EINDHOVEN UNIVERSITY OF TECHNOLOGY

Optimal morphs of planar orthogonal drawings

Citation for published version (APA):
Goethem, A. V., Speckmann, B., & Verbeek, K. (2022). Optimal morphs of planar orthogonal drawings. *Journal of Computational Geometry*, *13*(1), 263-297. https://doi.org/10.20382/jocg.v13i1a11

Document license:
CC BY

DOI:
10.20382/jocg.v13i1a11

Document status and date:
Published: 20/04/2022

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

Download date: 05. Oct. 2023

OPTIMAL MORPHS OF PLANAR ORTHOGONAL DRAWINGS*

Arthur van Goethem,[†] *Bettina Speckmann,*[‡] *and Kevin Verbeek.*[§]

Abstract. We describe an algorithm that morphs between two planar orthogonal drawings Γ_I and Γ_O of a graph G, while preserving planarity and orthogonality. Necessarily drawings Γ_I and Γ_O must be equivalent, that is, there exists a homeomorphism of the plane that transforms Γ_I into Γ_O. Our morph uses a linear number of linear morphs (linear interpolations between two drawings) and preserves linear complexity throughout the process, thereby answering an open question from Biedl et al. (ACM Transactions on Algorithms, 2013).

Our algorithm first *unifies* the two drawings to ensure an equal number of (virtual) *bends* on each edge. We then interpret bends as vertices which form obstacles for so-called *wires*: horizontal and vertical lines separating the vertices of Γ_O. We can find corresponding wires in Γ_I that share topological properties with the wires in Γ_O. The structural difference between the two drawings can be captured by the *spirality* s of the wires in Γ_I, which guides our morph from Γ_I to Γ_O. We prove that $s = O(n)$ and that $s + 1$ linear morphs are always sufficient to morph between two planar orthogonal drawings, even for disconnected graphs.

1 Introduction

Continuous morphs of planar drawings have been studied for many years, starting as early as 1944, when Cairns [5] showed that there exists a planarity-preserving continuous morph between any two (compatible) triangulations that have the same outer triangle. These results were extended by Thomassen [9] in 1983, who gave a constructive proof of the fact that two compatible straight-line drawings of any planar graph can be morphed into each other while maintaining planarity. The resulting algorithm to compute such a morph takes exponential time (just as Cairns' result). Thomassen also considered the orthogonal setting and showed how to morph between two rectilinear polygons with the same turn sequence. For planar straight-line drawings the question was settled by Alamdari et al. [1], following work by Angelini et al. [2]. They showed that $O(n)$ uni-directional linear morphs are sufficient to morph between any compatible pair of planar straight-line drawings of a graph with n vertices while preserving planarity. Kleist et al. [7] have shown that the corresponding morph can be found in $O(n^{1+\frac{\omega}{2}})$ time, where ω is the matrix multiplication exponent.

In this paper we consider the orthogonal setting, that is, we study planarity-preserving morphs between two planar orthogonal drawings Γ_I and Γ_O with maximum complexity n, of

*Preliminary results have been presented at SoCG 2018 [11] and GD 2019 [10]. Bettina Speckmann and Kevin Verbeek are partially supported by the Dutch Research Council (NWO) under project no. 639.023.208 (B.S.) and no. 639.021.541 (K.V.).

[†] *ASML*, avgoethem@hotmail.com

[‡] *TU Eindhoven*, b.speckmann@tue.nl

[§] *TU Eindhoven*, k.a.b.verbeek@tue.nl

a graph G. Here the complexity of an orthogonal drawing is defined as the number of vertices and bends. All intermediate drawings must remain orthogonal. This immediately implies that the results of Alamdari et al. [1] do not apply, since they do not preserve orthogonality. Biedl et al. [4] described the first results in this setting, for so-called *parallel* drawings, where every edge has the same slope in both drawings. They showed how to morph between two parallel drawings using $O(n)$ linear morphs while maintaining parallelity and planarity. More recently, Biedl et al. [3] showed how to morph between two planar orthogonal drawings using $O(n^2)$ linear morphs, while preserving planarity, orthogonality, and linear complexity. In this paper we improve this bound further to $O(n)$ linear morphs. This bound is tight, based on the lower bound for straight-line graphs proven by Alamdari et al. [1].

Paper Outline. In Section 2 we give preliminary definitions and explain how to create a *unified* graph G: we add additional vertices to ensure Γ_I and Γ_O are orthogonal straight-line drawings of the unified graph G. The complexity of Γ_I and Γ_O is still bounded by $O(n)$ after the unification process.

Our main tool are so-called *wires* which are introduced in Section 3 (see also Figure 2). Wires capture the horizontal and vertical order of the vertices. Specifically, we consider a set of horizontal and vertical lines that separate the vertices of Γ_O. If we consider the vertices of Γ_O as obstacles, then these wires define homotopy classes with respect to the vertices of G (for the embedding of G shared by Γ_I and Γ_O). These homotopy classes can be represented by orthogonal polylines (called wires) in Γ_I using orthogonal shortest and lowest paths as defined by Speckmann and Verbeek [8].

Intuitively our morph is simply straightening the wires in Γ_I using the *spirality* (the difference between the number of left and right turns) of the wires as a guiding principle. In Section 4 we show how this approach leads more or less directly to a linear number of linear morphs. However, the complexity of the intermediate drawings created by this algorithm might increase to $\Theta(n^3)$.

In the remainder of the paper we show how to "batch" intermediate morphs. We argue solely based on sets of wires, hence the results apply to both connected and disconnected graphs. In particular, in Section 5 we show how to combine all intermediate morphs that act on segments of spirality s into one single linear morph. Hence we need only s linear morphs to morph from Γ_I to Γ_O. In Section 5 we also show that each linear morph can be performed between two straight-line drawings, thereby reducing intermediate complexity and bounding the overall complexity of all intermediate drawings by $\Theta(n^2)$. Finally, in Section 6 we further refine the approach to also maintain linear complexity, but it comes at the cost of a single additional linear morph. The final morph preserves planarity, orthogonality, and linear complexity while using only $s + 1$ linear morphs.

We implemented our algorithm and believe that the resulting morphs are natural and visually pleasing. A short movie is available online[1]. We suggest that the reader first considers the video to form an intuition of the constructed morphs.

[1] See `https://youtu.be/JhrgFGTiB5c`.

2 Preliminaries

Orthogonal drawings. A *drawing* Γ of a graph $G = (V, E)$ is a mapping from every vertex $v \in V$ to a unique point $\Gamma(v)$ in the Euclidean plane and from each edge (u, v) to a simple curve in the plane starting at $\Gamma(u)$ and ending at $\Gamma(v)$. A drawing is *planar* if no two curves intersect in an internal point, and no vertices intersect a curve in an internal point. In a *straight-line drawing* every edge is represented by a single line-segment. A drawing is *orthogonal* if each edge is mapped to an orthogonal polyline consisting of horizontal and vertical segments meeting at *bends*. The *complexity* of an orthogonal drawing is the number of vertices and bends. Two planar drawings Γ and Γ' are *equivalent* if there exists a homeomorphism of the plane that transforms Γ into Γ'.

A *linear morph* of two straight-line drawings Γ and Γ' can be described by a continuous linear interpolation of all vertices and bends, which are connected by straight segments. For each $0 \leq t \leq 1$ there exists an intermediate drawing Γ_t where each vertex v is drawn at $\Gamma_t(v) = (1-t)\Gamma_v + t\Gamma'_v$ ($\Gamma_0 = \Gamma$ and $\Gamma_1 = \Gamma'$). A linear morph *maintains planarity* (orthogonality, linear complexity), if every intermediate drawing Γ_t is planar (orthogonal, of linear complexity).

Slides. Biedl et al. [3] introduced *slides* as a particular type of linear morph that operates on the segments of the drawing. A *zigzag* consists of three consecutive segments with endpoints α, β, γ and δ, where β forms a left turn and γ a right turn or vice versa (see Figure 1(a)). Assume for now that β is a left turn and γ a right turn. Let $\mathcal{V}$ be the set of all vertices and bends of the drawing that are right of or on the ray originating at β and passing through α, or strictly left of the ray originating at γ and passing through δ. We exclude γ from $\mathcal{V}$. The corresponding region is shaded in Figure 1. Note that there are no bends and vertices on the vertical segment itself as the drawing is planar. A *zigzag-eliminating slide* is a linear morph that straightens a zigzag by shifting all vertices and bends in $\mathcal{V}$ by the vector $\gamma - \beta$ (see Figure 1(b)). A zigzag-eliminating slide is a particular linear morph that by construction maintains planarity during the morph.

In the case where β is a right turn and γ a left turn let $\mathcal{V}$ be the set of all vertices and bends that are *left* of or on the ray originating at β and passing through α, or strictly *right* of the ray originating at γ and passing through δ. Once again we exclude γ from $\mathcal{V}$.

A *bend-creating slide* is a linear morph that introduces a zigzag in a horizontal or vertical line (see Figure 1(c)). For similar reasoning it also maintains planarity.

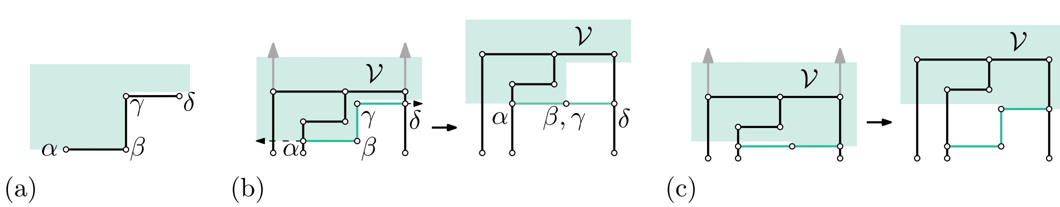


Figure 1: (a) A horizontal *zigzag*. (b) A *zigzag-eliminating* slide is a linear morph that straightens a zigzag. (c) A *bend-creating* slide is a linear morph that introduces a zigzag.

Homotopic paths. Our algorithm relies on the concept of *wires* separating the vertices of the drawings. Wires are linked up between different drawings via their homotopy classes. We consider the vertices of a drawing as the set of obstacles B. Let $\pi_1, \pi_2 \colon [0,1] \to \mathbb{R}^2 \setminus B$ be two paths in the plane avoiding the vertices. We say that π_1 and π_2 are *homotopic* ($\pi_1 \sim_h \pi_2$) if they have the same endpoints and there exists a continuous function avoiding B that deforms π_1 into π_2. That is, there exists a function $H : [0,1] \times [0,1] \to \mathbb{R}^2$ such that

- $H(0,t) = \pi_1(t)$ and $H(1,t) = \pi_2(t)$ for all $0 \leq t \leq 1$.
- $H(s,0) = \pi_1(0) = \pi_2(0)$ and $H(s,1) = \pi_1(1) = \pi_2(1)$ for all $0 \leq s \leq 1$.
- $H(\lambda, t) \notin B$ for all $0 \leq \lambda \leq 1$, $0 \leq t \leq 1$.

