Maintenance optimization and spare parts management in data-integrated environments

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Maintenance optimization and spare parts management in data-integrated environments

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Het onderzoek dat in dit proefschrift wordt beschreven is uitgevoerd in overeenstemming met de TU/e Gedragscode Wetenschapsbeoefening.

Contents

1

Introduction

Capital goods are systems that are used by businesses for providing services and products to their customers (van Houtum and Kranenburg, 2015, p.1-2). Some examples of capital goods are MRI scanners at hospitals, milking robots at farms, lithography machines that are used to produce chips, and trains. Many business operations cannot proceed without a functioning capital good. Downtime of a capital good results in high direct and indirect costs to the business owners.

We focus on an example of capital goods: MRI scanners at hospitals. Imagine you made an appointment for an MRI scan one month ago and the day you go to the hospital, the MRI scanner is not functioning. Therefore, your appointment is rescheduled to the next month because all slots are already booked until the upcoming month. If you think about this situation from the point of view of the hospital, there are many patients whose appointments should be rescheduled. Also, MRI scans are important in diagnosing diseases; therefore, postponing them also has an external cost to the patient's well-being. As we demonstrate in this example, an unplanned down of a capital good is costly to many parties. The longer the downtime, the more patients are affected by the failure of the MRI scanner. Therefore, the costs of the failure of an MRI scanner are increasing as a function of the duration of the down.

Capital goods are systems that have many components. Maintenance operations mostly consist of replacing defective, failed components and components that are about to fail (Arts, 2017). Corrective maintenance upon failure and preventive maintenance are two common maintenance strategies used in practice. Corrective maintenance tasks aim to bring a failed or faulty system into fully working condition again. On the other hand, preventive maintenance tasks aim to maintain the system preventively before observing a failure. Usually, the cost of corrective maintenance actions is higher than the cost of preventive maintenance actions because corrective maintenance costs consist of downtime cost, the cost of failure, and the subsequent emergency replacement cost in addition to the cost of the component. In this dissertation, we mainly focus on systems that are subject to failures because of failing components. In order to repair the system, the failed component has to be replaced by a new one.

Preventive maintenance policies can be classified into two as *condition-based* and *usage-based* maintenance policies (Arts, 2017). Under a condition-based policy, preventive maintenance decisions are taken based on the state of the component. The state of the component can be observed with continuous monitoring or with periodic inspections (Arts, 2017). Under usage-based maintenance, preventive maintenance takes place based on the usage of the component. Time is one of the most common measures that is used in practice to determine the usage of the component. We refer the reader to Ahmad and Kamaruddin (2012) for a more detailed overview of condition-based and time-based maintenance strategies. Under *time-based* maintenance, preventive maintenance takes place based on failure time analysis (Ahmad and Kamaruddin, 2012). The age of a component is an important indicator of the remaining lifetime. Age-based maintenance, introduced by Barlow and Hunter (1960), is a commonly used policy for determining when to perform maintenance activities. Under this policy, a preventive maintenance action is applied at a predetermined age. If a failure happens before the predetermined age, a corrective maintenance action is applied. We refer the reader to Jardine and Tsang (2013) and de Jonge and Scarf (2020) for a comprehensive overview of age-based maintenance policies.

After-sales services are important to keep capital goods up and running. After-sales service commitments are usually offered to the owners of capital goods by maintenance service providers. Maintenance services can be provided by an original equipment manufacturer (OEM) or a third-party maintenance service provider. We refer to them as *service providers* in the rest of this dissertation. The agreed commitments are formalized in service level agreements (SLA) that are part of the service contracts (Topan et al., 2020). A service provider is responsible for maintenance services and spare parts management. Maintenance providers aim to minimize the total cost of their operations while fulfilling the conditions of SLAs with their customers. The operational cost of maintenance activities may involve the costs of replacing spare parts, engineer visits, preventive maintenance actions, regular and emergency shipments, and keeping spare parts in the inventory. In this dissertation, we focus on the research objectives from the service provider's point of view.

Maintenance activities involve uncertainty by nature because most components in systems fail randomly. Estimating a failure, the time of failure, or the type of components that cause the system failure are important for determining the optimal maintenance decision that minimizes operational costs. Most of the time, the service provider must make decisions regarding maintenance activities and spare parts shipments to the customer in real-time. Real-time decision-making requires real-time coordination between different parties and data flows. Some of these parties are multiple customers at different locations, service engineers, and spare part warehouses. Industry 4.0 technologies enable the storage of a large amount of data, continuous monitoring, and real-time data transfer. Internet of things (IoT) technologies such as sensors, databases, and systems that are connected to the internet or a local network create data-integrated environments for today's production and service facilities.

Integrating Industry 4.0 and IoT technologies enable more efficient maintenance policies that result in economic and environmental benefits (Tortorella et al., 2021). For example, the number of unnecessary preventive replacements can be decreased by better failure predictions made by sensor readings. This will decrease the number of spare parts that are consumed and scrapped. However, there are barriers to integrating new technology into existing business practices (Tortorella et al., 2021). One of the barriers is the industry's lack of knowledge on adopting the new technology. Motivated by this barrier, this research aims to create knowledge on decision-making in data-integrated environments for maintenance optimization and spare parts management. Another barrier is the cost of the adoption of new technologies. This research also aims to provide insights to practitioners on the

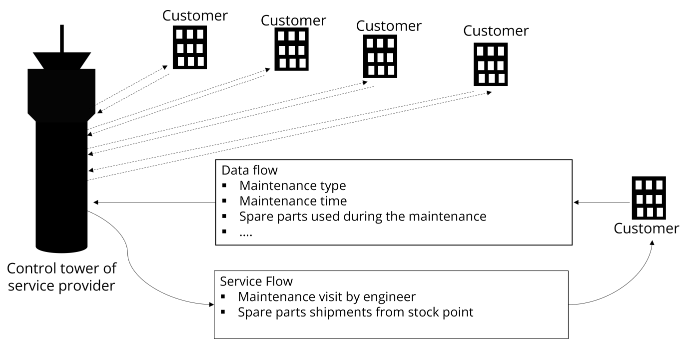


Figure 1.1: The data flow between a service provider and its customers.

(cost) benefits of adopting these new technologies. Different parties (i.e., customers, local warehouses, service engineers) are often involved, and multiple resources (i.e., spare parts, engineers) are controlled in maintenance planning and spare parts management processes. For the purpose of managing different parties and controlling the resources, the service provider needs to coordinate different data flows in real-time. The concept of *service control towers* (SCT) has been developed (Song et al., 2020) as a central decision-making actor that coordinates different data flows, provides communication between different parties, and controls resources. In this dissertation, we consider one service control tower that belongs to the service provider.

The service control tower collects data regarding the maintenance activities (i.e., time of maintenance, type of maintenance) and spare parts used during these activities from maintenance service reports for multiple customers. They use data to improve future maintenance services. For example, time and type of maintenance activities can be used to schedule the next planned maintenance activity by estimating the remaining lifetime of a critical component. Similarly, the type of parts used to resolve previous failures can provide information about future demand for that type of spare part in a similar maintenance case. An example of the data flow between the service provider and its customers via an SCT can be seen in Figure 1.1.

However, these estimations involve some amount of uncertainty. The notion of uncertainty is important in data-integrated decision-making because estimations affect the decision, and therefore also the operational costs. Uncertainty may have different sources. So-called *epistemic* and *aleatoric* uncertainty are the two possible sources of uncertainty. Epistemic uncertainty refers to the uncertainty caused by the lack of data or knowledge. In order to predict outcomes or future events, it is common to build predictive models based on historical data. Examples of predictive models in a maintenance context are the models estimating the component failures or the remaining lifetime of a component. Epistemic uncertainty focuses on the uncertainty that is caused by the predictive model itself. This means epistemic uncertainty can be reduced by collecting more data and learning. Aleatoric uncertainty refers to randomness or variability in the outcome by the nature of a stochastic event. Stochasticity in the process cannot be reduced by any additional information (Hüllermeier and Waegeman, 2021). For example, a component can have a lifetime distribution such that it fails at age 3 with 0.4 probability and it fails at age 4 with 0.6 probability. This uncertainty is caused by the randomness of the failure event itself and cannot be reduced by learning. In the rest of this dissertation, we refer to epistemic uncertainty as *model uncertainty* and aleatoric uncertainty as *statistical uncertainty*.

1.1. Maintenance optimization under model uncertainty

We focus on the model uncertainty in the domain of maintenance optimization specifically for the lifetime distribution of the components. The life cycle of a capital good has five phases: (1) determining the needs and requirements in the market, (2) design of the capital good, (3) production of the capital good, (4) exploitation, and (5) disposal of the capital good (van Houtum and Kranenburg, 2015, p.1-2). Uncertainty regarding the maintenance decision is at its highest when a capital good is newly designed because many components are also used for the first time and there is a lack of historical data for the lifetime of these components.

Traditionally, maintenance literature assumes that the lifetime distribution follows a *known* probability distribution and does not consider the availability of relevant data that is necessary to use a maintenance decision-making model in real-life applications (de Jonge and Scarf, 2020). However, in reality, there may exist uncertainty concerning the lifetime distribution itself due to lack of data, affecting the optimal replacement time; see Bunea and Bedford (2002); de Jonge et al. (2015a); Fouladirad et al. (2018); de Jonge and Scarf (2020).

One example of uncertainty in lifetime distributions is provided by van Wingerden (2019) for lithography systems produced by ASML. When newly designed lithography systems are introduced, some of the components are used for the first time, therefore there is not any lifetime data at the beginning of the lifespan. Martinetti et al. (2017) describe a case where maintenance service responsibilities of a specific type of train are transferred from the component supplier to the Netherlands railways. However, the data obtained from the component supplier is fragmented at the beginning of the lifespan. This forms another reason for uncertainty with respect to the lifetime distribution of the components.

When historical data is not available for the lifetime of a certain component, the estimation of the lifetime distribution is mainly based on expert opinions and technical data obtained from suppliers (van Wingerden, 2019, p.1-15). However, obtaining detailed data from suppliers can be costly. Suppliers may only share partial information because of strategic reasons (Martinetti et al., 2017). It is also not straightforward to obtain the exact lifetime distribution from technical specifications. These problems can lead to heterogeneous expert opinions on the true lifetime distribution and result in uncertainty in the lifetime distribution.

1.1.1 Uncertainty in the lifetime distribution

The source of uncertainty in the lifetime distribution may vary. In the literature, one type of lifetime model uncertainty is *parameter uncertainty*, where it is assumed that the parameters of a lifetime model are unknown (see Drent et al. 2020, van Staden et al. 2022, Deprez et al. 2022, Walter and Flapper 2017, Fouladirad et al. 2018).

Another type of model uncertainty is *population heterogeneity*. Under population heterogeneity, there exist multiple population types for the components. The lifetime distribution is known for each population type individually. However, the uncertainty in the true population type induces uncertainty in the lifetime distribution.

In practice, the characteristics of the components can be heterogeneous, representing the different expert opinions, the varying quality from different suppliers, production at different manufacturing lines, or different causes of failure (Jiang and Jardine, 2007). Population heterogeneity is a commonly studied topic in the literature because of its practical relevance (see Jiang and Jardine 2007; Scarf et al.

2009; Scarf and Cavalcante 2010, 2012; de Jonge et al. 2015a; Cavalcante et al. 2018; van Oosterom et al. 2017; Abdul-Malak et al. 2019).

In Chapters 2 and 3, as the source of lifetime model uncertainty, we focus on population heterogeneity for the components that are used for replacements. Please note that we consider a 'time-to-failure' setting as the failure model. Therefore, we use lifetime distribution and time-to-failure distribution interchangeably in Chapters 2 and 3. We assume that a component is always supplied from the same population (see de Jonge et al. 2015a). For example, for the lifetime distribution of a component, two experts can have two different estimates, where in reality only one of them may be the true distribution. Another example of population heterogeneity is having different types of populations in the market, where each supplier always provides from the same population. For example, due to different production qualities, there can be two populations in the market: a weak and a strong population. A weak component is more likely to fail earlier than a strong component. Each supplier has components from either a weak or strong population in their stocks. When a maintenance service provider makes an agreement with one of the suppliers throughout the lifespan of a technical system, the components will be always ordered from the same supplier, while the population type is unknown to the service provider.

Please note that in the literature, the population heterogeneity may also imply that there is an *unknown* mixture of populations for the components, where in each replacement, a component is supplied from any of these populations (see van Oosterom et al. 2017 and Abdul-Malak et al. 2019). For example, parts that are printed with two different printing options can have different reliability levels (Lolli et al., 2022).

1.1.2 Resolving the uncertainty by learning

Model uncertainty may mean either the uncertainty in model parameters or the true model itself. But it is usually possible to formulate estimations on model parameters or the true model. These uncertain estimations can be referred to as *belief*. A belief can be updated as new data is collected, which is referred to as *learning*. Therefore, it is possible to resolve model uncertainty by learning. According to Powell and Ryzhov (2012), 'any learning problem is a belief model'.

Frequentist and Bayesian views are two learning approaches. The frequentist approach only uses historical data to estimate the unknown parameters without assuming any prior information on the belief or model parameters. On the other hand, the Bayesian approach allows incorporating prior information on belief and can update the belief with the available data (Powell and Ryzhov, 2012).

In Chapters 2 and 3, we focus on sequential decision-making problems with Bayesian learning. 'Exploration-exploitation' trade-off and 'data pooling' are two concepts that are addressed related to the learning problem.

Exploration-exploitation trade-off: In the context of a replacement problem with model uncertainty in the lifetime distribution, the replacement decisions affect the belief. If a replacement is done earlier than a failure, the real lifetime cannot be observed exactly. This data point is right-censored (i.e., the time we observe is less than or equal to the lifetime of the component). Therefore, the data is less informative to learn the true population type. If we wait until a failure, then we observe the true lifetime. In this way, the true population type can be learned earlier, but we incur a higher corrective maintenance cost. This trade-off is known as *exploration-exploitation trade-off* in optimal learning literature. Exploration means explicitly considering the impact of current actions on future outcomes and being willing to take risky actions in the short term with the aim of getting a higher cost reduction in the longer term. On the other hand, exploitation refers to selecting the action that minimizes the cost in the short term by ignoring the impact of the current action on the information to be revealed in the future (Dezza et al., 2017; Powell and Ryzhov, 2012).

Data pooling: Most of the time, multiple capital goods are dedicated to similar business processes. These capital goods can be located at the same location such as multiple milking robots at a farm or at different locations such as multiple MRI machines at multiple hospitals. Each system has an identical critical component that needs to be replaced. It is possible to collect failure data from each system to learn the lifetime distribution. Collecting data from similar systems to increase the number of data points is referred to as *data pooling* (Gupta and Kallus, 2022). A service provider offers maintenance services to many similar systems and it can pool data from these systems. It is also possible to optimize the maintenance costs jointly for multiple systems because there is only one decision maker for the replacement decision, the service provider.

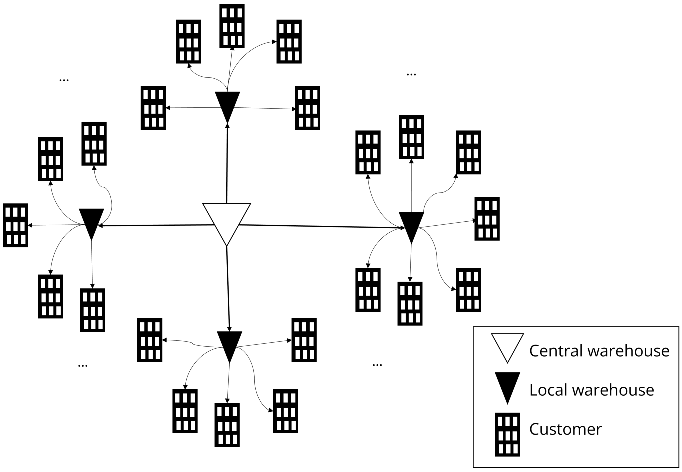


Figure 1.2: An example two-echelon spare parts inventory network with regular shipments.

1.2. Spare parts management with advance demand information

Complex technical systems contain multiple components. In many cases, a system failure is caused by a failed component that should be replaced. This means each failure may create a demand for spare parts that must be satisfied by the service provider. In this section, we focus on spare parts inventory management problems.

1.2.1 Spare parts inventory control problem

Typically, a service provider has many customers located at geographically dispersed locations. Usually, spare parts are stored at many local warehouses and a central warehouse. If possible, customers' demand for spare parts is satisfied from a local warehouse or a warehouse. In case of a stockout, customers' demand may be backordered, or satisfied either from another local warehouse by lateral shipment or by an emergency shipment from the central warehouse or a third-party supplier (van Houtum and Kranenburg, 2015). In Figure 1.2, we provide an example of two-echelon spare parts inventory management network. For these networks, spare parts inventory replenishment is an important problem to prevent long system downtimes, large costs of inventory holding, and emergency shipments. In this sense, it is an inventory control problem.

In general, inventory control problems mainly focus on the questions of how often the inventory status should be observed, when the inventory replenishment takes place, and how large the replenishment order should be (Silver, 1981). Inventory at local warehouses is reviewed usually either continuously or periodically (Zhang et al., 2021). Some common inventory control policies under continuous review are (s, S) and (s, Q) policies. Under the (s, S) policy, when the inventory position drops to or below s, you place an order to increase the inventory position to the order-up-to level S. Under the (s, Q) policy, when the inventory level drops to or below s, the inventory is replenished with a fixed order quantity Q. A common policy for periodic review is (R, S). An order is placed periodically at every R time unit to complete the inventory position to the order-up-to level S. For more details on inventory control and spare part inventory policies please see Silver (1981) and Zhang et al. (2021).

In Chapter 4, we focus on a periodic review and periodic spare parts inventory replenishment problem for a single warehouse. In this network, regular and emergency shipments are used.

1.2.2 Repair kit problem

Each local service point is responsible for the delivery of spare parts to its customers in case of a system failure. A system failure can be the result of a software failure, which requires no spare parts but a system reboot might be sufficient. It can be caused by a failure of only one critical component or multiple components. Therefore, it is crucial for the service provider to determine which set of spare parts should be shipped to customers to repair the system.

The set of spare parts carried by a service engineer during a maintenance visit is called a *kit*. Before the service engineer visits the failed system, it is not precisely known which spare parts are needed to repair the system. If all required parts are present in the kit, the system can be repaired immediately, otherwise, follow-up action is needed. This problem is known as *repair kit problem* (RKP) in the literature. The RKP was first introduced by Smith et al. (1980) 'for optimizing multi-item inventories for the repair of field equipment based on (inventory) holding costs and the probability of job completion without stockout'.

It is possible that the failures of different components in a system are independent or, alternatively, there is failure dependency between the components. For example, due to excessive heating in a certain region, multiple components can be broken. Another example is that a breaking component can break other components that are in contact with it. Please see Olde Keizer et al. (2017) for more details on dependencies between components.

In Chapter 5, we focus on an optimal spare part recommendation problem for corrective maintenance, which is similar to classical RKP. For this problem, we assume that it is possible to have failure dependency between components.

1.2.3 Uncertainty in spare parts demand

In order to deliver the required spare parts to the customer, it is important to integrate data regarding demand predictions into decision-making. After the initial phase in the lifetime of a capital good, data becomes abundant. There are various predictive models and prediction methods for spare parts demand in the literature (see Pinçe et al. 2021; Syntetos et al. 2012). Different sources of data can be used as input for these predictive models, i.e., historical demand data for spare parts for similar maintenance cases, sensor readings based on the condition of the components, and failure codes generated by the system in the case of failure.

Demand forecasts, early or advance orders, or any signal providing information regarding future demands can be viewed within the concept of advance demand information (ADI) (Karaesmen, 2013). There may be various sources for spare parts demand uncertainty. For example, the number of spare parts or the type of stock-keeping units (SKUs) required to resolve a maintenance case can be unknown to the decision-maker during the planning period. If the predictions provided by ADI are equal to the realized demand, we refer to it as *perfect* ADI. However, in practice, the predictive models do not resolve the uncertainty fully. Therefore, most of the time we have *imperfect* ADI.

Most repair kit problems in the literature assume that the service provider does not have any information regarding the source of failure reported by the customer before a diagnostic visit (Rippe and Kiesmüller, 2023). However, the internet of things and developed sensor technologies thanks to Industry 4.0, provide opportunities for monitoring the condition of a system remotely and generating failure codes for possible causes of failures. Machine learning and natural language processing techniques enable to compare data from different sources, i.e., textual engineer reports, sensor readings, system log data, spare part usage data, etc. and generate predictions for spare parts demand (see Rippe and Kiesmüller 2022, 2023; Grishina et al. 2020).

In Chapters 4 and 5, we consider capital goods after the initial part of their lifespan, and therefore, data is abundant. Thus, we do not consider an explicit learning problem. Even though the data is abundant, the predictive models are not always 100% perfect. In Chapter 4, we investigate how the quality of prediction models affects replenishment order decisions and the long-run expected costs related to the spare parts inventory. In Chapter 5, we consider only statistical uncertainty but not model uncertainty.

In Section 1.3, we discuss further details of problem descriptions and research objectives for capital goods in data-integrated environments in the context of maintenance optimization and spare parts management.

1.3. Problem descriptions and research objectives

In the first part of the research, we consider newly designed systems with a fixed lifespan. Each system has a critical component that is subject to random failures and requires replacement. A system cannot function if this component does not work. The aim is to preventively replace a component before it fails in order to prevent the costlier corrective maintenance upon failure. Therefore, the data regarding the lifetime distribution of the component is important to determine the optimal replacement time. Since this component is in use for the first time, there is no historical data regarding the lifetime of the component. We assume there are two different populations from which these components can come: a weak and a strong population. Components are always provided from the same population but the true population is unknown. There is an initial belief of the probability of the component coming from the weak population. This belief is updated in a Bayesian way with the data obtained during the system's lifespan. We formulate a sequential decision-making problem with Bayesian learning. We formulate two research objectives (RO) for this problem as follows.

Research objective 1: Balance exploration and exploitation optimally for age-based maintenance with Bayesian learning.

For RO 1, we consider an age-based maintenance policy to schedule the next optimal preventive maintenance time.

Research objective 2: Investigate the effect of data pooling and joint optimization for multiple single-component systems on maintenance costs.

For RO 2, we consider a replacement policy such that at the beginning of every time step we take either 'do a preventive replacement' or 'do nothing' action for each system.

In the second part of the research, we focus on spare parts management for capital goods. First, we consider multiple technical systems with a single SKU. The spare parts for this SKU are stored at a local warehouse. The inventory in the local warehouse is replenished periodically. During a period, a predictive algorithm generates signals for possible failures. However, it is possible that a signal is not generated for some of the failures (undetected failures) and some signals do not result in failures (false signals). The time between the signals and failures is called *demand lead time*. When a failure happens, the component is replaced from the stock of the local warehouse directly. Stocks at the local warehouse are replenished by the central warehouse periodically. In case of a stockout, an emergency shipment takes place from the central warehouse directly. The model reliability is quantified by *precision* and *sensitivity*. The precision of a predictive model denotes the fraction of true signals among all signals. The sensitivity of a predictive model denotes the fraction of detected failures among all failures. In the worst-case scenario, sensitivity, precision, and demand lead time are zero. We formulate the research objective for this problem as follows.

Research objective 3: Determine how the quality of failure predictions and the duration of demand lead time affects the spare part stock and replenishment costs.

Next, we consider a spare part recommendation problem similar to RKP where multiple SKUs might be required to resolve a corrective maintenance case. When a technical system fails, the customer informs the service provider regarding the failure with a failure code. This code is compared against historical maintenance cases to match similar cases. A predictive algorithm generates a list with the total set of spare parts used in similar cases and estimates the probability for each subset of spare parts that precisely this subset is required to fix the system. The service provider is responsible for sending an engineer for a diagnostic visit and making a separate shipment of spare parts to resolve this maintenance case. When an engineer makes a diagnosis of the system failure, all parts needed to fix the system are known with certainty. It is also possible that no spare parts would be required. If parts shipped simultaneously with the diagnostic visit are not used in repair, they are sent back to the warehouse. If the system can be fixed with the parts sent to the customer, the engineer fixes the system and the maintenance case is closed. If there are parts required to repair the system but not shipped to the customer during the diagnostic visit, another visit takes place the next day, where all remaining required parts are shipped from the warehouse to the customer. Based on the operational costs (i.e., shipment costs for spare parts, the cost of returning unused parts back, the cost of an additional visit if the maintenance case is not resolved during the diagnostic visit), and the probabilities generated by the predictive algorithm, the service provider needs to decide the set of spare parts that will be sent to the customer during the diagnostic visit. We develop the third research objective as follows.

Research objective 4: Develop a spare parts recommendation model based on historical data and operational costs.

1.4. Methodologies

In order to address ROs 1 and 2, we formulate partially observable Markov decision process (POMDP) models with Bayesian learning. POMDP is a suitable methodology to optimally balance the exploration and exploitation in learning the true population type because POMDPs are sequential decision-making models where the effect of the current action on the system is considered until the end of the time horizon. In a POMDP, we have a belief state and other state variables (e.g., the system's remaining lifespan, and the component's age). An action is taken according to the state variables and the lifetime probability distribution of the component. After the action, an observation is made (e.g. a failed or a functioning

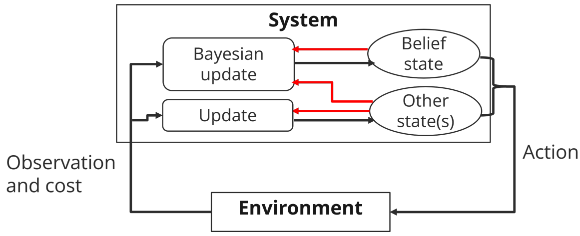


Figure 1.3: An overview of a general POMDP model with Bayesian updating.

component, or time between two replacements and the type of replacement). As a result of the action and observation, an immediate cost incurs. Based on the observations, and prior value of state variables, the state variables are updated. The belief state is updated by using Bayes' rule. We adopt the Bayesian learning approach because it is suitable when historical data does not exist or is very scarce. Bayesian learning has been used for multiple maintenance problems, see van Oosterom et al. (2017), Droguett and Mosleh (2008), de Jonge et al. (2015a), Dayanik and Gürler (2002), Elwany and Gebraeel (2008), and Walter and Flapper (2017). In Figure 1.3, an overview of a general POMDP model with Bayesian updating is demonstrated.

In order to address RO 3, we model a spare parts inventory management problem. We aim to optimize the replenishment decisions with the objective of minimizing the long-run average cost per period. For this reason, we built a discrete-time Markov decision process (MDP) model. Similar to the POMDPs, MDPs are also sequential decision-making models. This means we consider multiple sequential periods in our decision-making rather than a policy that minimizes the average costs only for a single period (i.e., a myopic policy). However, an MDP model considers the effect of the on-hand inventory level at the end of a period for the replenishment decision of the next period.

Finally, in order to address RO 4, we want to find the optimal set of spare parts by minimizing the costs of a maintenance case with multiple possible SKUs. This is a cost-minimization problem with multiple binary decision variables. We formulated a binary integer linear program to model this problem. An integer programming model involves decision variables that only can take integer values. In a binary integer linear programming (ILP) model, the decision variables only can take 0 (no) or 1 (yes) as values. All functions, including the objective function and constraints, are linear in a linear programming model. For more details of linear programming models please see Dantzig and Thapa (1997).

1.5. Contributions and outline of the dissertation

In this section, we provide an outline of the dissertation and the contribution of each chapter. We address RO 1 in Chapter 2. Chapter 2 is on the topic of '*exploration and exploitation in age-based maintenance*', which is based on Dursun et al. (2022b). We address RO 2 in Chapter 3. Chapter 3 is on '*data pooling and joint optimization for multiple systems*', which is based on Dursun et al. (2022c). We address RO 3 in Chapter 4, where we study a problem on '*spare parts replenishment at local warehouses with ADI*'. Chapter 4 is based on Dursun et al. (2022a). We address RO 4 in Chapter 5, where we investigate '*selecting the set of spare parts for corrective maintenance*' topic. Chapter 5 is based on Dursun et al. (2022d). Finally, in Chapter 6, we revisit the research objectives and provide the conclusions and future research directions for this dissertation.

This dissertation makes contributions in the area of maintenance optimization and spare parts management for systems in data-integrated environments. We provide a general overview of the characteristics of the models for each chapter in Table 1.1.

Now, we provide the contribution of each chapter in detail. In Chapter 2, we develop an age-based maintenance model for a system with a finite lifespan and a critical component under population heterogeneity. To the best of our knowledge, this is the first study that balances exploration and exploitation optimally for this specific problem. We build a POMDP model with Bayesian learning to minimize the total maintenance cost over the lifespan of the system.

We analytically characterize the structure of the optimal cost as a function of the belief regarding the component coming from a weak population and the remaining lifespan of the system. By numerical analysis, we compare the optimal policy against two benchmark heuristics from the literature, namely a *myopic policy* and a *threshold policy*. A myopic policy minimizes the average expected cost per maintenance cycle. Therefore, it does not take the effect of an action on the next cycles into account. A threshold policy, which has been proposed by de Jonge et al. (2015a), investigates the effect of exploration by considering a threshold on

Table 1.1: Overview of the characteristics of the models for each chapter.

	Chapter 2	Chapter 3	Chapter 4	Chapter 5
1. Problem domain				
Maintenance optimization	X	X		
Spare parts management			X	X
2. Role of data				
Resolving population heterogeneity in lifetime distribution	X	X		
Estimating the demand of spare parts			X	X
3. Methodology				
POMDP (with Bayesian learning)	X	X		
ILP				X
MDP			X	
4. Number of periods				
Single				X
Multiple	X	X	X	
5. Number of SKUs				
Single	X	X	X	
Multiple				X
6. Planning Horizon				
Finite	X	X		
Infinite			X	X

the belief regarding the population type, however, it does not optimally balance the exploration and exploitation on learning. By comparing the optimal policy against the benchmark heuristics, we generate insights about the benefit of following the optimal policy. We introduce a lower bound function on the optimal cost. This function denotes the costs when the true population type is known. Thus, we calculate the value of resolving the uncertainty on the true population type at the beginning of the system lifespan.

In Chapter 3, we consider multiple single-component systems with finite lifespans. We investigate the effect of data pooling and joint optimization on the maintenance cost of these systems under population heterogeneity. To the best of our knowledge, this research is the first to study this specific problem. We formulate a POMDP with Bayesian updating to find the optimal replacement policy that minimizes the expected total cost of maintenance throughout the lifespan of the system. We compare the cost of the optimal policy per system against two benchmark heuristics that follow a single system optimal policy, with and without data pooling, respectively. First, we consider a special case with deterministic lifetimes and provide insights on the effect of the number of systems on the benefit of exploration.

Then, we generate managerial insights on the effect of data pooling and joint optimization with comprehensive numerical analysis. We show in our numerical analysis that the majority of the cost reduction is due to data pooling, and a relatively small part is due to optimizing the preventive replacement decisions for multiple systems jointly. We examine the effect of input parameters on this result with an extensive sensitivity analysis. We investigate the effect of the number of systems on the benefits of data pooling. We show that the cost reduction due to data pooling can be up to 5.6% for two systems and up to 14.8% for 20 systems. Furthermore, the reduction in the cost per system decreases as a function of the number of systems, i.e., we obtain 40% of potential cost reduction by pooling data from two systems, 70% from five systems, and 80% from 10 systems. As the number of systems that pool the data increases, the cost of maintenance converges to the cost of maintenance under perfect information about the true population type for the same lifespan. These results show practitioners that data integration between only a few systems may help to obtain a large amount of potential cost reduction.

In Chapter 4, we consider a single SKU spare parts inventory problem for a local warehouse that supports multiple systems. Signals generated for upcoming failures constitute a form of imperfect ADI. We formulate an MDP model to find the optimal replenishment decisions that minimize the long-run average cost per period. We generate managerial insights about the effect of precision and sensitivity of the signals and the demand lead time on optimal costs and order-up-to levels. One important managerial insight for the developers of the predictive model is as follows. Sensitivity and the demand lead time affect the optimal costs only through their product, for a given precision level. This means both sensitivity and demand lead time are equally important to reduce cost Furthermore, a low value in sensitivity (demand lead time) will decrease the effectiveness of a high value in demand lead time (sensitivity) on cost reduction. A significant cost reduction compared to the worst case can be obtained for moderate values of precision. On the other hand, it is required to have high values of sensitivity and demand lead time for a significant reduction in optimal costs. We observe that, under a perfect sensitivity and a perfect demand lead time, 30% perfectness in precision (i.e., precision is equal to 30% of the perfect precision) brings a 70% reduction in optimal costs compared to the worst-case optimal cost. Contrary to this, under perfect precision, 70% perfectness in the product of sensitivity and demand lead time (i.e., the product is equal to 70% of the product of perfect sensitivity and perfect demand lead time) brings only a 30% reduction in optimal costs compared to the worst-case optimal cost. Finally, with respect to the spare part stock levels, we find that even with high-quality predictive models, the spare parts inventory stocks will not disappear completely.

In Chapter 5, we consider a set of spare part selection problems for corrective maintenance cases. We formulate an ILP model to find the optimal set of spare parts based on the probability estimation of a set of spare parts that might be required and operational costs by minimizing the total expected cost of a maintenance case. We allow failure dependency between SKUs in the modeling. We formulate a new model for the so-called spare parts recommendation problem described in Grishina et al. (2020). Additionally, we derive the optimal policy structure for problem instances with one or two SKUs and we obtain partial results for the optimal policy structure for problem instances with three or more SKUs. We compare the optimal policy against the existing heuristic policies that are used in practice by a company that manufactures and services high-tech capital goods. The optimal number of SKUs that minimizes the total expected cost of resolving a maintenance case is not fixed. Our model finds directly the optimal set of SKUs that should be shipped to the customer.

2

Exploration and exploitation in age-based maintenance

2.1. Introduction

In this chapter, we consider a system with a critical component that fails randomly and requires maintenance during the lifespan of the system. Motivated by the fact that capital goods often have a planned duration of service at the time of acquisition (Jiang, 2009; Nakagawa and Mizutani, 2009; Cheng et al., 2012; Lugtigheid et al., 2008), we assume that the system lifespan is finite and known. The service control tower (SCT) of the service provider is responsible to schedule a preventive replacement for the critical component at a given age when it applies age-based maintenance. In case a failure happens before the scheduled preventive replacement, the SCT is responsible for applying a corrective replacement immediately. After a replacement, either preventive or corrective, the component is 'good-as-new'.

We consider that there are two population types (i.e., a weak and a strong type) and all components necessary for replacements come from one of these two types. But, the true population type is unknown. There exists an initial belief on the true population, and this belief is updated over the course of the lifespan of the system by using the data obtained under age-based maintenance. Specifically, every time that a failure or preventive maintenance is performed, the age of the current component in the system is used to update the belief on the population type with Bayes' rule.

The use of Bayesian updates for uncertain model parameters is also useful to address the trade-off between *exploration* and *exploitation* in a sequential decision-making problem. In this chapter, we aim to strike the optimal balance between *exploration* and *exploitation* in making age-based replacement decisions under population heterogeneity. Further, we compare the optimal policy to two existing heuristic policies: the myopic policy and the threshold policy (de Jonge et al., 2015a).

Specifically, we address the following research questions: 1) How is the structure of the optimal age-based maintenance policy for a system with a finite lifespan under population heterogeneity? 2) Under which scenarios is the total cost of the optimal policy much lower than the cost of the existing heuristic policies? 3) What is the value of resolving the uncertainty on population heterogeneity? 4) How does the structure of the optimal policy differ from existing heuristic policies in terms of exploration and exploitation? In order to answer these questions, we build a discrete-time partially observable Markov decision process (POMDP) model that includes a continuous belief state on the true type of the component population and Bayesian updates of the belief state.

Our choice for adopting a discrete-time model is motivated by real-life situations, where actions are taken at discrete time points (e.g., a failure that occurs during a time period can only be fixed by a replacement with a new component that arrives at the beginning of the next period).

We can summarize the contributions of this study as follows. To the best of our knowledge, this work is the first to study the optimal balance of exploration and exploitation for a system with a finite lifespan and a critical component under population heterogeneity. We analytically show the structure of the optimal policy as a function of the belief state and the time until the end of the system lifespan. Furthermore, we establish a lower bound function on the optimal cost in order to quantify the value of resolving the uncertainty on the true population type at the beginning of the system lifespan. By performing a comprehensive numerical analysis, we show that both the population heterogeneity and the time until the end of the system lifespan have a significant effect on the structure of the optimal policy. Specifically, the instances where the strongest exploration takes place under the optimal policy are observed when the variance of the time-to-failure distribution is low and typically when the initial belief of a component coming from a strong population is sufficiently high. For these instances, we also show that the optimal policy learns the true type of the component population much faster and more accurately than the myopic policy, which does not consider the exploration-exploitation trade-off. Furthermore, we characterize when it is most beneficial to resolve the uncertainty on the population type (i.e., to learn the component type immediately) by using a lower bound function on the optimal cost.

The remainder of this chapter is organized as follows. Section 2.2 presents the relevant literature, and Section 2.3 provides a formal description of the model. The mathematical formulation of the POMDP model is presented in Section 2.4. Section 2.5 presents the analytical results obtained from our analysis of the POMDP model, and Section 2.6 characterizes the lower bound function on the optimal cost. Section 2.7 presents our numerical analysis and insights. Section 2.8 concludes the chapter.

2.2. Literature review

In this section, we review the previous studies in the maintenance literature addressing uncertainty in a failure model. We organize our review by first categorizing these studies based on whether their maintenance policies perform exploration or not. Next, we will focus on the studies with exploration and further classify them with respect to type of maintenance policy: age-based policies and condition-based policies. An overview of all related studies is shown in Table 2.1.

Among the studies that address uncertainty in a failure model, a major distinction is based on whether the current action considers the performance in the future in a dynamic way. We refer to policies which perform such an exploration as *forward-looking* policies. Otherwise, we clasify a policy as a *myopic* policy (e.g., Mazzuchi and Soyer 1996; Dayanik and Gürler 2002; Fouladirad et al. 2018; Coolen-Schrijner and Coolen 2004, 2007; Laggoune et al. 2010; Walter and Flapper 2017; Elwany and Gebraeel 2008; de Jonge et al. 2015b). A myopic policy updates the unknown failure model or its parameters with the most recent data but without explicitly considering the decisions to be made in the future. To the best of our knowledge, the number of studies that consider failure model uncertainty and uses a forward-looking policy, is limited (de Jonge et al., 2015a; van Oosterom et al., 2017; Abdul-Malak et al., 2019; Drent et al., 2020).

Table 2.1: Summary of the previous studies addressing failure model uncertainty.

	Mazzuchi and Soyer (1996)	Dayanik and Gürler (2002)	Coolen-Schrijner and Coolen (2004)	Coolen-Schrijner and Coolen (2007)	Elwany and Gebraeel (2008)	Laggoune et al. (2010)	de Jonge et al. (2015a)	de Jonge et al. (2015b)	van Oosterom et al. (2017)	Walter and Flapper (2017)	Fouladirad et al. (2018)	Abdul-Malak et al. (2019)	Drent et al. (2020)	**Current Work**
1. Type of Maintenance														
Age-based	X	X	X	X		X	X	X			X		X	X
Condition-based					X				X	X		X		
2. Failure Model														
Degradation State														
Discrete State (2 states)	X		X	X		X	X	X		X	X	X	X	X
Discrete State (3 states)		X												
Discrete State (>3 states)									X					
Continuous State					X									
Continuity of Time														
Discrete Time									X			X		X
Continuous Time	X	X	X	X	X	X	X	X		X	X		X	
3. Type of Model Uncertainty														
Parameter Uncertainty	X	X			X	X		X		X	X		X	
Population Heterogeneity							X		X			X		X
Distributional Uncertainty			X	X										
4. Time Horizon														
Finite		X			X		X			X			X	X
Infinite	X		X	X		X		X	X		X	X	X	
5. Type of Policy														
Myopic	X	X	X	X	X	X		X		X	X			
Forward-looking							X		X			X	X	X
6. Update of Model														
Frequentist			X	X		X					X			
Bayesian	X	X			X		X	X	X	X		X	X	X

The four studies on forward-looking policies can be further classified based on the type of maintenance policy. de Jonge et al. (2015a) and Drent et al. (2020) consider an age-based maintenance policy, while van Oosterom et al. (2017) and Abdul-Malak et al. (2019) follow a condition-based maintenance policy. Thus, we now focus further on Drent et al. (2020) and de Jonge et al. (2015a).

There is a number of aspects that distinguishes our research from Drent et al. (2020). First, we assume that the components always come from the same supplier who has a single population (either a weak population or a strong population) but the type of population is unknown. On the other hand, Drent et al. (2020) assume that the components come from a supplier who has a mix of components with different reliability levels. This main difference can also be described as follows. In Drent et al. (2020), each realization can be drawn from a different probability distribution. However, we consider a problem where the lifetime realizations in all cycles come from one specific distribution, but it is not known what that distribution is. Second, Drent et al. (2020) work on a component replacement problem under an age-based policy where the scale parameter of a Weibull time-to-failure distribution is unknown, while we assume a general time-to-failure distribution. Furthermore, we explicitly consider a finite, fixed lifespan for the system, while Drent et al. (2020) consider an age-based maintenance model with a finite number of maintenance cycles.

Our work comes closest to de Jonge et al. (2015a), who study the same problem. However, the main difference is that we propose an age-based maintenance policy that *optimally* balances exploration and exploitation. While de Jonge et al. (2015a) also address the trade-off between exploration and exploitation, they only provide a heuristic policy to do so. Additionally, our optimal policy explicitly considers the finite time horizon effect, while the policies of de Jonge et al. (2015a) do not take into account that the time horizon is finite with the claim that the error is acceptable if the length of the time horizon is large compared to the mean time between failures. We also formally derive a lower bound function that allows us to quantify the maximum amount that a decision-maker wants to pay to learn the true population type (this corresponds to the so-called perfect information case; this is considered in de Jonge et al. (2015a), but they do not show explicitly that this constitutes a lower bound for the optimal cost function).

2.3. Model description

We consider a system with a finite lifespan. The lifespan refers to the time from the beginning of operating the system until it is taken out of service. The planning horizon is equal to the lifespan of the system and we let the planning horizon consist of discrete time steps. Without loss of generality, we scale time such that the length of a time step is one time unit. The length of the planning horizon is expressed in the number of time steps and is denoted by $L \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers. The system has a critical component that fails randomly. If a failure occurs during the x-th time step after the installation of a new component, then the lifetime of the component is considered to be x. We let X denote the corresponding discrete random variable that represents the lifetime of the component. The failure that occurs during a time step is fixed by replacing the failed component with a new one at the end of this time step. We note that our discrete-time model can also be used to approximate a continuous-time model by taking the duration of a time step sufficiently small compared to the lifetime of a component.

An age-based replacement policy is applied. This means that a component is scheduled to be preventively replaced at a predetermined time with a cost C_p. If the component fails before reaching this predetermined time, then it is correctively replaced at cost C_f. We assume $C_p < C_f$ because the cost of corrective replacement includes the costs associated with an unplanned breakdown in addition to the costs related to a replacement. The time interval between two consecutive replacements is defined as a *cycle*. The beginning of a cycle is referred to as a decision moment because the next preventive replacement time needs to be planned at the beginning of each cycle. We let $\tau_n > 0$ denote the time interval until the next planned maintenance at cycle $n \in \mathbb{N}$. An example of the planning horizon and decision moments is shown in Figure 2.1, where τ_1 and τ_2 are both equal to 3. In the first cycle, no failure is observed in the first three time steps, and the planned replacement is performed at the end of the third time step. In the second cycle, a failure occurs in the second time step, the component is replaced correctively at the end of that time step, and a new cycle starts at the beginning of the sixth time step; and so on.

Let z_n denote the remaining lifespan of the system at the beginning of cycle n. Notice that z_1 (i.e., the remaining lifespan of the system at the beginning of the first

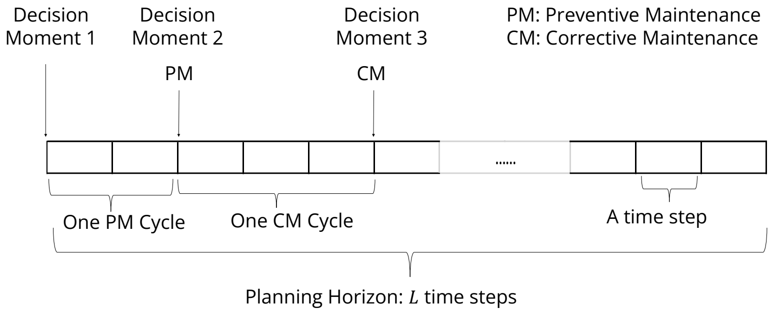


Figure 2.1: Illustration of timeline for maintenance activities.

cycle) is equal to L. If τ_n is chosen to be equal to z_n, it means no replacement activity is planned during the remaining lifespan of the system. When the system reaches its end of the lifespan (i.e. $z_n = 0$), the maintenance activities are terminated and no cost occurs at this moment or later because the system goes out of service at that moment.

We assume that there are two types of populations for the critical component: a weak and a strong population. That is, when a component is needed for a replacement, it comes from either a weak population or a strong population. In our context, we adopt the interpretation of de Jonge et al. (2015a) for population heterogeneity, where the components used for replacements always come from the same population, either a weak or a strong population, but the true type of the population is unknown and there is a belief associated with it. This is the case, for example, when the components are always ordered from the same supplier and the supplier always provides components from the same population, while the decision maker does not know whether the chosen supplier provides weak or strong components.

At any moment, we let $p \in [0, 1]$ denote the belief that the components belong to a weak population (i.e., the probability that the components always come from the weak population). Let X_j denote the lifetime random variable for component type j, where $j = 1$ refers to the weak type and $j = 2$ refers to the strong type. Therefore, under belief variable p, the lifetime random variable X satisfies

$$X = \begin{cases} X_1 & \text{with probability } p, \\ X_2 & \text{with probability } 1 - p. \end{cases}$$

For each population type, the time-to-failure distribution is assumed to be known. Since we have a discrete-time model, X_j is defined as a discrete random variable for $j \in \{1, 2\}$. Let $P_j(\cdot)$ and $F_j(\cdot)$ denote the probability mass function (pmf) and the cumulative distribution function (cdf) of the random variable X_j for $j \in \{1, 2\}$. We let $P(\cdot)$ and $F(\cdot)$ denote the pmf and cdf of the random variable X. For a given p, it holds that $P(x) = pP_1(x) + (1 - p)P_2(x)$ and $F(x) = pF_1(x) + (1 - p)F_2(x)$ for $x \in \mathbb{N}$.

We suppose that there is an initial belief $p_1 \in [0, 1]$, which represents the probability that the components belong to the weak population at the beginning of the planning horizon. For example, this belief can be set as the proportion of the suppliers providing weak components. For $n > 1$, we let $p_n \in [0, 1]$ denote the belief at the beginning of the n-th cycle, representing the probability that the components belong to the weak population conditional on the data collected in the previous $n - 1$ cycles. The belief at the beginning of cycle $n + 1$ is obtained by updating the belief p_n with the data collected at the end of the n-th cycle. We let (t_n, d_n) denote the data collected at the end of the n-th cycle. Specifically, t_n represents the length of the n-th cycle (i.e., $t_n = \min\{x_n, \tau_n\}$) with x_n the realization of the random variable X in the n-th cycle, and $d_n = 1$ and $d_n = 0$ denote that the n-th cycle ends with a corrective replacement and preventive replacement, respectively. The objective of the decision maker is to determine the optimal age-based policy that minimizes the expected total cost during the whole planning horizon.

2.4. Mathematical formulation

In this section, we provide a mathematical formulation of the model described in Section 2.3. Specifically, Section 2.4.1 characterizes the expected total cost of a given age-based replacement policy. This characterization can be used for the performance evaluation of any given policy. Section 2.4.2 proposes a POMDP model to determine an optimal age-based replacement policy that minimizes the expected total cost.

2.4.1 Performance evaluation of a preventive replacement policy

In our model formulation, the decision maker updates the belief variable at the end of each cycle by using the most recent belief variable and the information from the last cycle. To be specific, suppose that the decision maker starts the maintenance cycle n with the belief variable p_n and observes the data (t_n, d_n) in cycle n. It follows from Bayes' rule that the belief p_{n+1} at the end of the n-th cycle (or equivalently at the beginning of cycle $n+1$) can be recursively written as

$$p_{n+1} = g(p_n, t_n, d_n) = \begin{cases} \frac{p_n P_1(t_n)}{p_n P_1(t_n) + (1-p_n) P_2(t_n)} & \text{if } d_n = 1, \\ \frac{p_n [1-F_1(t_n)]}{p_n[1-F_1(t_n)] + (1-p_n)[1-F_2(t_n)]} & \text{if } d_n = 0. \end{cases} \tag{2.1}$$

for $n \in \mathbb{N}$. In Equation (2.1), notice that two types of events are distinguished. If $d_n = 1$ (i.e., the n-th cycle ends with a failure followed by a corrective replacement), it is known that t_n is exactly equal to the realized value of X in cycle n. Therefore, for $d_n = 1$, the likelihood of this realization is written as $P_j(t_n)$ for type j. On the other hand, if $d_n = 0$ (i.e., the n-th cycle ends with a preventive replacement), it is known that the realized value of X would have been larger than t_n. Therefore, for $d_n = 0$, the likelihood of this realization is written as $\sum_{i=t_n+1}^{\infty} P_j(i)$, or equivalently as $1 - F_j(t_n)$, for type j. By considering p_n as the Bayesian prior probability that the population is of weak type, Equation (2.1) follows from Bayes' rule. We refer to Gelman et al. (2013) for more details on Bayesian updating.

The remaining lifespan of the system at the beginning of cycle $n+1$, which is denoted as z_{n+1}, can be obtained recursively (by using the cycle length t_n realized in the n-th cycle) as

$$z_{n+1} = \begin{cases} z_n - t_n & \text{for } z_n > 0, \\ z_n & \text{for } z_n = 0. \end{cases} \tag{2.2}$$

for $n \in \mathbb{N}$, starting with the initial value $z_1 = L$ (i.e., the remaining lifespan of the system in the beginning of the first cycle is equal to the system lifespan L). Recall that t_n is the length of the n-th cycle and equal to $\min\{x_n, \tau_n\}$, where x_n denotes the realization of the lifetime random variable and τ_n denotes the preventive replacement age at cycle n. Since the cycle length, t_n is a function of the lifetime random variable, the variable z_{n+1} is random as well for each $n > 1$ (z_1 is equal to L by definition and hence is deterministic). In Equation (2.2), the case $z_n > 0$ represents that the end of the system lifespan has not been reached yet at the beginning of the n-th cycle. Thus, the remaining lifespan of the system decreases by the realization of the length of the n-th cycle. Notice that $z_n = 0$ means that the system has already reached the end of its lifespan at the beginning of n-th cycle and no planning needs to be done at (and beyond) the n-th cycle.

We let $c_j(\tau_n, z_n)$ denote the expected cycle cost for component type j, which is given by

$$c_j(\tau_n, z_n) = \begin{cases} F_j(\tau_n)C_f + (1 - F_j(\tau_n))C_p & \text{if } \tau_n < z_n, \\ F_j(\tau_n)C_f & \text{if } \tau_n = z_n. \end{cases} \tag{2.3}$$

In Equation (2.3), notice that the expected cycle cost does not include the cost of a possible preventive replacement for the case $\tau_n = z_n$. If τ_n is chosen equal to z_n, it means that preventive replacement is scheduled at the end of the lifespan of the system. However, if the component does not fail until then and the system indeed reaches the end of its lifespan, then there is no preventive replacement needed as the system goes out of service at that moment. Therefore, the expected cycle cost only includes the cost of a possible corrective replacement for the case $\tau_n = z_n$.

For a given age-based replacement policy π, let $\tau^\pi(p_n, z_n)$ denote the planned time interval until the next preventive replacement for a cycle starting with belief variable p_n and remaining system lifespan z_n. The expected total cost for the entire lifespan of the system under policy π can be characterized as

$$\sum_{n=1}^{\infty} \mathbb{E}[C(p_n, z_n, \tau^\pi(p_n, z_n))], \tag{2.4}$$

where

$$C(p_n, z_n, \tau^\pi(p_n, z_n)) = \begin{cases} p_n c_1(\tau^\pi(p_n, z_n), z_n) + (1 - p_n)c_2(\tau^\pi(p_n, z_n), z_n) & \text{if } z_n > 0, \\ 0 & \text{if } z_n = 0. \end{cases} \tag{2.5}$$

Notice that $z_{n'} = 0$ for all $n' \geq n$ once $z_n = 0$ for a given n. Further, notice that, besides z_n, also the belief variable p_n is a random variable for $n > 1$ since the lifetime of the component is random. The expectation in (2.4) is taken with respect to the randomness in the belief and remaining lifespan variables.

2.4.2 POMDP formulation

We formulate the problem of finding the optimal preventive replacement policy as a POMDP model, where the true type of the components is unknown but represented by a belief variable. We take a decision at the beginning of each cycle and the state at that moment is described by (p, z) where $p \in [0, 1]$ denotes the belief that components belong to the weak population and $z \in \{0, 1, ..., L\}$ denotes the remaining lifespan of the system. Notice that z belongs to a finite set because we have a discrete-time model for a system with a finite lifespan. It is important to note that the two state variables p and z capture all information from past maintenance cycles, meaning that it is sufficient for the decision maker to define a policy as a function of these two variables. Let $V(p, z)$ denote the total cost until the end of the planning horizon under the optimal policy when the current state is (p, z). It holds that $V(p, 0) = 0$ for all $p \in [0, 1]$, since $(p, 0)$ is an absorbing state, representing the end of system lifespan, and the system stays in this absorbing state forever at no cost.

