ &\int_{\Omega}\{-\operatorname{div}\{[f\mathbf{n}]_{\mathcal{H}}\times\operatorname{rot}\Phi\}+\operatorname{rot}\operatorname{rot}\Phi\cdot[f\mathbf{n}]_{\mathcal{H}}\}\,d\mathbf{x}=\\ &\int_{\partial\Omega}\operatorname{div}\Phi f\,d\sigma_{\mathbf{x}}=0\end{aligned}$$

$\mathbb{L}_2(\partial\Omega)$ is a Hilbert space, so $\operatorname{div}\Phi=0$ in $\mathbb{L}_2(\partial\Omega)$. $\operatorname{div}\Phi$ is harmonic so $\operatorname{div}\Phi=0$ in $\mathbb{L}_2(\Omega)$. But then we have that $\Phi\in\mathcal{H}_0$

3. $=$

Set $\chi=\mathcal{N}(\mathbf{0};f)$. Then by definition $\frac{\partial\chi}{\partial\mathbf{n}}=f$ at $\partial\Omega$. So,

$$f_{\mathcal{H}}=\mathbf{n}_{\mathcal{H}}\cdot\nabla\chi+2\,\mathcal{D}([\nabla_{\mathcal{H}}]\text{Hessian}(\chi))$$

Use this together with

$$[f\mathbf{n}]_{\mathcal{H}}=f_{\mathcal{H}}\mathbf{n}_{\mathcal{H}}+2\,\mathcal{D}(\nabla\mathbf{n}_{\mathcal{H}}[\nabla f_{\mathcal{H}}]^T)$$

and the result is obtained form point 2 $\square$.

Note that

- For any $\Phi \in \mathcal{H}_1$ we have that

$$\left.\begin{array}{l}\operatorname{div}\Phi = -\operatorname{div}[f\mathbf{n}]_{\mathcal{H}} \\ \operatorname{rot}\Phi = -\operatorname{rot}[f\mathbf{n}]_{\mathcal{H}}\end{array}\right\} \Rightarrow \|\Phi\|_{\mathbf{V}^{(3)}} = \|[f\mathbf{n}]_{\mathcal{H}}\|_{\mathbf{V}^{(3)}} =^{5} \int_{\partial\Omega} f\operatorname{div}[f\mathbf{n}]_{\mathcal{H}} d\sigma_{\mathbf{x}} \tag{5.14}$$

  So,

$$\|\Phi\|_{\mathbf{V}^{(3)}} = \|[f\mathbf{n}]_{\mathcal{H}}\|_{\mathbf{V}^{(3)}} \leq \|f\|_{\partial\Omega}\|\operatorname{div}[f\mathbf{n}]_{\mathcal{H}}\|_{\partial\Omega} \tag{5.15}$$

- The following relations hold

$$\begin{array}{l}[f\mathbf{n}]_{\mathcal{H}} = f_{\mathcal{H}}\mathbf{n}_{\mathcal{H}} + 2\mathcal{D}([\nabla\mathbf{n}_{\mathcal{H}}](\nabla f_{\mathcal{H}})^T) \\ \operatorname{div}[f\mathbf{n}]_{\mathcal{H}} = \nabla f_{\mathcal{H}}\cdot\mathbf{n}_{\mathcal{H}} + f_{\mathcal{H}}\operatorname{div}\mathbf{n}_{\mathcal{H}} + 2\operatorname{div}\mathcal{D}([\nabla\mathbf{n}_{\mathcal{H}}](\nabla f_{\mathcal{H}})^T)\end{array} \tag{5.16}$$

- $\mathcal{H}_0$ and $\mathcal{H}_1$ are closed subspaces of a Hilbert space and therefore they are Hilbert spaces themselves.

- Although equality 3. looks quite tedious in the general case, it is quite useful in the ball-case. The main reason for this is that we have $\mathbf{n}_{\mathcal{H}} = \mathbf{x}$ in the ball case. If we take for instance a square $[-a,a]\times[-a,a]$ we have that:

$$\mathbf{n}_{\mathcal{H}} = \frac{1}{a\cosh(n\pi)}\begin{pmatrix} x\sin\left(\frac{n\pi y}{a}\right)\cosh\left(\frac{n\pi x}{a}\right) \\ y\sin\left(\frac{n\pi x}{a}\right)\cosh\left(\frac{n\pi y}{a}\right)\end{pmatrix}$$

  and things become rather complicated.

- In chapter 5.3.2 theorem 5.19 it will be shown that the above definition of $\mathbf{h}_1[\chi]$ corresponds to that one in chapter 3.

Let's derive an alternative expression for the inner product:

$$\begin{aligned}(\Phi,\Psi)^{\Omega}_{\mathbf{V}^{(3)}} &= \int_{\Omega} \operatorname{div}\Phi\operatorname{div}\Phi + \Phi\cdot\operatorname{rotrot}\Psi + \operatorname{div}\{\Phi\times\operatorname{rot}\Psi\}\, d\tau \\ &= -\int_{\Omega} \Phi\Delta\Psi + \operatorname{div}\{\operatorname{rot}\Psi\times\Phi\}\, d\tau \\ &= -\int_{\Omega} \Phi\Delta\Psi\, d\tau + \int_{\partial\Omega} (\operatorname{rot}\Psi\times\Phi)\cdot\mathbf{n}\, d\sigma\end{aligned} \tag{5.17}$$

This implies:

$$(\underline{\mathrm{Harm}}^{(3)}(\Omega))^{\perp} = \{\Phi\in\mathbf{V}^{(3)} | \forall_{\Psi\in\underline{\mathrm{Harm}}^{(3)}(\Omega)} : \int_{\partial\Omega} (\Phi\times\operatorname{rot}\Psi)\cdot\mathbf{n}\, d\sigma = 0\} \tag{5.18}$$

5.1.1 The space $\mathbb{L}_2(\Omega)/\mathbb{R}$

In Chapter 2 we have shown that p satisfies solving operator equation $\operatorname{div}\mathcal{D}\nabla p = \operatorname{div}\mathbf{a}_{\mathcal{H}}$. Note that the pressure p is determined up to a constant. So, in order to obtain a unique solution it is natural to take the quotient space $\mathbb{L}_2(\Omega)/\mathbb{R}$ as the domain on which $\operatorname{div}\mathcal{D}\nabla$ acts. Each element in $\mathbb{L}_2(\Omega)/\mathbb{R}$ can be represented by a function orthogonal to the constant-subspace, i.e. :

$$\mathbb{L}_2(\Omega)/\mathbb{R} \equiv \{p\in\mathbb{L}_2(\Omega) \mid (1,p)_{\mathbb{L}_2(\Omega)} = \int_{\Omega} p\, d\mathbf{x} = 0\} \tag{5.19}$$

[5] To obtain this equality one must apply respectively $\operatorname{div}(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\operatorname{rot}\mathbf{a} - \mathbf{a}\cdot\operatorname{rot}\mathbf{b}$, $\operatorname{rotrot} = \nabla\operatorname{div} - \Delta$ and $\operatorname{div}\phi\mathbf{v} = \phi\operatorname{div}\mathbf{v} + \nabla\phi\cdot\mathbf{v}$

We equip $\mathbb{L}_2(\Omega)/\mathbb{R}$ with the restriction of $(\cdot,\cdot)_{\mathbb{L}_2(\Omega)}$ to $\mathbb{L}_2(\Omega)/\mathbb{R} \times \mathbb{L}_2(\Omega)/\mathbb{R}$. Then $\mathbb{L}_2(\Omega)/\mathbb{R}$ is by (5.19) a closed subspace of a Hilbert space and is therefore a Hilbert space itself.

Although that in the case $h = 0$, $\mathbf{f} = \mathbf{0}$ we have that p is harmonic we are not only interested in the harmonic subspace $\mathbb{L}_2^{\text{harm}}(\Omega)$. The reason for this is mainly a geometrical one. It will turn out that the space orthogonal to $\mathbb{L}_2^{\text{harm}}(\Omega)$ is the eigen space according to eigenvalue -1 of div$\mathcal{D}\nabla$.

Theorem 5.6 *Let Ω be a Lipschitz open bounded set in $\mathbb{R}^n$. Then the divergence operator maps $\overset{\circ}{\mathbf{H}}_1(\Omega)$ onto the space $\mathbb{L}_2(\Omega)/\mathbb{R}$*

Proof
As we will show in theorem (5.16) operator div$\mathcal{D}$grad is an isomorphism from $\mathbb{L}_2(\Omega)/\mathbb{R}$ onto itself. Since $\mathcal{R}(\mathcal{D}\text{grad} \subset \overset{\circ}{\mathbf{H}}_1(\Omega)$, we have that div : $\overset{\circ}{\mathbf{H}}_1(\Omega) \to \mathbb{L}_2(\Omega)/\mathbb{R}$ is surjective. □

Define

$$\mathbf{T}^{(3)} = \{\Phi \in V^{(3)} \mid T_0\Phi = 0\} \tag{5.20}$$

Then $\mathbf{T}^{(3)}$ is a closed linear subspace of $(\underline{\text{Harm}}^{(3)}(\Omega))^{\perp}$ as follows from (5.17). Since $\mathcal{N}(T_0) \subset \mathcal{N}(T_{\mathbf{n}})$, we must have $\mathbf{T}^{(3)} = \overset{\circ}{\mathbf{H}}_1(\Omega)$!

The next theorem shows that $\overset{\circ}{\mathbf{H}}_1(\Omega)$ is in fact the entire space $(\underline{\text{Harm}}^{(3)}(\Omega))^{\perp}$.

Theorem 5.7 *The space $(\underline{Harm}^{(3)}(\Omega))^{\perp}$ equals $\overset{\circ}{\mathbf{H}}_1(\Omega)$.*

Proof
Equation (5.18) implies $(\underline{\text{Harm}}^{(3)}(\Omega))^{\perp} \supset \{\Phi \in \mathbf{V}^{(3)} \mid T_{\mathbf{n}}\Phi = 0\}$. It remains to be shown that $\underline{\text{Harm}}^{(3)}(\Omega) \oplus \overset{\circ}{\mathbf{H}}_1(\Omega) = \mathbf{V}^{(3)}$.
Denote the harmonic subspace of $\mathbf{H}_1(\Omega)$ in $\mathbf{H}_1(\Omega)$ with $\mathbf{H}_1^{\text{Harm}}(\Omega)$. Next we show that

$$\left(\mathbf{H}_1^{\text{Harm}}(\Omega)\right)^{\perp} = \overset{\circ}{\mathbf{H}}_1(\Omega)$$

Let $\mathbf{h}, \mathbf{g} \in \mathbf{H}^1(\Omega)$, with $\Delta\mathbf{h} = \mathbf{0}$. Then by using Greens first identity

$$\begin{aligned}
&(\mathbf{h},\mathbf{g})_{\mathbf{H}^1(\Omega)} = 0 \Leftrightarrow \\
&(\int_\Omega \left[\sum_{k=1}^{n} \nabla h_k \cdot \nabla g_k\right] d\mathbf{x} = \\
&\int_{\partial\Omega} \sum_{k=1}^{n} T_0(g_k)\, T_{\mathbf{n}}(\nabla h_k)\, d\sigma_{\mathbf{x}}
\end{aligned}$$

The vector function $\mathbf{g}$ has to be orthogonal to all possible $\mathbf{h} \in \mathbf{H}^1(\Omega)$ which are harmonic. In particular to $\mathbf{h} = \mathcal{N}(\mathbf{0},\mathbf{f})$, with $\mathbf{f}$ an arbitrary element of $\mathbf{L}_2(\partial\Omega)$. The completeness of $\mathbf{L}_2(\partial\Omega)$ gives us that the restriction of $\mathbf{g}$ to the boundary *must* be zero. $\mathbf{H}_0^1(\Omega)$ is the kernel of the continuous operator $T_0 : \mathbf{H}^1(\Omega) \mapsto W_2^{\frac{1}{2}}(\partial\Omega)$ and therefore closed. So, $\mathbf{H}^1(\Omega) = \mathbf{H}_0^1(\Omega) \oplus \{\mathbf{h} \in \mathbf{H}^1(\Omega) \mid \Delta\mathbf{h} = \mathbf{0}\}$ and from this we conclude that $\overset{\circ}{\mathbf{H}}_1(\Omega) \oplus \underline{\text{Harm}}^{(3)}(\Omega) = \mathbf{V}^{(3)}$ □.

Combine Theorem 5.6 with the above theorem, then div : $(\underline{\text{Harm}}^{(3)}(\Omega))^{\perp} \mapsto \mathbb{L}_2(\Omega)$ is surjective .

Next, we will examine the surjectivity of div more general :

Let $g \in C(\Omega)$, not constant, then $\operatorname{div}\Phi = g$ if $\Phi = -(\nabla\mathcal{N}(g))$. Note that $\Phi \in C^1(\Omega)$ is indeed tangent at the boundary and that $\Delta\Phi = 0 \Rightarrow \Delta g = 0$. Note that $C(\Omega)$ is dense in $\mathbb{L}_2(\Omega)/_\mathbb{R}$ and that the tangent vector functions of $\mathbf{C}^1(\Omega)$ form a dense subset in $\mathbf{V}^3$. See, Duvaut and Lions [5] p.354 Lemme 6.1.
Therefore, div : $\mathbf{V}^3 \mapsto \mathbb{L}_2(\Omega)/_\mathbb{R}$ is surjective. Theorem 5.6 has already shown this, but from the above it follows that we don't necessarily take our originals from $(\underline{\text{Harm}}^{(3)}(\Omega))^\perp$. The next theorem shows that every element p in $\mathbb{L}_2^{\text{harm}}(\Omega)/_\mathbb{R}$ can be written as $p = \operatorname{div}\Phi$ with $\Phi \in \underline{\text{Harm}}^{(3)}(\Omega)$.

Theorem 5.8 *The divergence operator div :* $\underline{\mathit{Harm}}^{(3)}(\Omega) \mapsto \overset{harm}{\mathbb{L}_2(\Omega)}/_\mathbb{R}$ *is surjective.*

Proof
Use the orthogonal splitting introduced in 5.33, further on in this report. Above we have seen that div : $\mathbf{V}^{(3)} \mapsto \mathbb{L}_2(\Omega)/_\mathbb{R}$ is surjective, so we must show that for all $\Phi \in (\underline{\text{Harm}}^{(3)}(\Omega))^\perp$ with $\operatorname{div}\Phi \neq 0$ there exists a $\Psi \in \underline{\text{Harm}}^{(3)}(\Omega)$ such that $\operatorname{div}\Phi = \operatorname{div}\Psi$.
According to (5.32) $(\underline{\text{Harm}}^{(3)}(\Omega))^\perp$ can be split: $(\underline{\text{Harm}}^{(3)}(\Omega))^\perp = \mathcal{R}(\mathcal{C}) \oplus \mathcal{N}(\mathcal{C})$. Of course, we are only interested in elements in $(\underline{\text{Harm}}^{(3)}(\Omega))^\perp$ which have non-zero divergence. According to (5.32) they are exactly the elements of the image space of $\mathcal{C}$.
Let $\Phi \in \underline{\text{Harm}}^{(3)}(\Omega)$. Observe $\mathcal{C}\Phi$. We must now look for an element in $\underline{\text{Harm}}^{(3)}(\Omega)$ such that its divergence equals $\operatorname{div}\mathcal{C}\Phi$. It follows from (5.3) that this element is given by $-(\mathcal{B} + I)\Phi$, which is indeed harmonic if Φ is harmonic. □

5.2 Operator $\mathcal{B} = (\text{grad}\mathcal{N} - \mathcal{D}\text{grad})\text{div}$

Operator $\mathcal{B}$ acts on $\mathbf{V}^{(3)}$. Especially, we are interested how it acts on the harmonic subspace $\underline{\text{Harm}}^{(3)}(\Omega)$, since in operator equation (2.16) $\mathcal{B}$ works on $\alpha_\mathcal{H}$ which is both harmonic and tangent. Note that $\Delta\mathcal{D} = \Delta\mathcal{N} = -I$ implies that $\mathcal{B}$ maps $\mathbf{V}^{(3)}$ into $\underline{\text{Harm}}^{(3)}(\Omega)$.

First, we observe some simple but fundamental equations. One can easily verify that

$$\begin{aligned} &\mathcal{B} = \mathcal{F} - \mathcal{C} \,, \\ &\mathcal{F}^2 = -\mathcal{F} \,, \\ &\mathcal{C}\mathcal{F} = -\mathcal{C} \text{ and} \\ &\mathcal{C}\mathcal{B} = -\mathcal{C}(\mathcal{C}+1) \end{aligned} \tag{5.21}$$

from which follows that

$$\begin{aligned} &\mathcal{F}\mathbf{f} = -\mathbf{f} &&\Rightarrow \mathcal{B}\mathbf{f} = -\mathbf{f} \,, \\ &\mathcal{B}\mathbf{f} = \mu\mathbf{f} &&\Rightarrow \mathcal{C}(\mathcal{C}\mathbf{f}) = -(\mu+1)\mathcal{C}\mathbf{f} \,, \\ &-\sigma(\mathcal{B}+I) = -\sigma(\mathcal{B}) - 1 \subset \sigma(\mathcal{C}) \,, \\ &\sigma(\mathcal{F}) = \{0, -1\} \,. \end{aligned} \tag{5.22}$$

Theorem 5.9 *Operator* $\mathcal{B} = (\nabla\mathcal{N} - \mathcal{D}\nabla)\nabla\cdot$ *is self adjoint, negative semi definite*[6] *and bounded on* $\underline{\mathit{Harm}}^{(3)}(\Omega)$ *according to inner product* $(\cdot,\cdot)_{\mathbf{V}^{(3)}}$.

[6] negative definite on $\mathcal{H}_1$

Proof
Let Φ and $\Psi \in \mathbf{V}^{(3)}$. Straightforward derivations yield :

$$\begin{aligned}
(\mathcal{B}\Phi,\Psi)_{\mathbf{V}^{(3)}} &= \int_\Omega \operatorname{div}\{(\nabla\mathcal{N}-\mathcal{D}\nabla)\nabla\cdot\Phi\ \}\operatorname{div}\Phi - \operatorname{rot}\{\mathcal{D}\nabla\nabla\cdot\Phi\}\cdot\operatorname{rot}\Psi d\tau \\
&= -\int_\Omega \operatorname{div}\Phi\operatorname{div}\Psi + \operatorname{div}\{\mathcal{D}\nabla(\nabla\cdot\Phi)\}\operatorname{div}\Psi + \mathcal{D}\nabla(\nabla\cdot\Phi)\cdot\operatorname{rotrot}\Psi d\tau \\
&= -\int_\Omega \operatorname{div}\Phi\operatorname{div}\Psi - (\mathcal{D}\nabla\nabla\cdot\Phi)\cdot\Delta\Psi d\tau \\
&= -(\operatorname{div}\Phi,\operatorname{div}\Psi)_{\mathbf{L}_2(\Omega)} + \int_\Omega \mathcal{C}\Phi\cdot\Delta\Psi d\tau
\end{aligned} \tag{5.23}$$

Some remarks according to the second equality:

- $\operatorname{div}(\mathbf{v}\times\mathbf{w}) = \mathbf{w}\cdot\operatorname{rot}\mathbf{v} - \mathbf{v}\cdot\operatorname{rot}\mathbf{w}$
- Choose $\mathbf{v} = \operatorname{rot}\Psi$ and $\mathbf{w} = \mathcal{D}\nabla\operatorname{div}\Phi$
- Use Gauss divergence theorem and the fact that $\mathcal{D}f$ is zero at the boundary.

Some remarks according to the third equality:

- $\operatorname{rotrot} = \operatorname{graddiv} - \Delta$
- $\operatorname{div}(f\mathbf{v}) = f\operatorname{div}\mathbf{v} + \nabla f\cdot\mathbf{v}$
- Choose $\mathbf{v} = \mathcal{D}\nabla\operatorname{div}\Phi$ and $f = \operatorname{div}\Psi$
- Again, use Gauss divergence theorem and the fact that $\mathcal{D}f$ is zero at the boundary.

The final result is now obvious. □

Further remarks about and identities with operator $\mathcal{B}$:

- The following equality holds

$$((I+\mathcal{B})\Phi,\Psi)_{V^{(3)}} = \int_\Omega \operatorname{rot}\Phi\cdot\operatorname{rot}\Psi + \mathcal{C}\Phi\cdot\Delta\Psi\, d\tau\,. \tag{5.24}$$

- Let $p, q \in \operatorname{Harm}(\Omega)$ and $\Phi,\Psi \in \underline{\operatorname{Harm}}^{(n)}(\Omega)$ such that $p = \operatorname{div}\Phi$ and $q = \operatorname{div}\Psi$. Then by (5.24) and (5.3) we obtain

$$\begin{aligned}
&(\nabla\cdot\mathcal{D}\nabla p, q)_{\mathbf{L}_2(\Omega)} = \\
&\textstyle\int_\Omega \operatorname{rot}\mathcal{B}\Phi\cdot\operatorname{rot}\Psi d\tau = \\
&((\mathcal{B}+I)\mathcal{B}\Phi,\Psi)_{\mathbf{V}^{(3)}}\ .
\end{aligned} \tag{5.25}$$

- Let Φ and $\Psi \in \mathbf{V}^{(3)}$ not necessarily harmonic then:

$$\begin{aligned}
&(\mathcal{B}(I+\mathcal{B})\Phi,\Psi)_{\mathbf{V}^{(3)}} = \\
&\int_\Omega \operatorname{rot}\mathcal{B}\Phi\cdot\operatorname{rot}\Psi + \mathcal{C}\mathcal{B}\Phi\cdot\Delta\Psi d\tau = \\
&\textstyle\int_\Omega \operatorname{rot}\mathcal{B}\Phi\cdot\operatorname{rot}\Psi - (\mathcal{C}+I)\mathcal{C}\Phi\cdot\Delta\Psi d\tau
\end{aligned} \tag{5.26}$$

  and further,

$$(\nabla\cdot\mathcal{D}\nabla p, q)_{\mathbf{L}_2(\Omega)} = ((\mathcal{B}+I)\mathcal{B}\Phi,\Psi)_{\mathbf{V}^{(3)}} + \int_\Omega \mathcal{C}^2\Phi\cdot\Delta\Psi\, d\tau\,. \tag{5.27}$$

- Using Green's second identity one can simplify (5.23) to:

$$(\mathcal{B}\Phi, \Psi)_{\mathbf{V}^{(3)}} = -\int\limits_{\partial\Omega} \Psi \cdot \frac{\partial \mathcal{C}\Phi}{\partial \mathbf{n}}\, d\sigma = -\int\limits_{\partial\Omega} \sum_{k=1}^{n} \psi_k \frac{\partial \mathcal{C}\phi_k}{\partial \mathbf{n}}\, d\sigma \tag{5.28}$$

- From Theorem 5.9 we obtain a simple expression for the operator norm of $\mathcal{B}$:

$$\|\mathcal{B}\|_{V^{(3)}} = \sup_{\Phi \in \underline{\mathrm{Harm}}^{(3)}(\Omega)} \frac{1}{1+\frac{\|\mathrm{rot}\,\Phi\|^2_{\mathbf{L}_2(\Omega)}}{\|\mathrm{div}\,\Phi\|^2_{\mathbf{L}_2(\Omega)}}} \tag{5.29}$$

Theorem 5.10 *Operator $\mathcal{F} = grad\mathcal{N} div$ is self adjoint on $\mathbf{V}^{(3)}$, operator $\mathcal{C} = \mathcal{D}grad\, div$ is self adjoint on $(\underline{Harm}^{(3)}(\Omega))^{\perp}$ and for all $\Phi \in \mathbf{V}^{(3)}$*

$$\|\mathcal{F}\Phi\|^2 = \|\mathcal{B}\Phi\|^2 + \|\mathcal{C}\Phi\|^2 \,. \tag{5.30}$$

Proof
Let $\Phi, \Psi \in \mathbf{V}^{(3)}$.
Then easy and direct calculations show that

$$(\mathcal{F}\Phi, \Psi)_{\mathbf{V}^{(3)}} = -(\mathrm{div}\,\Phi, \mathrm{div}\,\Psi)_{\mathbf{L}_2(\Omega)} = (\Phi, \mathcal{F}\Psi)_{\mathbf{V}^{(3)}}$$

Equality 5.30 is Pythagoras, which can be applied because of

$$\begin{array}{ll} (\mathcal{C}\Phi, \Psi) = 0 & \text{if } \Delta\Psi = 0 \\ (\mathcal{C}\Phi, \mathcal{B}\Psi) = 0 & \text{for all } \Phi, \Psi \in V \end{array} \tag{5.31}$$

Let $\Phi, \Psi \in (\underline{\mathrm{Harm}}^{(3)}(\Omega))^{\perp}$.
Then

$$(\mathcal{C}\Phi, \Psi) = (\mathcal{F}\Phi, \Psi) - (\mathcal{B}\Phi, \Psi) = (\mathcal{F}\Phi, \Psi)$$

So from the fact that $\mathcal{F}$ is (in particular) self-adjoint on $(\underline{\mathrm{Harm}}^{(3)}(\Omega))^{\perp}$ it follows by the above equality that $\mathcal{C}$ is self-adjoint on $(\underline{\mathrm{Harm}}^{(3)}(\Omega))^{\perp}$ □.

Further on in this report we will show that the image space $\mathcal{R}(\mathcal{C})$ is closed. However, this results in a orthogonal splitting in $(\underline{\mathrm{Harm}}^{(3)}(\Omega))^{\perp} = \overset{\circ}{\mathbf{H}}_1(\Omega)$:

$$\underline{\mathrm{Harm}}^{(3)}(\Omega)^{\perp} = \mathcal{N}(\mathcal{C}) \oplus \mathcal{R}(\mathcal{C}) \,. \tag{5.32}$$

The nil space of $\mathcal{C}$, $\mathcal{N}(\mathcal{C})$ equals $\{\Phi \in (\underline{\mathrm{Harm}}^{(3)}(\Omega))^{\perp} \mid \mathrm{div}\,\Phi = 0\}$, since for all $\Phi \in (\underline{\mathrm{Harm}}^{(3)}(\Omega))^{\perp}$:
$\mathcal{C}\Phi = 0 \Rightarrow \nabla\mathrm{div}\,\Phi = 0 \Rightarrow \mathrm{div}\,\Phi = \lambda$, for some constant $\lambda \in \mathbb{R}$. It follows by

$$\int\limits_{\Omega} \mathrm{div}\,\Phi\, d\mathbf{x} = \int\limits_{\partial\Omega} \Phi \cdot \mathbf{n}\, d\sigma_{\mathbf{x}} = 0$$

that $\mathrm{div}\,\Phi = \lambda = 0$.

An orthogonal splitting for the space $\mathbf{V}^{(3)}$ is now obtained:

Theorem 5.11 *The space $\mathrm{V}^{(3)}$ can be split*

$$\mathbf{V}^{(3)} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{L}_0 \oplus \mathcal{L}_1 \tag{5.33}$$

In which

$$\begin{array}{l} \mathcal{H}_0 = \{\Phi \in \underline{Harm}^{(3)}(\Omega) \mid div\,\Phi = 0\} = \mathcal{N}(\mathcal{B}) \\ \mathcal{H}_1 = \mathcal{H}_0^{\perp} = \mathcal{R}(\mathcal{B}) \\ \mathcal{L}_0 = \{\Phi \in (\underline{Harm}^{(3)}(\Omega))^{\perp} \mid div\,\Phi = 0\} = \mathcal{N}(\mathcal{C}) \\ \mathcal{L}_1 = \mathcal{L}_0^{\perp} = \mathcal{R}(\mathcal{C}) \end{array}$$

Proof
The splitting in $(\underline{\text{Harm}}^{(3)}(\Omega))^\perp$ is already treated.
It remains to show that

1. $\mathcal{N}(\mathcal{B}) = \mathcal{H}_0$
2. $\mathcal{R}(\mathcal{B}) = \mathcal{H}_1$

Remind that for any pair Φ, $\Psi \in \underline{\text{Harm}}^{(3)}(\Omega)$:

$$(\mathcal{B}\Phi, \Psi)_{\mathbf{V}^{(3)}} = (\mathcal{F}\Phi, \Psi)_{\mathbf{V}^{(3)}}$$

So if $\mathcal{B}$ and $\mathcal{F}$ are restricted to $\underline{\text{Harm}}^{(3)}(\Omega)$, the following relation holds:[7]

$$\begin{aligned}\mathcal{N}(\mathcal{B}) &= \mathcal{R}^\perp(\mathcal{B}) \cap \underline{\text{Harm}}^{(3)}(\Omega) \\ &= \mathcal{R}^\perp(\mathcal{F}) \cap \underline{\text{Harm}}^{(3)}(\Omega) \\ &= \mathcal{N}(\mathcal{F}) \cap \underline{\text{Harm}}^{(3)}(\Omega) = \mathcal{H}_0\end{aligned}$$

From this it follows that $\mathcal{H}_1 = \overline{\mathcal{R}(\mathcal{B})}$.
According to Theorem 5.12 we have that $\mathcal{F}$ is an orthogonal projection on $\mathcal{H}_1 \oplus \mathcal{L}_1$ and by $\mathcal{R}(-\mathcal{F}) = \mathcal{N}(I + \mathcal{F})$ one easily sees that $\mathcal{R}(\mathcal{F})$ is closed. $\mathcal{R}(\mathcal{B})$ equals the range of the orthogonal projection of $\mathcal{R}(\mathcal{F})$ on the closed subspace $\underline{\text{Harm}}^{(3)}(\Omega)$ and is therefore also closed. So,

$$\mathcal{H}_1 = \overline{\mathcal{R}(\mathcal{B})} = \mathcal{R}(\mathcal{B}) \qquad \square$$

Using the above splitting and the surjective theorem (5.8) a correspondence between $\text{div}\mathcal{D}\text{grad} : \mathbb{L}_2^{\text{harm}}(\Omega)/\mathbb{R} \mapsto \mathbb{L}_2^{\text{harm}}(\Omega)/\mathbb{R}$ and $B + I : \underline{\text{Harm}}^{(3)}(\Omega) \mapsto \underline{\text{Harm}}^{(3)}(\Omega)$ becomes clear:

$$\Phi \in \mathcal{H}_1 \leftrightarrow p \in \mathbb{L}_2(\Omega)/\mathbb{R}$$

The correspondence is unique and is given by $-\text{div}\,\Phi = p$. One can easily verify that the following relation holds:

$$\Phi \leftrightarrow p \Rightarrow (B+I)\Phi \leftrightarrow \text{div}\mathcal{D}\nabla p \tag{5.34}$$

Therefore we write:

$$(B+I) \leftrightarrow \text{div}\mathcal{D}\nabla \tag{5.35}$$

Consequently we have that

$$\sigma(B+I) = \{1\} \cup -\sigma(\text{div}\mathcal{D}\nabla) \tag{5.36}$$

The mapping $-\text{div} : \mathcal{H}_1 \mapsto \mathbb{L}_2^{\text{harm}}(\Omega)/\mathbb{R}$ is continuous, but its inverse is probably *not.* Equation (5.14) and estimate (5.15) give an indication for this negative assumption. (NB. There is no trace operator from $\mathbb{L}_2(\Omega)$ to $\mathbb{L}_2(\partial\Omega)$)

Using Theorem (5.6) one can do the same as above with $\mathcal{L}_1$ and $\mathbb{L}_2^{\text{harm}}(\Omega)/\mathbb{R}$. The following theorem reveals some geometrical aspects of our structure.