Since the homotopic relation is an equivalence relation, every path belongs to a *homotopy class*. The *geometric intersection number* of a pair of paths π_1, π_2 is the minimum number of intersections between any pair of paths homotopic to π_1, respectively π_2. Freedman, Hass, and Scott proved the following theorem[2].

Theorem 1 (from [6] Theorem 3.3)**.** *Let M^2 be a closed, Riemannian 2-manifold, and let $\sigma_1 \subset M^2$ and $\sigma_2 \subset M^2$ be two shortest loops of their respective homotopy classes. If $\pi_1 \sim_h \sigma_1$ and $\pi_2 \sim_h \sigma_2$, then the number of crossings between σ_1 and σ_2 is at most the number of crossings between π_1 and π_2.*

In other words, the number of crossings between two loops of fixed homotopy classes are minimized by the shortest respective loops. This theorem can easily be extended to paths instead of loops, if we can consider the endpoints of the paths as obstacles. For orthogonal paths the shortest path is not uniquely defined, however using *lowest paths* the theorem still holds. An orthogonal path is *lowest* if it is shortest with respect to its homotopy class and each staircase subpath is as low as possible (the staircase subpath follows the lower envelope of all homotopic staircase paths between the endpoints). Refer to [8, Lemma 6] for details.

Conventions. Two equivalent drawings Γ and Γ' may not have the same complexity (number of vertices and bends) as the orthogonal polylines in Γ and Γ' representing the same edge may have a different number of segments. To simplify the discussion of our algorithm, we first ensure that each edge has the same number of segments in Γ and Γ'. This can be achieved by subdividing segments, creating additional virtual bends. Next, we replace all bends by vertices. As a result, all edges of the graph are represented by straight segments in both Γ and Γ'. We call the resulting graph the *unification* of Γ and Γ'. If the maximum complexity of Γ and Γ' is $O(n)$ then clearly the complexity of the unification of Γ and Γ' is $O(n)$.

We say that two planar drawings Γ and Γ' of a unified graph are *similar* if the horizontal and vertical order of the vertices is the same in both drawings.

Observation 1. *A planar orthogonal drawing can be morphed to a similar planar orthogonal drawing using a single linear morph while maintaining planarity.*

Proof. We can introduce a planarity violation only if two vertices swap in the horizontal or vertical order, which cannot happen during a linear morph between two similar drawings. □

[2] Reformulated (and simplified) to suit our notation rather than the more involved notation in [6].

3 Wires

In the following we show how to morph an orthogonal planar drawing Γ_I of $G = (V, E)$ to another orthogonal planar drawing Γ_O of G while maintaining planarity and orthogonality. We assume that Γ_I and Γ_O are *equivalent*, that G is the unification of Γ_I and Γ_O, and that G contains n vertices. To morph Γ_I to Γ_O, our main strategy is to first make Γ_I similar to Γ_O, after which we can morph Γ_I to Γ_O using a single linear morph (by Observation 1). To capture the difference between Γ_I and Γ_O we use two sets of *wires*. The *lr-wires* $W_\rightarrow$, going from left to right through the drawings, capture the vertical order of the vertices in comparison to the vertical order in Γ_O. The *tb-wires* $W_\downarrow$, going from top to bottom through the drawings, capture the horizontal order of the vertices with respect to Γ_O.

3.1 Basic properties

Since we want to match the horizontal and vertical order of vertices in Γ_O, the wires $W_\rightarrow$ and $W_\downarrow$ are simply horizontal and vertical lines in Γ_O, respectively, separating any two consecutive coordinates used by vertices (see Figure 2(a)). Now assume that we have a planar orthogonal morph from Γ_I to Γ_O (the existence of such a morph follows from [3]). If we were to apply this morph in the reverse direction on the wires in Γ_O, we end up with another set of wires in Γ_I with the following properties (see Figure 2(b)):

P1 Two wires are non-crossing if they both belong to $W_\rightarrow$ or $W_\downarrow$ and cross exactly once otherwise.

P2 The order of the wires in $W_\rightarrow$ ($W_\downarrow$) is the same as in Γ_O and the same vertices are between consecutive wires.

P3 The wires cross exactly the same sequence of edges and links as in Γ_O.

These properties follow directly from the fact that a planar morph cannot introduce or remove any crossings, and thus these properties are invariant under planar morphs. We refer to a set of wires in Γ_I that has the above properties as an *equivalent* set of wires (to

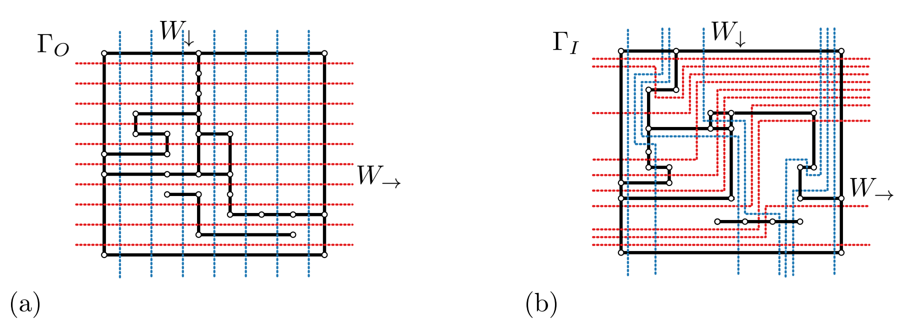


Figure 2: (a) The set $W_\rightarrow$ of lr-wires (red) and the set $W_\downarrow$ of tb-wires (blue) in the output drawing Γ_O. (b) Equivalent wires in the input drawing Γ_I.

the wires in Γ_O). Interestingly, any equivalent set of wires can be used to construct a planar morph from Γ_I to Γ_O. We first use a planar morph to straighten the wires. Then, by **P1** and **P2**, for the resulting drawing Γ the wires for a grid where each cell contains at most one vertex. By **P3** the wires also intersect the same sequence of edges and links as in Γ_O. As all wires are straight-line we conclude that Γ is similar to Γ_O. Hence, we can eliminate all bends in a single morph by combining individual morphs per cell. For each cell we morph all bends (and the vertex) to the center of the cell. The resulting drawing is similar to Γ_O and has no bends, and thus we can finish the planar morph with a single linear morph.

In the following we assume that we are given an equivalent set of wires in Γ_I to the set of straight-line wires in Γ_O. To keep the distinction between wires and edges clear, we refer to the horizontal and vertical segments of wires as *links*. It directly follows that every set of wires in Γ_I equivalent to straight-line wires in Γ_O fulfills the following properties:

Observation 2.

- *Every wire crosses an edge at most once.*
- *Every edge is crossed by wires from either $W_\downarrow$ or $W_\rightarrow$.*
- *All wires crossing an edge cross it in the same direction and the order of intersections matches the horizontal (vertical) order of the wires in Γ_O.*

Proof. As edges are straight lines in Γ_O, each edge can only be crossed once by a wire. As edges are straight-line segments and wires straight lines in Γ_O, each edge is either crossed by wires from $\Gamma_\downarrow$ or wires from $\Gamma_\rightarrow$ and every wire crosses the edge in the same direction. As equivalent wires in Γ_I cross the same sequence of edges and separate the vertices identically (and wires from the same set are non-crossing) the intersection of the edges in Γ_I and Γ_O must be identical. □

Without loss of generality we also make two assumptions to simplify the exposition.

Assumption 1. *No two links overlap in more than a single point in Γ_I.*

If two links overlap in a segment along both links then by carefully ε-perturbing the links we may prevent the overlap.

Assumption 2. *A bounding box surrounds the drawing in both Γ_I and Γ_O and each wire starts (ends) on this bounding box at the same place in the same orientation in Γ_I and Γ_O.*

Our goal is to straighten the wires. As even a single wire in Γ_I may have $\Omega(n^2)$ links (see Figure 3), it is inefficient to straighten the wires one link at a time. Instead we consider the spirality of the wires. For a wire $w \in W_\rightarrow$, let $\ell_1 \dots \ell_k$ be the links of w in order from left to right. Let b_i be the orientation of the bend between ℓ_i and ℓ_{i+1}, where $b_i = 1$ for a left turn, $b_i = -1$ for a right turn, and $b_i = 0$ otherwise. The *spirality* of a link ℓ_i is defined as $s(\ell_i) = \sum_{j=1}^{i-1} b_j$. By definition the spirality of ℓ_1 is 0, and by construction the spirality of ℓ_k is also 0. The *spirality* of a wire is defined as the maximum absolute spirality over all its links. The spirality of wires in $W_\downarrow$ is defined analogously, going from top to bottom.

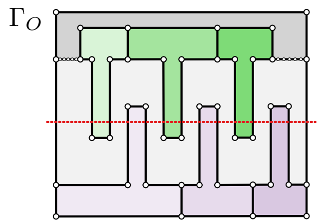


Figure 3: The complexity of a wire can be $\Omega(n^2)$. To satisfy property **P3**, the wire must spiral through the same polygon a linear number of times. The spirality is still $O(n)$.

Spirality is closely related to the number of convex and reflex corners in a simple closed orthogonal curve. Particularly, if we traverse a wire as part of the boundary of a closed orthogonal curve then for each convex corner (left turn) the spirality of the link traversed increases by one and for each reflex corner (right turn) the spirality decreases by one. Thus, when we trace a wire from a link with spirality s to a link with spirality t, then the curve has $t - s$ more convex than reflex corners along the traced segment. Symmetrically, if we traverse a wire backwards starting from a link with spirality s' to a link with spirality t', then the curve has $s' - t'$ more convex than reflex corners. We repeatedly use this fact together with the fact that a simple closed orthogonal curve has four more convex corners than reflex corners, to prove the spirality of links in the wires.

Lemma 1. *If a wire $w \in W_{\rightarrow}$ and a wire $w' \in W_{\downarrow}$ cross in links ℓ_i and ℓ'_j, then $s(\ell_i) = s(\ell'_j)$.*

Proof. By property **P1** w and w' cross exactly once. Consider the bounding box B that contains the complete drawing and intersects w and w' in ℓ_1 respectively ℓ'_1. The wires w and w' subdivide B into four simple faces (see Figure 4). Consider the top-left face. Since the face is simple and orthogonal, it contains four more convex corners than reflex corners. Two convex corners are at the intersection of w and w' with B, and one is a corner of B.