In our model, it is guaranteed to reach an absorbing state in a finite number of cycles, which is similar to a stochastic shortest path problem. Therefore, there is a unique optimal cost vector that satisfies the Bellman equations (Bertsekas, 1995). The Bellman equations are given by

$$V(p, z) = \min_{\tau \in \{1,2,...,z\}} \tilde{V}(p, z, \tau) \tag{2.6}$$

for all $p \in [0, 1]$ and $z \in \{1, ..., L\}$, where

$$\begin{aligned} \tilde{V}(p, z, \tau) \quad = \quad & C(p, z, \tau) + \sum_{x=1}^{\tau} V(g(p, x, 1), z - x)\Big(pP_1(x) + (1 - p)P_2(x)\Big) \qquad (2.7) \\ & + V(g(p, \tau, 0), z - \tau)\Big(p(1 - F_1(\tau)) + (1 - p)(1 - F_2(\tau))\Big). \end{aligned}$$

In the rest of the chapter, the function $V(p, z)$ in Equation (2.6) is also referred to as the value function. In Equation (2.7), the term $C(p,z,\tau)$ is the expected cost in the upcoming cycle, starting with state variables (p,z) and a preventive replacement scheduled at component age τ. The second term on the right-hand side of Equation (2.7) considers the scenarios where the upcoming cycle ends with a corrective replacement (i.e., the time-to failure realization x is less than or equal to the preventive replacement age τ), and takes the weighted average of the value function at all the possible states the system can be at the start of the next cycle, where the weights are equal to the likelihood of each possible state. Finally, the last term on the right-hand side of Equation (2.7) considers the scenarios where the upcoming cycle ends with a preventive replacement (i.e., the time-to failure realization x is greater than the preventive replacement age τ), and is equal to the value function evaluated at the state after the preventive replacement (i.e., at the state with updated belief $g(p,\tau,0)$ and remaining lifespan $z-\tau$) multiplied with the probability of the cycle ending with preventive replacement. The sum of these three terms on the right-hand side of Equation (2.7) represents the expected cost of a policy, which takes the preventive replacement decision τ at state (p,z) and follows the optimal policy thereafter.

Example 2.1 Assume that the maintenance cost parameters are equal to: $C_f = 10$ and $C_p = 1$. Further, let X_1 follow a discrete uniform distribution with support $\{1,2\}$ and X_2 a discrete uniform distribution with support $\{1,2,\ldots,10\}$. Then the probability mass functions are as follows:

$$P_1(x) = \begin{cases} \frac{1}{2} & \text{for } x \in \{1,2\}, \\ 0 & \text{for } x \geq 3. \end{cases}$$

$$P_2(x) = \begin{cases} \frac{1}{10} & \text{for } x \in \{1,2,\ldots,10\}, \\ 0 & \text{for } x \geq 11. \end{cases}$$

Applying the POMDP formulation leads to the following functions for $V(p,1)$ and $V(p,2)$:

$$V(p,1) = V(p,1,1) = C_f(pF_1(1) + (1-p)F_2(1)) = 1 + 4p,\ 0 \leq p \leq 1.$$

$$V(p,2) = \min\{\tilde{V}(p,2,1), \tilde{V}(p,2,2)\} = \min\{2\frac{9}{10} + 7\frac{3}{5}p, 2\frac{1}{10} + 10\frac{2}{5}p\}$$

$$= \begin{cases} 2\frac{1}{10} + 10\frac{2}{5}p & \text{for } 0 \leq p \leq \frac{2}{7}, \\ 2\frac{9}{10} + 7\frac{3}{5}p & \text{for } \frac{2}{7} \leq p \leq 1. \end{cases}$$

.

Remark 2.1 It is also possible to consider the beginning of each time step as a decision epoch, and to choose either the replace or do-nothing action at each decision epoch. This leads to an equivalent POMDP formulation. This equivalent formulation is described in Appendix 2.C. In that appendix, we also show that this alternative formulation leads to the same total cost function for Example 2.1. The POMDP formulation as given here has two advantages compared to the alternative formulation: (1) The current formulation updates the belief state only at the end of a maintenance cycle (not at the end of each time step). In a numerical experiment, we compared calculations under the current formulation with the calculations under the alternative formulation. In both cases, the same optimal cost is obtained, but, because of the more frequent updates of the continuous belief state, the alternative formulation requires a finer discretization level to obtain equally accurate results as for the current formulation. (2) The optimal policy under the current formulation denotes directly for each decision epoch at which maintenance is executed, when the next preventive maintenance action should take place. The optimal policy under the alternative formulation contains this information as well, but additional calculations are needed to retrieve that information.

Remark 2.2 Please notice that we assumed two populations in this chapter. It would also be possible to assume K distinct populations that our components might be coming from. Then the state description becomes $(\mathbf{p}, z)$, where $\mathbf{p} = (p_1, \ldots, p_K)$ and $\sum_{k=1}^{K} p_k = 1$. That is, the belief state becomes a vector instead of a single parameter (i.e., each point on the vector corresponds to a particular belief of having a certain population type). The Bayesian update function then also returns a vector:

$$\begin{aligned} \mathbf{g}(\mathbf{p}, z, d) &= \Big(\frac{p_1(P_1(t)d + (1 - F_1(t))(1-d))}{\sum_{k=1}^{K} p_k(P_k(t)d + (1 - F_k(t))(1-d))}, \ldots, \\ &\quad \frac{p_K(P_K(t)d + (1 - F_1(t))(1-d))}{\sum_{k=1}^{K} p_k(P_k(t)d + (1 - F_k(t))(1-d))} \Big). \end{aligned}$$

2.5. Structural analysis

In order to derive the structural properties of the value function, we make an ordering assumption regarding the time-to-failure random variables of the component types. Specifically, we assume that the time-to-failure random variable X_2 for the strong type is greater than the time-to-failure random variable X_1 for the weak type in the sense of likelihood ratio order, where the likelihood ratio order is defined as follows.

Definition 2.1 (Shaked and Shanthikumar, 2007). If $\mathbb{P}\{Y_2 = j\}\mathbb{P}\{Y_1 = i\} \geq \mathbb{P}\{Y_1 = j\}\mathbb{P}\{Y_2 = i\}$ for all $i \leq j$, then Y_2 is greater than Y_1 in likelihood ratio order, written as $Y_2 \geq_{lr} Y_1$.

Assumption 2.1 $X_2 \geq_{lr} X_1$.

Assumption 2.1 is equivalent to saying that the ratio $\mathbb{P}\{X_2 = x\}/\mathbb{P}\{X_1 = x\}$ is non-decreasing in $x \in \mathbb{N}$, which reflects the fact that the time-to-failure realizations are more likely to be high for strong components than for the weak components. Before proceeding with our analysis, we first provide some implications of Assumption 2.1.

Lemma 2.1 *(i)* $\mathbb{P}\{X_1 = x\}\mathbb{P}\{X_2 > x\} \geq \mathbb{P}\{X_2 = x\}\mathbb{P}\{X_1 > x\}$, $\forall x \in \mathbb{N}$.

(ii) $\mathbb{P}\{X_2 > x\} \geq \mathbb{P}\{X_1 > x\}, \forall x \in \mathbb{N}$.
(iii) $\sum_{x=1}^{\infty} \phi(x)\mathbb{P}\{X_2 = x\} \geq \sum_{x=1}^{\infty} \phi(x)\mathbb{P}\{X_1 = x\}$ *for any non-decreasing function* $\phi(\cdot)$.

Lemma 2.1(i) and Lemma 2.1(ii) show that the likelihood ratio ordering implies hazard-rate ordering and the usual stochastic order, respectively, and Lemma 2.1(iii) follows from the usual stochastic ordering of the random variables X_2 and X_1; see Shaked and Shanthikumar (2007) for more details. We next provide some properties of the Bayesian update function $g(p,t,d)$ that has been characterized in Equation (2.1). These properties will be needed for establishing the monotonicity properties of the value function later in this section.

Lemma 2.2 *(i) For each given* $t \in \mathbb{N}$ *and* $d \in \{0,1\}$, $g(p,t,d)$ *is a non-decreasing function of* p. *(ii)* $g(p,t,1)$ *is a non-increasing function of* t *for each given* $p \in [0,1]$. *(iii)* $g(p,t,1) \geq g(p,t,0)$ *for each given* $p \in [0,1]$ *and* $t \in \mathbb{N}$.

In Lemma 2.2(i), it is shown that the updated belief variable increases as the given belief increases. Lemma 2.2(ii) considers the updated belief in the case of a corrective replacement, and shows that the updated belief gets smaller when the realization of the cycle duration gets longer. Finally, Lemma 2.2(iii) shows that the updated belief at the end of a cycle will always be lower if the cycle ends with a corrective replacement instead of a preventive replacement, at a given prior belief and cycle length. In other words, under the same p and t values, a cycle ending with corrective replacement will lead to a higher updated belief of having a weak population compared to a cycle ending with a preventive replacement.

Before we start our analysis of the value function, we first derive some properties for the cost function $C(p,z,\tau)$, which represents the expected cycle cost as characterized in Equation (2.5).

Lemma 2.3 *(i) $C(p,z,\tau)$ is a non-decreasing function of z for a fixed p and τ.*

(ii) $C(p,z,\tau)$ is a non-decreasing, linear function of p for a fixed z and τ.

For a fixed planned replacement interval, Lemma 2.3(i) shows that the expected cycle cost becomes larger when there is a longer time left until the end of the system lifespan. This can be explained by the non-decreasing likelihood of a corrective replacement as the remaining lifespan increases. Lemma 2.3(ii) shows that the expected cycle cost becomes larger as it becomes more likely that the population of the components is of the weak type.

We introduce a new function that we need in the structural analysis of the value function. This new function is denoted with $V^l(p,z)$, and it represents the minimum expected total cost for a system that is currently at state (p,z) and for which the cost accounting stops at either the end of the lifespan or after l cycles, whichever of these time moments comes first. When $l=0$, it means that the cost accounting stops immediately and thus it holds that $V^0(p,z)=0$ for all $p \in [0,1]$ and $z \in \{0,1,\ldots,L\}$. For $l \geq 1$, the function $V^l(p,z)$ can be obtained recursively. It holds that

$$V^{l+1}(p,z) = \min_{\tau \in \{1,2,\ldots,z\}} \tilde{V}^{l+1}(p,z,\tau)$$

for $l \in \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of nonnegative integers, and

$$\begin{aligned}\tilde{V}^{l+1}(p,z,\tau) &= C(p,z,\tau) + \sum_{x=1}^{\tau} V^l(g(p,x,1),z-x)(pP_1(x) + (1-p)P_2(x)) \\ &+ V^l(g(p,\tau,0),z-\tau)\Big(p(1-F_1(\tau) + (1-p)(1-F_2(\tau))\Big) \quad (2.8)\end{aligned}$$

for all $p \in [0,1]$ and $z \in \{1,\ldots,L\}$. Notice that a system at state (p,z) reaches the end of its lifespan in at most z cycles. Therefore, it follows from the definition of the function $V^l(p,z)$ that $V(p,z) = V^l(p,z)$ for $z \leq l$. The idea behind our analysis is to show first the structural properties of the function $V^l(p,z)$ in Theorem 2.1 and then to use these properties for establishing the properties of the value function $V(p,z)$ in Corollary 2.1.

Theorem 2.1 *For each $l = 0,\ldots,L$, the following results hold:*

(i) $V^l(p,z)$ is a non-decreasing function of z for each given $p \in [0,1]$.

(ii) $V^l(p,z)$ is a non-decreasing and concave function of p for each given $z \in \{0,1,\ldots,L\}$.

Theorem 2.1(i) shows that the minimum expected cost of a system that reaches the end of its lifespan in at most l cycles becomes higher as the remaining lifespan gets larger. This is intuitive because maintenance costs accumulate over time, and the longer the planning horizon, the more the total cost. Theorem 2.1(ii) shows that the same function has monotonicity with respect to the belief variable. That is, as the belief on having a weak population increases, the minimum expected cost of a system having at most l cycles left in its lifespan becomes larger. This follows from the higher likelihood of failure and associated costs for weak population. Furthermore, the concavity implies that the rate of increase in this cost becomes smaller as the belief increases.

It is known from Theorem 2.1 that the function $V^L(p,z)$ is non-decreasing in z for a fixed p, and it is also non-decreasing and concave in p for a fixed z. Since we already know $V(p,z) = V^L(p,z)$ for all $z \leq L$, the structural properties of the function $V^L(p,z)$ also hold for $V(p,z)$. We formalize this result in the following corollary.

Corollary 2.1 *(i) $V(p,z)$ is a non-decreasing function of $z \in \{0,1,\ldots,L\}$ for each given $p \in [0,1]$.*
(ii) $V(p,z)$ is a non-decreasing and concave function of $p \in [0,1]$ for each given $z \in \{0,1,\ldots,L\}$.

2.6. Lower bound function

In this section, we present a lower bound on the minimum expected cost under the optimal policy. This lower bound function is obtained by assuming that the preventive replacement moments are determined under the assumption that the true population type of the components is known. This is referred to as the *perfect information* case. The optimal age-based maintenance policy under a finite planning horizon with a *known* lifetime distribution has been studied by Lugtigheid et al. (2008) and Belyi et al. (2017). They consider a single-component system (as we do) to find an optimal preventive maintenance schedule for minimizing the total cost. Both of these papers and the model we introduce in this section (to establish a lower bound function to the optimal cost under failure model uncertainty) assume the failure model is known, and therefore do not offer a mechanism to learn the failure model sequentially over time. For each component type, the optimal age-based replacement policy under perfect information can be found by a dynamic program with a single state variable z, representing the remaining lifespan. Specifically, the value function for component type $j \in \{1,2\}$ is given by

$$W_j(z) = \min_{\tau \in \{1,2,\ldots,z\}} \left(c_j(\tau,z) + \sum_{x=1}^{\tau} P_j(x) W_j(z-x) + (1 - F_j(\tau)) W_j(z-\tau) \right)$$

for $z \in \{1,2,...,L\}$ and $W_j(0) = 0$. Similar to our analysis in Section 2.5, we define a function $W_j^l(z)$ as the minimum expected total cost for a system that is currently at state z and for which the cost accounting stops at either the end of the lifespan or after l cycles, whichever of these time moments comes first. It holds that $W_j^0(z) = 0$ for all $z \in \{0,1,...,L\}$. Furthermore, it holds that

$$W_j^{l+1}(z) = \min_{\tau \in \{1,2,\ldots,z\}} \left(c_j(\tau,z) + \sum_{x=1}^{\tau} P_j(x) W_j^l(z-x) + (1 - F_j(\tau)) W_j^l(z-\tau) \right)$$

for $z \in \{1, 2, ..., L\}$ and $l \in \mathbb{N}_0$. Lemma 2.4 shows the monotonicity of the function $W_j^l(z)$ for each component type j.

Lemma 2.4 *For each $l \in \{1, \ldots, L\}$, $W_j^l(z)$ is a non-decreasing function of z for $j \in \{1, 2\}$.*

It follows from the definition of the function $W_j^l(z)$ that $W_j(z) = W_j^l(z)$ for any l such that $z \leq l$. Thus, the monotonicity of the function $W_j^L(z)$ also holds for the value function $W_j(z)$ of component type j in the perfect-information case. This result is formalized in Corollary 2.2.

Corollary 2.2 *$W_j(z)$ is a non-decreasing function of z for $j \in \{1, 2\}$.*

We introduce a function $W(p, z)$ defined as

$$W(p, z) = pW_1(z) + (1 - p)W_2(z) \tag{2.9}$$

for $p \in [0, 1]$ and $z \in \{0, ..., L\}$. The function $W(p, z)$ can be interpreted as the expected total cost if the belief on having a weak population is equal to p at state z, and the information on the true type is revealed immediately with all preventive replacement decisions set optimally by knowing the true component type in the remaining lifespan z.

Theorem 2.2 *$W(p, z) \leq V(p, z)$ for all $p \in [0, 1]$ and $z \in \{0, ..., L\}$.*

Theorem 2.2 shows that the function in (2.9) is always less than or equal to the minimum expected cost under the optimal policy, and thus it forms a lower bound function. This lower bound function corresponds to knowing the population of components with certainty. There are certain benefits of knowing how the optimal policy is under this perfect information case. First, we can compare the optimal policy to the lower bound policy in order to see how uncertainty affects the decision-making. The numerical results will be discussed in Section 2.7.1. Second, $V(p, z) - W(p, z)$ represents the value of uncovering the true population type in state (p, z). This can be interpreted as the maximum cost that the policy maker wants to pay in order to learn the true population. This is further elaborated in our numerical study in Section 2.7.4. In real life applications, an inspection might reveal the true component type with a certain cost. The difference between the lower bound and optimal cost determines how much we would be willing to pay for a possible inspection.

2.7. Numerical study

In order to solve the Bellman equations in (2.6), we first discretize the belief space and apply a dynamic programming algorithm with backward recursion. The details of our discretization scheme and a pseudocode of the solution algorithm can be found in the Appendix 2.A. In our numerical analysis, we set the discretization level, which is denoted with Δ_p in Algorithm 1, at 0.0025. This discretization level is chosen because it is a sufficiently small value, i.e., the results reported in this section do not change noticeably if the discretization level is further reduced.

For the numerical analysis, we assume that the lifetime random variable of each component type has a discrete Weibull distribution. We denote the pmf and cdf of a discrete Weibull distribution with $P(x;\lambda,k)$ and $F(x;\lambda,k)$, respectively, where $\lambda > 0$ is the scale parameter and $k > 0$ is the shape parameter. Specifically, it follows that $P(x,\lambda,k) = \exp\left[-\left(\frac{x-1}{\lambda}\right)^k\right] - \exp\left[-\left(\frac{x}{\lambda}\right)^k\right]$ and $F(x,\lambda,k) = 1 - \exp\left[-\left(\frac{x}{\lambda}\right)^k\right]$ for $x \in \mathbb{N}$. This discrete Weibull distribution can be derived from the well-known continuous Weibull distribution. To be specific, let $\hat{X}$ denote a continuous Weibull random variable with cumulative distribution function $F_{\hat{X}}(x) = 1 - \exp[-(x/\lambda)^k]$ for $x \geq 0$ given the scale parameter λ and shape parameter k. Similar to Chakraborty (2015), the pmf follows from defining $P(x,\lambda,k)$ as $F_{\hat{X}}(x) - F_{\hat{X}}(x-1)$ for $x \in \mathbb{N}$. We assume that the shape parameter is the same for both component types but the scale parameters are different. The scale parameter λ is equal to 10 for the weak type and it is equal to 20 for the strong type. We consider two possible values for the shape parameter: $k \in \{5, 10\}$. In Table 2.2, some distributional properties of the resulting discrete Weibull distributions are provided. To address our research questions, it appears that we can follow a similar test bed as in de Jonge et al. (2015a). The remainder of this section is organized as follows. In Section 2.7.1, the optimal policy structure will be discussed when the component type is perfectly known. In Section 2.7.2, the optimal policy and cost structure are presented for a specific problem instance under uncertainty in the component type. In Section 2.7.3, two different policies from the literature will be introduced as benchmarks

Table 2.2: Distributional properties for discrete Weibull.

Scale (λ)	Shape (k)	Expectation	Variance	Coefficient of Variation
10	5	9.682	4.506	0.219
20	5	18.863	17.775	0.224
10	10	10.014	1.393	0.118
20	10	19.527	5.324	0.118

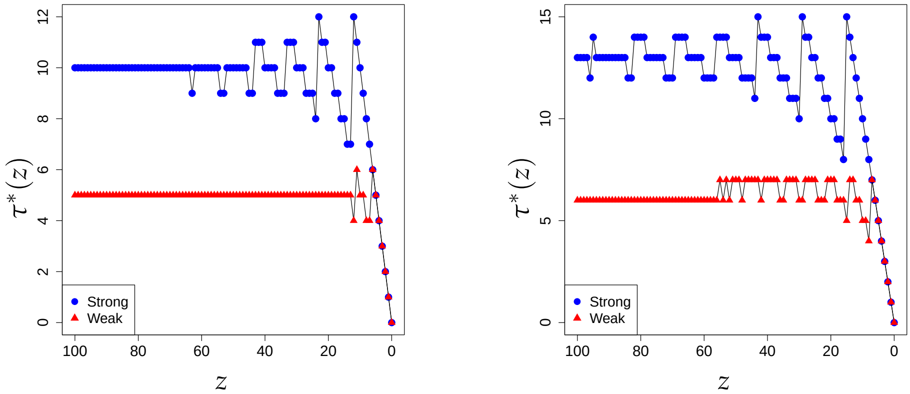
for the optimal policy. In Section 2.7.4, the costs under the optimal policy and the benchmark policies and the lower bound on the optimal cost are compared to each other. Finally, in Section 2.7.5, the structure of the optimal policy is compared with the structures of the benchmark policies in order to provide managerial insights on the exploration and exploitation trade-off in maintenance planning.

2.7.1 Perfect information case

In this section, we elaborate on how the optimal policy would look like if the decision maker had known the component type perfectly; i.e., when there is no population heterogeneity. We consider a problem instance with $C_p = 0.1$, $C_f = 1$, and $L = 100$. For each component type, Figure 2.2 plots the optimal action (i.e., the optimal amount of time until the next planned replacement) as a function of the state variable z. In Figure 2.2(a) and Figure 2.2(b), we consider the case with shape parameter k equal to 5 and 10, respectively. In both figures, $\tau^*(z)$ denotes the optimal policy under a given z for the perfect information case.

Remark 2.3 In numerical experiments, we implicitly assume that the discretization level for time is equal to $\Delta_t = 1$ (i.e., the length of each time step is one time unit), where Δ_t represents the discretization level for a continuous time Weibull distribution. Please note that we would see a continuum of values as optimal actions when Δ_t would go to 0. However, the overall fluctuating behavior of optimal actions would still exist. We can interpret these fluctuations as structural fluctuations that are caused by the distributional properties of the time-to-failure random variable and the finiteness of the planning horizon; see also the numerical experiment in Appendix 2.D, in which we study the effect of a smaller time step.

Notice that the remaining time until the end of lifespan, z, is the only state variable



(a) $k = 5$ *(b)* $k = 10$

Figure 2.2: Optimal actions under perfect information for $C_p = 0.1$, $C_f = 1$ and $L = 100$.

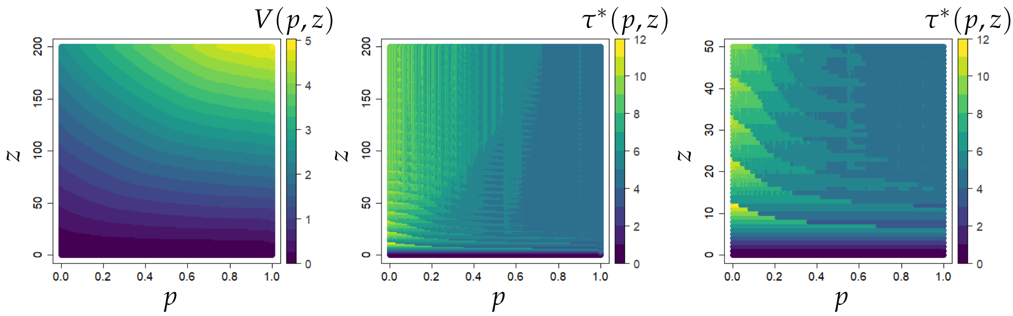
under the perfect information case. We observe in Figure 2.2 that the behavior of the optimal action can be split into three different patterns with respect to z: stable, fluctuating, and linear. When the system is far from the end of its lifespan (i.e., large z values), for both component types the optimal maintenance age $\tau^*(z)$ is stable at the value that minimizes the long-run average cost rate, obtained by using renewal theory under the assumption of an infinite time horizon. For example, we observe in Figure 2.2(a) with large values of z that the optimal action $\tau^*(z)$ is stable at 5 for the weak component, and it is stable at 10 for the strong component. This means that when z is large, the effect of the finite horizon is not observed in the optimal actions.

Optimal actions start to fluctuate as time passes (z becomes smaller). For example, in Figure 2.2(a) for the weak component, fluctuations take place between $z = 15$ and $z = 8$. When the system is close to the end-of-lifespan (z gets close to 0), the optimal actions are equal to z (i.e., between $z = 7$ and $z = 0$ in Figure 2.2(a)). An optimal action being equal to z means 'no replacement' (i.e., plan to do nothing in the rest of the system lifespan). The fluctuations in the figure represent a transition phase between stable optimal actions in the beginning of the lifespan (large z values) and 'no replacement' actions near to the end-of-lifespan (small z values). This can be explained by the finiteness of the planning horizon, referred to as the finite horizon effect. In the beginning of the lifespan, we obtain the action that minimizes the long-run expected cost rate as the optimal action. When the behavior of optimal actions starts fluctuating, the optimal policy tries to balance the lengths of the remaining cycles. While balancing, at certain points it becomes optimal to aim at one cycle less when you come closer to the end of the horizon. At those points, the optimal threshold value $\tau^*(z)$ jumps from a relatively low to a relatively high value. At the end of the lifespan (i.e., $z = 0$) we do not have a replacement action and the value function is equal to zero. As the system approaches the end of its lifespan, the optimal policy selects 'no replacement' as the optimal action because the model considers that the remaining lifespan of the system is not sufficiently long to justify a replacement of the component, thus, avoiding an unnecessary preventive maintenance cost.

Additionally, in Figure 2.2(b), where $k = 10$ (i.e., the lifetime of each component is less variable compared to $k = 5$), the optimal actions and the fluctuations in optimal actions are greater than their counterparts in Figure 2.2(a), where $k = 5$, for both strong and weak parts. Lower variability in a component's lifetime means that it becomes more predictable what will happen till the end of the planning horizon. We observe that this leads to a longer time interval with fluctuating optimal actions. Finally, when we compare the weak and strong parts with the same shape parameter, we see in both Figure 2.2(a) and (b) that the fluctuations are less for the weak population and the optimal actions for the weak population cannot exceed the optimal actions for the strong population.

2.7.2 Analysis of POMDP model for optimal cost and optimal action

The objective of this section is to investigate the structure of the optimal policy when there is population heterogeneity (i.e., the correct component type is unknown). This helps us to answer the first research question introduced in Section 2.1. For this purpose, we focus on a specific problem instance with $L = 200$, $k = 5$, $C_p = 0.1$, $C_f = 1$, and analyze the structure of the optimal cost and optimal policy via the POMDP model. Figure 2.3(a) and Figure 2.3(b) illustrate the value function and the optimal policy, respectively, with $\tau^*(p, z)$ denoting the optimal action at state (p, z). The results for the value function are consistent with Theorem 2.1 presented in Section 5. In particular, the monotonicity of $V(p, z)$ with respect to p and z as well as the concavity of $V(p,z)$ with respect to p can be observed in Figure 2.3(a).



(a) Optimal costs *(b)* Optimal actions *(c)* Optimal actions for $z \leq 50$

Figure 2.3: Optimal costs and actions under the POMDP model for $C_p = 0.1$, $C_f = 1$, $L = 200$ and $k = 5$.

It is important to note in Figure 2.3(b) that the optimal action $\tau^*(p,z)$ is not monotone with respect to p and z. However, there exists patterns in the optimal action. For a more clear observation of these patterns, Figure 2.3(c) provides a closer view of the optimal policy from Figure 2.3(b) for $z \leq 50$. Specifically, at a fixed p value, we observe a repetitive pattern in optimal actions if p is sufficiently small: there are bell-shaped structures, especially as the system approaches the end of its lifespan. In this bell-shaped pattern, the optimal action shows cycles of first increasing then decreasing behavior as z changes. Notice that having a strong component and having a weak component under the perfect information case corresponds to $p = 0$ and $p = 1$, respectively, in the POMDP model. Also recall from Section 2.7.1 that we observe more fluctuations for the strong component in the perfect information case. This explains why there are more fluctuations in Figure 2.3(b) for small values of p (i.e., as the components are more likely to be coming from the strong population for small p).

2.7.3 Benchmark policies

In this section, we introduce two alternative policies from the literature and use them as a benchmark for the optimal policy of the POMDP model to generate insights on the benefits of the optimal policy. To be specific, we first provide the definition of the so-called myopic policy and then the threshold policy (de Jonge et al., 2015a).

Myopic Policy. The myopic policy (MP) aims to minimize the long-run expected cost per unit time. It is a commonly used policy in maintenance literature. In our context, it uses the current estimate of the belief variable, say $\hat{p}$, and determines the time of the next planned replacement by the following equation:

$$\tau_{\mathrm{MP}}(\hat{p}) = \arg\min_{\tau} \left\{ \hat{p} \frac{C_f F_1(\tau) + C_p(1 - F_1(\tau))}{\sum_{x=1}^{\tau}(1 - F_1(x))} + (1 - \hat{p}) \frac{C_f F_2(\tau) + C_p(1 - F_2(\tau))}{\sum_{x=1}^{\tau}(1 - F_2(x))} \right\}. \tag{2.10}$$

It is referred to as a myopic policy because it ignores the effect of an action in the current cycle on the information collected in the following cycles (i.e., how the current action affects the belief variable in the future). The myopic policy is also used by de Jonge et al. (2015a) as a benchmark to their proposed heuristic. Notice that the myopic policy does not consider the finiteness of the planning horizon, while our problem has a finite planning horizon. However, it still serves as a natural benchmark due to its low computational complexity and relevance when the planning horizon is sufficiently larger than the expected time between failures, as stated in de Jonge et al. (2015a).

Threshold Policy. The threshold policy (TP) is proposed by de Jonge et al. (2015a). The aim of this policy is to incorporate the effect of an action on the update of the belief variable, and hence, to perform better in the future at the expense of performing less good in the short term. This is done by deliberately postponing a preventive replacement if the belief variable p is below a certain threshold. Specifically, if $p \leq \omega$, where ω is the threshold value, the time until the next planned replacement is selected as τ_2, given by

$$\tau_2 = \arg\min_{\tau} \frac{C_f F_2(\tau) + C_p(1 - F_2(\tau))}{\sum_{x=1}^{\tau}(1 - F_2(x))}. \tag{2.11}$$

On the other hand, if $p > \omega$, the time until next planned replacement is set equal to the value of τ_{MP}. This policy is implemented by using different values of the threshold, and the total expected cost (in the finite planning horizon) is obtained via simulation or the Bellman equations for each threshold value. The threshold value that leads to the lowest cost is referred to as the optimal threshold, denoted by ω^* (de Jonge et al., 2015a). Notice that the value of τ_2 is the time until the next planned replacement that minimizes the long run expected cost rate if the component is certainly of the strong type. Intuitively, the threshold policy delays the preventive maintenance moment (by pretending that the component is of the strong type) when the probability that the component is of the weak type is sufficiently small. It is important to note that the threshold policy aims to address the exploration-exploitation trade-off, but it is not necessarily the optimal policy. Also, notice that the threshold policy is a function of the belief variable p, hence, we denote the resulting action (i.e., time until next replacement) under the threshold policy with $\tau_{\mathrm{TP}}(p)$.

2.7.4 Numerical results

The objective of this section is to compare the minimum cost under the optimal policy with the cost associated with the benchmark policies and the lower bound on the optimal cost. Recall that p_1 represents the initial belief that the population is of the weak type at the beginning of the system lifespan. In our experiments, we consider $p_1 \in \{0.25, 0.5, 0.75\}$, meaning that the performance comparison will be made for systems that start their lifespan at different initial beliefs. Furthermore, we let $L \in \{100, 200\}$, $k \in \{5, 10\}$, $C_f = 1$, and $C_p \in \{0.05, 0.1, 0.2\}$. This setup leads to 36 problem instances in our experiment (see Table 2.3). It is important to note that this testbed mimics the testbed of de Jonge et al. (2015a). As a slight difference, we use discrete versions of the lifetime distributions considered in de Jonge et al. (2015a) after multiplying their scale parameters by 10 (this is to scale the time and make the lifetime realizations of the components suitable for our discrete-time problem setting). Additionally, we calculate the expected cost of any given policy via the Bellman equations while de Jonge et al. (2015a) use simulation to approximate the costs of the myopic and threshold policies. Therefore, the results reported in Table 2.3 are exact.

In Table 2.3, we compare the performance of the optimal policy with the performance of the myopic policy and threshold policy. This will enable us answering the second research question introduced in Section 2.1 (i.e., under which scenarios is the total cost of optimal policy (OP) much lower than the cost of existing heuristic policies?). The expected cost under the myopic policy is denoted with $V_{\mathrm{MP}}(p_1, L)$. The difference and the relative difference in the expected costs under the myopic policy and the optimal policy are denoted by $\Delta_{\mathrm{MP}} = V_{\mathrm{MP}}(p_1, L) - V(p_1, L)$ and

$\Delta_{\mathrm{MP}}^{\mathrm{rel}} = \Delta_{\mathrm{MP}}/V(p_1, L)$, respectively. We denote the expected cost under the optimal threshold policy with $V_{\mathrm{TP}}(p_1, L)$ and the corresponding optimal threshold with ω^*. The difference and the relative difference in the expected costs under the threshold policy and the optimal policy are denoted by $\Delta_{\mathrm{TP}} = V_{\mathrm{TP}}(p_1, L) - V(p_1, L)$ and $\Delta_{\mathrm{TP}}^{\mathrm{rel}} = \Delta_{\mathrm{TP}}/V(p_1, L)$. Furthermore, Table 2.3 compares the expected cost under the optimal policy with its lower bound $W(p_1, L)$, which was obtained in Section 2.6. This comparison will help us answering the third research question introduced in Section 2.1 (i.e., determining the value of resolving the uncertainty in the population heterogeneity). The difference and relative difference between the expected cost under the optimal policy and its lower bound are denoted by $\Upsilon_{\mathrm{LB}} = V(p_1, L) - W(p_1, L)$ and $\Upsilon_{\mathrm{LB}}^{\mathrm{rel}} = \Upsilon_{\mathrm{LB}}/V(p_1, L)$, respectively.

An immediate observation from Table 2.3 is that the expected costs for MP, TP and OP decrease in k and increase in p_1, C_p and L (i.e., the monotonicity of the expected cost for OP with respect to the belief variable and the remaining time until the end of the horizon was already proven in Section 2.5).

We start our detailed analysis by comparing OP with MP in order to assess the benefit of using the optimal policy instead of the myopic policy. The difference in expected costs can be interpreted as the value of optimally balancing exploration and exploitation rather than acting only in a myopic way without any exploration in maintenance planning. We make the following observations: (1) When looking at the values of $\Delta_{\mathrm{MP}}^{\mathrm{rel}}$ as a function of p_1, we generally obtain the highest values for $p_1 = 0.25$ (the only exception is seen in the instances 34-36). That is, if the population is more likely to be of the strong type, then it becomes more beneficial to follow the optimal policy instead of the myopic policy. (2) As C_p increases from 0.05 to 0.2, Δ_{MP} increases (except for instances 31 and 34). In our experimental design, having a large C_p value means that the cost of preventive maintenance approaches the cost of corrective maintenance, implying that exploration becomes relatively less expensive (i.e., because the corrective maintenance becomes relatively less costly). This results in learning the true type of the population earlier, leading to higher $\Delta_{\mathrm{MP}}^{\mathrm{rel}}$ values. (3) $\Delta_{\mathrm{MP}}^{\mathrm{rel}}$ is higher for $k = 10$ compared to $k = 5$ (with the exception of instances 25 and 31, yet with a small difference). That is, we observe that as the variability in the time-to-failure distribution of the component decreases (i.e., when the shape parameter k increases from 5 to 10), the optimal policy becomes more beneficial compared to the myopic policy. (4) We observe that the value of Δ_{MP} for

Table 2.3: Comparison of policies under different scenarios.

No.	L	C_p	k	p_1	$V(p_1, L)$	Δ_{MP}	Δ_{TP}	Υ_{LB}	$\Delta^{\text{rel}}_{\text{MP}}$	$\Delta^{\text{rel}}_{\text{TP}}$	$\Upsilon^{\text{rel}}_{\text{LB}}$	ω^*
1	100	0.05	5	0.25	1.071	0.006	0.006	0.186	0.6%	0.6%	17.3%	0
2	100	0.05	5	0.5	1.250	0.002	0.002	0.179	0.2%	0.2%	14.3%	0
3	100	0.05	5	0.75	1.387	0.004	0.004	0.129	0.3%	0.3%	9.3%	0
4	100	0.1	5	0.25	1.721	0.063	0.037	0.215	3.7%	2.1%	12.5%	0.3
5	100	0.1	5	0.5	2.123	0.018	0.018	0.298	0.8%	0.8%	14.0%	0
6	100	0.1	5	0.75	2.334	0.006	0.006	0.188	0.3%	0.3%	8.1%	0
7	100	0.2	5	0.25	2.765	0.152	0.063	0.221	5.5%	2.3%	8.0%	0.15
8	100	0.2	5	0.5	3.403	0.103	0.049	0.314	3.0%	1.4%	9.2%	0.45
9	100	0.2	5	0.75	3.932	0.062	0.047	0.297	1.6%	1.2%	7.5%	0.5
10	100	0.05	10	0.25	0.768	0.020	0.020	0.228	2.7%	2.7%	29.6%	0
11	100	0.05	10	0.5	0.839	0.008	0.008	0.183	1.0%	1.0%	21.8%	0
12	100	0.05	10	0.75	0.867	0.003	0.003	0.095	0.3%	0.3%	10.9%	0
13	100	0.1	10	0.25	1.222	0.209	0.018	0.227	17.1%	1.5%	18.6%	0.4
14	100	0.1	10	0.5	1.536	0.038	0.038	0.318	2.5%	2.5%	20.7%	0
15	100	0.1	10	0.75	1.610	0.047	0.047	0.167	2.9%	2.9%	10.4%	0
16	100	0.2	10	0.25	1.984	0.444	0.115	0.166	22.4%	5.8%	8.4%	0.25
17	100	0.2	10	0.5	2.551	0.377	0.080	0.325	14.8%	3.1%	12.7%	0.5
18	100	0.2	10	0.75	2.934	0.102	0.102	0.300	3.5%	3.5%	10.2%	0
19	200	0.05	5	0.25	2.079	0.068	0.029	0.262	3.3%	1.4%	12.6%	0.3
20	200	0.05	5	0.5	2.545	0.011	0.011	0.354	0.4%	0.4%	13.9%	0
21	200	0.05	5	0.75	2.810	0.030	0.030	0.246	1.1%	1.1%	8.8%	0
22	200	0.1	5	0.25	3.405	0.140	0.048	0.297	4.1%	1.4%	8.7%	0.35
23	200	0.1	5	0.5	4.143	0.141	0.071	0.394	3.4%	1.7%	9.5%	0.3
24	200	0.1	5	0.75	4.764	0.026	0.026	0.376	0.6%	0.6%	7.9%	0
25	200	0.2	5	0.25	5.589	0.158	0.101	0.312	2.8%	1.8%	5.6%	0.25
26	200	0.2	5	0.5	6.768	0.135	0.070	0.399	2.0%	1.0%	5.9%	0.45
27	200	0.2	5	0.75	7.841	0.167	0.131	0.380	2.1%	1.7%	4.9%	0.6
28	200	0.05	10	0.25	1.376	0.201	0.028	0.256	14.6%	2.0%	18.6%	0.3
29	200	0.05	10	0.5	1.721	0.028	0.028	0.365	1.7%	1.7%	21.2%	0
30	200	0.05	10	0.75	1.777	0.018	0.018	0.186	1.0%	1.0%	10.5%	0
31	200	0.1	10	0.25	2.315	0.547	0.072	0.229	23.6%	3.1%	9.9%	0.25
32	200	0.1	10	0.5	2.952	0.211	0.066	0.422	7.1%	2.2%	14.3%	0.5
33	200	0.1	10	0.75	3.300	0.026	0.026	0.325	0.8%	0.8%	9.8%	0
34	200	0.2	10	0.25	3.990	0.453	0.135	0.181	11.4%	3.4%	4.5%	0.25
35	200	0.2	10	0.5	4.952	0.763	0.117	0.347	15.4%	2.4%	7.0%	0.5
36	200	0.2	10	0.75	5.880	0.194	0.111	0.478	3.3%	1.9%	8.1%	0.75

$L = 200$ is higher than its counterpart for $L = 100$ (with the exception of instances 15 and 33). That is, if there is more time to exploit the information that is obtained by exploration, the optimal policy performs better than the myopic policy. In fact, we observe that the highest values of $\Delta^{\text{rel}}_{\text{MP}}$ are obtained at instances where $L = 200$ with $p_1 = 0.25$. Specifically, the value of $\Delta^{\text{rel}}_{\text{MP}}$ can be as high as 23.6% (see instance 31).

In Table 2.4, we can see the average of $\Delta^{\text{rel}}_{\text{MP}}$ and $\Delta^{\text{rel}}_{\text{TP}}$ values for subsets of instances with specific values of L, k, C_p and p_1. We see that $\Delta^{\text{rel}}_{\text{MP}}$ is higher for $L = 200$ than

Table 2.4: Average of relative differences under parameters L, k, C_p and p_1.

L	$\Delta_{\text{MP}}^{\text{rel}}$	$\Delta_{\text{TP}}^{\text{rel}}$	k	$\Delta_{\text{MP}}^{\text{rel}}$	$\Delta_{\text{TP}}^{\text{rel}}$	C_p	$\Delta_{\text{MP}}^{\text{rel}}$	$\Delta_{\text{TP}}^{\text{rel}}$	p_1	$\Delta_{\text{MP}}^{\text{rel}}$	$\Delta_{\text{TP}}^{\text{rel}}$
100	4.6%	1.8%	5	2.0%	1.1%	0.05	2.3%	1.1%	0.25	9.3%	2.3%
200	5.5%	1.6%	10	8.1%	2.3%	0.1	5.6%	1.7%	0.5	4.4%	1.5%
						0.2	7.3%	2.5%	0.75	1.5%	1.3%

its value for $L = 100$. But it is the other way around for $\Delta_{\text{TP}}^{\text{rel}}$ values. Both $\Delta_{\text{MP}}^{\text{rel}}$ and $\Delta_{\text{TP}}^{\text{rel}}$ are higher for $k = 10$ than for $k = 5$. As C_p increases in the set of $\{0.05, 0.1, 0.2\}$, also $\Delta_{\text{MP}}^{\text{rel}}$ and $\Delta_{\text{TP}}^{\text{rel}}$ increase. Finally, as p_1 increases in the set of $\{0.25, 0.5, 0.75\}$, $\Delta_{\text{MP}}^{\text{rel}}$ and $\Delta_{\text{TP}}^{\text{rel}}$ decrease.

We next investigate the benefit of following the optimal policy instead of the threshold policy. The relative difference $\Delta_{\text{TP}}^{\text{rel}}$ in expected cost can be interpreted as the percentage gain obtained from optimally balancing exploration and exploitation instead of following the heuristic TP to strike this balance. We observe in Table 2.3 that $\Delta_{\text{TP}}^{\text{rel}}$ varies from 0.2% to 5.8%. That is, following the threshold policy instead of the optimal policy can lead to a cost increase of 5.8%.

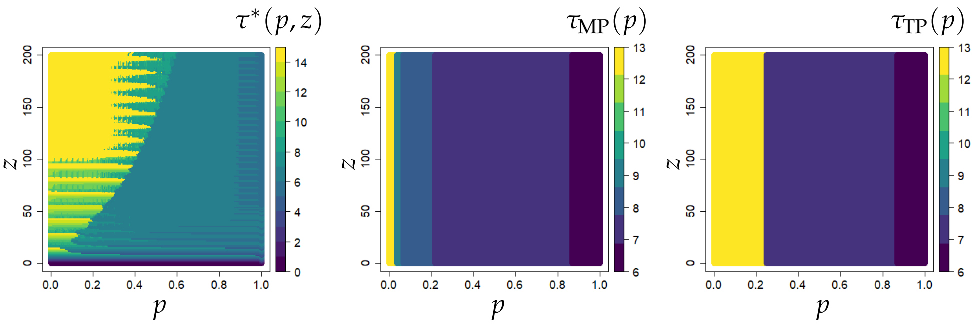
Remark 2.4 It is known that the heuristic policies MP and TP do not explicitly model the end of the planning horizon in obtaining the replacement age, while our optimal policy does. As part of our side experiments, we investigated how adapting these heuristics to a finite-horizon setting (by limiting the action space based on the remaining lifespan of the system, i.e., by limiting the action space to $\tau \in \{1, \ldots, z\}$) would affect their performance. By using the same test bed, we observed for both MP and TP that, when they are adapted to a finite-horizon setting, the decrease in the relative distance to the optimal cost is very small (in 90% of the instances), i.e., adapting the heuristics to a finite-horizon setting has a limited effect on their performance. For more details on this side experiment, see Appendix 2.E .

Finally, we investigate the value of knowing the true type of the components. Specifically, we focus on the reduction in the expected cost when the population uncertainty is resolved at the beginning of the planning horizon. Recall that this reduction is given by the value of Y_{LB}, and it can be argued that Y_{LB} is the maximum amount someone would be willing to pay for an inspection activity to reveal the true type of the component. We make the following observations: (1) $\text{Y}_{\text{LB}}^{\text{rel}}$ is typically higher for $k = 10$ than $k = 5$. That is, we observe that as the variability in the lifetime of the component is low, it generally becomes more beneficial to resolve the population heterogeneity at the beginning of the system lifespan. (2) $Y_{\text{LB}}^{\text{rel}}$ is lower for $L = 200$ compared to $L = 100$. That is, in a longer lifespan of the system, there is more time to exploit the knowledge obtained through learning under the optimal policy, and thus, the relative gain from following the optimal policy is higher (i.e., the performance of the optimal policy is closer to the performance of the policy that learns the true type as soon as the system starts its operation). (3) $Y_{\text{LB}}^{\text{rel}}$ is highest for an initial belief $p_1 = 0.5$ (except for instances 1-3 and 10-12), i.e. when we have the highest uncertainty about the population type. This can be interpreted as the situation where it is most important to learn the true component type by alternative means such as an inspection. (4) For larger values of C_p, $Y_{\text{LB}}^{\text{rel}}$ is lower. This can be argued as a result of exploration becoming relatively less expensive with an increasing value of C_p.

2.7.5 Comparison of policy structures

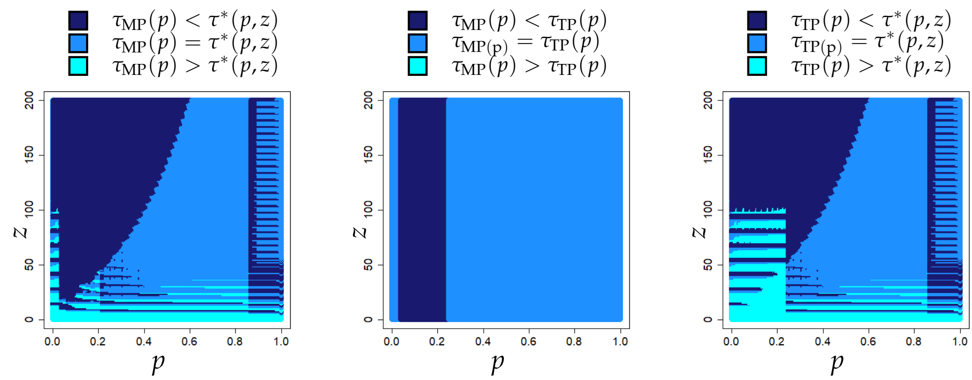
In this section, the objective is to compare the structure of the optimal policy with the structure of MP and TP in order to address the fourth research question introduced in Section 2.1 (i.e. how does the structure of the optimal policy differ from existing heuristics in terms of exploration and exploitation). For this purpose, we focus on the case with $C_p = 0.1$, $k = 10$ and $L = 200$, where the values of $\Delta_{\text{MP}}^{\text{rel}}$ and $\Delta_{\text{TP}}^{\text{rel}}$ are relatively high. Figure 2.4 illustrates the actions under the optimal policy and the actions under MP and TP. As it is seen from Figure 2.4, OP adapts its actions according to the remaining lifespan of the system where actions under MP and TP are constant with respect to the remaining lifespan. This is mainly because OP dynamically updates the action space with respect to the remaining lifespan and it considers the effect of an action on the future cycles during decision making.

For an easier interpretation of Figure 2.4, we plot the relationship between the actions of different policies in Figure 2.5. Specifically, Figure 2.5(a) shows in what states the optimal action is greater, equal to, and less than the action under the myopic policy. Similarly, Figure 2.5(b) compares the actions of the threshold and myopic policies. It can be argued that exploration takes place at a state when the optimal action is greater than the action under MP at that state; i.e., the time until the next planned replacement is longer to better learn the true type of the component population. We observe that exploration typically takes place when



(a) Optimal Actions *(b)* MP Actions *(c)* TP Actions ($\omega^* = 0.25$)

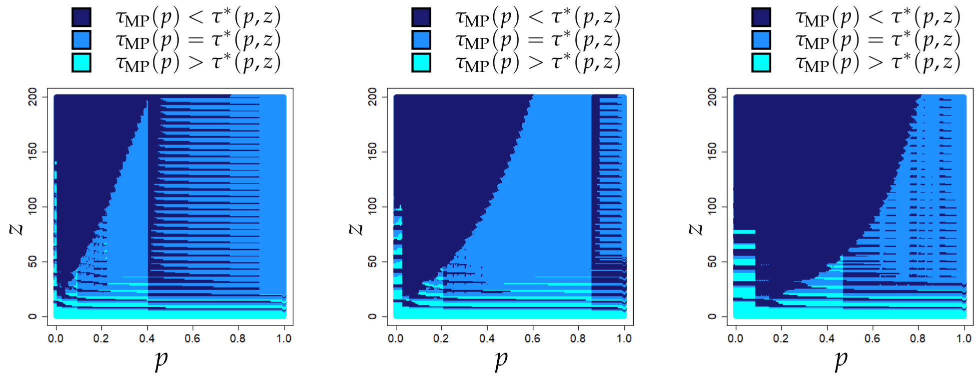
Figure 2.4: Illustration of the actions under the optimal policy, MP and TP for $C_p = 0.1$, $L = 200$, and $k = 10$ (e.g., $z = 200$ and $p_1 = 0.25$ correspond to the instance 31).



(a) Comparison of $\tau^*(p,z)$ and $\tau_{\mathrm{MP}}(p)$ *(b)* Comparison of $\tau_{\mathrm{TP}}(p)$ and $\tau_{\mathrm{MP}}(p)$ *(c)* Comparison of $\tau^*(p,z)$ and $\tau_{\mathrm{TP}}(p)$

Figure 2.5: Comparison of the actions under the optimal policy and the TP with the actions under the MP for $C_p = 0.1$, $L = 200$, and $k = 10$.

the belief that the components come from the weak population is small enough for both OP and TP. By comparing Figure 2.5(a) and Figure 2.5(b), we observe that the optimal policy performs exploration at a larger number of states than TP does. Also, Figure 2.5(c) directly compares the actions of the optimal and threshold policies. As the system gets closer to the end of its lifespan (i.e., as z decreases), the exploration decreases under the optimal policy. This can be associated with the ability of the optimal policy to explicitly take the end of the lifespan into account. To be specific, the optimal policy exploits the belief already learned close to the end



(a) $C_p = 0.05$ *(b)* $C_p = 0.1$ *(c)* $C_p = 0.2$

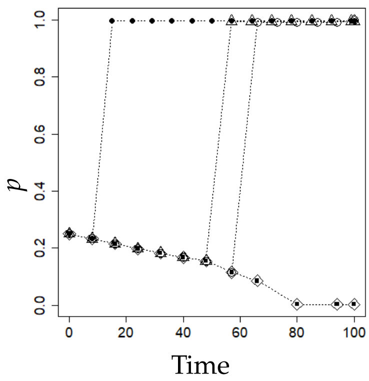
Figure 2.6: Comparison of the actions under the optimal policy with the actions under the MP for $L = 200$ and $k = 10$.

of the lifespan, while the threshold policy does not change its policy structure over time. This is mainly because the threshold policy postpones maintenance activities (for exploration purpose) based on a belief threshold that is fixed throughout the lifespan. In Figure 2.6, we illustrate the states where the exploration takes place under the optimal policy by considering different C_p values. Figure 2.6 shows that exploration takes place at a larger number of states when C_p is relatively higher. This is in line with our earlier interpretation of exploration being relatively cheaper with an increasing value of C_p.

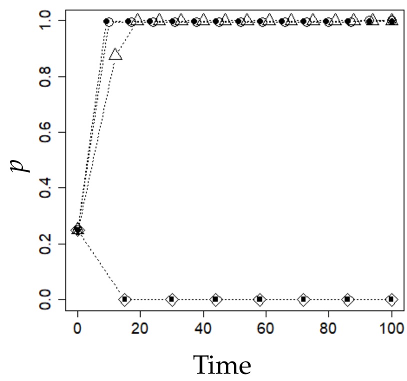
In the remainder of this section, we investigate how the belief variable evolves under the optimal policy by performing simulation experiments. By comparison with the evolution of the belief variable under the myopic policy, we aim to shed further light on the role of exploration in learning the true type of the component population. We choose to focus on instances 16 and 17 because $\Delta^{\mathrm{rel}}_{\mathrm{MP}}$ takes relatively large values in these instances. In each simulation run, we start with the initial belief p_1 (i.e., equal to 0.25 and 0.5 for instances 16 and 17, respectively) and generate a large number of a series of realizations from the time-to-failure distribution of a given population type. Specifically, we perform a total of 10000 simulation runs, and the proportion of the simulation runs where the time-to-failure realizations are generated from the weak population is equal to p_1; e.g., for instance, 16 with $p_1 = 0.25$, we performed 2500 simulation runs by generating the time-to-failure realizations for a weak component and 7500 simulation runs by generating these realizations for a strong component; each simulation run includes a sufficiently large number of time-to-failure realizations so that the entire lifespan is covered in each simulation run.

We apply the actions from the myopic policy and the actions from the optimal policy against the same realizations of the failure times in each simulation run. Notice that the actions under these two policies are potentially different, leading to different realizations of the cycle lengths, and hence a potentially different evolution of the belief variable p. Figure 2.7 illustrates examples on the evolution of the belief variable over the system lifespan, by showing five sample paths of the updated belief variables under both the myopic and optimal policies for the instances 16 and 17.