Theorem 5.12 *Operator $-\mathcal{F} = -\text{grad}\mathcal{N}\text{div}$ is the orthogonal projection on $\mathcal{H}_1 \oplus \mathcal{L}_1$ along $\mathcal{H}_0 \oplus \mathcal{L}_0$. Operator $(I + \mathcal{F}) = (I + \text{grad}\mathcal{N}\text{div})$ is the orthogonal projection operator on $\mathcal{H}_0 \oplus \mathcal{L}_0$ along $\mathcal{H}_1 \oplus \mathcal{L}_1$. The norm $\|\cdot\|_{\mathbf{V}^{(3)}}$ can be written as*

$$\|\Phi\|^2_{\mathbf{V}^{(3)}} = \|\mathcal{F}\Phi\|^2_{\mathbf{V}^{(3)}} + \|(I+\mathcal{F})\Phi\|^2_{\mathbf{V}^{(3)}}$$

[7]Remind that B is self adjoint on $\underline{\text{Harm}}^{(3)}(\Omega)$ and $\mathcal{F}$ is self adjoint on $\mathbf{V}^{(3)}$

which is in fact pythagoras.
The operators $\pm 2(\mathcal{F}+\frac{1}{2}I)$ are orthogonal and symmetric. They have the property

$$\begin{aligned} div \pm 2(\mathcal{F}+\tfrac{1}{2}I)\Phi &= \mp div\,\Phi \\ rot \pm 2(\mathcal{F}+\tfrac{1}{2}I)\Phi &= \pm rot\,\Phi \end{aligned} \tag{5.37}$$

Proof

$$\begin{array}{ll} (-\mathcal{F})^2 = -\mathcal{F} & \Rightarrow (I+\mathcal{F})^2 = (I+\mathcal{F}) \\ -\mathcal{F}^* = -\mathcal{F} & \Rightarrow (I+\mathcal{F})^* = I+\mathcal{F} \\ \mathcal{N}(-\mathcal{F}) = \{\Phi \in \mathbf{V}^{(3)} \mid \operatorname{div}\Phi = 0\} & \text{and } \mathcal{R}(\mathcal{F}+I) = \mathcal{N}(-\mathcal{F}) \end{array}$$

$-\mathcal{F}^2 = -\mathcal{F} \Rightarrow (\pm(2\mathcal{F}+I))^2 = I$. Finally note that

$$\begin{aligned} \|\mathcal{F}\Phi\|^2 &= \int_\Omega (\operatorname{div}\Phi)^2 d\mathbf{x} \\ \|I+\mathcal{F}\Phi\|^2 &= \int_\Omega \|\operatorname{rot}\Phi\|^2 d\mathbf{x} \end{aligned}$$

The rest is straightforward. □

5.3 Operator div$\mathcal{D}$grad

Before we examine operator div$\mathcal{D}$grad : $\mathbb{L}_2(\Omega)/\mathbb{R} \to \mathbb{L}_2(\Omega)/\mathbb{R}$ we describe an abstract mathematical structure, which turns out to be a generalization of the composed mapping div$\mathcal{D}$grad.

Theorem 5.13 *(Lax-Milgram) Let $a(\cdot,\cdot)$ be a coërcieve bounded bilinear functional on a Hilbert space H. Let G be a continuous linear functional on H, then there exists a unique $u \in H$ such that $a(u,\cdot) = G(\cdot)$.*

Proof
The mapping $y \mapsto a(x,y)$, with $x \in H$ fixed, is a continuous linear functional on H. Using Riesz' theorem it follows that there exists a unique $v_x \in H$ such that $a(x,y) = (v_x, y)$. Moreover, the operator $x \mapsto v_x$ is bounded on H. So there exists a unique $A \in B(H)$ such that $a(u,\cdot) = (Au,\cdot)$
Since G is assumed to be a continuous linear functional on H it follows again by Riesz' Theorem that there exists a unique $w \in H$ such that $G(\cdot) = (w,\cdot)$.
So, it remains te be proven that

$$\forall_{w \in H} \exists! u \in H[Au = w]$$

or shortly, A is a bijection.

- A is injective
  since, by coërcivity of a it follows
  $\exists_{K>0}\forall_{x\in H}[K\|x\|^2 \le |a(x,x)| = |(Ax,x)| \le \|Ax\|\|x\|$
- A is surjective,
  this is by some steps,
  - First we show that $R(A)$ is closed:
    Let $\{x_n\} \subset H$ and $y \in H$ such that $Ax_n \mapsto y$.
    Now, by $\|x_n - x_m\| \le \frac{1}{K}\|A(x_n - x_m)\|$,
    H is a Hilbert space and A is continuous it follows that $y \in R(A)$.
  - Next we show $R^\perp(A) = 0$:
    Suppose $\exists_{z \in H}[z \in R^\perp(A)$.]
    Then in particular $(Az, z) = 0$.
    So, together with $|(Az,z)| = |a(z,z)| \ge K(z,z)$ this implies $z = 0$. □

Theorem 5.14 *Let H be a Hilbert space with respect to inner product $(\cdot,\cdot)_H$. Let V be a normed linear space with inner product $(\cdot,\cdot)_V$ and let V' be its dual. Let there be a unitary operator $\mathcal{R} : V' \to V$ such that every continuous linear functional can be written*[8]

$$\langle F, \phi \rangle_{V'V} = (RF, \phi)_V.$$

Let there be a bounded linear operator $\mathcal{P} : H \to V'$ such that

$$\|\mathcal{P}f\|_{V'} \geq c\|f\|_H \text{ for all } f \text{ in } H \text{ for a certain } c > 0.$$

Let $\mathcal{P}'$ be the dual operator of $\mathcal{P}$.
Then we have that the composed operator $\mathcal{P}'\mathcal{R}\mathcal{P} : H \to H$ satisfies

$$(f, \mathcal{P}'\mathcal{R}\mathcal{P}f)_H = (\mathcal{P}f, \mathcal{P}f)_{V'} \geq c^2\|f\|_H^2.$$

Therefore, by Lax-Milgram's theorem, operator $\mathcal{P}'\mathcal{R}\mathcal{P}$ has a bounded inverse $(\mathcal{P}'\mathcal{R}\mathcal{P})^{-1}$ and the solution of the operator equation $\mathcal{P}'\mathcal{R}\mathcal{P}f = g$ can be written $f = (\mathcal{P}'\mathcal{R}\mathcal{P})^{-1}g$.

Proof
First note that by definition of the dual operator we have (after identification of H' with H)

$$\langle \mathcal{P}f, \phi \rangle_{V'V} = \langle f, \mathcal{P}'\phi \rangle_{HH} = (f, \mathcal{P}'\phi)_H \quad \text{for all } f \in H \text{ and } \phi \in V.$$

Let $f, g \in H$. Then we have

$$\begin{aligned} (g, \mathcal{P}'\mathcal{R}\mathcal{P}f)_H &= \langle \mathcal{P}g, \mathcal{R}\mathcal{P}f \rangle_{V'V} \\ &= (\mathcal{R}\mathcal{P}g, \mathcal{R}\mathcal{P}f)_V \\ &= (\mathcal{P}g, \mathcal{P}f)_{V'} \ . \end{aligned} \tag{5.38}$$

If we combine the above with the assumption on $\mathcal{P}$ we obtain

$$(f, \mathcal{P}'\mathcal{R}\mathcal{P})_H = (\mathcal{P}f, \mathcal{P}f)_{V'} \geq c^2\|f\|_H^2$$

Further, note that $\mathcal{R}$ is unitary and therefore bounded and $\mathcal{P}$ is bounded implies that $\mathcal{P}'$ is bounded. Therefore the composite mapping $\mathcal{P}'\mathcal{R}\mathcal{P}$ is bounded. The rest follows by theorem 5.13. □

Example:
Let $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear operator. Then $\mathcal{A}$ is injective if and only if there exists a positive constant C such that $\|A\mathbf{x}\| \geq C\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$. (NB. Denote the left inverse of $\mathcal{A}$ by $\mathcal{A}^{-1}$ then $\|\mathbf{x}\| = \|\mathcal{A}^{-1}\mathcal{A}\mathbf{x}\| \leq \|A^{-1}\|\|A\mathbf{x}\|$)
Taking $\mathcal{P} = \mathcal{A}$, $\mathcal{R} = \mathcal{I}_m$ the identity mapping on $\mathbb{R}^n$. Then by the above theorem we have that $\mathcal{A}$ is injective implies that $A^T A$ is bijective.

Next we apply theorem 5.14 to operator div$\mathcal{D}\nabla$. We will need the following theorem

Theorem 5.15

1. *Let Ω be a bounded Lipschitz open set in $\mathbb{R}^n$.*
   If a distribution p has all his first derivatives $D_i\, p \quad 1 \leq i \leq n$ in $\mathbb{L}_2(\Omega)$, then $p \in \mathbb{L}_2(\Omega)$ and

   $$\|p\|_{\mathbb{L}_2(\Omega)/\mathbb{R}} \leq c(\Omega)\|grad\, p\|_{\mathbb{L}_2(\Omega)}$$

[8] Note that if V is a Hilbert space with respect to inner product $(\cdot,\cdot)$, then $\mathcal{R}$ is the Riesz mapping

2. *If a distribution p has all its first derivatives $D_i\, p$, , $1 \le i \le n$, in $\mathbb{H}_{-1}(\Omega)$, then $p \in \mathbb{L}_2(\Omega)$ and*

$$||p||_{\mathbb{L}_2(\Omega)/\mathbb{R}} \le c(\Omega)||grad\, p||_{\mathbf{H}_{-1}(\Omega)}$$

Proof

The first point is proved in Deny & Lions [13] for a bounded star-shaped open set Ω. In our case , because of this result, p is $\mathbb{L}_2$ on every sphere contained in Ω with its closure. A bounded Lipschitz open set is "locally star-shaped". This means that each point $x_{j \in \partial\Omega}$, has an open neighborhood $\mathcal{O}_j$, such that $\mathcal{O}'_j = \Omega \cap \mathcal{O}_j$ is star- shaped with respect to one of its points. On all these sets $\mathcal{O}'_j$ we have that $p \in \mathbb{L}_2(\mathcal{O}'_j)$. Since a finite number of these balls and sets $\mathcal{O}'_j$ cover Ω the result follows.

For a proof of the second statement, see J.Nečas [6] □

Theorem 5.16 *Let Ω be bounded Lipschitz open set. Operator divDgrad : $\mathbb{L}_2(\Omega)/_{\mathbb{R}} \mapsto \mathbb{L}_2(\Omega)/_{\mathbb{R}}$ is bijective. Its inverse is bounded. i.e. divDgrad is an isomorphism. Therefore the solution of operator equation*

$$div\mathcal{D}grad\; p = div\, a_{\mathcal{H}} + div\mathcal{D}\boldsymbol{f} + h$$

is (unique) in $\mathbb{L}_2(\Omega)$
if $div\, a_{\mathcal{H}}$ and $h \in \mathbb{L}_2(\Omega)$ and if $\boldsymbol{f} \in \mathbf{H}^{-1}$.
The solution is given by

$$p = (div\mathcal{D}grad)^{-1}\; [div\, a_{\mathcal{H}} + div\mathcal{D}\boldsymbol{f} + h]\; .$$

Proof

We apply Theorem 5.14. Operator grad $: \mathbb{L}_2(\Omega)/\mathbb{R} \to \mathbf{H}_{-1}(\Omega)$ is bounded. Moreover, the dual operator of grad : $\mathbb{L}_2(\Omega)/\mathbb{R} \to \mathbf{H}_{-1}(\Omega)$ is given by
$-$div : $\overset{\circ}{\mathbf{H}}_1(\Omega) \to \mathbb{L}_2(\Omega)/\mathbb{R}$. (NB. $\overset{\circ}{\mathbf{H}}_1(\Omega)' = \mathbf{H}_{-1}(\Omega)$)
Since, by div $f\,\mathbf{v} = f$ div $\mathbf{v}$ + grad $f \cdot \mathbf{v}$ we have

$$-\int_\Omega \phi \mathrm{div}\, \mathrm{v}\, dx = \int_\Omega \mathrm{grad}\, f \cdot \mathrm{v}\, dx$$

Further by Theorem 5.15 we have

$$||\mathrm{grad}\, p||_{\mathbf{H}_{-1}(\Omega)} \ge c(\Omega)||p||_{\mathbb{L}_2(\Omega)/\mathbb{R}}$$

So in order to apply Theorem 5.14 with

$$\begin{aligned} &\mathcal{P} = \mathrm{grad}\ , \\ &\mathcal{P}' = -\mathrm{div}\ , \\ &\mathcal{R} = \mathcal{D}\ , \\ &V = \overset{\circ}{\mathbf{H}}_1(\Omega)\ , \\ &V' = \mathbf{H}_{-1}(\Omega)\ , \\ &H = \mathbb{L}_2(\Omega)/\mathbb{R}\ . \end{aligned}$$

It remains to show that $\mathcal{D} : \mathbf{H}_{-1}(\Omega) \mapsto \overset{\circ}{\mathbf{H}}_1(\Omega)$ is a unitary mapping such that:

$$\langle F, \phi\rangle = (\mathcal{D}F, \Phi)_{\mathbf{H}_1(\Omega)} \text{ for all } F \in \mathbf{H}_{-1}(\Omega) \text{ and } \Phi \in \overset{\circ}{\mathbf{H}}_1(\Omega). \tag{5.39}$$

Observe the Gelfand Triple

$$\overset{\circ}{\mathbf{H}}_1(\Omega) \hookrightarrow \mathbf{L}_2(\Omega) \hookrightarrow \mathbf{H}_{-1}(\Omega)$$

Because Ω is bounded the Poincaré inequality see 6.19 holds. So an appropriate norm on $\overset{\circ}{\mathbf{H}}_1(\Omega)$ is given by $\|f\| = (\nabla f, \nabla f)_{\mathbf{L}_2(\Omega)}$

$$\|\mathbf{f}\|^2 = (\nabla \mathbf{f}, \nabla \mathbf{f})_{\mathbf{L}_2(\Omega)} = \sum_{i=1}^{n} (\nabla f_i, \nabla f_i)_{\mathbf{L}_2(\Omega)} \qquad (\mathbf{f} = (f_1, \dots, f_n)) .$$

In particular we have by using Green's first identity:

$$\begin{aligned} \forall_{\mathbf{f} \in \mathbf{L}_2(\Omega)} \forall_{\Phi \in \mathcal{D}(\Omega)} \quad : \ & (\mathcal{D}\mathbf{f}, \Phi)_{\mathbf{H}_0^1(\Omega)} = (\nabla \mathcal{D}\mathbf{f}, \nabla \Phi)_{\mathbf{L}_2(\Omega)} = \\ & -(\Delta \mathcal{D}\mathbf{f}, \Phi)_{\mathbf{L}_2(\Omega)} = (\mathbf{f}, \Phi)_{\mathbf{L}_2(\Omega)} \end{aligned} \tag{5.40}$$

Now let $\mathbf{f} \in \mathbf{H}_{-1}(\Omega)$.
Observe the linear functional $\Lambda_{\mathbf{f}} : \overset{\circ}{\mathbf{H}}_1(\Omega) \mapsto \mathbb{R}$ given by $\Lambda_{\mathbf{f}}(\Phi) = (\mathcal{D}\mathbf{f}, \Phi)_{\overset{\circ}{\mathbf{H}}_1(\Omega)}$.

$\Lambda_{\mathbf{f}}$ is continuous and therefore an element of $\mathbf{H}_{-1}(\Omega)$ which is the dual space of $\overset{\circ}{\mathbf{H}}_1(\Omega)$. In the Gelfand-Triple elements in $\mathbf{H}_{-1}(\Omega)$ are represented by the $(\cdot, \cdot)_{\mathbf{L}_2(\Omega)}$ inner product. We have seen (5.40) that *if* $\mathbf{f} \in \mathbf{L}_2(\Omega)$ then $(\mathcal{D}\mathbf{f}, \Phi)_{\mathbf{H}_0^1(\Omega)} = (\mathbf{f}, \Phi)_{\mathbf{L}_2(\Omega)}$.
Therefore, since the embedding $\mathbf{L}_2(\Omega) \hookrightarrow \mathbf{H}_{-1}(\Omega)$ is dense, equation (5.39) follows.

So we conclude that $-\text{div}\mathcal{D}\nabla$ is an isomorphism from $\mathbb{L}_2(\Omega)/\mathbb{R}$ onto itself. □

Note that:

- $\mathcal{D}$ is unitary and bounded even under smaller assumptions to the boundary. The Poincaré lemma is also valid when Ω is bounded in some direction[9]
- By theorem 17.12 in Wloka[2] we have that there exists a Green solution operator G. The next theorem shows that operator div$\mathcal{D}$grad is self-adjoint, therefore the Green operator G is also self-adjoint.
- By the above theorem, equality (5.24), $\sigma(B + I) = \{1\} \cup -\sigma(\text{div}\mathcal{D}\text{grad})$ and the fact that div$\mathcal{D}$grad is negative definite it follows that

  $$\sigma(\,\text{div}\mathcal{D}\text{grad}\,|_{\mathbb{L}_2^{\text{harm}}}(\Omega)) \subset [-1, -c^2) \, .$$

  And from Theorem 5.18 (yet to come) we even have

  $$\sigma(\,\text{div}\mathcal{D}\text{grad}\,|_{\mathbb{L}_2^{\text{harm}}}(\Omega)) \subset (-1, -c^2) \, .$$

- $\mathcal{R}(\mathcal{C}) = \mathcal{R}(\mathcal{D}\text{grad}\,\text{div}\,)$ is closed for the same reason as for the closedness of div$\mathcal{D}\nabla$.
- For more information about Gelfand Triples, see Section 6.4.

Theorem 5.17 *Operator div$\mathcal{D}$grad is self adjoint and negative definite on* $\mathbb{L}_2(\Omega)/\mathbb{R}$.

Proof
Using $\text{div}\,\phi\mathbf{v} = \phi\text{div}\,\mathbf{v} + \text{grad}\phi \cdot \mathbf{v}$ one gets

$$(\nabla \cdot \mathcal{D}\nabla p, q)_{\mathbf{L}_2(\Omega)} = -\int_{\Omega} \nabla q \cdot \mathcal{D}\nabla p dx \tag{5.41}$$

The next equations show that $\mathcal{D}$ is positive definite and self adjoint. From this the final result follows.

[9] Ω lies within a slab whose boundary is two hyperplanes which are orthogonal to this direction. The minimal distance between such a pair of hyperplanes is called the thickness of Ω in the corresponding direction.

Let $f \in \mathbf{H}_{-1}(\Omega)$ such that $\mathbf{f} = \nabla p$, for a certain $p \in \mathbb{L}_2(\Omega)/\mathbb{R}$.
Then $\mathbf{f} \neq \mathbf{0}$ and by Greens first identity,

$$\begin{aligned}&\int_\Omega f\mathcal{D}f d\mathbf{x} = \\ &-\int_\Omega \Delta\mathcal{D}f\mathcal{D}f d\mathbf{x} = \\ &\int_\Omega \|\nabla\mathcal{D}f\|^2 d\mathbf{x} > 0\ ,\end{aligned}$$

since $\mathcal{D}\mathbf{f}$ can never be constant $\neq 0$.
Let $\mathbf{f}, \mathbf{g} \in \mathbf{H}^{-1}(\Omega)$ such that $\mathbf{f} = \nabla p$ and $\mathbf{g} = \nabla q$, for some $p,\ q \in \mathbb{L}_2(\Omega)$, then by Greens second identity we have

$$\begin{aligned}&\int_\Omega f\mathcal{D}g - g\mathcal{D}f d\mathbf{x} = \\ &\int_\Omega -\Delta\mathcal{D}f\mathcal{D}g + \Delta\mathcal{D}g\mathcal{D}f d\mathbf{x} = \\ &\int_{\partial\Omega} \mathcal{D}f\frac{\partial\mathcal{D}g}{\partial n} - \mathcal{D}g\frac{\partial\mathcal{D}f}{\partial n} d\mathrm{x} = 0 \qquad \square\end{aligned}$$

Note that the equation (5.27) and theorem 5.9 and theorem (5.8) already implied that div$\mathcal{D}$grad is self adjoint on $\mathbb{L}^{harm}(\Omega)$.
Because $\overline{\mathcal{R}(A)} = \mathcal{N}(A^*)^\perp$ for any bounded operator on a Hilbert space we have that div$\mathcal{D}$grad has a dense image in $\mathbb{L}_2(\Omega)/\mathbb{R}$. However, by theorem 5.16 we already have that the image space of div$\mathcal{D}$grad is the whole space $\mathbb{L}_2(\Omega)_\mathbb{R}$ and in particular closed. Note that any self operator $\mathcal{A}$ acting on a Hilbert Space with the property that there exists a $\lambda \in \mathbb{R}$ such that div$\mathcal{D}$grad $- \lambda I$ is compact has that same property. So we wonder whether there exists a λ such that div$\mathcal{D}$grad $- \lambda I$ is compact, since that would imply that operator div$\mathcal{D}$grad has a complete set of eigen functions. Note that by Riesz-Schauder's spectral theorem, see [2] p.166-167 this λ should be a density point of the spectrum of operator div$\mathcal{D}$grad. By theorem we must have $\lambda > c^2$. In the unit ball case we found that $\lambda = \frac{1}{2}$.

Because div$\mathcal{D}$grad is self adjoint and negative definite and the original space $\mathbb{L}_2(\Omega)/\mathbb{R}$ is a Hilbert space an explicit expression for the norm of div$\mathcal{D}$grad is given by:

$$\|\mathrm{div}\mathcal{D}\mathrm{grad}\| = \frac{-(\mathrm{div}\mathcal{D}\mathrm{grad}\, p, p)}{(p,p)_{\mathbb{L}_2(\Omega)}} \tag{5.42}$$

If div$\mathcal{D}$grad has a complete set of eigen functions, then $\|\mathrm{div}\,\mathcal{D}\mathrm{grad}\| = \sup\limits_{\lambda\in\sigma(\mathrm{div}\,\mathcal{D}\nabla)} -\lambda$.

Using equation (5.25) we obtain if $p = \mathrm{div}\,\Phi$, with $\Delta p = 0$

$$\frac{(\mathrm{div}\mathcal{D}\mathrm{grad}\, p, p)_{\mathbb{L}_2(\Omega)}}{(p,p)_{\mathbb{L}_2(\Omega)}} = \frac{((B+I)B\Phi, \Phi) + \int\limits_\Omega C^2\Phi\cdot\Delta\Phi dx}{(B\Phi,\Phi) + \int\limits_\Omega C\Phi\cdot\Delta\Phi dx} \tag{5.43}$$

From theorem 5.8 it follows that we can put $\Delta\mathrm{div}\,\Phi = \mathbf{0}$. So,

$$\|\,\mathrm{div}\mathcal{D}\mathrm{grad}\,\| = \sup_{\Phi\in\mathcal{H}_1} \frac{((B+I)B\Phi,\Phi)_{\mathbb{V}(3)}}{(B\Phi,\Phi)_{\mathbb{V}(3)}} \tag{5.44}$$

Next we will try to derive an expression for an integral kernel for operator div$\mathcal{D}$grad : $\mathbb{L}_2(\Omega) \mapsto \mathbb{L}_2(\Omega)$, inspired by the solution formula for the Dirichlet problem. Remind that this formula can obtained in a sloppy way by substituting $G(\mathbf{x},\mathbf{y}) = S(\mathbf{x},\mathbf{y}) + g(\mathbf{x},\mathbf{y})$ (with g harmonic such that $g(\cdot,\mathbf{y})$ compensates the fundamental solution[10] $S(\cdot,\mathbf{y})$ at the boundary) in the third identity of Green.

[10]The point-wise notation $S(\mathbf{x},\mathbf{y})$ might be a bit deceptive, in strict sense S is a distribution in $\mathcal{D}'(\mathbb{R}^n)$ with non compact support. However, the corollary of theorem 6.7 states that S is in fact an infinitely differentiable function outside the origin.

Write

$$W(\mathbf{x},\mathbf{y}) = \nabla S(\mathbf{x},\mathbf{y}) + h(\mathbf{x},\mathbf{y}) \tag{5.45}$$

In which S denotes the fundamental solution and h is a harmonic function that compensates $\nabla S(\mathbf{x},\mathbf{y})$ at the boundary. We have that

$$\text{div}\mathcal{D}\text{grad}\, p = \text{div}\, W * p$$

So

$$(\text{div}\mathcal{D}\text{grad}\, p)(\mathbf{x}) = -p(\mathbf{x}) + \int_\Omega \text{div}\, h(\mathbf{y},\mathbf{x}) p(\mathbf{y}) d\mathbf{y} \tag{5.46}$$

In which $\text{div}\, h(\cdot,\mathbf{x})$ is harmonic, therefore:

$$E_{-1}(\text{div}\mathcal{D}\text{grad}) \supset (\mathbb{L}_2^{\text{harm}}(\Omega))^\perp \tag{5.47}$$

Now the question arises whether these are the only eigen functions with eigen value -1. This suspicion comes from (5.46) and the fact that[11]

$$p = q \text{ A.E. on } \Omega \Leftrightarrow p = q \in \mathbb{L}_2(\Omega)$$

So,

$$\begin{aligned} &p \in E_{-1} \Leftrightarrow \textstyle\int_\Omega \text{div}\, h(\mathbf{y},\mathbf{x}) p(\mathbf{y})\, d\mathbf{y} = 0 \text{ A.E.} \Leftrightarrow \\ &\textstyle\int_\Omega \nabla(S(\mathbf{y},\mathbf{x}))_{\mathcal{H}} - (\nabla S(\mathbf{y},\mathbf{x}))_{\mathcal{H}} \cdot \nabla p(\mathbf{y})\, d\mathbf{y} = 0 \text{ A.E.} \end{aligned} \tag{5.48}$$

The next theorem gives an answer to our question from a different point of view.

Theorem 5.18 *The eigen space of eigen value -1 of operator* $div\mathcal{D}\nabla : \mathbb{L}_2(\Omega)/\mathbb{R} \mapsto \mathbb{L}_2(\Omega)/\mathbb{R}$ *is given by*

$$E_{-1}(\, div\mathcal{D}grad) = (\mathbb{L}_2^{harm}(\Omega))^\perp \tag{5.49}$$

Proof
$\supset$ is already shown.
Let $p \in E_{-1}(\text{div}\mathcal{D}\nabla)$ and suppose $\Delta p = 0$.
$p \in \mathbb{L}_2^{\text{harm}}(\Omega)/\mathbb{R}$, then $p = -\text{div}\,\Phi$, for a certain $\Phi \in \underline{\text{Harm}}^{(3)}(\Omega)$.
We have

$$\text{div}\mathcal{D}\nabla p + p = 0 \Leftrightarrow \text{div}\mathcal{B}\Phi = 0$$

Because $\mathcal{N}(B) = \mathcal{H}_0$ and $\mathcal{R}(B) = \mathcal{H}_1$ we have that

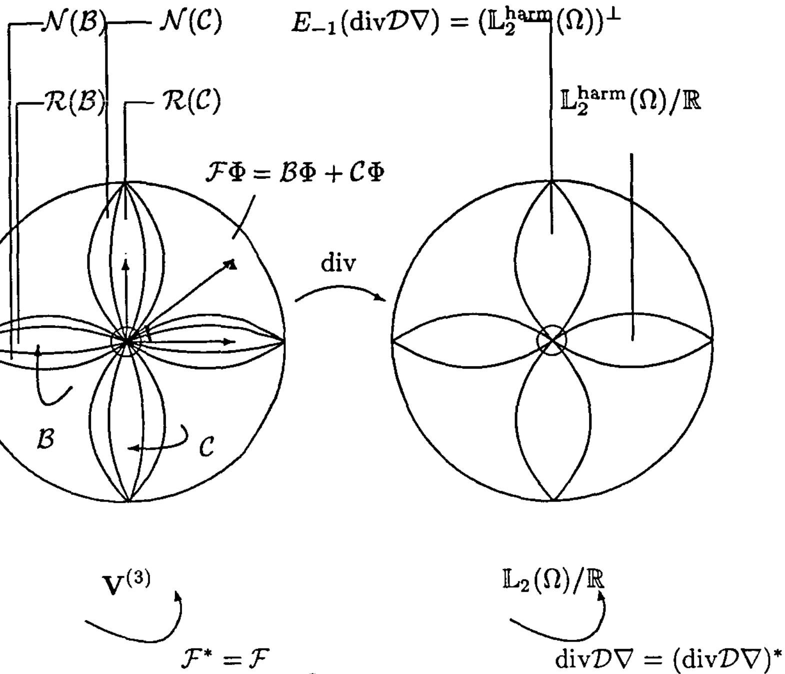
$$\begin{aligned} &\text{div}\mathcal{B}\Phi = 0 \Rightarrow \mathcal{B}\Phi \in \mathcal{H}_0 \cap \mathcal{H}_1 = \{\mathbf{0}\} \Rightarrow \\ &\Phi \in \mathcal{N}(B) \Rightarrow -\text{div}\,\Phi = p = 0 \end{aligned}$$

So we have that there doesn't exist harmonic eigen functions with eigenvalue -1
□

[11]NB. Čebysev : $\mu(\{\mathbf{x} \in \Omega \mid |f(\mathbf{x})| \geq \frac{1}{n}\}) \leq n \int_\Omega |f| d\mu = 0$

5.3.1 Geometrical picture

In this section a geometrical picture is drawn, in order to give a short illustration of the general theory in this chapter.



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5.3.2 Example: Spectra in the unit ball case

In this section we will look how the general theory of this chapter fits at the ball case. The general theory has until now only dealt with the three dimensional case. Nevertheless, in this chapter we will take the dimension n arbitrary in this paragraph.