By definition the spirality of ℓ_1 and ℓ'_1 is zero. Thus if the spirality of ℓ_i is x, then w contains x more convex than reflex corners. Symmetrically if the spirality of ℓ_j is y, then w' contains y less convex than reflex corners. As the intersection of ℓ_i and ℓ'_j also forms a convex corner, using a double-counting argument we get that $x - y + 4 = 4$. But then $x = y$. ☐

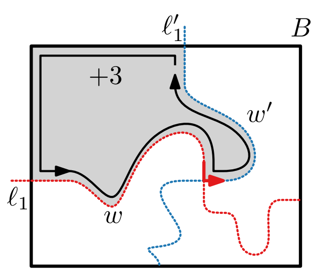


Figure 4: As ℓ_1 and ℓ'_1 have spirality zero and a closed orthogonal curve has four more convex than reflex corners, the crossing links of w and w' have equal spirality.

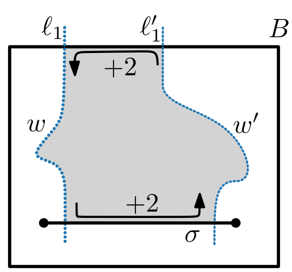


Figure 5: As ℓ_1 and ℓ'_1 have spirality zero and w and w' must have equally many more left turns than right turns, the two links crossing σ have the same spirality.

Lemma 2. *All links intersecting the same segment of an edge have the same spirality.*

Proof. Consider a segment σ of an edge e. Assume without loss of generality that e is horizontal in Γ_O. Only wires from $W_\downarrow$ intersect e (and thus σ) and all wires cross e (σ) in the same direction. Consider two consecutive wires $w, w' \in W_\downarrow$ intersecting σ. Let B be the bounding box of the drawing intersecting w and w' in ℓ_1 respectively ℓ'_1. By definition the spirality of ℓ_1 and ℓ'_1 is zero. The area enclosed by w, w', σ, and B between the intersection with ℓ_1 and ℓ'_1 forms a simple closed orthogonal curve (see Figure 5). A counter-clockwise tour of the curve has two convex turns at σ and two convex turns at B, all remaining turns are caused by w and w'. If x and x' are the spiralities of w and w' when intersecting σ, then using the fact that the polygon has four more convex than reflex corners and a double-counting argument similar to Lemma 1 we get $x + 2 - x' + 2 = 4$, and thus $x = x'$. □

3.2 Spirality bound

In Section 4 we show that we can straighten a set of wires using only $O(k)$ linear morphs, if the spirality of each wire is bounded by k. It is therefore pertinent to bound the spirality of an equivalent set of wires. Let Γ_I and Γ_O be two equivalent planar orthogonal drawings of a graph G. We show that we can find an equivalent set of wires in Γ_I with spirality $O(n)$ with respect to the straight-line wire-grid in Γ_O.

Below we show that we can choose homotopy classes for the wires in Γ_I incrementally, first for the lr-wires and then for the tb-wires, while maintaining the correct intersection pattern and hence equivalence with Γ_O. For each of the resulting equivalence classes we add the shortest wire to the set of wires. It remains to argue that the resulting set of wires has spirality $O(n)$, despite the interdependence of the homotopy classes and despite the fact that the arrangement of drawing and wires can have super-linear complexity. We consider only wires in $W_\rightarrow$ and links with positive spirality. Structurally identical symmetric results for wires from $W_\downarrow$ and for links with negative spirality can be setup. For a wire w let $w[\ell_i]$ be the partial wire consisting of links $\ell_1, \dots, \ell_i$.

Lemma 3. *Let ℓ_i be a horizontal link of a wire $w \in W_\rightarrow$ with even spirality, and let L be a vertical line crossing ℓ_i. The lowest link of $w[\ell_i]$ crossing L has spirality 0 or -2 and the highest link of $w[\ell_i]$ crossing L has spirality 0 or 2.*

Proof. Let ℓ_l be the lowest link from $w[\ell_i]$ crossing L. We can create a simple closed counter-clockwise orthogonal curve (see Figure 6) by (1) first going down from ℓ_1 along the bounding box, (2) going right along the bounding box until reaching L, (3) going up until reaching ℓ_l, and (4) following $w[\ell_i]$ backwards until reaching the starting point of the curve. The curve has four more convex than reflex corners, and it contains 3 convex corners by construction. Let x be the contribution of the turn at ℓ_l (which can be left $(+1)$ or right (-1)). Then we get that $3 + x - s(\ell_l) = 4$, so $s(\ell_l) = x - 1$, directly implying $s(\ell_l) \in \{0, -2\}$.

Let ℓ_h be the highest link from $w[\ell_i]$ crossing L. We create a simple closed counter-clockwise orthogonal curve by (1) following the wire from ℓ_1 to ℓ_h, (2) going up until the bounding box, (3) going left to the corner of the bounding box, (4) going down to ℓ_1. Let x be the contribution of the turn at ℓ_h. We get $s(\ell_h) + x + 3 = 4$, implying $s(\ell_h) \in \{0, 2\}$. □

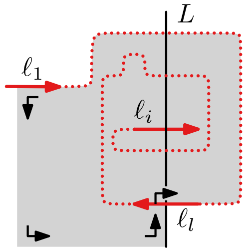


Figure 6: A partial wire $w[\ell_i]$ upto link ℓ_i and a vertical line L crossing ℓ_i. As a counter-clockwise tour of the gray region increases spirality by four, the lowest link of $w[\ell_i]$ crossing L must have spirality at most 0.

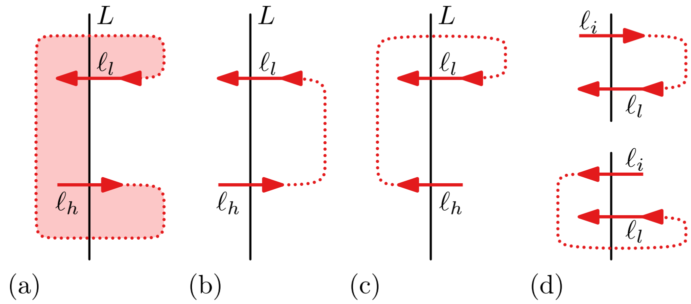


Figure 7: (a) The origin of ℓ_h cannot be enclosed by the wire from ℓ_h to ℓ_l as then the wire needs to cross L between ℓ_h and ℓ_l. (b,c) There are exactly two configurations for link ℓ_h ($h < l$) crossing directly below left-oriented link ℓ_l (for partial wire $w[\ell_l]$). Either the spirality of ℓ_h is two smaller or four larger than that of ℓ_l. (d) Two configurations for link ℓ_i ($i < l$) crossing directly above left-oriented link ℓ_l.

Lemma 4. *Let ℓ_l be a left-oriented link with spirality s of a wire $w \in W_\rightarrow$ and L be a vertical line crossing ℓ_l. If $w[\ell_l]$ (the wire upto ℓ_l) crosses L below ℓ_l then the highest link from $w[\ell_l]$ that crosses L below ℓ_l has spirality $s-2$ or $s+4$. If $w[\ell_l]$ crosses L above ℓ_l then the lowest link from $w[\ell_l]$ that crosses L above ℓ_l has spirality $s-4$ or $s+2$.*

Proof. Let ℓ_h be the highest (horizontal) link from $w[\ell_l]$ that crosses L lower than ℓ_l. Thus $w[\ell_l]$ (and moreover $w[\ell_h]$) cannot cross through L between ℓ_l and ℓ_h. But then the sub-wire from ℓ_h to ℓ_l cannot enclose the origin of ℓ_h as this would force $w[\ell_h]$ to intersect L between ℓ_l and ℓ_h (see Figure 7(a)). Thus if ℓ_h is right-oriented then the wire from ℓ_h to ℓ_l followed by the segment along L from the intersection with ℓ_l to the intersection with ℓ_h must form a simple closed counter-clockwise orthogonal curve (see Figure 7(b)). As this curve has two left turns at L, there must be two more left turns than right turns along the wire from ℓ_h to ℓ_l, leading to $s(\ell_h) = s(\ell_l) - 2$. If ℓ_h is left-oriented, then the segment along L from the intersection with ℓ_h to the intersection with ℓ_l followed by the wire backwards from ℓ_l to ℓ_h forms a simple closed orthogonal curve and we get $s(\ell_h) = s(\ell_l) + 4$ (see Figure 7(c)).

Symmetrically for the lowest link ℓ_i from $w[\ell_l]$ crossing L above ℓ_l it follows through a case distinction that if ℓ_i is left-oriented then $s(\ell_i) = s(\ell_l) - 4$ and if ℓ_i is right-oriented then $s(\ell_i) = s(\ell_l) + 2$ (see Figure 7(d)). □

Lemma 5. *Let ℓ_r be a right-oriented link with spirality s of a wire $w \in W_\rightarrow$ and L be a vertical line crossing ℓ_r. If $w[\ell_r]$ crosses L below ℓ_r then the highest link from $w[\ell_r]$ that crosses L below ℓ_r has spirality $s+2$ or $s-4$. If $w[\ell_r]$ crosses L above ℓ_r then the lowest link from $w[\ell_r]$ that crosses L above ℓ_r has spirality $s+4$ or $s-2$.*

Proof. Using a similar case distinction as Lemma 4 but mirroring the cases. □

Lemma 6. *For each right-oriented link $\ell_{\rightarrow}$ of a wire $w \in W_{\rightarrow}$ with spirality $s > 0$ and each vertical line L crossing $\ell_{\rightarrow}$ there exists a subsequence S of $\Omega(s)$ links of w crossing L, such that the spiralities of the links in sequence are $(0, 2, 4, \ldots, s-2, s)$, and when ordered top-to-bottom along L form the sequence $(2, 6, 10, \ldots, s-2, s, s-4, \ldots, 4, 0)$.*

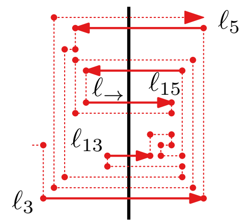


Figure 8: Lemma 6 for a link $\ell_{\rightarrow}$ and sequence $S = (\ell_3, \ell_5, \ell_{13}, \ell_{15}, \ell_{\rightarrow})$.