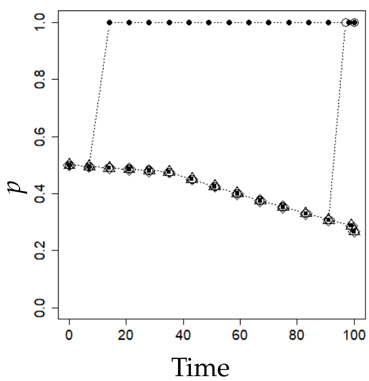
Recall that, under perfect information, the belief variable needs to be equal to 1 when the true component type is weak, and it needs to be equal to 0 when the true component type is strong. In our simulation experiments, if the updated belief variable gets a minimum value of 0.99 when the true component type is weak or if it gets a maximum value of 0.01 when the true component type is strong, we say that the belief variable has converged to its true value. We observe that the belief variable converges faster under the optimal policy than the myopic policy in most of the simulation runs, even though both policies start with the same initial belief. Specifically, for the problem instance 16, the belief variable converges to its true value in 92.84% of the sample paths under the optimal policy (i.e., in 9284 simulation runs out of 10000). Furthermore, for the problem instance 16, the belief variable under the optimal policy converges to its true value on average in 15.47 time units for weak type (i.e., since the half-width of 95% confidence intervals turns out to be less than 0.01 time units when the number of simulation runs is equal to 10000, we only report the average values from our simulation experiments in the remainder of this section). The corresponding value for the strong type is 15.06 time units. On the other hand, the belief variable converges to its true value in 92.78 % of sample paths under the myopic policy. While this value is only slightly different than for the optimal policy, we observe that belief variable converges to its true value on average in 40.01 time units when the true population is weak and in 80.09 time units when the true population is strong, which are both substantially longer than their counterparts under the optimal policy.



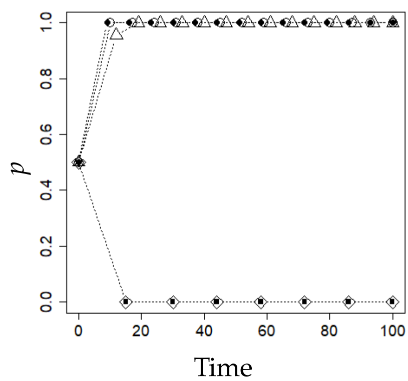
(a) Instance 16, MP



(b) Instance 16, OP



(c) Instance 17, MP



(d) Instance 17, OP

Figure 2.7: Evolution of the belief variable p in five simulation sample paths under the myopic policy (MP) and under the optimal policy (OP); a specific sample path in each figure is denoted with a distinct symbol.

For problem instance 17, we observe that the belief variable converges to its true value in 97.08% of the sample paths under the optimal policy. We also observe that the belief variable converges to its true value on average in 10.55 time units when the true population is weak and in 15.06 time units when the true population is strong. On the other hand, under the myopic policy, the belief variable reaches to its true value only in 33.47% of the sample paths. Furthermore, under the myopic policy, we observe that the belief variable converges to its true value on average in

58.62 time units when the true population is weak, while it fails to converge in any of the sample paths when the true population is strong.

2.8. Conclusion

In this research, we have developed an optimal age-based maintenance policy under population heterogeneity in a finite-lifespan setting. The initial belief about the true population type is updated in a Bayesian way, and a POMDP model is formulated. The model enables us to optimally balance exploration (i.e., deliberately delaying the preventive replacement moments to better learn the unknown population type for components) and exploitation (i.e., focusing only on the current maintenance cycle without considering the future maintenance cycles) in our setting. We compare the performance of the optimal policy to two heuristic policies (namely, the myopic policy and the threshold policy) from the literature. In our numerical experiments, we obtain that the threshold policy is up to 5.8% more costly compared to the optimal policy. The myopic policy is up to 23.6% more costly compared to the optimal policy. We also show that the true component type is learned much faster and more accurately under the optimal policy due to the effective use of exploration.

2.A. Solution algorithm

In this section, we provide an outline of the backward recursion algorithm that we use to solve the Bellman equations provided in Equation (2.6). Since the belief state is continuous, we perform a discretization of the belief space with Δ_p as the discretization step. Algorithm 1 presents the pseudocode for the solution algorithm, where the updated belief state is adjusted in accordance with the value of Δ_p with $\lceil \cdot \rceil$ denoting the ceiling function that rounds a real number to above. The data set generated for numerical experiments and the algorithm can be found at https://git.io/JzEc1.

Algorithm 1 Backward Recursion Algorithm

Initialize $z = 1$
$V(p, 0) \leftarrow 0 \; \forall p \in \{0, \Delta_p, 2\Delta_p, ..., 1\}$
while $z < L + 1$ **do**
  for $p \in \{0, \Delta_p, 2\Delta_p, ..., 1\}$ **do**
    for $\tau \in \{1, ..., z\}$ **do**

$$\begin{aligned} \tilde{V}(p,z,\tau) \quad \leftarrow \quad & C(p,z,\tau) + \sum_{x=1}^{\tau} V(\left\lceil g(p,x,1)\frac{1}{\Delta_p}\right\rceil \Delta_p, z-x)(pP_1(x) + (1-p)P_2(x)) \\ & + V(\left\lceil g(p,\tau,0)\frac{1}{\Delta_p}\right\rceil \Delta_p, z-\tau)\Big(p(1-F_1(\tau) + (1-p)(1-F_2(\tau))\Big) \end{aligned}$$

    end for
    $V(p,z) \leftarrow \min_\tau\{\tilde{V}(p,z,\tau)\}$

    $\tau^*(p,z) \leftarrow \operatorname{argmin}_\tau\{\tilde{V}(p,z,\tau)\}$

  end for
  $z = z + 1$

end for
  return $V(p,z), \tau^*(p,z), \; \forall (p,z)$

2.B. Proofs

Proof of Lemma 2.1

(i) Since $X_2 \geq_{lr} X_1$ by Assumption 2.1, it is known that $\mathbb{P}\{X_2 = x\}/\mathbb{P}\{X_1 = x\}$ is non-decreasing in x. Thus, it follows that

$$\mathbb{P}\{X_2 = x\} \geq \mathbb{P}\{X_1 = x\}\frac{\mathbb{P}\{X_2 = t\}}{\mathbb{P}\{X_1 = t\}}$$

for $x \geq t$. Consequently, it can be written that

$$\begin{aligned} \frac{\mathbb{P}\{X_2 = t\}}{\sum_{i=t+1}^{\infty} \mathbb{P}\{X_2 = i\}} \quad & \leq \quad \frac{\mathbb{P}\{X_2 = t\}}{\sum_{i=t+1}^{\infty} \mathbb{P}\{X_1 = i\}\frac{\mathbb{P}\{X_2 = t\}}{\mathbb{P}\{X_1 = t\}}} \\ & = \quad \frac{\mathbb{P}\{X_1 = t\}}{\sum_{i=t+1}^{\infty} \mathbb{P}\{X_1 = i\}} \end{aligned}$$

for any $t \in \mathbb{N}$. Since $\sum_{i=t+1}^{\infty} \mathbb{P}\{X_j = i\} = \mathbb{P}(X_j > t)$ for $j \in \{1,2\}$, the result follows.

(ii) By using the arguments in the proof of Lemma 2.1(i), we first note that the inequality

$$\mathbb{P}\{X_1 = x\}\mathbb{P}\{X_2 \geq x\} \geq \mathbb{P}\{X_2 = x\}\mathbb{P}\{X_1 \geq x\} \tag{2.12}$$

also holds for all $x \in \mathbb{N}$. By using conditional probabilities, it can be written that

$$\begin{aligned}
\mathbb{P}(X_2 \geq x) &= \prod_{i=0}^{x-2} \mathbb{P}(X_2 \geq x-i | X_2 \geq x-i-1) \\
&= \prod_{i=0}^{x-2} \left(1 - \frac{\mathbb{P}(X_2 = x-i-1)}{\mathbb{P}(X_2 \geq x-i-1)}\right) \\
&\geq \prod_{i=0}^{x-2} \left(1 - \frac{\mathbb{P}(X_1 = x-i-1)}{\mathbb{P}(X_1 \geq x-i-1)}\right) \\
&= \mathbb{P}(X_1 \geq x),
\end{aligned}$$

for $x \in \{2,3,\ldots\}$, where the inequality follows from (2.12). The result is equivalent to $\mathbb{P}(X_2 > x) \geq \mathbb{P}(X_1 > x)$ for all $x \in \mathbb{N}$.

(iii) Notice that Lemma 2.1(ii) implies the usual stochastic dominance. Thus, the claim follows from Shaked and Shanthikumar (2007), p.4, 1.A.7. □

Proof of Lemma 2.2

(i) For $d = 0$, in order to prove that $g(p,t,0)$ is a non-decreasing function of p, we need to show that the inequality

$$\frac{p'[1-F_1(t)]}{p'[1-F_1(t)] + (1-p')[1-F_2(t)]} \leq \frac{p''[1-F_1(t)]}{p''[1-F_1(t)] + (1-p'')[1-F_2(t)]} \tag{2.13}$$

holds for any $0 \leq p' < p'' \leq 1$ and at a given $t \in \mathbb{N}$. We first note that the denominators in (2.13) could be zero if $F_2(t) = F_1(t) = 1$, but that contradicts with ending a cycle with preventive maintenance after t time steps (i.e., to have $d = 0$). So, both denominators of (2.13) are well-defined. Since the terms $1 - F_1(t)$ and $1 - F_2(t)$ are always non-negative, it is easy to verify that the inequality (2.13) is equivalent to $p' \leq p''$, which always holds because it is known that $p' < p''$. So, $g(p,t,0)$ is a non-decreasing function of p.

For $d = 1$, in order to prove that $g(p, t, 1)$ is a non-decreasing function of p we need to show that the inequality

$$\frac{p' P_1(t)}{p' P_1(t) + (1 - p') P_2(t)} \leq \frac{p'' P_1(t)}{p'' P_1(t) + (1 - p'') P_2(t)} \tag{2.14}$$

holds for any $0 \leq p' < p'' \leq 1$ and at a given $t \in \mathbb{N}$. Similar to our observation above, the denominators in (2.14) could be zero if $P_1(t) = P_2(t) = 0$, but that contradicts with ending a cycle with corrective maintenance after t time steps (i.e., to have $d = 1$). So, both denominators in (2.14) are well-defined. Since the terms $P_1(t)$ and $P_2(t)$ are always non-negative, the inequality (2.14) is equivalent to $p' \leq p''$, which holds because $p' < p''$. So, $g(p, t, 1)$ is also a non-decreasing function of p.

(ii) We first note that $g(0, t, 1) = \frac{0 P_1(t)}{0 P_1(t) + 1 P_2(t)} = 0$ and $g(1, t, 1) = \frac{1 P_1(t)}{1 P_1(t) + 0 P_2(t)} = 1$ for all $t \in \mathbb{N}$. Thus, the result trivially holds for $p = 0$ and $p = 1$. For $p \in (0, 1)$, we rewrite $g(p, t, 1)$ as

$$g(p, t, 1) = \frac{p P_1(t)}{p P_1(t) + (1 - p) P_2(t)} = \frac{\frac{p P_1(t)}{p P_1(t)}}{\frac{p P_1(t)}{p P_1(t)} + \frac{(1-p) P_2(t)}{p P_1(t)}} = \frac{1}{1 + \frac{(1-p)}{p} \frac{P_2(t)}{P_1(t)}}.$$

Since $X_2 \geq_{lr} X_1$ by Assumption 2.1, $P_2(t)/P_1(t)$ is a non-decreasing function of t. Therefore, $g(p, t, 1)$ is a non-increasing function of t at any $p \in (0, 1)$.

(iii) For all $t \in \mathbb{N}$, notice that $g(0, t, 1)$ and $g(0, t, 0)$ are both equal to zero, and $g(1, t, 1)$ and $g(1, t, 0)$ are both equal to one. Thus, the result holds for $p = 0$ and $p = 1$. For $p \in (0, 1)$, the result also follows by noting that, for all $t \in \mathbb{N}$,

$$\begin{aligned} g(p, t, 1) &= \frac{1}{1 + \frac{(1-p)}{p} \frac{P_2(t)}{P_1(t)}} \\ &\geq \frac{1}{1 + \frac{(1-p)}{p} \frac{1 - F_2(t)}{1 - F_1(t)}} = g(p, t, 0), \end{aligned}$$

where the inequality holds because $P_1(t)(1 - F_2(t)) \geq P_2(t)(1 - F_1(t))$, $\forall t$, by Lemma 2.1(i). □

Proof of Lemma 2.3

(i) For $z = 0$, it is known from (2.5) that $C(p, 0, \tau) = 0$ for any p and τ. For $z > 0$, it also follows from (2.5) that the function $C(p,z,\tau)$ is a weighted average of the functions $c_1(\tau,z)$ and $c_2(\tau,z)$. For a given value of $p \in [0,1]$ and $\tau \in \{1,2,\ldots,L\}$, notice from (2.3) that the functions $c_1(\tau,z)$ and $c_2(\tau,z)$ are defined at the values of $z \in \{\tau,\tau+1,\ldots,L\}$. For $z = \tau$, $c_1(\tau,z) = F_1(\tau)C_f$. For $z > \tau$, $c_1(\tau,z) = F_1(\tau)C_f + (1 - F_1(\tau))C_p$. Since $(1 - F_1(\tau))C_p \geq 0$, $c_1(\tau,z)$ is a non-decreasing function in z. Similarly, it can be shown that $c_2(\tau,z)$ is a non-decreasing function in z. The result follows because a weighted average of non-decreasing functions is also non-decreasing.

(ii) For $z = 0$, it is known from (2.5) that $C(p,0,\tau) = 0$ for any p at a given τ. So, the result trivially holds. Suppose $z > 0$. For a fixed $z > 0$ and τ, taking the partial derivative of $C(p,z,\tau)$ with respect to p leads to

$$\frac{\partial C(p,z,\tau)}{\partial p} = \begin{cases} [C_f - C_p][F_1(\tau) - F_2(\tau)] & \text{if } \tau < z, \\ C_f[F_1(\tau) - F_2(\tau)] & \text{if } \tau = z. \end{cases}$$

Since $F_1(\tau) \geq F_2(\tau)$ for all τ (Lemma 2.1(ii)) and $C_f > C_p$, it always holds that $\partial C(p,z,\tau)/\partial p$ is constant and non-negative, and hence, the result follows. □

Proof of Theorem 2.1

(i) We use induction as the proof technique. We first note that $V^0(p,z) = 0, \forall(p,z)$. Thus, for $l = 0$, it is correct that the function $V^0(p,z)$ is non-decreasing in z for any given value of p. Next, as the induction hypothesis, for $l = k > 0$, we assume that $V^k(p,z)$ is a non-decreasing function of z for any p. Under this assumption, our objective is to show that for $l = k+1$, $V^{k+1}(p,z) = \min_{\tau\in\{1,2,\ldots,z\}} \tilde{V}^{k+1}(p,z,\tau)$ is also a non-decreasing function in z for any p.

We already know from Lemma 2.3(i) that $C(p,z,\tau)$ is a non-decreasing function in z. It follows from the induction hypothesis that the function $V^k(g(p,x,d(x)),z-x)$ is non-decreasing in z because the term $z - x$ is non-decreasing in z. Similarly, $V^k(g(p,\tau,d(x)),z-\tau)$ is also a non-decreasing function in z. Since the sum of non-decreasing functions is non-decreasing, it follows that the function $\tilde{V}^{k+1}(p,z,\tau)$ is non-decreasing in z for any p at a fixed τ. The minimum of non-decreasing functions is also non-decreasing. Therefore, $V^{k+1}(p,z)$ is also a non-decreasing function in z. This concludes the proof.

(ii) The proof consists of two parts. We first show the monotonicity and then the concavity.

Monotonicity with respect to p:

Similar to part (i), we use induction as the proof technique to show that $V^l(p,z)$ is a non-decreasing function of p. Since $V^0(p,z) = 0, \forall(p,z)$, it is correct for $l = 0$ that the function $V^l(p,z)$ is non-decreasing in p for any given value of z. Next, as the induction hypothesis, we assume for $l = k > 0$ that $V^k(p,z)$ is a non-decreasing function in p for any z. Under this assumption, our objective is to show that $V^{k+1}(p,z) = \min_{\tau \in \{1,2,\ldots,z\}} \tilde{V}^{k+1}(p,z,\tau)$ is also a non-decreasing function in p for any z. By reorganizing (2.8), we obtain

$$\begin{aligned}\tilde{V}^{k+1}(p,z,\tau) \quad = \quad & C(p,z,\tau) + \sum_{x=1}^{\infty} V^k(g(p, \min\{x,\tau\}, d(x)), z - \min\{x,\tau\}) \\ & (pP_1(x) + (1-p)P_2(x)),\end{aligned} \tag{2.15}$$

where $d(x) = 1$ for $1 \leq x \leq \tau$, and $d(x) = 0$ for $x > \tau$. We already know from Lemma 2.3(ii) that $C(p,z,\tau)$ is a non-decreasing function in p. Next, we will show that $\sum_{x=1}^{\infty} V^k(g(p, \min\{x,\tau\}, d(x)), z - \min\{x,\tau\})(pP_1(x) + (1-p)P_2(x))$ is also a non-decreasing function in p. This is equivalent to showing that

$$\begin{aligned}&\sum_{x=1}^{\infty} V^k(g(p', \min\{x,\tau\}, d(x)), z - \min\{x,\tau\})(p'P_1(x) + (1-p')P_2(x)) \\ &\quad \leq \sum_{x=1}^{\infty} V^k(g(p'', \min\{x,\tau\}, d(x)), z - \min\{x,\tau\})(p''P_1(x) + (1-p'')P_2(x))\end{aligned} \tag{2.16}$$

for any p', p'', where $0 \leq p' < p'' \leq 1$.

It follows from the induction hypothesis that $V^k(p,z)$ is a non-decreasing function in p for any z. Also, by Lemma 2.2(i), $g(p,t,d(x))$ is a non-decreasing function in p for any t. Therefore, $V^k(g(p, \min\{x,\tau\}, d(x)), z - \min\{x,\tau\})$ is a non-decreasing function in p. Consequently, it follows that

$$V^k(g(p', \min\{x,\tau\}, d(x)), z - \min\{x,\tau\}) \leq V^k(g(p'', \min\{x,\tau\}, d(x)), z - \min\{x,\tau\}) \tag{2.17}$$

for any $0 \leq p' < p'' \leq 1$ and $x \in \mathbb{N}$. Inequality (2.17) implies that the inequality

$$\sum_{x=1}^{\infty} V^k(g(p', \min\{x,\tau\}, d(x)), z - \min\{x,\tau\})(p'P_1(x) + (1-p')P_2(x))$$
$$\leq \sum_{x=1}^{\infty} V^k(g(p'', \min\{x,\tau\}, d(x)), z - \min\{x,\tau\})(p'P_1(x) + (1-p')P_2(x)) \tag{2.18}$$

holds for any $0 \leq p' < p'' \leq 1$. As the next step, we note that

$$p'F_1(x) + (1-p')F_2(x) \leq p''F_1(x) + (1-p'')F_2(x)$$

for all $x \in \mathbb{N}$ at any $0 \leq p' < p'' \leq 1$ because we already know from Lemma 2.1(ii) that $F_1(x) \geq F_2(x)$ for all $x \in \mathbb{N}$. That is, the random variable with pmf $p'P_1(\cdot) + (1-p')P_2(\cdot)$ is larger than the random variable with pmf $p''P_1(\cdot) + (1-p'')P_2(\cdot)$ in the sense of usual stochastic dominance (Shaked and Shanthikumar, 2007). So, if $V^k(g(p, \min\{x,\tau\}, d(x)), z - \min\{x,\tau\})$ is non-increasing in x for any p, then by Lemma 2.1(iii), it holds that

$$\sum_{x=1}^{\infty} V^k(g(p'', \min\{x,\tau\}, d(x)), z - \min\{x,\tau\})(p'P_1(x) + (1-p')P_2(x))$$
$$\leq \sum_{x=1}^{\infty} V^k(g(p'', \min\{x,\tau\}, d(x)), z - \min\{x,\tau\})(p''P_1(x) + (1-p'')P_2(x)), \tag{2.19}$$

and the inequality (2.16) holds from combining inequalities (2.18) and (2.19) (and hence the function $\tilde{V}^{k+1}(p,z,\tau)$ in (2.15) is non-decreasing in p). Therefore, in the remainder of the proof, we focus on showing that $V^k(g(p, \min\{x,\tau\}, d(x)), z - \min\{x,\tau\})$ is non-increasing in x for any p.

For a fixed τ, we first note that the function $g(p, \min\{x,\tau\}, d(x))$ is non-increasing in x. This can be verified as follows: If $x \leq \tau$, then $d = 1$ and we know $g(p,x,1)$ is non-increasing in x from Lemma 2.2(ii). If $x > \tau$, on the other hand, then $d = 0$, and $g(p,\tau,0)$ is constant. Since we know from Lemma 2.2(iii) that $g(p,\tau,0) \leq g(p,\tau,1)$ for any τ, it can be concluded that $g(p, \min\{x,\tau\}, d(x))$ is non-increasing in x. From Theorem 2.1(i), we know that $V^k(p,z)$ is a non-decreasing function in z for a fixed p. As a result, $V^k(g(p, \min\{x,\tau\}, d(x)), z - \min\{x,\tau\})$ is non-increasing in x for any p

since both the functions $g(p, \min\{x, \tau\}, d(x))$ and $z - \min\{x, \tau\}$ are non-increasing in x.

Since $\tilde{V}^{k+1}(p, z, \tau)$ is non-decreasing in p, $V^{k+1}(p, z)$ is also non-decreasing in p because the minimum of non-decreasing functions is also non-decreasing. This concludes the proof.

Concavity with respect to p:

We use induction as the proof technique to show $V^l(p, z)$ is a concave function of p. Since $V^0(p, z) = 0$ for all p at a given z, it is correct for $l = 0$ that the function $V^l(p, z)$ is concave in p. As the induction hypothesis, we assume that $V^l(p, z)$ is concave in p for $l = k > 0$. We need to show that $V^{k+1}(p, z)$ is concave in p. Since the minimum of concave functions is concave, we only need to prove that $\tilde{V}^{k+1}(p, z, \tau)$ is concave. To be specific, according to Equation (2.8), if we show that $C(p, z, \tau)$, $\sum_{x=1}^{\tau} V^k(g(p, x, 1), z - x)(pP_1(x) + (1 - p)P_2(x))$ and $V^k(g(p, \tau, 0), z - \tau)\big(p(1 - F_1(\tau)) + (1 - p)(1 - F_2(\tau))\big)$ are concave, then the concavity of $\tilde{V}^{k+1}(p, z, \tau)$ holds.

To start with, we note that $C(p, z, \tau)$ is linear in p (see Lemma 2.3(ii)), so it is concave.

Next, we want to show that $V^k(g(p, x, 1), z - x)(pP_1(x) + (1 - p)P_2(x))$ is concave in p for any $x \leq \tau$. Note that

$$g(p, x, 1) = \frac{pP_1(x)}{pP_1(x) + (1 - p)P_2(x)}.$$

So we can write for $0 \leq p' < p'' \leq 1$ and $\alpha \in [0, 1]$ that

$$\begin{aligned}
&g(\alpha p' + (1-\alpha)p'', x, 1) \\
&= \frac{(\alpha p' + (1-\alpha)p'')\, P_1(x)}{(\alpha p' + (1-\alpha)p'')P_1(x) + (1 - \alpha p' - (1-\alpha)p'')P_2(x)} \\
&= \alpha \frac{p' P_1(x)}{(\alpha p' + (1-\alpha)p'')P_1(x) + (1 - \alpha p' - (1-\alpha)p'')P_2(x)} \\
&\quad + (1-\alpha) \frac{p'' P_1(x)}{(\alpha p' + (1-\alpha)p'')P_1(x) + (1 - \alpha p' - (1-\alpha)p'')P_2(x)} \\
&= \frac{\alpha\, (p' P_1(x) + (1-p')P_2(x))}{\alpha\, (p' P_1(x) + (1-p')P_2(x)) + (1-\alpha)(p'' P_1(x) + (1-p'')P_2(x))} g(p', x, 1) \\
&\quad + \frac{(1-\alpha)\, (p'' P_1(x) + (1-p'')P_2(x))}{\alpha\, (p' P_1(x) + (1-p')P_2(x)) + (1-\alpha)(p'' P_1(x) + (1-p'')P_2(x))} g(p'', x, 1)
\end{aligned} \tag{2.20}$$

Furthermore, it can be written that

$$
\begin{aligned}
& V^k(g(\alpha p' + (1-\alpha)p'', x, 1), z - x) \qquad (2.21)\\
& \quad \cdot \big((\alpha p' + (1-\alpha)p'')P_1(x) + (1 - \alpha p' - (1-\alpha)p'')P_2(x)\big)\\
&= V^k\Bigg(\frac{\alpha\,(p'P_1(x) + (1-p')P_2(x))\,g(p',x,1)}{\alpha(p'P_1(x) + (1-p')P_2(x)) + (1-\alpha)(p''P_1(x) + (1-p'')P_2(x))} \qquad (2.22)\\
& \quad + \frac{(1-\alpha)\,(p''P_1(x) + (1-p'')P_2(x))\,g(p'',x,1)}{\alpha(p'P_1(x) + (1-p')P_2(x)) + (1-\alpha)(p''P_1(x) + (1-p'')P_2(x))}, z - x\Bigg)\\
& \quad \cdot \big((\alpha p' + (1-\alpha)p'')P_1(x) + (1 - \alpha p' - (1-\alpha)p'')P_2(x)\big),
\end{aligned}
$$

where the equality in (2.22) follows from replacing $g(\alpha p' + (1-\alpha)p'', x, 1)$ in (2.21) with the right-hand side of Equation (2.20). Let

$$
\tilde{\alpha} \triangleq \frac{\alpha\,(p'P_1(x) + (1-p')P_2(x))}{\alpha(p'P_1(x) + (1-p')P_2(x)) + (1-\alpha)(p''P_1(x) + (1-p'')P_2(x))}.
$$

Also notice that

$$
\begin{aligned}
& (\alpha p' + (1-\alpha)p'')P_1(x) + (1 - \alpha p' - (1-\alpha)p'')P_2(x)\\
& \qquad = \alpha(p'P_1(x) + (1-p')P_2(x)) + (1-\alpha)(p''P_1(x) + (1-p'')P_2(x)).
\end{aligned}
$$

Then, the right hand side of (2.22) can be rewritten as

$$
\begin{aligned}
& V^k\left(\tilde{\alpha} \cdot g(p',x,1) + (1-\tilde{\alpha}) \cdot g(p'',x,1), z - x\right) \qquad (2.23)\\
& \quad \cdot \left(\alpha(p'P_1(x) + (1-p')P_2(x)) + (1-\alpha)(p''P_1(x) + (1-p'')P_2(x))\right).
\end{aligned}
$$

It is known from the induction hypothesis that $V^k(p,z)$ is concave in p. Thus, it holds that

$$
\begin{aligned}
& V^k\left(\tilde{\alpha} \cdot g(p',x,1) + (1-\tilde{\alpha}) \cdot g(p'',x,1), z - x\right)\\
& \geq \tilde{\alpha} \cdot V^k\left(g(p',x,1), z - x\right) + (1-\tilde{\alpha}) \cdot V^k\left(g(p'',x,1), z - x\right)
\end{aligned}
$$

Since $\alpha(p'P_1(x) + (1-p')P_2(x)) + (1-\alpha)(p''P_1(x) + (1-p'')P_2(x)) \geq 0$, it also holds that

$$
\begin{aligned}
& V^k\left(\tilde{\alpha} \cdot g(p',x,1) + (1-\tilde{\alpha}) \cdot g(p'',x,1), z - x\right) \qquad (2.24)\\
& \quad \cdot \left(\alpha(p'P_1(x) + (1-p')P_2(x)) + (1-\alpha)(p''P_1(x) + (1-p'')P_2(x))\right)
\end{aligned}
$$

$$
\begin{aligned}
&\geq \Big(\tilde{\alpha} \cdot V^k\left(g(p',x,1),z-x\right) + (1-\tilde{\alpha}) \cdot V^k\left(g(p'',x,1),z-x\right)\Big) \\
&\qquad \cdot \left(\alpha(p'P_1(x) + (1-p')P_2(x)) + (1-\alpha)(p''P_1(x) + (1-p'')P_2(x))\right). \\
&= \alpha\left(p'P_1(x) + (1-p')P_2(x)\right) V^k\left(g(p',x,1),z-x\right) \\
&\qquad + (1-\alpha)\left(p''P_1(x) + (1-p'')P_2(x)\right) V^k\left(g(p'',x,1),z-x\right).
\end{aligned}
$$

By combining (2.21), (2.23), and (2.24), we obtain that

$$
\begin{aligned}
&V^k(g(\alpha p' + (1-\alpha)p'',x,1),z-x) \\
&\qquad \cdot \left((\alpha p' + (1-\alpha)p'')P_1(x) + (1-\alpha p' - (1-\alpha)p'')P_2(x)\right) \\
&\geq \alpha\left(p'P_1(x) + (1-p')P_2(x)\right) V^k\left(g(p',x,1),z-x\right) \\
&\qquad + (1-\alpha)\left(p''P_1(x) + (1-p'')P_2(x)\right) V^k\left(g(p'',x,1),z-x\right).
\end{aligned}
$$

This means that the function $V^k(g(p,x,1),z-x)(pP_1(x) + (1-p)P_2(x))$ is concave in p. Since the sum of concave functions is concave, the function

$$
\sum_{x=1}^{\tau} V^k(g(p,x,1),z-x)(pP_1(x) + (1-p)P_2(x))
$$

is also concave in p for any $\tau \leq z$. Finally, we need to show that $V^k(g(p,\tau,0),z-\tau)\big(p(1-F_1(\tau)) + (1-p)(1-F_2(\tau))\big)$ is concave in p. The proof is similar to the previous one. Note that

$$
g(p,\tau,0) = \frac{p(1-F_1(\tau))}{p(1-F_1(\tau)) + (1-p)(1-F_2(\tau))}
$$

So we can write for $0 \leq p' < p'' \leq 1$ and $\alpha \in [0,1]$,

$$
\begin{aligned}
&g(\alpha p' + (1-\alpha)p'',\tau,0) \\
&= \frac{\alpha g(p',\tau,0)\,(p'(1-F_1(\tau)) + (1-p')(1-F_2(\tau)))}{\alpha(p'(1-F_1(\tau)) + (1-p')(1-F_2(\tau)))) + (1-\alpha)(p''(1-F_1(\tau)) + (1-p'')(1-F_2(\tau)))} \\
&\quad + \frac{(1-\alpha)g(p'',\tau,0)\,(p''(1-F_1(\tau)) + (1-p'')(1-F_2(\tau)))}{\alpha(p'(1-F_1(\tau)) + (1-p')(1-F_2(\tau))) + (1-\alpha)(p''(1-F_1(\tau)) + (1-p'')(1-F_2(\tau)))}
\end{aligned}
\tag{2.25}
$$

By Equation (2.25) and from the concavity of $V^k(p,z)$ in p, the following inequality holds:

$$
\begin{aligned}
&V^k(g(\alpha p' + (1-\alpha)p'',\tau,0),z-\tau) \cdot \left((\alpha p' + (1-\alpha)p'')(1-F_1(\tau)) + (1-\alpha p' - (1-\alpha)p'')(1-F_2(\tau))\right) \\
&= V^k(g(\alpha p' + (1-\alpha)p'',\tau,0),z-\tau) \\
&\quad \cdot \left(\alpha(p'(1-F_1(\tau)) + (1-p')(1-F_2(\tau))) + (1-\alpha)(p''(1-F_1(\tau)) + (1-p'')(1-F_2(\tau)))\right)
\end{aligned}
$$

$$= V^k\Bigg(\frac{g(p',\tau,0)\alpha(p'(1-F_1(\tau))+(1-p')(1-F_2(\tau)))}{\alpha(p'(1-F_1(\tau))+(1-p')(1-F_2(\tau))))+(1-\alpha)(p''(1-F_1(\tau))+(1-p'')(1-F_2(\tau)))}$$

$$+\frac{g(p'',\tau,0))(1-\alpha)(p''(1-F_1(\tau))+(1-p'')(1-F_2(\tau)))}{\alpha(p'(1-F_1(\tau))+(1-p')(1-F_2(\tau)))+(1-\alpha)(p''(1-F_1(\tau))+(1-p'')(1-F_2(\tau)))},z-\tau\Bigg)$$

$$\cdot\big(\alpha(p'(1-F_1(\tau))+(1-p')(1-F_2(\tau)))+(1-\alpha)(p''(1-F_1(\tau))+(1-p'')(1-F_2(\tau)))\big)$$

$$\geq \alpha V^k(g(p',\tau,0),z-\tau)\,(p'(1-F_1(\tau))+(1-p')(1-F_2(\tau))$$

$$+(1-\alpha)V^k(g(p'',\tau,0),z-\tau)\,(p''(1-F_1(\tau))+(1-p'')(1-F_2(\tau)))$$

This proves $V^k(g(p,\tau,0),z-\tau)\big(p(1-F_1(\tau))+(1-p)(1-F_2(\tau))\big)$ is concave in p for a fixed τ. Since sum of the concave functions is concave, $V^{k+1}(p,z)$ is concave in p. This concludes the proof. □

Proof of Lemma 2.4

We use induction as the proof technique. Let $j \in \{1,2\}$. We first note that $W_j^0(z) = 0, \forall z$. Therefore, for $l = 0$, $W_j^l(z) = 0$ is non-decreasing in z. Next, for $l = k > 0$, we assume that $W_j^k(z)$ is a non-decreasing function of z. Under this assumption, we aim to show that for $l = k+1$, $W_j^{k+1}(z) = \min_{\tau\in\{1,2,\ldots,z\}} \tilde{W}_j^{k+1}(z,\tau)$ is also non-decreasing function of z, where $\tilde{W}_j^{k+1}(z,\tau) = (c_j(z,\tau) + \sum_{x=1}^{\tau} P_j(x)W_j^l(z-x) + (1-F_j(\tau))W_j^l(z-\tau))$.

From the proof of Lemma 2.3, we know that $c_j(z,\tau)$ is a non-decreasing function of z for a fixed τ. $W_j^k(z-x)$ and $W_j^k(z-\tau)$ are non-decreasing functions of z under the assumption that $W_j^k(z)$ is a non-decreasing function in z because $z-x$ and $z-\tau$ are non-decreasing functions in z. Therefore $W_j^k(z-x)$ and $W_j^k(z-\tau)$ are non-decreasing functions in z. $\tilde{W}_j^{k+1}(z,\tau)$ is a non-decreasing function in z for a fixed τ because sum of non-decreasing functions are non-decreasing. The minimum of non-decreasing functions is also a non-decreasing function. Thus, $W_j^{k+1}(z)$ is a non-decreasing function in z. This concludes the proof. □

Proof of Theorem 2.2

We first note that $V(0,z) = W_2(z)$ for all $z \in \{0,1,\ldots,L\}$. This is because if $p = 0$ then the updated belief variable $g(p,t,d) = 0$ for all t and d. That is, if the belief variable is equal to 0, it always stays at 0. On the other hand, if $p = 1$ then the updated belief variable $g(p,t,d) = 1$ for all t and d. That is, if the belief variable is equal to 1, it always stays at 1. This implies that $V(1,z) = W_1(z)$ for all $z \in$

$\{0,1,\ldots,L\}$. Consequently, it holds that

$$\begin{aligned} V(p,z) &\geq pV(1,z) + (1-p)V(0,z) \\ &= pW_1(z) + (1-p)W_2(z) \\ &= W(p,z) \end{aligned}$$

for all $p \in [0,1]$ and $z \in \{0,1,\ldots,L\}$, where the inequality follows from the concavity of the value function $V(p,z)$ in p at a fixed z. □

2.C. Alternative POMDP formulation

We describe the alternative POMDP formulation in this appendix. In this formulation we consider the beginning of each time period as a decision epoch with two possible actions (i.e., 'replace' or 'do nothing').

Decision epochs: A decision is made at the beginning of each time period. As introduced earlier, we let $z \in \{0,1,2,\ldots,L\}$ denote the number of remaining time periods in the planning horizon.

States: The state of the system is described by (p,y), where $p \in [0,1]$ denotes the current belief that the component population is of the weak type and $y \in \{0,1,...,L\}$ denotes the age of the component in the system. Note that the age of the component can be as low as 0 (i.e., the component is new) and as high as L (i.e., the component which is installed at the beginning of the lifespan has survived the entire lifespan of the system).

Actions: At each decision epoch and at any state, there are two possible actions: 'do a preventive replacement' and 'do nothing'. These actions are denoted with $a = 1$ and $a = 0$, respectively.

State Transitions & Rewards: Suppose the do-nothing action is taken (i.e., $a = 0$). The system starts the time period with the existing component, and there are two possibilities: either the component fails (denoted with $d = 0$) or it stays in the working condition (denoted with $d = 1$). In case of a failure, which occurs with probability

$$P(X_j \leq (y+1)|X_j > y) = \frac{P(X_j \leq y+1, X_j > y)}{P(X_j > y)} = \frac{F_j(y+1) - F_j(y)}{1 - F_j(y)} = \frac{P(X_j = y+1)}{1 - F_j(y)}$$

for population type j, the component is replaced correctively at cost C_f and the next period starts with a new component at age 0. If there is no failure, which occurs with probability

$$P(X_j > (y+1)|X_j > y) = \frac{1 - F_j(y+1)}{1 - F_j(y)}$$

for population type j, the age of the component increases by one.

Next, suppose that the replace action is taken (i.e., $a = 1$). The replacement is immediate at cost C_p, and the system starts the current time period with a new component at age zero. Similar to the case with the no-replacement action, the component either fails (i.e., denoted with $d = 0$) or it stays in the working condition (i.e., denoted with $d = 1$) in that period. The failure occurs with probability $P(X_j = 1)$ for population type j, the component is replaced correctively at cost C_f and the next period starts with a new component at age 0. On the other hand, if the component does not fail and, the age of the component increases by one with probability $P(X_j > 1)$ for population type j.

At the end of each period (i.e., after the realization of d), Bayes' rule can be used to update the belief variable p. Specifically, the updated belief, which we denote by the function $g(p, y, d)$, is given by

$$p \leftarrow g(p, y, d) = \begin{cases} \frac{pP(X_1 \leq (y+1)|X_1 > y)}{pP(X_1 \leq (y+1)|X_1 > y) + (1-p)P(X_2 \leq (y+1)|X_2 > y)} & \text{if } d = 1, \\ \frac{pP(X_1 > (y+1)|X_1 > y)}{pP(X_1 > (y+1)|X_1 > y) + (1-p)P(X_2 > (y+1)|X_2 > y)} & \text{if } d = 0. \end{cases} \tag{2.26}$$

By using the notation for the cdf and pmf of the random variables X_1 and X_2 introduced in the main text, the updated belief in (2.26) can be rewritten as

$$g(p, y, d) = \begin{cases} \frac{p \frac{P_1(y+1)}{1-F_1(y)}}{p \frac{P_1(y+1)}{1-F_1(y)} + (1-p) \frac{P_2(y+1)}{1-F_2(y)}} & \text{if } d = 1, \\ \frac{p \frac{1-F_1(y+1)}{1-F_1(y)}}{p \frac{1-F_1(y+1)}{1-F_1(y)} + (1-p) \frac{1-F_2(y+1)}{1-F_2(y)}} & \text{if } d = 0. \end{cases}$$

The state variable y (i.e., the age of the component) is updated as follows depending on the action a and the realized event d:

$$y \leftarrow \begin{cases} y+1 & \text{if } a = 0 \text{ and } d = 0, \\ 0 & \text{if } a = 0 \text{ and } d = 1, \\ 1 & \text{if } a = 1 \text{ and } d = 0, \\ 0 & \text{if } a = 1 \text{ and } d = 1. \end{cases}$$

Bellman Optimality Equations: Let $V_z(p, y)$ denote the optimal cost until the end of the planning horizon when there are z time periods remaining in the planning horizon and the current state is (p, y). We assume that $V_0(p, y) = 0$ for all $p \in [0, 1]$ and $y \in \{0, 1, ..., L\}$.

The Bellman optimality equations are given by

$$V_z(p, y) = \min_{a \in \{0,1\}} \tilde{V}_z(p, y, a) \tag{2.27}$$

for all $p \in [0, 1]$, $y \in \{0, 1, \ldots, L\}$ and $z \in \{1, \ldots, L\}$, where

$$\begin{aligned} \tilde{V}_z(p, y, 0) &= (C_f + V_{z-1}(g(p, y, 1), 0))\Big(p\frac{P_1(y+1)}{1 - F_1(y)} + (1-p)\frac{P_2(y+1)}{1 - F_2(y)})\Big) \\ &+ V_{z-1}(g(p, y, 0), y+1)(\Big(p\frac{1 - F_1(y+1)}{1 - F_1(y)} + (1-p)\frac{1 - F_2(y+1)}{1 - F_2(y)}\Big))\Big) \end{aligned}$$

and

$$\begin{aligned} \tilde{V}_z(p, y, 1) &= C_p + (C_f + V_{z-1}(g(p, 0, 1), 0))\Big(pP_1(1) + (1-p)P_2(1))\Big) \\ &+ V_{z-1}(g(p, 0, 0), 1)(p(1 - F_1(1)) + (1-p)(1 - F_2(1)). \end{aligned}$$

Example 2.2 Now, we apply the alternative POMDP formulation to Example 1. We obtain:

$$V_1(p, 0) = min\{\tilde{V}_1(p, 0, 0), \tilde{V}_1(p, 0, 1)\} = \{1 + 4p, C_p + \tilde{V}_1(p, 0, 0)\} = 1 + 4p.$$

Please notice that this function $V_1(p,0)$ is equal to the function $V(p,1)$ of the original formulation. Next, we obtain:

$$\begin{aligned} V_1(p,1) &= \min\{\tilde{V}_1(p,1,0), \tilde{V}_1(p,1,1)\} = \min\{1\frac{1}{9} + 8\frac{8}{9}p, 2+4p\} \\ &= \begin{cases} 1\frac{1}{9} + 8\frac{8}{9}p & \text{for } 0 \le p \le \frac{2}{11}, \\ 2+4p & \text{for } \frac{2}{11} \le p \le 1. \end{cases} \end{aligned}$$

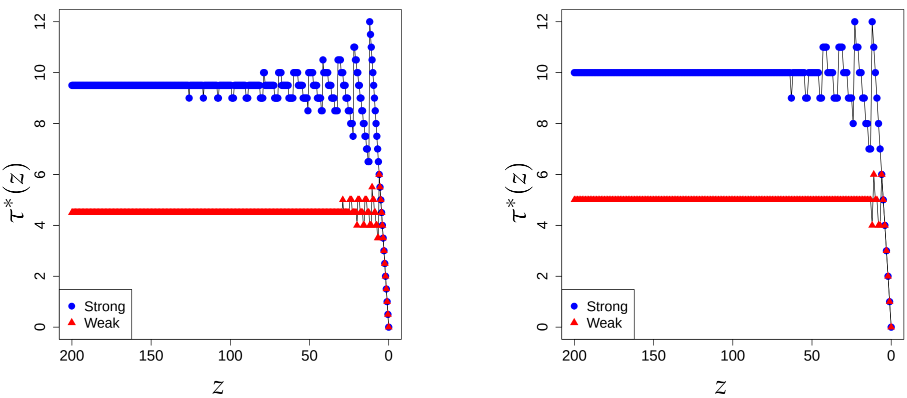
$$\begin{aligned} \tilde{V}_2(p,0,1) &= C_p + \tilde{V}_2(p,0,0) > \tilde{V}_2(p,0,0). \\ V_2(p,0) &= \min\{\tilde{V}_2(p,0,0), \tilde{V}_2(p,0,1)\} = \tilde{V}_2(p,0,0) \\ &= \begin{cases} 2\frac{1}{10} + 10\frac{2}{5}p & \text{for } 0 \le p \le \frac{2}{7}, \\ 2\frac{9}{10} + 7\frac{3}{5} & \text{for } \frac{2}{7} \le p \le 1. \end{cases} \end{aligned}$$

The function $V_2(p,0)$ is equal to V(p,2) of the formulation in Section 2.3, which shows that the total optimal cost remains the same under the alternative POMDP formulation.

2.D. Effect of smaller time steps on the structure of the optimal policy

In this appendix, we show the effect of the discretization level of time on the structure of the optimal policy, the perfect information case (see also Remark 2.3). We illustrate how the optimal actions compare under two distinct discretization levels for the same continuous Weibull distribution. We compare the optimal actions at discretization level $\Delta_t = 1$ with the optimal actions at discretization level $\Delta_t = 0.5$ (see Figures 2.8 and 2.9). If we select a smaller time step, we keep the length of the time horizon expressed in time units at the same value. For discretization level $\Delta_t = 0.5$, we say that the length of one period is equal to 0.5 time units. Therefore, the set for z becomes $\{0, 0.5, 1, ..., L\}$. We also redefine the pmf of the discretized Weibull distribution as follows: $P(x, \lambda, k) = F_{\hat{X}}(x) - F_{\hat{X}}(x - 0.5)$ for $x \in \{0.5, 1, 1.5, \dots\}$.

Please observe that the jumps in the optimal actions for $\Delta_t = 0.5$ are smaller than for $\Delta_t = 1$ at the majority of the time steps. However, the large jumps that we see in Figures D.8(b) and D.9(b) remain equal in size (when the optimal actions are expressed in time units). The last jump before the end of the horizon is obtained because until a given time point, you plan for a last preventive replacement, while after that time point, you plan for no further preventive replacement. The second but last jump before the end of the horizon is obtained because until a given time point, you plan for two preventive replacements in the rest of the time horizon, while after that point, you plan for one last preventive replacement; and so on. If the time step Δ_t would be further reduced, we can expect that these large jumps stay coming back in the pattern of the optimal actions.

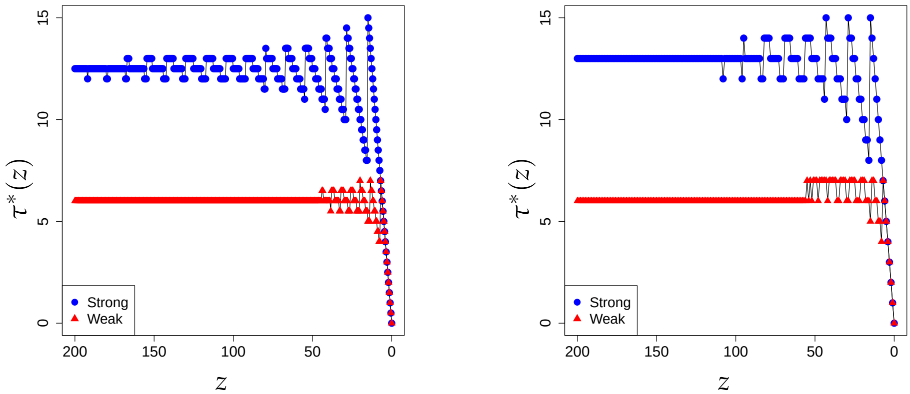


(a) $\Delta_t = 0.5$ *(b)* $\Delta_t = 1$

Figure 2.8: Optimal actions under perfect information for $C_p = 0.1$, $C_f = 1$, $k = 5$ and $L = 200$.

2.E. A comparison of infinite and finite horizon approaches for heuristics

In this appendix, we show the experiment results mentioned in Remark 2.4. We introduced $V_{\mathrm{MP}}(p_1, L)$ and $V_{\mathrm{TP}}(p_1, L)$ as the finite-horizon (i.e., L-period) costs under MP and TP, respectively, obtained under the implicit assumption that the planning horizon is infinite. By explicitly considering the remaining lifespan of the system, we adapt the range for τ such that $\tau \in \{0, ..., z\}$. We denote the finite-



(a) $\Delta_t = 0.5$

(b) $\Delta_t = 1$

Figure 2.9: Optimal actions under perfect information for $C_p = 0.1$, $C_f = 1$, $k = 10$ and $L = 200$.

horizon (i.e., L-period) costs under these finite-horizon adjusted formulations of the myopic and threshold policies with $V_{\mathrm{MP_L}}(p_1, L)$ and $V_{\mathrm{TP_L}}(p_1, L)$, respectively. In Table 3, we report $\Delta^{\mathrm{rel}}_{\mathrm{MP}}$ and $\Delta^{\mathrm{rel}}_{\mathrm{TP}}$. Now, we additionally report the values "$\Delta^{\mathrm{rel}}_{\mathrm{MP_L}}$" and "$\Delta^{\mathrm{rel}}_{\mathrm{TP_L}}$" next to them in Table 2.5 (please note that $\Delta^{\mathrm{rel}}_{\mathrm{MP_L}} = \frac{V_{\mathrm{MP_L}}(p_1,L)-V(p_1,L)}{V(p_1,L)}$ and $\Delta^{\mathrm{rel}}_{\mathrm{TP_L}} = \frac{V_{\mathrm{TP_L}}(p_1,L)-V(p_1,L)}{V(p_1,L)}$). Comparing the values of "$\Delta^{\mathrm{rel}}_{\mathrm{MP_L}}$" with "$\Delta^{\mathrm{rel}}_{\mathrm{MP}}$" shows the effect of incorporating the end of the planning horizon in the myopic policy. Similarly, comparing the values of "$\Delta^{\mathrm{rel}}_{\mathrm{TP_L}}$" with "$\Delta^{\mathrm{rel}}_{\mathrm{TP}}$" shows the effect of incorporating the end of the planning horizon in the threshold policy.

As it can seen from Table 2.5, only for instances 7, 16 and 17, these differences are larger than 1%, and for all the other instances, they are below 1%. This shows that incorporating the end-of-horizon effect in the heuristics indeed affects the performance of the heuristics, but our analysis also shows that this effect is limited for most of the instances.

Table 2.5: The effect of incorporating the end of the planning horizon for MP and TP

No.	L	C_p	k	p_0	$\Delta^{\mathrm{rel}}_{\mathrm{MP_L}}$	$\Delta^{\mathrm{rel}}_{\mathrm{MP}}$	$\Delta^{\mathrm{rel}}_{\mathrm{TP_L}}$	$\Delta^{\mathrm{rel}}_{\mathrm{TP}}$
1	100	0.05	5	0.25	0.5%	0.6%	0.5%	0.6%
2	100	0.05	5	0.5	0.1%	0.2%	0.1%	0.2%
3	100	0.05	5	0.75	0.2%	0.3%	0.2%	0.3%
4	100	0.1	5	0.25	3.5%	3.7%	1.7%	2.1%
5	100	0.1	5	0.5	0.7%	0.8%	0.7%	0.8%
6	100	0.1	5	0.75	0.1%	0.3%	0.1%	0.3%
7	100	0.2	5	0.25	2.4%	5.5%	0.8%	2.3%
8	100	0.2	5	0.5	2.7%	3.0%	1.2%	1.4%
9	100	0.2	5	0.75	1.1%	1.6%	1.1%	1.2%
10	100	0.05	10	0.25	3.3%	2.7%	3.2%	2.7%
11	100	0.05	10	0.5	1.0%	1.0%	1.0%	1.0%
12	100	0.05	10	0.75	0.3%	0.3%	0.3%	0.3%
13	100	0.1	10	0.25	17.0%	17.1%	1.1%	1.5%
14	100	0.1	10	0.5	3.3%	2.5%	3.3%	2.5%
15	100	0.1	10	0.75	2.7%	2.9%	2.7%	2.9%
16	100	0.2	10	0.25	22.0%	22.4%	2.4%	5.8%
17	100	0.2	10	0.5	10.9%	14.8%	1.4%	3.1%
18	100	0.2	10	0.75	3.1%	3.5%	3.1%	3.5%
19	200	0.05	5	0.25	3.2%	3.3%	1.3%	1.4%
20	200	0.05	5	0.5	0.4%	0.4%	0.4%	0.4%
21	200	0.05	5	0.75	1.0%	1.1%	1.0%	1.1%
22	200	0.1	5	0.25	3.2%	4.1%	1.1%	1.4%
23	200	0.1	5	0.5	2.5%	3.4%	1.6%	1.7%
24	200	0.1	5	0.75	0.4%	0.6%	0.4%	0.6%
25	200	0.2	5	0.25	2.0%	2.8%	0.9%	1.8%
26	200	0.2	5	0.5	1.8%	2.0%	0.8%	1.0%
27	200	0.2	5	0.75	2.1%	2.1%	1.6%	1.7%
28	200	0.05	10	0.25	14.6%	14.6%	2.0%	2.0%
29	200	0.05	10	0.5	1.6%	1.7%	1.6%	1.7%
30	200	0.05	10	0.75	1.0%	1.0%	1.0%	1.0%
31	200	0.1	10	0.25	23.5%	23.6%	3.0%	3.1%
32	200	0.1	10	0.5	7.0%	7.1%	2.1%	2.2%
33	200	0.1	10	0.75	0.6%	0.8%	0.6%	0.8%
34	200	0.2	10	0.25	11.2%	11.4%	3.2%	3.4%
35	200	0.2	10	0.5	15.2%	15.4%	2.0%	2.4%
36	200	0.2	10	0.75	3.3%	3.3%	1.4%	1.9%

3

Data pooling and joint optimization for multiple systems

3.1. Introduction

In this chapter, we consider a similar problem setting as in Chapter 2. In Chapter 2, we focus on only one system whereas, in this chapter, we consider multiple newly designed single-component systems with a fixed lifespan. We approach this problem from the perspective of a service provider of multiple systems at different locations. MRI machines located in different hospitals or milking robots located at different farms are examples of such systems. These systems are largely identical. Hence, the service provider can pool the data from all systems to address the data scarcity problem at the beginning of systems' lifespan and optimize the replacement decisions jointly for these systems.

The components in all systems are subject to the same random failure mechanism. The aim is to preventively replace a component before it fails in order to prevent the costlier corrective maintenance upon failure. We build a discrete-time partially observable Markov decision process (POMDP) model to find the optimal replacement policy for a critical component that occurs in multiple identical systems with the objective of minimizing the total cost during the whole lifespan. Similarly

Table 3.1: Literature review on (partially observable) Markov decision models under failure model uncertainty.

Source of Failure model uncertainty	No. of systems	Failure model: Time-to-failure	Failure model: Degradation
Parameter uncertainty	Single	Drent et al. (2020)	
	Multiple	van Staden et al. (2022)	Drent (2022) (p.139-155)
Population heterogeneity	Single	Chapter 2	van Oosterom et al. (2017) Abdul-Malak et al. (2019)
	Multiple	**This paper**	

to Chapter 2, this is a sequential learning problem for which we apply a solution approach (POMDP) that balances the trade-off between *exploration* and *exploitation* optimally.