Although that the operators $\mathcal{B}$, $\mathrm{div}\mathcal{D}\nabla$ and $\mathcal{B}+I$ are bounded and self adjoint on $\underline{\mathrm{Harm}}^{(n)}(B_n)$ and they have a complete set of eigen functions (NB. spherical harmonics!), they are not compact:

$$\begin{aligned}
&\sigma(\mathcal{B}) = \{0\} \cup \{-\tfrac{1}{2} - \tfrac{(n-2)}{4m+2(n-2)}\}_{m\in\mathbb{N}} \\
&\sigma(\mathcal{B}+\mathcal{I}) = \{1\} \cup \{\tfrac{1}{2} - \tfrac{(n-2)}{4m+2(n-2)}\}_{m\in\mathbb{N}} \\
&\sigma(\mathcal{B}+\tfrac{1}{2}\mathcal{I}) = \{\tfrac{1}{2}\} \cup \{-\tfrac{(n-2)}{4m+2(n-2)}\}_{m\in\mathbb{N}} \\
&\sigma(\mathrm{div}\mathcal{D}\nabla) = \{-\tfrac{1}{2} + \tfrac{(n-2)}{4m+2(n-2)}\}_{m\in\mathbb{N}}
\end{aligned} \tag{5.50}$$

By the Riesz Schauder theorem the only density point in the spectrum of a compact operator on a Hilbert space is 0. For a formal proof see Wloka[2]p.166-168 . If one also assumes that the operator is self adjoint, it can be shown in an easier way:
The eigen functions are then orthogonal. Each orthogonal row in a Hilbert Space converges weakly to zero because of Bessel's inequality. Finally note that a compact operator A maps a weakly convergent sequence on a strongly converging sequence, i.e.

$$\mathbf{x}_n \rightharpoonup \mathbf{x} \Rightarrow A\mathbf{x}_n \mapsto A\mathbf{x}$$

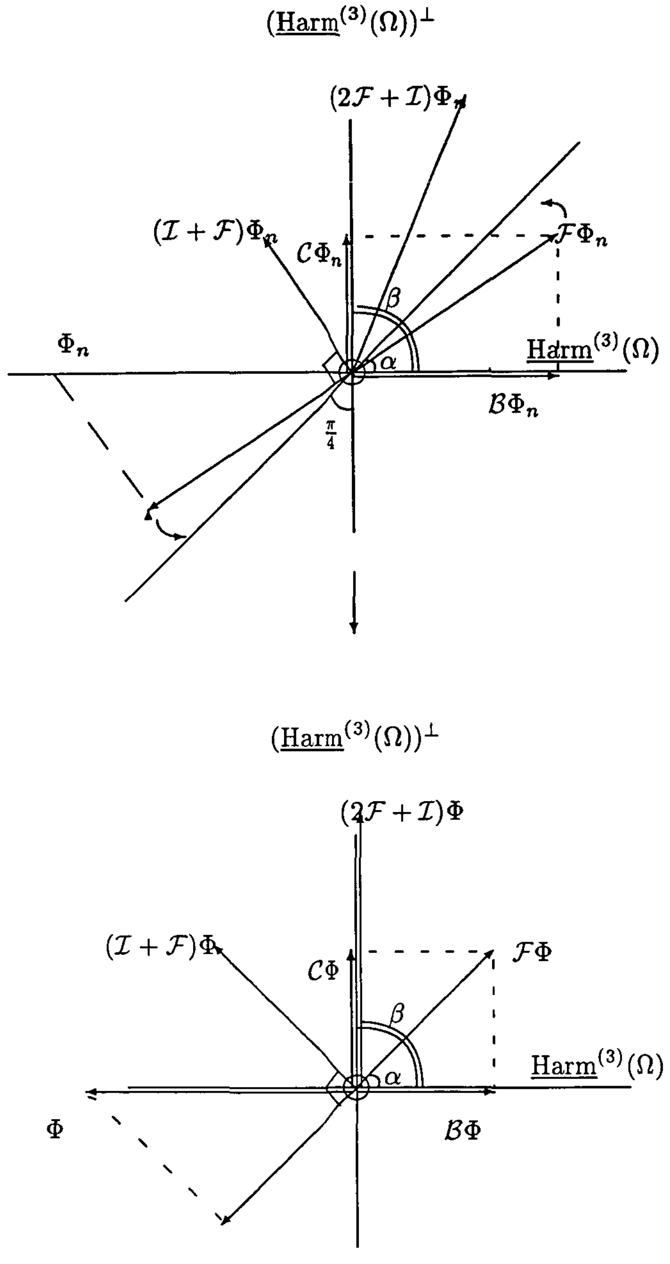
So, the only possible compact operator is $\mathcal{B} + \frac{1}{2}\mathcal{I}$. It is indeed compact as follows from the fact that every Hilbert-Schmidt operator is a compact operator. For the same reason div$\mathcal{D}\nabla - \frac{1}{2}I$ is compact.
Now the question arises whether this compactness can be shown in the general case. Note that any compact self adjoint operator on a Hilbert space has a *complete* set of eigen functions and we would like to proof that such a set exists. Until now we were unable to give an answer. However, there are remarkable geometrical features according to this mysterious half that might help to find an answer:

Let $\{\Phi_n\}_{n\in\mathrm{N}}$ be the complete set of eigen functions of $\mathcal{B}$. Let $\{\lambda_n\}_{n\in\mathrm{N}}$ be the corresponding eigen values. Let $\alpha_n = \angle(-\mathcal{F}\Phi_n, \Phi_n)$. Let $\beta_n = \angle((2\mathcal{F} + I)\Phi_n, \Phi_n)$

Then the following statements are equivalent:

$$
\begin{aligned}
&\lambda_n \uparrow -\tfrac{1}{2} \Leftrightarrow \\
&\frac{\|(I+\mathcal{F})\Phi_n\|^2_{\mathrm{V}(3)}}{\|\mathcal{F}\Phi_n\|^2_{\mathrm{V}(3)}} = \frac{\|\mathrm{rot}\,\Phi_n\|^2_{\mathrm{V}(3)}}{\|\mathrm{div}\,\Phi_n\|^2_{\mathrm{V}(3)}} \uparrow 1 \Leftrightarrow \\
&\alpha_n \uparrow \tfrac{\pi}{4} \\
&\beta_n \uparrow \tfrac{\pi}{2}
\end{aligned} \tag{5.51}
$$

The next figure illustrates this:



figure

In chapter 3 we worked with $\mathbf{h}_0[\phi]$ and $\mathbf{h}_1[\chi]$, with $\Delta\chi = \Delta\phi = 0$. We saw that every element Φ of $\underline{\mathrm{Harm}}^{(n)}(B_n)$ can be written as $\mathbf{h}_0[\phi] + \mathbf{h}_1[\chi]$ in a unique matter. Next it will be shown that this splitting is indeed a special case of the general splitting $\mathcal{H}_0 \oplus \mathcal{H}_1 = \underline{\mathrm{Harm}}^{(3)}(\Omega)$:

Theorem 5.19 *If $\Omega = B_3$ then the definition of h_0 and h_1 given by (3.2) is indeed* *a special case of the more general definition in theorem 5.5. i.e.*

1. $\mathcal{H}_0 = \{\boldsymbol{x} \times \nabla\phi \mid \Delta\phi = 0$
2. $\mathcal{H}_1 = \{\nabla\chi - (\mathcal{E}\chi)\boldsymbol{x} + \frac{1}{2}(\|\boldsymbol{x}\|^2 - 1)\nabla[(\mathcal{E} + \frac{1}{2})^{-1}\mathcal{E}]\chi \mid \Delta\chi = 0\}$

Proof

First see theorem 5.5.

1. On the ball $\mathbf{n} = \mathbf{x}$ so we have indeed
$\mathbf{x} \times \nabla\phi - \nabla\mathcal{N}(0, (\mathbf{n} \times \mathbf{x}) \cdot \nabla\phi) = \mathbf{x} \times \nabla\phi$
2. We have that $F[\chi] = \mathbf{n}_{\mathcal{H}} \cdot \nabla\chi + 2\mathcal{D}(\text{trace}\{[\nabla \mathbf{n}_{\mathcal{H}}]\text{Hessian}(\chi)\} = \mathcal{E}\chi$. So,

$$\begin{aligned} \mathbf{h}_1[\chi] &= \nabla\chi - \mathcal{E}\chi\mathbf{x} - 2\mathcal{D}(\nabla\mathcal{E}\chi) \\ &= \nabla\chi - \mathcal{E}\chi\mathbf{x} + \tfrac{1}{2}(\|\mathbf{x}\|^2 - 1)\nabla(\mathcal{E} + \tfrac{1}{2})^{-1}\mathcal{E}\chi \qquad \square \end{aligned}$$

The next theorem is a test for and a special case of theorem 5.8:

Theorem 5.20 *Operator div :* $\underline{Harm}^{(n)}(B_n) \mapsto Harm(B_n)$ *is surjective in the ball-case.*

Proof

Let $g \in \text{Harm}(\Omega)$. Then $g = \sum_{m=0}^{\infty} \lambda_m Q_m$ converging uniform on compact subsets of B_n with $Q_m \in \text{HarmHomPol}_n(B_n)$. Therefore, without loss of generality we assume that $g \in \text{HarmHomPol}_n(B_n)$. Define $\Psi \in \underline{\text{Harm}}^{(n)}(B_n)$ by:

$$\Psi(\mathbf{x}) = \frac{2m + n - 2}{(n + m)(2m + n - 2) - 2m}(\mathbf{x}g + 2\mathcal{D}\nabla g)$$

Straight calculations show that

$$\begin{aligned} &\text{div}\,\Psi = Q_m \\ &\Delta\Psi = \mathbf{0} \qquad \square \end{aligned}$$

Note that the relation

$$\sigma(\mathcal{B}+\mathcal{I}) = -\sigma(\nabla \cdot \mathcal{D}\nabla) \cup \{1\} \tag{5.52}$$

indeed holds in the ball case.

5.4 The two dimensional case

The two dimensional case is almost analogue to the three dimensional case. The only problem that must be solved is the fact that there is (again) no cross product and as a consequence operator rot must be changed. In the three dimensional case it was shown that the inner product $(\cdot, \cdot)_V$ indeed generated a norm. The main reason for this was that in a simply connected region with smooth boundary we have that $\text{rot}\,\Phi = 0$ implies $\Phi = \text{grad}F$ for some scalar function F, because of Stokes Theorem. It is not difficult to construct such a function in $\mathbb{R}^2$:

$$\begin{aligned} \oint_C \Phi \cdot \mathbf{t}ds &= \oint_K \Phi \cdot d\mathbf{x} = \\ \int_S (\tfrac{\partial\Phi_1}{\partial y} - \tfrac{\Phi_2}{\partial x})dxdy &= \int_S \text{div}\,\sigma\Phi dxdy \end{aligned} \tag{5.53}$$

Where σ is the 2×2 matrix:

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.54}$$

So introduce in the two dimensional case the inner product $(\cdot, \cdot)_{\mathbf{V}^{(2)}}$ defined on on $\mathbf{V}^{(2)}$ given by:

$$(\Phi, \Psi)_{\mathbf{V}^{(2)}} = \int_\Omega \{\text{div}\,\Phi\ \text{div}\,\Psi + \text{div}\,\sigma\Phi\ \text{div}\,\sigma\Psi\}\ d\tau \tag{5.55}$$

Theorem 5.21 *Operator $\mathcal{B}$ is self adjoint, negative definite and bounded (by 1) on $\underline{Harm}^{(2)}(\Omega)$ according to inner product $(\cdot,\cdot)_{V^{(2)}}$*

Proof
Let Φ and $\Psi \in \mathbf{V}^{(2)}$. Straightforward derivations yield :

$$\begin{aligned}
(\mathcal{B}\Phi,\Psi)_{V^{(2)}} &= \int_\Omega [\operatorname{div}\{(\nabla\mathcal{N}-\mathcal{D}\nabla)\nabla\cdot\Phi\}\operatorname{div}\Psi - \operatorname{div}\sigma\{\mathcal{D}\nabla\nabla\cdot\Phi\}\cdot\operatorname{div}\sigma\Psi]\,d\tau \\
&= -\int_\Omega [\operatorname{div}\Phi\ \operatorname{div}\Psi + \operatorname{div}\{\mathcal{D}\nabla(\nabla\cdot\Phi)\}\operatorname{div}\Psi + \operatorname{div}\sigma\{\mathcal{D}\nabla\nabla\cdot\Phi\}\cdot\operatorname{div}\sigma\Psi]\,d\tau \\
&= -\int_\Omega [\operatorname{div}\Phi\ \operatorname{div}\Psi + \operatorname{div}\{\mathcal{D}\nabla(\nabla\cdot\Phi)\}\operatorname{div}\Psi - \operatorname{graddiv}\{\sigma\Psi\}\cdot\sigma\mathcal{D}(\nabla\nabla\cdot\Phi)]\,d\tau \\
&= -\int_\Omega [\operatorname{div}\Phi\ \operatorname{div}\Psi + \operatorname{div}\{\mathcal{D}\nabla(\nabla\cdot\Phi)\}\operatorname{div}\Psi - [\operatorname{grad}\operatorname{div}\{\Psi\}\sigma + \Delta\sigma\Psi]\cdot\sigma\mathcal{D}(\nabla\nabla\cdot\Phi)]\,d\tau \\
&= -\int_\Omega [\operatorname{div}\Phi\ \operatorname{div}\Psi + \operatorname{div}\{\mathcal{D}\nabla(\nabla\cdot\Phi)\}\operatorname{div}\Psi - [-\operatorname{grad}\operatorname{div}\{\Psi\} + \Delta\Psi]\cdot\mathcal{D}(\nabla\nabla\cdot\Phi)]\,d\tau \\
&= -\int_\Omega [\operatorname{div}\Phi\ \operatorname{div}\Psi - (\mathcal{D}\nabla\nabla\cdot\Phi)\cdot\Delta\Psi]\,d\tau \\
&= -(\operatorname{div}\Phi,\operatorname{div}\Psi)_{\mathbb{L}_2(\Omega)} + \int_\Omega \mathcal{C}\Phi\cdot\Delta\Psi\,d\tau
\end{aligned} \tag{5.56}$$

Some remarks:

- $\operatorname{div}(f\mathbf{v}) = f\operatorname{div}\mathbf{v} + \operatorname{grad}f\cdot\mathbf{v}$ in combination with Gauss divergence theorem and the fact that the image of the Dirichlet operator is zero at the boundary is used twice, in the third and sixth equality above.
- $\operatorname{grad}\operatorname{div}\sigma\mathbf{v} = \operatorname{graddiv}\cdot\sigma + (\Delta\sigma\mathbf{v})$
- $\sigma^2 = -I$ and $\sigma^T = -\sigma$
- As expected the result is the same as in $\mathbb{R}^3$.

The final result is now obvious. □

In the next theorem we show that $(\cdot,\cdot)_{\mathbf{V}^{(2)}}$ has a remarkable property according to conformal mappings.

Theorem 5.22 *Let Ω be a simply connected open subset of $\mathbb{R}^2$. Let $f:\mathbb{C}\mapsto\mathbb{C}$ be a conformal mapping, such that the corresponding function $\boldsymbol{u}:\mathbb{R}^2\mapsto\mathbb{R}^2$ maps Ω onto S. Set $\boldsymbol{u}=(u,v)$. Let $\Phi\in\underline{Harm}^{(2)}(\Omega)$ and define*

$$\Phi^* = \Phi\circ\boldsymbol{u}^{-1} \tag{5.57}$$

. i.e.
Describe points in Ω by (x,y) coordinates and describe points in S by (u,v) coordinates, then by (5.57) we have

$$\begin{aligned}
\Phi^*(u,v) &= \Phi(x(u,v),y(u,v)) \\
\Phi(x,y) &= \Phi^*(u(x,y),v(x,y))
\end{aligned}$$

Then S is simply connected and Φ^* is a harmonic function which is tangent to ∂S. Moreover,

$$(\Phi,\Psi)^{\Omega}_{V^{(2)}} = (\Phi^*,\Psi^*)^{S}_{V^{(2)}} \tag{5.58}$$

Proof
S is simply connected, since any conformal mapping maps simply connected regions in its domain onto simply connected in its range.
One can show by using direct computation and the Cauchy-Riemann equations that

$$\Delta_{u,v}\Phi^*(u,v) = \left((x_u)^2+(y_v)^2\right)\,[\Delta_{x,y}\Phi(x,y)]|_{\mathbf{x}=\mathbf{x}(u,v)}$$

From which follows that Φ^* is again harmonic. Φ^* is tangent because of the angle preserving property of conformal mappings.
In the derivation of equality (5.58) we shall use the next equations which are a direct result of the chain rule and the Cauchy-Riemann equations:

$$\begin{aligned} &\operatorname{div}\Phi(x,y) = \operatorname{div}_{(x,y)}\Phi^*(u(x,y),v(x,y)) = \\ &\operatorname{trace}\left\{ \left(\begin{array}{cc} \frac{\partial\Phi_1^*}{\partial u} & \frac{\partial\Phi_1^*}{\partial v} \\ \frac{\partial\Phi_2^*}{\partial u} & \frac{\partial\Phi_2^*}{\partial v} \end{array} \right)\Bigg|_{\mathbf{u}=\mathbf{u}(x,y)} \left(\begin{array}{cc} u_x & u_y \\ -u_y & u_x \end{array} \right) \right\} = \\ &(\operatorname{div}_{(u,v)}\Phi^*|_{\mathbf{u}=\mathbf{u}(x,y)})u_x + (\operatorname{div}_{(u,v)}\sigma\Phi^*|_{\mathbf{u}=\mathbf{u}(x,y)})u_y \end{aligned} \tag{5.59}$$

Analogue to the above:

$$\operatorname{div}\sigma\Phi(x,y) = (\operatorname{div}_{(u,v)}\Phi^*|_{\mathbf{u}=\mathbf{u}(x,y)})u_y - (\operatorname{div}_{(u,v)}\sigma\Phi^*|_{\mathbf{u}=\mathbf{u}(x,y)})u_x \tag{5.60}$$

First the special case $\Psi = \Phi$ will be shown.

$$\begin{aligned} (\Phi,\Phi)_{V^{(2)}} &= \int_\Omega \operatorname{div}\Phi\operatorname{div}\Phi + \operatorname{div}\sigma\Phi\operatorname{div}\sigma\Phi \; d\tau_{\mathbf{x}} \\ &= \int_\Omega (\operatorname{div}\Phi^*|_{\mathbf{u}=\mathbf{u}(x,y)})^2(u_x)^2 + (\operatorname{div}\Phi^*|_{\mathbf{u}=\mathbf{u}(x,y)})^2(u_y)^2 + \\ &\quad 2\operatorname{div}\Phi^*|_{\mathbf{u}=\mathbf{u}(x,y)}\operatorname{div}\sigma\Phi^*|_{\mathbf{u}=\mathbf{u}(x,y)}\, u_x u_y \\ &+ \int_\Omega (\operatorname{div}\Phi^*|_{\mathbf{u}=\mathbf{u}(x,y)})^2(u_y)^2 + (\operatorname{div}\Phi^*|_{\mathbf{u}=\mathbf{u}(x,y)})^2(u_x)^2 + \\ &\quad -2\operatorname{div}\Phi^*|_{\mathbf{u}=\mathbf{u}(x,y)}\operatorname{div}\sigma\Phi^*|_{\mathbf{u}=\mathbf{u}(x,y)}\, u_y u_x \;\; d\tau_{\mathbf{x}} \\ &= \int_\Omega [(\operatorname{div}\Phi^*|_{\mathbf{u}=\mathbf{u}(x,y)})^2 + (\operatorname{div}\Phi^*|_{\mathbf{u}=\mathbf{u}(x,y)})^2]\det\left(\begin{array}{cc} u_x & u_y \\ -u_y & u_x \end{array} \right) d\tau_{\mathbf{x}} \\ &= \int_S \operatorname{div}\Phi^*\operatorname{div}\Phi^* + \operatorname{div}\sigma\Phi^*\operatorname{div}\sigma\Phi^* \; d\tau_{\mathbf{u}} \end{aligned} \tag{5.61}$$

Using polarization we obtain the final result:

$$(\Phi,\Psi)^\Omega_{V^{(2)}} = \frac{1}{4}(||\Phi+\Psi||^\Omega_{V^{(2)}} + ||\Phi-\Psi||^\Omega_{V^{(2)}}) = (\Phi^*,\Psi^*)^S_{V^{(2)}} \qquad \square \tag{5.62}$$

Note that in general $\mathcal{B}\Psi^* = \mathcal{B}(\Psi\circ\mathbf{u}) \neq (\mathcal{B}\Psi)\circ\mathbf{u}$. If it is a equality It would have implied that the spectrum of B is the same for the two different domains. This is most likely to be untrue. For example the unit disk and the half space can be mapped conformally onto each other by using a GLA. The spectrum of $\mathcal{B}$ in the disk case is discrete and in the half space case the spectrum of $\mathcal{B}$ is continu. However, the spectrum of div$\mathcal{D}$grad $= -\text{div}(\mathcal{B}+I)$ is exactly the same (namely,$\{0, -\frac{1}{2}\}$).

6 Sobolev spaces

This chapter includes a short survey of the theory of Sobolev spaces. First, the spaces $\mathbb{H}_s$ will be introduced. They are defined on $\mathbb{R}^n$ and therefore Fourier transformation and the Plancherel theorem can be used in order to avoid the use of generalized derivatives. The spaces $\mathbb{H}_s$ are interesting from a theoretical point of view, but in practice we are more interested in regions which are bounded or unbounded subsets of $\mathbb{R}^n$. Therefore, the Sobolev spaces $\mathcal{W}_{l,2}(\Omega)$ are introduced accordingly. If $\Omega = \mathbb{R}^n$ the spaces $\mathcal{W}_{l,2}(\Omega)$ and $\mathcal{W}_{l,2}(\Omega)$ coincide. In the final paragraph of this chapter the application of Gelfand triples on Stokes problems is examined.
First introduce some notation and preliminaries:

multi-index notation
Given a multi-index $\alpha \in \mathbb{N}^n$ we write:
$|\alpha| = \sum_{k=1}^{n} \alpha_k$
$D^\alpha = \prod_{k=1}^{n} \left(\frac{\partial}{\partial x_k}\right)^{\alpha_k}$
$D_\alpha = (i)^{-|\alpha|} D^\alpha$
$D^\alpha e^{\mathbf{t}} = D^\alpha e^{t_1 x_1 + \cdots + t_n x_n} = t_1^{\alpha_1} \cdots t_n^{\alpha_n} e^{\mathbf{t}} = \mathbf{t}^\alpha e^{\mathbf{t}}$
$D_\alpha e_{\mathbf{t}} = D_\alpha e^{i\mathbf{t}\cdot\mathbf{x}} = D_\alpha e^{i(t_1 x_1 + \cdots + t_n x_n)} = (i)^{|\alpha|}(i)^{-|\alpha|}\mathbf{t}^\alpha e_{\mathbf{t}} = \mathbf{t}^\alpha e_{\mathbf{t}}$

Test functions and distributions
The support of a function $f : \Omega \to \mathbb{C}$ is given by:
$\text{supp}(f) = \overline{\{x \in \Omega \mid f(x) \neq 0\}}$
Denote the test function space on an open space Ω by:
$\mathcal{D}(\Omega) = \{\phi \in C^\infty(\Omega) \mid \text{supp}(\phi) \subset \Omega \text{ and } \text{supp}(\phi) \text{ is compact}\}$

The space $\mathcal{D}(\Omega)$ is equipped with the local convex topology (see Appendix ??) generated by the semi-norms $q_N : \mathcal{D}(\Omega) \to \mathbb{R}$ defined by :
$q_N(f) = \sup_{|\alpha| \le N} \sup_{\|\mathbf{x}\| \le N} |(D_\alpha f)(\mathbf{x})|$,with $N \in \mathbb{N}$.
The dual space of $\mathcal{D}(\Omega)$ is denoted by $\mathcal{D}'(\Omega)$
and elements in $\mathcal{D}'(\Omega)$ are called distributions in Ω.

Note that a linear functional Λ on $\mathcal{D}(\Omega)$ is a distribution in Ω
if and only if to every compact $K \subset \Omega$ there exists a nonnegative N
and a constant $C < \infty$ such that the inequality
$|\Lambda\phi| \le C\, q_N(\phi)$
holds for every $\phi \in \mathcal{D}(K)$.

Functions as distributions
Suppose f is locally integrable. Define the linear functional $\Lambda_f : \mathcal{D}(\Omega) \to \mathbb{R}$ by
$\Lambda_f(\phi) = \int_\Omega f(x)\phi(x)\, dx$,
then $\Lambda_f \in \mathcal{D}'(\Omega)$, since for all compact subsets K of Ω
$|\Lambda_f(\phi)| \le \|f\|_{\mathbb{L}_1(K)} \cdot q_0(f)$

Differentiation of distributions
If α is a multi-index and $\Lambda \in \mathcal{D}'(\Omega)$, the formula

$$(D^\alpha\Lambda)(\phi) = (-1)^{|\alpha|}\Lambda(D^\alpha\phi) \qquad [\phi \in \mathcal{D}(\Omega)]$$

defines a linear functional $D^\alpha\Lambda$ on $\mathcal{D}(\Omega)$, since
$|(D^\alpha\Lambda)(\phi)| \leq C\, q_N(D^\alpha\phi) \leq C\, q_{N+|\alpha|}$.

Distributional derivatives of functions
The αth distribution derivative of a locally integrable function $f : \Omega \to \mathbb{C}$ is, by definition, the distribution $D^\alpha\Lambda_f$. The obvious consistency problem is whether the equation
$D^\alpha\Lambda_f = \Lambda_{D^\alpha f}$ i.e.
$(-1)^{|\alpha|} \int_\Omega f(x)(D^\alpha\phi)(x)\, dx = \int_\Omega (D^\alpha f)(x)\phi(x)\, dx$ for all $\phi \in \mathcal{D}(\Omega)$
always holds under these conditions.
If $f \in C^N(\Omega)$, integration by parts leads without difficulty to this equation, but in general this consistency equation may be false.

The space $\mathcal{S}_n$
Define the space $\mathcal{S}_n$ by
$\mathcal{S}_n = \{f \in C^\infty(\mathbb{R}^n) | \forall_{N\in\mathbb{N}} : \sup_{|\alpha|\leq N} \sup_{\mathbf{x}\in\mathbb{R}^n} (1 + ||\mathbf{x}||^2)^N \mid (D_\alpha f)(\mathbf{x})| < \infty\}$

Fourier transformation
Fourier transformation will be denoted by:
$\hat{f}(\mathbf{y}) = [\mathcal{F}(f)](\mathbf{y})$
The support of a distribution on Ω is defined by:
$\mathrm{supp}(\Lambda) = \Omega \setminus \bigcup_{W \text{open}} \{W \subset \Omega \mid \forall_{\phi\in\mathcal{D}(W)} : \Lambda\phi = 0\}$
The Fourier transform of a distribution is defined by:
$\hat{u}(\phi) = u(\hat{\phi})$ (6.1)

Remarks:

- According to the definition of the Fourier transform of a distribution we remark:
  To every $f \in \mathbb{L}_1(\mathbb{R}^n)$ corresponds to a distribution Λ_f. Note that the definitions of Fourier transform of resp. f and Λ_f coincide (as they should): $(\hat{\Lambda}_f)(\phi) = \Lambda_f(\hat{\phi}) = \int f\hat{\phi} = \int \hat{f}\phi = (\Lambda_{\hat{f}})(\phi)$. Moreover, the same argument is valid if $f \in \mathbb{L}_2(\mathbb{R}^n)$.
- Elements of $\mathcal{S}_n$ are often called *rapidly decreasing functions.* They are defined by the property that $P \cdot D_\alpha f$ is a bounded function for every polynomial P and every multi-index α. Since this is true with $(1 + ||\mathbf{x}||)^N P(\mathbf{x})$ in place of $P(\mathbf{x})$ it follows that every $P \cdot D_\alpha f$ lies in $L_1(\mathbb{R}^n)$.
- The space $\mathcal{S}_n$ is equipped with the local convex topology (See appendix ??) generated by the semi-norms $p_N : \mathcal{S}_n \to \mathbb{R}$ defined by :

$$p_N(f) = \sup_{|\alpha|\leq N} \sup_{\mathbf{x}\in\mathbb{R}^n} (1+||\mathbf{x}||^2)^N \mid (D_\alpha f)(\mathbf{x})| \text{ ,with } N \in \mathbb{N}.$$

6.1 The Spaces $\mathbb{H}_s$

Given $s \in \mathbb{R}$ define a positive measure on $\mathbb{R}^n$ by setting:

$$d\mu_s(\mathbf{y}) = (1+||\mathbf{y}||^2)^s\, d\mathbf{y} \tag{6.2}$$

If $f \in \mathbb{L}_2(\mu_s)$ then according to theorem 6.2 f is a tempered distribution. A tempered distribution is a distribution u_L that can be written as $u_L = L \circ i$ where $L \in \mathcal{S}_n'$ and $i : \mathcal{D}(\mathbb{R}^n) \mapsto \mathcal{S}_n$ is the identity mapping.
From $\overline{\mathcal{D}(\mathbb{R}^n)} = \mathcal{S}_n$ (See theorem 6.1) and the fact that i is continuous (on a compact set $K \subset \mathbb{R}^n$ (the function $\mathbf{x} \mapsto 1+|\mathbf{x}|^2)^N$ is bounded for all N.) it follows that u_L is indeed continuous and that $L \mapsto u_L$ is an isomorphism. Therefore it is customary to identify u_L with L. The tempered distributions are then precisely the members of $\mathcal{S}_n'$.