Proof. Consider a vertical line L through $\ell_{\rightarrow}$. We inductively find the desired subsequence S of w by constructing it in reverse order starting from $\ell_{\rightarrow}$. Let $\ell^i \in S$ be the unique link with spirality $0 \leq i \leq s$ in S. We maintain the following stronger induction hypothesis for $0 \leq t \leq s$ and $t \pmod 2 = 0$. The first two sub-hypotheses form our claim.

IH1 A subsequence S_t of w exists where the links have spiralities $(t, t+2, \ldots, s-2, s)$.

IH2 When the links from S_t are ordered top-to-bottom along L the resulting sequence is a monotone increasing sequence of all left-oriented links (with spirality $s(\ell_i) \pmod 4 = 2$), followed by a monotone decreasing sequence of all right-oriented links (with spirality $s(\ell_j) \pmod 4 = 0$).
(Specifically this gives rise to the sequence $(t+2, t+6, \ldots, s-2, s, s-4, \ldots, t+4, t)$ when $t \pmod 4 = 0$ or $(t, t+4, t+8, \ldots, s-2, s, s-4, \ldots, t+6, t+2)$ when $t \pmod 4 = 2$.)

IH3 For every two links ℓ^u and ℓ^{u-2} from S_t, where $u \pmod 4 = 2$ and $t < u \leq s$, link ℓ^{u-2} is the highest link with spirality $u-2$ from $w[\ell^u]$ that crosses L below ℓ^u.

IH4 For every two links ℓ^u and ℓ^{u-2} from S_t where $u \pmod 4 = 0$ and $t < u \leq s$, link ℓ^{u-2} is the lowest link with spirality $u-2$ from $w[\ell^u]$ that crosses L above ℓ^u.

In the base case we use $S_s = (\ell_{\rightarrow})$ which trivially satisfies all requirements. For the inductive step we may assume a subsequence $S_t = (\ell^t, \ldots, \ell_{\rightarrow})$ of w exists meeting the requirements.

Case 1 ℓ^t is left-oriented: $(t \pmod 4 = 2)$
As ℓ^t has spirality $t \pmod 4 = 2$, by **IH2** link ℓ^t is part of the initial monotone increasing subsequence in S_t when ordered top-to-bottom along L. Moreover, also by **IH2**, as ℓ^t has the smallest spirality from all selected links in S_t, link ℓ^t must be the first link in S_t and hence be the highest link that crosses L. We show that (1) a link from $w[\ell^t]$ that crosses L below ℓ^t and that has spirality $t-2$ exists and (2) the highest such link can be added to S_t while maintaining all properties of the induction hypothesis.

We first prove that a link with the desired properties exists (1). Starting with $\ell_i = \ell^t$ we repeat the following search. Find the highest (horizontal) link from $w[\ell_i]$ that crosses L lower than ℓ_i. By Lemma 3 such a link must exist while $s(\ell_i) \geq 2$. Let this link be ℓ_j. If $s(\ell_j) \neq t-2$ then repeat the search with $\ell_i = \ell_j$. If $s(\ell_j) = t-2$ then, as $\ell_j \in w[\ell_i] \subset w[\ell^t]$, link ℓ_j has the desired properties. It remains to show that such a link must be found.

Assume for contradiction that no link of spirality $t-2$ was found. As each next link found occurs strictly lower along L the search must terminate. By Lemma 3 the lowest link

CC BY

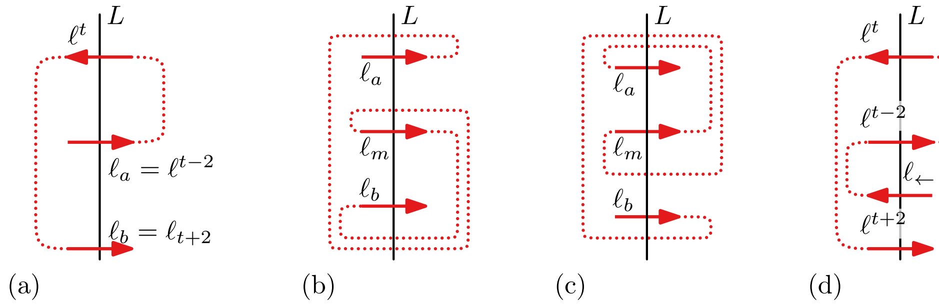


Figure 9: (a) The highest link ℓ^{t-2} with spirality $t-2$ from $w[\ell^t]$ crosses L above ℓ^{t+2}. (b) Potential situation in which $s(\ell_a) \leq s-2$, $s(\ell_b) \geq s+2$, $\ell_a \in w[\ell_b]$ and $s(\ell_m) \geq s+2$. (c) Potential situation in which $\ell_b \in w[\ell_a]$ and $s(\ell_m) \leq s-2$. (d) Link $\ell_\leftarrow$ has spirality s, hence ℓ^t is not the lowest link from $w[\ell^{t+2}]$ crossing L above ℓ^{t+2}, contradicting **IH4**.

of $w[\ell_i]$ that crosses L has spirality at most 0. Thus the search must terminate with a link of spirality at most 0. Moreover, by Lemma 4 if $s(\ell_i) \pmod 4 = 2$ then link ℓ_j has spirality $s(\ell_i) - 2$ or $s(\ell_i) + 4$. Similarly, by Lemma 5, if $s(\ell_i) \pmod 4 = 0$ then link ℓ_j has spirality $s(\ell_i) + 2$ or $s(\ell_i) - 4$. But then it must be that at some point during the search $s(\ell_i) = t$ and $s(\ell_j) = t - 4$ as initially the spirality is $t \geq 2$ and at termination the spirality is at most 0. Contradiction as $t \pmod 4 = 2$ and thus by Lemma 4 $s(\ell_j) \geq t - 2$.

As a link from $w[\ell^t]$ exists that crosses L below ℓ^t and that has spirality $t-2$, the highest link ℓ^{t-2} with these properties also exists. Adding ℓ^{t-2} to S_t trivially maintains **IH1, IH3, IH4**. We show that it also maintains **IH2**. Particularly we show that ℓ^{t-2} crosses L lower than *all* links from S_t. If $S_t = \{\ell^t\}$ this is trivially true. Otherwise, by **IH1** there exists a link ℓ^{t+2} in S_t with spirality $t+2$. Link ℓ^{t+2} is right-oriented and, as it has the smallest spirality of all right-oriented links in S_t, it is the link from S_t that crosses L in the lowest point. We show ℓ^{t-2} crosses L in an even lower point.

Assume for contradiction that ℓ^{t-2} crosses L above ℓ^{t+2} (see Figure 9(a)). We maintain an interval on L bounded from above by a link ℓ_a that crosses L and from below by a link ℓ_b that crosses L. We show that within this interval a link from the subwire $w[\ell^{t+2}]$ with spirality t exists. Initially $\ell_a = \ell^{t-2}$ and $\ell_b = \ell^{t+2}$. We have $s(\ell_a) = t-2$ and $s(\ell_b) = t+2$, and both ℓ_a and ℓ_b occur on w before (or at) ℓ^{t+2}. During our search we maintain stronger conditions. Particularly, $s(\ell_a) \leq t-2$ if $s(\ell_a) \pmod 4 = 0$ and $s(\ell_a) \leq t-4$ otherwise. Moreover, $s(\ell_b) \geq t+2$ if $s(\ell_b) \pmod 4 = 0$ and $s(\ell_b) \geq t+4$ otherwise.

We make a case distinction on whether $\ell_a \in w[\ell_b]$ or $\ell_b \in w[\ell_a]$. Initially $\ell_a \in w[\ell_b]$.

Case 1a $\ell_a \in w[\ell_b]$: (see Figure 9(b))
If $s(\ell_b) \pmod 4 = 0$ (ℓ_b is right-oriented) then by Lemma 5 the lowest link from $w[\ell_b]$ crossing L above ℓ_b has spirality $s(\ell_b) - 2$ or $s(\ell_b) + 4$. If $s(\ell_b) \pmod 4 = 2$, and thus $s(\ell_b) \geq t+4$, then by Lemma 4 this link has spirality $s(\ell_b) - 4$ or $s(\ell_b) + 2$. In each case the spirality is at least t. As $s(\ell_a) \leq t-2$, there must be a different link ℓ_m from $w[\ell_b]$ that intersects L between ℓ_a and ℓ_b. We have either $s(\ell_m) = t$, or $s(\ell_m) \geq t+2$ and $s(\ell_m) \pmod 4 = 0$, or $s(\ell_m) \geq t+4$ and $s(\ell_m) \pmod 4 = 2$. If $s(\ell_m) \geq t+2$ then let $\ell_b = \ell_m$.

Case 1b $\ell_b \in w[\ell_a]$: (see Figure 9(c))
If $s(\ell_a) \pmod 4 = 0$ then by Lemma 5 the highest link from $w[\ell_a]$ crossing L below ℓ_a has spirality $s(\ell_a) + 2 \leq t$ or $s(\ell_a) - 4 \leq t$. If $s(\ell_a) \pmod 4 = 2$, and thus $s(\ell_a) \leq t - 4$, then by Lemma 4 this link has spirality $s(\ell_a) + 4 \leq t$ or $s(\ell_a) - 2 \leq t$. As $s(\ell_b) \geq t + 2$, there must be a different link ℓ_m from $w[\ell_a]$ that intersects L between ℓ_a and ℓ_b, such that either $s(\ell_m) = t$, or $s(\ell_m) \leq t - 2$ and $s(\ell_m) \pmod 4 = 0$, or $s(\ell_m) \leq t - 4$ and $s(\ell_m) \pmod 4 = 2$. If $s(\ell_m) \leq t - 2$ then let $\ell_a = \ell_m$.

As we can repeat case 1a and 1b as long as $s(\ell_a) \neq t$ and $s(\ell_b) \neq t$, and as ℓ_a and ℓ_b have fewer links crossing L between them after each iteration, at some point we must have that either $s(\ell_a) = t$ or $s(\ell_b) = t$. As both ℓ_a and ℓ_b are part of $w[\ell^{t+2}]$, and as ℓ_a and ℓ_b both cross L lower (or at) ℓ^{t-2}, we reach a contradiction with the fact that by **IH4** ℓ^t is the lowest link from $w[\ell^{t+2}]$ with spirality t that crosses L above ℓ^{t+2} (e.g., see Figure 9(d)).