We position our research in the area of *optimal* learning with (partially observable) Markov decision process models under failure model uncertainty. To the best of our knowledge, the number of studies in this area is limited; see van Oosterom et al. (2017), Abdul-Malak et al. (2019), Drent et al. (2020), Chapter 2, van Staden et al. (2022), Drent (2022) (p.139-155). A classification of these studies is provided in Table 3.1. Here, we classify the papers first according to the source of failure model uncertainty, which can be either parameter uncertainty or population heterogeneity. Then, we classify the papers according to the number of systems that they consider, which can be either a single system or multiple systems. Finally, we classify the papers based on the type of failure models. By a "time-to-failure" failure model, we mean that there are only two degradation states (i.e., good-as-new and failed) where the time until failure can follow any probability distribution. On the other hand, a "degradation" failure model refers to a situation with more than two degradation states (i.e., not just good-as-new and failed but also intermediate states such as defective) and a Markov chain modeling the transition from one degradation state to the next.

As stated above, the work in this chapter also considers population heterogeneity under the assumption of a time-to-failure model but extends the work of Chapter 2 to multiple machines. Among the works that consider parameter uncertainty, Drent et al. (2020) also focus on an age-based replacement policy for a single system and follow a Bayesian approach to learn an unknown parameter of a specific form of a probability distribution for the lifetime of components. Similar to our work, Drent (2022) (p.139-155) apply data pooling from multiple systems in a Bayesian way to resolve the failure model uncertainty. However, the source of uncertainty in their failure model is parameter uncertainty. By focusing on the case with parameter uncertainty and a time-to-failure model, van Staden et al. (2022) minimize the expected maintenance costs of a system over its finite contract period and prescribe scheduled preventive maintenance interventions based on a frequentist 'first predict' then 'optimize' approach.

Another related study is Deprez et al. (2022). This paper is not positioned in Table 3.1 because its analysis is not based on optimal policies. However, Deprez et al. (2022) is related to our work in its objective to build a myopic-type data-driven maintenance policy for multiple systems by using historical data coming from different systems. Deprez et al. (2022) aim to optimize the number of preventive maintenance interventions for a machine during a finite time horizon. Different from our work, they assume system heterogeneity (i.e. data is pooled from machines operating in non-identical conditions) and adopt a frequentist estimation approach. In our research, we resolve failure model uncertainty with Bayesian updating and focus on the optimal policy by building a POMDP model.

In this research, we investigate the potential benefit of learning the true population type from multiple similar systems at the same time. We define three policies in order to quantify this benefit and generate further managerial insights. Specifically, Policy I is the optimal policy for multiple systems with data pooling. Policy II applies the policy which is known to be optimal in a single-system setting with pooling the data from multiple systems (i.e. updating the belief regarding to the true population type by using all the data coming from multiple systems). Policy III also uses the optimal policy for a single-system setting but without data pooling (in this case, we use the POMDP formulation as described in Remark 2.1 and obtain the optimal policy for a single system from that formulation).

We address the following research questions: (1) How does the structure of the optimal policy for a multi-system setting with data pooling (Policy I) look like and how does it differ from the optimal single-system policy (Policy III)? (2) What is the benefit of jointly optimizing the replacement policies of all systems with data pooling instead of separately optimizing the replacement policy of each system without data pooling (i.e. comparing Policy I and Policy III)? (3) How much of this benefit is due to data pooling (i.e. comparing Policy II and III) and how much of this reduction is due to jointly optimizing the actions for multiple systems (i.e. comparing Policy I and II)? (4) What is the effect of the number of systems, the length of the lifespan, the coefficient of variation of the time-to-failure distribution, the cost of maintenance activities, and the initial belief on the costs?

To the best to our knowledge, our research is the first to study how the true population type can be learned optimally in a setting with multiple systems and population heterogeneity. This research aims to provide insights on the potential cost benefits of adopting technologies that enable data pooling. In our numerical experiments, we show that cost reduction due to data pooling is up to 5.6% for two systems and up to 14.8% for 20 systems. The majority of the cost reduction is due to data pooling, and a relatively small part is due to optimizing the preventive replacement decisions for multiple systems jointly. As the number of systems that pool the data increases, the cost of maintenance converges to the cost of maintenance under perfect information about the true population type. Also, the reduction in the cost per system as a function of the number of systems is higher for low numbers of systems and becomes smaller as the number of systems increases.

The remainder of this chapter is organized as follows. In Section 3.2 and Section 3.3, the problem description and the mathematical formulation of the POMDP model are provided, respectively. Section 3.4 presents the benchmark policies that will be compared against the optimal policy of the POMDP model. Section 3.5 provides structural results for a special case and presents insights based on this special ase. Section 3.6 presents the results and insights from our computational experiments. Finally, Section 3.7 concludes the chapter.

3.2. Problem description

We consider n single-component systems. Let $i \in \{1, \ldots, n\}$ denote the index of a system. It is known that the systems are taken out of service at the same time, and we refer to the time until that moment as the lifespan of the systems. The time horizon of the problem is set equal to the lifespan of the systems, and we let this time horizon consist of discrete time periods of equal length. Without loss of generality, we scale time such that the length of each time period is one time unit.

The length of the time horizon is expressed in the number of time periods and is equal to $L \in \mathbb{N}$, where $\mathbb{N}$ is the set of positive integers. Each system has a critical component which fails randomly and independent of the components in the other systems. If a failure occurs during the x-th time period after the installation of a new component, then the component is replaced at the end of that time period and we say the lifetime of the component is x. We let X denote the corresponding discrete random variable for the lifetime of the component. At the beginning of each time period, an action is taken for each system which is either to replace the component preventively with a cost C_p or to do nothing with no cost. If a component fails before reaching the next time period, then it is correctively replaced at cost C_f. If a component is replaced preventively at the beginning of a period, the system starts that period with a new component. It holds that $C_p < C_f$ because the cost of corrective replacement includes the costs associated with a breakdown in addition to the costs related to a replacement. When a system reaches the end of the lifespan, the maintenance activities are terminated and no cost occurs at this moment or later because all systems go out of service at that moment.

We assume that there are two populations for the components: a weak and a strong population. The components always come from the same population. However, the true population type is unknown. We let p denote the belief that the components belong to a weak population (i.e., the probability that the components always come from the weak population) and $\hat{p}$ denotes the initial belief at the beginning of the lifespan.

Let X_j denote the lifetime random variable for component type j, where $j = 1$ refers to the weak type and $j = 2$ refers to the strong type. Therefore, under the belief variable p, the lifetime random variable X satisfies

$$X = \begin{cases} X_1 & \text{with probability } p, \\ X_2 & \text{with probability } 1 - p. \end{cases}$$

For each population, the time-to-failure distribution is assumed to be known. The objective of the decision maker is to determine the optimal replacement policy that minimizes the expected total cost over the time horizon.

3.3. Mathematical formulation

In this section, we provide a mathematical formulation of the problem described in Section 3.2.

Decision epochs: A decision is made at the beginning of each time period. We let $z \in \{0, 1, 2, \ldots, L\}$ denote the number of time periods remaining in the time horizon. Note that $z = 0$ corresponds to the end of the time horizon (i.e., the moment the final time period is completed).

States: At each decision epoch, the state is described as follows: $p \in [0, 1]$ is the current belief that the component population is of the weak type, and $y_i \in \{0, 1, \ldots\}$ denotes the age of the component in system i. Let $\mathbf{y}$ represent the age vector for the components of all the systems: $\mathbf{y} = (y_1, \ldots, y_n)$. The state of the model is represented by $(p, \mathbf{y})$.

Actions: At each decision epoch and at any state, there are two possible actions for each system: 'do a preventive replacement' and 'do nothing'. These actions are denoted with $a_i = 1$ and $a_i = 0$, respectively, for system i. Let $\mathbf{a}$ denote the action vector $(a_1, \ldots, a_n)$, representing the actions for all the systems. Let $\mathcal{A}$ denote the set of all possible action vectors. (Please note that Chapter 2 formulate the replacement decisions as a scheduling problem; i.e., the actions describe when to perform the preventive replacement of a component at the moment that a new component starts its operation in the system. This way of describing the actions would lead to a much larger action space than the alternative way used in this chapter.)

State Transitions & Rewards: Suppose that the do-nothing action is taken for system i (i.e., $a_i = 0$). The system starts the time period with the existing component, and there are two possibilities: either the component fails (denoted with $d_i = 1$) or it stays in the working condition (denoted with $d_i = 0$). The components in all systems are subject to the same time-to-failure distribution, but failures occur independently. Time-to-failure is represented by the random variable $X^{(i)}$ for system i. In case of a failure, which occurs with probability

$$P(X_j^{(i)} \leq y_i + 1 | X_j^{(i)} > y_i) = \frac{P(X_j^{(i)} \leq y_i + 1, X_j^{(i)} > y_i)}{P(X_j^{(i)} > y_i)} = \frac{P(X_j^{(i)} = y_i + 1)}{P(X_j^{(i)} > y_i)}$$

for population type j, the component installed in system i is replaced correctively at cost C_f and the next period starts with a new component at age 0. If there is no failure, which occurs with probability

$$P(X_j^{(i)} > y_i + 1 | X_j^{(i)} > y_i) = \frac{P(X_j^{(i)} > y_i + 1)}{P(X_j^{(i)} > y_i)}$$

for population type j, the age of the component in system i increases by one.

Next, suppose that the replace action is taken for system i (i.e., $a_i = 1$). The replacement is immediate at cost C_p, and the system starts the current time period with a new component at age zero. Similar to the case with the do-nothing action, the component either fails (denoted with $d_i = 0$) or it stays in the working condition (denoted with $d_i = 1$) in that period. If the component in system i fails, which occurs with probability $P(X_j^{(i)} = 1)$ for population type j, the component is replaced correctively at cost C_f and the next period starts with a new component at age 0. On the other hand, if the component does not fail, which occurs with probability $P(X_j^{(i)} > 1)$ for population type j, the age of the component in system i increases by one.

The updated age vector $\hat{\mathbf{y}} = (\hat{y}_1, \dots, \hat{y}_n)$ under action $\mathbf{a}$ will be $\hat{\mathbf{y}} = \mathbf{y} \circ (\mathbf{e} - \mathbf{a})$ with '$\circ$' denoting the element-wise multiplication (Hadamard product) of two vectors and $\mathbf{e}$ denoting a vector with n elements where all elements are equal to 1. We let $\mathbf{d} = (d_1, \dots, d_n)$ denote the vector of observations for all the systems. Note that there are 2^n possible distinct realizations of $\mathbf{d}$. Let $D = \{\mathbf{d}^1, \dots, \mathbf{d}^{2^n}\}$ denote the set of all these possible realizations. The probability that observation $\mathbf{d}$ occurs when the state is $(p, \hat{\mathbf{y}})$ is denoted by $F(p, \hat{\mathbf{y}}, \mathbf{d})$:

$$F(p, \hat{\mathbf{y}}, \mathbf{d}) = p l_1(\hat{\mathbf{y}}, \mathbf{d}) + (1 - p) l_2(\hat{\mathbf{y}}, \mathbf{d}) \tag{3.1}$$

where the expression $l_j(\hat{\mathbf{y}}, \mathbf{d})$, which represents the likelihood of observing $\mathbf{d}$ at the age vector $\hat{\mathbf{y}}$ for population type j, can be calculated as

$$l_j(\hat{\mathbf{y}}, \mathbf{d}) = \prod_{i=1}^{n} P(X_j^{(i)} \leq \hat{y}_i + 1 | X_j^{(i)} > \hat{y}_i)^{d_i} P(X_j^{(i)} > \hat{y}_i + 1 | X_j^{(i)} > \hat{y}_i)^{1-d_i}$$

for $j \in \{1, 2\}$. At the end of each period (i.e., after the realization of $\mathbf{d}$), Bayes' rule can be used to update the belief variable p. Specifically, the updated belief variable, which we denote as the function $g(p, \hat{\mathbf{y}}, \mathbf{d})$, is given by

$$g(p, \hat{\mathbf{y}}, \mathbf{d}) = \frac{pl_1(\hat{\mathbf{y}}, \mathbf{d})}{pl_1(\hat{\mathbf{y}}, \mathbf{d}) + (1-p)l_2(\hat{\mathbf{y}}, \mathbf{d})}.$$

Finally, the age vector $\hat{\mathbf{y}}$ is updated to $(\hat{\mathbf{y}} + \mathbf{e}) \circ \mathbf{d}$.

Bellman Optimality Equations: Let $V_z(p, \mathbf{y})$ denote the optimal cost until the end of the time horizon with z time periods remaining in the time horizon given that the current state is $(p, \mathbf{y})$. It holds that $V_0(p, \mathbf{y}) = 0$ for all $p \in [0, 1]$ and $y_i \in \{0, 1, \dots\}$ with $i \in \{1, \dots, n\}$.

The Bellman optimality equations are given by

$$V_z(p, \mathbf{y}) = \min_{\mathbf{a} \in \mathcal{A}} \tilde{V}_z(p, \mathbf{y}, \mathbf{a}) \tag{3.2}$$

for all $z \in \{1, ..., L\}$, where

$$\begin{aligned} \tilde{V}_z(p, \mathbf{y}, \mathbf{a}) \quad = \quad & \sum_{i=1}^{n} C_p a_i \qquad (3.3) \\ + \quad & \sum_{m=1}^{|D|} \Big(\sum_{i=1}^{n} C_f d_i^m + V_{z-1}\Big(g(p, \mathbf{y} \circ (\mathbf{e} - \mathbf{a}), \mathbf{d}^m), ((\mathbf{y} \circ (\mathbf{e} - \mathbf{a})) + \mathbf{e}) \circ \mathbf{d}^m\Big)\Big) \\ & \quad \times F(p, \mathbf{y} \circ (\mathbf{e} - \mathbf{a}), \mathbf{d}^m) \end{aligned}$$

for all $p \in [0, 1]$ and $y_i \in \{0, 1, \dots\}$ with $i \in \{1, \dots, n\}$. In the rest of the chapter, the function $V_z(p, \mathbf{y})$ in Equation (3.2) is also referred to as the value function. We refer to the resulting optimal policy for this multi-system setting with data pooling (i.e., observations collected from all systems are used to update the belief state) as Policy I. We denote the actions under this policy by $\mathbf{a}_{\mathrm{I}}^*(p, \mathbf{y}, z) = (a_{\mathrm{I}}^*(p, y_1, z), \dots, a_{\mathrm{I}}^*(p, y_n, z))$ for a given $(p, \mathbf{y}, z)$. The algorithm to solve the Bellman equations is provided in Appendix 3.B.

Remark 3.1 Please notice that we assumed two populations in this section and Section 3.2. It would also be possible to assume K distinct populations that our components might be coming from. Then the state description becomes $(\mathbf{p}, \mathbf{y})$, where $\mathbf{p} = (p_1, \dots, p_K)$ and $\sum_{k=1}^{K} p_k = 1$. That is, the belief state becomes a vector instead of a single parameter (i.e., each point on the vector corresponds to a particular belief of having a certain population type). The Bayesian update function then also returns a vector:

$$\mathbf{g}(\mathbf{p}, \hat{\mathbf{y}}, \mathbf{d}) = \left(\frac{p_1 l_1(\hat{\mathbf{y}}, \mathbf{d})}{\sum_{k=1}^{K} p_k l_k(\hat{\mathbf{y}}, \mathbf{d})}, \ldots, \frac{p_K l_K(\hat{\mathbf{y}}, \mathbf{d})}{\sum_{k=1}^{K} p_k l_k(\hat{\mathbf{y}}, \mathbf{d})} \right).$$

3.4. Benchmark policies

In this section, we provide the benchmark policies to compare against Policy I and each other in order to address the research questions introduced in Section 3.1. We start in Section 3.4.1 with the optimal policy when the true population type is certainly known. Section 3.4.2 and Section 3.4.3 describe the benchmark policies obtained by using the optimal policy of a single system without and with data pooling, respectively.

3.4.1 Perfect information setting

We consider a setting where the true population type is assumed to be known, referred to as the perfect information setting. The difference between the cost obtained under the perfect information setting and the optimal cost under Policy I constitutes a base cost for the maximum amount the decision maker is willing to pay to resolve the population heterogeneity. We formulate an MDP model to determine the optimal policy in this setting. Note that the belief state is not needed anymore since the true population type is known, and only the ages of the components in the systems are considered as the state variables. Let $W_z^{(j)}(\mathbf{y})$ denote the optimal cost until the end of the time horizon with z time periods remaining in the time horizon given that the current state is $\mathbf{y}$ for population type j. We present Bellman optimality equations for a single system because an optimal policy for each system can be calculated independently, and therefore, it holds that $W_z^{(j)}(\mathbf{y}) = \sum_{i=1}^{n} W_z^{(j)}(y_i)$. The Bellman optimality equations for a single system are given by

$$W_z^{(j)}(y) = \min_{a \in \{0,1\}} \tilde{W}_z^{(j)}(y, a),$$

where

$$\begin{aligned}
\tilde{W}_z^{(j)}(y,0) &= (C_f + W_{z-1}(0))P(X_j \leq y+1 | X_j > y) \\
&+ W_{z-1}(y+1)P(X_j > y+1 | X > y), \\
\tilde{W}_z^{(j)}(y,1) &= C_p + (C_f + W_{z-1}(0))P(X \leq 1) + W_{z-1}(1)P(X > 1),
\end{aligned}$$

and $W_0^{(j)}(y) = 0$ for $z \in \{1, ..., L\}$ and $y \in \{0, 1, ...\}$.

We introduce a function $W_L(\hat{p}, \mathbf{y})$ defined as

$$W_L(\hat{p}, \mathbf{y}) = \hat{p} W_L^{(1)}(\mathbf{y}) + (1 - \hat{p}) W_L^{(2)}(\mathbf{y}) \tag{3.4}$$

for $\hat{p} \in [0, 1]$, for all $y_i \in \{0, 1, ...\}$ where $i \in \{1, \ldots, n\}$. The function $W_L(\hat{p}, \mathbf{y})$ can be interpreted as follows. Suppose that the true component type is unknown and an initial belief $\hat{p}$ is available at the beginning of the lifespan as in the original problem. However, different from the original problem, suppose that the true component type is immediately revealed to the decision maker just before the lifespan starts, and systems are operated in their entire lifespan by following the optimal replacement policy corresponding to that true population type. The function $W_L(\hat{p}, \mathbf{y})$ represents the expected cost under this scenario just before the true population type is revealed to the decision maker. Consequently, the function $V_L(\hat{p}, \mathbf{y}) - W_L(\hat{p}, \mathbf{y})$ can be interpreted as the expected benefit of resolving the uncertainty in the true population type (or the cost of not knowing the true population type) at the beginning of the lifespan under the initial belief $\hat{p}$ and age $\mathbf{y}$.

3.4.2 Single-system optimal policy without data pooling (Policy III)

For convenience in presentation, we introduce the single-system optimal policy without data pooling (Policy III) before introducing the single-system optimal policy with data pooling (Policy II) as the optimal actions of Policy III will be used by Policy II.

For Policy III, we assume that the true population type is learned without data pooling. This means that each system is considered as isolated from others so that the replacement decision of the component in a system is independent of the other systems and the belief on the true population type is updated by only using the data collected from that particular system. Thus, each system can be analyzed separately and we formulate the value function for a single system. The age of the component in the system is y, the action space is $\{0,1\}$ and the set of observation vectors reduces to $\{0,1\}$. We define the Bayesian update function for this policy as follows:

$$\tilde{g}(p,y,d) = \begin{cases} \dfrac{pP(X_1 \leq y+1|X_1 > y)}{pP(X_1 \leq y+1|X_1 > y) + (1-p)P(X_2 \leq y+1|X_2 > y)} & \text{if } d = 1, \\ \dfrac{pP(X_1 > y+1|X_1 > y)}{pP(X_1 > y+1|X_1 > y) + (1-p)P(X_2 > y+1|X_2 > y)} & \text{if } d = 0. \end{cases}$$

The Bellman optimality equations are given by

$$V_z^{\text{III}}(p,y) = \min_{a \in \{0,1\}} \tilde{V}_z^{\text{III}}(p,y,a)$$

for all $z \in \{1,...,L\}$, where

$$\begin{aligned} \tilde{V}_z^{\text{III}}(p,y,0) &= (C_f + V_{z-1}^{\text{III}}(\tilde{g}(p,y,1),0)) \\ &\quad \times \Big(pP(X_1 \leq y+1|X_1 > y) + (1-p)P(X_2 \leq y+1|X_2 > y)\Big) \\ &\quad + V_{z-1}^{\text{III}}(\tilde{g}(p,y,0),y+1) \\ &\quad \times \Big(pP(X_1 > y+1|X_1 > y) + (1-p)P(X_2 > y+1|X_2 > y)\Big), \\ \tilde{V}_z^{\text{III}}(p,y,1) &= C_p + (C_f + V_{z-1}^{\text{III}}(\tilde{g}(p,0,1),0))\Big(pP(X_1 \leq 1) + (1-p)P(X_2 \leq 1)\Big) \\ &\quad + V_{z-1}^{\text{III}}(\tilde{g}(p,0,0),1)\Big(pP(X_1 > 1) + (1-p)P(X_2 > 1)\Big) \end{aligned}$$

and $V_0^{\text{III}}(p,y) = 0$ for all $p \in [0,1]$ and $y \in \{0,1,\dots\}$. The optimal policy that is obtained via these Bellman equations is referred to as Policy III. We denote the optimal action under Policy III for state (p,y) and the remaining number of time periods z with $a^*_{\text{III}}(p,y,z)$. We also denote $\mathbf{a}^*_{\text{III}}(p,\mathbf{y},z) = (a^*_{\text{III}}(p,y_1,z),\dots,a^*_{\text{III}}(p,y_n,z))$ as the optimal policy vector for n systems for a given $(p,\mathbf{y})$ under Policy III.

3.4.3 Single-system optimal policy with data pooling (Policy II)

In this section, we describe our benchmark policy (called Policy II) that is obtained by applying the optimal action of Policy III in each system independently from other systems, while updating the belief variable by using the data collected from all the systems. We let $\mathbf{a}_{\mathrm{II}}(p, \mathbf{y}, z)$ denote the action taken by Policy II at state $(p, \mathbf{y})$ and remaining number of time periods z. Thus, it follows that

$$\mathbf{a}_{\mathrm{II}}(p, \mathbf{y}, z) = \mathbf{a}^*_{\mathrm{III}}(p, \mathbf{y}, z) = (a^*_{\mathrm{III}}(p, y_1, z), \ldots, a^*_{\mathrm{III}}(p, y_n, z)).$$

The value function under this policy is given as

$$V^{\mathrm{II}}_z(p, \mathbf{y}) = \tilde{V}^{\mathrm{II}}_z(p, \mathbf{y}, \mathbf{a}_{\mathrm{II}}(p, \mathbf{y}, z))$$

for all $z \in \{1, ..., L\}$, where

$$\begin{aligned}
\tilde{V}^{\mathrm{II}}_z(p, \mathbf{y}, \mathbf{a}_{\mathrm{II}}(p, \mathbf{y}, z)) \quad = \quad & \sum_{i=1}^{n} C_p a^*_{\mathrm{III}}(p, y_i, z) + \sum_{m=1}^{|D|} \Big[\sum_{i=1}^{n} C_f d_i^m \\
+ \quad & V^{\mathrm{II}}_{z-1}\Big(g(p, \mathbf{y} \circ (\mathbf{e} - \mathbf{a}_{\mathrm{II}}(p, \mathbf{y}, z)), \mathbf{d}^m), \\
& (\mathbf{y} \circ (\mathbf{e} - \mathbf{a}_{\mathrm{II}}(p, \mathbf{y}, z)) + \mathbf{e}) \circ \mathbf{d}^m \Big) \Big] \\
\times \quad & F(p, \mathbf{y} \circ (\mathbf{e} - \mathbf{a}_{\mathrm{II}}(p, \mathbf{y}, z)), \mathbf{d}^m)
\end{aligned}$$

and $V^{\mathrm{II}}_0(p, \mathbf{y}) = 0$ for all $p \in [0, 1]$ and $y_i \in \{0, 1, \ldots\}$ for $i \in \{1, \ldots, n\}$. Notice that Policy II follows exactly the same actions as Policy III, however, it uses the data collected from all the systems in a period to update the belief variable at the end of that period.

3.5. Structural results for a special case with deterministic lifetimes

In this section, we introduce a special case with a deterministic lifetime distribution for each population. For this special case, we can derive analytical results because in this setting the population type is learned perfectly after one 'do nothing' action for a component with an age that is one time unit less than the deterministic lifetime of a weak component (in that case, you find out in the upcoming time period whether you have a weak component or a strong component). We limit the number of scenarios such that at most two replacements would be needed for the weak and one replacement would be needed for a strong component type. The proofs regarding the propositions in this section can be found in Appendix 3.A.

Assumption 3.1 We assume we have n (> 1) systems. A component that comes from the weak population fails at age t (≥ 4) and a component from the strong population fails at age $2t$. We let $2t < L < 3t - 3$. The systems are newly installed, therefore, $\mathbf{y} = \mathbf{0}$ at the beginning of time horizon. Finally, we assume that $\hat{p} = 0.5$ and $2C_p < C_f$.

We introduce a new policy to examine the effect of joint optimization on the total expected cost.

Definition 3.1 (Perfect learning policy) We define a 'perfect learning policy' as follows: when the initial components reach age $t - 1$ (i.e. at $z = L - (t - 1)$), we apply the 'do nothing' action for one of the systems and we apply the 'do a preventive replacement' action for all other systems.

By this policy, we learn the true population perfectly at $z = L - t$ and we limit the risk of failure (i.e. high cost of corrective maintenance) of multiple systems to only one system (e.g. risking failure for only one milking machine). After that, the optimal policy is applied.

Proposition 3.1 *The total expected cost under the perfect learning policy for n systems is* $[\frac{3}{2}n - \frac{1}{2}]C_p + \frac{1}{2}C_f$.

We see that the preventive cost part ($[\frac{3}{2}n - \frac{1}{2}]C_p$) increases linearly with the number of systems, but the corrective cost part ($\frac{1}{2}C_f$) is a constant function of n and it is shared among all systems. From $z = L - t$ till the end of the lifespan, the optimal policy can be followed for either the weak or the strong population.

Proposition 3.2 *The total expected cost under Policy III for n systems is* $2nC_p$.

The total expected cost under Policy III increases with the number of systems.

Proposition 3.3 *It is never optimal to apply the 'do nothing' action to more than one system when the initial components reach age $t-1$.*

Proposition 3.3 shows that risking a failure is not desirable for more than one system (e.g. risking the failure of two milking machines) under this special case.

Proposition 3.4 *(i) The relative difference of Policy III with respect to the perfect learning policy is $\frac{(n+1)C_p - C_f}{(3n-1)C_p + C_f}$. (ii) If $(n+1) > \frac{C_f}{C_p}$, then Policy I is the same as the perfect learning policy. Otherwise, Policy I is the same as the Policy III.*

Proposition 3.4 shows that the perfect learning policy is cheaper than Policy III if and only if $(n+1) > \frac{C_f}{C_p}$. This implies that exploration is only beneficial for large enough values of n. As n goes to infinity, this relative difference goes to $\frac{1}{3}$. For very large values of n, the relative cost increase, when using the non-optimal Policy III instead of the optimal Policy I (the perfect learning policy in that case), can become $\frac{1}{3}$.

3.6. Numerical results

In this section, we present our numerical study to address the research questions introduced in Section 3.1. First, we determine a base instance which we introduce in Section 3.6.1. We address research question (1) in Section 3.6.2 by showing the structure of Policy I and compare it against the structure of Policy III. in Section 3.6.3, we compare the total expected cost per system under each policy for a test bed of 36 instances in order to answer research questions (2) and (3). Finally, we answer research question (4) in Section 3.6.4 by executing a sensitivity analysis to study how the total expected cost per system and the relative cost difference between policies change with respect to the input parameters L, k, $\hat{p}$, C_p and n.

3.6.1 Base instance

For our numerical analysis, we assume that the lifetime random variable of each population type has a right-truncated discrete Weibull distribution. This distribution is derived from the well-known continuous Weibull distribution with scale parameter λ and shape parameter k by first truncating it at a value u and then by defining the probability mass function $P(X = x)$ as $P(X \geq x) - P(X \geq x - 1)$ for $x \in \{1, 2, \ldots, u\}$, where $P(x \geq x) = \frac{1-\exp[-(x/\lambda)^k]}{1-\exp[-(u/\lambda)^k]}$. We assume that the shape parameter is the same for both population types but the scale parameters are different. The scale parameter λ is equal to 6 for the weak type and it is equal to 12 for the strong type. For the shape parameter, we select $k = 5$. The value of u is chosen equal to 22; it holds that $\exp[-(u/\lambda)^k]$ becomes negligible for $u > 22$ under all the selected values of λ and k. The mean and coefficient of variation of the lifetime of the weak type are equal to 6.009 and 0.215, respectively. For the strong type, these values are equal to 11.518 and 0.221. This completes the description of the lifetime distributions for the base instance. We set the other input parameters as $n = 2$, $L = 75$, $C_p = 0.1$, $C_f = 1$ and $\hat{p} = 0.5$ in the base instance. Notice that for C_p and C_f, only their relative difference matters. Hence, we can choose $C_f = 1$ during all experiments w.l.o.g.

3.6.2 The structure of Policy I and its comparison against Policy III

In this section, we answer the first research question by investigating the structure of the optimal policy (Policy I), and comparing this against the structure of Policy III. Let $\mathbf{a}^*_{\mathrm{I}}(p, \mathbf{y}, z)$ denote the optimal action under Policy I at state variables p and $\mathbf{y}$ and remaining lifespan z.

In Figure 3.1, we see how the optimal policy structure changes with respect to p and y_1 for some fixed values of y_2. The optimal policy for the first system is not affected by the age of the second system for most values of p and y_1 (the same behavior has also been observed for all other values of z). As p increases, the optimal action may become 'replace' because the time-to-failure is stochastically smaller for the weak population. Similarly, as y_1 increases, the optimal action becomes 'replace'. This is because of the increasing failure rate of the time-to-failure distribution, making a failure more likely as the age of a component increases.

In Figure 3.2, we fix the value of p and observe the change in the optimal policy with respect to y_1 and y_2. Note that Figure 3.2(b) corresponds to the base instance with initial belief 0.5. We see that the optimal action is almost symmetrical for the two systems. For small values of the age of a component, the optimal action is 'do

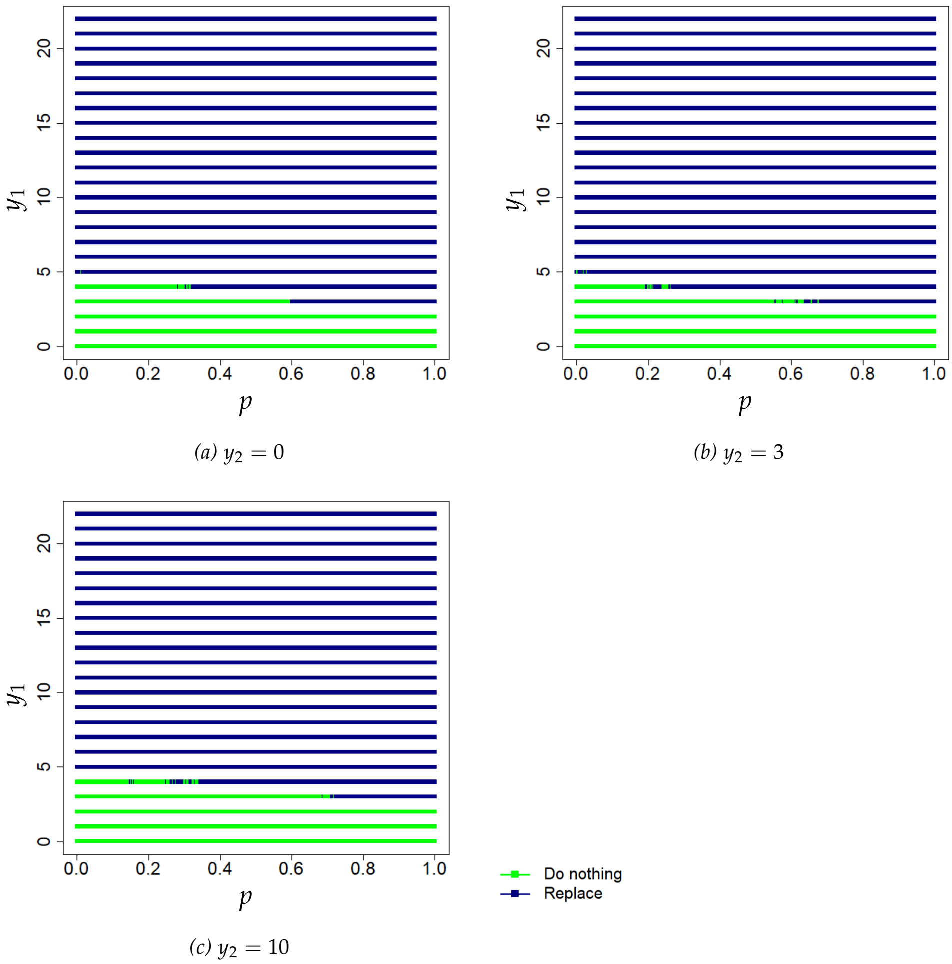
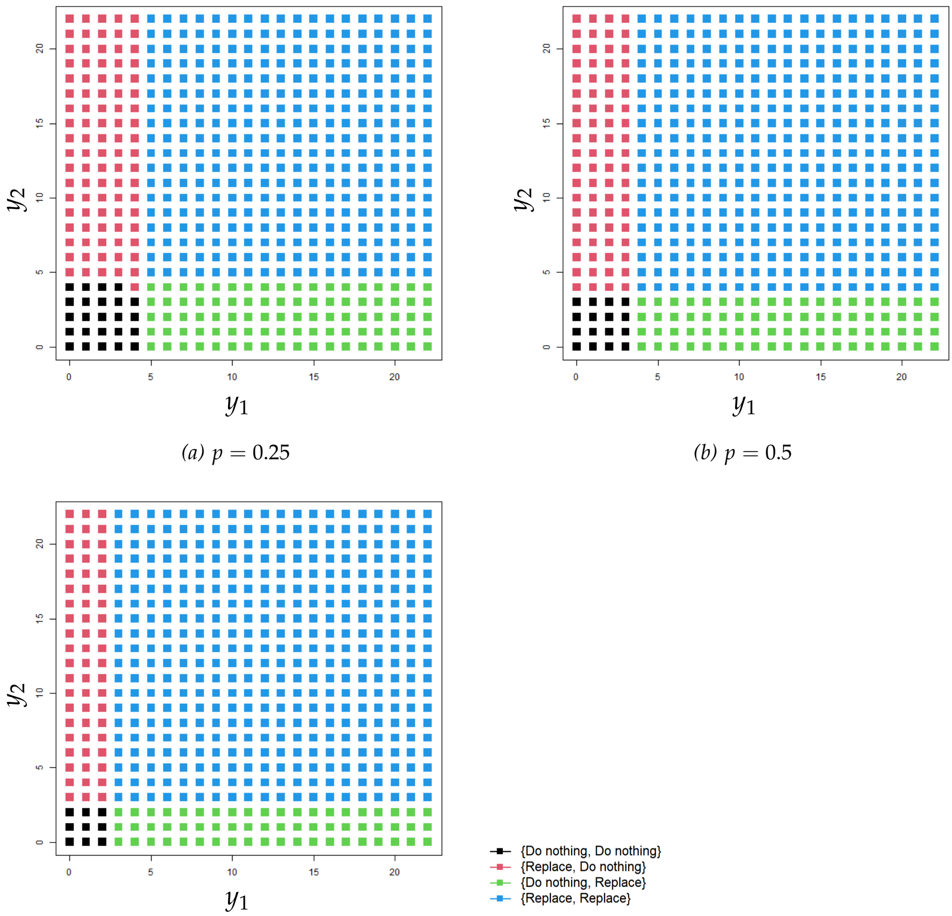


Figure 3.1: Optimal action $\mathbf{a}^*_{\mathrm{I}}(p, \mathbf{y}, 75)$ with $\mathbf{y} = (y_1, y_2)$.

nothing'. After an age limit, it becomes 'replace'. Additionally, for larger values of p the 'do nothing' area becomes smaller and the 'replace' area becomes larger.

In Figure 3.3, the change of the structure of optimal actions of Policy I with respect to z is shown for a particular belief state (i.e., when the belief state is equal to 0.5 at each z value). We observe that the optimal actions are the same for $z = \{50, 60, 75\}$ (see Figure 3.2 for $z = 75$). For $z = 20$, only a small area differs due to the end of lifespan effect.



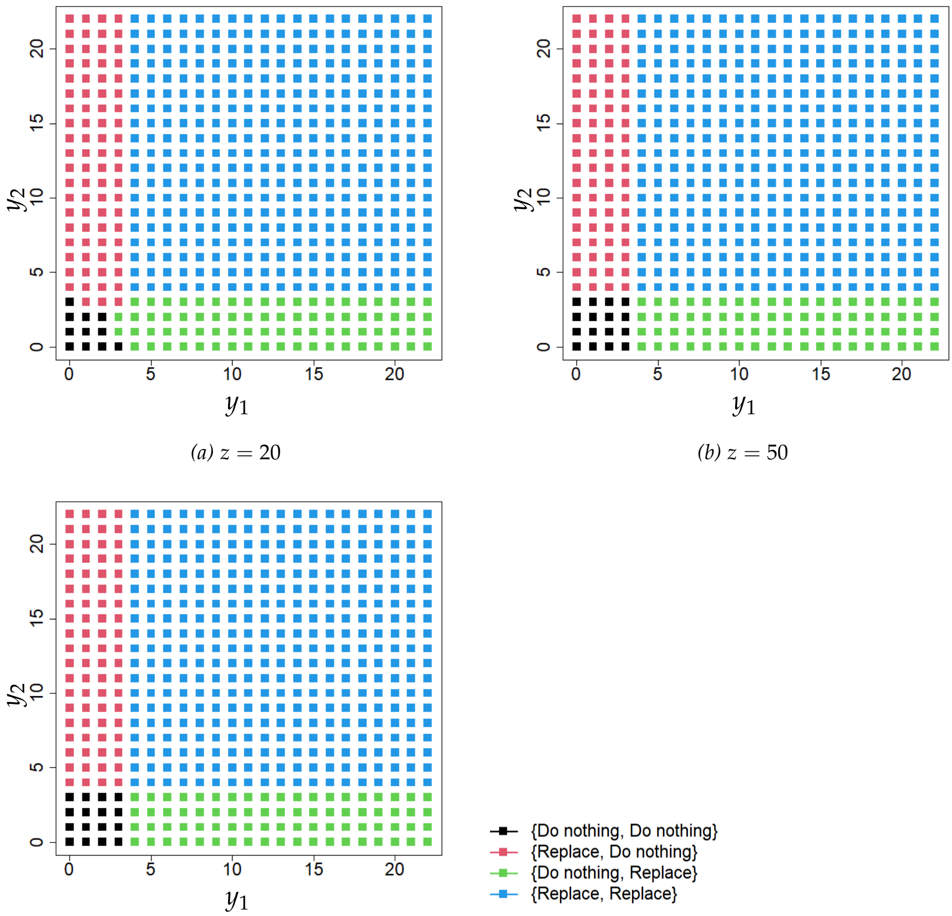
(a) $p = 0.25$

(b) $p = 0.5$

(c) $p = 0.75$

Figure 3.2: Optimal action $\mathbf{a}_{\mathrm{I}}^*(p, \mathbf{y}, 75)$ with $\mathbf{y} = (y_1, y_2)$.

In the implementation of our solution approach for the base instance, we note that there are 158,739,675 combinations of the variables p, (y_1, y_2) and z (see Appendix 3.B for our solution approach including the details on the discretization of the belief space). Only in 1.9% of these combinations, the action under Policy I differs from the action under Policy III (the cost difference between these policies will be provided in Section 3.6.3). In order to visualize at which states the two policies differ, we introduce a metric defined as $\sum_{z=1}^{L} \mathbb{1}_{\mathbf{a}_{\mathrm{I}}^*(p,\mathbf{y},z) \neq \mathbf{a}_{\mathrm{III}}^*(p,\mathbf{y},z)}$ for a given



(a) $z = 20$

(b) $z = 50$

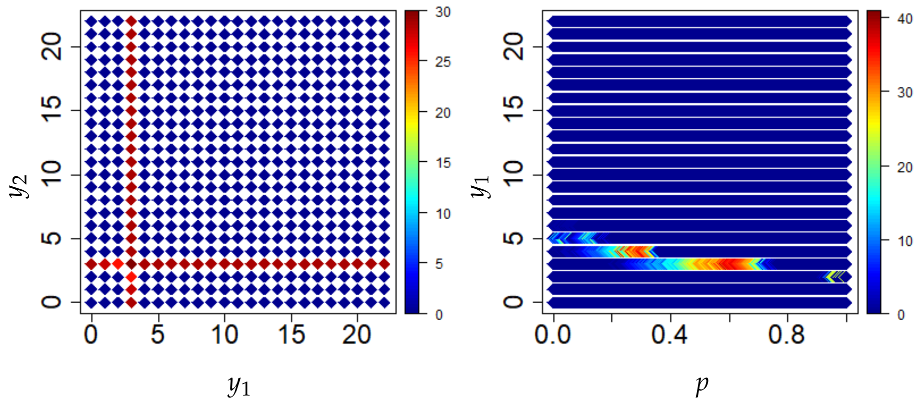
(c) $z = 60$

Figure 3.3: Optimal action $\mathbf{a}_{\mathrm{I}}^{*}(0.5, \mathbf{y}, z)$ with $\mathbf{y} = (y_1, y_2)$.

$(p, \mathbf{y})$, where

$$\mathbb{1}_{\mathbf{a}_{\mathrm{I}}^{*}(p,\mathbf{y},z) \neq \mathbf{a}_{\mathrm{III}}^{*}(p,\mathbf{y},z)} = \begin{cases} 1, & \text{if } \mathbf{a}_{\mathrm{I}}^{*}(p, \mathbf{y}, z) \neq \mathbf{a}_{\mathrm{III}}^{*}(p, \mathbf{y}, z) \\ 0, & \text{if } \mathbf{a}_{\mathrm{I}}^{*}(p, \mathbf{y}, z) = \mathbf{a}_{\mathrm{III}}^{*}(p, \mathbf{y}, z). \end{cases}$$

In Figure 3.4, we visualize this metric, representing the number of times the Policy I and Policy III take a different action in a particular state state p and $\mathbf{y}$ over all the possible z values. That is, if the metric at state $(p, \mathbf{y})$ is equal to zero, it means that the actions of Policy I and III are the same for all z values at this particular state. In Figure 3.4, we see that the part of the state space where the actions of Policy I and Policy III are different is limited to only a limited number of states. In particular, the actions are different around specific age values which constitutes an age-limit between 'do nothing' and 'replace' actions and this limit seems to be different for Policy I and III.



(a) $p = 0.5$ *(b)* $y_2 = 0$

Figure 3.4: The number of times Policy I and Policy III take a different action for a particular state p and (y_1, y_2) over all the possible z values.

3.6.3 Comparison of the expected cost per system under Policy I against Policy II and Policy III

In this section, we address research questions (2) and (3) by comparing the expected cost per system associated with each policy against each other. For this purpose, we generate a test bed of 36 instances. For this test bed, we consider the parameter values as follows: $\hat{p} \in \{0.25, 0.5, 0.75\}$, $L \in \{75, 150\}$, $k \in \{3, 5\}$, $C_f = 1$, and $C_p \in \{0.05, 0.1, 0.2\}$. We continue to use the truncated discrete Weibull lifetime distribution with the scale parameters as in the base instance.

We let $C_{\text{I}} = \frac{V_L(\hat{p},\mathbf{0})}{n}$ and $C_{\text{II}} = \frac{V_L^{\text{II}}(\hat{p},\mathbf{0})}{n}$ denote the expected cost per system under Policy I and Policy II, respectively, starting with a new set of components at age zero. For notational convenience, we also introduce $C_{\text{III}} = V_L^{\text{III}}(\hat{p}, 0)$ to denote the expected cost under Policy III. Note that C_{III} is not normalized with respect to n, as it already represents the expected cost for a single system. In Table 3.2, we denote the difference $C_{\text{II}} - C_{\text{I}}$ with $\Delta_{\text{II}-\text{I}}$ and the relative difference $\frac{C_{\text{II}}-C_{\text{I}}}{C_{\text{I}}}100\%$ with $\Delta^{\text{rel}}_{\text{II}-\text{I}}$ to compare Policy I and II. Similarly, we define $\Delta_{\text{III}-\text{I}} = C_{\text{III}} - C_{\text{I}}$ and $\Delta^{\text{rel}}_{\text{III}-\text{I}} = \frac{C_{\text{III}}-C_{\text{I}}}{C_{\text{I}}}100\%$ to compare Policy I and III, and we define $\Delta_{\text{III}-\text{II}} = C_{\text{III}} - C_{\text{II}}$ and $\Delta^{\text{rel}}_{\text{III}-\text{II}} = \frac{C_{\text{III}}-C_{\text{II}}}{C_{\text{II}}}100\%$ to compare Policy II and III.

Table 3.2 shows that, for 34 out of 36 instances, the relative difference $\Delta^{\text{rel}}_{\text{II}-\text{I}}$ between Policy I and Policy II is less than or equal to 0.20%. For the two remaining instances, it is less than or equal to 0.80%. This quantifies the maximum benefit that can be obtained via jointly making the replacement decisions for multiple systems. The relative difference $\Delta^{\text{rel}}_{\text{III}-\text{II}}$ between Policy II and III is less than or equal to 1% for six out of 36 instances. It is greater than 1% and smaller than 2% for 19 out of 36 instances, and it is is greater than 2% for the remaining 11 instances. Policy III is up to 5.6% costlier than Policy II. This is the maximum benefit that can be obtained due to data pooling in this test bed. Due to the small difference in the expected cost per system between Policy I and Policy II, the statistics for $\Delta^{\text{rel}}_{\text{III}-\text{I}}$ are similar to $\Delta^{\text{rel}}_{\text{III}-\text{II}}$. Considering that the structures of the policies are mostly similar for Policy I and Policy III, we can conclude that the reduction in costs by using Policy I instead of Policy III is mostly due to the data pooling rather than jointly making the replacement decisions for multiple systems.

3.6.4 Sensitivity analysis

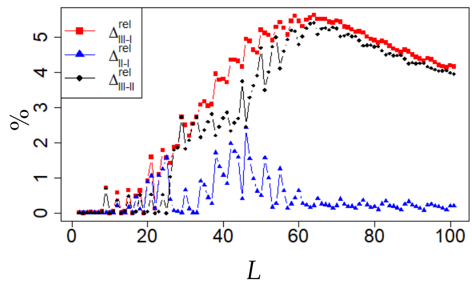
In this section, we perform a sensitivity analysis around the base instance to answer research question (4). We show how the costs and relative cost differences are affected with respect to varying values of the input parameters L, k, C_p, $\hat{p}$ and n. When we study the effect of a specific input parameter, we assume all the other input parameters are the same as in the base instance.

Effect of the lifespan. In Figure 3.5(a), we show how the relative difference in costs changes with respect to L. We observe that the relative differences $\Delta^{\text{rel}}_{\text{III}-\text{I}}$ and $\Delta^{\text{rel}}_{\text{III}-\text{II}}$ are the largest for $50 \leq L \leq 75$, and the relative difference $\Delta^{\text{rel}}_{\text{II}-\text{I}}$ is the largest for $35 \leq L \leq 50$. The relative difference between policies gets smaller for the values of L larger than 75. This is intuitive because a long lifespan leads to learning the population type accurately, and beyond some point, the effect of not knowing the

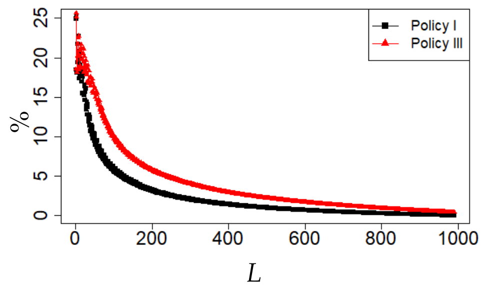
Table 3.2: Comparison of the policies for the instances in the test bed

Instance	L	C_p	k	$\hat{p}$	C_{I}	$\Delta_{\mathrm{III-I}}$	$\Delta_{\mathrm{II-I}}$	$\Delta_{\mathrm{III-II}}$	$\Delta^{\mathrm{rel}}_{\mathrm{III-I}}$	$\Delta^{\mathrm{rel}}_{\mathrm{II-I}}$	$\Delta^{\mathrm{rel}}_{\mathrm{III-II}}$
1	75	0.05	3	0.25	2.05	0.04	0.00	0.04	2.0%	0.0%	2.0%
2	75	0.05	3	0.5	2.48	0.05	0.00	0.05	2.0%	0.0%	2.0%
3	75	0.05	3	0.75	2.84	0.01	0.00	0.01	0.4%	0.0%	0.4%
4	150	0.05	3	0.25	4.06	0.08	0.01	0.07	2.0%	0.2%	1.7%
5	150	0.05	3	0.5	4.9	0.10	0.01	0.09	2.0%	0.2%	1.8%
6	150	0.05	3	0.75	5.66	0.06	0.00	0.06	1.1%	0.0%	1.1%
7	75	0.1	3	0.25	3.12	0.06	0.00	0.06	1.9%	0.0%	1.9%
8	75	0.1	3	0.5	3.79	0.07	0.00	0.07	1.8%	0.0%	1.8%
9	75	0.1	3	0.75	4.4	0.05	0.00	0.05	1.1%	0.0%	1.1%
10	150	0.1	3	0.25	6.23	0.10	0.01	0.09	1.6%	0.2%	1.4%
11	150	0.1	3	0.5	7.54	0.13	0.01	0.12	1.7%	0.1%	1.6%
12	150	0.1	3	0.75	8.79	0.11	0.01	0.10	1.3%	0.1%	1.1%
13	75	0.2	3	0.25	4.61	0.08	0.00	0.08	1.7%	0.0%	1.7%
14	75	0.2	3	0.5	5.58	0.11	0.00	0.11	2.0%	0.0%	2.0%
15	75	0.2	3	0.75	6.5	0.08	0.00	0.08	1.2%	0.0%	1.2%
16	150	0.2	3	0.25	9.3	0.10	0.00	0.1	1.1%	0.0%	1.1%
17	150	0.2	3	0.5	11.19	0.15	0.00	0.15	1.3%	0.0%	1.3%
18	150	0.2	3	0.75	13.04	0.12	0.00	0.12	0.9%	0.0%	0.9%
19	75	0.05	5	0.25	1.25	0.07	0.00	0.07	5.6%	0.0%	5.6%
20	75	0.05	5	0.5	1.57	0.01	0.01	0.00	0.6%	0.6%	0.0%
21	75	0.05	5	0.75	1.76	0.00	0.00	0.00	0.0%	0.0%	0.0%
22	150	0.05	5	0.25	2.46	0.08	0.00	0.08	3.3%	0.0%	3.3%
23	150	0.05	5	0.5	3.04	0.13	0.00	0.13	4.3%	0.0%	4.3%
24	150	0.05	5	0.75	3.53	0.02	0.00	0.02	0.6%	0.0%	0.6%
25	75	0.1	5	0.25	2.04	0.08	0.00	0.08	3.9%	0.0%	3.9%
26	75	0.1	5	0.5	2.50	0.13	0.01	0.12	5.2%	0.4%	4.8%
27	75	0.1	5	0.75	2.91	0.02	0.02	0.00	0.7%	0.7%	0.0%
28	150	0.1	5	0.25	4.06	0.10	0.00	0.10	2.5%	0.0%	2.5%
29	150	0.1	5	0.5	4.92	0.15	0.01	0.14	3.0%	0.2%	2.8%
30	150	0.1	5	0.75	5.73	0.16	0.00	0.16	2.8%	0.0%	2.8%
31	75	0.2	5	0.25	3.37	0.08	0.00	0.08	2.4%	0.0%	2.4%
32	75	0.2	5	0.5	4.15	0.12	0.00	0.12	2.9%	0.0%	2.9%
33	75	0.2	5	0.75	4.86	0.11	0.01	0.10	2.3%	0.2%	2.1%
34	150	0.2	5	0.25	6.82	0.09	0.01	0.08	1.3%	0.1%	1.2%
35	150	0.2	5	0.5	8.31	0.15	0.01	0.14	1.8%	0.1%	1.7%
36	150	0.2	5	0.75	9.75	0.14	0.01	0.13	1.4%	0.1%	1.3%

true population type disappears (i.e., policies converge to the policy under perfect information setting).



(a) Relative difference in expected cost per system.



(b) Relative difference between the expected cost per system of a policy and the expected cost for a single system under perfect information.

Figure 3.5: Effect of the lifespan length L on costs; $L = 75$ corresponds to the base instance.

To better see this, in Figure 3.5(b), we compare the expected costs under Policy I and Policy III against the cost under perfect information. For a long lifespan such as $L = 1000$, the costs of both policies approach the cost under perfect information. This means both policies (with and without data pooling) learn the true population type well for a sufficiently large lifespan. We further see that the cost of Policy I converges to the perfect information cost earlier than Policy III. This can be interpreted as the effect of data pooling under Policy I.

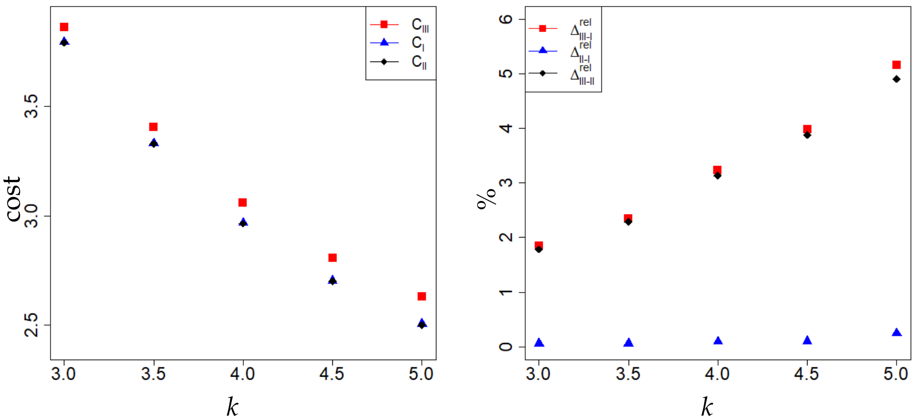
Effect of the coefficient of variation of the time-to-failure distributions. We consider that the shape parameter k of the time-to-failure distribution (which is common for both population types) takes values from the set $\{3, 3.5, 4, 4.5\}$ in addition to 5, which was the value of k in the base instance. As the shape parameter k increases, the expectation of the time-to-failure distribution increases a little bit while its variance (and hence coefficient of variation) decreases significantly. Table 3.3 provides the distributional properties of the time-to-failure distributions corresponding to the selected parameters.