Theorem 6.1 *The space $\mathcal{D}(\mathbb{R}^n)$ is dense in $\mathcal{S}_n$.*

Proof
Let f be in $\mathcal{S}_n$. Choose $\psi \in \mathcal{D}(\mathbb{R}^n)$ so that $\psi = 1$ on the unit ball of $\mathbb{R}^n$ and put

$$f_r(\mathbf{x}) = f(\mathbf{x})\psi(r\mathbf{x}) \qquad \mathbf{x} \in \mathbb{R}^n, r > 0$$

Then $f_r \in \mathcal{D}(\mathbb{R}^n)$. If P is a polynomial and α is a multi-index, then

$$P(\mathbf{x})\, D^\alpha(f-f_r)(\mathbf{x}) = P(\mathbf{x}) \sum_{\beta \le \alpha} c_{\alpha\beta} (D^{\alpha-\beta} f)(\mathbf{x})\, r^{|\beta|} D^\beta[1-\psi(r\mathbf{x})].$$

If $||\mathbf{x}|| \le 1/r$ then because our choice of ψ we have that $D^\beta[1-\psi(r\mathbf{x})] = 0$. Since every element in $\mathcal{S}_n$ is infinitely continuous differentiable we have that $P \cdot D^{\alpha-\beta} f \in C_0(\mathbb{R}^n)$ for all $\beta \le \alpha$. It follows that the above sum tends to 0, uniformly on $\mathbb{R}^n$, when $r \to 0$. Thus $f_r \to f$ in $\mathcal{S}_n$. $\Box$

Theorem 6.2 *If $g \in \mathbb{L}_2(\mu_s)$ then g is a tempered distribution.*

Proof
The linear functional $\Lambda : \mathcal{S}_n \mapsto \mathbb{R}$ defined by $\Lambda f = \int_{\mathbb{R}^n} fg\, d\mathbf{x}$ is continuous:
Using Cauchy-Schwarz gives us

$$|\Lambda f| \le ||g||^{\frac{1}{2}}_{\mathbb{L}_2(\mu_s)} \left\{ \int_{\mathbb{R}^n} |(1+||\mathbf{x}||^2)^{\frac{s}{2}} f(x)|^2 dx \right\}^{\frac{1}{2}} \le ||g||^{\frac{1}{2}}_{\mathbb{L}_2(\mu_s)} C^{\frac{1}{2}} \sup_{\mathbf{x}\in\mathbb{R}^n} |(1+||\mathbf{x}||^2)^M f(\mathbf{x})|$$

In which M is taken sufficiently large such that

$$C = \int_{\mathbb{R}^n} (1+||\mathbf{x}||^2)^{s-2M)} < \infty \qquad \Box$$

Definition 6.1 *The Sobolev space $\mathbb{H}_s$ is defined by*

$$\mathbb{H}_s = \{u \in \mathcal{S}_n' \mid \hat{u} \in \mathbb{L}_2(\mu_s)\} \tag{6.3}$$

and is equipped with the norm:

$$||u||_s = \left(\int_{\mathbb{R}^n} |\hat{u}(y)|^2\, d\mu_s(y) \right)^{\frac{1}{2}} \tag{6.4}$$

Theorem 6.3 *The Fourier transform is a continuous linear mapping of $\mathcal{S}_n$ into $\mathcal{S}_n$.*

For a proof see Rudin[1] p.168 theorem 7.4.

Theorem 6.4 *The Fourier transform is a continuous linear bijective mapping of $\mathcal{S}_n'$ onto $\mathcal{S}_n'$ of period 4, whose inverse is also continuous.*[12]

[12] The topology to which this statement refers is the weak* topology that $\mathcal{S}_n$ induces on $\mathcal{S}_n'$

Proof
Let W be a neighborhood of 0 in $\mathcal{S}'_n$. Then by definition of the weak* topology (see ??) there exist $\epsilon > 0$, $k \in \mathbb{N}$ and $\phi_i \in \mathcal{S}_n$ such that the neighborhood

$$N(0, \epsilon, \phi_1, \ldots, \phi_k) = \{u \in \mathcal{S}'_n \mid \forall_{i \in \{1,\ldots,k\}} \mid |u(\phi_i)| < \epsilon\}$$

is a subset of W.
Define

$$V = \{u \in \mathcal{S}'_n \mid \forall_{i \in \{1,\ldots,k\}} |u(\hat{\phi}_i)| < \epsilon\}$$

Then because of the fact that $\hat{\phi}_i \in \mathcal{S}_n$ for $i = 1, \ldots, n$ (see theorem 6.3) , V is again a neighborhood of 0 in $\mathcal{S}'_n$.
Moreover, because of (6.1) we see that the Fourier transform $\mathcal{F}$ maps V into W. This proves the continuity of $\mathcal{F}$.
Another direct consequence of (6.1) is that $\mathcal{F}$ has period 4 on $\mathcal{S}'_n$, since it has period 4 on $\mathcal{S}_n$. Hence $\mathcal{F}$ is one-to-one and onto.
$\mathcal{F}^{-1} = \mathcal{F}^3$ implies that $\mathcal{F}^{-1}$ is continuous. □

Theorem 6.5 *The space $\mathbb{H}_s$ is both isomorphic and isometric to the space $\mathbb{L}_2(\mu_s)$.*

Proof
From definition of $\mathbb{H}_s$ and from Theorem 6.2 it follows that the Fourier transform maps $\mathbb{H}_s \subset S'_n$ into $\mathbb{L}_2(\mu_s) \subset S'_n$, i.e.

$$\mathcal{F}(\mathbb{H}_s) \subset \mathbb{L}_2(\mu_s) \tag{6.5}$$

Theorem 6.4 states that the Fourier transform is an isomorphic mapping from S'_n onto itself.
So, it remains to be shown that $\mathcal{F}(\mathbb{H}_s) = \mathbb{L}_2(\mu_s)$. This follows by (6.5) and $\mathcal{F}^{-1}(\mathbb{L}_2(\mu_s)) \subset \mathbb{H}_s$.
Finally, note that (6.4) implies that $\mathcal{F}$ is isometric. □

Note that : $\mathbb{H}_0 = \mathbb{L}_2$ because of the Plancherel theorem.

Because of Riesz representation theorem, an isomorphism between Hilbert spaces yields an isomorphism between their dual spaces:

$$\mathbb{L}_2^* = \mathbb{L}_2 \Leftrightarrow \mathbb{L}_2^*(\mu_{-s}) = \mathbb{L}_2(\mu_s) \Leftrightarrow \mathbb{H}_s^* = \mathbb{H}_{-s} \tag{6.6}$$

Further note that

if $t < s$ then $\mathbb{L}_2(\mu_s) \subset \mathbb{L}_2(\mu_t)$ and therefore $\mathbb{H}_s \subset \mathbb{H}_t$

The union X of all spaces $\mathbb{H}_s$ is therefore a vector space. A linear operator $\lambda : X \mapsto X$ is said to have *order* t if the restriction of Λ to each $\mathbb{H}_s$ is a continuous mapping of $\mathbb{H}_s$ into $\mathbb{H}_{s-t}$.

Examples:

1. Let $t \in \mathbb{R}$ then the mapping $u \mapsto v$ given by

$$\hat{v}(\mathrm{y}) = (1+|\mathrm{y}|^2)^{\frac{t}{2}}\hat{u}(\mathrm{y}) \qquad (\mathrm{y} \in \mathbb{R}^n) \tag{6.7}$$

is a linear isometry of $\mathbb{H}^s$ onto $\mathbb{H}^{s-t}$ and is therefore an operator of order t. Its inverse has order $-t$.

2. If $b \in \mathbb{L}^\infty$ the mapping $u \mapsto v$ given by $\hat{v} = b\hat{u}$ is an operator of order 0.

3. For every multi-index α, the operator D_α is an operator of order $|\alpha|$, since (NB. $|\widehat{D_\alpha u}(y)| = |y^\alpha||\hat{u}(y)| \leq (1+|y|^2)^{\frac{|\alpha|}{2}}|\hat{u}(y)|$ and therefore $||D_\alpha u||_{s-|\alpha|} \leq ||u||_s$)

Definition 6.2 *Let Ω be open in $\mathbb{R}^n$. A distribution $u \in \mathcal{D}'(\Omega)$ is said to be locally $\mathbb{H}^s$ if there corresponds to each point $x \in \Omega$ a distribution $v \in \mathbb{H}^s$ such that $u = v$ in some neighborhood ω of x. This means that $u(\phi) = v(\phi)$ for all $\phi \in \mathcal{D}(\omega)$.*

In theorem 6.6 we will see that if the local behaviour of a distribution is known, then it is possible to describe a distribution globally.

Theorem 6.6 *Suppose Γ is an open cover of an open set $\Omega \subset \mathbb{R}^n$ and suppose that for each $\omega \in \Gamma$ there exists a distribution $\Lambda_\omega \in \mathcal{D}'(\omega)$ such that*

$$\Lambda_{\omega'} = \Lambda_{\omega''} \text{ in } \omega' \cap \omega'' \tag{6.8}$$

whenever $\omega' \cap \omega'' \neq \emptyset$.
Then there exists a unique $\Lambda \in \mathcal{D}'(\Omega)$ such that

$$\forall_{\omega \in \Gamma} : \Lambda = \Lambda_\omega \text{ in } \omega \tag{6.9}$$

Proof
See Rudin[1] Theorem 6.20 p.147
The next theorem has some very important consequences. For a proof of this theorem the reader is referred to literature, since it lies far beyond the scope of this report.

Theorem 6.7 *Assume Ω is an open set in $\mathbb{R}^n$ and*

- *L is a linear elliptic differential operator in Ω, of order $N \geq 1$, with coefficients in $C^\infty(\Omega)$*
- *u and v are distributions $\mathcal{D}'(\Omega)$ that satisfy*

$$Lu = v \text{ on } \Omega \tag{6.10}$$

- *and v is locally $\mathbb{H}_s$.*

Then u is locally $\mathbb{H}_{s+N}$.

Proof
See Rudin[1] Theorem 8.12 p.201-203

Remark: Rudin[1] also shows that u is locally in $\mathbb{H}_s$ if and only if $D_\alpha u$ is locally in $\mathbb{L}_2$ for every α with $|\alpha| \leq s$ (Theorem 8.11 Rudin[1] Theorem 8.11 p.200-201). By Sobolevs lemma (see 6.15) it now follows that if $s > p + \frac{n}{2}$, then there exists a function $f_0 \in C^p(\Omega)$ such that $f_0(x) = f(x)$ for almost every $x \in \Omega$.
This means that if $v \in C^\infty(\Omega)$, then every solution u of (6.10) belongs to $C^\infty(\Omega)$. Moreover, if L is elliptic differential operator in $\mathbb{R}^n$, with constant coefficients, and E is the fundamental solution of L, then we have that, except at the origin, E is infinitely differentiable function. Since outside the origin the equation $LE = \delta_0$ reduces to $LE = 0$.

Theorem 6.8 *Every distribution with compact support ($u \in \mathcal{D}(\mathbb{R}^n)$) lies in a certain $\mathbb{H}_s$*

Proof
See Folland[11] Theorem 6.30.
corollary
The Dirac distribution $\delta_{\mathbf{a}}$ lies in some $\mathbb{H}_s$. Note that $s < 0$. In the next theorem we will show that any $s < -\frac{n}{2}$ will do.

Theorem 6.9 *Let $a \in \mathbb{R}^n$. The Dirac distribution $\delta_a \in \mathbb{H}_s$ for $s < -\frac{n}{2}$*

Proof
Let $s = 1 + \delta$, with $\delta > 0$. We will show that $\delta_{\mathbf{x}} \in \mathbb{H}_s^*$. (NB. see (6.6)
For $u \in \mathbb{H}_s \cap \mathcal{D}(\Omega)$ we have:

$$\begin{aligned} u(\mathbf{a}) &= \langle \delta_{\mathbf{a}}, u \rangle \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \hat{u}(\mathbf{y}) e^{i\mathbf{y}\cdot\mathbf{a}}\, dy \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \sqrt{1 + \|\mathbf{y}\|^{2(1+\delta)}}\, \hat{u}(\mathbf{y}) \frac{e^{i\mathbf{y}\cdot\mathbf{a}}}{\sqrt{1+\|\mathbf{y}\|^{2(1+\delta)}}}\, d\mathbf{y} \\ &\leq \frac{1}{(2\pi)^{\frac{n}{2}}} \left\| (1 + \|\mathbf{y}\|^{2(1+\delta)}) \hat{u}(\mathbf{y}) \right\|_{\mathbb{L}_2} \left\| \frac{1}{\sqrt{(1+\|\mathbf{y}\|^{2(1+\delta)})}} \right\|_{\mathbb{L}_2} \\ &\leq C \, \|\hat{u}(\mathbf{y})\|_{\mathbb{H}_s} \end{aligned} \tag{6.11}$$

An additional remark according to the constant C above:
If $s > \frac{n}{2}$ we have

$$\| \frac{1}{\sqrt{1 + \|\mathbf{y}\|^{2s}}} \|_{\mathbb{L}_2}^2 = \int_{\mathbb{R}^n} \frac{1}{1 + \|\mathbf{y}\|^{2s}}\, d\mathbf{y} = \omega_n \int_0^\infty \frac{r^{n-1}}{1 + r^{2s}} < \infty. \qquad \square \tag{6.12}$$

6.2 The Sobolev spaces $W_{l,2}(\Omega)$

Definition 6.3 *The Sobolev spaces* $W_{l,2}(\Omega)$ *are defined by*
$W_{l,2}(\Omega) = \{\phi \in \mathbb{L}_2(\Omega) | \forall_{|\alpha| \leq l} \mid D^\alpha \phi \in \mathbb{L}_2(\Omega)\}$,
where D^α *denote the distributional derivatives as introduced in the preliminaries of this chapter.*
Give $W_{l,2}(\Omega)$ *the inner product*

$$(\phi, \psi)_l = \sum_{|\alpha| \leq l} (D^\alpha \phi, D^\alpha \psi)_{\mathbb{L}_2(\Omega)}$$

Theorem 6.10 *The Sobolev space* $W_{l,2}(\Omega)$ *is a separable Hilbert space, hence* $W_{l,2}(\Omega)$ *has a countable basis.*

Proof
First check the completeness.
Let $\{\phi_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $W_{l,2}(\Omega)$.
Let α be a multi-index. Because

$$||D^\alpha f||_{\mathbb{L}_2(\Omega)} \leq ||f||_{W_2^l(\Omega)} \tag{6.13}$$

the sequence $\{D^\alpha \phi_n\}$ is a Cauchy sequence in $\mathbb{L}_2(\Omega)$ and therefore convergent to a certain ϕ^α.
It remains to be shown that

$$D^\alpha \phi^0 = \phi^\alpha \text{ for all } |\alpha| \leq l \tag{6.14}$$

Let $\psi \in \mathcal{D}(\Omega)$. Then on the one hand

$$\int_\Omega D^\alpha \phi_n(\mathrm{x}) \psi(\mathrm{x}) dx \to \int_\Omega \phi^\alpha(\mathrm{x}) \psi(\mathrm{x}) dx \tag{6.15}$$

and on the other hand

$$\begin{aligned} \int_\Omega D^\alpha \phi_n(\mathrm{x})\ \psi(\mathrm{x}) dx &= (-1)^{|\alpha|} \int_\Omega \phi_n(\mathrm{x})\ D^\alpha \psi(\mathrm{x}) dx \to (-1)^{|\alpha|} \int_\Omega \phi^0(\mathrm{x})\ D^\alpha \psi(\mathrm{x}) dx \\ & \int_\Omega D^\alpha \phi^0(\mathrm{x})\ \psi(\mathrm{x}) dx \end{aligned} \tag{6.16}$$

So because of the uniqueness of a limit (6.14) follows by (6.15) and (6.16). As a result the Cauchy sequence $\{\phi_n\}$ is convergent and the completeness is now shown.
$W_{l,2}(\Omega)$ is separable because it is a subspace of $\mathbb{L}_2(\Omega)$ which is separable.[13] □
$W_{l,2}(\Omega)$ has the following density property:

$$W_{l,2}(\Omega) = \overline{\{f \in W_{l,2}(\Omega) \mid \text{supp} f \text{ is bounded }\}} \tag{6.17}$$

For a proof, see (Wloka[2] p.65).

Definition 6.4 *We denote the closure of* $\mathcal{D}(\Omega)$ *in* $W_{l,2}(\Omega)$ *as the space* $\overset{\circ}{\mathcal{W}}_{l,2}(\Omega)$

$$\overset{\circ}{\mathcal{W}}_{l,2}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W_{l,2}} \tag{6.18}$$

$\overset{\circ}{\mathcal{W}}_{l,2}(\Omega)$ is a closed subspace of the separable Hilbert space $W_{l,2}(\Omega)$ and therefore it is again a separable Hilbert space. In the next theorem we obtain a nice density property.

[13] Let $\{h_k\}_{k \in \mathbb{N}}$ be dense in $\mathbb{L}_2(\Omega)$. Then the sequence $\{g_k\}$, where g_k is the orthogonal projection of h_k on $W_{l,2}(\Omega)$, is dense in $W_{l,2}(\Omega)$

Theorem 6.11

$$\overset{\circ}{\mathcal{W}}_{l,2}(\Omega) = \overline{\{\phi \in W_2^l(\Omega) \mid supp\ \phi \subsetneqq \Omega\}}^{W_2^l} \tag{6.19}$$

Proof
Since by definition $\mathcal{D}(\Omega)$ is dense in $\overset{\circ}{\mathcal{W}}_{l,2}(\Omega)$ we only need to show that any $\phi \in W_2^l(\Omega)$ can be approximated by functions from $\mathcal{D}(\Omega)$.
Let $\psi \in W_2^l(\Omega)$ with $\text{supp}(\psi) \subsetneqq \Omega$.
Let $\epsilon > 0$. The *regularisation* of ψ, ψ_ϵ is given by

$$\psi_\epsilon = h_\epsilon * \psi = \int_{\mathbb{R}^n} \psi(\mathbf{y}) h_\epsilon(\mathbf{x}-\mathbf{y}) d\mathbf{y}$$

where h_ϵ is defined by

$$h_\epsilon(\mathbf{x}) = \frac{1}{\epsilon^n} \frac{h(\mathbf{x}/\epsilon)}{\int_{\mathbb{R}^n} h(\mathbf{x}) d\mathbf{x}} \tag{6.20}$$

and $h : \mathbb{R}^n \mapsto \mathbb{R}$ is given by:

$$h(\mathbf{x}) = \begin{cases} 0 & |\mathbf{x}| \geq 1 \\ e^{\frac{-1}{1-|\mathbf{x}|^2}} & |\mathbf{x}| < 1 \end{cases}$$

Note that supp $h_\epsilon = \overline{B_{0,\epsilon}}$ and that $||h_\epsilon||_{\mathbb{L}_1(\mathbb{R}^n)} = 1$.
The following statements can be shown about regularisations:

1. $\phi \in \mathbb{L}_1(\Omega)$ with compact support $K \Rightarrow$
   supp $\phi_\epsilon \subset \{\mathbf{x} \in \mathbb{R}^n \mid d(\mathbf{x}, K) \leq \epsilon\}$ which is again compact.
   For ϵ sufficiently small : $\phi_\epsilon \in \mathcal{D}(\Omega)$
2. $\phi \in \mathbb{L}_p(\Omega) \Rightarrow \phi_\epsilon \mapsto \phi \quad (\epsilon \downarrow 0)$ in $\mathbb{L}_p$ sense.
3. $\phi \in C(\Omega) \Rightarrow \phi_\epsilon \mapsto \phi \quad (\epsilon \downarrow 0)$ uniform.
4. $\phi \in W_2^l(\Omega) \Rightarrow \phi_\epsilon \mapsto \phi \quad (\epsilon \downarrow 0)$ in $\mathbb{W}_2^l$ sense.

For the proof of these statements see (Wloka[2] theorem 1.3 p.9-10 and theorem 3.3 p.66-67).
If $\text{supp}(\psi)$ is compact then use statement four and the final remark in statement one so that
$\psi_\epsilon \mapsto \psi$ in $W_{l,2}(\Omega)$

But unfortunately, $\text{supp}(\psi)$ need not be compact.

Therefore we multiply ψ with a blob-function $k_N \quad N \in \mathbb{N}$, with

$$\begin{cases} 0 \leq k_N \leq 1 \\ k_N \in \mathcal{D}(\mathbb{R}^n) \\ k_N(x) = 1 \quad \forall_{||x|| \leq N} \\ \sup_{|\alpha| \leq l} \sup_{x \in \Omega} \sup_{N \in \mathbb{N}} |(D_\alpha k_N)(x)| < \infty \end{cases}$$

Since, $\text{supp}(\psi\, k_N) \subsetneqq \Omega$ is compact we can apply the above strategy on $\psi\, k_N$. It remains to proof that $\psi\, k_N \to \psi$ if $N \to \infty$ in $W_{l,2}(\Omega)$.

We have $\lim_{N\to\infty} (\psi\, k_N) = \psi$ in $\mathbb{L}_2(\Omega)$, since we have that ψ and $k_N \psi \quad (N \in \mathbb{N})$ are integrable, $\lim_{N\to\infty} \psi(x)\, k_N(x) = \psi(x)$ pointwise and

$$\psi(x)\, k_N(x) \leq \psi(x) \text{ for all } x \in \Omega \text{ and } N \in \mathbb{N},$$

so that Lebesgue's dominated convergence principle can be applied:

$$\lim_{N\to\infty} \|\psi - k_N\,\psi\|^2_{\mathbb{L}_2(\Omega)} = \lim_{N\to\infty} \int_\Omega |\psi|^2 (1-k_N)^2 = 0\,.$$

We must also show that for all $|\alpha| \le l$:

$$\lim_{N\to\infty} D^\alpha(\psi\,k_N) = D^\alpha\psi \text{ in } \mathbb{L}_2(\Omega). \tag{6.21}$$

We will first show

$$\lim_{N\to\infty} \partial_i(\psi\,k_N) = \lim_{N\to\infty} \partial_i(\psi)\,k_N + \lim_{N\to\infty} \psi\,(\partial_i k_N) = \partial_i\psi \text{ in } \mathbb{L}_2(\Omega). \tag{6.22}$$

Again using Lebesgue's dominated convergence principle, with $\partial_i\psi$ in stead of ψ we have:

$$\lim_{N\to\infty} (\partial_i\psi) k_N = \partial\psi \text{ in } \mathbb{L}_2(\Omega)\,.$$

Now (6.22) follows since again by the dominated convergence principle (NB. $1_{B^c_n}(x) \to 0$ pointwise),

$$\lim_{N\to\infty} \int_{\|x\|>N} |\psi(x)|^2\,dx = \lim_{N\to\infty} \int_{\mathbb{R}^n} |\psi(x)|^2 1_{B^c_N} = 0.$$

For general $|\alpha| < l$ (i.e. (6.21)) we use Leibnitz rule and the term with all derivatives applied to ψ gives the right limit, the others all vanish. □

The equalities (6.19) and (6.17) imply that $W_{l,2}(\mathbb{R}^n) = \overset{\circ}{W}_{l,2}(\mathbb{R}^n)$ because a subspace of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.
It can be shown that

$$\overline{C^\infty(\Omega) \cap W_{l,2}(\Omega)} = W_{l,2}(\Omega) \tag{6.23}$$

See for instance Wloka[2] theorem 3.6 p.69-72 or Ziemer[9] theorem 2.3.2, p.54-55

Note that:

- In some books, take for instance (Yosida[3] p.57 and p.41) $\mathbb{H}_s(\Omega)$ is defined as the completion of the pre-Hilbert space

  $$\hat{H}_s(\Omega) = \{f \in C^s(\Omega) \mid \|f\|_{W_{s,2}(\Omega)} < \infty\}.$$

  Equation (6.23) shows that such a definition leads (in general to a proper) subspace of $W_{l,2}(\Omega)$. In the sequel we will *not* use the definition for $\mathbb{H}_s(\Omega)$ from above. See definition (6.5).

- The approximating space $C^\infty(\Omega) \cap W_{l,2}(\Omega)$ admits functions that are not smooth across the boundary of Ω and therefore it is natural to ask whether it is possible to approximate functions in $W_{l,2}(\Omega)$ by a nicer space, say

  $$C^\infty(\overline{\Omega}) \cap W_{l,2}(\Omega) \tag{6.24}$$

  In general, this is seen to be false:
  Consider the domain Ω defined as an n-dimensional ball with its equatorial $(n-1)$-plane deleted. The function u definedby $u \equiv 1$ on the top half-ball and $u \equiv -1$ on the bottom half-ball is clearly an element of $W_{l,2}(\Omega)$ that cannot be closely approximated by an element in (6.24). The difficulty here is that the domain lies on both sides of the boundary. If the domain lies on both sides of part of its boundary. If Ω possesses the segment property, it has been shown in Adams[10] Theorem 3.18, that the space (6.24) is then dense in $W_{l,2}(\Omega)$.

The spaces $W_{l,2}(\mathbb{R}^n)$ and $\mathbb{H}_l$,with $l \in \mathbb{N}$ are equivalent (in set theoretical and topological sence)as will be shown by the next theorem.

Theorem 6.12 *For $l \geq 0$ the Fourier transformation $\mathcal{F} : W_{l,2}(\mathbb{R}^n) \mapsto \mathbb{L}_2(\mu_l)$ is a (topological) isomorphism.*
Therefore, the norms $\|\cdot\|_{\mathbb{H}^l}$ and $\|\cdot\|_{W^{l,2}}$ on $W^{l,2}(\mathbb{R}^n)$ are equivalent.

Proof
Using Parseval one gets:

$$\|\phi\|^2_{W^l_2} = \sum_{|s|\leq l} \int_{\mathbb{R}^n} |D^s\phi|^2 d\mathbf{x} = \int_{\mathbb{R}^n} |\hat{\phi}|^2 \sum_{|s|\leq l} |\mathbf{y}^s|^2 d\mathbf{y} \tag{6.25}$$

The result now follows from the inequalities

$$\frac{1}{2^{2l}}(1+|\mathbf{y}|^2)^l \leq \sum_{|s|\leq l} |\mathbf{y}^{2s}| \leq (1+|\mathbf{y}|^2)^l \qquad \Box \tag{6.26}$$

In order to find such an equivalence between H_s and $W_{l,2}$ on a " arbitrary " region (As we will see a condition must be imposed to boundary) we must define $H_s(\Omega)$ first:

Definition 6.5 *The space $\mathbb{H}_s(\Omega)$ is defined by*

$$\mathbb{H}_s(\Omega) = \{\phi|_\Omega \mid \phi \in \mathbb{H}_s\}$$

equiped with norm:
$$\|\phi\|_{\mathbb{H}_s(\Omega)} = \inf_{\phi^c \in \mathbb{H}_s} \|\phi^c\|_{\mathbb{H}_s}. \tag{6.27}$$

where the infimum is taken over all distributions $\phi^c \in \mathbb{H}^s$, whose restriction to Ω gives the element ϕ, that is $R_\Omega \phi^c = \phi \in \mathbb{H}^s(\Omega)$.

Theorem 6.13 *The space $\mathbb{H}_s(\Omega)$ is complete.*

Proof
Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathbb{H}_s(\Omega)$ with[14] $\sum_{n=1}^{\infty} \|\phi_n - \phi_{n+1}\|_{\mathbb{H}_s(\Omega)} < \infty$
Then for all $n \in \mathbb{N}$ there exist an extension ϕ^c_n of ϕ_n such that

$$\|\phi^c_n - \phi^c_{n+1}\|_{\mathbb{H}_s} \leq 2\,\|\phi_n - \phi_{n+1}\|_{\mathbb{H}_s(\Omega)}.$$

Then

$$\sum_{n=1}^{\infty} \|\phi^c_n - \phi^c_{n+1}\|_{\mathbb{H}_s} < \infty.$$

So $\{\phi^c_n\}$ is a Cauchy sequence in the Hilbert space $\mathbb{H}_s$ and therefore convergent to, say ψ. Set $\phi_0 = \psi|_\Omega$. Since $(\|\phi_n - \phi_0\|_{\mathbb{H}_s(\Omega)} \leq \|\phi^c_n - \psi\|_{\mathbb{H}_s})$, the sequence $\{\phi_n\}$ converges to the restriction of ϕ_0 $\quad\Box$.

Definition 6.6 *The space $\mathring{\mathbb{H}}_l(\Omega)$ is defined by*

$$\mathring{\mathbb{H}}_l(\Omega) = \overline{\mathcal{D}(\Omega)}^{\mathbb{H}_s} \tag{6.28}$$

[14] This property can be assumed without loss of generality since there can always be found a subsequence that has this property. And it is easy to see that if a Cauchy sequence has a convergent subsequence then it converges to the same limit.

Note that if $f \in \mathcal{D}(\Omega)$ then f can be extended to $\mathbb{R}^n$ by zero. This means that the restriction of the norm $\|\cdot\|_{\mathcal{H}_s(\Omega)}$ to $\mathcal{D}(\Omega)$ is equivalent to the restriction of the norm $\|\cdot\|_{W_{s,2}(\Omega)}$ to $\mathcal{D}(\Omega)$. Therefore,

$$\overset{\circ}{\mathcal{W}}_{l,2}(\Omega) \simeq \overset{\circ}{\mathbb{H}}_l(\Omega) \tag{6.29}$$

Note that the extension by zero can not be applied in order to show that in general regions[15] $\mathbb{H}_s(\Omega) \simeq W_{s,2}(\Omega)$. So, we need other extensions. To construct them we first need to specify their left inverses which are called restriction operators.

Elements from $W_{l,2}(\Omega)$ respectively $\mathbb{H}_s(\Omega)$ are distributions, therefore the restriction operator $R^{\Omega}_{\Omega'}$ from a larger to a smaller set $(\Omega' \subset \Omega)$ are always defined. Obviously the restriction operator $R^{\Omega}_{\Omega'} : W_{l,2}(\Omega) \mapsto W_{l,2}(\Omega)$ is continuous.

Definition 6.7 *Let $\Omega' \subset \Omega \subset \mathbb{R}^n$ be open. An extension operator from Ω' to Ω is a linear operator*

$$F^{\Omega}_{\Omega'} : W_{l,2}(\Omega') \mapsto W_{l,2}(\Omega)$$

with the property

$$R^{\Omega}_{\Omega'} \circ F^{\Omega}_{\Omega'} = I_{W_{l,2}(\Omega')}$$

Theorem 6.14 *For the space $W_{l,2}(\Omega)$, $l \in \mathbb{N}$, let there exist a continuous extension operator $F^{\mathbb{R}^n}_{\Omega} : W_{l,2}(\Omega) \mapsto W_{l,2}(\mathbb{R}^n)$. Then*

$$W_{l,2}(\Omega) \simeq \mathbb{H}_l(\Omega) \tag{6.30}$$

both set theoretically and topologically.

Proof

First we show $\mathbb{H}_l(\Omega) \subset W_{l,2}(\Omega)$. Any ϕ in $\mathbb{H}_l(\Omega)$ has by definition an extension ϕ^c to $\mathbb{H}_l(\mathbb{R}^n)$ which equals $W_{l,2}(\mathbb{R}^n)$ according to theorem 6.12 and $\phi^c \in W_{l,2}(\mathbb{R}^n)$ implies that $\phi \in W_{l,2}(\Omega)$.

So, $\mathbb{H}_l(\Omega)$ consists of those functions $\phi \in W_{l,2}(\Omega)$ that have at least one extension $\phi^c \in W_{l,2}(\mathbb{R}^n)$. Since, by assumption we may put $\phi^c = F^{\mathbb{R}^n}_{\Omega}$, we have that $\mathbb{H}_l(\Omega) = W_{l,2}(\Omega)$ set theoretically.

Obviously, we have that

$$\|\phi\|_{\mathbb{H}_l(\Omega)} \leq \|F^{\mathbb{R}^n}_{\Omega}\phi\|_{W^l_2(\mathbb{R}^n)}$$

and from the continuity of $F^{\mathbb{R}^n}_{\Omega}$ it follows that there exists a $C > 0$ such that

$$\|F^{\mathbb{R}^n}_{\Omega}\phi\|_{W_{l,2}(\mathbb{R}^n)} \leq C\|\phi\|_{W_{l,2}(\Omega)}.$$

So, we have that

$$C\|\phi\|_{W_{l,2}(\Omega)} \geq \|\phi\|_{\mathbb{H}_l(\Omega)}$$

On the other hand the continuity of the restriction operator $R^{\mathbb{R}^n}_{\Omega}$ gives

$$\|\phi\|_{W_{l,2}(\Omega)} \leq \|R^{\mathbb{R}^n}_{\Omega}\ \phi^c\| \leq \|\phi^c\|_{W_{l,2}(\mathbb{R}^n)}$$

So, after taking the infimum we obtain

$$\|\phi\|_{W_{l,2}(\Omega)} \leq \|\phi\|_{\mathbb{H}_l(\Omega)} \qquad \square$$

Next we state a fundamental theorem according to Sobolev Spaces.