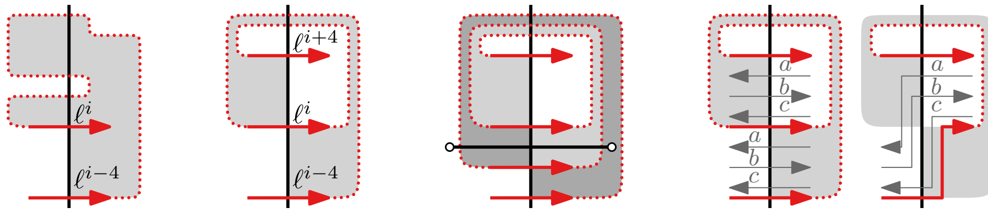
Case 2 ℓ^t is right-oriented: ($t \pmod 4 = 0$)
This case is symmetric to case 1, we abbreviate the argument for conciseness. As ℓ^t has spirality $t \pmod 4 = 0$, by **IH2** the link ℓ^t is part of the monotone decreasing subsequence in S_t when ordered top-to-bottom along L. Moreover ℓ^t is the lowest link that crosses L. Let ℓ_j be the lowest (horizontal) link from $w[\ell^t]$ that crosses L higher than ℓ^t. By Lemma 5 we have $s(\ell_j) = t - 2$ or $s(\ell_j) = t + 4$. If $s(\ell_j) = t - 2$ then the claim is proven. Otherwise, we continue the upwards search from ℓ_j. As by Lemma 3 the highest link crossing L has spirality at least 0, we can repeat the upwards search until a link of spirality $t - 2$ is found. As we continuously search earlier along the wire this link must also be part of $w[\ell^t]$.

Adding the lowest link ℓ^{t-2} with these properties to S_t trivially maintains **IH1, IH3, IH4**. We show that it also maintains that ℓ^{t-2} crosses L higher than *all* other links from S_t (**IH2**). Assume for contradiction that ℓ^{t-2} crosses L below the previously highest link ℓ^{t+2}. Then a link ℓ_m must exist that crosses L between ℓ^{t-2} and ℓ^{t+2} because as a consequence of Lemma 4 ℓ^{t-2} cannot be the highest link from $w[\ell^{t+2}]$ crossing below ℓ^{t+2}. Using ℓ_m we can repeatedly shrink the search interval along L until we find a link with spirality t that is part of $w[\ell^{t+2}]$ and that crosses L higher than ℓ^t contradicting **IH3**. ☐

Let $\ell_{\rightarrow}$ be a right-oriented link on a wire w and w.l.o.g. let $s > 0$ be the spirality of $\ell_{\rightarrow}$. Further, let L be a vertical line through $\ell_{\rightarrow}$ and S a subsequence from w with the properties guaranteed by Lemma 6. Finally, let $\ell^i \in S$ be the unique link with spirality $0 \leq i \leq s$ in S. We define the *i-core* for S (for $4 \leq i \leq s$ and $i \pmod 4 = 0$) as the region enclosed by the wire w from the intersection between ℓ^{i-4} and L to the intersection between ℓ^i and L and the line segment along L connecting them (see Figure 10(a)). We define a *layer* of S as the difference between the i-core and the $(i+4)$-core, for $4 \leq i \leq s-4$ and $i \pmod 4 = 0$ (see Figure 10(b)). There are $\Theta(s)$ layers surrounding a link of spirality $s \geq 8$.

Lemma 7. *An equivalent set of lr-wires with spirality $O(n)$ exists in Γ_I with respect to a straight-line wire-grid in Γ_O.*

Proof. We prove the statement constructively by induction on the size of the constructed set. Assume a set $\mathcal{S}$ of k lr-wires with spirality $O(n)$ exists, that is equivalent to a subset $\mathcal{T}$ of the wires from $W_{\rightarrow}$ in Γ_O. Moreover, assume that each wire in $\mathcal{S}$ is shortest with respect to the previously inserted wires in Γ_I. In the base case $\mathcal{S} = \mathcal{T} = \emptyset$.



(a) (b) (c) (d)

Figure 10: (a) The i-core of a spiral for a link $\ell^i \in S$ (gray). (b) The i-layer of the spiral (gray). (Shape simplified for exposition.) (c) Each edge can cross through (at most) the two layers directly adjacent to the crossing with L. (d) A layer cannot only contain wires as then all wires can be shortened.

For the inductive step consider an arbitrary wire $w_O \in W_\rightarrow$ from Γ_O where $w_O \notin \mathcal{T}$. Find the Euclidean shortest lr-wire w in Γ_I such that $\mathcal{S} \cup w$ is equivalent to $\mathcal{T} \cup w_O$. Consider a right-oriented link $\ell_\rightarrow \in w$ with maximum absolute spirality s. Based on Lemma 6 we conclude that there are $\Theta(s)$ layers surrounding $\ell_\rightarrow$. We bound the number of layers surrounding $\ell_\rightarrow$ to $O(n)$. Thus, it must also be that $s = O(n)$ and the spirality of w is $O(n)$. To achieve this we classify the layers of the spiral by their containment of the drawing: (1) a layer contains a vertex, or (2) a layer is crossed by an edge, or (3) a layer contains no part of the drawing (but may contain wires).

There are at most $O(n)$ layers that contain a vertex of the drawing. If a layer contains no vertex but does intersect an edge, then that edge must cross through the layer. As each edge is crossed at most once by w (Observation 2), an edge can only exit the layer on one side by crossing through w. Hence each edge crossing a layer must cross L exactly once. Each edge can at most cross through the (two) layers directly adjacent to the crossing with L in this way (see Figure 10(c)). Thus only $O(n)$ layers intersect an edge.

We now show that these are all layers, by showing every layer must contain a part of G. Assume for contradiction a layer R exists that does not contain any part of G. The boundary of R is formed by w and two straight-line segments along L. We refer to the two segments along L as the *gates* of the layer.

The layer R may still contain (subsections of) lr-wires, including w itself. Lr-wires do not cross and hence must enter and leave R through the gates. As R contains no part of the drawing and the lr-wires are each shortest with respect to the previously inserted wires they cannot consecutively enter and leave R through the same gate. Moreover, as lr-wires do not cross or self-intersect, the order of the wires at both gates must be identical.

Disconnect all lr-wires at the gates of R. Also disconnect w at the lower link adjacent to each gate. Remove all disconnected components. Reconnect the remaining parts locally along L ensuring no crossings occur and all wires visit the remaining links in the same order (see Figure 10(d)). All wires crossing R are shortened by this. Contradiction, because particularly the rerouted wire that was inserted first was not shortest with respect to the previously inserted wires. We conclude there are at most $O(n)$ layers, and thus the maximum absolute spirality of any link of w and thereby w is $O(n)$. □

Theorem 2. *Let Γ_I and Γ_O be two unified planar orthogonal drawings of a (potentially disconnected) graph G. There exists a set of wires in Γ_I that is equivalent to the straight-line wires in Γ_O and that has spirality $O(n)$.*

Proof. By Lemma 7 we can insert all lr-wires with spirality $O(n)$. By Lemma 1 intersecting links have equal spirality. Thus when a tb-wire intersects a lr-wire it has spirality $O(n)$ and we can consider the regions between two consecutive lr-wires independently. Within the region between two consecutive lr-wires no pair of wires intersect. Using the same proof as Lemma 7 we conclude that within the region between two lr-wires the spirality of each tb-wire may increase (decrease) by at most $O(n)$, but the spirality will be $O(n)$ again when crossing the second lr-wire. Thus the maximum spirality of the tb-wires is also $O(n)$. □

4 Basic algorithm using a linear number of morphs

We now describe our algorithm to morph Γ_I to Γ_O using $O(s) = O(n)$ linear morphs, where s is the spirality of an equivalent set of wires in Γ_I. The complexity of the intermediate drawing may rise to $O(n^3)$. In Section 5 we refine the algorithm by batching linear morphs to reduce the required number of linear morphs to s. Finally, in Section 6 we describe an extension to the algorithm to ensure the complexity of intermediate drawings remains $O(n)$ at the cost of a single extra linear morph. Thus using $s + 1$ linear morphs we maintain planarity, orthogonality, and linear complexity of the drawing during the morph.

For our analysis of the initial spirality we required Γ_I and Γ_O to be straight-line drawings of the unified graph. For the morph itself we let go of this stringent requirement. During the morph we introduce bends in the edges to change their orientation while maintaining orthogonality. We will show that the spirality of the wires only decreases during the morph.

The idea of the algorithm is to reduce the maximum spirality of the wires using only $O(1)$ linear morphs. Then, by Theorem 2 we need only $O(n)$ linear morphs to straighten the wires. The resulting drawing can be made similar to Γ_O, by adding additional bends in Γ_O, after which we can morph the resulting drawing to Γ_O using a single linear morph (Observation 1). We show how to reduce the spirality of wires in $W_\rightarrow$ without increasing the spirality of wires in $W_\downarrow$ and vice versa. In the description below, we limit ourselves to straightening the wires in $W_\rightarrow$.

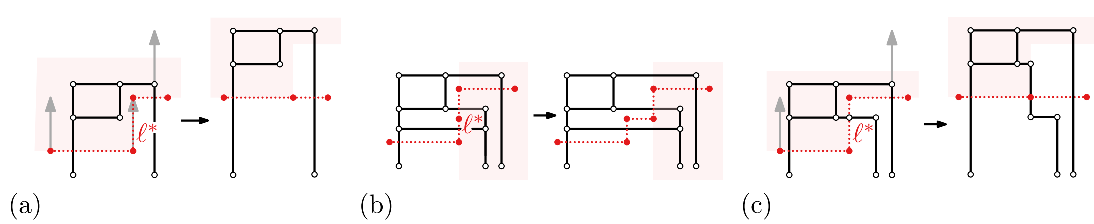


Figure 11: Different slide-types used on the wires. **(a)** The slide from [3] executed on a wire. **(b)** A bend-introducing slide splits the link without increasing the overall spirality. **(c)** A single crossing edge (link) causes the introduction of new bends in the edge (link).

To straighten the wires we use zigzag-eliminating slides (see Section 2). Consider a link ℓ^* with maximum absolute spirality. We consider two cases. In the first case ℓ^* is not crossed by any link or segment. As ℓ^* is a link with maximum absolute spirality, the links before and after ℓ^* are on opposite sides of the line through ℓ^* and hence ℓ^* is the middle link of a zigzag in the wire. Performing a zigzag-eliminating slide on ℓ^* eliminates ℓ^* and consequently may lower the spirality of the wire containing ℓ^*.

In the second case, ℓ^* is crossed by one or more links or segments. We describe two additional steps to account for this by introducing additional bends in the wires.

Step 1: (only if ℓ^* intersects multiple segments and/or links)
We split ℓ^* into several links each crossing exactly one segment/link by performing a bend-introducing slide on ℓ^* inbetween each pair of crossings (see Figure 11(b)). We can pick a direction for the bend-creating slides to ensure only links are created with an absolute spirality smaller than or equal to the maximum absolute spirality.