Table 3.3: Properties of time-to-failure distributions (with u equal to 22).

Shape (k)	Scale (λ)	Expectation	Variance	Coefficient of Variation
3	6	5.858	1.969	0.336
3	12	11.189	3.867	0.346
3.5	6	5.898	1.733	0.294
3.5	12	11.294	3.424	0.303
4	6	5.938	1.553	0.261
4	12	11.377	3.065	0.269
4.5	6	5.975	1.410	0.236
4.5	12	11.451	2.776	0.242
5	6	6.009	1.294	0.215
5	12	11.518	2.540	0.221

Figure 3.6 shows how the shape parameter k affects the expected cost per system under each policy and the relative cost differences between policies. As k increases (i.e. the coefficient of variation decreases), we observe that the expected cost per system under all policies decreases and the relative difference between the costs of Policy I and Policy III increases (the relative difference between the costs of Policy II and Policy III behaves similarly). This also shows that the benefit of data sharing is higher as k increases. The relative difference between the costs of Policy I and II continues to be small and does not vary much.

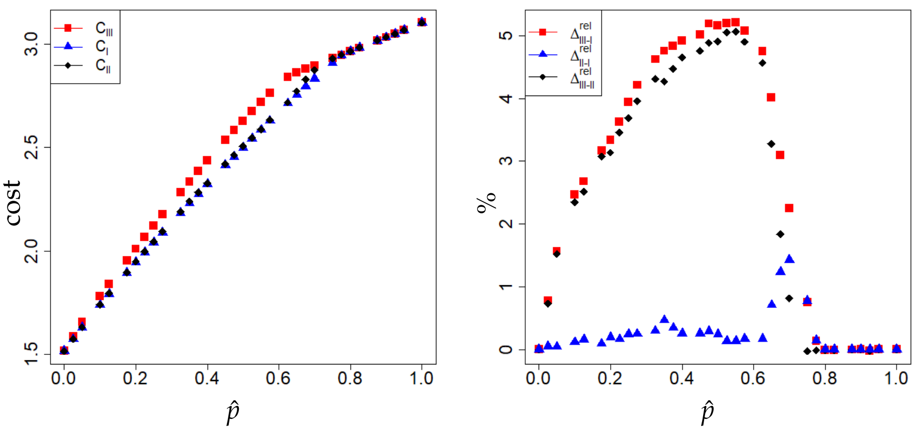
Effect of initial belief. We choose $\hat{p} \in \{0, 0.025, \ldots, 0.975, 1\}$ for the sensitivity analysis with respect to the initial belief $\hat{p}$. Figure 3.7 shows how the expected cost per system associated with each policy and the relative differences between them change with respect to the initial belief on the true population type. We observe that the expected cost per system increases for all the policies as $\hat{p}$ increases. This is due to the fact that the expected number of failures is higher when the components come from the weak population rather than the strong population. Therefore, it results in more replacement activities and increases the expected cost per system. The relative cost difference between policies III-I and III-II is the largest when the



(a) Expected cost per system *(b)* Relative difference in expected cost per system

Figure 3.6: Effect of the shape parameter k on costs; $k = 5$ corresponds to the base instance.

uncertainty for the true type of population is high (i.e., for $\hat{p}$ around 0.5). This is when the benefit obtained by resolving the population-type uncertainty early (via data pooling) is also high.

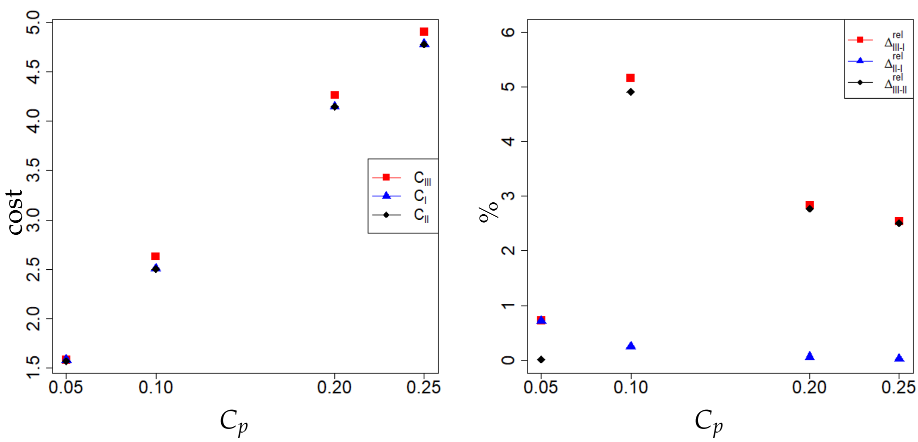


(a) Expected cost per system *(b)* Relative difference in expected cost per system.

Figure 3.7: Effect of the initial belief on costs; $\hat{p} = 0.5$ corresponds to the base instance.

Effect of the cost of preventive maintenance. We choose $C_p \in \{0.05, 0.1, 0.2, 0.25\}$

to study the effect of preventive replacement cost C_p on the expected costs (see Figure 3.8). Naturally, the expected cost under each policy increases as the cost of preventive replacement increases. Figure 3.8 also shows that the relative difference in costs between Policies II and III and between Policies I and III also decreases as the cost of preventive replacement increases (excluding the case $C_p = 0.05$). The instance with $C_p = 0.05$ is an exception because policies require a larger lifespan to learn the true population type when the preventive replacement cost is that low (i.e., the preventive maintenance is so inexpensive that the components are always replaced preventively in an early phase of their lifetime, preventing to distinguish between weak and strong population with the historical data). Thus, we end up with three policies that perform very closely.

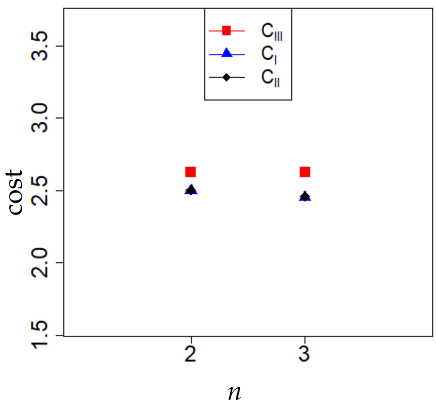


(a) Expected cost per system. *(b)* Relative difference in expected cost per system.

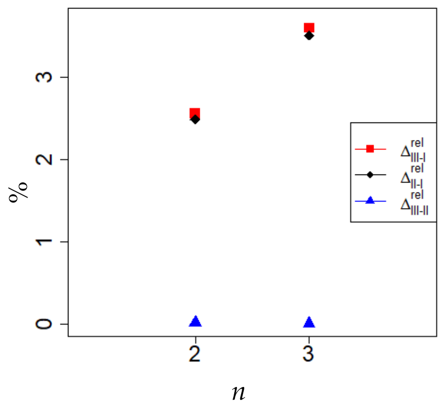
Figure 3.8: Effect of the preventive maintenance cost on the costs; $C_p = 0.1$ corresponds to the base instance.

Effect of the number of systems. We compare the costs associated with the three policies in Figure 3.9 for $n \in \{2, 3\}$. We observe that the expected cost per system under Policies I and II slightly decreases when we go from $n = 2$ to $n = 3$. On the other hand, the relative cost differences between policies I and III and between policies II and III increase in this case.

In order to better see how the number of systems affect the decrease in expected cost per system for Policies I and II (the decrease in cost is because of increased data pooling with higher number of systems under these policies), we increase n



(a) Expected cost per system.



(b) Relative difference in expected cost per system.

Figure 3.9: Effect of the number of systems n on costs; $n = 2$ corresponds to the base instance.

up to 20. As expected, the state space of the POMDP becomes too large to be able to efficiently solve the Bellman equations characterized in Section 3.3 in order to obtain Policy I. Therefore, we do not report the performance of Policy I from now on for n greater than 2. Instead, we use the optimal policy under the perfect-information setting (see Section 3.4.1) as a benchmark to quantify the maximum benefit that could have been obtained by Policy I. Notice that the optimal expected cost in the perfect-information setting can be interpreted as a lower bound on the expected cost under Policy I. Thus, it allows us to quantify the maximum reduction in expected cost per system via data pooling.

We let C_{PI} denote the expected cost for a single system in the perfect-information setting at the beginning of the lifespan and at belief state $\hat{p}$ and component age zero, i.e., $C_{\mathrm{PI}} = W_L(\hat{p}, 0)$. Furthermore, we introduce the notation $\phi_\omega = C_\omega - C_{\mathrm{PI}}$ and $\phi_\omega^{\mathrm{rel}} = \frac{C_\omega - C_{\mathrm{PI}}}{C_{\mathrm{PI}}} 100\%$ to denote the difference and the relative difference between the expected cost per system under policy $\omega \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ and the expected cost for a single system in the perfect-information setting, respectively. In Table 3.4, we list these differences for the test bed of Section 3.6.3.

The relative difference $\phi_{\mathrm{I}}^{\mathrm{rel}}$ between Policy I and the perfect information setting is greater than 1% and less than 3% for 13 out of 36 instances. For the remaining instances, it is greater than 3%. The largest relative difference between Policy I

Table 3.4: Comparison of the costs of Policy I, II and III against the cost under perfect-information setting ($n = 2$).

Instance	ϕ_I	ϕ_{II}	ϕ_{III}	ϕ_I^{rel}	ϕ_{II}^{rel}	ϕ_{III}^{rel}
1	0.11	0.11	0.15	5.7%	5.7%	7.7%
2	0.14	0.14	0.19	6.0%	6.0%	8.1%
3	0.10	0.10	0.11	3.6%	3.6%	4.0%
4	0.13	0.14	0.21	3.3%	3.6%	5.3%
5	0.17	0.18	0.27	3.6%	3.8%	5.7%
6	0.14	0.14	0.20	2.5%	2.5%	3.6%
7	0.11	0.11	0.17	3.7%	3.7%	5.6%
8	0.15	0.15	0.22	4.1%	4.1%	6.0%
9	0.13	0.13	0.18	3.0%	3.0%	4.2%
10	0.12	0.13	0.22	2.0%	2.1%	3.6%
11	0.17	0.18	0.30	2.3%	2.4%	4.1%
12	0.16	0.17	0.27	1.9%	2.0%	3.1%
13	0.10	0.10	0.18	2.2%	2.2%	4.0%
14	0.14	0.14	0.25	2.6%	2.6%	4.6%
15	0.14	0.14	0.22	2.2%	2.2%	3.5%
16	0.11	0.11	0.21	1.2%	1.2%	2.3%
17	0.15	0.15	0.30	1.4%	1.4%	2.7%
18	0.15	0.15	0.27	1.2%	1.2%	2.1%
19	0.11	0.11	0.18	9.6%	9.6%	15.8%
20	0.16	0.17	0.17	11.3%	12.1%	12.1%
21	0.09	0.09	0.09	5.4%	5.4%	5.4%
22	0.12	0.12	0.20	5.1%	5.1%	8.5%
23	0.18	0.18	0.31	6.3%	6.3%	10.8%
24	0.14	0.14	0.16	4.1%	4.1%	4.7%
25	0.13	0.13	0.21	6.8%	6.8%	11.0%
26	0.19	0.20	0.32	8.2%	8.7%	13.9%
27	0.20	0.22	0.22	7.4%	8.1%	8.1%
28	0.15	0.15	0.25	3.8%	3.8%	6.4%
29	0.21	0.22	0.36	4.5%	4.7%	7.6%
30	0.23	0.23	0.39	4.2%	4.2%	7.1%
31	0.11	0.11	0.19	3.4%	3.4%	5.8%
32	0.16	0.16	0.28	4.0%	4.0%	7.0%
33	0.15	0.16	0.26	3.2%	3.4%	5.5%
34	0.10	0.11	0.19	1.5%	1.6%	2.8%
35	0.15	0.16	0.30	1.8%	2.0%	3.7%
36	0.14	0.15	0.28	1.5%	1.6%	2.9%

and the perfect information setting is 11.3% at instance 20. This means that the population heterogeneity leads to an 11.1% increase in expected cost (i.e., this can be interpreted as the cost of not knowing the true population type). If Policy III is used instead of Policy I, this additional cost can be up to 15.8%.

The expected total cost of Policy I and II are close to each other in all numerical experiments. However, the size of the discretized state space that belongs to the POMDP model associated with Policy I becomes much larger than the one that corresponds to Policy II as the number of systems increases. We evaluate the expected cost per system under Policy II for $n \in \{5, 10, 15, 20\}$ via simulation and compare it against the expected costs per system under Policy III and the perfect information setting. For this purpose, we perform 5000 simulation runs and report the results in Table 3.5. We only report the average value from these simulations because the half width of the 95% confidence interval (CI) built around the average value is not greater than 0.00 for any instance.

In Table 3.5, we see that as the number of systems increases the expected cost per system comes closer to the cost under perfect information. When there are 20 systems, for five out of 36 instances, the relative difference $\Delta^{\text{rel}}_{\text{III}-\text{II}}$ is more than 2% and less than 3%. For 17 of these instances, it is greater than or equal to 3% and less than or equal to 5.3%. For the remaining instances, it is between 5.6% to 14%. That is, the largest benefit obtained by data pooling is for 20 systems and equal to 14%. For 20 systems, when we compare the cost of Policy II against the cost under perfect information, we see that relative difference $\phi^{\text{rel}}_{\text{II}}$ is less than 1% for 27 out of 36 instances. It is greater than or equal to 1% and less than or equal to 1.7% for the remaining instances. That is, the cost under data pooling from 20 systems differs from the cost under perfect information by no more than 1.7%. As the number of systems increases, the total expected cost per system decreases for Policy II and gets closer to the cost under perfect information. This shows that even when the replacement decisions are made independently for multiple systems, data pooling is effective in cost reduction.

Note that the maximum cost reduction that can be obtained by data pooling is $C_{\text{III}} - C_{\text{PI}}$. We refer to this as *potential cost reduction*. The actual reduction obtained by data pooling is $C_{\text{III}} - C_{\text{II}}$. We introduce the performance measure γ, defined as $\frac{C_{\text{III}} - C_{\text{II}}}{C_{\text{III}} - C_{\text{PI}}} 100\%$, to quantify the percentage of the potential cost reduction obtained by data pooling. We calculate γ for each instance based on the average cost obtained from simulation. In Figure 3.10, we show the distribution of these γ values for $n \in \{5, 10, 15, 20\}$. On average, data pooling from five systems achieves around 70% of the potential cost reduction. This is on average above 90% for 20 systems.

Finally, we show the effect of the number of systems on the expected cost per system

Table 3.5: Comparison of the costs of Policy II and III against the cost under perfect-information setting ($n > 2$)

		C_{II}					$n = 20$	
Instance	C_{III}	$n = 5$	$n = 10$	$n = 15$	$n = 20$	C_{PI}	$\Delta^{\mathrm{rel}}_{\mathrm{III-II}}$	$\phi^{\mathrm{rel}}_{\mathrm{II}}$
1	2.09	2.00	1.97	1.96	1.96	1.94	6.7%	1.0%
2	2.53	2.42	2.38	2.37	2.37	2.34	6.9%	1.2%
3	2.85	2.81	2.78	2.77	2.76	2.74	3.2%	0.8%
4	4.14	3.98	3.95	3.96	3.95	3.93	4.9%	0.4%
5	5.00	4.80	4.77	4.75	4.75	4.73	5.3%	0.4%
6	5.72	5.59	5.56	5.56	5.54	5.52	3.3%	0.3%
7	3.18	3.06	3.03	3.03	3.02	3.01	5.2%	0.4%
8	3.86	3.72	3.67	3.67	3.66	3.64	5.6%	0.4%
9	4.45	4.34	4.30	4.29	4.29	4.27	3.8%	0.4%
10	6.33	6.17	6.14	6.13	6.12	6.11	3.4%	0.2%
11	7.67	7.45	7.41	7.39	7.39	7.37	3.8%	0.2%
12	8.90	8.69	8.67	8.65	8.64	8.63	3.0%	0.2%
13	4.69	4.56	4.54	4.52	4.52	4.51	3.8%	0.2%
14	5.69	5.50	5.48	5.46	5.45	5.44	4.5%	0.1%
15	6.58	6.42	6.39	6.38	6.36	6.36	3.4%	0.0%
16	9.40	9.24	9.22	9.21	9.19	9.19	2.2%	0.0%
17	11.34	11.09	11.07	11.06	11.05	11.04	2.6%	0.1%
18	13.16	12.95	12.92	12.90	12.91	12.89	2.0%	0.1%
19	1.32	1.21	1.17	1.17	1.16	1.14	14.0%	1.6%
20	1.58	1.50	1.45	1.44	1.43	1.41	10.2%	1.7%
21	1.76	1.74	1.71	1.69	1.69	1.67	4.2%	1.1%
22	2.54	2.40	2.38	2.36	2.35	2.34	8.2%	0.3%
23	3.17	2.96	2.92	2.90	2.89	2.86	9.6%	1.2%
24	3.55	3.47	3.43	3.42	3.41	3.39	4.2%	0.5%
25	2.12	1.98	1.95	1.93	1.93	1.91	9.9%	1.0%
26	2.63	2.41	2.37	2.35	2.34	2.31	12.4%	1.3%
27	2.93	2.81	2.76	2.75	2.74	2.71	7.1%	1.0%
28	4.16	3.98	3.94	3.93	3.92	3.91	6.0%	0.3%
29	5.07	4.81	4.76	4.75	4.73	4.71	7.1%	0.5%
30	5.89	5.63	5.57	5.54	5.54	5.50	6.3%	0.7%
31	3.45	3.30	3.29	3.27	3.28	3.26	5.3%	0.5%
32	4.27	4.06	4.01	4.01	4.00	3.99	6.9%	0.1%
33	4.97	4.78	4.74	4.72	4.72	4.71	5.3%	0.3%
34	6.91	6.76	6.73	6.73	6.73	6.72	2.7%	0.1%
35	8.46	8.24	8.19	8.18	8.17	8.16	3.6%	0.1%
36	9.89	9.67	9.63	9.62	9.62	9.61	2.8%	0.1%

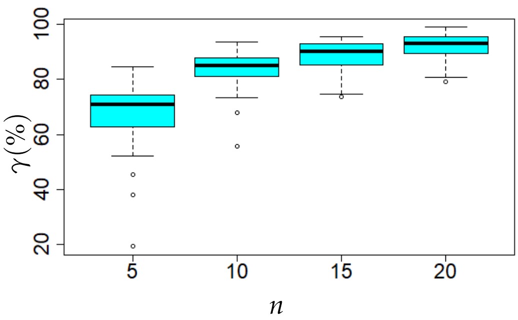


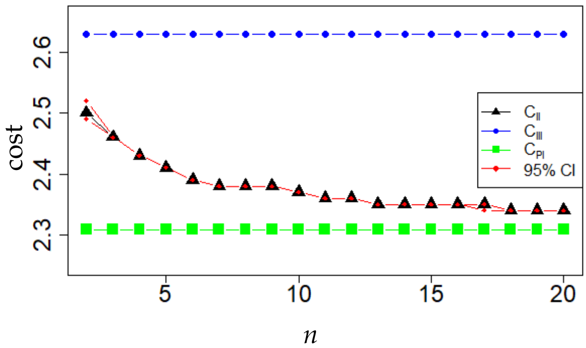
Figure 3.10: Percentage of the potential cost reduction obtained by Policy II.

in Figure 3.11, where all input parameters are the same as in the base instance but the value of n varies. We report the costs for $n \in \{2, 3, \ldots, 20\}$. In Figure 3.11(a), we visualize how the expected cost per system decreases for Policy II as a function of n. Clearly, the marginal cost reduction due to data pooling is non-increasing in n. In Figure 3.11(b), we visualize the value of the percentage of the potential cost reduction achieved by data pooling.

There are two main insights on the benefit of data pooling. First, the Bayesian updates are the same for all policies, therefore the number of data points required for learning the true population would also be the same. However, by pooling data from multiple systems, we can obtain the same number of data in a shorter time span. Second, as the number of systems increases, the cost of learning (exploration) per system decreases (similarly to the special case analyzed in Section 3.5). Therefore, we exploit the learned information regarding the population type by a higher number of systems.

3.7. Conclusion

We have studied the optimal replacement policy for multiple single-component systems with a fixed lifespan under population heterogeneity. For this purpose, we built a POMDP model with Bayesian updating. We investigated the benefit of data pooling and jointly making the component replacement decisions on the expected cost per system. As a benchmark to the optimal policy, we introduced



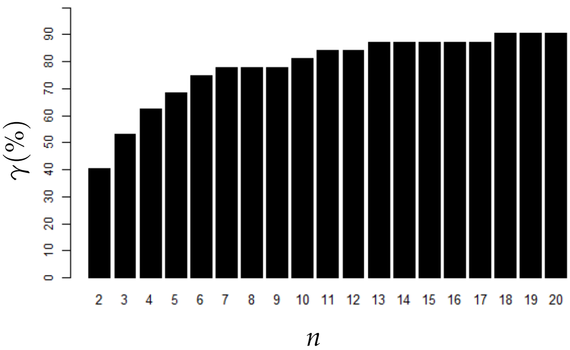


Figure 3.11: Effect of the number of systems on the expected cost per system for Policy II (left) and on the percentage of the potential cost reduction obtained by Policy II (right).

two other policies that allow us to quantify these benefits. We further introduced a policy under which the true population type is known; this policy gives an upper bound on the maximum benefit that can be gained by data pooling. For a test bed with 36 problem instances for two systems, the maximum reduction in total expected cost per system that is obtained via joint optimization is 0.80% and the maximum reduction in total expected cost per system that is obtained via data pooling is 5.6%. These results indicate that data pooling is more effective reducing the expected costs than jointly making the replacement decisions. Considering the computational complexity of the POMDP model for multiple systems and with data pooling, applying the policy by considering the optimal policy of a single system but with data pooling is a favorable approach in practice with many systems. We investigated how the reduction in expected cost per system increases as the number of systems increases. For 20 systems, the maximum cost reduction obtained by data pooling is up to 14% (the upper bound for the cost reduction in this particular instance is 15.8%). Insights on the cost benefits of data pooling also provide managerial insights to practitioners regarding their investments in adopting new technologies that enable data pooling.

3.A. Proofs

Under Assumption 3.1, the Bayesian update function simplifies to

$$g(p, \hat{\mathbf{y}}, \mathbf{d}) \quad = \quad \begin{cases} p & \text{if } 0 \leq \hat{y}_i \leq t-2 \text{ and } d_i = 0 \text{ for all } i \in \{1, \ldots, n\}, \\ 1 & \text{if } \hat{y}_i = t-1 \text{ and } d_i = 1 \text{ for one or more systems } i \in \{1, \ldots, n\}, \\ 0 & \text{if } \hat{y}_i = t-1 \text{ and } d_i = 0 \text{ for one or more systems } i \in \{1, \ldots, n\}, \\ 0 & \text{if } \hat{y}_i \geq t, \text{ for any } d_i \in \{0,1\} \text{ for at least one } i \in \{1, \ldots, n\}, \end{cases}$$

$p \in [0, 1]$. Please note that, $g(0, \hat{\mathbf{y}}, \mathbf{d}) = 0$ and $g(1, \hat{\mathbf{y}}, \mathbf{d}) = 1$. Please also note that for each system i, it holds that age $\hat{y}_i$ with $0 \leq \hat{y}_i \leq t-2$ or $t \leq \hat{y}_i \leq 2t-2$ implies $d_i = 0$, age $\hat{y}_i = t-1$ allows $d_i = 0$ and $d_i = 1$, and age $\hat{y}_i = 2t-1$ implies $d_i = 1$ (hence, not all combinations of the vectors $\hat{\mathbf{y}}$ and $\mathbf{d}$ are possible).

For any system i, if $y_i \leq t-2$ or $t \leq y_i \leq 2t-2$, the optimal action is 'do nothing' for that system because the probability of failure during the next time period is equal to zero. In this age interval, no cost incurs. For any system i, if $y_i = 2t-1$, the optimal action is 'do a preventive replacement' for that system because the probability of failure during the next time period is equal to one. In this case, a cost C_p is incurred.

Under the perfect learning policy, after the true population is learned at $z = L - t$, the perfect learning policy follows the perfect information setting. If the component is coming from the weak population, the age will be zero for the failed component, and it should be replaced only one more time after $t-1$ time units (i.e., at $z = L - 2t - 1$). At this moment, the age of all preventively replaced components will be one. They should be replaced only one more time after $t-2$ time units (i.e.,

$z = L - 2t - 2$). If the true population type is strong, the age of the component for which the 'do nothing' action was applied will be t and it should be replaced one time after $t-1$ time units (i.e., at $z = L - 2t - 1$). The age of the preventively replaced components will be one. We do not need to replace these components again because the component will fail $2t - 1$ time units later (which is after the end of the lifespan of the system).

Proof of Proposition 3.1

We apply one 'do nothing' action when the components in the systems reach age $t-1$. Let vector $\mathbf{e}_i$ be the vector with the i^{th} element equal to one and with all other elements equal to zero. Let us assume that the 'do nothing' action is applied for a given system i at $z = L - (t-1)$. Then we obtain the following total expected cost

$$\begin{aligned}
\tilde{V}_{L-(t-1)}\Big(0.5, (t-1)\mathbf{e}, \mathbf{e} - \mathbf{e}_i\Big) &= (n-1)C_p + 0.5(C_f + V_{L-(t-1)-1}(1, \mathbf{e} - \mathbf{e}_i)) \\
&+ 0.5V_{L-(t-1)-1}(0, \mathbf{e} + (t-1)\mathbf{e}_i) \\
&= (n-1)C_p + 0.5(C_f + nC_p) + 0.5C_p,
\end{aligned}$$

where $V_{L-t}(1, \mathbf{e} - \mathbf{e}_i) = nC_p$ and $V_{L-t}(0, \mathbf{e} + (t-1)\mathbf{e}_i) = C_p$. □

Proof of Proposition 3.2

The expected total cost under Policy III for the lifespan of a system is

$$\begin{aligned}
V_L^{\text{III}}(0.5, 0) &= V_{L-(t-1)}^{\text{III}}(0.5, t-1) \\
&= \min\{\tilde{V}_{L-(t-1)}^{\text{III}}(0.5, t-1, 1), \tilde{V}_{L-(t-1)}^{\text{III}}(0.5, t-1, 0)\} \\
&= \min\{C_p + V_{L-t}^{\text{III}}(0.5, 1), 0.5[C_f + V_{L-t}^{\text{III}}(1, 0)] + 0.5V_{L-t}^{\text{III}}(0, t)\} \\
&= \min\{C_p + \min\{C_p, 0.5C_f\}, 0.5[C_f + C_p] + 0.5C_p\} \\
&= \min\{\min\{2C_p, 0.5C_f + C_p\}, 0.5C_f + C_p\} \\
&= \min\{2C_p, 0.5C_f + C_p\} = C_p + \min\{C_p, 0.5C_f\},
\end{aligned}$$

where $V_{L-t}^{\text{III}}(0, t) = C_p$, $V_{L-t}^{\text{III}}(1, 0) = C_p$, and the following result is used in the third step

$$\begin{aligned}
V_{L-t}^{\text{III}}(0.5, 1) &= V_{L-t-1}^{\text{III}}(0.5, 2) = V_{L-t-2}^{\text{III}}(0.5, 3) = \cdots = V_{L-(2t-2)}^{\text{III}}(0.5, t-1) \\
&= \min\{\tilde{V}_{L-(2t-2)}^{\text{III}}(0.5, t-1, 1), \tilde{V}_{L-(2t-2)}^{\text{III}}(0.5, t-1, 0)\} \\
&= \min\{C_p + V_{L-(2t-1)}^{\text{III}}(0.5, 1), 0.5[C_f + V_{L-(2t-1)}^{\text{III}}(1, 0)] + 0.5V_{L-(2t-1)}^{\text{III}}(0, t)\}
\end{aligned}$$

$$= \min\{C_p, 0.5C_f\}.$$

Please also note that $V^{\text{III}}_{L-(2t-1)}(0.5, 1) = V^{\text{III}}_0(0.5, L-2t+2) = 0$, because $L-2t+2 < t-1$. Additionally, $V^{\text{III}}_{L-(2t-1)}(1, 0) = 0$, and $V^{\text{III}}_{L-(2t-1)}(0, t) = 0$. Under Assumption 3.1, the formula for the total expected cost under Policy III for n systems simplifies to $2nC_p$. □

Proof of Proposition 3.3

We choose a policy such that we apply the 'do nothing' action to n' (> 0) systems when the components reach age $t-1$ for the first time. We denote a vector $\mathbf{e}_{n'}$ where its i^{th} elements corresponding to these selected n' systems are equal to one and the rest is equal to zero. In this case, we have the following total expected cost at the beginning of the lifespan of the systems.

$$\begin{aligned}
\tilde{V}_{L-(t-1)}\Big(0.5, (t-1)\mathbf{e}, \mathbf{e}-\mathbf{e}_{n'}\Big) &= (n-n')C_p + 0.5(n'C_f + V_{L-(t-1)-1}(1, \mathbf{e}-\mathbf{e}_{n'})) \\
&+ 0.5V_{L-(t-1)-1}(0, \mathbf{e}+(t-1)\mathbf{e}_{n'}) \\
&= (n-n')C_p + 0.5(n'C_f + nC_p) + 0.5n'C_p \\
&= 1.5nC_p + 0.5n'(C_f - C_p),
\end{aligned}$$

We see that $1.5nC_p + 0.5n'(C_f - C_p)$ is a non-decreasing function of n'. Therefore, a policy where $n' > 1$ is never optimal. □

Proof of Proposition 3.4

The difference between the expected total costs of Policy III and the perfect learning policy is $2nC_p - \Big([\frac{3}{2}n - \frac{1}{2}]C_p + \frac{1}{2}C_f\Big) = \frac{1}{2}[(n+1)C_p - C_f]$ for n systems. The relative difference of Policy III with respect to the perfect learning policy is equal to $\frac{\frac{1}{2}[(n+1)C_p - C_f]}{[\frac{3}{2}n-\frac{1}{2}]C_p + \frac{1}{2}C_f} = \frac{(n+1)C_p - C_f}{(3n-1)C_p + C_f}$. It holds that $\lim_{n\to\infty} \frac{(n+1)C_p - C_f}{(3n-1)C_p + C_f} = \frac{1}{3}$.

Suppose $(n+1) > \frac{C_f}{C_p}$. Under Policy I (= the optimal policy), the 'do nothing' action is taken at $z = L, L-1, \ldots, L-(t-2)$. At $z = L-(t-1)$, it is never optimal to apply the 'do nothing' action for more than one system (cf. Proposition 3.3). Hence, it is optimal to apply preventive maintenance for n or $n-1$ systems. If preventive maintenance is applied for $n-1$ systems at $z = L-(t-1)$, then one follows the perfect learning policy at $z = L-(t-1)$ and also during the rest of the lifespan.

Now, let us first apply preventive maintenance for n systems at $z = L-(t-1)$, the optimal policy after that. We apply the 'do nothing' action for $z = L - t, \dots, L - (2t - 3)$. In this case, the next decision time is at $z = L - (2t - 2)$. With the same reasoning as for Proposition 3.3, at time $z = L - (2t - 2)$, we obtain that it is never optimal to apply the 'do nothing' action for more than one system. If we apply the 'do nothing' action only for 1 system at this moment, then the expected total cost is $nC_p + (n-1)C_p + 0.5[C_f + V_{L-(2t-1)}(1, \mathbf{e} - \mathbf{e}_i)] + 0.5V_{L-(2t-1)}(0, \mathbf{e} + (t-1)\mathbf{e}_i) = (2n-1)C_p + 0.5C_f$. If we follow Policy III at $z = L - (t-1)$ and $z = L - (2t-2)$, the expected total cost is equal to $2nC_p$. Under Assumption 3.1 ($2C_p < C_f$), $(2n-1)C_p + 0.5C_f > 2nC_p$. Therefore, if preventive maintenance is applied for n systems at time $z = L - (t-1)$, then one follows Policy III at $z = L - (t-1)$ and also during the rest of the lifespan.

Hence Policy I is equal to either the perfect learning policy or Policy III during the whole lifespan. If $(n+1) > \frac{C_f}{C_p}$, then the perfect learning policy is cheaper than Policy III, and thus Policy I is equal to the perfect learning policy. Otherwise, Policy I is equal to Policy III.

□

3.B. Solution approach

In this section, we provide an outline of the backward recursion algorithm that we use to solve for the Bellman optimality equations characterized in Section 3.3. Since the belief state is continuous, we apply a grid-based discretization approach with linear interpolation of the value function (see Algorithm 2).

We denote the discretization level of the belief state by Δ_p. The belief is rounded up by using the function $g_{RU}(p, \mathbf{y}, \mathbf{d})$ and rounded down by using the function $g_{RD}(p, \mathbf{y}, \mathbf{d})$. The round-up function is calculated as $g_{RU}(p, \mathbf{y}, \mathbf{d}) = \left\lceil g(p, \mathbf{y}, \mathbf{d}) \frac{1}{\Delta_p} \right\rceil \Delta_p$ and the round-down function is calculated as $g_{RD}(p, \mathbf{y}, \mathbf{d}) = \left\lfloor g(p, \mathbf{y}, \mathbf{d}) \frac{1}{\Delta_p} \right\rfloor \Delta_p$, where $\lceil \cdot \rceil$ is the ceiling function that gives the least integer greater than or equal to its argument as output, and $\lfloor \cdot \rfloor$ is the floor function that gives the greatest integer less than or equal to its argument as output. In order to approximate the updated belief and the value function, linear interpolation is used (Hauskrecht, 2000). For this purpose, the ratio corresponding to the distance of

$g_{RD}(p, \mathbf{y}, \mathbf{d})$ to the updated belief $g(p, \mathbf{y}, \mathbf{d})$ is calculated as $\alpha = \frac{g(p,\mathbf{y},\mathbf{d}) - g_{RD}(p,\mathbf{y},\mathbf{d})}{\Delta_p}$. The ratio that corresponds to the distance of $g_{RU}(p, \mathbf{y}, \mathbf{d})$ to the updated belief is then equal to $1 - \alpha$. We approximate the value function at the updated belief state $g(p, \mathbf{y}, \mathbf{d})$ as

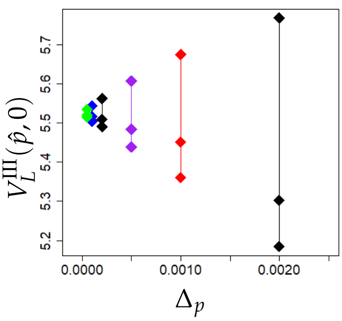
$$\hat{V}_z(g(p, \mathbf{y}, \mathbf{d}), \mathbf{y}) = \alpha V_z^{\text{appr}}(g_{RD}(p, \mathbf{y}, \mathbf{d}), \mathbf{y}) + (1 - \alpha) V_z^{\text{appr}}(g_{RU}(p, \mathbf{y}, \mathbf{d}), \mathbf{y}),$$

where $V_z^{\text{appr}}(p, \mathbf{y})$ is a function that represents an approximation of the optimal expected cost for systems with the component-age vector $\mathbf{y}$ when the remaining number of periods in the lifespan is z and the belief state is $p \in \{0, \Delta_p, 2\Delta_p, \ldots, 1\}$. Note that the original value function V_z takes any value of belief in $[0, 1]$ as an input while the function V_z^{appr} only takes the belief values from a discrete belief space determined by Δ_p.

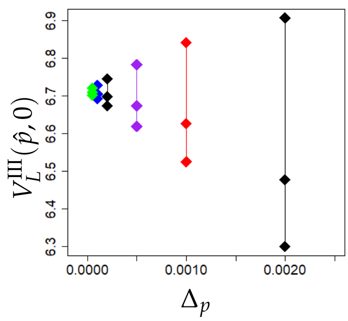
It is important to choose the discretization parameter Δ_p small enough to have a good approximation. The finer the discretization level is (in other words, as Δ_p decreases), the better the approximation is. Note that the number of states is equal to $\left(\frac{1}{\Delta_p} + 1\right)(u + 1)^n$. As Δ_p decreases, the number of states increases, which increases the memory requirement and solution time. The number of states increases exponentially under Policy I. During the value iteration, every possible policy is evaluated for each possible state. The number of states also determines the time complexity of the algorithm. For Policy III the number of states is equal to $\left(\frac{1}{\Delta_p} + 1\right)(u + 1)$, which does not depend on the number of systems. Please note that under Policy II, we apply directly Policy III but we update the Bayesian function jointly for each system. In numerical experiments, we have 92,023 states (required 2.11 GB RAM space) for $n = 1$ (Policy III), 2,116,529 states (required 2.23 GB RAM space) for $n = 2$ and 48,680,167 states for $n = 3$. We needed to reduce the number of states for $n = 3$ by removing the states that are never visited during the value iteration. This required a long preprocessing of the data beforehand. After preprocessing, we have 18,716,828 states (required 10.02 GB RAM space). The experiments are done on a computer with an Ubuntu Desktop 20.04 operating system, 64 GB RAM, and 16 core AMD EPYC processor (1996.249 MHz). The computation times for $L = 75$ of the value iteration algorithm are 17.430 sec for $n = 1$, 405.915 sec for $n = 2$ (Policy I) and 944.929 sec for $n = 3$ (Policy I). However, please note that data preparation times before the value iteration are not reported here. The preparation times increase significantly as the number of systems increase. Please also note that computation times increase linearly as a function of L, and they are insensitive for the C_p, C_f, k and $\hat{p}$ values.

Figure 3.12 shows the behavior of the approximate value function as a function of Δ_p for $C_p = 0.1$, $k = 5$, $L = 200$ and $n = 1$. In Figure 3.12, there are three points for each Δ_p value. The middle point shows the approximate value of $V_L^{\text{III}}(\hat{p}, 0)$ returned by the algorithm when the interpolation of the approximate value function is performed as described above. The lower point shows the approximated value of $V_L^{\text{III}}(\hat{p}, 0)$ if the algorithm were implemented with $\alpha = 1$, and the upper point shows the approximated value of $V_L^{\text{III}}(\hat{p}, 0)$ if it were implemented with $\alpha = 0$.

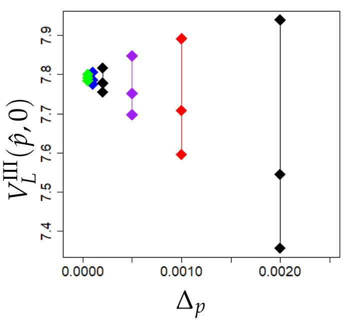
For our numerical experiments, $\Delta_p = 0.00025$ is selected due to fact that the change in approximation compared to $\Delta_p = 0.00005$ is sufficiently small (for instance, it is less than 0.3% for $C_p = 0.1$, $k = 5$, $L = 200$, $y = 0$ and $n = 1$). Notice that the rounding error accumulates through backward recursion, therefore, the error is largest for $L = 200$.



(a) $\hat{p} = 0.25$



(b) $\hat{p} = 0.5$



(c) $\hat{p} = 0.75$

Figure 3.12: Behavior of $V_L^{\mathrm{III}}(\hat{p},0)$ with respect to Δ_p for $C_p = 0.1$, $k = 5$, $L = 200$ and $n = 1$.

Algorithm 2 Backward Recursion Algorithm with Linear Interpolation of Belief State

---

Initialize $z = 1$

$V_0^{\text{appr}}(p, \mathbf{y}) \leftarrow 0\ \forall \mathbf{y}$ and $p \in \{0, \Delta p, 2\Delta p, \dots, 1\}$.

while $z < L + 1$ **do**

  for all $\mathbf{y}$ and $p \in \{0, \Delta p, 2\Delta p, \dots, 1\}$ **do**

    for all $\mathbf{a} \in \mathcal{A}$ **do**

$$\begin{aligned}\tilde{V}_z^{\text{appr}}(p, \mathbf{y}, \mathbf{a}) &= \sum_{i=1}^{n} C_p a_i \\ &+ \sum_{m=1}^{|D|} \Big[\Big(\sum_{i=1}^{n} C_f d_i^m + \hat{V}_{z-1}(g(p, (\mathbf{y} \circ (\mathbf{e} - \mathbf{a}), \mathbf{d}^m), (\mathbf{y} \circ (\mathbf{e} - \mathbf{a}) + \mathbf{e}) \circ \mathbf{d}^m) \Big) \\ &\quad F(p, \mathbf{y} \circ (\mathbf{e} - \mathbf{a}), \mathbf{d}^m) \Big]\end{aligned}$$

    end for

    $\hat{V}_z(p, \mathbf{y}) \leftarrow \min_{\mathbf{a}}\{\tilde{V}_z^{\text{appr}}(p, \mathbf{y}, \mathbf{a})\}$.

    $\mathbf{a}_{\text{I}}^*(p, \mathbf{y}, z) \leftarrow \operatorname{argmin}_{\mathbf{a}}\{\tilde{V}_z^{\text{appr}}(p, \mathbf{y}, \mathbf{a})\}$

  end for

  $z = z + 1$

end while

return $\hat{V}_z(p, \mathbf{y})$ as the approximation of the original value function $V_z(p, \mathbf{y})$ and the corresponding optimal policy as , $\mathbf{a}_{\text{I}}^*(p, \mathbf{y}, z),\ \forall (p, \mathbf{y}, z)$.

4

Spare parts replenishment at local warehouses with ADI

4.1. Introduction

In this chapter, we investigate a spare part replenishment problem. We consider a setting with multiple technical systems that have a single critical stock-keeping unit (SKU). These systems are supported by a local warehouse with spare parts. The inventory at the local warehouse is replenished with periodic reviews. When a system requires a spare part, it is directly shipped from the local warehouse. If there is not any available spare part, a part can be delivered via an emergency shipment from the central warehouse. However, the technical system is then down for a longer time, which is costly, and the emergency shipment itself is generally also expensive. However, keeping stock for spare parts results in inventory holding costs, which involves opportunity costs, warehousing costs and/or costs of spare parts becoming obsolete (van Wingerden, 2019, p.1-15). Prediction of demand for spare parts becomes crucial to balance the trade-off between inventory holding costs and costs of emergency shipments. A signal can be generated in advance of a failure. Based on the total number of signals, replenishment takes place at the beginning of a period. These signals constitute advance demand information (ADI) for the spare parts stock (see Hariharan and Zipkin (1995) and Karaesmen (2013) for more details on ADI).

The ideal situation for the maintenance of technical systems is that all failures are predicted, no false predictions are generated, and the predictions are made sufficiently far in advance. In that case, for all upcoming failures, a spare part can be sent to the system from a central location and the failing component can be replaced from the stock immediately. There would be no expensive local stocks of spare parts and no emergency shipments. However, ADI is not always perfect in practice. Signals are imperfect. That means false positive signals (i.e., a signal not leading to a failure) and false negative signals (i.e., unpredicted failures) are possible. We refer to the time between the generation of the signal and the actual failure as the *demand lead time*. The fraction of signals that are true positive is called *precision*, and the fraction of failures for which a signal is generated is called *sensitivity*. In the worst case, we have no predictions at all (zero precision), no useful signals (zero sensitivity), or each failure happens at the same moment as when the signal is generated (zero demand lead time).

The Internet of Things and Artificial Intelligence (AI) can bring us closer to the ideal situation of having perfect ADI. But how close do we need to be to that ideal situation in order to have sufficiently low spare parts and low costs? In this chapter, we mainly investigate this problem.

We formulate a Markov decision process model with precision, sensitivity, and demand lead time as input parameters. We derive an optimal policy for the spare parts inventory that minimizes the long-run average cost per period. Next, we compare the optimal costs for a given precision, sensitivity, and delay time against the optimal costs in the worst-case situation. Subsequently, we analyze how the optimal costs and the optimal spare parts stock reduce as precision, sensitivity, and demand lead time approach from the worst case to the ideal case.

We summarize the main findings of this study as follows: (1) For a given precision level, the optimal costs depend on the sensitivity and the demand lead time only through the product of these two terms (see Proposition 4.1). (2) The Pareto principle holds for precision, e.g., 30% perfectness in precision (i.e., precision is equal to 30% of the perfect precision) brings 70% reduction in optimal costs compared to the worst case optimal costs. (3) The opposite of the Pareto principle holds for the product of sensitivity and demand lead time, e.g., 70% perfectness in the product of sensitivity and demand lead time (i.e., the product is equal to 70% of the product of perfect sensitivity and perfect demand lead time) brings only 30% reduction in optimal costs compared to the worst case optimal costs. (4) We analyze the combined effect of precision and the product of sensitivity and demand lead time on optimal costs and inventory levels. To obtain a significant cost reduction, having a high sensitivity and a high demand lead time is needed while the precision can be moderate.

We organize the rest of the chapter as follows. In Section 4.2, we provide the literature review on the related work. In Section 4.3, we present the model description. In Section 4.4, we provide a reduction of the main problem (cf. main finding (1)). In Section 4.5, we formulate an MDP model for this reduced problem. In Section 4.6, we analyze two special cases of the reduced problem. This section is followed by computational experiments and sensitivity analysis in Section 4.7. Finally, in Section 4.8, we conclude the chapter.

4.2. Literature review

There are two streams of literature on inventory control problems related to our work with imperfect ADI. The first stream concerns single-item, infinite-horizon inventory control problems with imperfect ADI. The second stream focuses on spare parts inventory control problems with ADI that is obtained from condition-based monitoring of the machines installed in the field.

In the first literature stream, we have the following papers: van Donselaar et al. (2001), Thonemann (2002), Tan et al. (2007), Liberopoulos and Koukoumialos (2008), Tan (2008), Tan et al. (2009), Gayon et al. (2009), Benjaafar et al. (2011), Song and Zipkin (2012), Bernstein and DeCroix (2015), Topan et al. (2018), Zhu et al. (2020). van Donselaar et al. (2001) study the effect of imperfect ADI for inventory systems in a project-based supply chain. Thonemann (2002) investigates the effect of sharing imperfect ADI within the supply chain on average costs, mean basestock levels and variations of the production quantities for a multi-echelon system. Results are shown for the value of ADI as a function of order probability and information quality, and ADI for single items is compared to aggregate ADI for a group of items. Tan et al. (2007) consider an inventory control problem where imperfect ADI signals are generated for future demands. They show that the optimal policy is of the order-up-to type and the order level is a function of the number of imperfect ADI signals. Liberopoulos and Koukoumialos (2008) investigate how the uncertainty in ADI affects the performance of a make-to-stock supplier. They assume a single-item system with two customer classes. The first customer class does not provide any ADI and places immediate orders. The second customer class provides cancelable reservations on a requested due date. Tan (2008) considers a demand forecasting problem in a make-to-stock system. Information provided by customers on their future demand, which is subject to change in future, constitutes imperfect ADI for the decision maker. Tan et al. (2009) consider an inventory problem with two customer classes having different priorities. Available stock is reserved for the future demand of preferred customers at the expense of losing the current orders of lower class customers. Future demand estimations of preferred customers constitutes imperfect ADI for the decision maker. Gayon et al. (2009) study an inventory-production system with multiple customer classes where customers provide imperfect ADI on the due date of their orders. Benjaafar et al. (2011) consider a production-inventory system where customers provide imperfect ADI for their orders. In this problem, customers provide updates on their orders but the times between the consecutive updates are random. Song and Zipkin (2012) analyze a capacity/inventory planning problem under imperfect ADI for a single product with seasonal demand. Bernstein and DeCroix (2015) consider a multi-product system where the decision maker receives imperfect signals for the demand volume (i.e., signals for the total aggregate demand) or mixed demand (i.e., signals revealing information about the market shares for each product). Topan et al. (2018) focus on a spare parts management problem with imperfect ADI and the option of returning inventory. Zhu et al. (2020) assume a single-item, periodic-review setting for a spare parts management problem under imperfect ADI. They use planned maintenance tasks to forecast spare part demand.

Among these studies, Song and Zipkin (2012), Gayon et al. (2009), Benjaafar et al. (2011), Topan et al. (2018), and Zhu et al. (2020) assume that demands are lost or satisfied via emergency shipments in stockout situations. Our study comes closest to Topan et al. (2018). They consider a single-item, single-location, periodic-review, infinite-horizon inventory control problem, where imperfect signals are generated for future demands. Signals are generated for a fraction of all demands, signals can be false, and the actual demand occurs a stochastic time after the signal was generated. They derive the structure of the optimal ordering and return policy, and they show the value of the imperfect ADI in a computational experiment. We assume a simpler model, but explicitly characterize how the optimal costs behave as a function of the precision, sensitivity and the demand lead time of the imperfect demand signals.

The second stream of literature is about condition monitoring in spare parts management. The condition of a component can be used to predict when the component fails. In that way, also advance demand information is obtained. Within this second stream, we have the following studies: Deshpande et al. (2006), Li and Ryan (2011), Lin et al. (2017), Eruguz et al. (2018). Deshpande et al. (2006) use the part-age information to model the degradation of aircraft spare parts at the U.S. Coast Guard. Li and Ryan (2011) exploit the real-time condition monitoring information for the inventory control of spare parts. They assume a Wiener process as the degradation model. Lin et al. (2017) consider a single critical component of multiple installed machines in the field. The installed components follow a Markov degradation process and that information is used for optimizing the spare parts inventory. Eruguz et al. (2018) study an integrated maintenance and spare part optimization problem for moving assets where the degradation level of a single critical component is observable. They model the degradation of the component by a continuous-time Markov chain. These studies all assume that the condition of the components can be observed perfectly. Hence, these papers implicitly assume that both precision and sensitivity are perfect. Only the moments that the failures occur are uncertain.

To the best of our knowledge, our study is the first that provides analytical insights on the effect of precision, sensitivity, and the demand lead time on the optimal costs and inventory levels in a spare parts management problem setting.

4.3. Model description

In this section, we provide the detailed description of the model to address the problem introduced in Section 4.1. We consider a setting where a significant number of technical systems is supported by a local warehouse that keeps spare parts on stock. These spare parts are needed to execute maintenance actions. The technical

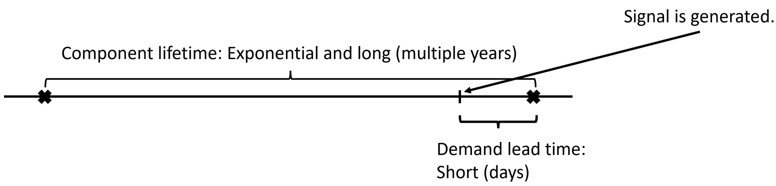


Figure 4.1: Generation of predictions.

systems are operated during a time horizon that is assumed to be infinite. The local warehouse is part of a service network consisting of a central warehouse and multiple local warehouses.

We focus on a single critical component that is part of all technical systems. All components are identical and have the same failure behavior. We assume that a component has an exponentially-distributed lifetime. A certain amount of time before a failure (i.e., the end of a component's lifetime), a signal is generated by a predictive model. After this signal generation, the component still functions, but it is known that the component may fail soon. The time from the signal generation until the end of the lifetime is referred to as demand lead time and is denoted by D. The demand lead time is deterministic and relatively short compared to the lifetime of the component (Figure 4.1).

We assume that the technical systems operate continuously. Therefore, interrupting their operation for a preventive replacement is equally expensive as a corrective replacement. Hence, we assume that replacements are only executed when a component fails. The technical systems are at a close distance from the local warehouse and we assume that a spare part is provided to a technical system within such a short time that it does not cause extra downtime of the technical system.

Spare parts of the critical component are kept in stock at the local warehouse. The local warehouse is replenished periodically (e.g., every week). Hence, we divide the time horizon in periods of length one, and the periods are numbered as $0, 1, \ldots$. The beginning of a period t is called time t. The local service point is replenished by the central warehouse, which is assumed to have ample stock. The corresponding replenishment lead time is short and is assumed to be 0. Hence, ready-for-use parts are ordered at the beginning of each period t, and they arrive immediately.

The generated signals for upcoming failures are subject to false positives and false negatives. Let $p \in [0,1]$ denote the precision of the generated signals, and let $q \in [0,1]$ denote the sensitivity. If $p = 1$ and $q = 1$, we have perfect signals. If $p = 0$ or $q = 0$, the signals are useless.

If the demand lead time D is at least one period, there is at least one replenishment moment during the demand lead time and at the last replenishment moment a part can be ordered via a regular replenishment (and that part can be used for a corrective replacement as soon as the failure occurs). This implies that having a value of D that is larger than 1 is equally good as having D equal to 1. Therefore, without loss of generality, we limit ourselves to values of $D \in [0,1]$. If $D = 1$, there is always precisely one replenishment moment during the demand lead time of a failing component and that moment is used to order a spare part in order to replace the component as soon as the failure occurs. If $D = 0$, the signal and the corresponding failure of a component occur at the same time, and hence the generated signals are useless.

In each period t, we have the following order of events. First, at the beginning of the period, based on the active number of signals (collected during the time interval $(t - D, t)$) and the on-hand spare parts stock, a replenishment order for ready-for-use spare parts is placed and delivered. Next, during the whole period, replacements of failed parts are executed. Replacements are also executed for failures of systems for which no signal was generated. The replacements can be executed without any delay as long as spare parts are on stock. When the on-hand stock is 0 and a failure occurs, a spare part is delivered from the central warehouse via an emergency shipment. This leads to a short delay for the execution of the replacement and hence to a downtime cost for the involved system. In addition, we have an extra cost for the emergency shipment. The corresponding costs are captured in the cost factor c_{em}. For every failure, a spare part is sent to the local warehouse via a regular replenishment or via an emergency shipment. Hence, the unit costs for the delivered parts are constant under any reasonable policy. Hence, these costs are excluded in our model (they are also excluded in the cost factor c_{em}). For parts that are on stock at the end of a period, we have inventory holding costs c_{h} per part. The order of events in period t is summarized in Figure 4.2.

Notice that the moments at which failures of technical systems occur are independent of each other and the spare parts provisioning. Given the lifetime distribution that we have, the failures for all technical systems together occur according to a

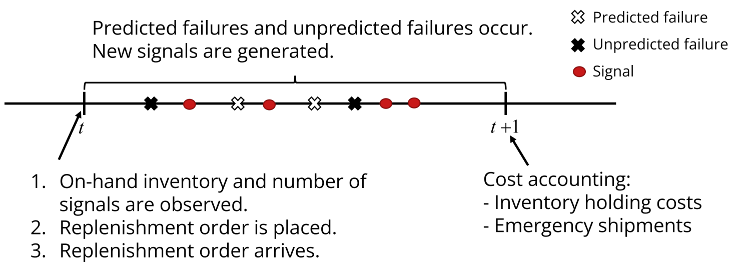


Figure 4.2: Order of events in period t.