Theorem 6.15 *(Sobolev's lemma). Let Ω be an open subset of $\mathbb{R}^n$. Let $\sigma \geq 0$ such that $u \in W^k_2(\Omega)$ and $k > \frac{n}{2} + \sigma$. Then, for any open subset $\Omega' \subset \Omega$ such that $\overline{\Omega'}$ is a compact subset of Ω, there exists a function $w \in C^{\sigma}(\Omega')$ such that $u(x) = w(x)$ for almost every $x \in \Omega'$.*

Proof

See Rudin[1] Theorem 7.25 p.186-187 or Yosida[3] p.174-175.

[15] The spaces $\mathbb{H}_s(\Omega)$ can be genuine subspaces of $W_{l,2}(\Omega)$, see [??].

6.3 Compact embeddings - Ehrlings lemma

Let V, W be vector spaces such that V is dense in W. If the identity map $i : V \to W$ is compact, i.e. every bounded sequence in V consists a subsequence which is convergent in W, we say the embedding $V \hookrightarrow W$ is compact. Note that this means that the unit ball in V is a compact set in W.

Theorem 6.16 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $l_2 < l_1$; $l_1, l_2 \in \mathbb{N}$. Then the embedding $\mathring{W}_{l_1,2}(\Omega) \hookrightarrow \mathring{W}_{l_2,2}(\Omega)$ is compact. If Ω also satisfies the uniform cone condition then the embedding $W_{l_1,2}(\Omega) \hookrightarrow W_{l_2,2}(\Omega)$ is also compact.*

Proof
For a proof of the first statement see Wloka[2] Theorem 7.1 p.113.
For the compactness of $W_{l_1,2}(\Omega) \hookrightarrow W_{l_2,2}(\Omega)$ (under the above conditions) see, Wloka[2] Theorem 7.2 p.114, Corollary 5.1 p.100 and Theorem 5.4 p.95-96.
A sketch of the second proof:
Under the above conditions on Ω there exists a continuous extension operator $F_{\Omega}^{\Omega_\epsilon}$ in which Ω_ϵ is some epsilon neighborhood of Ω. Consider the scheme

$$\mathring{W}_{l,2}(\Omega) \underset{F_{\Omega}^{\Omega_\epsilon}}{\longrightarrow} \mathring{W}_{l_1,2}(\Omega_\epsilon) \hookrightarrow \mathring{W}_{l_2,2}(\Omega_\epsilon) \underset{R_{\Omega}^{\Omega_\epsilon}}{\longrightarrow} W_{l_2,2}(\Omega) \tag{6.31}$$

By the first part of the proof we have that the embedding $\hookrightarrow$ is compact. A composition of a compact operator and continuous operators is again compact. Therefore, the composite map $W_{l_1,2}(\Omega) \hookrightarrow W_{l_2,2}(\Omega)$ is compact.

We now want to prove Ehrling's lemma. First the abstract version:

Theorem 6.17 *Let X_1, X_2, X_3 be normed spaces, $A : X_1 \to X_2$ compact, $T : X_2 \to X_3$ a continuous injection. Then for each $\epsilon > 0$ there exists a constant $c(\epsilon)$ with*

$$||Ax||_2 \leq \epsilon||x||_1 + c(\epsilon)||TAx||_3 \qquad \text{for all } x \in X_1. \tag{6.32}$$

Proof
We assume that (6.32) fails for some $\epsilon_0 > 0$. Then for each $n \in \mathbb{N}$ there exists some $\mathbf{x}_n \in X_1$ (without loss of generality let $||\mathbf{x}_n||_1 = 1$) with

$$||A\mathbf{x}_n||_2 > \epsilon_0 + n||TA\mathbf{x}_n||_3 \tag{6.33}$$

Since A is continuous we have $||A\mathbf{x}_n||_2 \leq ||A||$ for all $n \in \mathbb{N}$. Therefore we have

$$||TA\mathbf{x}_n||_3 < \frac{||A||}{n} - \frac{\epsilon_0}{n}.$$

As a result

$$TA\mathbf{x}_n \to 0 \text{ as } n \to \infty \tag{6.34}$$

On the other hand by assumption $\{A\mathbf{x}_n\}_{n\in\mathbb{N}}$ is relatively compact. Hence there exists a subsequence $\{A\mathbf{x}_{n_l}\}_{l\in\mathbb{N}}$ which converges to some element $\mathbf{y}$ in X_2. Since T is continuous we have that $TA\mathbf{x}_{n_l} \to T\mathbf{y}$ as $l \to \infty$. From (6.34) it follows that $T\mathbf{y} = 0$. T was assumed to be injective, so $\mathbf{y} = 0$. i.e. $A\mathbf{x}_n \to 0$ contradicting (6.33). □

corollary (Ehrling's lemma)
Let Ω be bounded and satisfy the uniform cone condition. Let k, l be in $\mathbb{N}$ such

that $1 \leq k \leq l$.
If we take in particular

$$X_1 = W_{l,2}(\Omega)$$
$$X_2 = W_{l-k,2}(\Omega)$$
$$X_3 = W_{0,2}(\Omega) = \mathbb{L}_2(\Omega)$$

and take in particular for A and T respectively the embedding[16] $W_{l,2}(\Omega) \hookrightarrow W_{l-k,2}(\Omega)$ and the embedding $W_{l-k,2}(\Omega) \hookrightarrow \mathbb{L}_2(\Omega)$, then for each $\epsilon > 0$ there exists a constant $c(\epsilon)$, so that for all $\phi \in W_{l,2}(\Omega)$ we have

$$||\phi||_{W_{l-k,2}(\Omega)} \leq \epsilon ||\phi||_{W_{l,2}(\Omega)} + c(\epsilon)||\phi||_{\mathbb{L}_2(\Omega)}$$

We can use Ehrling's lemma to introduce a new equivalent norm on the space $W_{l,2}(\Omega)$ (or $\overset{\circ}{W}_{l,2}(\Omega)$ if you like) :

Theorem 6.18 *Let Ω be bounded and satisfy the uniform cone condition and let $l \in \mathbb{N}$. The norms defined by the equations*

$$||\phi||^2_{W_{l,2}(\Omega)} = \sum_{|s|\leq l} ||D^s\phi||^2_{\mathbb{L}_2(\Omega)}$$

and

$$||\phi||^2 = ||\phi||^2_{\mathbb{L}_2(\Omega)} + \sum_{|s|=l} ||D^s\phi||^2_{\mathbb{L}_2(\Omega)}$$

on $W_{l,2}(\Omega)$ are equivalent.

Proof
It is clear that $||\phi|| \leq ||\phi||_{W_{l,2}(\Omega)}$ for all $\phi \in W_{l,2}(\Omega)$. By Ehrling's lemma we have for $\epsilon = \frac{1}{2}$ there exists some $c = c(\frac{1}{2})$ with

$$||\phi||_{W_{l-1,2}(\Omega)} \leq \frac{1}{2}||\phi||_{W_{l,2}(\Omega)} + c||\phi||_{\mathbb{L}_2(\Omega)},$$

from which it follows that

$$\begin{aligned} ||\phi||^2_{W_{l-1,2}(\Omega)} &\leq 2\left(\tfrac{1}{4}||\phi||^2_{W_{l,2}(\Omega)} + c^2||\phi||^2_{\mathbb{L}_2(\Omega)}\right) \\ &= \tfrac{1}{2}||\phi||^2_{W_{l-1,2}(\Omega)} + \tfrac{1}{2}\sum_{|s|=l} ||D^s\phi||^2_{\mathbb{L}_2(\Omega)} + 2c^2||\phi||^2_{\mathbb{L}_2(\Omega)}. \end{aligned}$$

So,

$$||\phi||^2_{W_{l-1,2}(\Omega)} \leq 4c^2||\phi||^2_{\mathbb{L}_2(\Omega)} + \sum_{|s|=l} ||D^s\phi||^2_{\mathbb{L}_2(\Omega)}$$

and therefore we obtain

$$\begin{aligned} ||\phi||^2_{W_{l,2}(\Omega)} &= ||\phi||^2_{W_{l-1,2}(\Omega)} + \sum_{|s|=l} ||D^s\phi||^2_{\mathbb{L}_2(\Omega)} \leq 4c^2||\phi||^2_{\mathbb{L}_2(\Omega)} + 2\sum_{|s|=l} ||D^s\phi||^2_{\mathbb{L}_2(\Omega)} \\ &\leq \max\{2, 4c^2\}\, ||\phi||^2 \text{ for all } \phi \in W_{l,2}(\Omega) \qquad \Box \end{aligned}$$

In sloppy language one could say that it is allowed to omit the middle terms in the Sobolev norms. The next paragraph shows whether omitting the $\mathbb{L}_2$-norm part is allowed as well.

[16]The embedding $W_{l,2}(\Omega) \hookrightarrow W_{l-k,2}(\Omega)$ is compact according to theorem 6.16.

6.3.1 Poincaré inequalities

Theorem 6.19 *(first Poincaré inequality) Let Ω be bounded. Then there exists a constant c dependent only on the diameter of Ω, such that for all $\phi \in \mathring{\mathcal{W}}_{1,2}(\Omega)$*

$$||\phi||_{W_{1,2}(\Omega)} \leq c \sum_{|s|=l} \int_{\Omega} |D^s\phi(x)|^2 \, dx \,. \tag{6.35}$$

Proof
By definition we have that $\mathcal{D}(\Omega)$ are dense in $\mathring{\mathcal{W}}_{1,2}(\Omega)$. Therefore we only need to show (6.35) for $\phi \in \mathcal{D}(\Omega)$. Using integration by parts we obtain the following estimate for $\phi \in \mathcal{D}(\Omega)$:

$$\begin{aligned} ||\phi||^2_{\mathbb{L}_2(\omega)} &= \tfrac{1}{n} \sum_{i=1}^{n} \int_{\Omega} |\phi(\mathrm{x})|^2 \cdot 1 \cdot dx = -\tfrac{1}{n} \sum_{i=1}^{n} \int_{\Omega} \tfrac{\partial(\phi(\mathrm{x})\overline{\phi(\mathrm{x})})}{\partial x_i} x_i \, d\mathrm{x} \\ &= -\tfrac{1}{n} \sum_{i=1}^{n} \int_{\Omega} \tfrac{\partial \phi}{\partial x_i} \overline{\phi(\mathrm{x})} \, x_i \, d\mathrm{x} - \tfrac{1}{n} \sum_{i=1}^{n} \int_{\Omega} \phi(\mathrm{x}) \tfrac{\partial \overline{\phi}}{\partial x_i} x_i \, d\mathrm{x} \,. \end{aligned}$$

Let $\Omega \subset \{\mathrm{x} \mid |x_i| \leq d$ for a certain $d > 0$. We apply the Schwarz inequality:

$$||\phi||^2_{\mathbb{L}_2(\Omega)} \leq \frac{2}{n} \sum_{i=1}^{n} \int_{\Omega} \left|\frac{\partial \phi}{\partial x_i}\right| \, |\phi(\mathrm{x})| \, |x_i| \, d\mathrm{x} \leq \frac{2d}{n} ||\phi||_{\mathbb{L}_2(\Omega)} \sum_{i=1}^{n} \left(\int_{\Omega} \left|\frac{\partial \phi}{\partial x_i}\right| \, d\mathrm{x} \right)^{1/2} .$$

After manipulation this gives

$$||\phi||_{\mathbb{L}_2(\Omega)} \leq \frac{2d}{\sqrt{n}} \left[\sum_{i=1}^{n} \int_{\Omega} \left| \frac{\partial \phi}{\partial x_i} \right|^2 \, d\mathrm{x} \right]^{1/2} .$$

From this finally follows

$$||\phi||^2_{W_{1,2}(\Omega)} = ||\phi||^2_{\mathbb{L}_2(\Omega)} + \sum_{|s|=1} ||D^s\phi||^2_{\mathbb{L}_2(\Omega)} \leq \left(\frac{4d^2}{n} + 1 \right) \sum_{|s|=1} ||D^s\phi||^2_{\mathbb{L}_2(\Omega)}. \qquad \square$$

remark
The first Poincaré inequality also holds in regions which are bounded in only one direction. For example suppose $|x_1| \leq d$, then we have

$$||\phi||^2_{\mathbb{L}_2(\Omega)} = \int_{\Omega} |\phi(\mathrm{x})|^2 \cdot 1 \, d\mathrm{x} = -\int_{\Omega} \frac{\partial(\phi\overline{\phi})}{\partial x_1} x_1 \, d\mathrm{x} \leq 2d \, ||\phi||^2_{\mathbb{L}_2(\Omega)} \left[\int_{\Omega} \left| \frac{\partial \phi}{\partial x_1} \right|^2 \, d\mathrm{x} \right]^{1/2} ,$$

from which the further proof follows.

Theorem 6.20 *(second Poincaré inequality) Let Ω be bounded and satisfy the uniform cone condition. Then for all $\phi \in W_{l,2}(\Omega)$ we have the inequality*

$$||\phi||^2_{W_{l,2}(\Omega)} \leq c \left[\sum_{|s|=l} \int_{\Omega} |D^s\phi|^2 \, dx + \sum_{|s|<l} \left| \int_{\Omega} D^s\phi \, dx \right|^2 \right] . \tag{6.36}$$

Note that in case $l = 1$ equation (6.36) is equivalent to:

$$||\phi||^2_{W_{1,2}(\Omega)} \leq c \left[||\nabla\phi||^2_{\mathbb{L}_2(\Omega)} + |(\phi, 1)_{\mathbb{L}_2(\Omega)}|^2 \right]$$

Proof
We first proof the second poincaré inequality for $l = 1$ and then easily generalize this proof to the case where l is arbitrary.
We assume that (6.36) is not correct, then there exists a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset W_{1,2}(\Omega)$ with

$$1 = \|\phi_n\|^2_{W_{1,2}(\Omega)} > n\left[\|\nabla\phi_n\|^2_{\mathbf{L}_2(\Omega)} + |(\phi_n, 1)^2_{\mathbf{L}_2(\Omega)}|\right] \tag{6.37}$$

from which it follows that

$$D^s\phi_n \to 0 \text{ in } \mathbf{L}_2(\Omega) \tag{6.38}$$

for every multi-index s with $|s| = 1$.
By theorem 6.16 we have that $W_{1,2}(\Omega) \hookrightarrow \mathbb{L}_2(\Omega)$ is compact. As a result the sequence $\{\phi_n\}_{n\in\mathbb{N}}$ has a subsequence $\{\phi_{n_l}\}_{l\in\mathbb{N}}$ which converges to say ϕ in $\mathbb{L}_2(\Omega)$. From this together with (6.38) it follows that $\{\phi_{n_l}\}_{l\in\mathbb{N}}$ is a Cauchy sequence in $W_{1,2}(\Omega)$. The space $W_{1,2}(\Omega)$ is complete and therefore the sequence $\{\phi_{n_l}\}_{l\in\mathbb{N}}$ must converge in $W_{1,2}(\Omega)$. From (6.38) it follows that the limit must be ϕ and that $D_s\phi \equiv 0$ for all multi-index s with $|s| = 1$. From this we conclude that ϕ must be a constant (A.E.), contradicting (6.37).

The general case,i.e. $l \in \mathbb{N}$ arbitrary, can be proved analogue. We remark to this end that

- $D^s\phi \equiv 0$ for $s = l$ implies that ϕ is a polynomial of degree $\leq l - 1$.
- Use the embedding $W_{l,2}(\Omega) \hookrightarrow W_{l-1,2}(\Omega)$ in stead of $W_{1,2}(\Omega) \hookrightarrow \mathbb{L}_2(\Omega)$ (which is also compact according to theorem 6.16). □

6.4 Gelfand triples

Before giving the definition(s) of a Gelfand triple we state the following theorem:

Theorem 6.21 *Let X and Y be Banach spaces and $A : X \mapsto Y$ a continuous linear operator from X into Y. Then*

$$\textit{im } (A) \textit{ is dense in } Y \textit{ if and only if } A' \textit{ is injective.} \tag{6.39}$$

Proof
$\Rightarrow$:
Let $y' \in Y'$. Then

$$A'y' = 0 \Leftrightarrow \forall x \in X : \langle y', Ax\rangle = \langle A'y', x\rangle = 0.$$
So, $\overline{im(A)} = Y \Rightarrow y' = 0$

$\Leftarrow$:
Let $\overline{im(A)} \neq Y$. Then by the Hahn-Banach theorem there exists $0 \neq y' \in Y'$ with $\forall_{x\in X}\langle y', Ax\rangle = 0$. Then $\langle A'y, x\rangle = 0$ for all $x \in X$, i.e. $A'y' = 0$. Contradiction to the fact that A' is injective. □

Definition 6.8 *The first definition of a Gelfand triple:*
Let V be a reflexive Banach space ($V'' = V$). Let H be a Hilbert space. Suppose $V \hookrightarrow_i H$ is a continuous injective and dense embedding.
Then the scheme

$$V \hookrightarrow_i H \hookrightarrow_{i'} V' \tag{6.40}$$

is called a Gelfand triple.

Note that:

- Theorem 6.21 implies that:

$$\mathcal{N}(i) = \{0\} \Leftrightarrow \overline{\mathrm{im}(i')} = V' \qquad (6.41)$$
$$\mathcal{N}(i') = \{0\} \Leftrightarrow \overline{\mathrm{im}(i)} = V$$

- i continu means $\exists_{c>0}\forall_{x\in V} : \|ix\|_H \leq C\|x\|_V$. So if an equivalent norm on V is used we can take $C = 1$. Then $\|i'\| = \|i\| \leq 1$ and so,

$$\forall_{x\in V}\|(i'\circ i)(x)\|_{V'} \leq \|ix\|_H \leq \|x\|_V$$

  or if one identifies $i\,x$ with x and $i'\,h$ with h

$$\forall_{x\in V}\|x\|_{V'} \leq \|x\|_H \leq \|x\|_V \qquad (6.42)$$

- Since H' is identified with H one has[17]

$$\forall_{x\in V}\forall_{h\in H} :< i'h, x >_V = (h, ix)_H \qquad (6.43)$$

  so $\|i'h\|_{V'} \leq \|h\|_H$

- Because $\overline{\mathrm{im}(i')} = V'$ any linear functional $x \mapsto \langle x', x\rangle$ on V is the (uniform) limit of functionals of the form $x \mapsto (h, ix)_H$, since :
  If $\{h_n\}_{n\in\mathbb{N}} \subset H$ such that $\lim_{n\to\infty} i'h_n = x'$, then

$$< x', x >_V = \lim_{n\to\infty}(h_n, ix)_H \qquad (6.44)$$

Finally, note that in a Gelfand triple (6.40) we have that i' is continuous injective and dense, $|(h, ix)_H| = |\langle i'h, x\rangle_V| \leq \|x\|_V\|i'h\|_{V'}$ for all $h \in H$ and $x \in V$, and $\|i'h\|_{V'} = \sup_{x\in V\setminus\{0\}} \frac{|\langle i'h,x\rangle|}{\|x\|_V} = \sup_{x\in V\setminus\{0\}} \frac{|(h,ix)|}{\|x\|_V}$ for all $h \in H$. The next theorem gives an abstract version of the reverse of the last remark, i.e. it could have been used as an alternative version of a Gelfand triple.

Theorem 6.22 *Let X_1, X_{-1} be reflexive Banach spaces such that*

$$X_1 \hookrightarrow_{i_1} H \hookrightarrow_{i_{-1}} X_{-1},$$

with i_1 and i_{-1} continuous dense injections such that

$$\exists_{c_1>0}\forall_{x\in X_1}\forall_{h\in H} : |(h, i_1x)_H| \leq c_1\|x\|_1\|i_{-1}h\|_{-1}\ , \qquad (6.45)$$

$$\exists_{c_2>0}\forall_{h\in H} : \|i_{-1}h\|_{-1} \leq c_2 \sup_{0\neq x\in X_1} \frac{|(h, i_1x)|}{\|x\|_1}\ . \qquad (6.46)$$

Then there exists a unique[18] *isomorphism $\Phi : X_{-1} \to X_1'$ such that*

$$[\Phi(i_{-1}(h))](x) = (h, i_1(x))_H \textit{ for all } h \in H \textit{ and } x \in X_1.$$

Moreover, $i_1' = \Phi \circ i_{-1}$.

Remark: If one uses Φ to identify X_{-1} with X_1' then $X_1 \hookrightarrow_{i_1} H \hookrightarrow_{i_{-1}} X_{-1}$ is a Gelfand triple.

[17]The correspondence is given by $h \leftrightarrow F \leftrightarrow F(\cdot) = (h, \cdot)$. So, $(h, ix) = F(ix) = \langle i'F, x\rangle_V = \langle i'h, x\rangle_V$.

[18]Note that $\overline{\mathrm{Im}(i_{-1})} = X_{-1}$ and $\overline{\mathrm{Im}(i_1')} = X_1'$ together with (6.43) imply the uniqueness of Φ.

Proof
See Wloka[2] p.264.

Finally, by using the above theorem a third approach to the structure of Gelfand triples can be made:

Theorem 6.23 *Let X_1 be a reflexive Banach space. Let H be a Hilbert space. Let $X_1 \hookrightarrow_{i_1} H$ be dense continuous and injective. Introduce a second norm on H:*

$$||h||_{-1} = \sup_{x \in X_1} \frac{|(h, i_1 x)|}{||x||_V} \tag{6.47}$$

Denote the completion of H in the norm $||\cdot||_{-1}$ as X_{-1}. Then there exists a unique[19] isomorphism $\Phi : X_{-1} \to X_1'$ such that $[\Phi(h)](x) = (h, i_1 x)$ for all $h \in H$ and $x \in X_1$.

Remark: If one uses Φ to identify X_{-1} with X_1' then $X_1 \hookrightarrow_{i_1} H \hookrightarrow_{i_{-1}} X_{-1}$ is a Gelfand triple.

Proof
Apply theorem 6.22. This can be done because the canonical embedding in the completion i_{-1} is by definition dense, continuous and injective. Further note that equations (6.45) and (6.46) are indeed true (take $c_1 = c_2 = 1$) by definition of $||\cdot||_{-1}$ □

Concrete examples of Gelfand triples:

The following embeddings are injective, continuous and dense therefore they can be extended to Gelfand triples:
Let $l \in \mathbb{N}$[20]

$$\mathbb{H}^l \hookrightarrow \mathbb{H}^0$$
$$\mathbb{H}^l(\Omega) \hookrightarrow \mathbb{H}^0(\Omega)$$
$$W_2^l(\Omega) \hookrightarrow \mathbb{L}_2(\Omega)$$

6.5 The Gelfand triple $\mathbb{H}_2 \hookrightarrow \mathbb{L}_2 \hookrightarrow \mathbb{H}_{-2}$

The regions of Stokes problems are not equal to $\mathbb{R}^n$, therefore studying the Gelfand Triple $\mathbb{H}_2 \hookrightarrow \mathbb{L}_2 \hookrightarrow \mathbb{H}_{-2}$ can only result in theoretical ideas. As noticed in the last paragraphs, there exist a connection between $\mathbb{H}_2$ and $W_{2,2}(\Omega)$. In $\mathbb{H}_2$ (region $\mathbb{R}^n$) it is possible to use Fourier transform.

As was shown in the previous paragraph $\mathbb{H}_s$ is isometrical isomorphic to $\mathbb{L}_2(\mu_s)$. The norm on $\mathbb{H}_2$ is given by:

$$\begin{aligned} ||u||^2_{\mathbb{H}_2} &= \int_{\mathbb{R}^n} |\hat{u}|^2 d\mu_2(\mathrm{y}) \\ &= \int_{\mathbb{R}^n} |\hat{u}(\mathrm{y})|^2 (1 + ||\mathrm{y}||^2)^2 d\mathrm{y} \\ &= \int_{\mathbb{R}^n} |\mathcal{F}\{(1 + |\Delta|)u\}|^2 d\mathrm{y} \\ &= \int_{\mathbb{R}^n} \overline{u}\; u + 2\mathrm{Re}\{\overline{u}\Delta u\} + \Delta u\; \Delta u dx \end{aligned} \tag{6.48}$$

[19] Uniqueness of Φ follows by $\overline{H} = X_{-1}$ (By definition of a completion).
[20] $l \in \mathbb{R}^+$ is also correct, but in this report there is no need to look at those Sobolev spaces

Note that restricted to the Harmonic subspace this norm equals the L_2 norm and that because of (6.26) an equivalent norm on $\mathbb{H}_2$ is given by:

$$||u||^2 = ||u||^2_{\mathbb{L}_2(\Omega)} + ||\Delta u||^2_{\mathbb{L}_2(\Omega)}. \tag{6.49}$$

6.5.1 Projections on harmonic subspaces

Keeping example 6.7 in mind, we see that $(I + |\Delta|)$ is in fact the isometric mapping from $\mathbb{H}_2$ onto $\mathbb{L}_2$ and that $(I + |\Delta|)^{-1}$ is the isometric mapping from $\mathbb{L}_2$ onto $\mathbb{H}_2$. Note that both mappings become the identity if they are restricted to harmonic distributions. This means that both the embedding $\mathbb{H}_{-2} \hookrightarrow \mathbb{L}_2$ and the embedding $\mathbb{L}_2 \hookrightarrow \mathbb{H}_2$ are not compact. From Theorem 6.9 follows that $\delta_{\mathbf{a}}$ is a continuous linear functional on $\mathbb{H}_2$ if ($n \leq 3$!!), therefore it has a Riesz representant $K(\mathbf{a}, \cdot)$ in $\mathbb{H}_2$, which is called the reproducing kernel. Note that because of the mean value theorem for harmonic functions, $\delta_{\mathbf{a}}$ is also a linear functional on Harm$(\mathbb{R}^n)$:

$$|\delta_{\mathbf{a}} u| = |u(\mathbf{a})| = \frac{1}{\mu(B_{\mathbf{a},R})} \int_{B_{\mathbf{a},R}} u(\mathbf{x})\, 1_{B_{\mathbf{a},R}}(\mathbf{x})\, d\mathbf{x} \leq \frac{1}{\sqrt{\mu(B_{\mathbf{a},R})}} ||u||_{\mathbb{L}_2(B_{\mathbf{a},R})} \leq C\, ||u||_{\mathbb{L}_2(B_{\mathbf{a},R})} \tag{6.50}$$

for all $R > 0$ and a certain $C > 0$.

In order to distinguish the two delta functionals, we write an extra index $\mathcal{H}$ to stress that it acts on Harm$(\mathbb{R}^n)$ in stead of on $\mathbb{H}_2$: $\delta^{\mathcal{H}}_{\mathbf{a}}$. Its Riesz representant will be denoted by $K^{\mathcal{H}}(\mathbf{a}, \cdot)$. This reproducing kernel can be used to derive projection operators $P^0 : \mathbb{L}_2 \mapsto \mathbb{L}_2^{harm}$ and $P : \mathbb{H}_2 \mapsto \mathbb{L}_2^{harm}$:

$$\begin{aligned} (P^0 f)(\mathbf{a}) &= (K^{\mathcal{H}}(\mathbf{a}, \cdot), f(\cdot))_{\mathbb{L}_2} \\ (Pf)(\mathbf{a}) &= (K^{\mathcal{H}}(\mathbf{a}, \cdot), f(\cdot))_{\mathbb{H}_2} \end{aligned} \tag{6.51}$$

Note that P^0 is in fact an extension of P since $\Delta K^{\mathcal{H}}(\mathbf{a}, \cdot) = 0$.

Observe the dual operator P' : which is defined by

$$\forall_{f \in \mathbb{H}^{-2}} \forall_{\phi \in \mathbb{H}^2} < P'F, \phi > = < F, P\phi > \tag{6.52}$$

The connection between P',P and P^0 *in the* $\mathbb{R}^n$ *($n \leq 3$) case*, can be derived in a more direct way by using the isometrical mappings $(I + \Delta)$ and $(I + \Delta)^{-1}$:

$$\begin{aligned} P' &= (I + |\Delta|) P^0 (I + |\Delta|)^{-1} \\ P &= (I + |\Delta|)^{-1} P^0 (I + |\Delta|) \end{aligned} \tag{6.53}$$

Since,

$$\begin{aligned} (P'f, \phi)_{\mathbb{H}_{-2}} &= ((I + |\Delta|)^{-1} P'f, (I + |\Delta|)^{-1}\phi)_{\mathbb{H}_0} \\ &= ((I + |\Delta|)^{-1} P'(I + |\Delta|)(I + |\Delta|)^{-1} f, (I + |\Delta|)^{-1}\phi)_{\mathbb{H}_0} \\ &= (P^0 (I + |\Delta|)^{-1} f, (I + |\Delta|)^{-1}\phi) \end{aligned} \tag{6.54}$$

Generalization to $\mathbb{H}_s \hookrightarrow \mathbb{L}_2 \hookrightarrow \mathbb{H}_s$

Finally, note that the results of the last two paragraphs can be generalized to the Gelfand triple $\mathbb{H}_s \hookrightarrow \mathbb{L}_2 \hookrightarrow \mathbb{H}_s$ with $s = 2k < \frac{n}{2}$, $k \in \mathbb{N}$ taking notice of the following modifications/remarks:

- We must impose $s < \frac{n}{2}$, since we want to apply theorem 6.9.

- The norm on $\mathbb{H}_s$ is given by

$$\begin{aligned}
\|u\|^2_{\mathbb{H}_s} &= \int_{\mathbb{R}^n} |\hat{u}|^2 \, d\mu_2(\mathbf{y}) \\
&= \int_{\mathbb{R}^n} |\hat{u}(\mathbf{y})|^2 (1+\|\mathbf{y}\|^2)^s \, d\mathbf{y} \\
&\asymp \int_{\mathbb{R}^n} |\hat{u}(\mathbf{y})|^2 (1+\|\mathbf{y}\|^{2s}) \, d\mathbf{y} \\
&= \int_{\mathbb{R}^n} |\hat{u}(\mathbf{y})|^2 + \left(\|\mathbf{y}\|^{2k} \, |\hat{u}(\mathbf{y})|\right)^2 d\mathbf{y} \\
&= (u,u)_{\mathbb{L}_2(\mathbb{R}^n)} + (\Delta^k u, \Delta^k u)_{\mathbb{L}_2(\mathbb{R}^n)}
\end{aligned} \tag{6.55}$$

- $(I+|\Delta|^k)$ is the isometric mapping from $\mathbb{H}_{2k}$ onto $\mathbb{L}_2$.
In stead of projecting onto $\mathbb{L}_2^{\mathrm{harm}}(\mathbb{R}^n)$ we can project on the subspace $\{f \in \mathbb{L}_2(\mathbb{R}^n) \,|\, \Delta^k = 0\}$. For instance, if $k=2$ the bi-harmonic subspace, which again is invariant of the isometric mappings $(I+|\Delta|^k)$ and $(I+|\Delta|^k)^{-1}$.