Step 2:
Note that ℓ^* intersects exactly one segment (link) (u, v). Before executing the zigzag-eliminating slide we first introduce two bends in (u, v) at the intersection of (u, v) and ℓ^* (see Figure 11(c)). Without loss of generality assume that ℓ^* is a vertical link. We symbolically offset the first bend to be left of ℓ^* and the second to be right of ℓ^*. Segment (u, v) is now split into three segments, the first and last being completely on one side of the zigzag and the middle being a degenerate segment. Upon performing the linear slide all three segments will stay (degenerately) horizontal or vertical during the slide. A zigzag-eliminating slide eliminates ℓ^* and may thereby lower the maximum absolute spirality of the wire containing ℓ^*. In the next part we show that these slides also maintain the spirality of all other wires.

To reduce the number of linear morphs, we combine all slides of the same type into a single linear morph. For all links with the same spirality, all bend-creating slides are combined into one linear morph, and all zigzag-eliminating slides are combined into another linear morph. For convenience of argument, links with positive spirality and links with negative spirality are combined into separate linear morphs. Thus, using at most 4 linear morphs, we can reduce the maximum absolute spirality of all wires in $W_{\rightarrow}$ by one.

Analysis. We first show that performing slides on links in $W_{\rightarrow}$ does not have adverse effects on wires in $W_{\downarrow}$. This is easy to see for bend-creating slides, as wires in $W_{\rightarrow}$ and wires in $W_{\downarrow}$ never overlap in more than a point.

Lemma 8. *Performing a zigzag-eliminating slide on a link with maximum absolute spirality in $W_{\rightarrow}$ does not increase the spirality of a wire in $W_{\downarrow}$.*

Proof. Let ℓ^* be the middle link of the zigzag causing the zigzag-eliminating slide. The zigzag-eliminating slide can only change a wire w' in $W_{\downarrow}$ if ℓ^* crosses a link ℓ' in w'. By Lemma 1 $s(\ell') = s(\ell^*)$ before the linear slide is performed. The slide does not change the spirality of any existing link in w' as the introduction of a left and right bend in w' has an overall neutral effect on the spiralities of w'. A new link is created however and this link crosses the link formed by the merging of the link before and after ℓ^*, which both have absolute spirality $|s(\ell^*)|-1$. By Lemma 1 the new link in w' must also have absolute spirality $|s(\ell^*)| - 1$, and thus the spirality of w' has not been increased. □

Lemma 9. *Multiple bend-creating or zigzag-eliminating slides on links of the same spirality in $W_\rightarrow$ can be combined into a single linear morph that maintains planarity and orthogonality.*

Proof. As bend-creating slides and zigzag-eliminating slides operate similarly, we restrict our argument to the latter. As all zigzag-eliminating slides operate on links of the same spirality, they are either all horizontal or all vertical. Without loss of generality, assume that all zigzags are horizontal. Then all vertices in the drawing are moved only vertically and vertical edges remain vertical. Furthermore, since we introduce bends at edges that intersect the middle segment of zigzags, horizontal edges are either subdivided or remain horizontal during the linear morph. As the horizontal order of vertices and bends is maintained, planarity can only be violated if there exists a vertical line where the vertical order is changed. However, by construction of the slides, on any vertical line points with higher y-coordinates are moved up at least as far as points with lower y-coordinates, maintaining the vertical order. □

Theorem 3. *Let Γ_I and Γ_O be two orthogonal planar drawings of G, where G is the unification of Γ_I and Γ_O, and Γ_I and Γ_O are equivalent. Then we can morph Γ_I to Γ_O using $O(n)$ linear morphs, where n is the number of vertices of G.*

Proof. Let $W_\rightarrow$ and $W_\downarrow$ be an equivalent set of wires for Γ_I, with respect to the straight-line wire-grid in Γ_O, having maximum spirality $O(n)$. By Theorem 2 such a set exists. By Lemma 8 the wires in $W_\rightarrow$ ($W_\downarrow$) can be straightened without affecting the spirality of the wires in $W_\downarrow$ ($W_\rightarrow$). By Lemma 9, we can reduce the maximum spirality of all wires in $W_\rightarrow$ ($W_\downarrow$) by one using at most four linear morphs. Hence, all wires can be straightened with at most $O(n)$ linear morphs. Afterwards, the resulting drawing Γ is similar to Γ_O except for additional bends. After adding matching bends in Γ_O, we can use a single linear morph to morph Γ to Γ_O (Observation 1). □

5 Batching linear morphs

In the previous section we have shown that a morph that maintains planarity and orthogonality exists that consists of $O(n)$ linear morphs. However, the intermediate complexity of the drawing could increase to $O(n^3)$ as each of the $O(n)$ morphing steps may reduce $O(n)$ links each of which crosses $O(n)$ edges, thus adding an additional $O(n^2)$ bends per step of the morph. Before we show how to also maintain $O(n)$ complexity of the drawing (Section 6), we first further strengthen the result. The proof of Theorem 3 implies that a morph between two unified planar orthogonal drawings Γ_I and Γ_O exists using $O(s)$ linear morphs, where s is the spirality of Γ_I. In this section we show how to effectively combine consecutive linear slides, resulting in a planarity and orthogonality preserving morph using only s linear morphs.

In this section we initially consider the previously defined morph without any merging of linear slides. Thus each linear morph is directly defined by a single zigzag-eliminating or bend-introducing slide. This sequence of linear slides can be encoded by a sequence of drawings, starting with Γ_I and ending with Γ_O, such that every consecutive pair of drawings is caused by a linear slide. For notational convenience let $\Gamma_i \twoheadrightarrow \Gamma_j$ indicate that Γ_i occurs before Γ_j during the morph and $\Gamma_i \underline{\twoheadrightarrow} \Gamma_j$ that $\Gamma_i \twoheadrightarrow \Gamma_j$ or $\Gamma_i = \Gamma_j$.

Let the spirality of a drawing, with respect to a given set of wires, be defined as the maximum spirality over all wires. Note that spirality of a drawing Γ is always *relative to* another drawing Γ' *and* dependent on the set of wires selected for Γ. There may be multiple potential sets of equivalent wires in Γ for a given drawing Γ'. Still, whenever the drawing Γ' and the matching set of wires are clear from the context, then by abuse of notation we will speak of the spirality of Γ. Unless stated otherwise, we consider spirality relative to Γ_O.

Let an *iteration* of the original morph consist of all linear slides that jointly reduce spirality of the drawing by one. Let the first drawing of iteration s be the first drawing in the original morph with spirality s and the last drawing of iteration s be the first drawing with spirality $s-1$. Consecutive iterations overlap in exactly one drawing. These drawings in the overlap of iterations are the intermediate steps of the final morph. Within this section let $\Gamma_I \dashrightarrow \Gamma_a \rightarrow \Gamma_b \dashrightarrow \Gamma_O$, where Γ_a is the first drawing with spirality s and Γ_b is the first drawing with spirality $s-1$.

We prove that within one iteration for any two vertices the relative order along at most one axis can be changed. We use this to show that the spirality of Γ_a with respect to Γ_b is one. We then show that we can also find a drawing Γ'_b, based on Γ_b, such that Γ_a has spirality one to Γ'_b, Γ'_b has spirality $s-1$, *and* Γ'_b is a straight-line drawing. Finally we show that any two drawings Γ, Γ' where Γ has spirality one relative to Γ' can be linearly interpolated without violating planarity of the drawing during the morph. Together this implies we can reduce spirality by one using only a single linear morph while maintaining a straight-line intermediate drawing (though each individual morph may temporarily increase complexity by $O(n^2)$). Consequently using only s linear morphs we can morph Γ_I into Γ_O while maintaining planarity and orthogonality.

As bend-introducing slides (Section 4, Figure 11(b)) can trivially be offset to not have any vertices along the vertical (horizontal) link that is split, we can prevent them from changing the x or y-order of any pair of vertices in the drawing. Therefore, we leave them out of consideration in this section.

5.1 Staircases

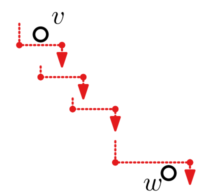
Consider two distinct vertices v and w of the drawing. Define an *x-inversion* (*y-inversion*) of v and w between Γ_a and Γ_b when the sign (+,−,0) of $v.x - w.x$ ($v.y - w.y$) differs in Γ_a and Γ_b. In that case we say two vertices are *x-inverted* (*y-inverted*), or simply *inverted.* Two vertices v and w are *separated* in a drawing by a link ℓ when they are both in the vertical (horizontal) strip spanned by ℓ, and v and w are on opposite sides of ℓ.

Lemma 10. *Let Γ_a be a drawing and Γ_b be obtained from Γ_a by a zigzag-removing slide along link ℓ. If two vertices v and w are inverted, then v and w were separated by ℓ in Γ_a.*

Proof. W.l.o.g. assume ℓ is vertical and the spirality is positive (see Figure 12). Let $\mathcal{V}$ be the set of vertices moved by a zigzag-removing slide on ℓ. If $v, w \in \mathcal{V}$ or $v, w \notin \mathcal{V}$ then v, w are moved equally in the same direction and cannot have been inverted. Hence either $v \in \mathcal{V}$ or $w \in \mathcal{V}$; assume $v \in \mathcal{V}$. All vertices in $\mathcal{V}$ move up by the length of ℓ. To be inverted we need that initially $w.y > v.y$, but also that $w \notin \mathcal{V}$. Then v and w must be separated by ℓ. □

(a) (b)

Figure 12: (a) Motion of the vertices in $\mathcal{V}$ (gray) defined by the (horizontally extended) zigzag containing ℓ. (b) To change the order of v and w along the y-axis, both must be in the horizontal strip defined by ℓ (blue) and separated by ℓ.



A *downward horizontal staircase* is a sequence of horizontal links where: (1) the left-endpoints are x-monotone increasing and strictly y-monotone decreasing, (2) the projection on the x-axis intersects for a pair if and only if they are consecutive in the sequence, and (3) all links have positive spirality. Two vertices v and w are *separated* by a downward staircase if v is in the vertical strip spanned by the first link of the staircase and above it and w is in the vertical strip spanned by the last link and below it. Similar concepts can be defined for upward horizontal staircases and for vertical staircases.