Poisson process with a constant rate. That rate is denoted by λ (> 0). Correct signals will be generated for a fraction q of these failures. That means that correct signals (true positives) occur according to a Poisson process with rate $q\lambda$. Let $\hat{\lambda}$ be the rate of the Poisson process with which signals arrive; i.e., $\hat{\lambda}$ is the rate with which true positives and false positives arrive. The rate with which true positives arrive is $p\hat{\lambda}$. Since we already know the true positives arrive at rate $q\lambda$, it follows that $\hat{\lambda} = \frac{q\lambda}{p}$ (we take this rate equal to ∞ when $p = 0$).

At the beginning of each period, we have to decide how many parts must be replenished. The objective is to minimize the long-run average costs per period, which consist of inventory holding costs and costs related to emergency shipments (recall that the latter costs include system downtime costs). The minimal costs are denoted by $\tilde{C}(p, q, D)$. We include the precision p, sensitivity q, and demand lead time D as parameters because later we are interested in how these minimal costs behave as a function of p, q, and D.

We require the assumption of exponential lifetime distribution to provide analytical and numerical insights. But we would like to add a remark on the lifetime distribution because we assume that a signal can be generated before a failure.

Remark 4.1 The exponential lifetime distribution and a deterministic demand lead time as assumed in this section can be seen as a variant of the so-called *delay time model* in the literature. In a delay time model, we can consider a setting with a long exponential time-to-defect and a short deterministic delay time until failure. When the time-to-defect is too long (e.g., 2 years) compared to a short delay time (e.g., 3 days), the lifetime distribution looks like it is exponential. Hence, our assumptions are almost the same as assuming a delay time model with a long exponential time-to-defect and a short deterministic delay time.

4.4. Reduction of the main problem

In this section, we reduce the main problem with a cost function having three parameters (i.e. p, q and D) into a problem with a cost function that has two parameters (i.e. p and r, where $r = qD$). During a period t, a signal that occurs in the first $1 - D$ time units will result in a failure before the end of the period (in case of a true positive) or it vanishes before the end of the period (in case of a false positive). For the signals that occur in the last D time units, the failure occurs in period $t + 1$. For a fraction q of these failures, a signal is generated, and the required spare parts can be ordered at the beginning of period $t + 1$. Overall, for all failures occurring in period $t + 1$, qD is the fraction for which signals are generated in period t and $1 - qD$ is the fraction for which no signals are generated (see Figure 4.3).

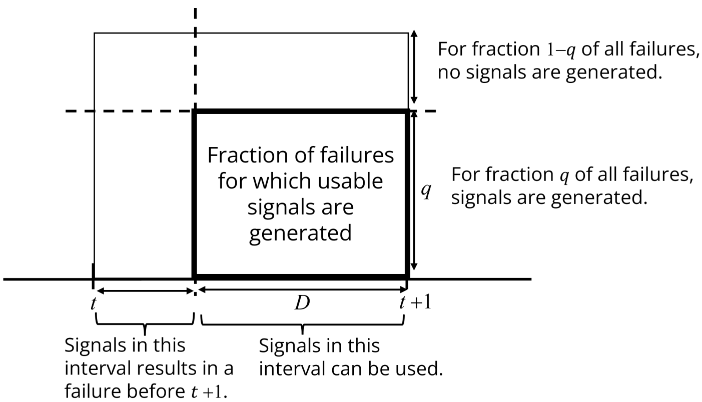


Figure 4.3: Fraction of usable signals.

The above reasoning shows that the number of predicted failures in a given period $t + 1$ is Poisson distributed with rate $qD\lambda$, and the number of unpredicted failures in that period $t + 1$ is Poisson distributed with rate $(1 - qD)\lambda$. This latter amount is denoted by X^{u}. For the predicted failures, the corresponding number of signals in the preceding period t is Poisson distributed with rate $qD\lambda/p$. These signals are all active at the beginning of period $t+1$. We denote this Poisson distributed amount by X^{s}.

Let us now consider the dynamics in period $t+1$ and how the demand behaves in that period. At the beginning of that period, we have a number of active signals that is a realization of X^{s}. Let us denote this amount by a. For a given a, the number of predicted failures in period $t+1$ is Binomially distributed with a trials and success probability p. This Binomially distributed amount is denoted by $X^{\mathrm{p}}(a)$. The number of unpredicted failures in period $t+1$ is given by X^{u}. Hence, the total demand in period $t+1$ equals $X^{\mathrm{p}}(a)+X^{\mathrm{u}}$. The parameters of the distributions of $X^{\mathrm{p}}(a)$ and X^{u} only depend on q and D via their product qD. Because q and D play no role in other aspects of our inventory model, this leads to the following proposition.

Proposition 4.1 *For each $p \in [0,1]$, the optimal policy and optimal costs $\tilde{C}(p,q,D)$ only depend on the product of the sensitivity q and demand lead time D.*

This proposition implies that a sensitivity $q=\alpha \in [0,1]$ and demand lead time $D=\beta \in [0,1]$ lead to the same optimal policy and optimal costs as a sensitivity $q=\beta$ and demand lead time $D=\alpha$. That is, it is equally important to have a high value for the sensitivity q as having a high value for the demand lead time D. Based on Proposition 4.1, the optimal costs function $\tilde{C}(p,q,D)$ is simplified to the function $C(p,r)$ with $r=qD$ (notice that r represents the fraction of failures for which usable signals are generated).

4.5. MDP formulation

In this section, we provide the MDP formulation for the reduced problem presented in Section 4.4. We need this MDP formulation because decisions in subsequent periods depend on each other. This can be seen as follows. At the beginning of a period, the stock will be increased to a certain amount, and with that amount the demands X^{u} and $X^{\mathrm{p}}(a)$ have to be covered. The larger the number of signals, the larger the level to which the inventory position will be increased, but if this level is chosen relatively high and the number of realized demands from the active signals is low, then a relatively large stock is left at the end of the period, and that may lead to a larger stock than desired in the next period. That means that a simple myopic policy that minimizes the costs in the current period will not be optimal.

For the MDP, the state at the beginning of a period is described by (y, a), where y is the on-hand stock and a is the number of active signals. The state space is given by $\mathcal{S} = \{(y, a) \mid y, a \in \mathbb{N}_0\}$. At the beginning of a period, based on the state (y, a), the on-hand stock is increased by a replenishment order. We describe the action by the level $z \geq y$ to which the inventory position is increased. The replenishment order arrives immediately, and thus the on-hand stock becomes also immediately equal to z. Given action z, the direct expected costs are

$$d(z,a) = \sum_{x=0}^{z} (z-x) P\{X^{\mathrm{u}} + X^{\mathrm{p}}(a) = x\} c_{\mathrm{h}} + \sum_{x=z+1}^{\infty} (x-z) P\{X^{\mathrm{u}} + X^{\mathrm{p}}(a) = x\} c_{\mathrm{em}}.$$

If the total demand is x, then the on-hand stock at the beginning of the next period is $(z-x)^+$. The number of active signals $\hat{a}$ at the beginning of the next period is a realization of X^{s}. This results in the following formulas for the n-period costs $V_n(y, a)$:

$$V_{n+1}(y,a) = \min_{z \geq y} \hat{V}_{n+1}(z,a), \quad (y,a) \in \mathcal{S}, \tag{4.1}$$

where

$$\begin{aligned} \hat{V}_{n+1}(z,a) \quad &= \quad d(z,a) + \sum_{\hat{a}=0}^{\infty} P\{X^{\mathrm{s}} = \hat{a}\} \Bigg(P\{X^{\mathrm{u}} + X^{\mathrm{p}}(a) \geq z\} V_n(0, \hat{a}) \\ &\qquad + \sum_{x=0}^{z-1} P\{X^{\mathrm{u}} + X^{\mathrm{p}}(a) = x\} V_n(z-x, \hat{a}) \Bigg) \end{aligned}$$

and $V_0(y, a) = 0$ for all $(y, a) \in \mathcal{S}$. Appendix 4.B presents how the contributions of the inventory holding costs and emergency shipment costs to $V_n(y, a)$ are calculated by using the MDP formulation. The optimal costs $C(p, r)$ are obtained by $C(p, r) = \lim_{n\to\infty} \frac{V_n(0,0)}{n}$. In order to see the effect of p and r on the optimal costs, we compare $C(p, r)$ with respect to the worst-case situation where $p = 0$ (i.e., we have no predictions at all) and $r = 0$ (i.e., we have no useful signals and each failure happens at the same moment as when the signal is generated). Optimal costs under the worst-case situation provide an upper bound on the costs. For this purpose, we define $\hat{C}(p, r) = \frac{C(p,r)}{C(0,0)}$. Then, $\hat{C}(0, 0) = 1$ and $\hat{C}(p, r)$ denotes how close we are to the worst-case situation at each point (p, r). For example, $\hat{C}(p, r) = 0.8$ means that we have 80% of the costs associated with the worst-case situation.

4.6. Special cases

In this section, we study two special cases of our model and provide analytical and numerical results. For this purpose, we first define a base instance with parameters $\lambda = 0.2$ (failures/demands per period), $c_{\mathrm{h}} = 1$ (Euro per part per period), and $c_{\mathrm{em}} = 10^4$ (Euro per emergency shipment). Please note that the emergency shipment cost also includes the system downtime costs. The value for λ is a common value for the demand rate at a local warehouse for one component. The value for c_{h} can be chosen w.l.o.g.; a value of 1 Euro per part per period would correspond to roughly 50 Euro per part per year and that could correspond to a part with a price of 250-500 Euro. The value of c_{em} includes the costs for executing an emergency shipment and the costs for extra downtime of the involved system while it is waiting for the delivery of the part. Normally an emergency shipment takes multiple hours and then the downtime costs can be significant. Hence, 10,000 Euro as costs for an emergency shipment is a quite common number in practice.

4.6.1 Special case 1: Perfect precision

Consider the model of Section 4.4 for the special case with precision $p = 1$. We then know that every active signal at the beginning of a period will result in an actual failure and it is optimal to take one part on stock per active signal. The number of unpredicted demands X^{u} is Poisson distributed with rate $(1-r)\lambda$. The optimal amount of stock for the unpredicted failures is like the optimal base stock level in a basic model with only unpredicted failures. This basic model is described in Appendix 4.A. In this case, we have a basic model instance with a Poisson distributed demand with rate $(1-r)\lambda$, and with cost parameters c_{h} and c_{em} for inventory holding and emergency shipments. The optimal base stock level is denoted by $S^*((1-r)\lambda)$, and this denotes the optimal stock for the unpredicted failures. For the predicted and unpredicted failures together, it is optimal to increase the on-hand stock to $z^*(y,a) = a + S^*((1-r)\lambda)$, when being in state (y,a) at the beginning of a period. If this rule is followed in every period, then the on-hand stock y at the beginning of a period will never exceed $S^*((1-r)\lambda)$ and hence is never larger than $a + S^*((1-r)\lambda)$. This leads to part (1) of the following lemma. Parts (2) and (3) of this lemma follow directly from Lemma A1.

Lemma 4.1 *For precision $p = 1$, it holds that:*

1. *It is optimal to increase the on-hand stock to $z^*(y,a) = a + S^*((1-r)\lambda)$ at the beginning of each period when being in state (y,a). The base stock level $S^*((1-r)\lambda)$ is non-increasing as a function of r;*
2. *The base stock level $S^*((1-r)\lambda)$ equals 0 if and only if $(1-r)\lambda \le ln(1+(c_h/c_{em}))$;*
3. *If $(1-r)\lambda \le ln(1+(c_h/c_{em}))$, then the optimal costs are equal to $(1-r)\lambda c_{em}$.*

In the best case, $r = 1$. Then all failures are predicted and for each failure a part can be ordered at the first order moment after a signal occurs. In that case, the optimal costs are equal to zero and no parts have to be kept on stock for unpredicted failures. These observations lead to the following corollary.

Corollary 4.1 *For precision $p = 1$, it holds that: If in addition $r = 1$, then the optimal costs are equal to 0 and $S^*((1-r)\lambda) = 0$.*

Next, we investigate the behavior of the optimal costs and the average on-hand inventory (at the end of a period) for the base instance. In Figure 4.5(a-b), we show how $\hat{C}(1,r)$ and $C(1,r)$ change as a function of r. Figure 4.5(a-b) suggests that costs are non-increasing and piecewise convex functions as a function of r. We also see an inverse Pareto principle: 70% perfectness for r leads to only a 35% reduction in optimal costs. In Figure 4.5(c), we illustrate the share of emergency shipment costs and inventory holding costs in $C(1,r)$ as a function of r. We see that costs of emergency shipments decrease as r increases until a certain point. This can be explained by the average on-hand inventory which is depicted in Figure 4.4(d). For low values of r, the costs of emergency shipments are relatively high. As r increases, the costs of emergency shipments decrease until the point where the base stock level $S^*((1-r)\lambda)$ is decreased from 3 to 2. At that point, the average on-hand inventory decreases with a large jump, and the costs of emergency shipments increase. After that point, a similar behavior is obtained until a second jump point, and that behavior is also obtained in the interval between that second jump point and $r = 1$.

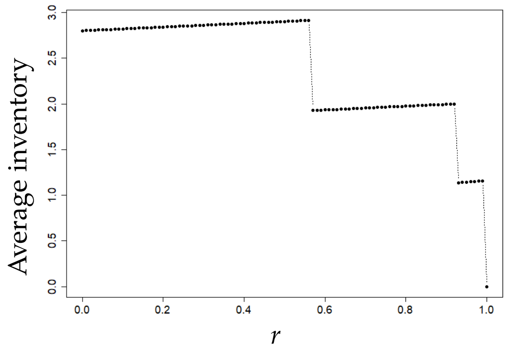
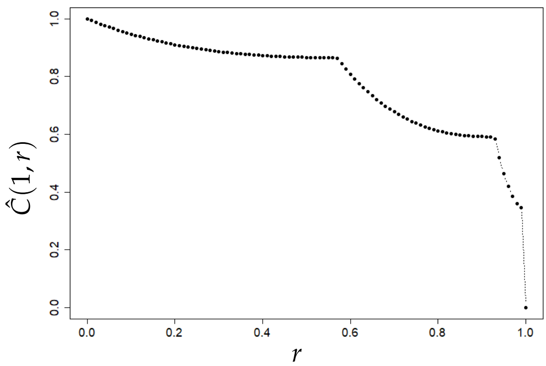
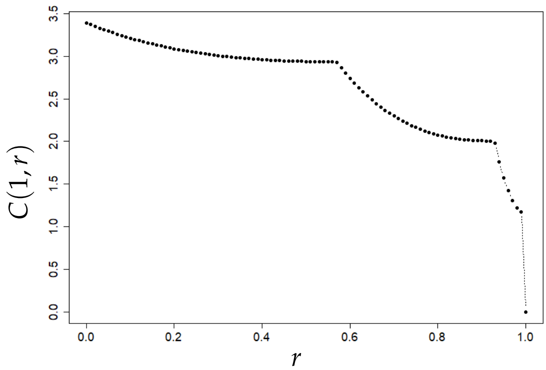


Figure 4.4: Average on-hand inventory as a function of r for the base instance with $p = 1$.

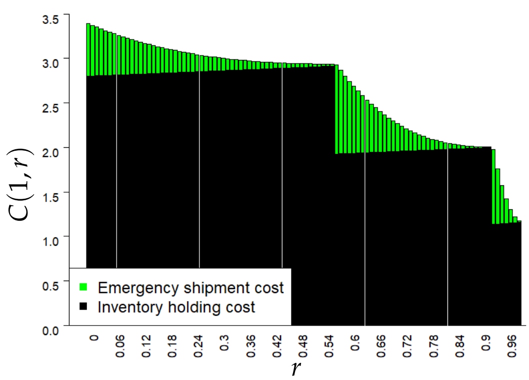
In Figure 4.6, we provide the optimal order-up-to levels $z^*(y, a)$ for the base instance with two different values of r. The optimal order-up-to level $z^*(y, a)$ is increasing as a function of a for a given y, and the other way around. We also see that, in all states, $z^*(y, a)$ is smaller for $r = 0.6$ than for $r = 0.5$ (which is in line with Lemma 4.1(1)).



(a) $\hat{C}(1,r)$ as a function of r.

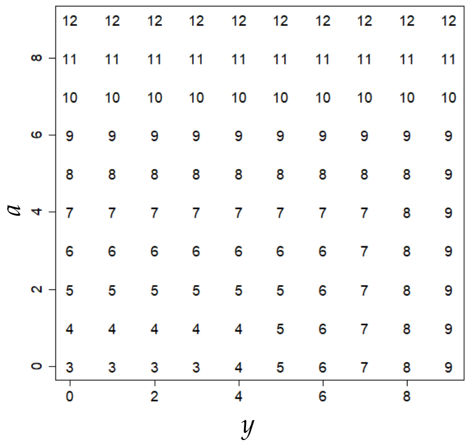


(b) $C(1,r)$ as a function of r.

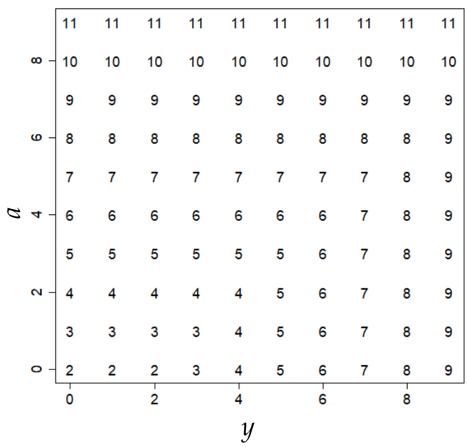


(c) Share of inventory holding and emergency shipment costs in $C(1,r)$ as a function of r.

Figure 4.5: Change in costs for the base instance with $p = 1$.



(a) For $p = 1$ and $r = 0.5$.



(b) For $p = 1$ and $r = 0.6$.

Figure 4.6: Optimal actions $z^*(y, a)$ for the base instance.

4.6.2 Special case 2: Perfect sensitivity and perfect timing of predictions

In this special case, we assume both a perfect sensitivity and a perfect demand lead time (i.e. $r = 1$). We again investigate the behavior of the optimal costs and the average on-hand inventory for the base instance. In Figure 4.8(a-b), we observe that $\hat{C}(p, 1)$ and $C(p, 1)$ are non-increasing in p. We further note that the decreasing behavior of the cost functions is different than what we observed in Figure 4.5(a-b). Specifically, the optimal costs decrease fastly for low values of p, while they decrease slowly for large values of p. Further, we now observe a Pareto principle: 30% perfectness for the precision p brings 70% reduction for the optimal costs.

We illustrate the share of emergency shipment costs and inventory holding costs in $C(p, 1)$ as a function of p in Figure 4.8(c) and the average on-hand inventory in Figure 4.7. For very low values of p, the costs of emergency shipments are relatively high. After that, they quickly decrease to zero. The inventory holding costs and the average on-hand inventory are first non-decreasing for very low values of p and they decrease on the rest of the interval [0,1].

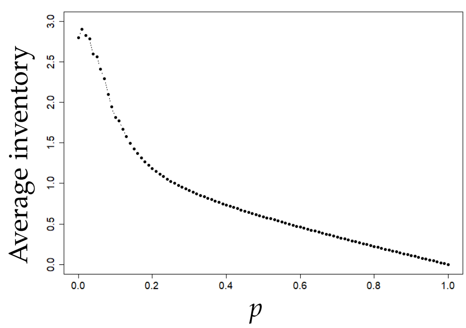
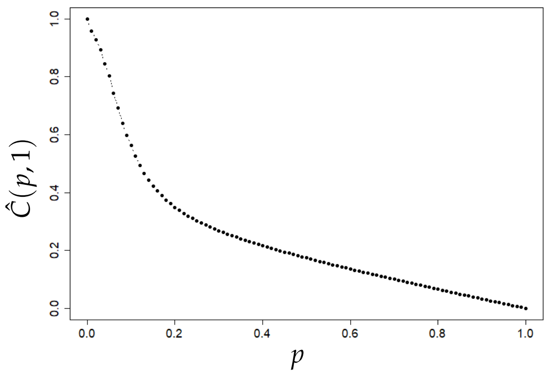
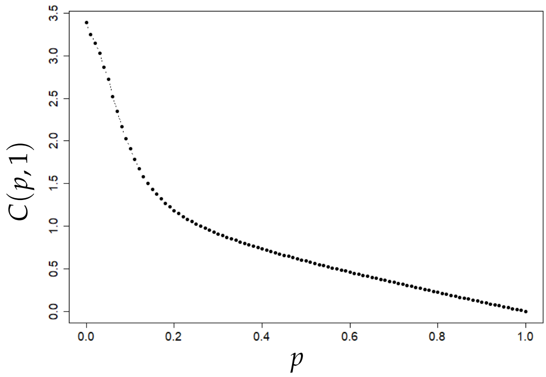


Figure 4.7: Average on-hand inventory as a function of p for the base instance with $r = 1$.

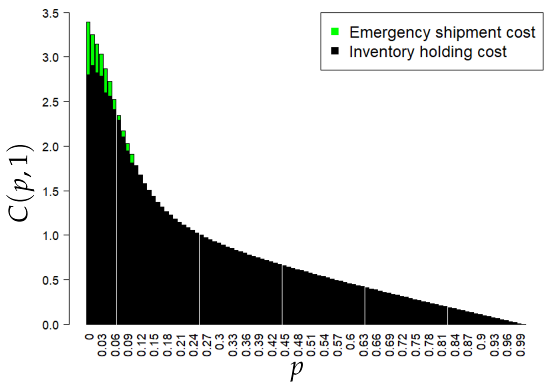
In Figure 4.9, we provide the optimal order-up-to levels $z^*(y, a)$ with respect to on-hand inventory levels y and the number of active signals a. Even though we have a relatively low level of p (i.e. $p = 0.5$), it holds that $z^*(y, a) = \min\{a, y\}$ at all points in this figure. For all active signals at the beginning of a period, a spare part is taken on stock, which explains having zero emergency shipment costs in this case (see Figure 4.8(c)).



(a) $\hat{C}(p,1)$ as a function of p.



(b) $C(p,1)$ as a function of p.



(c) Share of inventory holding and emergency shipment costs in $C(p,1)$ as a function of p.

Figure 4.8: Change in costs and inventory levels for the base instance with $r = 1$.

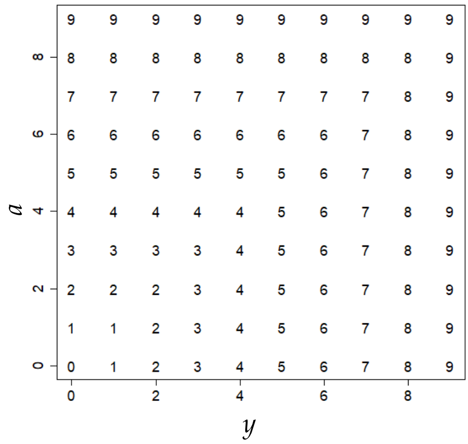


Figure 4.9: Optimal actions $z^*(y, a)$ for the base instance with $p = 0.5$ and $r = 1$.

4.7. Computational experiments

In this section, we provide the results of our computational experiments for varying values of p and r. We also perform a sensitivity analysis on the parameters of the base instance.

4.7.1 Computational experiments for the base instance for a general precision and sensitivity

The goal of this section is to generate further insights on the effect of a general p and r on $\hat{C}(p, r)$, the average on-hand inventory, and the average number of emergency shipments (per period) for the base instance. In Table 4.1, we observe how $\hat{C}(p, r)$ changes with respect to p and r for the base instance. For a constant $r > 0$, $\hat{C}(p, r)$ decreases in p. We observe the Pareto principle in each row of Table 4.1. For example, for $r = 1$, $\hat{C}(p, r)$ decreases 65% when p is only 20% of the perfect level. This effect can be seen more clearly in Figure 4.10(a). We also see that the larger r, the stronger $\hat{C}(p, r)$ decreases as a function of p. On the other hand, for a constant $p > 0$, we see the inverse Pareto principle. For example, for $p = 1$, in order to achieve an about 40% decrease in $\hat{C}(p, r)$ (i.e., $\hat{C}(p, r)$ equal to 59.2%), the value of r should be 90%. This behavior can also be observed in Figure 4.10(b). Table 4.1

and Figure 4.10 also show that a large value for r is needed to obtain a signifcant reduction in optimal costs, while for p a moderate value suffices.

Table 4.1: $\hat{C}(p,r)(\%)$ for the base instance.

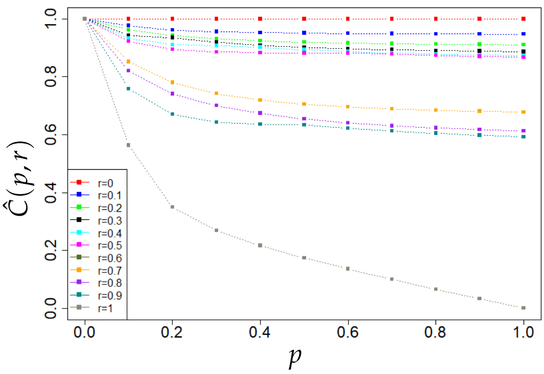
$r \backslash p$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
0.1	100.0	97.7	96.1	95.6	95.3	95.1	95.0	94.9	94.8	94.8	94.7
0.2	100.0	96.2	94.3	93.1	92.4	91.9	91.7	91.4	91.3	91.2	91.1
0.3	100.0	94.3	93.5	91.9	90.8	90.2	89.7	89.3	89.1	88.8	88.7
0.4	100.0	93.3	91.1	90.8	90.2	89.3	88.7	88.2	87.8	87.5	87.3
0.5	100.0	92.2	89.4	88.6	88.3	88.2	88.1	87.8	87.4	87.0	86.7
0.6	100.0	89.0	85.4	83.3	82.2	81.7	81.4	81.2	81.1	81.0	80.9
0.7	100.0	85.2	78.1	74.2	71.9	70.5	69.6	68.9	68.4	68.1	67.8
0.8	100.0	82.1	74.1	70.0	67.4	65.4	64.0	63.0	62.3	61.7	61.3
0.9	100.0	75.8	67.0	64.3	63.6	63.4	62.3	61.3	60.5	59.8	59.2
1	100.0	56.3	34.9	26.9	21.6	17.4	13.6	10.0	6.6	3.2	0.0

In Table 4.2, we show how the average on-hand inventory changes as a function of p and r. In the worst case scenario, the average on-hand inventory is 2.80 units. It is a non-monotonic function of p for a fixed r and a non-monotonic function of r for a fixed p. In general, the average on-hand inventory is non-increasing as a function of p and r.

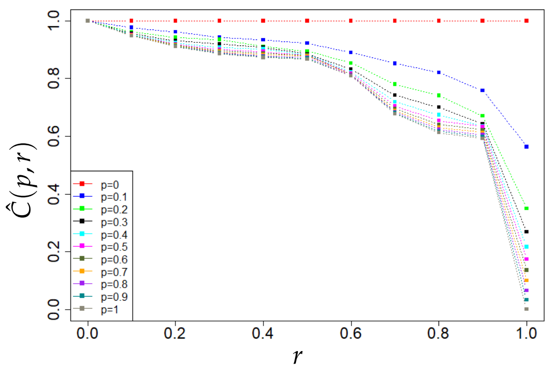
Table 4.2: Average on-hand inventory for the base instance.

$r \backslash p$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
0	2.80	2.80	2.80	2.80	2.80	2.80	2.80	2.80	2.80	2.80	2.80
0.1	2.80	2.98	2.89	2.86	2.85	2.84	2.83	2.83	2.83	2.82	2.82
0.2	2.80	2.86	2.98	2.92	2.90	2.88	2.87	2.86	2.85	2.84	2.84
0.3	2.80	2.92	2.84	2.98	2.94	2.92	2.90	2.89	2.88	2.87	2.86
0.4	2.80	2.99	2.86	2.83	2.98	2.95	2.93	2.91	2.90	2.89	2.88
0.5	2.80	2.88	2.89	2.85	2.83	2.82	2.82	2.93	2.92	2.91	2.90
0.6	2.80	2.61	2.38	2.20	2.11	2.05	2.01	1.98	1.96	1.95	1.93
0.7	2.80	2.72	2.35	2.26	2.16	2.09	2.04	2.01	1.99	1.97	1.95
0.8	2.80	2.36	2.40	2.24	2.20	2.13	2.08	2.04	2.01	1.99	1.97
0.9	2.80	2.45	2.10	1.96	1.90	1.87	2.07	2.04	2.04	2.01	2.00
1	2.80	1.81	1.18	0.91	0.73	0.59	0.46	0.34	0.22	0.11	0.00

In Table 4.3, we show how the average number of emergency shipments behaves as a function of p and r. In the worst case, the average number of emergency shipments is 0.59×10^{-4}. Due to the relatively high cost of an emergency shipment, the average number of emergency shipments is in general low under the optimal policy. Similar to the average on-hand inventory, the average number of emergency shipments is a non-monotonic function of p and r. We see that the average number of emergency shipments can be less than or equal to 0.1×10^{-4} for $p \geq 0.1$ with



(a) Change in $\hat{C}(p,r)$ as a function of p for varying values of r.



(b) Change in $\hat{C}(p,r)$ as a function of r for varying values of p.

Figure 4.10: Combined effect of p and r on $\hat{C}(p,r)$.

$r = 1$ and for $p \geq 0.6$ with $r \geq 0.8$. Again, in Table 4.3, we see that focusing on having large values of r is more crucial than focusing on having large values of p for a low average number of emergency shipments.

Finally, in Figure 4.11, we show the values of the optimal order-up-to level $z^*(y,a)$ as a function of the state variables a and y for $p = 0.8$ and $r = 0.8$. The value of $z^*(y,a)$ is at least 2 in all states (y,a). In fact, we observe that $z^*(y,a) = \min\{2 + a, y\}$ at all points in this figure. That means that in all states one spare part is taken on stock for each active signal and that a stock of (at least) 2 is kept for the unpredicted failures.

Table 4.3: Average number of emergency shipments ($\times 10^{-4}$) for the base instance.

$r \backslash p$	**0**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**1**
0	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.59	0.59
0.1	0.59	0.33	0.37	0.38	0.38	0.39	0.39	0.39	0.39	0.39	0.39
0.2	0.59	0.40	0.22	0.23	0.23	0.24	0.24	0.24	0.25	0.25	0.25
0.3	0.59	0.28	0.33	0.14	0.14	0.14	0.14	0.14	0.15	0.15	0.15
0.4	0.59	0.17	0.23	0.24	0.08	0.08	0.08	0.08	0.08	0.08	0.08
0.5	0.59	0.24	0.14	0.15	0.16	0.17	0.17	0.04	0.05	0.04	0.04
0.6	0.59	0.40	0.52	0.63	0.68	0.72	0.75	0.77	0.79	0.80	0.81
0.7	0.59	0.18	0.30	0.26	0.28	0.30	0.31	0.33	0.33	0.34	0.34
0.8	0.59	0.42	0.11	0.14	0.09	0.09	0.09	0.10	0.10	0.10	0.10
0.9	0.59	0.12	0.17	0.22	0.25	0.28	0.04	0.04	0.01	0.01	0.01
1	0.59	0.10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

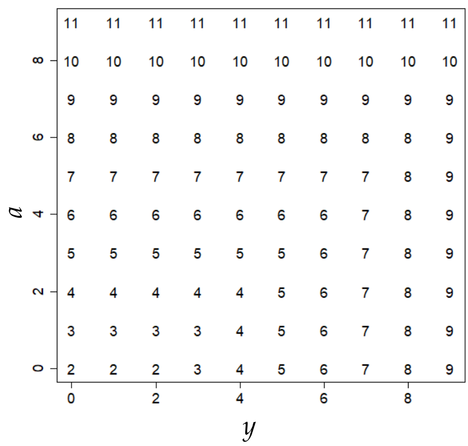


Figure 4.11: Optimal action $z^*(y, a)$ for the base instance, where $p = 0.8$ and $r = 0.8$.

4.7.2 Sensitivity analysis

In this section, we perform a sensitivity analysis for the parameters λ and c_{em} by considering $\lambda \in \{0.1, 0.2, 0.5\}$ and $c_{\text{em}} \in \{10^2, 10^4, 10^6\}$. In Table 4.4, we show how $\hat{C}(p, r)$ changes for varying values of λ and c_{em} as a function of r and p. Similar to our earlier observations for the base instance, we see that the Pareto principle holds for an optimal costs reduction in terms of p and an inverse Pareto principle holds in terms of r for different values of λ and c_{em}. Observations regarding the Pareto and inverse Pareto principles are insensitive to the failure rate and the cost of an emergency shipment.

Table 4.4: $\hat{C}(p,r)$ (%) for varying values of λ and c_{em} as a function of r and p.

	$\lambda = 0.1$ and $c_{em} = 10^2$					$\lambda = 0.2$ and $c_{em} = 10^2$					$\lambda = 0.5$ and $c_{em} = 10^2$				
$r \backslash p$	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**
0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
0.25	100.0	91.7	89.2	88.3	87.8	100.0	99.6	99.2	99.1	99.1	100.0	92.5	89.8	88.7	88.1
0.5	100.0	88.7	83.3	81.3	80.3	100.0	82.4	77.5	75.6	74.7	100.0	86.3	81.2	77.5	75.2
0.75	100.0	81.1	80.0	78.6	77.1	100.0	76.0	67.6	63.7	61.6	100.0	79.4	69.1	65.5	64.1
1	100.0	58.8	35.9	17.1	0.0	100.0	53.0	30.7	14.6	0.0	100.0	61.5	33.7	15.8	0.0

	$\lambda = 0.1$ and $c_{em} = 10^4$					$\lambda = 0.2$ and $c_{em} = 10^4$					$\lambda = 0.5$ and $c_{em} = 10^4$				
$r \backslash p$	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**
0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
0.25	100.0	89.5	88.8	88.7	88.6	100.0	93.0	90.9	90.1	89.7	100.0	92.2	89.8	88.8	88.3
0.5	100.0	77.3	74.7	73.8	73.3	100.0	88.9	88.2	87.6	86.7	100.0	86.3	83.9	83.5	82.1
0.75	100.0	72.9	70.6	69.0	68.2	100.0	73.6	67.3	65.0	63.8	100.0	76.1	70.2	66.4	64.0
1	100.0	27.8	17.0	8.1	0.0	100.0	30.3	17.4	8.3	0.0	100.0	38.3	19.5	9.1	0.0

	$\lambda = 0.1$ and $c_{em} = 10^6$					$\lambda = 0.2$ and $c_{em} = 10^6$					$\lambda = 0.5$ and $c_{em} = 10^6$				
$r \backslash p$	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**
0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
0.25	100.0	99.3	99.3	99.3	99.1	100.0	92.1	91.1	90.8	90.7	100.0	91.0	89.6	88.6	88.1
0.5	100.0	83.3	81.4	80.8	80.5	100.0	86.3	83.4	82.2	81.6	100.0	83.3	79.5	77.6	76.7
0.75	100.0	78.0	76.9	75.8	75.2	100.0	72.6	68.0	66.4	65.7	100.0	73.2	67.1	64.1	62.6
1	100.0	20.6	12.5	6.0	0.0	100.0	21.0	12.1	5.7	0.0	100.0	27.2	13.8	6.5	0.0

In Table 4.5, we do a sensitivity analysis for $C(p,r)$ for varying values of λ and c_{em}. We see that $C(p,r)$ is non-decreasing both in the failure rate and in the cost per emergency shipment for any (p,r) pair. Similarly, for the worst case (i.e., $p = 0$ or $r = 0$), the average on-hand inventory is non-decreasing both in λ and in c_{em} (see Table 4.6). The average number of emergency shipments in the worst-case scenario is non-monotonic in λ and c_{em}. However, we cannot directly observe a monotonic behavior for the average on-hand inventory (see Table 4.6) and the average number of emergency shipments (see Table 4.7) for varying values of λ and c_{em}.

Table 4.5: $C(p, r)$ for varying values of λ and c_{em} as a function of r and p.

	$\lambda = 0.1$ and $c_{\mathrm{em}} = 10^2$					$\lambda = 0.2$ and $c_{\mathrm{em}} = 10^2$					$\lambda = 0.5$ and $c_{\mathrm{em}} = 10^2$				
$r \backslash p$	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**
0	1.39	1.39	1.39	1.39	1.39	1.92	1.92	1.92	1.92	1.92	2.70	2.70	2.70	2.70	2.70
0.25	1.39	1.27	1.24	1.23	1.22	1.92	1.91	1.91	1.91	1.91	2.70	2.49	2.42	2.39	2.38
0.5	1.39	1.23	1.16	1.13	1.11	1.92	1.58	1.49	1.45	1.44	2.70	2.33	2.19	2.09	2.03
0.75	1.39	1.13	1.11	1.09	1.07	1.92	1.46	1.30	1.23	1.18	2.70	2.14	1.86	1.77	1.73
1	1.39	0.82	0.50	0.24	0.00	1.92	1.02	0.59	0.28	0.00	2.70	1.66	0.91	0.42	0.00
	$\lambda = 0.1$ and $c_{\mathrm{em}} = 10^4$					$\lambda = 0.2$ and $c_{\mathrm{em}} = 10^4$					$\lambda = 0.5$ and $c_{\mathrm{em}} = 10^4$				
$r \backslash p$	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**
0	2.94	2.94	2.94	2.94	2.94	3.39	3.39	3.39	3.39	3.39	4.65	4.65	4.65	4.65	4.65
0.25	2.94	2.63	2.61	2.61	2.60	3.39	3.15	3.08	3.06	3.04	4.65	4.29	4.18	4.13	4.11
0.5	2.94	2.27	2.20	2.17	2.16	3.39	3.02	2.99	2.97	2.94	4.65	4.02	3.90	3.89	3.82
0.75	2.94	2.14	2.07	2.03	2.00	3.39	2.50	2.28	2.20	2.17	4.65	3.54	3.26	3.09	2.98
1	2.94	0.82	0.50	0.24	0.00	3.39	1.03	0.59	0.28	0.00	4.65	1.78	0.91	0.42	0.00
	$\lambda = 0.1$ and $c_{\mathrm{em}} = 10^6$					$\lambda = 0.2$ and $c_{\mathrm{em}} = 10^6$					$\lambda = 0.5$ and $c_{\mathrm{em}} = 10^6$				
$r \backslash p$	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**
0	3.98	3.98	3.98	3.98	3.98	4.88	4.88	4.88	4.88	4.88	6.57	6.57	6.57	6.57	6.57
0.25	3.98	3.95	3.95	3.95	3.94	4.88	4.49	4.45	4.43	4.42	6.57	5.98	5.88	5.82	5.78
0.5	3.98	3.32	3.24	3.22	3.20	4.88	4.21	4.07	4.01	3.98	6.57	5.47	5.22	5.09	5.03
0.75	3.98	3.10	3.06	3.02	2.99	4.88	3.54	3.32	3.24	3.20	6.57	4.80	4.40	4.21	4.11
1	3.98	0.82	0.50	0.24	0.00	4.88	1.03	0.59	0.28	0.00	6.57	1.78	0.91	0.42	0.00

Table 4.6: Average on-hand inventory for varying values of λ and c_{em} as a function of r and p.

	$\lambda = 0.1$ and $c_{\mathrm{em}} = 10^2$					$\lambda = 0.2$ and $c_{\mathrm{em}} = 10^2$					$\lambda = 0.5$ and $c_{\mathrm{em}} = 10^2$				
$r \backslash p$	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**
0	0.90	0.90	0.90	0.90	0.90	1.80	1.80	1.80	1.80	1.80	2.50	2.50	2.50	2.50	2.50
0.25	0.90	1.02	0.98	0.96	0.95	1.80	1.82	1.81	1.81	1.80	2.50	1.94	1.79	1.71	1.67
0.5	0.90	1.12	1.04	1.01	1.00	1.80	1.21	1.09	1.02	0.99	2.50	1.82	1.95	1.88	1.81
0.75	0.90	0.96	0.92	1.06	1.04	1.80	1.35	1.16	1.12	1.08	2.50	1.65	1.45	1.24	1.13
1	0.90	0.81	0.50	0.24	0.00	1.80	1.02	0.59	0.28	0.00	2.50	1.61	0.91	0.42	0.00
	$\lambda = 0.1$ and $c_{\mathrm{em}} = 10^4$					$\lambda = 0.2$ and $c_{\mathrm{em}} = 10^4$					$\lambda = 0.5$ and $c_{\mathrm{em}} = 10^4$				
$r \backslash p$	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**
0	2.90	2.90	2.90	2.90	2.90	2.80	2.80	2.80	2.80	2.80	4.50	4.50	4.50	4.50	4.50
0.25	2.90	2.00	1.95	1.94	1.93	2.80	2.98	2.90	2.87	2.85	4.50	3.90	3.75	3.67	3.63
0.5	2.90	2.08	2.00	1.97	1.95	2.80	2.86	2.82	2.93	2.90	4.50	3.78	3.61	3.79	3.75
0.75	2.90	1.94	2.04	2.00	1.98	2.80	2.28	2.11	2.01	1.96	4.50	3.16	3.08	3.01	2.89
1	2.90	0.82	0.50	0.24	0.00	2.80	1.03	0.59	0.28	0.00	4.50	1.78	0.91	0.42	0.00
	$\lambda = 0.1$ and $c_{\mathrm{em}} = 10^6$					$\lambda = 0.2$ and $c_{\mathrm{em}} = 10^6$					$\lambda = 0.5$ and $c_{\mathrm{em}} = 10^6$				
$r \backslash p$	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**
0	3.90	3.90	3.90	3.90	3.90	4.80	4.80	4.80	4.80	4.80	6.50	6.50	6.50	6.50	6.50
0.25	3.90	3.91	3.90	3.90	3.93	4.80	4.00	3.90	3.87	3.85	6.50	5.61	5.72	5.67	5.63
0.5	3.90	3.10	3.00	2.97	2.95	4.80	4.13	4.00	3.93	3.90	6.50	5.21	4.98	4.83	4.75
0.75	3.90	2.94	3.04	3.00	2.98	4.80	3.37	3.10	3.00	2.95	6.50	4.53	4.24	4.00	3.88
1	3.90	0.82	0.50	0.24	0.00	4.80	1.03	0.59	0.28	0.00	6.50	1.78	0.91	0.42	0.00

Table 4.7: Average number of emergency shipments for varying values of λ and c_{em} as a function of r and p.

	$\lambda = 0.1$ and $c_{em} = 10^2$ $(\times 10^{-2})$					$\lambda = 0.2$ and $c_{em} = 10^2$ $(\times 10^{-2})$					$\lambda = 0.5$ and $c_{em} = 10^2$ $(\times 10^{-2})$				
$r \backslash p$	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**
0	0.48	0.48	0.48	0.48	0.48	0.12	0.12	0.12	0.12	0.12	0.19	0.19	0.19	0.19	0.19
0.25	0.48	0.25	0.26	0.27	0.27	0.12	0.09	0.10	0.10	0.10	0.19	0.55	0.63	0.68	0.71
0.5	0.48	0.11	0.12	0.12	0.12	0.12	0.37	0.40	0.43	0.45	0.19	0.50	0.24	0.21	0.22
0.75	0.48	0.17	0.19	0.03	0.04	0.12	0.11	0.14	0.11	0.11	0.19	0.49	0.42	0.53	0.60
1	0.48	0.01	0.00	0.00	0.00	0.12	0.00	0.00	0.00	0.00	0.19	0.04	0.00	0.00	0.00

	$\lambda = 0.1$ and $c_{em} = 10^4$ $(\times 10^{-4})$					$\lambda = 0.2$ and $c_{em} = 10^4$ $(\times 10^{-4})$					$\lambda = 0.5$ and $c_{em} = 10^4$ $(\times 10^{-4})$				
$r \backslash p$	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**
0	0.04	0.04	0.04	0.04	0.04	0.59	0.59	0.59	0.59	0.59	0.15	0.15	0.15	0.15	0.15
0.25	0.04	0.63	0.66	0.67	0.68	0.59	0.17	0.18	0.19	0.19	0.15	0.39	0.43	0.46	0.48
0.5	0.04	0.19	0.19	0.20	0.20	0.59	0.15	0.17	0.04	0.04	0.15	0.23	0.29	0.10	0.07
0.75	0.04	0.20	0.03	0.03	0.03	0.59	0.22	0.17	0.19	0.20	0.15	0.38	0.18	0.08	0.09
1	0.04	0.00	0.00	0.00	0.00	0.59	0.00	0.00	0.00	0.00	0.15	0.00	0.00	0.00	0.00

	$\lambda = 0.1$ and $c_{em} = 10^6$ $(\times 10^{-6})$					$\lambda = 0.2$ and $c_{em} = 10^6$ $(\times 10^{-6})$					$\lambda = 0.5$ and $c_{em} = 10^6$ $(\times 10^{-6})$				
$r \backslash p$	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**	**0**	**0.25**	**0.5**	**0.75**	**1**
0	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.08	0.07	0.07	0.07	0.07	0.07
0.25	0.08	0.05	0.05	0.05	0.02	0.08	0.50	0.55	0.56	0.57	0.07	0.37	0.16	0.15	0.16
0.5	0.08	0.22	0.24	0.25	0.25	0.08	0.07	0.07	0.08	0.08	0.07	0.26	0.24	0.26	0.28
0.75	0.08	0.16	0.02	0.02	0.02	0.08	0.17	0.22	0.24	0.25	0.07	0.28	0.17	0.21	0.23
1	0.08	0.00	0.00	0.00	0.00	0.08	0.00	0.00	0.00	0.00	0.07	0.00	0.00	0.00	0.00

4.8. Conclusion

We have studied the spare parts inventory problem of a single critical component that is kept on stock in a single local warehouse. For upcoming failures of the component in the supported technical systems, signals are generated. For these signals, we distinguish the factors precision, sensitivity, and demand lead time, and we investigated how the average inventory and costs for inventory holding, emergency shipments, and system downtime depend on these three factors under optimal inventory control of the spare parts stock. This optimal control is obtained via a Markov decision process. Our investigation gives directions for the trade-off between precision, sensitivity, and demand lead time for developers of these signals. We found that optimal inventory control and optimal costs only depend on the sensitivity and the demand lead time via their product. This implies that, when developing signals, getting a high value for sensitivity is equally important as getting a high value for the demand lead time. Further, we found that both factors need to have a high value in order to get a significant reduction in optimal costs and average inventory in comparison to the situation without signals. For precision, a significant cost reduction is obtained for high values but also for moderate values. So, it is much better to develop signals with a moderate value for precision and high values for sensitivity and demand lead time than the other way around.

4.A. Basic model for unpredicted demand

In this section, we consider a basic model for unpredicted demand that we will use as a building block for the analysis of the general model of Section 4.3. For this basic model, everything is the same as for the model of Section 4.3, but we assume that no signals are generated (i.e., $q = 0$ can be assumed) and that failures occur according to a Poisson process with rate μ. We introduce a new parameter for this rate because we will use this basic model for different demand rates. For the emergency shipment costs and the inventory holding costs, we still use the cost parameters c_{em} and c_{h}.

For this basic model, the demand per period is denoted by X, which is Poisson distributed with rate μ. Hence,

$$P\{X = x\} = \frac{\mu^x}{x!} e^{-\mu}, \quad x \in \mathbb{N}_0.$$

Consider the replenishment decision at the beginning of period 0. The initial inventory level is 0. Suppose that the on-hand inventory is increased to S (by ordering S units), then the expected cost in period 0 is equal to

$$\tilde{G}(\mu, S) = \sum_{x=0}^{S} (S - x) P\{X = x\} c_{\text{h}} + \sum_{x=S+1}^{\infty} (x - S) P\{X = x\} c_{\text{em}}.$$

This function $\tilde{G}(\mu, S)$ is similar to the cost function for a newsvendor problem. It is convex as a function of S, and is minimized at the lowest S for which

$$P\{X \leq S\} \geq \frac{c_{\text{em}}}{c_{\text{em}} + c_{\text{h}}}.$$

This optimal S is denoted by $S^*(\mu)$. Let the corresponding minimal costs for period 0 be denoted by $G(\mu) = \tilde{G}(\mu, S^*(\mu))$.

Let us now look at the whole time horizon. It is not possible to get strictly lower expected costs per period than $\tilde{G}(\mu, S^*(\mu))$. By following a base stock policy with base stock level $S^*(\mu)$ (i.e., by increasing the on-hand inventory at the beginning of each period to $S^*(\mu)$), we get expected costs $\tilde{G}(\mu, S^*(\mu))$ in each period, and thus the resulting average costs per period are also equal to $\tilde{G}(\mu, S^*(\mu))$. That implies that the base stock policy with base stock level $S^*(\mu)$ is optimal, and the corresponding minimal costs are equal to $G(\mu) = \tilde{G}(\mu, S^*(\mu))$. The following results hold for $S^*(\mu)$ and $G(\mu)$.

Lemma A1 *1. $S^*(\mu)$ is non-decreasing as a function of μ.*

2. $S^*(\mu) = 0$ if and only if $\mu \leq ln(1 + (c_h/c_{em}))$.

3. If $\mu \leq ln(1 + (c_h/c_{em}))$, then $G(\mu) = \mu c_{em}$.

First, we prove Lemma A1(a). A rate μ leads to the optimal inventory level $S^*(\mu)$. Let us assume another rate $\mu + \delta$, where $\delta > 0$ is a small increment. Then $X_{\mu+\delta}$ stochastically dominates X_μ, where they respectively represent the random variables of a Poisson distribution with rate $\mu + \delta$ and μ. This results in $S^*(\mu)$

being the smallest value for which $P\{X_\mu \leq S^*(\mu)\} \geq \frac{c_{\mathrm{em}}}{c_{\mathrm{em}}+c_{\mathrm{h}}}$ and $S^*(\mu+\delta)$ being the smallest value for which $P\{X_{\mu+\delta} \leq S^*(\mu+\delta)\} \geq \frac{c_{\mathrm{em}}}{c_{\mathrm{em}}+c_{\mathrm{h}}}$. Because $X_{\mu+\delta}$ stochastically dominates X_μ, it holds that $P\{X_\mu \leq S\} \geq P\{X_{\mu+\delta} \leq S\}$ for all S. Therefore $P\{X_\mu \leq S^*(\mu+\delta)\} \geq P\{X_{\mu+\delta} \leq S^*(\mu+\delta)\} \geq \frac{c_{\mathrm{em}}}{c_{\mathrm{em}}+c_{\mathrm{h}}}$ and hence $S^*(\mu) \leq S^*(\mu+\delta)$. This proves that $S^*(\mu)$ is non-decreasing as a function of μ.

Next, we prove Lemma A1(b). It holds that $S^*(\mu) = 0$ if and only if

$$
\begin{aligned}
&P\{X \leq 0\} \geq \frac{c_{\mathrm{em}}}{c_{\mathrm{em}}+c_{\mathrm{h}}} \Leftrightarrow e^{-\mu} \geq \frac{c_{\mathrm{em}}}{c_{\mathrm{em}}+c_{\mathrm{h}}} \\
&\Leftrightarrow -\mu ln\left(\frac{c_{\mathrm{em}}}{c_{\mathrm{em}}+c_{\mathrm{h}}}\right) \Leftrightarrow \mu \leq ln\left(1+\frac{c_{\mathrm{h}}}{c_{\mathrm{em}}}\right).
\end{aligned}
$$

Lemma A1(c) follows directly from the observation that all demands are satisfied by an emergency shipment if no parts are kept on stock. □

4.B. Value iteration algorithm

In this section, we show how the inventory holding and emergency shipment costs under the optimal policy are calculated by solving the MDP formulation. Algorithm 3 provides a pseudocode of our algorithm. We let $d^{\mathrm{h}}(z,a) = \sum_{x=0}^{z}(z-x)P(X^u + X^{\mathrm{p}}(a) = x)c_{\mathrm{h}}$ denotes the direct expected costs of inventory holding and $d^{\mathrm{em}}(z,a) = \sum_{x=z+1}^{\infty}(z-x)P(X^u + X^{\mathrm{p}}(a) = x)c_{\mathrm{em}}$ denotes the direct expected costs of emergency shipments. The direct expected costs $d(z,a)$ are equal to the sum of the direct expected costs of inventory holding and the direct expected costs of emergency shipments, i.e., $d(z,a) = d^{\mathrm{h}}(z,a) + d^{\mathrm{em}}(z,a)$. By splitting direct expected costs into two, we can calculate the costs contribution of each into the total costs separately. For this purpose, we introduce $\hat{V}^{\mathrm{h}}_{n+1}(z,a)$ and $\hat{V}^{\mathrm{em}}_{n+1}(z,a)$ denoting the so-called value functions for inventory holding and emergency shipment costs, respectively. We calculate these function by a value iteration algorithm. At each iteration, we calculate $\hat{V}^{\mathrm{h}}_{n+1}(z,a)$ and $\hat{V}^{\mathrm{em}}_{n+1}(z,a)$ independently, then we sum them up to update the value of $\hat{V}_{n+1}(z,a)$ for all (z,a), and then we calculate the optimal value of $V_{n+1}(y,a)$ for all (y,a). The value iteration algorithm stops when the long-run average cost per period (i.e., value of $V_n(y,a)/n$) converges to a constant at some sufficiently large value of n. The convergence is checked by comparing the deviation of the average cost per-period in two subsequent steps of the algorithm to a small number ϵ (=0.01).

Finally, we define the costs of inventory holding and emergency shipments as

$$C^{\mathrm{h}}(p,r) = \lim_{n\to\infty} \frac{V_n^{\mathrm{h}}(0,0)}{n}$$

and

$$C^{\mathrm{em}}(p,r) = \lim_{n\to\infty} \frac{V_n^{em}(0,0)}{n},$$

respectively. If we divide $C^{\mathrm{h}}(p,r)$ by c_{h}, we obtain the average on-hand inventory level for a given (p,r). Similarly, if we divide $C^{\mathrm{em}}(p,r)$ by c_{em}, we obtain the average number of emergency shipments per period.