- The same relations as in hold, with $|\Delta|^k$ instead of $|\Delta|$.

6.6 The Gelfand triple $W_{2,2}(\Omega) \hookrightarrow \mathbb{L}_2(\Omega) \hookrightarrow [W_{2,2}(\Omega)]'$

Operator div$\mathcal{D}\nabla$ acts on $\mathbb{L}_2(\Omega)/\mathbb{R}$. In particular we are interested how it acts on the closed subspace $\mathbb{L}_2^{harm}(\Omega)/\mathbb{R}$. Our quest is to find/ to examine the reproducing kernel $K^{\mathcal{H}}(\mathbf{a},\cdot)$ and again see in what way it plays a role in projections on Harmonic subspaces in the three different spaces of the Gelfand triple $W_{2,2}(\Omega) \hookrightarrow \mathbb{L}_2(\Omega) \hookrightarrow [W_{2,2}(\Omega)]'$. Some remarks about this Gelfand triple:

- The space $[W_{2,2}(\Omega)]'$ is equipped with (dual) norm:

$$\|F\| = \sup_{\substack{\phi \in W_{2,2}(\Omega) \\ \|\phi\|_{W_{2,2}(\Omega)}=1}} |\langle F,\phi\rangle|$$

$W_{2,2}(\Omega)'$ is then complete.

- The embedding which is used in the Gelfand triple is *not* a compact embedding because the (infinite !) dimensional harmonic subspace is mapped on itself.

- Note that $W_{2,2}(\Omega)$ is a Hilbert space, so we can identify $W_{2,2}(\Omega)$ with its dual by using Riesz' theorem. But in Gelfand triples we don't represent functionals from $[W_{2,2}(\omega)]'$ by $(\cdot,\cdot)_{W_{2,2}(\Omega)}$ but by the product $(\cdot,\cdot)_{L_2(\Omega)}$. However for harmonic functions the two Riesz identifications are equivalent.

- $W_{2,2}(\Omega)'$ is the completion of $\mathbb{L}_2(\Omega)$ if we equip $\mathbb{L}_2(\Omega)$ with the norm

$$\|h\|_{-1} = \sup_{f \in W_2^2(\Omega)} \frac{|(h,\iota f)|}{\|f\|_{W_2^2(\Omega)}}$$

Note that analogue to the $\mathbb{R}^n$ case every harmonic function can be "reproduced" by $K^{\mathcal{H}}(\mathbf{x},\cdot)$ again:

$$\Delta h = 0 \Rightarrow \mathbf{x} \mapsto h(\mathbf{x}) = \;<\delta^{\mathcal{H}}_{\mathbf{x}}, h(\cdot)>= (K^{\mathcal{H}}(\mathbf{x},\cdot), h(\cdot)) \tag{6.56}$$

So, in order to find an integral kernel for div$\mathcal{D}\nabla - \lambda I$ we (only) need to find out how div$\mathcal{D}\nabla - \lambda I$ acts on the reproducing kernel.

6.6.1 Projections on harmonic subspaces

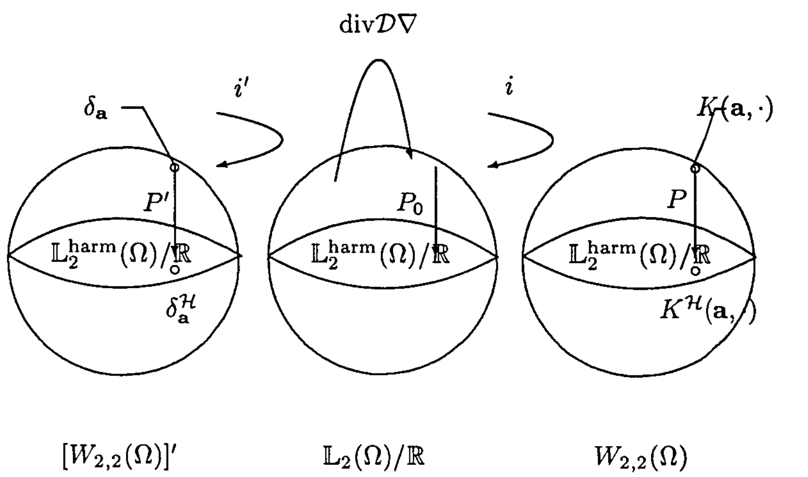
Let M and N be subspaces of a banach space V then P is a projection on M along N then P' is a projection on $N^0 = \{f \in V' \mid \forall_{x \in M} : < f, \mathbf{x} >= 0\}$ along M^0. In a Hilbert space $N^0 = N^\perp, M^0 = M^\perp$ and $P' = P^*$. Note that this implies that $P \in B(V)$ is an orthogonal projection if and only if P satisfies $P^* = P \wedge P^2 = P$.

Equation (6.51) is also valid for general Ω. So, again P_0 and P use the same reproducing kernel. Further note that P_0 and P are both *orthogonal* projections, because they map functions orthogonal to the harmonic subspace to zero. Therefore $P' = P$ and $P_0' = P_0$.
Further note that:

$$< P'\delta_{\mathbf{a}}, \phi >=< \delta_{\mathbf{a}}, P\phi >= (P\phi)(\mathbf{a}) = (K^{\mathcal{H}}(\mathbf{a},\cdot), \phi(\cdot))_{\mathbb{L}_2} \tag{6.57}$$

So, $P'\delta_{\mathbf{a}} = \delta_{\mathbf{a}}^{\mathcal{H}}$ and $PK(\mathbf{a},\cdot) = K^{\mathcal{H}}(\mathbf{a},\cdot)$ since, $\Delta F = 0 \Rightarrow$

$$(PK(\mathbf{a},\cdot), F)_{W_{2,2}(\Omega)} = (P'F, K(\mathbf{a},\cdot))_{W_{2,2}(\Omega)} = (F, K(\mathbf{a},\cdot)) =< \delta_{\mathbf{a}}, F >= F(\mathbf{a})$$



figuur Gelfand-triple $W_{2,2}(\Omega) \subset \mathbb{L}_2(\Omega) \subset W_{2,2}(\Omega)$

7 Some properties of Harmonic functions

In this chapter we observe a few important theorems about harmonic functions that have been used else in this report.
The complex vector space of all harmonic functions on Ω is denoted by Harm(Ω). One of the main properties of harmonic functions is the mean value problem, which as follows from theorem 7.2 actually characterizes harmonic functions.

Theorem 7.1 *Let $B \subset \Omega$ be a ball with centre $\mathbf{a}$ and radius $R > 0$. If $u \in Harm(\Omega)$ then*

$$u(a) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B} u(x)\, d\sigma_x$$

or, equivalently,

$$u(a) = \frac{1}{\omega_n R^{n-1}} \int_{\partial B} u(x)\, dx.$$

Here $\omega_k R^{n-1}$ denotes the total surface measure of the boundary ∂B of B.[21]

Proof
Let $B_0 \subset B$ be a ball with centre $\mathbf{a}$ and radius $R_0 < R$. Then with Greens second identity

$$\int_{\partial B \setminus B_0} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) d\sigma = \int_{B \setminus B_0} (u \Delta v - v \Delta u)\, d\mathbf{x}\,.$$

We take $u \in$ harm(Ω) and take

$$\mathbf{v}(\mathbf{x}) = \begin{cases} \frac{1}{2\pi} \log \frac{1}{\|\mathbf{x}-\mathbf{a}\|} & n = 2 \\ \frac{1}{\omega_n (n-2) \|\mathbf{x}-\mathbf{a}\|^{n-2}} & n > 2\,. \end{cases} \tag{7.1}$$

Since $v \in$ Harm($\Omega \setminus \{\mathbf{a}\}$), we find

$$\frac{1}{\omega_n R^{n-1}} \int_{\partial B} u\, d\sigma = \frac{1}{\omega_n R_0^{n-1}} \int_{\partial B_0} u\, d\sigma\,.$$

Now let $R_0 \to 0$. This yields the first formula. The second one follows by integrating over spheres. □

Remark: Although that $\mathbf{v}$ corresponds to the fundamental solution $S_{\mathbf{a}}$ (see the remark after theorem 6.7) we did not use $\Delta S_{\mathbf{a}} = -\delta_{\mathbf{a}}$.

Theorem 7.2 *Let $u \in C(\Omega)$, $\Omega \subset \mathbb{R}^n$, Ω open and suppose that for each $\mathbf{a} \in \Omega$ and each ball $B_{a,R} = \{x \mid \|x - a\| < R\}$ such that $B_{a,R} \subset \Omega$, $u(a) = \frac{n}{\omega_n R^n} \int_{B_{a,R}} u(x)\, dx$, then $u \in Harm(\Omega)$*

Proof
Consider an arbitrary ball $B \subset \Omega$. Let $w = u_{\mathcal{H}}$, i.e. the solution of the Dirichlet problem

$$\begin{cases} \Delta w = 0 & \text{in } B \\ w = v & \text{on } \partial B \end{cases}$$

[21] $\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}$.

Since w harmonic it has the mean value property and u has the mean value property by assumption. Therefore $u - w$ has the mean value property in B and we have $w - u = 0$ on ∂B. Therefore $w - u = 0$ on the whole of B. $\square$

Remark: The condition $u \in C(\Omega)$ can be weakened to $u \in L_{1,\mathrm{loc}}(\Omega)$.

The above theorem can often be used to show that a Harmonic subspace of a normed vector space is closed. For instance,

Theorem 7.3 *(Harnack's Theorem)*
Let $\{u_m\}_{m\in\mathbb{N}}$ be a sequence of harmonic functions in harm(Ω). Suppose that $u_m(x) \to u(x)$ uniformly on compact sets in Ω. Then also $u \in$ harm(Ω).

Proof
The equality $u_m(\mathbf{a}) = \frac{n}{\omega_n R^n} \int\limits_B u_m(\mathbf{x})\,d\mathbf{x}$ persists if $m \to \infty$. $\square$

Next we observe some properties of harmonic polynomials.

Denote the vector space of all polynomials on $\mathbb{R}^n$ of degree m with $\mathrm{Pol}_m(\mathbb{R}^n)$. A subspace of this vector space is given by the set of all k-homogeneous polynomials on$\mathbb{R}^n$ $(0 \le k \le m)$ which will be denoted by $\mathrm{HomPol}_k(\mathbb{R}^n)$. Remind that a k-homogeneous polynomial Q is a polynomial such that $Q(\lambda\mathbf{x}) = \lambda^k Q(\mathbf{x})$ and is therefore determined by its image on the unit sphere. The following relation holds,

$$\mathrm{Pol}_m(\mathbb{R}^n) = \bigoplus_{k=1}^{m} \mathrm{HomPol}_k(\mathbb{R}^n)\,. \tag{7.2}$$

For instance, if $n = 3$ then every $P \in \mathrm{Pol}_m$ can be written uniquely

$$p(x, y, z) = \sum_{k=0}^{m} \left(\sum_{l+i+j=k} a_{l,i,j} x^i y^j z^l \right).$$

Define $\mathrm{HarmPol}_m(\mathbb{R}^n)$ and $\mathrm{HarmHomPol}_m(\mathbb{R}^n)$ by

$$\begin{aligned}\mathrm{HarmPol}_m(\mathbb{R}^n) &= \mathrm{Pol}_m(\mathbb{R}^n) \cap \mathcal{N}(\Delta),\\ \mathrm{HarmHomPol}_m(\mathbb{R}^n) &= \mathrm{HomPol}_m(\mathbb{R}^n) \cap \mathcal{N}(\Delta).\end{aligned}$$

Elements of $\mathrm{HarmHomPol}_m(\mathbb{R}^n)$ are called spherical harmonics.

On the space Pol_m we introduce the inner product

$$(p, q) = p(\nabla)q(\mathbf{x})|_{\mathbf{x}=\mathbf{0}}\,.$$

Theorem 7.4 *In $HomPol_m(\mathbb{R}^n)$ we have*

$$HarmHomPol_m(\mathbb{R}^n)^{\perp} = ||x||^2 \cdot HomPol_{m-2}(\mathbb{R}^n). \tag{7.3}$$

Therefore, every $f \in HomPol_m(\mathbb{R}^n)$ can be split

$$f = f_m + ||x||^2 f_{m-2} + ||x||^4 f_{m-4} + \dots,$$

with all $f_j \in HarmHomPol_j(\mathbb{R}^n)$.

The dimension of $HomPol_m(\mathbb{R}^n)$ equals

$$d_m^n = dim\ HomPol_m(\mathbb{R}^n) \binom{n+m-1}{m} - \binom{n+m-3}{m-2}. \tag{7.4}$$

Proof
Define the polynomial s by $s(\mathbf{x}) = \|\mathbf{x}\|^2$.
If $f \in \text{HarmHomPol}(\mathbb{R}^n)$ then for all $q \in \text{HomPol}_{m-2}(\mathbb{R}^n)$

$$(q\,s, f) = q(\nabla)\,s(\nabla)f = q(\nabla)\Delta f = 0\,.$$

Conversely, let $f \perp s\cdot\text{HomPol}_{m-2}(\mathbb{R}^n)$. This means $\forall_{q\in\text{HomPol}_{m-2}(\mathbb{R}^n)} : q(\nabla)\,s(\nabla)f = 0$. Take $q = s(\nabla)f = \Delta f$, then $(\Delta f, \Delta f) = 0$ so we have $\Delta f = 0$.

Now let $f \in \text{HomPol}_m(\mathbb{R}^n)$. Denote the orthogonal projection of f on $\text{HomPol}_m(\mathbb{R}^n)$ by f_m and the orthogonal projection of f on $\text{HomPol}_m(\mathbb{R}^n)^\perp$ by $f_m^\perp$, then by (7.3) there exists a $\hat{f}_{m-2} \in$ such that

$$f = f_m + f_m^{bot} = f_m + \|\mathbf{x}\|^2 \hat{f}_{m-2}.$$

Repeat the same argument on $\hat{f}_{m-2}$ and we find $\hat{f}_{m-2} = f_{m-2} + \|\mathbf{x}\|\hat{f}_{m-4}$ etc.

The monomials $\{x_1^{\alpha_1}\cdot x_2^{\alpha_2}\cdots x_n^{\alpha_n}\}$ with $\sum_{k=1}^{n} \alpha_k = m$ form a basis for $\text{HomPol}_m(\mathbb{R}^n)$, therefore

$$\dim\ \text{HomPol}_m(\mathbb{R}^n) = \binom{m+n-1}{m}\,.$$

Together with the above (7.4) now follows. □

Because of Stone-Weierstrass theorem any continuous function on the sphere $S_R^{n-1} \subset \mathbb{R}^n$ can be uniformly approximated by a sequence of (restrictions to S_R^{n-1} of) polynomials. From theorem 7.4 and (7.2) it now follows that such an approximated sequence can be replaced by a sequence of spherical harmonics. As a consequence the restrictions of the elements of $\text{HarmHomPol}_m(\mathbb{R}^n)$ to S_R^{n-1} establish a dense linear subspace of $\mathbb{L}_2(S_R^{n-1})$. From now on we suppose that $\text{HarmHomPol}_m(\mathbb{R}^n)$ carries the $\mathbb{L}_2(S_1^{n-1})$-inner product $(f,g)_{\mathbb{L}_2(\partial B_n)}$. (NB. Notation: $S_1^{n-1} = \partial B_n$).

Gathering our results we have

$$\mathbb{L}_2(\partial B_n) = \bigoplus_{k=0}^{\infty} \text{HarmHomPol}_k(\mathbb{R}^n)\,. \tag{7.5}$$

Theorem 7.5 *Let $f \in \text{HarmHomPol}_k(\mathbb{R}^n)$. Let $a \in \mathbb{R}^n$*

1. *For each $0 < \epsilon < 1$, we have the estimation*

$$|f(a)|^2 \le \left[\frac{n}{\omega_n(2k+n)}\left(1+\frac{1}{\epsilon}\right)^n\right](1+\epsilon)^{2k}\|a\|^{2k} \int_{\partial B_n} |f(x)|^2\, d\sigma_x\,.$$

2. *And even, more subtle,*

$$|f(a)|^2 \le \frac{d_k^n}{\omega_n} \int_{\partial B_n} |f(x)|^2\, d\sigma_x\,, \qquad \|a\| = 1\,. \tag{7.6}$$

Proof
1. Since f is homogeneous it suffices to prove the result for $\mathbf{a} \in \mathbb{R}^n$, with $\|\mathbf{a}\| = 1$. From the mean value theorem

$$f(\mathbf{a}) = \frac{n}{\omega_n \epsilon^n} \int_{B_{\mathbf{a},\epsilon}} f(\mathbf{x})\, d\mathbf{x}$$

we find, applying Cauchy-Schwarz,

$$\begin{aligned}|f(\mathbf{a})|^2 &\leq \tfrac{n}{\omega_n \epsilon^2} \int\limits_{B_{\mathbf{a},\epsilon}} |f(\mathbf{x})|^2 \, d\mathbf{x} \\ &< \tfrac{n}{\omega_n \epsilon^2} \int\limits_{B_{0,1+\epsilon}} |f(\mathbf{x})|^2 \, d\mathbf{x} = \tfrac{n}{\omega_n \epsilon^2} \tfrac{(1+\epsilon)^{2k+n}}{2k+n} \int\limits_{B_{0,1}} |f(\mathbf{x})|^2 \, d\mathbf{x}\end{aligned}$$

2. See Müller[refmueller]. □

Note that the above inequalities can be used in order to obtain uniform and point wise convergence on a sphere once we have $\mathbb{L}_2$ convergence of a sequence spherical harmonics (with increasing degree). Note that the integral $\int\limits_{\partial B_n} |f(\mathbf{x})|^2 d\sigma$ in the right hand side equals $R_1^{-2k} \int\limits_{\|\mathbf{x}\|=R_1} |f(\mathbf{x})|^2 \, d\sigma$, for any $R_1 > 0$.

Theorem 7.6 *Let $f \in HarmHomPol_k(B_{0,R})$, $R > 0$. then there exists a unique expansion*

$$f = \sum_{k=0}^{\infty} f_k \qquad \text{with } f_k \in HarmHomPol_k(\mathbb{R}^n)$$

which converges uniformly on compact sets in $B_{0,R}$.
The $\mathbb{L}_2(\partial B_{0,R})$-norms $\|f_k\|_{\mathbb{L}_2(\partial B_{0,R})}$ of f_k satisfy

$$\forall_{r>R} : \sum_{k=0}^{\infty} r^{2k} \|f_k\|^2_{\mathbb{L}_2(\partial B_{0,R})} < \infty \tag{7.7}$$

or, equivalently

$$\forall_{r>R} : \sup_k r^k \|f_k\|_{\mathbb{L}_2(\partial B_{0,R})} < \infty \, .$$

Conversely, if a sequence $\{f_k\}_{k\in\mathbb{N}}$ of spherical harmonics satisfies (7.7) then the sum
$g(x) = \sum\limits_{k=0}^{\infty} f_k(x)$ exists at each $x \in B_R$ and $g \in Harm(B_R)$.

Proof
Because of the scaling properties of harmonic functions and spherical harmonics it is sufficient to show that, given $f \in \text{Harm}(\Omega)$, $R > 1$, the expansion result is valid on the closed n-dimensional unit ball $\overline{B_n}$.
First, note that $f|_{\partial B_n} = \sum\limits_{k=0}^{\infty} f_k|_{\partial B_n}$ in $\mathbb{L}_2(\partial B_n)$ sense; with $f_k \in \text{HarmHomPol}_k(\mathbb{R}^n)$.
Further, take $1 < R_1 < R$. Then

$$\begin{aligned}R_1^{n-1} \sum_{k=0}^{\infty} R_1^{2k} \|f_k\|^2_{\partial B_n} &= \sum_{k=0}^{\infty} R_1^{n+2k-1} \int\limits_{\|\mathbf{x}\|=1} |f_k(\mathbf{x})|^2 \, d\sigma = \\ &= \sum_{k=0}^{\infty} \int\limits_{\|\mathbf{x}\|=R_1} |f(\mathbf{x})|^2 \, d\sigma < \infty \, .\end{aligned}$$

By the first part of Theorem 7.5 we obtain (take ϵ such that $(1+\epsilon) < R_1$) that $\sum\limits_{k=0}^{\infty} f_k(\mathbf{x})$ converges uniform to $f(\mathbf{x})$ on ∂B_n. But then, because of the maximum principle $\left[f(\mathbf{x}) - \sum\limits_{k=0}^{N} f_k(\mathbf{x})\right] \to 0$, uniformly on B, as $N \to \infty$.

Conversely, if a sequence $\{f_k\}_{k\in\mathbb{N}}$ $\sum_{k=0}^{N} f_k$ is a Cauchy sequence in $\mathbb{L}_\infty(B_{R_1})$ on each ball B_{R_1} with $R_1 < R$. The estimate is as follows, take $R_1 < R_2 < R$;

$$\begin{aligned}\left|\sum_{k=N+1}^{M} f_k(\mathbf{x})\right| &\leq C_{n,\epsilon} \sum_{k=N+1}^{M} (1+\epsilon)^k R_1^k \|f_k\|_{\mathbb{L}_2(\partial B_n)} = \\ &= C_{n,\epsilon} \sum_{k=N+1}^{M} \frac{((1+\epsilon)R_1)^k}{R_2^k} R_2^k \|f_k\|_{\mathbb{L}_2(\partial B_n)} \\ &\leq C_{n,\epsilon} \left\{ \sum_{k=N+1}^{M} \frac{((1+\epsilon)R_1)^{2k}}{R_2^{2k}} \right\}^{\frac{1}{2}} \left\{ \sum_{k=0}^{\infty} R_2^{2k} \|f_k\|^2_{\mathbb{L}_2(\partial B_n)} \right\}^{\frac{1}{2}} \qquad \square.\end{aligned}$$

Remark: The expansion in the above theorem is valid on the unit ball. In the 3D-unit ball and disk case we used it. On other regions we don't have such a theorem. Even, finding a complete orthonormal $\mathbb{L}_2$ base for $\mathbb{L}_2^{\text{Harm}}(\Omega)$ on other regions seemed to be very difficult.

A Solution of Stokes problem on Disk

In this paragraph we will take $\mathbf{f} = \mathbf{0}$ and $h = 0$. Note, that in the general theory was shown how a generalization to problems with nonzero $\mathbf{f}$ and h can be made. According to theorem (3.7) the solution of the Stokes problem $P(\mathbf{0}, 0, \mathbf{a})$ on the disk B_2 is given by:

$$\begin{array}{l} p = -2\nabla \cdot \mathbf{a}_{\mathcal{H}} \\ \mathbf{v} = \mathbf{a}_{\mathcal{H}} + \frac{(1-\|\mathbf{x}\|^2)}{2}\nabla \mathcal{E}^{-1}\nabla \cdot \mathbf{a}_{\mathcal{H}} \end{array} \tag{A.1}$$

So, in order to construct the solution one (only) needs to calculate $\mathbf{a}_{\mathcal{H}}$ and its divergence. Although $\mathbf{a}_{\mathcal{H}}$ can be split in $\mathbf{h}_i$ $i = 1, 2, 3$, as was shown in theorem (F.2), we will derive $\mathbf{a}_{\mathcal{H}}$ in a direct way. Later, $\mathbf{h}_i$ $i = 1, 2, 3$, χ, ψ will be recognized.

$$\begin{array}{l} L_2(\partial B_2) = \bigoplus\limits_{k=0}^{\infty} \mathrm{HarmHomPol}_k(\mathbb{R}^2)\big|_{\partial B_2} \Rightarrow \\ \mathbf{a}(\mathbf{x}, \mathbf{y}) = \sum\limits_{n=-\infty}^{\infty} \{a_n \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} e^{in\theta} + b_n \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} e^{in\theta}\} \end{array} \tag{A.2}$$

Note that b_0 plays the role of c in general theory[22]. Therefore it is zero. Further remarks are :

- $(e^{in\theta})_{\mathcal{H}} = z^n \quad n \geq 0 \wedge z = x + iy$
- $(e^{in\theta})_{\mathcal{H}} = \overline{z}^n \quad n \leq 0$
- $(\overline{w})_{\mathcal{H}} = \overline{(w)_{\mathcal{H}}}$, where $\overline{w}$ stands for the conjugation of w.

As a result harmonic expansion from ∂B_2 to B_2 is a combination of analytic expansion and conjugation after analytic expansion. However,

$$\begin{array}{rl} \mathbf{a}_{\mathcal{H}} = & \frac{1}{2}\sum\limits_{n=1}^{\infty} a_n \left[\begin{pmatrix} i \\ 1 \end{pmatrix} z^{(n+1)} + \begin{pmatrix} -i \\ 1 \end{pmatrix} z^{(n-1)}\right] + \\ & \frac{1}{2}\sum\limits_{n=-\infty}^{-1} a_n \left[\begin{pmatrix} i \\ 1 \end{pmatrix} \overline{z}^{(|n|-1)} + \begin{pmatrix} -i \\ 1 \end{pmatrix} \overline{z}^{(|n|+1)}\right] + \\ & \frac{1}{2} a_0 \left[\begin{pmatrix} i \\ 1 \end{pmatrix} z + \begin{pmatrix} -i \\ 1 \end{pmatrix} \overline{z}\right] + \\ & \frac{1}{2}\sum\limits_{n=1}^{\infty} b_n \left[\begin{pmatrix} 1 \\ -i \end{pmatrix} z^{(n+1)} + \begin{pmatrix} 1 \\ i \end{pmatrix} z^{(n-1)}\right] + \\ & \frac{1}{2}\sum\limits_{n=\infty}^{-1} b_n \left[\begin{pmatrix} 1 \\ -i \end{pmatrix} \overline{z}^{(|n|-1)} + \begin{pmatrix} 1 \\ i \end{pmatrix} \overline{z}^{(|n|+1)}\right] \end{array} \tag{A.3}$$

Now, calculate $\nabla \cdot \mathbf{a}_{\mathcal{H}}$:

$$\begin{array}{rl} \nabla \cdot \mathbf{a}_{\mathcal{H}} = & \sum\limits_{n=-\infty}^{-1} (b_n - a_n i)(|n| + 1)\overline{z}^{|n|} + \\ & \sum\limits_{n=1}^{\infty} (b_n + a_n i)(n + 1)z^n \end{array} \tag{A.4}$$

[22]See, theorem F.2

Note that:

$$
\begin{cases}
\mathbf{h}_0 = \frac{1}{2}\, a_0 \left[\begin{pmatrix} i \\ 1 \end{pmatrix} z + \begin{pmatrix} -i \\ 1 \end{pmatrix} \overline{z} \right] \\[2ex]
\chi(x,y) = -(\mathcal{E}+1)^{-1}\mathcal{E}^{-1}\nabla\cdot\mathbf{a}_{\mathcal{H}} = \sum\limits_{n=1}^{\infty} \frac{-b_n-a_n i}{n} z^n + \sum\limits_{n=-\infty}^{-1} \frac{-b_n+a_n i}{|n|}\overline{z}^{|n|} \\
\text{which follows from (3.5).} \\[2ex]
\mathbf{h}_1[\chi] = \frac{1}{2}\sum\limits_{n=1}^{\infty}(b_n+a_n i)\begin{pmatrix} 1 \\ -i \end{pmatrix} z^{n+1} + \frac{1}{2}\sum\limits_{n=-\infty}^{-1}(b_n-a_n i)\begin{pmatrix} 1 \\ i \end{pmatrix}\overline{z}^{|n|+1} + \\
-\frac{1}{2}\sum\limits_{n=1}^{\infty}(b_n+a_n i)\begin{pmatrix} 1 \\ i \end{pmatrix} z^{n-1} - \frac{1}{2}\sum\limits_{n=-\infty}^{-1}(b_n-a_n i)\begin{pmatrix} 1 \\ -i \end{pmatrix}\overline{z}^{|n|-1} \\[2ex]
\psi(x,y) = \sum\limits_{n=1}^{\infty} b_n\, n\, z^n + \sum\limits_{n=-\infty}^{-1} b_n\, |n|\, \overline{z}^{|n|} \\[2ex]
\mathbf{h}_2[\psi] = \nabla\mathcal{E}^{-1}\psi = \sum\limits_{n=1}^{\infty} b_n z^{n-1}\begin{pmatrix} 1 \\ i \end{pmatrix} z^{n-1} + \sum\limits_{n=1}^{\infty} b_n \begin{pmatrix} 1 \\ -i \end{pmatrix}\overline{z}^{|n|-1}
\end{cases}
\tag{A.5}
$$

Next, calculate $\frac{1}{2}(1-\|\mathbf{x}\|^2)\nabla\mathcal{E}^{-1}\nabla\cdot\mathbf{a}_{\mathcal{H}}$:

$$
\begin{aligned}
\nabla\mathcal{E}^{-1}\nabla\cdot\mathbf{a}_{\mathcal{H}} = \; & \sum_{n=-\infty}^{-1}(b_n - i a_n)(|n|+1)\begin{pmatrix} 1 \\ -i \end{pmatrix}\overline{z}^{|n|-1} + \\
& \sum_{n=1}^{\infty}(b_n + a_n i)(|n|+1)\begin{pmatrix} 1 \\ i \end{pmatrix} z^{n-1} \qquad \Rightarrow
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
& \tfrac{1}{2}(1-\|\mathbf{x}\|^2)\nabla\mathcal{E}^{-1}\nabla\cdot\mathbf{a}_{\mathcal{H}} = \\
& \tfrac{1}{2}(\tfrac{1}{z}-\overline{z})\sum_{n=1}^{\infty}(b_n+a_n i)z^n\begin{pmatrix} 1 \\ i \end{pmatrix} + \\
& \tfrac{1}{2}(\tfrac{1}{\overline{z}}-z)\sum_{n=-\infty}^{-1}(b_n-a_n i)\overline{z}^n\begin{pmatrix} 1 \\ i \end{pmatrix} + \\
& \tfrac{1}{2}(1-|z|^2)\left\{ \begin{array}{l} \sum\limits_{n=1}^{\infty}(b_n+a_n i)nz^{n-1}\begin{pmatrix} 1 \\ i \end{pmatrix} + \\ \sum\limits_{n=-\infty}^{-1}(b_n-a_n i)|n|\overline{z}^{|n|-1}\begin{pmatrix} 1 \\ -i \end{pmatrix} \end{array} \right\}
\end{aligned}
\tag{A.7}
$$