Lemma 11. *Two vertices v and w whose order during a morph from Γ_a to Γ_b is x-inverted (y-inverted) before it optionally is y-inverted (x-inverted), are separated by a horizontal (vertical) staircase of maximum spirality links in Γ_a.*

Proof. Assume w.l.o.g. that only one inversion occurs and it occurs from Γ_{b-1} to Γ_b, otherwise consider the submorph from the start until the first slide causing an inversion. Assume that $v.x < w.x$, $v.y > w.y$ in all drawings from Γ_a to Γ_{b-1} and $v.x > w.x$, $v.y > w.y$ in Γ_b. We prove the claim inductively in backwards direction. For convenience of the argument we will treat v as the right endpoint of an initial link of the staircase and w as the left endpoint of a final link.

The base case is Γ_{b-1}, where v and w are separated by a single maximum absolute spirality link (Lemma 10). As $v.y > w.y$ and $v.x < w.x$ in Γ_{b-1} and $v.x > w.x$ in Γ_b, this link has positive spirality and trivially (together with the imaginary initial and final links) forms a downward staircase.

For the step let the sequence S compose a downwards staircase in Γ_i, where $\Gamma_a \rightarrow \Gamma_i \rightarrow \Gamma_{b-1}$. Let ℓ_{i-1} be the link in Γ_{i-1} of maximum absolute spirality that specified the linear slide from Γ_{i-1} to Γ_i. We define four rectangular regions A, B, C, D surrounding ℓ_{i-1} that partition the plane in Γ_{i-1}. In the case where ℓ_{i-1} is horizontal in Γ_{i-1} we define region A as the halfspace left of the left endpoint of ℓ_{i-1}, region D as the halfspace right of the right endpoint of ℓ_{i-1}, and the remaining strip is split in region B above ℓ_{i-1} and region C below (e.g., see Figure 13(a)). In the case when ℓ_{i-1} is vertical we define A to be above, D below, B left of, and C right of ℓ_{i-1}. During the linear slide from Γ_{i-1} to Γ_i all four regions are maintained. Moreover, two new regions F and G appear in Γ_i. As regions A, B, C, D are maintained and together contain all vertices, regions F and G do not contain any vertices.

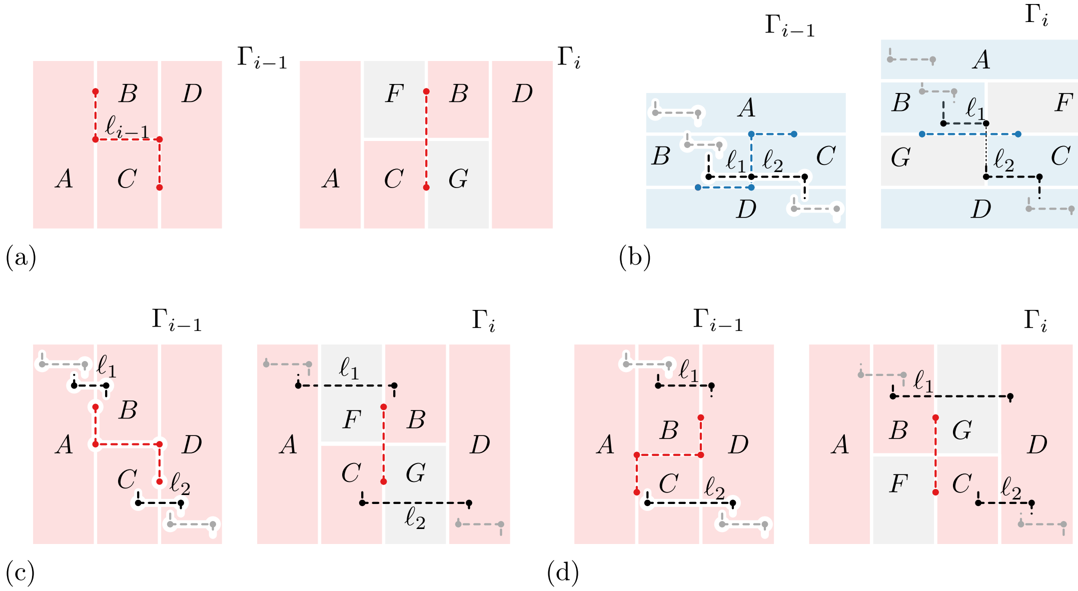


Figure 13: (a) Regions surrounding ℓ_{i-1} in Γ_{i-1} and the matching regions in Γ_i. Regions A, B, C, D are maintained between the two drawings. (b) A vertical slide may merge two links from staircase S in Γ_{i-1}. (c) If S is split in Γ_{i-1} then it can be extended to a staircase (white outline) by adding ℓ_{i-1}. (d) If non-adjacent links from S overlap in Γ_{i-1} then we can select a subsequence from S forming a staircase in Γ_{i-1} (white outline).

Assume S is not a downwards staircase in Γ_{i-1}, for otherwise we are done. A staircase is defined solely by the x- and y-order of the endpoints of the links. Thus, as S is not a downwards staircase, there must be at least two endpoints of links in S whose x- or y-order is different in Γ_{i-1} from Γ_i. The respective endpoints are separated by ℓ_{i-1} in Γ_{i-1}, for otherwise their order could not change (Lemma 10). Hence at least one link from S intersects region B and one link intersects region C. (As B and C are not horizontally or vertically adjacent in Γ_i these must be two *different* links.)

Let S_1 be the subsequence of S up to the last link ℓ_1 that intersects, or is contained in, region B. Let S_2 consist of the remaining links. The first link ℓ_2 of S_2 must intersect or be contained in region C. The endpoints of the links in S are monotone decreasing in y (respectively monotone increasing in x) in Γ_i. As region C is below (respectively right) of B, no link in S_1 intersects C and no links from S_2 intersect B. Hence, within S_1 and S_2 there are no changes in the x- and y-order of the endpoints. Any staircase properties must be broken by the interaction between S_1 and S_2. We make a case distinction on the orientation and spirality of ℓ_{i-1}.

Case 1: (ℓ_{i-1} is vertical – Figure 13(b))
Then ℓ_{i-1} must have positive spirality. Assume for contradiction that ℓ_{i-1} has negative spirality. Then in Γ_i, in mirror to the displayed case, the top of region B is at the same height as the bottom of region C. But then ℓ_1 must have a lower or equivalent y-coordinate to ℓ_2 in Γ_i. This contradicts the fact that S was a valid staircase in Γ_i.

Thus ℓ_{i-1} has positive spirality. As the x-projection of ℓ_1 and ℓ_2 touches (overlaps) in Γ_i and regions F and G contain no vertices, it must be that ℓ_1 ends at the right border of B and ℓ_2 starts at the left border of C. As link ℓ_{i-1} can only be crossed by a single edge, the right endpoint of ℓ_1 must be equal with the left endpoint of ℓ_2 in Γ_{i-1} forming a single link. The order of all other pairs of links is maintained as a vertical slide does not affect the x-order and on each vertical line the order of all points is maintained. We conclude that the sequence S is also a valid staircase in Γ_{i-1}.

Case 2: (ℓ_{i-1} is horizontal and has positive spirality – Figure 13(c))
A horizontal slide can falsify the x-monotonicity or the overlap of links. As S is a valid staircase in Γ_i the projection on the x-axis of any two non-adjacent links is non-overlapping. Then ℓ_1 and ℓ_2 cannot be fully contained in B respectively C as otherwise either ℓ_1 is contained in ℓ_2 or vice versa, and therefore at least one pair of non-adjacent links must overlap. Moreover, from S only ℓ_1 enters B and only ℓ_2 enters C. In Γ_{i-1} only the right endpoint of ℓ_1 and the left endpoint of ℓ_2 may be inverted. If so, then by Lemma 10 ℓ_1 and ℓ_2 are separated by ℓ_s. Thus (S_1, ℓ_s, S_2) forms a downwards staircase in Γ_{i-1}.

Case 3: (ℓ_{i-1} is horizontal and has negative spirality – Figure 13(d))
As S is a valid staircase in Γ_i link ℓ_1 must intersect both region B and D. As ℓ_2 intersects region C, and as only for adjacent links in S the projection on the x-axis can overlap, link ℓ_1 is the only link intersecting region B and D. Similarly only ℓ_2 intersects region C. However, in Γ_{i-1} the projection on the x-axis of several links from S_1 may overlap with the projection of ℓ_2. Select a subsequence from S satisfying all constraints by removing all links from S_1 after the first link from S_1 that overlaps ℓ_2. □

5.2 Inversions

We show that every pair of vertices is inverted along at most one axis during the morph from Γ_a to Γ_b. We then prove that Γ_a has spirality one relative to Γ_b. This will be used in the next sections to prove a single linear morph is sufficient to morph from Γ_a to Γ_b.

Lemma 12. *Two vertices v and w can be inverted along only one axis during any sub-morph of the morph from Γ_a to Γ_b.*

Proof. Suppose for a contradiction, that a pair of vertices v, w exists that is inverted along both axes during the submorph from Γ_i to Γ_j, where $\Gamma_a \twoheadrightarrow \Gamma_i \rightarrow \Gamma_j \twoheadrightarrow \Gamma_b$. Assume they are inverted along both axes exactly once, otherwise consider the smaller submorph where this is the case. W.l.o.g. let $v.x < w.x$, $v.y < w.y$ in Γ_i, $v.x < w.x$, $v.y > w.y$ in all drawings from Γ_{i+1} to Γ_{j-1}, and $v.x > w.x$, $v.y > w.y$ in Γ_j.

By Lemma 11, and the relative position of v and w, there exists a downwards staircase separating v and w in Γ_{i+1}. By Lemma 10 and the inversion of v, w from Γ_i to Γ_{i+1}, the morph from Γ_i to Γ_{i+1} is defined by a vertical maximum-spirality link separating v and w in Γ_i. The resulting vertical slide cannot break a downward staircase (see Case 1 in the proof of Lemma 11) and hence if a downwards staircase separating v and w was present in Γ_{i+1} then there is also a downwards staircase separating v and w in Γ_i. But then it must be that $v.y > w.y$ in Γ_i. Contradiction. □

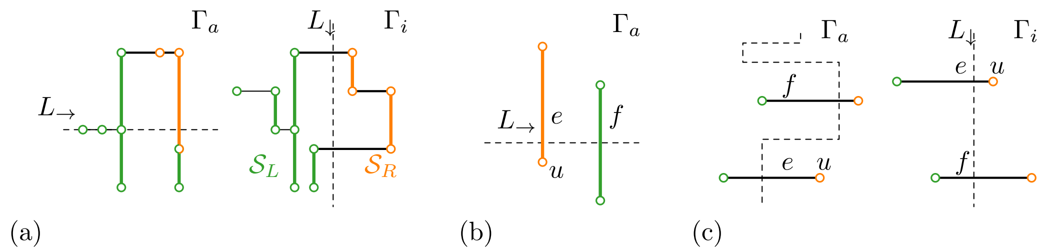


Figure 14: (a) Sets $\mathcal{S}_L$ and $\mathcal{S}_R$ in Γ_a and Γ_i. (b) A vertical edge $e \in \mathcal{S}_R$ cannot cross $L_\rightarrow$ left of a vertical edge $f \in \mathcal{S}_L$ as vertex u must be x- and y-inverted with one of the endpoints of f during the morph. (c) The y-monotone line cannot cross the edges in the wrong order as then vertex u must be x- and y-inverted with an endpoint of f.