Algorithm 3 Value Iteration Algorithm

Initialize $V_0(y,a) \leftarrow 0$, $V_0^{\mathrm{h}}(y,a) \leftarrow 0$, $V_0^{\mathrm{em}}(y,a) \leftarrow 0\,\forall(y,a) \in S$, $n = 0$, Stop=False

while Stop=False **do**

for $\forall(y,a) \in S$ **do**

for $\forall z \geq y$ **do**

$$\hat{V}_{n+1}^{\mathrm{h}}(z,a) = d^{\mathrm{h}}(z,a) + \sum_{\hat{a}=0}^{\infty} P\{X^{\mathrm{s}} = \hat{a}\}\Bigg(P\{X^{\mathrm{u}} + X^{\mathrm{p}}(a) \geq z\} V_n^{\mathrm{h}}(0,\hat{a}) + \sum_{x=0}^{z-1} P\{X^{\mathrm{u}} + X^{\mathrm{p}}(a) = x\} V_n^{\mathrm{h}}(z-x,\hat{a})\Bigg)$$

$$\hat{V}_{n+1}^{\mathrm{em}}(z,a) = d^{\mathrm{em}}(z,a) + \sum_{\hat{a}=0}^{\infty} P\{X^{\mathrm{s}} = \hat{a}\}\Bigg(P\{X^{\mathrm{u}} + X^{\mathrm{p}}(a) \geq z\} V_n^{\mathrm{em}}(0,\hat{a}) + \sum_{x=0}^{z-1} P\{X^{\mathrm{u}} + X^{\mathrm{p}}(a) = x\} V_n^{\mathrm{em}}(z-x,\hat{a})\Bigg)$$

$$\hat{V}_{n+1}(z,a) = \hat{V}_{n+1}^{\mathrm{h}}(z,a) + \hat{V}_{n+1}^{\mathrm{em}}(z,a)$$

end for

$V_{n+1}(y,a) \leftarrow \min_{z \geq y}\{\hat{V}(z,a)\}$

$z^*(y,a) \leftarrow \operatorname{argmin}_{z \geq y}\{\hat{V}(z,a)\}$

$V_{n+1}^{\mathrm{h}}(y,a) = \hat{V}_{n+1}^{\mathrm{h}}(z^*(y,a),a)$

$V_{n+1}^{\mathrm{em}}(y,a) = \hat{V}_{n+1}^{\mathrm{em}}(z^*(y,a),a)$

end for

if $n > 0$ and $\max_{\forall(y,a)\in S}\{|\frac{V_{n+1}(y,a)}{n+1} - \frac{V_n(y,a)}{n}|\} \leq \epsilon$ **then**
Stop=True

$n = n + 1$

end while

5

Selecting the set of spare parts for corrective maintenance

5.1. Introduction

In this chapter, we consider a spare part selection problem under imperfect advance demand information (ADI) and develop a decision-support model for the service control tower (SCT) of the service provider. The SCT is responsible for determining the set of spare parts that will be sent to the customer for corrective maintenance. The problem setting is as follows. Upon system failure, a customer reports it to the service provider and a maintenance case is opened by the service provider. These systems are installed at geographically dispersed locations. A service engineer conducts a diagnostic visit to a customer's location when a failure is observed in the system and reported to the service provider. At this visit, the engineer executes a maintenance process to bring the system into a fully functioning state. During the maintenance process, a set of spare parts may be required to repair the system. It is also possible that the system can be fixed without any part replacement. Spare parts are stocked in a central warehouse. Simultaneously with the diagnostic visit, a set of spare parts can be sent to the customer's location directly from this warehouse. The engineer diagnoses which spare parts are required to fix the system with certainty during a visit. If the system is fixed during the diagnostic visit, then the case is closed. If there is at least one spare part needed to resolve the maintenance case and the required part(s) are not sent during the diagnostic visit, a second engineer visit is scheduled. The required spare parts are sent to the customer's location from the central warehouse. On the other hand, if some shipped parts are not needed during the diagnostic visit, they are shipped back to the central warehouse. In this research, we focus on the problem of selecting the set of spare parts that will be shipped during the diagnostic visit.

If a set of spare parts is needed to resolve a failure, we can consider that there is a demand for this set of spare parts. Therefore, we consider a setting with imperfect ADI, where a probability estimate is available for each possible set of parts. In this setting, ADI is obtained as follows. The service provider stores information about successfully resolved maintenance cases and the demanded stock-keeping units (SKU) for resolving earlier failures. When there is a system failure, the customer shares information by specifying a failure code, such as a "high cryocompressor temperature" warning that the customer might get from a cooling system of purchased equipment. SKUs used in resolved maintenance cases that have similar failure codes could serve as a basis for predicting the spare parts that might be required for the resolution of a new maintenance case. We assume a predictive algorithm is available that generates a list of sets of spare parts and corresponding probabilities that a set of spare parts may be demanded. This algorithm is based on the failure codes and the spare part usage in the historically resolved maintenance cases.

Grishina et al. (2020) introduce a spare part recommendation framework by using natural text similarity metrics. Their framework has two steps: (i) The service provider enters the failure code as a query. Similar maintenance cases are retrieved from the database of resolved cases and a text similarity metric finds matching words between the failure code and cases in the database. A list of parts that are demanded in previous cases is generated with their usage frequency. (ii) Parts mentioned in these reports are ranked based on the frequency of part use. The service provider selects the set of spare parts that will be shipped to the customer from this ranked list.

In this study, we extend the spare part recommendation problem introduced by Grishina et al. (2020) by proposing an optimal selection model for the set of spare parts. Our model uses the operational costs (i.e., engineer's visit cost, spare part shipment costs, and send-back costs) and the corresponding probability for each set of spare parts that may be required. Our study does not aim to develop an algorithm that predicts the probability distribution on sets of required spare parts for corrective maintenance. The aim of this study is to find the set of spare parts that minimizes the total expected cost of a maintenance case. For this purpose, we formulate an integer linear programming model (ILP).

We compare the optimal policy against the two benchmark heuristics that have been used in practice by a high-tech capital goods service provider. We define the heuristic policies as follows. *(Policy 1)* During the diagnostic visit to the failed system, no parts are sent. The maintenance process starts with a diagnostic visit by an engineer to the customer site to determine which parts are needed for corrective maintenance. Then the required set of parts is brought on-site after which the failed system is repaired with a second engineer visit. *(Policy 2)* If a service provider fully trusts the part recommendation generated based solely on the frequency of part use, a fixed number of the top recommended parts can be shipped to the customer's site simultaneously with the diagnostic visit. The fixed number of top recommended parts can be optimized and if the value of this is equal to zero, then this policy is equivalent to Policy 1. These heuristics are easily implementable for practitioners however deciding on the fixed number of parts or choosing between Policy 1 and Policy 2 is still a challenge in terms of determining the least costly policy. The main motivation behind proposing an optimal policy for this problem is to overcome this challenge.

The main contributions of this chapter are as follows. (1) We formulate a new model for the so-called spare parts recommendation problem. (2) We derive the optimal policy structure for problem instances with one or two SKUs and we obtain some analytical results on the structure of the optimal policy structure for the problem instances with a general number of SKUs. (3) We compare the optimal policy against the two heuristic policies.

The remainder of the chapter is organized as follows. We discuss the studies in the literature that are relevant to our work in Section 5.2. We provide a detailed problem description in Section 5.3. Section 5.4 presents the ILP formulation. In Section 5.5 and Section 5.6, we present the results for structural analysis and numerical analysis, respectively. Section 5.7 concludes the chapter.

5.2. Literature review

The spare part recommendation problem described in Section 5.1 aims to bring the right parts on-site to avoid a second visit and minimize operational costs. Our spare part recommendation problem to determine the optimal kit is related to the so-called *repair kit problem* (RKP) in the literature (Teunter, 2006). In this section, we position our chapter in the RKP literature and we provide Table 5.1 as an overview of RKP literature and our work.

In RKP, the main trade-off is between the cost of holding parts in the kit and service level costs. There are two kinds of models in the literature to model this trade-off: cost models (introduced by Smith et al. (1980)) and service models (introduced by Graves (1982)). In cost models, the holding cost is minimized, where not completing a job during the first time is penalized with a cost (see Smith et al. (1980); Mamer and Smith (1982); Mamer and Shogan (1987); Teunter (2006); Bijvank et al. (2010); Saccani et al. (2017); Karabağ et al. (2020); Neves-Moreira et al. (2021)). In service models, the holding cost is minimized subject to a service level constraint (see Graves (1982); Mamer and Shogan (1987); Heeremans and Gelders (1995); Teunter (2006); Bijvank et al. (2010); Prak et al. (2017); Rippe and Kiesmüller (2022)). The first papers in the literature assume that a repair kit is used for a *single job* (Smith et al., 1980; Graves, 1982; Mamer and Smith, 1982; Mamer and Shogan, 1987). Heeremans and Gelders (1995) are the first to relax the single job assumption by introducing a *multi-job* model. In a multi-job model, multiple on-site visits can be done with the same kit (see, Heeremans and Gelders (1995); Teunter (2006); Bijvank et al. (2010); Saccani et al. (2017); Prak et al. (2017); Neves-Moreira et al. (2021); Rippe and Kiesmüller (2022)).

The first papers assume that at most one unit from each stock-keeping unit (SKU) is needed during a single job (Smith et al., 1980; Graves, 1982; Heeremans and Gelders, 1995). Mamer and Smith (1982) relaxed this assumption by introducing a *multi-unit* model. Mamer and Smith (1982) are also the first to relax the assumption of independency between the failure behavior of different SKUs by defining representative job types. When the failure behavior of SKUs is dependent, demand for these SKUs during a corrective maintenance visit is also dependent. Therefore, we refer to this as *demand dependency*. Similar to Mamer and Shogan (1987); Teunter (2006); Karabağ et al. (2020), we also assume demand dependency between SKUs. Different from Teunter (2006); Mamer and Smith (1982); Mamer and Shogan (1987); Karabağ et al. (2020), we do not define specific job types but consider the probability for each possible part combination that can appear for a maintenance case. We refer to this as a full dependency.

Our problem is an extension of the RKP. In our problem, the maintenance sites are located at geographically dispersed locations. Field engineers travel on-site without any parts, and parts are sent per job from a central warehouse. The service control tower decides which parts will be sent to the customer according to ADI provided to the service control tower and historical data. The list of SKUs and the probabilities of SKUs and the set of SKUs for resolving a maintenance case is specific to that case based on the provided ADI. Similar to our problem, Karabağ et al. (2020); Rippe and Kiesmüller (2022, 2023) study the RKP with ADI. The source of ADI is the sensors that monitor the condition of a subset of parts, in the research of Karabağ et al. (2020); Rippe and Kiesmüller (2022). The error code provided by the customer regarding the failure is the source of ADI in the paper of Rippe and Kiesmüller (2023). In their modeling, Karabağ et al. (2020); Rippe and Kiesmüller (2022, 2023) consider ADI for the demand probability distribution per SKU. They do not have an explicit assumption of demand dependency between SKUs. We consider ADI for the demand probability per set of SKUs. Additionally, our model takes retrieval cost, sending back cost of SKUs, and fixed delivery cost into account in the objective function. Similar to our objective function, Karabağ et al. (2020) include an emergency order cost (fixed transportation cost), and return (sending back) cost. Additionally, the fixed ordering (retrieval) cost per item is considered by Prak et al. (2017), fixed transportation cost is considered by Saccani et al. (2017), and sending back cost is considered by Rippe and Kiesmüller (2023).

5.3. Problem description

We consider a maintenance service provider who is fully responsible for addressing the failures encountered in the systems operated by its customers. Suppose that a customer reports a system failure to the service provider on day t. At this moment, the service provider creates a corrective-maintenance case in its case-management system and enters a text query into the system to determine the spare parts that have been used in similar maintenance cases. Suppose that in total N different

Table 5.1: Literature review.

	Smith et al. (1980)	Graves (1982)	Mamer and Smith (1982)	Mamer and Shogan (1987)	Heeremans and Gelders (1995)	Teunter (2006)	Bijvank et al. (2010)	Saccani et al. (2017)	Prak et al. (2017)	Karabağ et al. (2020)	Neves-Moreira et al. (2021)	Rippe and Kiesmüller (2022)	Rippe and Kiesmüller (2023)	**Current Work**
1. Number of jobs per tour														
Single	X	X	X	X						X			X	X
Multiple					X	X	X	X	X		X	X	X	
2. Number of units needed per SKU per job														
Single	X	X			X					X			X	X
Multiple			X	X		X	X	X	X		X	X		
3. Failure dependency between different SKUs														
No dependency	X	X			X		X	X	X		X	X	X	
Dependency via job types			X	X		X				X				
Full dependency														X
4. Model characteristic														
Service model		X		X	X	X	X		X			X		X
Cost model	X		X	X		X	X	X		X	X		X	X
5. Additional features														
Fixed transportation cost								X		X				X
Sending back cost										X			X	X
Sending missing items cost														X
6. Advance demand information per failure														
Probability per SKU										X		X	X	
Probability per set of SKUs														X

SKUs are demanded in matching cases. This means that the system failure can be caused by the (possibly joint) failure of N different SKUs, and the replacement of up to N SKUs may be needed to fix the system failure. We let $I = \{1, 2, \ldots, N\}$ denote the set of SKUs. We assume that at most one part is needed for each SKU to successfully complete the corrective maintenance of the system. The main challenge in practice is not knowing upfront which SKUs require replacement for the corrective maintenance of the system. The SKUs that require replacement can be certainly known only upon having a physical examination of the system by a service engineer. On the other hand, by considering the historical cases of system failures, the service provider can establish a probability distribution on the set of SKUs needed for corrective replacement. In the most general case, notice that there are 2^N possible sets of SKUs that require replacement during the corrective maintenance.

Let $M = \{0, \ldots, 2^N - 1\}$ denote the collection of indices for these sets. Specifically, we index each possible set with $m \in M$ and $\mathbf{s}_m = (s_{1m}, ..., s_{im}, \ldots, s_{Nm})$ denotes the binary vector indicating the SKUs needed in set m, where $s_{im} = 1$ denotes SKU i is in set m and $s_{im} = 0$ denotes otherwise. The indices are ordered such that they represent a situation where the binary vectors $\mathbf{s}_m$ are ordered lexicographically. If $N = 2$, for example, then $\mathbf{s}_0 = (0, 0)$, $\mathbf{s}_1 = (0, 1)$, $\mathbf{s}_2 = (1, 0)$, and $\mathbf{s}_3 = (1, 1)$. We let $\hat{p}_m$ denote the probability that the set of SKUs that require replacement during corrective maintenance is equal to set m. By using these probabilities, it is possible to calculate the probability of SKU i being needed during the corrective maintenance. We denote this probability with p_i, and it holds that

$$p_i = \sum_{m \in M} s_{im} \hat{p}_m, \ \forall i \in I. \tag{5.1}$$

Once the corrective-maintenance case is created and the probability distribution $\{\hat{p}_m\}_{m \in M}$, is established, the corrective maintenance process starts. In Figure 5.1, we provide a diagram that summarizes this corrective maintenance process. The execution of a corrective maintenance process by the service provider has two ingredients: the on-site visit of the service engineer and the shipment of spare parts to the system. The first on-site visit, referred to as the *diagnostic visit*, takes place in any case, meaning that an engineer must visit the system upon the receipt of a failure report from the customer. The cost of the engineer going to the site is fixed and denoted by D (€) (> 0). In parallel, a set of SKUs is identified and their spare parts are sent to the site (note that this can be an empty set, meaning that no spare part is sent to the customer). The selected spare parts are picked from the OEM warehouse. Retrieving one spare part for SKU i from the warehouse and including it in the shipment to the customer incurs a cost r_i (€/unit) (> 0).

The spare parts are shipped to the customer overnight as one batch with a fixed transportation cost F (€) (> 0), and they become available on the site at the beginning of day $t+1$. On day $t+1$, the diagnostic visit of the field service engineer also takes place. At that moment, the problem is identified and the failed parts are replaced with the corresponding spare parts.

If all required spare parts are already sent to the site (or it turns out the failure is not because of a failed component and thus no spare part is needed), the maintenance case is completed on day $t+1$. It is possible that not all the spare parts that are sent to the site are actually required on day $t+1$. The engineer collects and returns unused parts to the warehouse. However, an additional cost related to checking and possibly re-packing the unused parts arises. Also, the shipment of a part from the warehouse decreases the availability of that SKU by one unit until it is returned back to the warehouse. We introduce the cost parameter b_i (€/unit) (> 0) to capture these costs for SKU i. In such a return process, the parts are typically placed at a return point. Therefore, we do not have a fixed cost for returning parts; we only have variable return costs for each unit.

In case the system cannot be brought back to the functioning state during the diagnostic visit because not all the required spare parts are available, the missing parts are ordered on the same day (i.e., day $t+1$), based on the diagnosis of the service engineer. Similar to earlier, a variable cost of r_i for sending a spare part of SKU i from the warehouse to the customer, and the fixed transportation cost F must be charged. In addition, a follow-up visit must be performed by the service engineer for resolving the case during a second visit with the correct spare parts on the next day. The cost of the follow-up visit is the same as the cost of the diagnostic visit. On day $t+2$, the maintenance is completed with the second on-site visit.

The problem is to determine the optimal set of SKUs for which a spare part is sent to the customer site to make it available during the first diagnostic visit of the engineer. The objective is to minimize the expected total cost of resolving a reported failure. One extreme policy is that spare parts for all of the SKUs are sent to the site. This guarantees that the follow-up visit of the service engineer is not needed, but can incur a lot of retrieval and return costs for the unused spare parts. Another extreme policy is that no spare part is sent to the site, avoiding all unnecessary retrieval and return costs but this most likely leads to a costly follow-up visit. Both extreme policies can be optimal for specific problem instances, but generally, an in-between policy will be optimal. In the next section, we show how an optimal policy can be obtained.

5.4. ILP formulation

In this section, we provide an ILP model to identify the optimal solution for the problem described in Section 5.3.

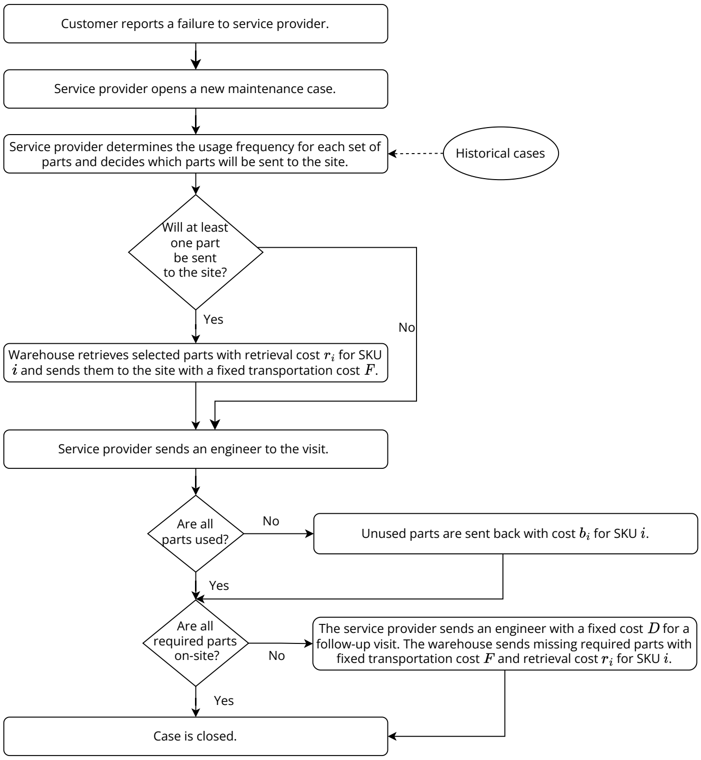


Figure 5.1: Process diagram of corrective maintenance.

Decision Variables

We let $\mathbf{x} = (x_1, \ldots, x_i, \ldots, x_N)$ denote the decision variables that indicate whether a spare part from a particular SKU is made available for the service engineer during the diagnostic visit. Specifically, $x_i = 1$ denotes that a spare part from SKU i is selected to be sent to the customer site so that it can be used during the diagnostic visit of the service engineer, and $x_i = 0$ otherwise. We introduce the binary variable z to indicate whether at least one part is sent, i.e., $z = 1$ means at least one SKU is chosen to make its spare part available during the diagnostic visit of the service engineer, and $z = 0$ means no SKU is chosen. We use $z \geq x_i, \forall i \in I$ as a constraint to indicate the relation between decision variables z and x_i.

To describe the costs related to a possible second visit, we consider all possible sets $m \in M$ that might be required to resolve the maintenance case. Here, we only consider sets of SKUs $m \in M$ with a strictly positive probability $\hat{p}_m$, and we denote these sets with $M' \subseteq M$. This reduces the number of variables and hence the computation time when solving the ILP. The vector $\mathbf{s}_m = (s_{1m}, \ldots, s_{Nm})$ defines a set of spare parts that can be shipped during the diagnostic visit. If for some $i \in I$, the part is needed (i.e. $s_{im} = 1$) but the part is not shipped for the diagnostic visit (i.e. $x_i = 0$), then a part of SKU i will be needed for the second engineer visit. Whether an SKU i should be shipped with a second engineer visit or not, is described by a binary variable u_{im}, for which we require $u_{im} \geq s_{im} - x_i$. If $s_{im} = 1$ and $x_i = 0$, then u_{im} will be forced to be 1 and $u_{im} = 0$ in all other cases. For example, suppose that $N = 2$ and a spare part from SKU 2 is not made available for the diagnostic visit. Then, $u_{21} = 1$ because we know $\mathbf{s}_1 = (0, 1)$ (i.e., SKU 2 is needed for corrective maintenance in scenario $m = 1$) but SKU 2 is not available during diagnostic visit. Finally, the variable $\hat{u}_m$ is 0 if $u_{im} = 0$ for all $i \in I$ and 1 otherwise. This variable denotes whether a second visit is needed if set m is the true set of parts. For the same example where SKU 2 is not made available for the diagnostic visit, $u_1 = 1$ and $u_3 = 1$ because $s_1 = (0, 1)$ and $s_3 = (1, 1)$ (i.e. SKU 2 is needed for corrective maintenance in scenarios $m = 1, 3$).

Objective function

Now, we introduce the objective function of the ILP model. The objective function has three parts. The first part, $\Big(Fz + \sum_{i \in I} r_i x_i\Big)$, represents the total cost related to the diagnostic visit which consists of the fixed transportation cost and variable transportation cost of parts. The fixed cost D is excluded because it always occurs and does not play a role in determining the optimal policy. The second part, $\Big(\sum_{i \in I} b_i x_i (1 - p_i)\Big)$, is the expected cost for returning parts. These costs are incurred for unused parts during the diagnostic visit. The third part, $\Big((D + F)(\sum_{m \in M'} \hat{u}_m \hat{p}_m) + \sum_{i \in I} r_i (1 - x_i) p_i\Big)$, is the expected cost for a second engineer visit if any required parts for maintenance are not brought during the diagnostic visit. The overall objective function is

$$C(\mathbf{x}) = \Big(Fz + \sum_{i \in I} r_i x_i\Big) + \Big(\sum_{i \in I} b_i x_i (1 - p_i)\Big) + \Big((D + F)(\sum_{m \in M'} \hat{u}_m \hat{p}_m) + \sum_{i \in I} r_i (1 - x_i) p_i\Big)$$

and it can be simplified as

$$C(\mathbf{x}) = Fz + \sum_{i \in I}(r_i + b_i)(1 - p_i)x_i + (D + F)(\sum_{m \in M'} \hat{u}_m \hat{p}_m) + \sum_{i \in I} r_i p_i.$$

For brevity in notation, we denote $r_i + b_i$ with c_i. We also remove the term $\sum_{i \in I} r_i p_i$ from the objective function because it is a constant. Then, the objective function is further simplified into

$$C(\mathbf{x}) = Fz + \sum_{i \in I} c_i(1 - p_i)x_i + (D + F)(\sum_{m \in M'} \hat{u}_m \hat{p}_m)$$

ILP model

The complete ILP model is formulated as follows:

$$\min_{\mathbf{x}} \quad C(\mathbf{x}) = Fz + \sum_{i \in I} c_i(1 - p_i)x_i + (D + F)(\sum_{m \in M'} \hat{u}_m \hat{p}_m)$$

s.t.

$$z \geq x_i, \forall i \in I \tag{5.2}$$

$$u_{im} \geq s_{im} - x_i, \forall i \in I, \forall m \in M' \tag{5.3}$$

$$\hat{u}_m \geq u_{im}, \forall i \in I, \forall m \in M' \tag{5.4}$$

$$x_i \in \{0, 1\}, \forall i \in I \tag{5.5}$$

$$z \in \{0, 1\} \tag{5.6}$$

$$u_{im} \in \{0, 1\}, \forall i \in I, \forall m \in M' \tag{5.7}$$

$$\hat{u}_m \in \{0, 1\}, \forall m \in M' \tag{5.8}$$

5.5. Structural analysis

In this section, we execute a structural analysis for the optimal policy. First, we have the analysis for $N = 1$, then for $N = 2$, and finally for a general N. Proofs of structural results can be found in Appendix 5.A.

5.5.1 Optimal policy structure for $N = 1$

For $N = 1$, $\mathbf{x} = (x_1)$ and we denote the optimal policy with x_1^*. We derive the optimal policy structure in Proposition 5.1.

Proposition 5.1 *For $N = 1$, the solution $x_1 = 0$ is optimal for $p_1 \in [0, \check{p}_1]$ and the solution $x_1 = 1$ is optimal for $p_1 \in [\check{p}_1, 1]$, where $\check{p}_1 = \frac{F+c_1}{D+F+c_1}$. Both actions are optimal at the threshold value $p_1 = \check{p}_1$.*

Proposition 5.1, says that a part of SKU 1 should be sent to the system for the diagnostic visit if and only if the probability p_1 that this part is needed exceeds the threshold $\check{p}_1$. The optimal policy structure for $N = 1$ is shown in Figure 5.2. As the cost of a second visit, D, increases, the optimality region for bringing a part of SKU 1 increases. For high values of $\frac{F+c_1}{D}$, $\check{p}_1$ is close to one ($\lim_{F+c_1 \to \infty} \check{p}_1 = \frac{F+c_1}{D+F+c_1} = 1$). For low values of $\frac{F+c_1}{D}$, $\check{p}_1$ is close to zero ($\lim_{F+c_1 \to 0} \check{p}_1 = \lim_{F+c_1 \to 0} \frac{F+c_1}{D+F+c_1} = 0$ and $\lim_{D \to \infty} \check{p}_1 = \lim_{D \to \infty} \frac{F+c_1}{D+F+c_1} = 0$).

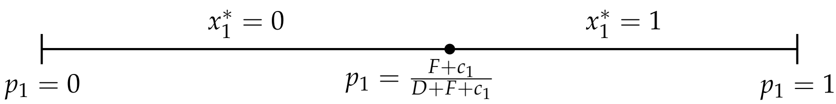


Figure 5.2: Optimal policy structure for $N = 1$.

5.5.2 Optimal policy structure for $N = 2$

In this section, we characterize the optimal policy structure for $N = 2$. We reformulate the demand distribution of SKUs as a bi-variate Bernoulli distribution to see the effect of demand dependency between the two SKUs more explicitly. Let X be a Bernoulli random variable that represents the quantity demanded from SKU 1 to resolve the maintenance case with mean $E[X] = P(X = 1) = p_1$. Let Y be a Bernoulli random variable that represents the quantity demanded from SKU 2 to resolve the case with mean $E[Y] = P(Y = 1) = p_2$. Then the covariance between X and Y is

$$\begin{aligned} \sigma_{1,2} &= E[(X - p_1)(Y - p_2)] \\ &= E[XY - Xp_2 - Yp_1 + p_1 p_2] \end{aligned}$$

$$= \quad E[XY] - p_1p_2 = \hat{p}_3 - p_1p_2, \tag{5.9}$$

where $E[XY] = P(X = 1, Y = 1) = \hat{p}_3$. Then, we can formulate $\hat{p}_3$ as $p_1p_2 + \sigma_{1,2}$. Please note that covariance $\sigma_{1,2}$ being equal to zero means X and Y are independent variables, i.e. there is no demand dependency between SKU 1 and SKU 2. Let $S = (X, Y)$ be a bi-variate Bernoulli variable. S can have four possible values: $\mathbf{s}_0 = (0, 0)$ (i.e. no parts are required for the maintenance activity), $\mathbf{s}_1 = (0, 1)$ (i.e. only SKU 2 is required for the maintenance activity), $\mathbf{s}_2 = (1, 0)$ (i.e. only SKU 1 is required for the maintenance activity) and $\mathbf{s}_3 = (1, 1)$ (i.e. both SKU 1 and SKU 2 are required for the maintenance activity). Then, we can characterize the probability distribution of S as (by using the relations between $\hat{p}_m$ and p_i and (5.9)).

$$P(S = \mathbf{s}_0) = \hat{p}_0 = (1 - p_1)(1 - p_2) + \sigma_{1,2}$$
$$P(S = \mathbf{s}_1) = \hat{p}_1 = (1 - p_1)p_2 - \sigma_{1,2}$$
$$P(S = \mathbf{s}_2) = \hat{p}_2 = p_1(1 - p_2) - \sigma_{1,2}$$
$$P(S = \mathbf{s}_3) = \hat{p}_3 = p_1p_2 + \sigma_{1,2},$$

where $\hat{p}_0 + \hat{p}_1 + \hat{p}_2 + \hat{p}_3 = 1$ and $1 \geq \hat{p}_0, \hat{p}_1, \hat{p}_2, \hat{p}_3 \geq 0$ (Teugels, 1990). Please also note that

$$\min\{p_2 - p_1p_2, p_1 - p_1p_2\} \geq \sigma_{1,2} \geq \max\{p_1 + p_2 - p_1p_2 - 1, -p_1p_2\} \tag{5.10}$$

because of the domain of $\hat{p}_0, \hat{p}_1, \hat{p}_2, \hat{p}_3$. The maximum possible value of $\sigma_{1,2}$ is 1 and the minimum possible value of $\sigma_{1,2}$ is -1.

As a next step, we want to show the structure of the optimal policy. We define six points and three functions in (p_1, p_2) to describe the optimal policy structure for $N = 2$.

Definition 5.1 (i) We define four points such that

- $(\underline{p}_1, \bar{p}_2) = \left(\frac{c_1}{D+F+c_1}, \frac{(F+c_2+(D+F)\sigma_{1,2})(D+F+c_1)}{(D+F+c_1)(D+F+c_2)-(D+F)c_1}\right)$,
- $(\bar{p}_1, \underline{p}_2) = \left(\frac{(F+c_1+(D+F)\sigma_{1,2})(D+F+c_2)}{(D+F+c_1)(D+F+c_2)-(D+F)c_2}, \frac{c_2}{D+F+c_2}\right)$,
- $\tilde{p}_1 = \frac{F+c_1+(D+F)\sigma_{1,2}}{D+F+c_1}$, and
- $\tilde{p}_2 = \frac{F+c_2+(D+F)\sigma_{1,2}}{D+F+c_2}$.

(ii) We define three functions in p_1 such that

- $f(p_1) = -\frac{F+c_2+(D+F)\sigma_{1,2}}{c_2+(1-p_1)(D+F)}$,
- $g(p_1) = \frac{(F+c_1+c_2+(D+F)\sigma_{1,2})-(D+F+c_1)p_1}{(D+F+c_2)-(D+F)p_1}$, and
- $h(p_1) = \frac{F+c_1+(D+F)\sigma_{1,2}-(D+F+c_1)p_1}{D+F}$.

Lemma 5.1 *For $\sigma_{1,2} = 0$, it holds that $0 < \underline{p}_1 < \tilde{p}_1 < \bar{p}_1$ and $0 < \underline{p}_1 < \tilde{p}_1 < 1$. Similarly, it holds that $0 < \underline{p}_2 < \tilde{p}_2 < \bar{p}_2$, and $0 < \underline{p}_2 < \tilde{p}_2 < 1$.*

Theorem 5.1 *For $N = 2$, we characterize the optimal policy for $\sigma_{1,2} = 0$.*

(i) Solution $(x_1, x_2) = (0,0)$ is optimal in region $R_1 = \{(p_1, p_2) \in [0,1]^2 | p_1 \in [0, \underline{p}_1], p_2 \le f(p_1)\} \cup \{(p_1, p_2) \in [0,1]^2 | p_1 \in [\underline{p}_1, \bar{p}_1], h(p_1) \le p_2 \le g(p_1)\}$.

(ii) Solution $(x_1, x_2) = (0,1)$ is optimal in region $R_2 = \{(p_1, p_2) \in [0,1]^2 | p_1 \in [0, \underline{p}_1], p_2 \ge f(p_1)\}$.

(iii) Solution is $(x_1, x_2) = (1,1)$ is optimal in region $R_3 = \{(p_1, p_2) \in [0,1]^2 | p_1 \in [\tilde{p}_1, \bar{p}_1], p_2 \le h(p_1)\} \cup \{(p_1, p_2) \in [0,1]^2 | p_1 \in [\bar{p}_1, 1], p_2 \le \underline{p}_2\}$.

(iv) Solution is $(x_1, x_2) = (1,0)$ is optimal in region $R_4 = \{(p_1, p_2) \in [0,1]^2 | p_1 \in [\underline{p}_1, \bar{p}_1], g(p_1) \le p_2\} \cup \{(p_1, p_2) \in [0,1]^2 | p_1 \in [\bar{p}_1, 1], p_2 \ge \underline{p}_2\}$.

In Figure 5.3, the optimal policy regions are shown for $\sigma_{1,2} = 0$, $F = 100$, $D = 100$, $c_1 = 20$ and $c_2 = 20$. In Figure 5.4, some examples to observe the effect of operational costs on the optimal policy regions are given. We see in Figure 5.4(a) that an increased cost of the second engineer visit D increases the region where $(x_1, x_2) = (1,1)$ is optimal. In Figure 5.4(b), it is shown that increasing the value of the fixed transportation cost F increases the region where $(x_1, x_2) = (0,0)$ is optimal. In Figures 5.4(c-d), we observe that the optimal policy regions for bringing only SKU i increases as the variable cost c_j of SKU j, $j \neq i$, gets closer to the values of F and D.

Finally, in Figure 5.5, we provide a numerical example for the change in the optimal policy regions under a positive demand dependency between the two SKUs (i.e. $\sigma_{1,2} > 0$). Please note we derive these regions by using Definition 5.1 (similarly to Theorem 5.1), however, the possible values of $\sigma_{1,2}$ is limited in the interval [-1,1]. This limits the possible values of p_1 and p_2 due to (5.10). The region where

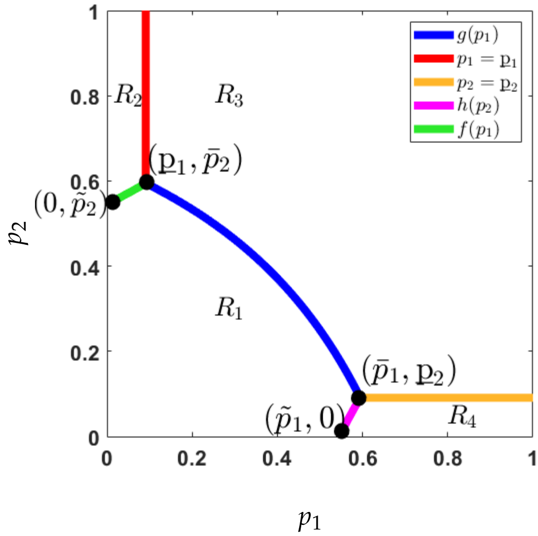
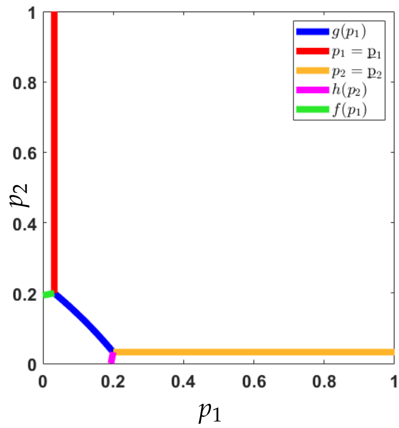


Figure 5.3: Optimal policy regions for $\sigma_{1,2} = 0$, $F = 100$, $D = 100$, $c_1 = 20$, $c_2 = 20$.

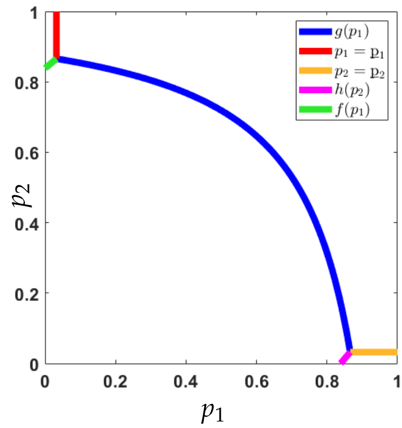
it is not possible to have the particular value for $\sigma_{1,2} > 0$ is shaded in Figure 5.5. In this example, we see that only solutions $(x_1, x_2) = (0, 0)$ and $(x_1, x_2) = (1, 1)$ can be optimal. Also, this figure shows that the curve that divides the optimality region of these solutions shifts towards the optimality region of $(x_1, x_2) = (1, 1)$ as the covariance increases from 0.05 to 0.1 for this particular example. When the covariance increases from 0.05 to 0.1, $(0, 0)$ tends to be optimal more often than $(1, 1)$. In this particular example, having a larger covariance makes having a diagnostic visit without part shipment more preferable.

5.5.3 Characterization of the dominating and dominated solutions for a general N value

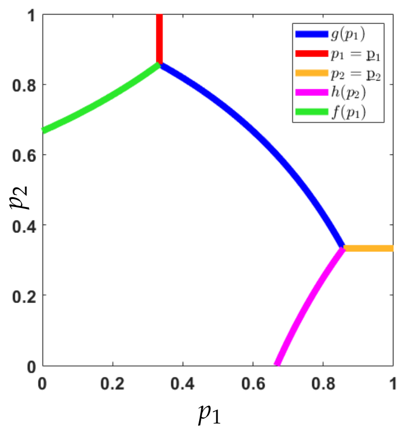
In this section, we show analytical results that determine the dominance between two neighboring solutions. Let us assume we decide to bring an SKU set $I' \subseteq I$ to the maintenance site for the diagnostic visit. Any set I'' that is equal to $I' \setminus \{l\}$, where $l \in I'$, or $I' \cup \{l\} \subseteq I$, where $l \in I \setminus I'$ is a neighbouring set of I'. We denote



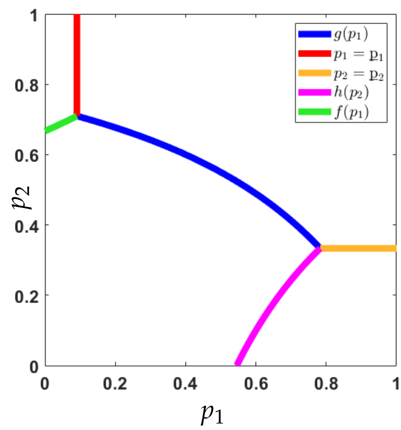
(a) $F = 100$, $D = 500$, $c_1 = 20$, $c_2 = 20$



(b) $F = 500$, $D = 100$, $c_1 = 20$, $c_2 = 20$

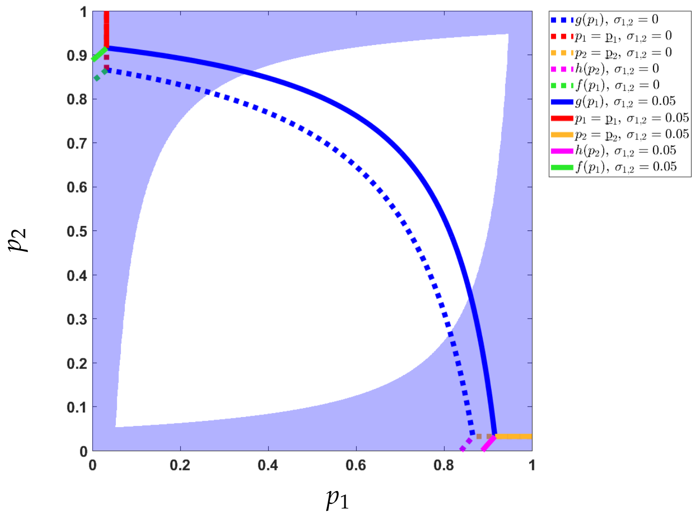


(c) $F = 100$, $D = 100$, $c_1 = 100$, $c_2 = 100$

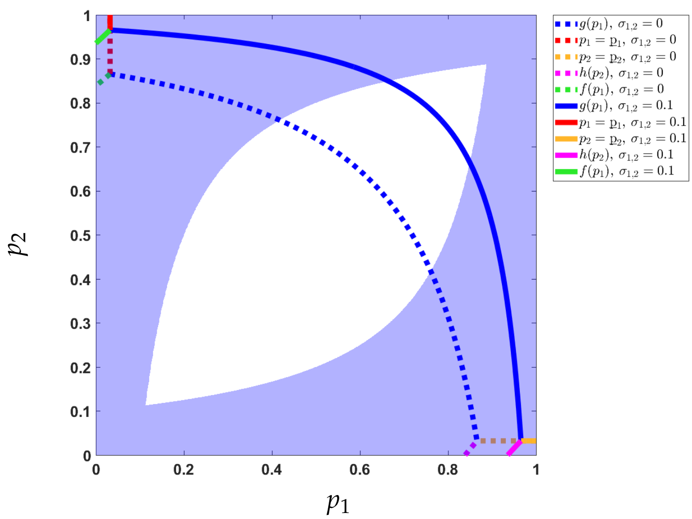


(d) $F = 100$, $D = 100$, $c_1 = 20$, $c_2 = 100$

Figure 5.4: Optimal policy regions for 2 SKUs with independent demand ($\sigma_{1,2} = 0$) (R_1, R_2, R_3 and R_4 are defined as the regions between the same colored lines as in Figure 5.3).



(a) $\sigma_{1,2} = 0.05$



(b) $\sigma_{1,2} = 0.1$

Figure 5.5: Optimal policy regions for $F = 100$, $D = 100$, $c_1 = 20$, $c_2 = 20$ with dependent demand (The shaded (purple) area denotes which combination of p_1 and p_2 are not possible. The white area denotes which combinations of p_1 and p_2 are possible under the given $\sigma_{1,2} > 0$. In the white area, below the continuous blue line the optimal policy is $(0, 0)$, and above the continuous blue line, the optimal policy is $(1, 1)$).

this solution with $\mathbf{x}_{I'} = (x'_1, \ldots, x'_N)$ where

$$x'_i = \begin{cases} 1 & \text{if } i \in I' \\ 0 & \text{if } i \in I \backslash I'. \end{cases}$$

Definition 5.2 A solution $\mathbf{x}_{I'}$ is dominated by another solution $\mathbf{x}_{I''}$ if $C(\mathbf{x}_{I'}) > C(\mathbf{x}_{I''})$. We refer to $\mathbf{x}_{I'}$ as dominated solution and $\mathbf{x}_{I''}$ as dominating solution.

The probability of a second engineer visit will be required under the solution $x_{I'}$ is $h_{I'} = \sum_{m \in M'} \hat{u}_m(I')\hat{p}_m$, where

$$\hat{u}_m(I') = \begin{cases} 0 & \text{if } I' \text{ contains all SKUs that are in } \mathbf{s}_m \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 5.2 *Let $l \in I$, $I' \subset I \backslash \{l\}$. It holds that $h_{I' \cup \{l\}} \leq h_{I'}$. It also holds that the difference, $h_{I'} - h_{I' \cup \{l\}}$, is a non-decreasing function of I' on $I \backslash \{l\}$.*

Lemma 5.2 shows that the difference in the probability of a second engineer visit for a set I' and $I' \cup \{l\}$ is greater than or equal to the difference in the probability of a second engineer visit for a subset of I' and the union of that subset with SKU $\{l\}$. Lemma 5.2 is helpful to provide results for the structure of the optimal policy for a general N number of SKUs.

Remark 5.1 Please note that $h_I = 0$ and $h_{I'} - h_I = p_i$, for any $I' = I \backslash \{i\}$. Additionally, the followings hold if the demands of SKUs are independent: $h_\emptyset - h_I = 1 - \prod_{i \in I}(1 - p_i)$. Similarly, for any set $I' \subseteq I$ and $I'' = I' \backslash \{i\}$, where $I', I'' \neq \emptyset$, $h_{I''} - h_{I'} = \left(\prod_{j \in I \backslash I''}(1 - p_j) \right)$. Finally, for a set $I' = \{i\}$ and a set $I'' = \emptyset$, $h_\emptyset - h_{I'} = \left(\prod_{j \in I \backslash \{i\}}(1 - p_j) \right)$.

Lemma 5.3 *We derive some conditions to determine dominating and dominated solutions.*

(i) Solution $\mathbf{x}_\emptyset$ dominates the solution $\mathbf{x}_I$ if and only if

$$F + \sum_{i \in I} c_i(1 - p_i) - (D + F)h_\emptyset \quad > \quad 0. \tag{5.11}$$

(ii) For any set $\{i\} \in I' \subseteq I$ *and* $I'' = I' \backslash \{i\}$, *where* $I', I'' \neq \emptyset$, $\mathbf{x}_{I''}$ *dominates* $\mathbf{x}_{I'}$ *if*

$$c_i(1 - p_i) - (D + F)(h_{I''} - h_{I'}) \quad > \quad 0. \tag{5.12}$$

As a special case, solution $\mathbf{x}_{I'}$, *where* $I' = I \backslash \{i\}$, *dominates solution* $\mathbf{x}_I$ *if*

$$\frac{c_i}{D + F + c_i} \quad > \quad p_i. \tag{5.13}$$

(iii) For a set $I' = \{i\}$ *and a set* $I'' = \emptyset$, $\mathbf{x}_{I''}$ *dominates* $\mathbf{x}_{I'}$ *if*

$$F + c_i(1 - p_i) - (D + F)(h_{\emptyset} - h_{I'}) \quad > \quad 0.$$

Lemma 5.3 provides some conditions to determine the better solution among two solutions. These conditions depend on the cost parameters and the estimated demand probabilities of SKUs and the sets of SKUs.

Theorem 5.2 *If the solution* $x_{I' \backslash \{i\}}$ *is a better solution than* $x_{I'}$, *then the solution* $x_{I'' \backslash \{i\}}$ *is a better solution than* $x_{I''}$, *where* $I'' \subset I'$ *and* $i \in I''$, *also holds.*

According to Theorem 5.2, if removing an SKU i from a set I' is better than not removing, this also holds for any subsets of I' that contain SKU i.

Corollary 5.1 *For an optimal solution* $\mathbf{x}^*$, *it will hold that* $x_i^* = 0$ *if* (5.13) *holds.*

Corollary 5.1 limits the solution space by only checking whether (5.13) holds or not, which can be easily implemented in practice.

5.6. Numerical analysis

In this section, we aim to show how the optimal policy and optimal cost change with respect to parameters $\hat{p}_m$, $\mathbf{c} = (c_1, \ldots, c_N)$, D, and F. Then, we compare the costs under the optimal policy against Policy 1 and Policy 2. Recall that no parts are sent during the diagnostic visit under Policy 1. If any part is required to fix the system failure, they are sent with an additional shipment and an additional engineer visit will take place. On the other hand, a fixed number of top recommended parts will

Table 5.2: Scenario settings for the spare parts demand distributions.

	$(p_1, p_2, \ldots, p_{10})$	Demand dependency	$\hat{p}_m$
A	$0.135, \forall i \in I$	Yes	0.09 for $m \in \{6, 24, 96, 384, 513\}$, 0.045 for $m \in \{1, 2, 4, 8, 16, 32, 64, 128, 256, 512\}$, 0.1 for $m = 0$, 0 otherwise
B	$0.135, \forall i \in I$	No	$\prod_{i \in I} \left(p_i s_{im} + (1 - p_i)(1 - s_{im}) \right)$ for $m \in \{0, \ldots, 2^{10} - 1\}$
C	(0.5, 0.45, 0.4, 0.35, 0.3, 0.25, 0.2, 0.15, 0.1, 0.05)	Yes	0.05 for $m = 293$, 0.1 for $m \in \{390, 448\}$, 0.2 for $m = 392$, 0.25 for $m \in \{544, 592\}$, 0 otherwise
D	(0.5, 0.45, 0.4, 0.35, 0.3, 0.25, 0.2, 0.15, 0.1, 0.05)	No	$\prod_{i \in I} \left(p_i s_{im} + (1 - p_i)(1 - s_{im}) \right)$ for $m \in \{0, \ldots, 2^{10} - 1\}$

be sent during the diagnostic visit from a list, where SKUs are sorted in decreasing order of p_i, $\forall I \in I$ under Policy 2.

In the numerical analysis, we consider instances with $N = 10$ SKUs. We use Gurobi 9.1.2 to solve the ILP models. For this purpose, we created two test beds, where $F \in \{25, 50, 100\}$, $D \in \{100, 200, 400\}$. For the first test bed, $\mathbf{c}$ is equal to $\mathbf{c}' = (20.13, 17.65, 10.51, 12.87, 10.49, 10.44, 14.38, 17.3, 14.5, 24.86)$; where each cost parameter is generated from a uniform distribution in the interval [10,25]. Four spare parts demand distribution scenarios are generated for the first test bed. We refer to them as scenarios A, B, C, and D with details provided in Table 5.2. One factor that may affect the optimal solution is the marginal demand probability distribution for SKUs. Therefore, in Scenarios A and B, we choose the marginal demand probabilities for SKUs identical. In Scenarios C and D, we choose varying values between 0.05 to 0.5 for the marginal demand probabilities. Another factor that may impact the optimal solution is the existence of demand dependency between SKUs. Therefore, demand dependency is added as a factor in the experimental design.

The optimal solution and optimal cost for each instance are given in Table 5.3. In 10 out of 36 instances, the optimal policy is the same as Policy 1. In 13 out of 36 instances, the optimal policy is equal to the 'send all parts' solution. For the remaining instances, the optimal number of parts that should be sent during the diagnostic visit is between seven and nine. We observe only 'send no parts' or 'send all parts' solutions under Scenarios A and B, where the marginal demand probabilities of SKUs are equal. The optimal solution is 'send no parts' when the second engineer visit cost D is relatively low in comparison to fixed transportation cost F, and the optimal solution is 'send all parts' when D is relatively high in comparison to F.

We observe other optimal solutions than 'send no parts' and 'send all parts' under Scenarios C and D, where the variance among the marginal probabilities for SKUs is high. The marginal probability of an SKU is an important factor to have the SKU in the optimal solution or not.

We also compare the optimal cost against the cost of Policy 1 and Policy 2. In Policy 1, $\mathbf{x} = \mathbf{0}$, i.e. this is the solution 'send no parts'. In Policy 2, we select the first k parts for the diagnostic visit from a list of SKUs that are ranked from the highest to the lowest based on the marginal probabilities (notice that we have already done this when defining Scenarios A, B, C, and D). We denote the cost of Policy 1 by C_1, the cost of Policy 2 for a given $k \in \{1, \ldots, N\}$ (referred to as Policy 2(k)) by $C_2(k)$, and the cost of the optimal policy by C_{opt}. The relative difference $\frac{C_1 - C_{\text{opt}}}{C_{\text{opt}}} 100\%$ is denoted by Δ_1 and the difference $\frac{C_2(k) - C_{\text{opt}}}{C_{\text{opt}}} 100\%$ is denoted by $\Delta_2(k)$.

The cost comparison between policies for instances 1-36 is given in Table 5.4. In this table, for each instance, we see that either Policy 1 or Policy 2(k) for a given k finds the optimal policy. However, one particular heuristic policy cannot find the optimal solution in all problem instances. The best heuristic policy (i.e., Policy 2(10)) is 12.2% costlier than the optimal policy on an average of 36 instances.

Next, we create a second test bed for instances 19-36 to see the effect of cost c_i on the optimal solution. In this new test bed, the cost vector $\mathbf{c}$ is equal to a new cost vector $\mathbf{c}''$, where the cost of SKU 4 is significantly higher than the cost of other SKUs. The only difference between vectors $\mathbf{c}'$ and $\mathbf{c}''$ is that c_4'' is equal to 121.87 instead of 12.87. In Table 5.5, we see the optimal solution and cost for instances 19-36 and $\mathbf{c} = \mathbf{c}''$. In the first test bed for instances 22 and 31, we see that SKU 4 is in the set of parts that are sent to the customer. We see that the heuristic policies cannot always find the optimal solution in the second test bed, where the cost of SKU 4 is high. We focus on instance 22 to investigate the effect of demand dependency on optimal costs. Under Scenario C, s_{544} involves SKUs 1, 4, 6 with $\hat{p}_{544} = 0.25$. We see that SKU 4 is not in the optimal set of spare parts due to the high cost of c_4. We also

Table 5.3: Optimal solution and cost for problem instances 1-36 and $\mathbf{c} = \mathbf{c}'$.