Now p and $\mathbf{v}$ can be calculated:

$$
p(x,y) = -2\sum_{n=1}^{\infty}(b_n+ia_n)(n+1)z^n - 2\sum_{n=-\infty}^{-1}(b_n-ia_n)(|n|+1)\overline{z}^{|n|}
\tag{A.8}
$$

$$
\begin{aligned}
\mathbf{v}(x,y) = & \tfrac{1}{2}\begin{pmatrix} (z-\overline{z}) \\ -i(z+\overline{z}) \end{pmatrix}\left\{\sum_{n=1}^{\infty}(b_n+a_n i)z^n + \sum_{n=-\infty}^{-1}(b_n-a_n i)\overline{z}^n\right\} + \\
& \tfrac{1}{2}(1-|z|^2)\left\{\sum_{n=1}^{\infty}(b_n+a_n i)nz^{n-1}\begin{pmatrix} 1 \\ i \end{pmatrix} + \sum_{n=-\infty}^{-1}(b_n-a_n i)|n|\overline{z}^{|n|-1}\begin{pmatrix} 1 \\ -i \end{pmatrix}\right\} + \\
& \tfrac{1}{2}\, a_0\left[\begin{pmatrix} i \\ 1 \end{pmatrix} z + \begin{pmatrix} -i \\ 1 \end{pmatrix}\overline{z}\right] + \\
& \sum_{n=-\infty}^{-1} b_n\begin{pmatrix} 1 \\ -i \end{pmatrix}\overline{z}^{n-1} + \sum_{n=1}^{\infty}\begin{pmatrix} 1 \\ i \end{pmatrix} z^{n-1}
\end{aligned}
\tag{A.9}
$$

B Solution of Stokes problem on 3D Unit Ball

Taking $n = 3$ in the general formula for the solution of $P(\mathbf{0}, 0, \mathrm{a})$ (3.16) yields:

$$\begin{aligned} p &= 2(\mathcal{E}+1)(\mathcal{E}+\tfrac{3}{2})(\chi) \\ \mathbf{v} &= \mathbf{h}_0[\phi] + \mathbf{h}_1[\chi] + \mathbf{h}_2[\psi] + \tfrac{(\|\mathbf{x}\|^2-1)}{2}\nabla(\mathcal{E}+\tfrac{1}{2})^{-1}(\mathcal{E}+1)(\mathcal{E}+\tfrac{3}{2})\chi \end{aligned} \tag{B.1}$$

Special cases:

- $\mathbf{a} = \mathbf{h}_0[Q_m] = \mathbf{x} \times \nabla Q_m \Rightarrow$
  $\mathbf{a}_{\mathcal{H}} = \mathbf{a}$, therefore

$$\begin{aligned} p &= 0 \\ \mathbf{v} &= \mathbf{x} \times \nabla Q_m \end{aligned} \tag{B.2}$$

- $\mathbf{a} = \nabla Q_m - mQ_m\mathbf{x} \Rightarrow$
  $\mathbf{a}_{\mathcal{H}} = \mathbf{h}_1[Q_m]$

$$\begin{aligned} p &= (2m+3)(m+1)Q_m \\ \mathbf{v} &= \nabla Q_m - mQ_m\mathbf{x} + \left\{\tfrac{m}{(m+\frac{1}{2})} + \tfrac{(m+\frac{3}{2})(m+1)}{(m+\frac{1}{2})}\right\}(1-\|\mathbf{x}\|)\nabla Q_m = \\ & Q_m\mathbf{x} + (m+3)(1-\|\mathbf{x}\|^2)\nabla Q_m \end{aligned} \tag{B.3}$$

- $\mathbf{a} = Q_m\mathbf{x} \Rightarrow$

$$\mathbf{a}_{\mathcal{H}} = \mathbf{h}_1[\frac{-Q_m}{m}] + \mathbf{h}_2[Q_m]$$

  According to (B.1) again gives:

$$\begin{aligned} p &= -\tfrac{(2m+3)(m+1)}{m}Q_m \\ \mathbf{v} &= Q_m\mathbf{x} + \left\{\tfrac{m}{2m(m+\frac{1}{2})} + \tfrac{(m+\frac{3}{2})(m+1)}{2m(m+\frac{1}{2})}\right\}(1-\|\mathbf{x}\|)\nabla Q_m = \\ & Q_m\mathbf{x} + \tfrac{(m+3)}{2m}(1-\|\mathbf{x}\|^2)\nabla Q_m \end{aligned} \tag{B.4}$$

- $\mathbf{a} = \mathbf{h}_2[mQ_m] = \nabla Q_m \Rightarrow$
  $\mathbf{a}_{\mathcal{H}} = \mathbf{a}$

$$\begin{aligned} p &= 0 \\ \mathbf{v} &= \nabla Q_m \end{aligned} \tag{B.5}$$

Note that :

$$\begin{aligned} & \Delta\phi = \Delta\chi = \Delta\psi = 0 \Rightarrow \\ & \phi(\mathbf{x}) = \sum_{m=0}^{\infty} a_m Q_m \text{ converging on compact sets in} B_3 \\ & \chi(\mathbf{x}) = \sum_{m=0}^{\infty} b_m Q_m \text{ converging on compact sets in} B_3 \\ & \psi(\mathbf{x}) = \sum_{m=0}^{\infty} c_m Q_m \text{ converging on compact sets in} B_3 \end{aligned} \tag{B.6}$$

C The space $\mathbf{V}^{(3)}$ (addendum)

In this chapter we will approach the space $\mathbf{V}^{(3)}$ differently. We will give an alternative (more sophisticated) definition of $\mathbf{V}^{(3)}(\Omega)$. The construction is by some steps.

Define the space[23]

$$\mathbf{M}^{(3)}(\Omega) = \{\Phi \in \mathbf{L}_2(\Omega) \mid \operatorname{div}\Phi \in \mathbb{L}_2(\Omega),\ \operatorname{rot}\Phi \in \mathbf{L}_2(\Omega)\}$$

Notice that the norm introduced on $\widetilde{\mathbf{V}}^{(3)}$ by (5.5) is not a norm on this space. Since elements of the subspace $\mathcal{K}$ of $\mathbf{M}^{(3)}(\Omega)$, defined by

$$\mathcal{K} = \{\Phi \in \mathbf{M}^{(3)} | \Phi = \nabla\psi \text{ with } \Delta\psi = 0\,,\}$$

satisfy $\operatorname{div}\Phi = 0$ and $\operatorname{rot}\Phi = \mathbf{0}$. Therefore, we equip the space $\mathbf{M}^{(3)}(\Omega)$ with the inner product $(\cdot,\cdot)_{\mathbf{M}^{(3)}(\Omega)}$ given by

$$(\Phi,\Psi)_{\mathbf{M}^{(3)}(\Omega)} = (\Phi,\Psi)_{\mathbb{L}_2(\Omega)} + (\operatorname{div}\Phi, \operatorname{div}\Psi)_{\mathbf{L}_2(\Omega)} + (\operatorname{rot}\Phi, \operatorname{rot}\Psi)_{\mathbf{L}_2(\Omega)}\,. \qquad \text{(C.7)}$$

Let $\mathbf{f} \in \mathbf{M}^{(3)}(\Omega)$. If $\mathbf{f}$ happens to be in $\widetilde{\mathbf{V}}^3(\Omega)$ then the restriction $\mathbf{f}|_{\partial\Omega}$ to the boundary makes sense. The restriction of an element f of $\mathbb{L}_2(\Omega)$ to the boundary doesn't make sense because the boundary has zero three dimensional measure. f represents a class of functions and any assertion on f should be representative independent.

However, the trace theorem 5.1 in chapter 6 shows us that if Ω is an open Lipschitz region in $\mathbb{R}^3$, then there exists a continuous linear operator $T_0 \in B(W_2^1(\Omega), \mathbb{L}_2(\partial\Omega))$ such that $T_0 u = u|_{\partial\Omega}$ for all $u \in W_2^1(\Omega) \cap C(\overline{\Omega})$. The kernel of this operator equals $\overset{\circ}{\mathcal{W}}_{1,2}(\Omega)$. See for instance, Wloka [2] theorem 8.9. The image space $T_0(W_2^1(\Omega))$ is a dense subspace of $\mathbb{L}_2(\partial\Omega)$ and is denoted by $W_2^{\frac{1}{2}}(\partial\Omega)$.[24] By theorem ??, there exists a continuous, linear extension operator $Z_0 : W_2^{\frac{1}{2}}(\partial\Omega) \mapsto W_2^1(\Omega)$ which is a right inverse of T_0. So, $T_0 \circ Z_0 = I_{W_2^{\frac{1}{2}}(\partial\Omega)}$. The next theorem shows a similar result for vector functions in $\mathbf{M}^{(3)}(\Omega)$.
Observe the Gelfand-triple

$$W_2^{\frac{1}{2}}(\partial\Omega) \hookrightarrow_i \mathbb{L}_2(\partial\Omega) \hookrightarrow_{i'} W_2^{-\frac{1}{2}}(\partial\Omega) \qquad \text{(C.8)}$$

By the general theory of Gelfand triples in chapter 7 we can regard the continuous extension of $(\cdot,\cdot)_{\mathbb{L}_2(\partial\Omega)}$ on $W_2^{-\frac{1}{2}}(\partial\Omega) \times W_2^{\frac{1}{2}}(\partial\Omega)$ as a new representation formula for the functionals from $W_2^{-\frac{1}{2}}(\partial\Omega)$.

Theorem C.1 *Let Ω be an open bounded Lipschitz domain in $\mathbb{R}^3$. Then there exists a linear continuous operator $T_n : \boldsymbol{M}^{(3)} \mapsto W_2^{-\frac{1}{2}}(\partial\Omega)$ such that*[25]*:*

$$\forall_{u\in[\mathcal{D}(\overline{\Omega})]^3} : [T_n u](\phi) = \int_{\partial\Omega} (u\cdot n)\phi\, d\sigma_x \qquad (\phi \in W_2^{\frac{1}{2}}(\partial\Omega)) \text{ and} \qquad \text{(C.9)}$$

$$\forall_{f\in M^{(3)}(\Omega)}\ \forall_{w\in W_2^1(\Omega)} : (f, \nabla w)_{\mathbb{L}_2(\Omega)} + (\mathit{div} f, w)_{\mathbb{L}_2(\Omega)} = \langle T_n f, T_0 w\rangle \qquad \text{(C.10)}$$

[23] Differentiation in distributional sense.

[24] In the sequel we will use this as the definition of $W_2^{\frac{1}{2}}(\partial\Omega)$. For the proper definition, see Wloka [2] p.61-62

[25] If $u \in [\mathcal{D}(\overline{\Omega})]^3$, then $T_n u = i'((\mathbf{u}\cdot\mathbf{n})|_{\partial\Omega})$.

Proof
Let $\phi \in W_2^{\frac{1}{2}}(\partial\Omega)$ and let $w \in W_2^1(\Omega)$ be such that $T_0 w = \phi$. Keeping the formula

$$\operatorname{div}(g\,\mathbf{v}) = g \operatorname{div}\mathbf{v} + \nabla g \cdot \mathbf{v}\,. \tag{C.11}$$

in mind we define for $\mathbf{f} \in M^{(3)}(\Omega)$

$$\begin{aligned} \Lambda_{\mathbf{f}}(w) &= \int_\Omega \operatorname{div}\mathbf{f}(\mathbf{x}) w(\mathbf{x}) + \mathbf{f}(\mathbf{x}) \cdot \nabla w(\mathbf{x}) d\mathbf{x} \\ &= (\mathbf{f}, \nabla w)_{\mathbf{L}_2(\Omega)} + (\operatorname{div}\mathbf{f}, w)_{\mathbf{L}_2(\Omega)} \end{aligned} \tag{C.12}$$

First, define the linear functional $\Upsilon_{\mathbf{f}} : W_2^{\frac{1}{2}}(\partial\Omega) \mapsto \mathbb{R}$ by

$$\Upsilon_{\mathbf{f}}(\phi) = (\Lambda_{\mathbf{f}} \circ Z_0)(\phi) = \Lambda_{\mathbf{f}}(Z_0(\phi))$$

We next show that $\Upsilon_{\mathbf{f}}(\phi) = \Lambda_{\mathbf{f}}(w)$ for all $w \in W_2^1(\Omega)$ with $T_0 w = \phi$.

Let w_1 and w_2 belong to $W_2^1(\Omega)$, with[26]

$$T_0 w_1 = T_0 w_2 \Leftrightarrow$$

and let $w = w_1 - w_2$. We must prove that

$$\Lambda_{\mathbf{f}}(w) = (\mathbf{f}, \nabla w)_{\mathbf{L}_2(\Omega)} + (\operatorname{div}\mathbf{f}, w)_{\mathbf{L}_2(\Omega)} = 0 \tag{C.13}$$

From $w \in W_2^1(\Omega)$, $T_0 w = 0$ and the fact that $\overset{\circ}{\mathcal{W}}_{1,2}(\Omega)$ is the kernel of T_0 it follows (by definition of $\mathcal{W}_{1,2}(\Omega)$) that

$$\exists_{w_k \in \mathcal{D}(\Omega)}\; w = \lim_{m\to\infty} w_k$$

Using (C.11) and Gauss one gets

$$\forall_{w_k \in \mathcal{D}(\Omega)} (\mathbf{f}, \nabla w_k)_{\mathbf{L}_2(\Omega)} + (\operatorname{div}\mathbf{f}, w_k)_{\mathbf{L}_2(\Omega)} = 0$$

And (C.13) follows as $k \to \infty$.
So the linear functional $\Upsilon_{\mathbf{f}}(\cdot)$ is independent of the choice of Z_0.

Next, we will show that it is continuous:
Using respectively the Cauchy-Schwarz inequality, and the continuity of Z_0 one obtains the estimate:

$$\begin{aligned} |\Upsilon_{\mathbf{f}}(\phi)| &\le (\|\mathbf{f}\|_{\mathbf{L}_2(\Omega)} + \|\operatorname{div}\mathbf{f}\|_{\mathbf{L}_2(\Omega)})\, \|Z_0\phi\|_{W_2^1(\Omega)} \\ &\le \|\mathbf{f}\|_{\mathbf{M}^{(3)}} \|Z_0\phi\|_{W_2^1(\Omega)} \le c_0 \|\mathbf{f}\|_{\mathbf{M}^{(3)}} \|\phi\|_{W^{\frac{1}{2}}(\partial\Omega)} \end{aligned} \tag{C.14}$$

The mapping $\Upsilon_{\mathbf{f}}(\cdot)$ is therefore continuous and It is clear that the mapping $\mathbf{f} \mapsto \Upsilon_{\mathbf{f}}$ is linear and by the above estimate it follows that this mapping, which will be denoted by $T_{\mathbf{n}}$, is continuous (!). Equation (C.10) is a special case (restriction to all $\phi \in W_2^{\frac{1}{2}}(\Omega)$ which can be written as $\phi = T_0 w$ with $w \in W_2^1$) of the definition of $T_{\mathbf{n}}$.

It remains to be shown that $T_{\mathbf{n}}\mathbf{f} =$ the restriction of $\mathbf{f} \cdot \mathbf{n}$ on $\partial\Omega$ in case $\mathbf{f} \in \mathcal{D}(\overline{\Omega})$:

Using (C.11) again one gets:

$$\begin{aligned} \Lambda_{\mathbf{f}}(w) &= \int_\Omega \operatorname{div}(\mathbf{f}w) d\mathbf{x} \\ &= \int_{\partial\Omega} w(\mathbf{f}\cdot\mathbf{n}) d\sigma_{\mathbf{x}} = \int_{\partial\Omega} T_0 w(\mathbf{f}\cdot\mathbf{n}) d\sigma_{\mathbf{x}} \\ &= \langle \mathbf{f}\cdot\mathbf{n}, T_0 w \rangle \end{aligned}$$

[26] $w = Z_0\phi \Rightarrow T_0 w = \phi$

Note that the second equality follows from the fact that any element in $\mathcal{D}(\overline{\Omega})$ can be restricted to the boundary.
From the fact that $T_0(W_2^1(\Omega))$ is dense in $W_2^{\frac{1}{2}}(\partial\Omega)$ it follows that $T_\mathbf{n}\mathbf{f} = \mathbf{f}\cdot\mathbf{n}|_{\partial\Omega}$ □

An alternative way to define $\mathbf{V}^{(3)}(\Omega)$ can now be given:

Definition C.1 *(Alternative definition $V^{(3)}(\Omega)$) Let $T_n : M^{(3)} \to W_2^{-\frac{1}{2}}(\partial\Omega)$ be a continuous operator such that (C.9) and (C.10) are satisfied. Then the space $V^{(3)}(\Omega)$ is defined as the nill space of this operator, i.e.*

$$V^{(3)}(\Omega) = \mathcal{N}(T_n) \ .$$

Note that:

- From $\mathcal{K} \cap \mathcal{N}(T_\mathbf{n}) = \{\mathbf{0}\}$ it (again) follows that on $V^{(3)}$ the norms $\|\cdot\|_{M^{(3)}}$ and $\|\cdot\|_{V^{(3)}}$ are equivalent. So, the restriction of operator $T_\mathbf{n}$ to $V^{(3)}$ remains continuous if we equip $V^{(3)}$ with norm $\|\cdot\|_{V^{(3)}}$ in stead of $\|\cdot\|_{M^{(3)}}$.
- From a practically point of view, we are only interested in vector functions tangent to the boundary, i.e. the nill space of $T_\mathbf{n}$. Therefore in comparison to the original construction of $V^{(3)}$, using theorem C.1 might be a bit exaggerated.

D div$\mathcal{D}$grad on a square

In this section we will examine operator div$\mathcal{D}$grad in case
$\Omega = \{(x,y) \in \mathbb{R}^2 \mid -a < x < a,\ -b < y < b\}$
we give some short results and some remarks why examining operator div$\mathcal{D}$grad on a square leads to difficulties.

First we will give some results and discuss the problems that arise on a square.

Theorem D.1 *Operator div$\mathcal{D}$grad lets the odd respectively even subspace of* $\mathbb{L}_2(\Omega)/\mathbb{R}$ *invariant.*

proof
Let p be odd, then grad p is even and since we have $\Delta\mathcal{D} = -I$, the function $\mathcal{D}$grad p is again even. Taking the divergence of an even function leads to an odd function again. □

In this section we only observe the case $h = 0$ and $\mathbf{f} = \mathbf{0}$. So we must solve div$\mathcal{D}$grad$p$ = div$\mathbf{a}_{\mathcal{H}}$. Note that by the same kind of arguments as in Theorem D.1 it follows that div$\mathcal{D}$grad$p$ is odd resp. even if and only if $\mathbf{a}$ is odd. resp. even.

The main problem is that in contrast with the unit ball, it isn't easy to find an orthogonal base for operator div$\mathcal{D}$grad. We couldn't even derive an orthogonal base for the space $\mathbb{L}^{harm}(\Omega)$. By the Gramm-Schmidt procédé it isn't difficult to construct an orthogonal set from the set

$$\Big\{\cosh\left(\tfrac{n\pi y}{a}\right)\sin\left(\tfrac{n\pi x}{a}\right),\ \sinh\left(\tfrac{m\pi y}{a}\right)\sin\left(\tfrac{m\pi x}{a}\right), \\ \cosh\left(\frac{k\pi y}{b}\right)\sin\left(\frac{k\pi y}{b}\right), \sinh\left(\frac{l\pi y}{a}\right)\sin\left(\tfrac{l\pi x}{a}\right)\Big\} \qquad (n,m,k,l \in \mathbb{N})\ ,\ \text{but}$$

of course, the span of this set will never be the entire space $\mathbb{L}_2^{\mathrm{harm}}(\Omega)$.

E div$\mathcal{D}$grad on a semi infinite strip

First, derive a formula for $\mathcal{D}\nabla p$ on a semi infinite strip. Define

$$S = \{(x,y) \in \mathbb{R}^2 \mid 0 < y < 1\}$$
$$\mathbf{V} = \mathcal{D}\nabla p$$
$$\mathbf{V}^{(\text{part})} = -\tfrac{1}{2}p(\mathbf{x})\mathbf{x}$$

Write $\mathbf{V} = \mathbf{V}^{(\text{part})} - \mathbf{W}$. As a result one gets:

$$\begin{cases} \Delta \mathbf{W} = 0 & \text{in S} \\ \mathbf{W}(0,y) = -\frac{1}{2}\begin{pmatrix} 0 \\ y \end{pmatrix} p(0,y) & y \in (0,1) \\ \mathbf{W}(x,1) = -\frac{1}{2}\begin{pmatrix} x \\ 1 \end{pmatrix} p(x,1) & x \in (0,\infty) \\ \mathbf{W}(x,0) = -\frac{1}{2}\begin{pmatrix} x \\ 0 \end{pmatrix} p(x,0) & x \in (0,\infty) \end{cases} \tag{E.1}$$

Although the solution of this problem can be obtained in a direct way, namely by using Fourier Sine Transform and separation of variables, we first try the use of conformal mapping and hope that this will reveals information between the operators $\nabla \cdot \mathcal{D}\nabla$ on the different regions.
Define $\Sigma : \mathbb{C} \mapsto \mathbb{C}$ by $\Sigma(z) = \cosh \pi z$.
Identify $\mathbb{R}^2$ with $\mathbb{C}$ by writing $z = x + iy$. Then Σ maps S conformally ($\Sigma^{-1}(w) = \frac{1}{\pi}\log(w + \sqrt{w^2-1})$) onto Ω (upper half space).

z-vlak w-vlak

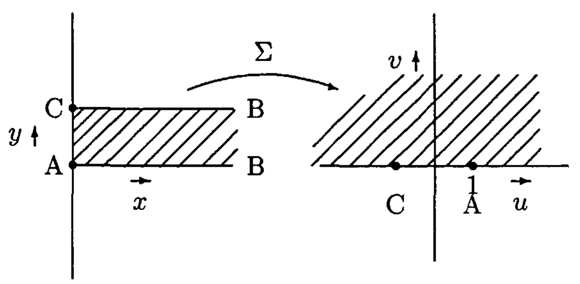


figure 1

Let $\rho : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be the function corresponding to Σ, then it is easy to see that ρ is given by

$$\rho(x,y) = \begin{pmatrix} \cosh \pi x \cos \pi y \\ \sinh \pi x \sin \pi y \end{pmatrix}.$$

Given $f : S \mapsto \mathbb{R}$ we'll define $f^* : \Omega \mapsto \mathbb{R}$ by $f^* = p \circ \rho^{-1}$. Note that $f^*(u(x,y), v(x,y)) = f(x,y)$. We have

$$\begin{array}{lll} \Delta \mathbf{W}^* = 0 & & \text{in } S \\ \mathbf{W}^*(u,0) = -\frac{1}{2}\begin{pmatrix} \frac{1}{\pi}\log(u+\sqrt{u^2-1}) \\ 0 \end{pmatrix} \mathbf{p}^*(u,0) & & \text{for } u > 1 \\ \mathbf{W}^*(u,0) = -\frac{1}{2}\begin{pmatrix} \frac{1}{\pi}\log(-u+\sqrt{u^2-1}) \\ 0 \end{pmatrix} \mathbf{p}^*(u,0) & & \text{for } u < -1 \\ \mathbf{W}^*(u,0) = -\frac{1}{2}\begin{pmatrix} 0 \\ \frac{1}{\pi}\arccos u \end{pmatrix} \mathbf{p}^*(u,0) & & \text{for } -1 < u < 1 \end{array} \tag{E.2}$$

In the previous paragraph, the solution of this problem was derived. See (4.1).

NB. Because one has to substitute cosine and sine hyperbolic functions in exponential expressions in Fourier transformations I do not think that using conformal mapping here, doesn't seem to enduce new insights.

Therefore, we solve **W** in a direct way:

Split

$$\mathbf{W} = \begin{pmatrix} I \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ J \end{pmatrix} + \begin{pmatrix} 0 \\ K \end{pmatrix}$$

In which I,J and K are the respective solutions of:

$$\begin{cases} \Delta I = 0 & \text{in } S \\ I(0,y) = 0 & y \in (0,1) \\ I(x,1) = -\frac{1}{2}xp(x,1) & x \in (0,\infty) \\ I(x,0) = -\frac{1}{2}xp(x,0) & x \in (0,\infty) \end{cases} \tag{E.3}$$

$$\begin{cases} \Delta J = 0 & \text{in } S \\ J(0,y) = 0 & y \in (0,1) \\ J(x,1) = -\frac{1}{2}p(x,1) & x \in (0,\infty) \\ J(x,0) = 0 & x \in (0,\infty) \end{cases} \tag{E.4}$$

$$\begin{cases} \Delta K = 0 & \text{in } S \\ K(0,y) = -\frac{1}{2}p(0,y)y & y \in (0,1) \\ K(x,1) = 0 & x \in (0,\infty) \\ K(x,0) = 0 & x \in (0,\infty) \end{cases} \tag{E.5}$$

Below a survey is given of the derivations of the solutions of these respective problems:

- Because it is obvious that this problem should be split in two and that the two different solutions can be derived in a completely similar way, we only give a short explanation of the derivation of one of them (i.e. take $I(x,0) = 0$).
  Use Fourier sine transform with respect to x ($x \to \xi$) and $\mathcal{F}_s(I) = \mathcal{F}(X)Y(y)$.
  Then:

  $$\mathcal{F}_s(\Delta I) = 0 \wedge Y(0) = 0,\ Y(1) = 1 \Rightarrow Y(y) = \frac{\sinh 2y\xi}{\sinh 2\xi}$$

  So the compleet solution of (E.3) is given by:

  $$\begin{aligned} I(x,y) = \quad & -\tfrac{1}{2}\sqrt{\tfrac{2}{\pi}}\int_0^\infty \Big\{[\mathcal{F}_s(x\ p(x,0))](\xi)\tfrac{\sinh 2\xi(1-y)}{\sinh 2\xi} + \\ & [\mathcal{F}_s(x\ p(x,1))](\xi)\tfrac{\sinh 2y\xi}{\sinh 2\xi}\Big\}\sin(\xi x)d\xi \end{aligned} \tag{E.6}$$

- Analogue to the "I problem ":

  $$J(x,y) = -\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty \left\{[\mathcal{F}_s(p(x,1))](\xi)\frac{\sinh 2y\xi}{\sinh 2\xi}\right\}\sin(\xi x)d\xi \tag{E.7}$$

- Using separation of variables one obtains:

  $$K(x,y) = \sum_{n=1}^{\infty} \gamma_n \sin(n\pi y)e^{-\frac{n\pi}{2}x} \tag{E.8}$$

With:

$$\gamma_n = -\frac{1}{2}\int_{-1}^{1} \eta\, p(0,\eta) \sin(n\pi\eta) d\eta \tag{E.9}$$

Now, $\text{div}\mathcal{D}\nabla p$ can be calculated.

$$\begin{aligned} \nabla \cdot \mathcal{D}\nabla p &= \text{div } -\tfrac{1}{2}\mathbf{x}p(\mathbf{x}) - \text{div}\,\mathbf{W} \\ &= -p - \tfrac{1}{2}\mathcal{E}p - \tfrac{\partial I}{\partial x} - \tfrac{\partial J}{\partial y} - \tfrac{\partial K}{\partial y} \end{aligned} \tag{E.10}$$

On $\Gamma_1 = \{(x,y) \in \mathbb{R}^2 \mid x > 0 \ \wedge \ y = 1\}$:

$$\text{div}\,\mathbf{W} = -\frac{1}{2}p(x,1) - \frac{1}{2}x\frac{\partial}{\partial x}p(x,1) + \frac{\partial J}{\partial y}(x,1) + \frac{\partial K}{\partial y}(x,1)$$

On $\Gamma_2 = \{(x,y) \in \mathbb{R}^2 | x > 0 \ \wedge \ y = 0\}$:

$$\text{div}\,\mathbf{W} = -\frac{1}{2}p(x,1) - \frac{1}{2}x\frac{\partial}{\partial x}p(x,1) + \frac{\partial J}{\partial y}(x,0) + \frac{\partial K}{\partial y}(x,0)$$

On $\Gamma_3 = \{(x,y) \in \mathbb{R}^2 \mid x = 0 \ \wedge \ y \in (0,1)\}$:

$$\text{div}\,\mathbf{W} = -\frac{1}{2}p(0,y) - \frac{1}{2}y\frac{\partial}{\partial y}p(0,y) + \frac{\partial I}{\partial x}(0,y)$$

$$\text{div } \mathbf{W}|_{\partial\Omega} = -\frac{1}{2}p - \frac{1}{2}\mathcal{E}p\Big|_{\partial\Omega} + \begin{cases} \frac{\partial J}{\partial y} + \frac{\partial K}{\partial y} & \text{at } \Gamma_1 \\ \frac{\partial I}{\partial x} & \text{at } \Gamma_2 \\ \frac{\partial J}{\partial y}\,\frac{\partial K}{\partial y} & \text{at } \Gamma_3 \end{cases} \tag{E.11}$$

Let H be the harmonic extension of the second term in the righthand side of equation (E.11), then

$$\text{div}\mathcal{D}\nabla p = -\frac{1}{2}p - H \tag{E.12}$$

It would be nice if H could be expressed in p in a way like was done in the ball case, but many attempts to this end were unsuccessful.

F Splitting of the Stokes problem.