Instance	F	D	**Scenario**	**Optimal Solution**	**Optimal Cost**
1	25	100	A	$(0,0,0,0,0,0,0,0,0,0)$	112.5
2	50	100	A	$(0,0,0,0,0,0,0,0,0,0)$	135.0
3	100	100	A	$(0,0,0,0,0,0,0,0,0,0)$	180.0
4	25	200	A	$(1,1,1,1,1,1,1,1,1,1)$	157.5
5	50	200	A	$(1,1,1,1,1,1,1,1,1,1)$	182.5
6	100	200	A	$(1,1,1,1,1,1,1,1,1,1)$	232.5
7	25	400	A	$(1,1,1,1,1,1,1,1,1,1)$	157.5
8	50	400	A	$(1,1,1,1,1,1,1,1,1,1)$	182.5
9	100	400	A	$(1,1,1,1,1,1,1,1,1,1)$	232.5
10	25	100	B	$(0,0,0,0,0,0,0,0,0,0)$	95.7
11	50	100	B	$(0,0,0,0,0,0,0,0,0,0)$	114.8
12	100	100	B	$(0,0,0,0,0,0,0,0,0,0)$	153.1
13	25	200	B	$(1,1,1,1,1,1,1,1,1,1)$	157.5
14	50	200	B	$(1,1,1,1,1,1,1,1,1,1)$	182.5
15	100	200	B	$(0,0,0,0,0,0,0,0,0,0)$	229.6
16	25	400	B	$(1,1,1,1,1,1,1,1,1,1)$	157.5
17	50	400	B	$(1,1,1,1,1,1,1,1,1,1)$	182.5
18	100	400	B	$(1,1,1,1,1,1,1,1,1,1)$	232.5
19	25	100	C	$(1,1,1,1,1,1,1,0,0,0)$	104.9
20	50	100	C	$(1,1,1,1,1,1,1,0,0,0)$	133.6
21	100	100	C	$(0,0,0,0,0,0,0,0,0,0)$	190.0
22	25	200	C	$(1,1,1,1,1,1,1,0,0,0)$	119.9
23	50	200	C	$(1,1,1,1,1,1,1,0,0,0)$	148.6
24	100	200	C	$(1,1,1,1,1,1,1,1,1,0)$	203.9
25	25	400	C	$(1,1,1,1,1,1,1,1,1,0)$	135.1
26	50	400	C	$(1,1,1,1,1,1,1,1,1,0)$	161.4
27	100	400	C	$(1,1,1,1,1,1,1,1,1,1)$	212.5
28	25	100	D	$(1,1,1,1,1,1,1,1,0,0)$	118.9
29	50	100	D	$(0,0,0,0,0,0,0,0,0,0)$	145.1
30	100	100	D	$(0,0,0,0,0,0,0,0,0,0)$	193.4
31	25	200	D	$(1,1,1,1,1,1,1,1,1,0)$	125.1
32	50	200	D	$(1,1,1,1,1,1,1,1,1,0)$	151.4
33	100	200	D	$(1,1,1,1,1,1,1,1,1,0)$	203.9
34	25	400	D	$(1,1,1,1,1,1,1,1,1,0)$	135.1
35	50	400	D	$(1,1,1,1,1,1,1,1,1,0)$	161.4
36	100	400	D	$(1,1,1,1,1,1,1,1,1,1)$	212.5

Table 5.4: Comparison of optimal policy against Policy 1 and 2 for problem instances 1-36 and $\mathbf{c} = \mathbf{c}'$.

Instance	Δ_1	$\Delta_2(k)$									
		$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
1	**0.0%**	32.7%	41.2%	34.4%	39.3%	32.4%	35.4%	31.4%	39.7%	35.8%	40.0%
2	**0.0%**	45.0%	51.3%	43.0%	46.2%	37.9%	39.6%	33.9%	39.9%	34.2%	35.2%
3	**0.0%**	60.2%	63.7%	53.8%	54.9%	45.0%	45.0%	36.9%	40.2%	32.2%	29.2%
4	28.6%	49.1%	52.4%	38.9%	39.5%	26.0%	25.3%	13.9%	17.0%	5.7%	**0.0%**
5	23.3%	54.1%	56.3%	42.8%	42.7%	29.2%	28.0%	16.3%	18.4%	6.7%	**0.0%**
6	16.1%	60.9%	61.6%	48.1%	47.1%	33.5%	31.7%	19.6%	20.2%	8.2%	**0.0%**
7	142.9%	157.7%	155.2%	124.6%	119.5%	88.8%	82.4%	53.9%	51.2%	22.8%	**0.0%**
8	121.9%	147.8%	145.0%	116.8%	111.8%	83.5%	77.3%	50.8%	47.9%	21.5%	**0.0%**
9	93.5%	134.4%	131.3%	106.2%	101.3%	76.1%	70.4%	46.7%	43.4%	19.8%	**0.0%**
10	**0.0%**	39.5%	49.9%	53.1%	57.4%	58.3%	57.8%	59.5%	61.9%	59.7%	64.6%
11	**0.0%**	53.9%	61.7%	63.2%	65.6%	64.9%	62.9%	62.3%	62.2%	57.8%	58.9%
12	**0.0%**	71.9%	76.4%	75.9%	75.8%	73.2%	69.2%	65.9%	62.5%	55.5%	51.9%
13	9.4%	31.1%	34.7%	33.5%	32.5%	29.0%	23.9%	19.3%	14.3%	5.7%	**0.0%**
14	4.9%	36.8%	39.4%	37.6%	36.0%	32.1%	26.6%	21.5%	15.8%	6.7%	**0.0%**
15	**0.0%**	46.3%	47.5%	45.0%	42.5%	37.9%	32.0%	26.0%	19.3%	9.5%	1.2%
16	106.6%	123.6%	121.9%	114.5%	106.3%	94.4%	79.7%	64.1%	46.3%	22.8%	**0.0%**
17	88.8%	116.7%	114.6%	107.6%	99.7%	88.5%	74.8%	60.1%	43.5%	21.5%	**0.0%**
18	64.6%	107.3%	104.7%	98.1%	90.8%	80.6%	68.2%	54.8%	39.5%	19.8%	**0.0%**
19	13.3%	46.6%	56.0%	62.0%	58.0%	35.2%	12.9%	**0.0%**	14.0%	14.6%	31.1%
20	6.7%	51.6%	58.9%	63.6%	58.6%	36.1%	13.8%	**0.0%**	11.0%	9.6%	21.6%
21	**0.0%**	57.9%	63.1%	66.4%	60.2%	37.8%	15.6%	0.6%	8.3%	4.7%	11.8%
22	78.3%	107.5%	115.7%	120.9%	109.1%	68.3%	27.9%	**0.0%**	12.3%	4.4%	14.7%
23	59.8%	100.2%	106.8%	111.0%	99.8%	62.7%	25.9%	**0.0%**	9.9%	1.9%	9.4%
24	39.8%	93.8%	98.5%	101.6%	91.0%	57.8%	24.9%	1.1%	8.3%	**0.0%**	4.2%
25	198.8%	224.8%	232.0%	236.6%	211.4%	138.2%	65.3%	11.0%	21.8%	**0.0%**	1.8%
26	164.9%	202.1%	208.1%	212.0%	189.3%	124.2%	59.3%	10.7%	19.8%	**0.0%**	0.7%
27	123.5%	175.3%	179.9%	182.9%	163.3%	107.9%	52.8%	11.1%	18.0%	0.7%	**0.0%**
28	1.7%	27.7%	30.2%	27.2%	23.0%	15.4%	6.7%	1.1%	**0.0%**	1.0%	15.6%
29	**0.0%**	38.0%	39.1%	35.3%	30.0%	21.6%	11.9%	4.8%	1.7%	0.9%	12.0%
30	**0.0%**	53.5%	53.0%	48.1%	41.3%	31.6%	20.6%	11.5%	5.9%	2.8%	9.8%
31	73.9%	96.1%	94.2%	85.0%	72.4%	54.8%	34.9%	18.0%	6.6%	**0.0%**	9.9%
32	59.7%	94.0%	91.5%	82.6%	70.5%	53.8%	34.9%	18.6%	7.1%	**0.0%**	7.3%
33	42.3%	91.5%	88.4%	79.8%	68.2%	52.5%	35.0%	19.2%	7.6%	**0.0%**	4.2%
34	204.3%	219.9%	210.3%	189.9%	162.5%	126.8%	86.9%	49.7%	20.2%	**0.0%**	1.8%
35	169.7%	197.8%	188.8%	170.6%	146.0%	114.1%	78.4%	45.0%	18.4%	**0.0%**	0.7%
36	127.6%	171.7%	163.7%	148.0%	126.7%	99.4%	68.9%	40.1%	16.8%	0.7%	**0.0%**
Average	57.4%	95.0%	96.9%	90.6%	83.1%	62.5%	44.6%	27.2%	24.8%	13.5%	12.2%

Table 5.5: Optimal solution and cost for problem instances 19-36 and $\mathbf{c} = \mathbf{c}''$.

Instance	Optimal Solution	Optimal Cost
19	$(0,0,0,0,0,0,0,0,0,0)$	118.8
20	$(0,0,0,0,0,0,0,0,0,0)$	142.5
21	$(0,0,0,0,0,0,0,0,0,0)$	190.0
22	$(1,1,1,0,1,0,1,0,0,0)$	183.7
23	$(1,1,1,1,1,1,1,0,0,0)$	220.8
24	$(1,1,1,1,1,1,1,1,1,0)$	276.0
25	$(1,1,1,1,1,1,1,1,1,0)$	207.3
26	$(1,1,1,1,1,1,1,1,1,0)$	233.5
27	$(1,1,1,1,1,1,1,1,1,1)$	284.6
28	$(0,0,0,0,0,0,0,0,0,0)$	120.9
29	$(0,0,0,0,0,0,0,0,0,0)$	145.1
30	$(0,0,0,0,0,0,0,0,0,0)$	193.5
31	$(1,1,1,0,1,1,1,1,1,0)$	192.9
32	$(1,1,1,1,1,1,1,1,1,0)$	223.5
33	$(1,1,1,1,1,1,1,1,1,0)$	276.0
34	$(1,1,1,1,1,1,1,1,1,0)$	207.3
35	$(1,1,1,1,1,1,1,1,1,0)$	233.5
36	$(1,1,1,1,1,1,1,1,1,1)$	284.6

see that SKU 6 is not in the optimal solution due to high demand dependency with SKU 4 and low marginal demand probability (i.e., $p_6 = 0.25$). On the other hand, SKU 1 is still in the optimal solution because its marginal demand probability is comparatively higher than SKU 4 and 6 (i.e., $p_1 = 0.5$). In the first test bed, $x_i^* = 1$ is for some i under instances 19, 20, and 28. However, in the second test bed, the optimal solution is 'send no parts' in these instances. When the shipment and the send-back cost is high for an SKU that has a high marginal demand probability, we observe a more conservative optimal solution (i.e., first having a diagnostic visit, then shipment of SKUs).

In Table 5.6, we see the comparison of the cost of the optimal policy against Policy 1 and 2. The optimal policy in the second test bed is equal to Policy 1 in more instances and equal to 'send all parts' in fewer instances than in the first test bed. For two problem instances, none of the heuristic policies finds the optimal policy in the second test bed. For instance 22, the best heuristic solution (i.e., Policy 2(7)) is 4.5% costlier than the optimal policy. For instance 31, the best heuristic solution (i.e., Policy 2(9)) is 2.3% costlier than the optimal policy. On average of 18 instances in the second test bed, the best heuristic policy (Policy 2(9)) is 17.8% costlier than the optimal policy.

Table 5.6: Comparison of cost of optimal policy against Policy 1 and 2 for problem instances 19-36 and $\mathbf{c} = \mathbf{c}''$.

Instance	Δ_1	$\Delta_2(k)$									
		$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$
19	**0.0%**	29.5%	37.7%	43.0%	99.2%	80.1%	60.4%	49.1%	61.4%	61.9%	76.5%
20	**0.0%**	42.2%	49.0%	53.4%	98.5%	78.2%	57.4%	44.4%	54.7%	53.3%	64.6%
21	**0.0%**	57.9%	63.1%	66.4%	97.5%	75.7%	53.6%	38.6%	46.3%	42.6%	49.8%
22	16.3%	35.4%	40.7%	44.1%	75.0%	49.1%	22.8%	4.5%	12.5%	7.4%	14.1%
23	7.6%	34.8%	39.2%	42.0%	66.6%	42.2%	17.4%	**0.0%**	6.7%	1.3%	6.3%
24	3.3%	43.2%	46.7%	48.9%	66.8%	42.7%	18.4%	0.8%	6.2%	**0.0%**	3.1%
25	94.8%	111.7%	116.4%	119.4%	137.1%	90.1%	42.5%	7.1%	14.2%	**0.0%**	1.2%
26	83.1%	108.8%	113.0%	115.7%	130.3%	85.8%	41.0%	7.4%	13.7%	**0.0%**	0.5%
27	66.9%	105.6%	109.0%	111.2%	121.5%	80.6%	39.4%	8.3%	13.5%	0.5%	**0.0%**
28	**0.0%**	25.6%	28.1%	25.1%	79.6%	73.2%	64.7%	59.1%	58.1%	59.1%	73.4%
29	**0.0%**	38.0%	39.1%	35.3%	78.8%	71.3%	61.7%	54.6%	51.4%	50.6%	61.7%
30	**0.0%**	53.5%	53.0%	48.1%	77.9%	68.9%	57.9%	48.8%	43.2%	40.1%	47.1%
31	12.9%	27.2%	26.0%	20.0%	48.6%	37.8%	24.9%	13.9%	6.6%	2.3%	8.7%
32	8.2%	31.4%	29.8%	23.7%	47.2%	36.4%	23.7%	12.6%	4.8%	**0.0%**	5.0%
33	5.1%	41.4%	39.2%	32.8%	49.9%	38.8%	25.8%	14.2%	5.6%	**0.0%**	3.1%
34	98.3%	108.5%	102.2%	89.0%	105.3%	82.6%	56.6%	32.4%	13.2%	**0.0%**	1.2%
35	86.4%	105.8%	99.7%	87.1%	100.4%	78.9%	54.2%	31.1%	12.7%	**0.0%**	0.5%
36	69.9%	102.8%	96.8%	85.1%	94.2%	74.2%	51.4%	30.0%	12.6%	0.5%	**0.0%**
Average	30.7%	61.3%	62.7%	60.6%	87.5%	65.9%	43.0%	25.4%	24.3%	17.8%	23.1%

5.6.1 Service level constraint extension

In SLAs, there might be constraints on the minimum service level for the service provider. We introduce a service level constraint

$$1 - \sum_{m \in M'} \hat{u}_m \hat{p}_m \geq \alpha, \tag{5.14}$$

to ensure the probability of resolving a maintenance case during the diagnostic visit is at least at the desired level α. Note that $\sum_{m \in M'} \hat{u}_m \hat{p}_m$ is the probability that all of the required parts are sent to the site. We let C^{α}_{opt} be the cost of the optimal solution with the service level constraint under a given α.

We denote the percentage difference between the optimal cost without a service level constraint and the optimal cost with a service level constraint under a given $\alpha > 0$ as $\Delta^{\alpha} = \frac{C^{\alpha}_{\text{opt}} - C_{\text{opt}}}{C_{\text{opt}}}\%$. In Table 5.7, we present Δ^{α} for instances 1-36 for $\alpha = 0.9$ and $\mathbf{c} = \mathbf{c}'$.

Table 5.7: Δ^{α} values for instances 1-36 under $\alpha = 0.9$ and $\mathbf{c} = \mathbf{c}'$.

1	2	3	4	5	6	7	8	9	10	11	12
41.4%	36.3%	30.0%	1.0%	0.9%	0.7%	1.0%	0.9%	0.7%	66.2%	60.3%	52.9%
13	14	15	16	17	18	19	20	21	22	23	24
1.0%	0.9%	1.9%	1.0%	0.9%	0.7%	15.8%	10.5%	5.3%	5.5%	2.7%	0.6%
25	26	27	28	29	30	31	32	33	34	35	36
1.0%	0.8%	0.6%	2.1%	1.8%	3.5%	1.0%	0.8%	0.6%	1.0%	0.8%	0.6%

A service constraint with a service level $\alpha = 0.9$ increases the optimal cost from 0.6% up to 66.2%. We choose instances 21 and 30 to show how the optimal solution and optimal cost behave as a function of $\alpha \in \{0, 0.5, 0.8, 0.9\}$, see Table 5.8. For both instances 21 and 30, we have the same cost parameters ($F = 100$ and $D = 100$) and the same distribution for marginal probabilities p_i (i.e., Scenarios C and D). However, the demand for SKUs is dependent in instance 21, where it is independent for instance 30. First, we see that demand dependency between SKUs affects the optimal solution. For instance, the number of SKUs in the optimal solution for instance 20 is less for $\alpha \in \{0.5, 0.8\}$ than the number of SKUs in the optimal solution for instance 30. Next, the optimal cost increases by 5.3% for instance 20, and 3.5% for instance 30 under a high service level (i.e., $\alpha = 0.9$) (please see Table 5.8).

Table 5.8: Optimal solution and cost of instances 21 and 30 under service constraint for varying values of α and $\mathbf{c} = \mathbf{c}'$.

α	**Instance**	**Optimal Solution**	**Optimal Cost**
0	21	$(0,0,0,0,0,0,0,0,0,0)$	190.0
0.5	21	$(1,1,1,1,1,1,1,0,0,0)$	192.4
0.8	21	$(1,1,1,1,1,1,1,0,0,0)$	192.4
0.9	21	$(1,1,1,1,1,1,1,1,1,0)$	200.2
0	30	$(0,0,0,0,0,0,0,0,0,0)$	193.5
0.5	30	$(1,1,1,1,1,1,1,1,1,0)$	200.2
0.8	30	$(1,1,1,1,1,1,1,1,1,0)$	200.2
0.9	30	$(1,1,1,1,1,1,1,1,1,0)$	200.2

5.7. Conclusion

In this chapter, we consider a spare part recommendation problem from the point of view of a maintenance service provider. We develop a decision support model for the selection of spare parts to resolve a maintenance case based on historical cases and operational costs. For this aim, we formulate an ILP model that minimizes the total cost of a maintenance case by selecting the optimal set of spare parts. We provide results on the structure of the optimal policy for the problem instances with one and two SKUs, and partial results for the structure of the optimal policy for problem instances with a general number of SKUs. Finally, we compare the optimal policy against two benchmark heuristics used in practice. In this comparison, we see that heuristics from practice cannot always find the optimal policy. This results in up to 12.2% and 17.8% costlier maintenance operations for the best heuristic policies compared to the optimal policy for the two defined test beds.

5.A. Proofs for structural results

Proof of Proposition 5.1

For $N = 1$, there are two solutions for the ILP: $x_1 = 1$ and $x_1 = 0$. The costs under these solutions are

$$\begin{aligned} C(1) &= F + c_1(1 - p_1) \\ C(0) &= (D + F)p_1. \end{aligned}$$

Note that $C(1)$ is strictly decreasing as a function of p_1 and $C(0)$ is strictly increasing. Let $\check{p}_1$ be the point where the cost under both solutions is equal. Then:

$$\begin{aligned} C(1) - C(0) = 0 \quad &\iff \quad F + c_1(1 - \check{p}_1) - (D + F)\check{p}_1 = 0 \\ &\iff \quad \check{p}_1 = \frac{F + c_1}{D + F + c_1} \end{aligned}$$

When $C(1) \geq C(0)$ (i.e. $\frac{F+c_1}{D+F+c_1} \leq \check{p}_1$), solution $x_1 = 0$ is optimal. When $C(1) < C(0)$ (i.e. $\frac{F+c_1}{D+F+c_1} \geq \check{p}_1$), solution $x_1 = 1$ is optimal.

□

Proof of Lemma 5.1

We assume that $\sigma_{1,2} = 0$. Then, $\tilde{p}_1 = \frac{F+c_1}{D+F_1+c_1}$, and $\bar{p}_1 = \frac{(F+c_1)(D+F+c_2)}{(D+F+c_2)(D+F+c_1)-(D+F)c_2} = \frac{F+c_1}{(D+F+c_1)-\frac{(D+F)c_2}{D+F+c_2}}$. Obviously, $0 < \frac{c_1}{D+F+c_1} < \frac{F+c_1}{D+F_1+c_1} < \frac{F+c_1}{(D+F+c_1)-\frac{(D+F)c_2}{D+F+c_2}}$. Hence, $0 < \underline{p}_1 < \tilde{p}_1 < \bar{p}_1$. Additionally, $0 < \underline{p}_1 < \tilde{p}_1 < 1$. It also holds that $0 < \underline{p}_2 < \tilde{p}_2 < \bar{p}_2$ and $0 < \underline{p}_2 < \tilde{p}_2 < 1$ from symmetry. □

Proof of Theorem 5.1

Assume that $\sigma_{1,2} = 0$ and $N = 2$. There are four possible solutions: $\mathbf{x}_0 = (x_1, x_2) = (0,0)$, $\mathbf{x}_1 = (x_1, x_2) = (0,1)$, $\mathbf{x}_2 = (x_1, x_2) = (1,0)$ and $\mathbf{x}_3 = (x_1, x_2) = (1,1)$. Let $\tilde{h}_m$ be the probability of a second engineer visit if combination $\mathbf{x}_m$ is chosen for the repair kit. $\tilde{h}_0 = p_1 + p_2 - p_1 p_2$, $\tilde{h}_1 = p_1$, $\tilde{h}_2 = p_2$ and $\tilde{h}_3 = 0$. From these probabilities we calculate costs under each solution (x_1, x_2) as

$$\begin{aligned}
C(0,0) &= (D+F)(p_1 + p_2 - p_1 p_2) \\
C(0,1) &= F + c_2(1 - p_2) + (D+F)p_1 \\
C(1,0) &= F + c_1(1 - p_1) + (D+F)p_2 \\
C(1,1) &= F + c_1(1 - p_1) + c_2(1 - p_2).
\end{aligned}$$

We define $\omega_1(p_1, p_2)$ as the cost difference between the solutions $(0,0)$ and $(1,1)$.

$$\begin{aligned}
\omega_1(p_1, p_2) &= C(0,0) - C(1,1) \\
&= (D+F)(p_1 + p_2 - p_1 p_2) - (F + c_1(1-p_1) + c_2(1-p_2)) \\
&= (D+F+c_1)p_1 + (D+F+c_2)p_2 - (p_1 p_2)(D+F) - (F + c_1 + c_2)
\end{aligned}$$

We define $\omega_2(p_1, p_2)$ as the cost difference between the solutions $(1,1)$ and $(1,0)$.

$$\begin{aligned}
C(1,1) - C(1,0) &= F + c_1(1-p_1) + c_2(1-p_2) - (F + c_1(1-p_1) + (D+F)p_2) \\
&= c_2 - p_2(D+F+c_2)
\end{aligned}$$

We define $\omega_3(p_1, p_2)$ as the cost difference between the solutions $(1,1)$ and $(0,1)$.

$$\begin{aligned}
C(1,1) - C(0,1) &= F + c_1(1-p_1) + c_2(1-p_2) - (F + c_2(1-p_2) + (D+F)p_1) \\
&= c_1 - p_1(D+F+c_1)
\end{aligned}$$

We define $\omega_4(p_1, p_2)$ as the cost difference between the solutions $(1,0)$ and $(0,0)$.

$$\begin{aligned}
C(1,0) - C(0,0) &= F + c_1(1-p_1) + (D+F)p_2 - ((D+F)(p_1 + p_2 - p_1 p_2)) \\
&= F + c_1 + (D+F)(p_1 p_2) - p_1(D+F+c_1)
\end{aligned}$$

We define $\omega_5(p_1, p_2)$ as the cost difference between the solutions $(0,0)$ and $(0,1)$.

$$\begin{aligned} C(0,1) - C(0,0) &= (F + c_2(1-p_2) + (D+F)p_1) - (D+F)(p_1 + p_2 - p_1 p_2) \\ &= F + c_2 + (D+F)(p_1 p_2) - p_2(D+F+c_2) \end{aligned}$$

We define $\omega_6(p_1, p_2)$ as the cost difference between the solutions $(1,0)$ and $(0,1)$.

$$\begin{aligned} C(0,1) - C(1,0) &= F + c_2(1-p_2) + (D+F)p_1 - (F + c_1(1-p_1) + (D+F)p_2) \\ &= (c_2 - c_1) - p_2(c_2 + D + F) + p_1(c_1 + D + F) \end{aligned}$$

i. Solution $\mathbf{x} = (0,0)$ is optimal if and only if $\omega_1(p_1, p_2) \leq 0$, $\omega_4(p_1, p_2) \geq 0$ and $\omega_5(p_1, p_2) \geq 0$.

ii. Solution $\mathbf{x} = (0,1)$ is optimal if and only if $\omega_3(p_1, p_2) \geq 0$, $\omega_5(p_1, p_2) \leq 0$, and $\omega_6(p_1, p_2) \leq 0$.

iii. Solution $\mathbf{x} = (1,1)$ is optimal if and only if $\omega_1(p_1, p_2) \geq 0$, $\omega_2(p_1, p_2) \leq 0$ and $\omega_3(p_1, p_2) \leq 0$.

iv. Solution $\mathbf{x} = (1,0)$ is optimal if and only if $\omega_2(p_1, p_2) \geq 0$, $\omega_4(p_1, p_2) \leq 0$, and $\omega_6(p_1, p_2) \geq 0$.

Please note that if $\omega_2(p_1, p_2) \geq 0$, then $\omega_5(p_1, p_2) \leq 0$. If $\omega_4(p_1, p_2) \leq 0$ and $\omega_5(p_1, p_2) \leq 0$, then $\omega_6(p_1, p_2) \geq 0$. So, solution $\mathbf{x} = (1,0)$ is optimal if and only if $\omega_2(p_1, p_2) \geq 0$ and $\omega_4(p_1, p_2) \leq 0$. Therefore, condition $\omega_6(p_1, p_2) \geq 0$ is redundant. Similarly, if $\omega_3(p_1, p_2) \geq 0$, then $\omega_4(p_1, p_2) \geq 0$. If $\omega_4(p_1, p_2) \geq 0$ and $\omega_5(p_1, p_2) \leq 0$, then $\omega_6(p_1, p_2) \leq 0$. Therefore, solution $\mathbf{x} = (0,1)$ is optimal if and only if $\omega_3(p_1, p_2) \geq 0$ and $\omega_5(p_1, p_2) \leq 0$. The condition $\omega_6(p_1, p_2) \leq 0$ is redundant.

In order to derive the structure of optimal policy, we determine the intersection points of $\omega_k(p_1, p_2) = 0, k \in \{1,2,3,4,5\}$. We first calculate the intersection of $\omega_2(p_1, p_2) = 0$ and $\omega_4(p_1, p_2) = 0$. If $\omega_2(p_1, p_2) = 0$, then $p_2 = \frac{c_2}{D+F+c_2}$. We first plug p_2 into $\omega_4(p_1, p_2) = 0$.

$$\begin{aligned} & F + c_1 - p_1\Big[(D + F + c_1) - \frac{c_2}{D+F+c_2}(D+F)\Big] = 0 \\ \iff \quad & (F + c_1)(D + F + c_2) \end{aligned}$$

$$
\begin{aligned}
& - \quad p_1\Big[(D+F+c_2)(D+F+c_1) - c_2(D+F)\Big] = 0 \\
\iff \quad & p_1 = \frac{(F+c_1)(D+F+c_2)}{(D+F+c_2)(D+F+c_1)-(D+F)c_2} = \frac{(F+c_1)}{(D+F+c_1)-\frac{(D+F)c_2}{D+F+c_2}}.
\end{aligned}
$$

We characterize the intersection point of $\omega_2(p_1,p_2) = 0$ and $\omega_4(p_1,p_2) = 0$ as $(\bar{p}_1, \underline{p}_2)$.

Next, we determine the intersection point of $\omega_1(p_1,p_2) = 0$ and $\omega_2(p_1,p_2) = 0$. We also plug the point $p_2 = \frac{c_2}{D+F+c_2}$ in $\omega_1(p_1,p_2) = 0$:

$$
\begin{aligned}
& (D+F+c_1)p_1 + \frac{(D+F+c_2)c_2}{D+F+c_2} - \frac{(D+F)c_2p_1}{D+F+c_2} - (F+c_1+c_2) = 0 \\
\iff \quad & (D+F+c_1)p_1 + c_2 - \frac{(D+F)c_2p_1}{D+F+c_2} - \sigma_{1,2}(D+F) - (F+c_1+c_2) = 0 \\
\iff \quad & (D+F+c_1)p_1 - \frac{(D+F)c_2p_1}{D+F+c_2} - (F+c_1) = 0 \\
\iff \quad & p_1\Big((D+F+c_1) - \frac{c_2}{D+F+c_2}(D+F)\Big) = F+c_1 \\
\iff \quad & p_1 = \frac{(F+c_1)(D+F+c_2)}{(D+F+c_2)(D+F+c_1)-(D+F)c_2}
\end{aligned}
$$

We characterize the intersection point of $\omega_1(p_1,p_2) = 0$ and $\omega_2(p_1,p_2) = 0$ as $(\bar{p}_1, \underline{p}_2)$. We see that $\omega_1(p_1,p_2) = 0$, $\omega_2(p_1,p_2) = 0$ and $\omega_4(p_1,p_2) = 0$ intersects at the same point. From symmetry, the same holds for the intersection point of $\omega_1(p_1,p_2) = 0$, $\omega_3(p_1,p_2) = 0$ and $\omega_5(p_1,p_2) = 0$. We characterize this point with $(\underline{p}_1, \bar{p}_2)$ where $(\underline{p}_1, \bar{p}_2) = \Big(\frac{c_1}{D+F+c_1}, \frac{(F+c_2)(D+F+c_1)}{(D+F+c_1)(D+F+c_2)-(D+F)c_1}\Big)$.

At point $(\bar{p}_1, \underline{p}_2)$ solutions (1,1), (1,0) and (0,0) are all optimal. At point $(\underline{p}_1, \bar{p}_2)$ solutions (1,1), (0,1) and (0,0) are all optimal. $\omega_4(p_1,p_2)$ cuts the p_1-axis at the point $\tilde{p}_1 = \frac{F+c_1}{D+F_1+c_1}$. $\omega_5(p_1,p_2)$ cuts the p_2-axis at $\tilde{p}_2 = \frac{F+c_2}{D+F+c_2}$. We describe these points in Definition 5.1 and we demonstrate them in Figure 5.6.

Finally, we derive Theorem 5.1 from the following results. First, we reformulate $\omega_5(p_1,p_2) = 0$ as $f(p_1) = p_2$, reformulate $\omega_1(p_1,p_2) = 0$ as $g(p_1) = p_2$ and reformulate $\omega_4(p_1,p_2) = 0$ as of $h(p_1) = p_2$ (please see Definition 5.1). Then we formulate the Theorem 5.1.

(i) We let R_1 define a region where $\omega_1(p_1,p_2) \le 0$, $\omega_4(p_1,p_2) \ge 0$ and

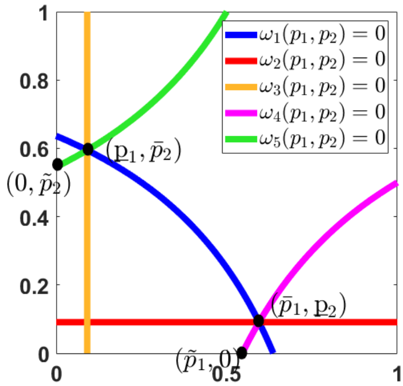


Figure 5.6: Points defined in Definition 5.1.

$\omega_5(p_1, p_2) \geq 0$. Therefore, the solution $(0, 0)$ is optimal in region R_1.

(ii) We let R_2 define a region where $\omega_5(p_1, p_2) \leq 0$ and $\omega_3(p_1, p_2) \geq 0$. In region R_2 the solution $(0, 1)$ is optimal.

(iii) We let R_3 define a region, where $\omega_3(p_1, p_2) \leq 0$, $\omega_1(p_1, p_2) \geq 0$, and $\omega_2(p_1, p_2) \leq 0$. The solution $(1, 1)$ is optimal in region R_3.

(iv) We let R_4 define the region, where $\omega_4(p_1, p_2) \leq 0$, and $\omega_2(p_1, p_2) \geq 0$. The solution $(1, 1)$ is optimal in the region R_4.

$\square$

Proof of Lemma 5.2

First, we show that $h_{I'} \geq h_{I' \cup \{l\}}$. Hence, we have to show

$$h_{I'} - h_{I' \cup \{l\}} = \sum_{m \in M'} (\hat{u}_m(I'_1) - \hat{u}_m(I'_1 \backslash \{l\}))\hat{p}_m \geq 0. \tag{5.15}$$

It holds that

$$\hat{u}_m(I') - \hat{u}_m(I' \cup \{l\}) = \begin{cases} 1 & \text{if SKU } l \text{ in } s_m \text{ and } I' \text{ contains all other SKUs from } \mathbf{s}_m, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, (5.15) holds. Next, we need to show that $h_{I'_1} - h_{I'_1 \cup \{l\}} \leq h_{I'_2} - h_{I'_2 \cup \{l\}}$ where

$I'_1 \subset I'_2 \subseteq I \backslash \{l\}$. This is equivalent to showing that

$$\sum_{m \in M'} (\hat{u}_m(I'_1) - \hat{u}_m(I'_1 \cup \{l\}))\hat{p}_m \leq \sum_{m \in M'} (\hat{u}_m(I'_2) - \hat{u}_m(I'_2 \cup \{l\}))\hat{p}_m.$$

It holds that

$$\hat{u}_m(I'_1) - \hat{u}_m(I'_1 \cup \{l\}) = \begin{cases} 1 & \text{if SKU } l \text{ in } s_m \text{ and } I'_1 \text{ contains all other SKUs from } \mathbf{s}_m, \\ 0 & \text{otherwise.} \end{cases}$$

$$\hat{u}_m(I'_2) - \hat{u}_m(I'_2 \cup \{l\}) = \begin{cases} 1 & \text{if SKU } l \text{ in } s_m \text{ and } I'_2 \text{ contains all other SKUs from } \mathbf{s}_m, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\hat{u}_m(I'_1) - \hat{u}_m(I'_1 \cup \{l\}) = 1$ implies that $\hat{u}_m(I'_2) - \hat{u}_m(I'_2 \cup \{l\}) = 1$ because $I'_1 \subset I'_2$. Hence, (5.16) will hold. □

Proof of Lemma 5.3

(i) Cost of bringing all parts is $C(\mathbf{x}_I) = F + \sum_{i \in I} c_i(1 - p_i)$. The cost of bringing no parts is $C(\mathbf{x}_\emptyset) = (D + F)h_\emptyset$. Their difference is $C(\mathbf{x}_I) - C(\mathbf{x}_\emptyset) = F + \sum_{i \in I} c_i(1 - p_i) - (D + F)h_\emptyset$. If this difference is greater than zero, (i.e. $F + \sum_{i \in I} c_i(1 - p_i) - (D + F)h_\emptyset > 0$), then the solution $\mathbf{x}_\emptyset$ dominates the solution $\mathbf{x}_I$.

(ii) Now, we write the condition for removing a part i from any set I', where the new set is $I'' = I' \backslash \{i\}$. The cost of solution, $\mathbf{x}_{I'}$ is $C(\mathbf{x}_{I'}) = F + \sum_{j \in I'} c_j(1 - p_j) + (D + F)h_{I'}$ and the cost of taking solution, $\mathbf{x}_{I''}$ is $C(\mathbf{x}_{I''}) = F + \sum_{j \in I''} c_j(1 - p_j) + (D + F)h_{I''}$. The solution $\mathbf{x}_{I''}$ dominates the solution $\mathbf{x}_{I'}$ if the following holds.

$$\begin{aligned} C(\mathbf{x}_{I'}) - C(\mathbf{x}_{I''}) &= F + \sum_{j \in I'} c_j(1 - p_j) + (D + F)h_{I'} \\ &- F - \sum_{j \in I''} c_j(1 - p_j) - (D + F)h_{I''} \\ &= c_i(1 - p_i) - (D + F)(h_{I''} - h_{I'}) > 0. \end{aligned}$$

As a special case, the condition for removing part i from the solution I is

$$C(\mathbf{x}_I) - C(\mathbf{x}_{I\setminus\{i\}}) = c_i(1-p_i) - (D+F)p_i > 0,$$

which can be rewritten to condition (5.12).

(iii) Let assume $I' = \{i\}$ and $I'' = \emptyset$. Solution $\mathbf{x}_\emptyset$ dominates the solution $\mathbf{x}'_I$ if the following holds.

$$C(x_{I'}) - C(x_\emptyset) \quad = \quad F + c_i(1-p_i) - (D+F)(h_\emptyset - h_{I'}) \geq 0,$$

where $h_\emptyset - h_{I'} > 0$.

□

Proof of Theorem 5.2

Let assume $I'\setminus\{i\}$ is a better solution than I'. $I'\setminus\{i\}$ is a better solution than I', if and only if $F\mathbb{1}_{\{I'\setminus\{i\}\neq\emptyset\}} + c_i(1-p_i) - (D+F)(h_{I'\setminus\{i\}} - h_{I'}) > 0$ holds. We need to show that $\mathbf{x}_{I''\setminus\{i\}}$ is a better solution than $\mathbf{x}_{I''}$, therefore, $C(\mathbf{x}_{I''}) - C(\mathbf{x}_{I''\setminus\{i\}}) > 0$, to complete the proof. Please note that $(h_{I'\setminus\{i\}} - h_{I'}) \geq (h_{I''\setminus\{i\}} - h_{I''})$ holds by Lemma 5.2.

$$\begin{aligned} C(\mathbf{x}_{I''}) - C(\mathbf{x}_{I''\setminus\{i\}}) &= F\mathbb{1}_{\{I''\setminus\{i\}\neq\emptyset\}} + c_i(1-p_i) - (D+F)(h_{I''\setminus\{i\}} - h_{I''}) \\ &\geq F\mathbb{1}_{\{I'\setminus\{i\}\neq\emptyset\}} + c_i(1-p_i) - (D+F)(h_{I'\setminus\{i\}} - h_{I'}) > 0. \end{aligned}$$

This shows that $\mathbf{x}_{I''\setminus\{i\}}$ is a better solution than $\mathbf{x}_{I''}$. □

Proof of Corollary 5.1

Let $\mathbf{x}^*$ be an optimal solution and suppose that (5.13) holds. Then, we need to show that $x_i^* = 0$. First consider a set I', where $i \in I'$. Then the cost of the solution $\mathbf{x}_{I'}$ is $C(\mathbf{x}_{I'}) = F + c_i(1-p_i) + \sum_{j\in I'\setminus\{i\}} c_j(1-p_j) + (D+F)h_{I'}$. In order to show that $x_i = 0$ is a better solution than $x_i = 1$, we need to show that $C(\mathbf{x}_{I'}) - C(\mathbf{x}_{I'\setminus\{i\}}) > 0$.

$$\begin{aligned} C(\mathbf{x}_{I'}) - C(\mathbf{x}_{I'\setminus\{i\}}) &= F\mathbb{1}_{\{I'\setminus\{i\}\neq\emptyset\}} + c_i(1-p_i) - (D+F)(h_{I'\setminus\{i\}} - h_{I'}) \\ &\geq F\mathbb{1}_{\{I'\setminus\{i\}\neq\emptyset\}} + c_i(1-p_i) - (D+F)p_i \\ &> F\mathbb{1}_{\{I'\setminus\{i\}\neq\emptyset\}} + c_i(1 - \frac{c_i}{c_i+D+F}) - (D+F)\frac{c_i}{c_i+D+F} \end{aligned}$$

$$= \quad F\mathbb{1}_{\{I'\setminus\{i\}\neq\emptyset\}} + \frac{c_i(D+F)}{c_i+D+F} - \frac{c_i(D+F)}{c_i+D+F} = F\mathbb{1}_{\{I'\setminus\{i\}\neq\emptyset\}} > 0.$$

Please note that from Remark 5.1 and Lemma 5.2 it holds that $(h_{I'\setminus\{i\}} - h_{I'}) \leq p_i$.
□

6 Conclusion

Capital goods require maintenance and part replacements in order to continue functioning. This dissertation focuses on problems in the domain of maintenance optimization and spare parts management for capital goods in data-integrated environments.

In Chapter 2, we considered a problem on 'exploration and exploitation in age-based maintenance'. In Chapter 3, we have studied an extension of the problem considered in Chapter 2, which is 'data pooling and joint optimization for multiple systems'. We studied a problem on 'spare parts replenishment at local warehouses with advance demand information(ADI)' in Chapter 4. We investigated the topic of 'selecting the set of spare parts for corrective maintenance' in Chapter 5. We addressed these problems in order to meet four research objectives. In this chapter, we revisit these research objectives and provide future research directions. Then, we elaborate on the potential implications of the main findings on businesses, society, and the environment.

6.1. Main findings and future research

Research objective 1: Balance exploration and exploitation optimally for age-based maintenance with Bayesian learning.

We address RO 1 in Chapter 2, where we focus on a newly designed system with a finite lifespan. There is uncertainty in the lifetime distribution of the critical component in the system. We consider an age-based replacement policy for this component and the population heterogeneity as the source of model uncertainty. We formulate a partially observable Markov decision process (POMDP) model with Bayesian learning to minimize the total cost of maintenance throughout the lifespan of the system. This model finds the optimal balance between the cost of learning and the cost of maintenance activities (i.e., the so-called exploration-exploitation trade-off).

The main findings of Chapter 2 are as follows. First, we characterize the behavior of the optimal cost as a function of state variables. Then, we compare the optimal policy against two benchmark heuristics, namely a myopic policy and a threshold policy. A myopic policy does not consider exploration in learning the true population type. A threshold policy considers the exploration in learning, however, it does not optimally balance the exploration and exploitation. Under the optimal policy, we observe the strongest exploration when the variance of the lifetime distribution is low and the belief that the component is coming from the strong population is high. In the instances where the exploration is high, the true population type is learned much faster than using the myopic policy. The myopic policy and the threshold policy are up to 23.6% and 5.8% costlier than the optimal policy, respectively.

Finally, we formulate a lower bound function that represents the optimal cost if the model uncertainty would immediately be resolved at a given state. The difference between the optimal cost under population heterogeneity and the lower bound function provides insights on the value of learning and the cost of resolving the uncertainty immediately (e.g., by inspection) for the service provider.

In Chapter 2, we consider an age-based replacement setting. A future research direction is to investigate a similar research objective for a condition-based maintenance setting. The problem can be formulated in a way that the degradation levels of two populations with different degradation processes can be monitored. The objective can be reformulated as learning the true population type while determining the optimal degradation threshold for the condition-based replacement.

Research objective 2: Investigate the effect of data pooling and joint optimization for multiple single-component systems on maintenance costs.

We address RO 2 in Chapter 3. We consider an extended version of the problem of Chapter 2 for multiple systems. We define three policies: the optimal policy for multiple systems with data pooling, the single-system optimal policy with data pooling, and the single-system optimal policy without data pooling. We build a POMDP model with Bayesian learning to find the optimal policy. This policy optimally balances the exploration and exploitation for the multiple-system setting.

The main findings of this research are as follows. We show that the cost reduction due to joint optimization can be up to 0.8% and due to data pooling up to 5.6% for two systems. We also investigate the effect of the number of systems on the costs. We show that as the number of systems increases, the cost of exploration (learning) per system decreases. This is due to the fact that learned information regarding the true population is exploited for a high number of systems. The maximum cost reduction due to the data pooling can be up to 14% for the problem instances with 20 systems, where the maximum potential reduction is 15.8% for this specific instance.

Investigating the effect of the number of systems on joint optimization is limited due to the increase in the number of states and actions as a function of the number of systems. A more detailed analysis is provided for joint optimization under a special case with deterministic lifetimes. Under this setting, the true population type can be learned perfectly by risking only one failure. Risking a failure (exploration) is not desirable for more than one system and exploration under the optimal policy is only beneficial for a large enough number of systems.

There might be cases where maintenance costs differ for different systems for various reasons (e.g., different numbers of users for each system). This results in heterogeneous systems. In future research, we can relax the assumption of identical systems by considering heterogeneous systems and we can investigate how much heterogeneous systems can benefit from data pooling and joint optimization.

Furthermore, it is possible to investigate the effect of the number of populations on the obtained insights.

Research objective 3: Determine how the quality of failure predictions and the duration of demand lead time affects the spare part stock and replenishment costs.

We address RO 3 in Chapter 4. We consider a spare parts inventory replenishment problem for a local warehouse, where a predictive model provides imperfect advance demand information (ADI) for the demand for spare parts. We build a Markov decision process model with precision and sensitivity of the predictive model and the demand lead time as input parameters.

We show that the optimal cost depends on the sensitivity and the demand lead time only through their product. From a practitioner's point of view, this means that a low value in one of them can be compensated by increasing the other (if possible). Additionally, the Pareto principle holds for precision under a fixed value of the product of sensitivity and demand lead time (e.g. 30% perfectness in precision provides a 70% reduction in optimal costs compared to the worst-case optimal costs). The opposite of the Pareto principle holds for the product of sensitivity and demand lead time under a fixed value of precision (e.g. 70% perfectness in the product of sensitivity and demand lead time provides a 30% reduction in optimal costs compared to the worst-case optimal costs). Furthermore, we show that in order to obtain a significant cost reduction, a moderate value of precision is sufficient, however high values of sensitivity and demand lead time are needed.

Finally, we show that in most cases (when input parameters are not perfect), we have average on-hand inventory levels greater than zero. Therefore, this shows that on-hand inventories will continue to exist even for having moderate or high-quality predictive models and high values of the demand lead time.

As future work, we can consider the following extensions for the problem studied in Chapter 4. Currently, we assume that systems are operating continuously. Therefore, the cost of preventive maintenance is equally high as the cost of corrective maintenance. In future research, we can consider other settings where this assumption does not hold (e.g., where systems are not operating during some time slots in the period). In such a setting, the costs of preventive replacements will be lower than the costs of corrective replacements because preventive replacements do not cause extra downtime. This also brings an extra challenge when the precision is less than one. In this case, having a preventive replacement cost that is less than the corrective replacement cost may result in unnecessary replacements. This leads to a trade-off between the cost of failure and the cost of unnecessary replacements. Another possible extension is to investigate the effect of stochastic lead times on the costs. Additionally, we can allow returns of spare parts from the local warehouse to the central warehouse, either for free or against a return cost.

Research objective 4: Develop a spare parts recommendation model based on historical data and operational costs.

We address RO 4 in Chapter 5. We consider a setting where the service provider is responsible for maintenance visits and spare part shipments, if needed, when a failure is reported for a system. A failure code provided by a customer is used to generate the sets of spare parts that may be required and a probability estimate for each possible set of parts. The output of this predictive model constitutes a form of imperfect ADI on the demand distribution of the sets of spare parts. We build an ILP model that finds the optimal set of spare parts that will be sent during the diagnostics visit. We provide the structure of the optimal policy for problems with one and two stock-keeping units (SKUs). We provide examples of how the cost parameters and the demand dependency between two SKUs affect the structure of the optimal policy.

We generate results on the structure of the optimal policy for problem instances with a general number of SKUs. We formulate conditions for removing an SKU from or adding an SKU to the selected set of spare parts. Furthermore, we compare the optimal policy against two benchmark heuristics used in practice. The first heuristic policy considers having a diagnostic visit only, then shipping the required parts to the customer with a second visit if necessary. The second heuristic policy considers shipping a fixed number of top-recommended parts during the diagnostic visit. This results in having a different policy for a different selection of this fixed number. We observe that one specific heuristic policy cannot find the optimal policy for all problem instances. By considering the average of all problem instances in a particular test bed, the best heuristic policy is 12.2% costlier than the optimal policy. In some settings (e.g., a high shipment cost for one particular SKU with a demand dependency between this SKU and the other SKUs), the average cost of the best heuristic among all problem instances can be 17.8% costlier than the optimal policy.

Finally, we extend the problem with a service level constraint. This constraint is useful if the service provider promises a minimum service level (i.e., the probability of resolving a failure during a diagnostic visit should be greater than a predetermined value) to its customers. We show that a high service level (i.e., 90%) may increase the costs up to 66.2%.

In Chapter 5, we assume the only source of uncertainty is due to data but we do not consider uncertainty in the predictive model itself. In reality, the predictive models are not perfect. It is also interesting to consider uncertainty in the predictive model on the demand probability distribution of the set of spare parts. As a future research direction, the effect of model uncertainty on the optimal costs and actions can be investigated.

6.2. Effects on businesses, society, and environment

Problems in this dissertation are motivated by practical problems encountered in industry. Improvements in IoT, sensor, and data storage technologies enable the collection of data easier than in the past. There is an abundance of data in terms of log data, and maintenance service data. The industry is in the phase where they preprocess the data and interpret the data. The second phase of the application is to develop decision-support tools based on this data (Olsen and Tomlin, 2020). Integrating new technologies into existing business practices requires an investment of new knowledge and economic investment. This dissertation is motivated by addressing these two barriers in front of the adoption. This dissertation aims to provide examples of decision-making models that will be used in decision-support tools and provide insight to the practitioners on the cost benefits of adopting the new technologies.

In order to provide real-life connections to the solutions provided in this dissertation, we adopt the service control tower (SCT) concept. Service control towers are a concept for managing service operations that involve many parties from a central place (Song et al., 2020). A service provider with multiple customers must coordinate different data flows and plan the resource allocation (i.e., service engineers and spare parts) for maintenance activities mostly in real-time. The models developed in this dissertation aim to provide a basis for the decision-support tools that are utilized in these conceptual SCTs. These models can be used by maintenance service providers that have customers in geographically dispersed locations. With efficient maintenance operations, businesses prevent unnecessary costs of downtime, maintenance costs, shipment cost of spare parts, and inventory holding costs.

Maintenance activities have also societal effects. Each capital good is used to provide either service or products. There is an end-party that uses these services or goods, e.g., patients going for an MRI scan, passengers using a train, or governments buying computer chips to produce IDs and passports. It is not always easy to quantify the cost of delay in accessing a service or a product for these parties. However, decreasing the downtime of capital goods improves the quality and timeliness of the services or the products that are received by these end parties.

Maintenance activities also affect spare parts usage and the useful lifespan of capital goods. Environmental resources (e.g., electricity, gas, metal, water) are used in the production and disposal of capital goods. Increasing the time that the capital good is exploited (the capital good that is useful) increases the productivity per unit of the used resources. Additionally, each spare parts shipment costs energy and emits CO_2. Efficient shipment of spare parts reduces the energy and carbon footprint of the process. Finally, if excess inventory is stocked or spare parts are shipped unnecessarily due to poor planning, there is a risk of these parts being scrapped or obsolete. This causes inefficient use of natural resources for both production and disposal of these parts. For the current models, it is possible to implicitly add the societal and environmental costs into the existing cost parameters. For example, we can add a cost for CO_2 emission into the fixed shipment cost in Chapter 5. It is also a possible future research direction to study the societal and environmental externalities of the maintenance activities explicitly.

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Summary

Maintenance optimization and spare parts management in data-integrated environments

Capital goods are crucial for the continuation of production and services. Many business operations are dependent on a functioning capital good. The efficiency of maintenance operations is important to keep these systems functioning. Maintenance service providers (SPs) are responsible to provide maintenance services and spare parts to their customers according to service level agreements. The first part of this dissertation (Chapters 2 and 3) is on maintenance optimization policies under the so-called time-to-failure model uncertainty and the second part of this dissertation (Chapters 4 and 5) is on spare parts management by using advance demand information (ADI) from the point of view of an SP of capital goods.

In Chapters 2 and 3, we consider newly designed systems with a fixed lifespan. In all systems, the same critical component occurs that is subject to random failures. The component can be replaced preventively to avoid a costly failure. This component always comes from either a weak population or a strong population. This is known as *population heterogeneity*. The true population type is unknown to the decision-maker, but there is a belief with respect to the probability of having

a weak population. This belief is updated with a Bayesian approach by using the data collected over the lifespan of the system.

In Chapter 2, we focus on a single system. We build a partially observable Markov decision process (POMDP) model to find the optimal replacement policy that minimizes the total cost over the lifespan of the system. It optimally balances the trade-off between *exploration*, i.e., the cost of learning the true population type (via deliberately delaying the preventive replacement time), and *exploitation*, i.e., the cost of maintenance activities. We generate insights on the optimal policy and its structure and we compare its performance with existing heuristic approaches from the literature, namely a *myopic policy* (i.e., a heuristic that does not consider exploration) and a *threshold heuristic* (i.e., a heuristic that considers exploration). In the numerical analysis, we observe that the true population type is learnt much faster under high exploration. Additionally, the myopic policy and the threshold policy are up to 23.6% and 5.8% costlier than the optimal policy, respectively.

In Chapter 3, we consider multiple identical systems. We investigate the effect of so-called *data pooling* by updating the belief with the data collected from all systems. We build a discrete-time POMDP model to find the optimal replacement policy which minimizes the expected total cost throughout the lifespan. We compare the cost per system under the optimal policy with the cost per system under two benchmark heuristics that follow the single-system optimal policy with and without data pooling, respectively. We investigate the effect of joint optimization and data pooling, and the number of systems on the cost per system. In our numerical experiments, we show that the cost reduction relative to the worst benchmark heuristic (i.e., the benchmark heuristic without data pooling) can be up to 5.6% for two systems, and this increases up to 14% for 20 systems.

In Chapter 4, we study multiple technical systems that are supported by a local stock point. We consider a single critical component that occurs in each system and is subject to random failure. After a failure, a replacement takes place. Signals for possible failures are generated by predictive models, which constitute ADI for the spare parts inventory. However, signals might be imperfect. We assume a periodic-review replenishment policy. We formulate a Markov decision process model to find the optimal inventory replenishment policy that minimizes the long-run average cost per period. We investigate the effect of *precision* (i.e., the fraction of signals that are true positive), *sensitivity* (i.e., the fraction of failures for which a signal is generated), and the demand lead time (i.e., the time between the signals and failures) on the optimal costs. We show that a significant cost reduction can be obtained for a moderate value of precision. For the sensitivity and demand lead time, you always need high values in order to get a significant cost reduction.

In Chapter 5, we consider an SP that organizes maintenance visits and spare part shipments to a customer when a failure code is reported. This code constitutes a form of ADI. We formulate a mixed integer linear programming model to find the optimal set of spare parts that will be sent to a customer site to resolve the failure. The optimal set minimizes the total costs consisting of shipment costs, costs for the return of parts that are not needed, and costs for a required second visit if the set does not contain all parts to solve the failure. We derive analytical results for the structure of the optimal policy. We compare the policy generated by our model to existing benchmark policies. In an extensive numerical study, we observe that one specific benchmark policy cannot find the optimal policy for all problem instances. The best benchmark policy is on average 12.2% costlier than the optimal policy.

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About the author

İpek Dursun was born in April 1994, Şişli (Turkey). She received her B.Sc. and M.Sc. degrees in Industrial Engineering at Boğaziçi University in İstanbul, Turkey in 2016 and 2018, respectively. She also received a minor degree in Economics at the same university in 2016. She worked as a research assistant between January 2017-September 2018 at the Department of Industrial Engineering at Boğaziçi University. Her master's thesis with the title *The impacts of manufacturers' capacity and pricing decisions on dynamic random-access memory market* was completed under the supervision of dr. Gönenç Yücel.

In November 2018, she started a Ph.D. project at the Department of Industrial Engineering and Innovation Sciences at Eindhoven University of Technology under the supervision of dr. Alp Akçay and prof.dr.ir. Geert-Jan van Houtum. She worked on *maintenance optimization and spare parts management for capital goods in data-integrated environments*. During her Ph.D., she participated in the EU project DayTiMe between November 2018-March 2022 and collaborated with partners from the industry. She collaborated with Anastasiia Grishina from Simula Research Laboratory and the University of Oslo, for the work of Chapter 5.

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İpek defends her Ph.D. thesis on 13 April 2023, at 16.00.