Let's see what happens, if $P(\mathbf{f}, h, \mathbf{a})$ is split:

$$P(\mathbf{f}, h, \mathbf{a}) = P(\mathbf{f}, h, 0) + P(\mathbf{0}, 0, \mathbf{a}) \tag{F.1}$$

Notice that this can only be done if $\int_\Omega h\, d\mathbf{x} = \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n}\, d\sigma_{\mathbf{x}} = 0$, since h and $\mathbf{a}$ of any Stokes problem must satisfy (1.2).One might think that the solution of $P(\mathbf{f}, h, 0)$ is given by

$$\begin{pmatrix} \mathbf{v} \\ p \end{pmatrix} = \begin{pmatrix} \mathcal{D}\mathbf{f} + \nabla\mathcal{N}(h; \mathbf{a}\cdot\mathbf{n}) + \nabla\mathcal{N}(\nabla \cdot \mathcal{D}\mathbf{f}) \\ -h - \nabla \cdot \mathcal{D}\mathbf{f} \end{pmatrix}$$

but unfortunately appearances are deceptive:
$\nabla\mathcal{N}(\nabla \cdot \mathcal{D}\mathbf{f})|_{\partial\Omega} + \nabla\mathcal{N}(h; \mathbf{a}\cdot\mathbf{n})|_{\partial\Omega} \neq 0$

So, a more convenient way of splitting is given by

$$\begin{pmatrix} \mathbf{v} \\ p \end{pmatrix} = \begin{pmatrix} \mathbf{v}_0 \\ p_0 \end{pmatrix} + \begin{pmatrix} \mathcal{D}\mathbf{f} + \nabla\mathcal{N}(\nabla \cdot \mathcal{D}\mathbf{f}) + \nabla\mathcal{N}(h; \mathbf{a}\cdot\mathbf{n}) \\ -h - \nabla \cdot \mathcal{D}\mathbf{f} \end{pmatrix} \tag{F.2}$$

, where

$$\begin{pmatrix} \mathbf{v} \\ p \end{pmatrix} \text{ is the solution of } P(\mathbf{f}, h, \mathbf{a})$$

and

$$\begin{pmatrix} \mathbf{v}_0 \\ p_0 \end{pmatrix} \text{ is the solution of } P(\mathbf{0}, 0, -\nabla\mathcal{N}(\nabla\cdot\mathcal{D}\mathbf{f}) - \nabla\mathcal{N}(h; \mathbf{a}\cdot\mathbf{n}) + \mathbf{a}).$$

Note that:

- Because the third argument in the Stokes problem in the righthand side is tangent, see (F.3), we only need to impose the condition $\int_\Omega h \, d\mathbf{x} = 0$. By the way, from this condition it also follows that $\mathcal{N}(h; \mathbf{a}\cdot\mathbf{n}) = \mathcal{N}h + \mathcal{N}(0; \mathbf{a}\cdot\mathbf{n})$.

$$\mathbf{n}\cdot(-\nabla\mathcal{N}(\nabla\cdot\mathcal{D}\mathbf{f}) - \mathcal{N}(h; \mathbf{a}\cdot\mathbf{n}) + \mathbf{a})_{\partial\Omega} = \mathbf{a}\cdot\mathbf{n} - \mathbf{a}\cdot\mathbf{n} = 0. \quad \text{(F.3)}$$

- If we observe the above splitting in the special case $\mathbf{f} = \mathbf{0}$ and $h = 0$, then we see that adding $\mathcal{N}(0; \mathbf{a}\cdot\mathbf{n})$ to $\mathbf{a}$ leads to an addition of $\mathcal{N}(0; \mathbf{a}\cdot\mathbf{n})$ to $\mathbf{v}$. (NB.div $\mathcal{N}(0; \mathbf{a}\cdot\mathbf{n}) = 0$ and $\Delta\mathcal{N}(0, \mathbf{a}\cdot\mathbf{n}) = 0$)
  This already reveals that the tangential of $\mathbf{a}$ plays a more important role than the normal part of $\mathbf{a}$.

G Fundamental mathematical subjects

This chapter gives some fundamental mathematical background according to the completion of a metric space and local convex topology. Notice that these subjects appear in respectively Chapter 5 and in the first paragraph of Chapter 6. In this chapter we do not use bold face notation as introduced in Chapter 1, because the abstract spaces in this chapter need not be vector spaces.

G.1 Completion of a metric space

In this chapter we will give the definition of a completion of a metric space and we will show in theorem G.1 that every metric space can be completed, i.e. each metric space is a dense subspace of a complete metric space.

Definition G.1 *Let (Y, d') and (X, d) be metric spaces.*
(Y, d') is the completion of (X, d) if there exists an isometric mapping $\Phi : X \to Y$ such that its image is dense in Y, i.e.

$$\exists_{\Phi: X \to Y} [\Delta(\Phi(x), \Phi(y)) = d(x, y)] \text{ and } \overline{\Phi(X)} = Y .$$

Next, we will do some preparations for theorem G.1:

- Let Y be a set and δ a semi-metric on Y. Define the equivalence relation

  $$y_1 \sim y_2 \Leftrightarrow \delta(y_1, y_2) = 0.$$

  Note that this indeed is an equivalence relation: It is obvious that $\sim$ is reflexive and symmetric and for all $y_1 \sim y_2$ and $y_2 \sim y_3$ we have $0 \leq \delta(y_1, y_3) \leq \delta(y_1, y_2) + \delta(y_2, y_3) = 0$, from which transitivity follows. Define $Y_u = Y/_\sim$, then this space can be equipped with the well-defined metric δ_u given by

  $$\delta_u([y_1], [y_2]) = \delta(y_1, y_2).$$

  This simple principle which is used for instance in the introduction of $\mathbb{L}_p$-spaces will also be used in theorem G.1.

- Let (X, d) be a metric space. Denote the set of all Cauchy sequences in X by $\mathcal{F}(X) \subset X^{\mathbb{N}}$. Elements in $\mathcal{F}(X)$ will be denoted by ξ, η and ζ and their n-th components will be denoted by $\xi(n)$, $\eta(n)$ and $\zeta(n)$.
  Let $\xi, \eta \in \mathcal{F}(X)$, then $\{d(\xi(n), \eta(n))\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Since,[27] $|d(\xi(n), \eta(n)) - d(\xi(m), \eta(m))| \leq |d(\xi(n), \xi(m)) + d(\xi(m), \eta(m)) + d(\eta(m), \eta(n)) - d(\xi(m), \eta(m))| \leq d(\xi(n), \xi(m)) + d(\eta(m), \eta(n))$.
  Since $\mathbb{R}$ is complete we can define a semi-metric Δ on $\mathcal{F}(X)$ by:

  $$\Delta(\xi, \eta) = \lim_{n \to \infty} d(\xi(n), \eta(n)). \qquad \text{(G.1)}$$

In theorem G.1 we will use the following lemma.

Lemma 1 *Let (R, d) be a metric space, and let A be a dense subset, then (R, d) is complete if every Cauchy sequence in A converges to a certain $x \in R$.*

Proof
Let $\{x_k\}_{k \in \mathbb{N}}$ be a Cauchy sequence in R. Then there exists a sequence $\{a_k\}_{k \in \mathbb{N}}$ in A such that $d(a_k, x_k) < \frac{1}{k}$. Therefore $\{a_k\}_{k \in \mathbb{N}}$ is a Cauchy sequence in A, since $d(a_k, a_l) < \frac{1}{k} + \frac{1}{l} + s(x_k, x_l)$. By assumption we now have that exists a $x \in \mathbb{R}$ such that $a_k \to x$ $(k \to \infty)$. From this we conclude by using $s(x_k, x) \leq \frac{1}{k} + s(a_k, x)$ that $x_k \to x$ $(k \to \infty)$. □

[27] without loss of generality assume $d(\xi(n), \eta(n)) > d(\xi(m), \eta(m))$.

Theorem G.1 *Every metric space (X, d) can be completed.*

Proof
For notation and preliminaries see the above remarks.
We will show that $\mathcal{F}(X)_u = \mathcal{F}(X)/_\sim$ with metric Δ_u is a complete metric space and that there is an isometric mapping Φ from X into $\mathcal{F}(X)_u$ such that $\Phi(X)$ is dense in $\mathcal{F}(X)_u$. We shall see that Φ is surjective if and only if X is complete.
Define $\Psi : X \to \mathcal{F}(X)$ by

$$\Psi(x) : n \mapsto x \quad (n \in \mathbb{N}) \quad \text{(constant sequence)}$$

Let $\xi \in \mathcal{F}(X)$. Suppose ξ is convergent with limit $x \in X$, then for $\eta \in \mathcal{F}(X)$ we have $\xi \sim \eta \Leftrightarrow \eta$ is convergent to the same limit x, since

$$d(\eta(n), x) \leq d(\xi(n), \eta(n)) + d(\xi(n), x) \text{ and}$$
$$d(\xi(n), \eta(n)) \leq d(\xi(n), x) + d(\eta(n), x) .$$

As a result $[\Psi(x)]$ is the equivalence class of all convergent sequences with limit x. Note that *if* X is complete, the mapping $x \mapsto [\Psi(x)]$ is bijective.

Define the mapping $\Phi : X \to \mathcal{F}(X)/_\sim$ by $\Phi(x) = [\Psi(x)]$. We will now show that Φ is an isometric mapping.
Let $x, y \in X$, then we have

$$\Delta_u(\Phi(x), \Phi(y)) = \Delta(\Psi(x), \Psi(y)) = \lim_{n\to\infty} d(x, y) = d(x, y) .$$

Next we will show that $\Phi(X)$ is dense in $\mathcal{F}(X)/_\sim$:

Let $\xi \in \mathcal{F}(X)$, i.e.
$\forall_{\epsilon>0}\exists_{n_0}\forall_{n,m>n_0} : d(\xi(n), \xi(m)) < \epsilon$
Define $x_m = \xi(m), m \in \mathbb{N}$
Let $\epsilon > 0$. Let $m > n_0$, then
$\Delta_u(\Phi(x_m), [\xi]) = \Delta(x_m, \xi) = \lim_{n\to\infty} d(\xi(m), \xi(n)) \leq \epsilon.$

Finally, we show that $(\mathcal{F}(X)_u, \Delta_u)$ is complete:

By lemma 1 and the above we only have to prove that every Cauchy sequence in $\mathcal{F}(X)_u$ is convergent.
Let $\{\Phi(x_k)\}_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{F}(X)$
Then $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in X.
Define $\xi \in \mathcal{F}(X)$ by $\xi(n) = x_n$. We have :
$\Delta_u(\Phi(x_k, [\xi])) = \lim_{n\to\infty} d(x_k, \xi(n)) = \lim_{n\to\infty} d(x_k, x_n)$
So, $\Phi(x_k) \to [\xi]$ $(k \to \infty)$ in $(\mathcal{F}(X), \Delta_u)$. □

G.2 Locally convex topologies

Let V be a vector space. Let $\mathcal{P}$ be a collection of semi-norms defined on V. Note that $\mathcal{P}$ need not be a finite set. Let $p_1, \ldots, p_N$ be in $\mathcal{P}$ and $\epsilon > 0$, then we define the neighborhood $N(0, p_1, \ldots, p_n, \epsilon)$ by:

$$N(0, p_1, \ldots, p_n, \epsilon) = \{x \in V \mid \forall_{i\leq n}[p_i(x-y) < \epsilon]\}. \tag{G.2}$$

Such a neighborhood is convex since for arbitrary $x, y \in V$ and $\lambda \in [0, 1]$ we have for all $i \in \{1, \ldots, N\}$:

$$p_i(\lambda x + (1-\lambda)\beta y) \leq |\lambda| p_i(x) + |1-\lambda| p_i(y) < \epsilon .$$

Let $x \in V$, then the neighborhood $N(x, p_1, \ldots, p_n, \epsilon)$ by

$$\begin{aligned} N(x, p_1, \ldots, p_n, \epsilon) &= \{x + y \mid y \in N(0, p_1, \ldots, p_n, \epsilon)\} \\ &= x + N(0, p_1, \ldots, p_n, \epsilon) \end{aligned}$$

Define the set

$$\mathcal{T} = \{\, U \subset V \;:\; \forall_{x \in U} \exists_{n \in \mathbb{N}} \exists_{p_1, \ldots, p_n \in \mathcal{P}} \exists_{\epsilon > 0} \, [x + N(p_1, \ldots, p_n) \subset U] \,\} \,. \tag{G.3}$$

Then $\mathcal{T}$ is a topology which in particular contains all neighborhoods defined by (G.2). Note that $\mathcal{T}$ is determined by $\mathcal{P}$.

Examples:

- In case $V = C(\mathbb{R})$, $p_n(f) = \sup_{x \in [-n,n]} |f(x)|$, for $n \in \mathbb{N}$ we have $N(p_1, \ldots, p_K, \epsilon) = N(p_K, \epsilon)$.
- The weak topology on a normed space E:

  $$\mathcal{P} = \{x \mapsto |f(x)| \;:\; f \in E'\} \,.$$

  Thus we have that a weak neighborhood of $0 \in E$ is given by:

  $$N(0, F_1, \ldots, F_n, \epsilon) = \{x \in E \mid |F_i(x)| < \epsilon \text{ for } i = 1 \ldots n\}$$

- The weak* topology on the dual (E') of a normed space E:

  $$\mathcal{P} = \{x \mapsto |f(x)| \;:\; x \in E\}\} \,.$$

  If E is reflexive, i.e. every functional $\hat{x}$ in E'' is given by $\hat{x}(F) = F(x)$ for some $x \in E$, we have that the definition of weak* topology and weak topology on E' coincide.
  A weak* neighborhood of $0 \in E'$ is given by:

  $$N(0, x_1, \ldots, x_n, \epsilon) = \{G \in E' \;:\; |G(x_i)| < \epsilon \text{ for } i = 1 \ldots n\} \,.$$

Theorem G.2 *The topological space $(V, \mathcal{T})$, with $\mathcal{T}$ given by G.3 is a Hausdorff space $\Leftrightarrow$ $\forall_{0 \neq x \in V} \exists_{p \in \mathcal{P}} [p(x) \neq 0]$*

Proof

$\Rightarrow$

Let $x \in V$. If $(V, \mathcal{T})$ is a Hausdorff space, then there exists a neighborhood $N(0, p_1, \ldots, p_N, \epsilon)$ such that x is not a member of this neighborhood. This means that there must exist a p_i such that $p_i(x) > \epsilon$ and therefore $p_i(x) \neq 0$.

$\Leftarrow$

Let $x \in V$. Put $\epsilon = p(x)$, then $N(0, p, \frac{1}{2}\epsilon)$ and $N(x, p, \frac{1}{2}\epsilon)$ are open sets containing respectively 0 and x such that $N(0, p, \frac{1}{2}\epsilon) \cap N(x, p, \frac{1}{2}\epsilon) = \emptyset$ □

Definition G.2 *Let V be a vector space and let $\mathcal{T}$ be a topology on V, then $(V, \mathcal{T})$ is called locally convex, if for all $x \in V$ and $U \in \mathcal{T}$ with $x \in U$ there exists a convex $W \in \mathcal{T}$ such that $x \in W \subset U$.*

It can be shown that every locally convex topological vector space $(V, \mathcal{T})$ is of the form (G.3).

G.3 Proof of theorem 5.1

Next we give a proof of a special case of theorem 5.1. For, simplicity we also suppose:

1. There exists a $\phi \in C^\infty(\Omega)$, with the property:

$$\exists_{\eta>0}\forall_{x\in\partial\Omega} : (\mathrm{grad}\phi\cdot\mathbf{n})(\mathbf{x}) \geq \eta. \tag{G.4}$$

2. The boundary $\partial\Omega$ is almost everywhere continuously differentiable.

Let $\phi \in C^\infty(\Omega)$ such that (G.4) is satisfied. Then we have by using Gauss' theorem

$$\eta \int_{\partial\Omega} |u|^2\, d\sigma \leq \int_{\partial\Omega} |u|^2 \mathrm{grad}\,\phi\cdot\mathbf{n}\, d\sigma = \int_{\Omega} \mathrm{div}\left[|u|^2 \mathrm{grad}\phi\right]\, d\mathbf{x} =$$

$$2\int_{\Omega} (\frac{1}{\epsilon}u)\,\epsilon\,\mathrm{grad}\,u\cdot\mathrm{grad}\,\phi\, d\mathbf{x} + \int_{\Omega} |u|^2 \Delta\phi\, d\mathbf{x}\,.$$

Apply Cauchy-Schwarz and we obtain:

$$\exists_{c>0}\forall_{\epsilon>0}\exists_{M>0} : \int_{\partial\Omega} |u|^2 d\sigma \leq C\left\{\epsilon\int_{\Omega} |\mathrm{grad}\,u|^2 d\mathbf{x} + M\int_{\Omega} |u|^2\, d\mathbf{x}\right\}.$$

So, in particular we have

$$\exists_{c_1>0} : \|u\|^2_{\mathbb{L}_2(\Omega)} \leq c_1\|u\|_{\mathbb{H}_1(\Omega)}\,.$$

By, taking a Cauchy sequence in $\mathcal{D}(\Omega)$ which converges to $\mathbf{u} \in \mathbb{H}_1(\Omega)$, it follows that the restriction of a function in $\mathbb{H}_1(\Omega)$ to the boundary makes sense, as long as this trace is considered as an element of $\mathbb{L}_2(\partial\Omega)$. □

H Existence and uniqueness of solutions of Stokes problems

In this chapter we will give a survey of how the Stokes equations are usually treated. For more detailed information, weaker assumptions to the region Ω etc. see Temam[7]. However, in the next chapters we will approach the Stokes equations in a different and unusual way. Note that this chapter doesn't connect with the other chapters in a direct sense.

Remark I.1
During this chapter we shall denote the Sobolev space $W_{2,1}(\Omega) = \{\phi \in \mathbb{L}_2(\Omega) \mid \forall_{|s|\leq 1} : D^s\phi \in \mathbb{L}_2(\Omega)\}$ by $\mathbb{H}_1(\Omega)$. Although this is common use we remark that (in strict sense) this is not correct. In chapter 6 we define the space $\mathbb{H}_1(\Omega)$ differently, see definition 6.5. In Theorem 6.14 the conditions are mentioned such that the spaces $\mathbb{H}_1(\Omega)$ and $W_{2,1}(\Omega)$ *are* the same.

We give the variational and minimization formulation of the Stokes problem and an existence and uniqueness result using the Lax-Milgram theorem, but first some preliminary results:

Definition H.1 *Define the test-space*

$$\mathcal{T} = \{u \in \mathcal{D}(\Omega) \mid div\, u = 0\}$$

Definition H.2 *Denote the closure of $\mathcal{T}$ in $\overset{\circ}{\mathbf{H}}_1(\Omega)$ by V, i.e.*

$$V = \overline{\mathcal{T}}^{\overset{\circ}{\mathbf{H}}_1(\Omega)}$$

and define[28]

$$\mathcal{L}_0 = \{\mathbf{u} \in \overset{\circ}{\mathbf{H}}_1(\Omega) \mid \operatorname{div} \mathbf{u} = 0\}$$

In a special case $\mathcal{L}_0$ and V coincide as will be shown in theorem H.3, but first we give some fundamental theorems.

Theorem H.1 *Let Ω be an open set of $\mathbb{R}^n$ and $f = \{f_1, \ldots, f_n\}$, $f_i \in \mathcal{D}'(\Omega), i = 1, \ldots, n$. A necessary and sufficient condition that*

$$\begin{array}{ll} f = grad\, p & \text{for some } p \text{ in } \mathcal{D}'(\Omega) \\ \text{is that} & \forall_{v\in\mathcal{T}} : <f, v> = 0 \end{array}$$

For a proof of this theorem the reader is referred to G. de Rahm [8].

Theorem H.2 *1. Let Ω be a bounded Lipschitz open set in $\mathbb{R}^n$. If a distribution p has all his first derivatives $D_i\, p$ $\; 1 \leq i \leq n$ in $\mathbb{L}_2(\Omega)$, then $p \in \mathbb{L}_2(\Omega)$ and*

$$||p||_{\mathbb{L}_2(\Omega)/\mathbb{R}} \leq c(\Omega)||grad\, p||_{\mathbb{L}_2(\Omega)}$$

2. If a distribution p has all its first derivatives $D_i\, p$, $\; , 1 \leq i \leq n$, in $\mathbf{H}_{-1}(\Omega)$, then $p \in \mathbb{L}_2(\Omega)$ and

$$||p||_{\mathbb{L}_2(\Omega)/\mathbb{R}} \leq c(\Omega)||grad\, p||_{\mathbf{H}_{-1}(\Omega)}$$

[28]The notation $\mathcal{L}_0$ might look strange. The purpose of this notation is to stress that this space is the same $\mathcal{L}_0$ that is considered in chapter 5. See theorem 5.11

For a proof of this theorem see Temam[7] p.15.

Theorem H.3 *Let Ω be an open bounded Lipschitz set. Then*

$$V = \mathcal{L}_0 = \{u \in \overset{\circ}{\mathbf{H}}_1(\Omega) \mid div\, u = 0\}$$

Proof
Let $\mathbf{u} \in V$. Then $\mathbf{u} = \lim_{m\to\infty} \mathbf{u}_m$ in $\overset{\circ}{\mathbf{H}}_1(\Omega)$. From $\operatorname{div}\mathbf{u}_m = 0$ it follows that $\operatorname{div}\mathbf{u} = 0$. So, $V \subset \mathcal{L}_0$
To prove that $V = \mathcal{L}_0$ we will show that any continuous linear functional L on $\mathcal{L}_0$ which vanishes on V is identically equal to zero:
We first observe that L admits a (non-unique) representation of the type:

$$L(\mathbf{v}) = \sum_{i=1}^{n} \langle l_i, v_i \rangle \qquad l_i \in \mathbf{H}_{-1}(\Omega) \tag{H.1}$$

Indeed $\mathcal{L}_0$ is a closed subspace of $\overset{\circ}{\mathbf{H}}_1(\Omega)$ and (by Hahn-Banach) any continuous linear functional on $\mathcal{L}_0$ can be extended to a continuous linear functional on $\overset{\circ}{\mathbf{H}}_1(\Omega)$ and such a form is of the same type as the form of the right-hand side of (H.1).
Now, the vector distribution $\mathbf{l} = (l_1, \ldots, l_n)$ belongs to $\mathbf{H}_{-1}(\Omega) = (\overset{\circ}{\mathbf{H}}_1(\Omega))'$ and $\langle \mathbf{l}, \mathbf{u} \rangle = 0$ for all $\mathbf{u} \in \mathcal{T}$. Theorem H.1 and H.2 are applicable and show that $\mathbf{l} = \nabla p$, $p \in \mathbb{L}_2(\Omega)$; thus

$$\langle l_i, v_i \rangle = \langle D_i p, v_i \rangle = -(p, D_i v_i)\ ,\ \forall v_i \in \overset{\circ}{\mathbf{H}}_1(\Omega)$$

Therefore L vanishes on $\mathcal{L}_0$, since for all $\mathbf{v} \in \mathcal{T}$

$$L(\mathbf{v}) = \sum_{i=1}^{n} \langle l_i, v_i \rangle = -(p, \operatorname{div}\mathbf{v}) = 0 \qquad \square$$

H.1 The homogeneous Stokes problem

In this section we will observe the homogeneous case of the Stokes problem $P(\mathbf{f}, 0, \mathbf{0})$ i.e.

$$\begin{cases} \Delta \mathbf{v} = \nabla p - \mathbf{f} & \text{in } \Omega \\ \nabla \cdot \mathbf{v} = 0 & \text{in } \Omega \\ \mathbf{v} = \mathbf{0} & \text{at } \partial\Omega \end{cases} \tag{H.2}$$

First we give the variational formulation of the homogeneous Stokes problem.

H.1.1 The variational problem

Define the bilinear form $a : V \times V \to \mathbb{R}$ by

$$a(\mathbf{u}, \mathbf{v}) = \operatorname{trace}([D\mathbf{u}][D\mathbf{v}]^T) = \sum_{i=1}^{n} (\nabla u_i, \nabla v_i) \tag{H.3}$$

Then the variational formulation of the homogeneous Stokes problem is given by:

$$\begin{cases} a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{\mathbb{L}_2(\Omega)} & , \forall_{\mathbf{v} \in V} \\ \mathbf{u} \in V \end{cases} \tag{H.4}$$

Theorem H.4 *Let Ω be a (0,1)-smooth region in $\mathbb{R}^n$. Then the following conditions are equivalent:*

1. *u satisfies (H.4)*

2. $\boldsymbol{u}$ *belongs to* $\overset{\circ}{\mathbf{H}}_1(\Omega)$ *and satisfies the homogeneous Stokes problem* $P(\boldsymbol{f}, 0, 0)$ *in the following weak sense:*

> *There exists a* $p \in \mathbb{L}_2(\Omega)$ *such that* $-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}$ *in the distributional sense in* Ω.
> $div\, \boldsymbol{u} = 0$ *in the distributional sense in* Ω.
> $T_0 \boldsymbol{u} = 0$

Proof
1. $\Rightarrow$ 2.
Let $\mathbf{u}$ satisfy 1. . Then $\mathbf{u}$ belongs to $\overset{\circ}{\mathbf{H}}_1(\Omega)$ and therefore $T_0 u_i = 0$ for $i = 1 \dots n$ in $W_2^{\frac{1}{2}}(\Omega)$. $u \in V$ implies (using theorem H.3) that $\operatorname{div} \mathbf{u} = 0$ in the distributional sense. From $a(u, v) = (\mathbf{f}, \mathbf{v})_{\mathbf{L}_2(\Omega)}$ it follows that

$$\langle -\Delta \mathbf{u} - \mathbf{f}, \mathbf{v} \rangle = 0 \qquad \forall_{\mathbf{v} \in \mathcal{T}}$$

Then by virtue of theorem H.1 and H.2, there exists some distribution $p \in \mathbb{L}_2(\Omega)$ such that
$-\Delta \mathbf{u} - \mathbf{f} = -\nabla p$ in the distributional sense.

2. $\Rightarrow$ 1.
Let $\mathbf{u}$ satisfy 2. . Then by theorem H.3 we have that $\mathbf{u} \in V$. By definition we have that $V = \overline{\mathcal{T}}^{\mathbf{H}_1(\Omega)}$. Therefore (by continuity) we only need to show $a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})_{\mathbb{L}_2(\Omega)}$ for all $\mathbf{v} \in \mathcal{T}$:
Let $\mathbf{v}$ be in $\mathcal{T}$.
We have that

$$(-\Delta \mathbf{u} - \mathbf{f} + \nabla p, \mathbf{v}) = 0$$

The term $(\nabla p, \mathbf{v})$ vanishes because of Gauss and $\operatorname{div}(p\, \mathbf{v}) = p \operatorname{div} \mathbf{v} + \nabla p \cdot \mathbf{v}$.
Using Greens first identity, the term $(-\Delta \mathbf{u}, \mathbf{v})$ becomes $a(u, v)$. So we have that $a(u, v) = (\mathbf{f}, \mathbf{v})_{\mathbb{L}_2(\Omega)}$ □

Theorem H.5 *Let* Ω *be bounded in some direction, then for any* $\boldsymbol{f} \in \mathbf{L}_2(\Omega)$*, the variational problem given by H.4 has a unique solution u.(The result is also valid if* f *is given in* $\mathbf{H}_{-1}(\Omega)$*)*

Proof
Because Ω is bounded in one direction we can apply the Poincaré inequality (See Theorem 6.20). So, in stead of the usual $\|\cdot\|_{\mathbb{H}_1}$ norm (induced to V) we can use an equivalent norm $\|\cdot\|_V$ given by $\|\mathbf{v}\|_V = \operatorname{trace}\left([D\mathbf{v}][D\mathbf{v}]^T\right)$. It is now trivial that the bilinear form $a(\cdot,\cdot)$ is coércive and continuous. Note that the space V is a closed subspace of a Hilbert space and thus a Hilbert space.
The linear functional $\mathbf{v} \mapsto (\mathbf{f}, \mathbf{v})_{\mathbb{L}_2(\Omega)}$ is continuous.
The result now follows by the Lax-Milgram theorem. □

H.1.2 The minimization problem

Theorem H.6 *The solution of the variational problem given by (H.4) is also the unique element of* V *such that*

$$E(u) \leq E(v) \quad \text{for all } v \in V \tag{H.5}$$

where

$$E(v) = \|v\|^2 - 2(f, v)$$

Proof
Let $\mathbf{u}$ be the solution of (H.4). Then as

$$||\mathbf{u}-\mathbf{v}||_V^2 \geq 0 \text{ for all } \mathbf{v} \in V$$

we have

$$||\mathbf{u}||_V^2 + ||\mathbf{v}||_V^2 - 2\,a(\mathbf{u},\mathbf{v}) \geq 0 \tag{H.6}$$

By (H.4) we have $a(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v})_{\mathbb{L}_2(\Omega)}$ from which it follows that

$$E(\mathbf{u}) = -||\mathbf{u}||_V^2 \tag{H.7}$$

Equality (H.6) and (H.7) imply (H.5).

Conversely, if $\mathbf{u} \in V$ satisfies (H.5), then for any $\mathbf{v} \in V$ and $\lambda \in \mathbb{R}$ one has

$$E(\mathbf{u}) \leq E(\mathbf{u}+\lambda\mathbf{v})$$

This may be reduced to

$$\lambda^2||\mathbf{v}||^2 + 2\lambda\, a(\mathbf{u},\mathbf{v}) - 2\lambda(\mathbf{f},\mathbf{v}) \geq 0 \qquad \text{for all } \lambda \in \mathbb{R}$$

This inequality can hold for each $\lambda \in \mathbb{R}$ only if

$$a(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v})_{\mathbb{L}_2(\Omega)} \qquad \square$$

H.2 The inhomogeneous Stokes problem

In this paragraph we observe the inhomogeneous Stokes problem $P(\mathbf{f}, h, \mathbf{a})$ as defined by the equation (1.1).

Theorem H.7 *Let Ω be a Lipschitz open bounded set in $\mathbb{R}^n$. Suppose $g \in \mathbb{L}_2(\Omega)$, $\boldsymbol{f} \in \mathbf{H}_{-1}$ and $\boldsymbol{a} \in \mathbf{H}_{1/2}(\partial\Omega)$ such that (1.2) is satisfied. Then there exists $\boldsymbol{u} \in \mathbf{H}_1(\Omega)$ and $p \in \mathbb{L}_2(\Omega)$ of the inhomogeneous Stokes problem $P(\boldsymbol{f}, h, \boldsymbol{a})$ (NB. replace $\boldsymbol{u} = \boldsymbol{a}$ on $\partial\Omega$ by $T_0\boldsymbol{u} = \boldsymbol{a}$). $\boldsymbol{u}$ is unique and p is unique up to the addition of a constant.*

Proof
Since $\mathbf{H}_{1/2}(\partial\Omega) = T_0\mathbf{H}_1(\Omega)$, there exists $\mathbf{u}_0 \in \mathbf{H}_1(\Omega)$, such that $T_0\mathbf{u}_0 = \mathbf{a}$. Then, from (1.2) and Gauss' law we have that

$$\int_\Omega (g - \operatorname{div}\mathbf{u}_0)\, d\mathbf{x} = 0.$$

The divergence operator div maps $\overset{\circ}{\mathbf{H}}_1(\Omega)$ *onto* the space $\mathbb{L}_2(\Omega)/\mathbb{R}$. See theorem 5.6. So there is a $\mathbf{u}_1 \in \overset{\circ}{\mathbf{H}}_1(\Omega)$ such that $\operatorname{div}\mathbf{u}_1 = g - \operatorname{div}\mathbf{u}_0$. Setting $\mathbf{v} = \mathbf{u} - \mathbf{u}_0 - \mathbf{u}_1$, the inhomogeneous Stokes problem reduces to a homogeneous Stokes problem for $\mathbf{v}$:

$$\begin{cases} -\Delta\mathbf{v} + \nabla p = \mathbf{f} - \Delta(\mathbf{u}_0 + \mathbf{u}_1) \in \mathbf{H}_{-1}(\Omega) \\ \operatorname{div}\mathbf{v} = 0 \qquad \text{in } \Omega \\ \mathbf{v} = \mathbf{0} \qquad \text{on } \partial\Omega \end{cases}$$

By theorem H.5 the existence and uniqueness of $\mathbf{v}$ and p now follows. $\square$

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