Lemma 13. *For any drawing* Γ_i *where* $\Gamma_a \longrightarrow \Gamma_i \underline{\rightarrow} \Gamma_b$*, each vertical (horizontal) straight-line wire in* Γ_i *not crossing a vertex has an equivalent* y*- (*x*-)monotone wire in* Γ_a*.*

Proof. Let $L_\downarrow$ be a vertical line in Γ_i not crossing any vertex. Line $L_\downarrow$ partitions the set of vertices and vertical edges in Γ_i into two subsets $\mathcal{S}_L, \mathcal{S}_R$ (see Figure 14(a)). Let $L_\rightarrow$ be an arbitrary horizontal line in Γ_a. We first prove that on $L_\rightarrow$ every element from $\mathcal{S}_L$ comes before every element from $\mathcal{S}_R$. It follows that a y-monotone wire must exist in Γ_a that correctly partitions the vertices and vertical edges.

Assume for contradiction that there exist elements $e \in \mathcal{S}_R$ and $f \in \mathcal{S}_L$, with $e.x < f.x$ in Γ_a and $e.x > f.x$ in Γ_i, that lie on a horizontal line in Γ_a. As e and f are x-inverted, they (or their endpoints) cannot be y-inverted between Γ_a and Γ_i (Lemma 12). Assume e, f are vertical edges in Γ_a and an endpoint u of e is in the horizontal strip defined by f (see Figure 14(b)). The case where e or f are vertices is analogous. As the morph is planar u cannot move through f while morphing to Γ_i. But then u changed in the y-order with one of the endpoints of f during the morph. Contradiction.

We now show that a y-monotone polyline separating $\mathcal{S}_L$ and $\mathcal{S}_R$ in Γ_a intersects the horizontal edges in the same order as L in Γ_b. Consider an arbitrary pair of horizontal edges e, f that is intersected by $L_\downarrow$ in this order in Γ_i. If e, f have the same vertical order in Γ_a then the claim trivially holds. Otherwise the end-points of e, f are y-inverted in Γ_a (see Figure 14(c)) and thus by Lemma 12 the x-order of the end-points is the same in Γ_a and Γ_i. Using the same argument as in the previous paragraph there must exist an endpoint of e and an endpoint of f that have also changed in the x-order during the morph. Contradiction. □

Lemma 14. *Drawing* Γ_a *has spirality one relative to* Γ_i*, where* $\Gamma_a \longrightarrow \Gamma_i \underline{\rightarrow} \Gamma_b$*.*

Proof. The spirality of Γ_a relative to Γ_i is one if there exists a wire grid of spirality one in Γ_a that is equivalent to a straight-line wire grid in Γ_i. A wire with spirality one is, by definition, equivalent to a monotone wire. Thus, it is sufficient to show a wire grid exists where each wire is monotone. To this end, consider a new straight-line wire grid in Γ_i. By Lemma 13 we can find an equivalent x- (y-) monotone wire in Γ_a for each straight wire in Γ_i. We show that there is also a set of monotone wires in Γ_a that form an equivalent set to the wires in Γ_i.

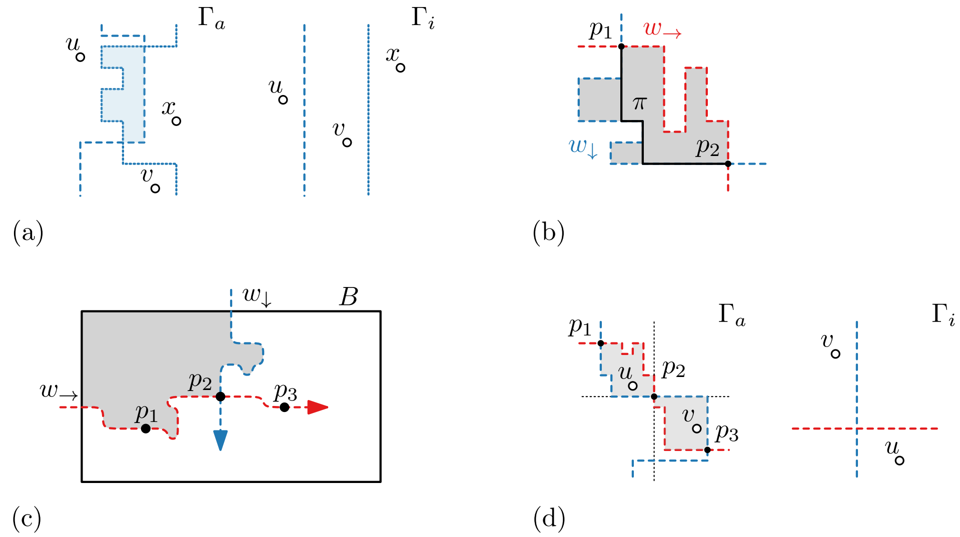


Figure 15: (a) The area enclosed by two y-monotone wires between the top-most two crossings (light blue) cannot contain vertices. (b) Each enclosed region contains an xy-monotone path π. (c) If $w_\downarrow$ crosses $w_\rightarrow$ first in p_2 then it must enter the gray area through p_1, but it cannot leave it. (d) An x-monotone lr-wire and a y-monotone tb-wire cannot cross three times.

Let W be a set of wires in Γ_a with the following properties. Firstly, every wire is monotone. Secondly, each wire is individually equivalent to a distinct straight wire in Γ_i. Thirdly, W has the minimum number of intersections over all sets satisfying the previous requirements. We show that W is equivalent to the wire-grid in Γ_i. That is, each pair of tb- (lr-) wires does not intersect and each pair of a tb- and a lr-wire intersect exactly once.

Assume for contradiction that a pair of y-monotone tb-wires intersect at least twice in W (see Figure 15(a)). Consider the top-most two intersections. The region enclosed by the wires cannot contain vertices as both wires partition the vertices equivalently to Γ_i. As the enclosed region is simple and every edge is intersected at most once by a single wire, the order in which edges are intersected along both sides is the same. Locally reroute both wires along the enclosed region to remove both intersections. Contradiction as W has the minimum number of intersections. Thus, each pair of y-monotone tb-wires in W does not intersect. A symmetric argument show no pair of x-monotone lr-wires can intersect.

Assume for contradiction that an x-monotone lr-wire $w_\rightarrow$ and a y-monotone tb-wire $w_\downarrow$ intersect more than once. As they must intersect an odd number of times, they intersect at least three times. Consider a region R enclosed between consecutive intersections p_1, p_2. Assume w.l.o.g. that $w_\rightarrow$ intersects $w_\downarrow$ left to right in p_1 (see Figure 15(b)). If R does not contain vertices, then consider the left-most, lowest x-monotone increasing path π through R. As the boundary right of π is y-monotone decreasing, π must also be y-monotone decreasing. Reroute both wires along π between p_1 and p_2 to remove the intersections. Contradiction as W has the minimum number of intersections. Each enclosed region must contain a vertex.

Consider the leftmost three consecutive intersections p_1, p_2, p_3 along $w_\rightarrow$. We have

$p_1.x \leq p_2.x \leq p_3.x$. Assume for contradiction $w_\downarrow$ crosses through p_2 first. Then $w_\rightarrow$ upto p_2, $w_\downarrow$ upto p_2, and the bounding box B surrounding the drawing (and wires) enclose a simple region (see Figure 15(c)). Wire $w_\downarrow$ enters this region through p_1, but cannot exit it without intersecting $w_\rightarrow$ somewhere before p_2. Contradiction, as p_1, p_2, p_3 are the first three intersections along $w_\rightarrow$. Similarly, if $w_\downarrow$ crosses through p_2 last then $w_\rightarrow$ between p_1 and p_3, and $w_\downarrow$ between p_1 and p_3 enclose a simple region, which $w_\downarrow$ enters through p_2. However, then there must be another intersection between $w_\downarrow$ and $w_\rightarrow$ occuring along $w_\rightarrow$ before p_2, which contradicts our assumption. Thus either $p_1.y \leq p_2.y \leq p_3.y$ or $p_1.y \geq p_2.y \geq p_3.y$.

Assume w.l.o.g. $p_1.y \geq p_2.y \geq p_3.y$ (see Figure 15(d)). The wires between these intersections enclose two disjoint regions R_1, R_2. Each region contains at least one vertex, let $u \in R_1$ and $v \in R_2$. Subdivide the plane into four axis-aligned quadrants at p_2. Region R_1 lies in the top-left quadrant and R_2 in the bottom-right quadrant. Thus, $u.x < v.x$ and $u.y > v.y$ in Γ_a. As the wires are equivalent to Γ_i, by construction $u.x > v.x$ and $u.y < v.y$ in Γ_i. Contradiction, as vertices u, v cannot be inverted along both axes (Lemma 12). Thus W cannot contain a lr-wire and a tb-wire that intersect each other more than once.

We conclude that W is an equivalent set of x- respectively y-monotone wires that matches the straight-line wire-grid in Γ_i. As all wires are monotone they have spirality one and thus, by definition, Γ_a has spirality one relative to Γ_i. □

The claim that Γ_a has spirality one relative to Γ_b follows directly by choosing $\Gamma_i = \Gamma_b$.

5.3 Simplification

In the previous subsection we showed that the spirality of Γ_a relative to Γ_b is one, where Γ_a is the initial drawing of an iteration and Γ_b the last drawing. Let s be the spirality of Γ_a and, therefore, $s-1$ the spirality of Γ_b. As the morph from Γ_a to Γ_b inserts additional bends in the edges the complexity of Γ_b may be superlinear. In this section we show that an alternative drawing Γ'_b exists that has linear complexity, has the same spirality as Γ_b, and such that Γ_a has spirality one relative to Γ'_b.

For future use (Section 6) we discuss the results in a more general case for a drawing Γ_h, where Γ_h is a drawing resulting from Γ_a by performing linear slides on a subset of the maximum absolute spirality links in Γ_a. This subset must contain at least one of the crossing links for each segment in Γ_a crossed by maximum absolute spirality links. Trivially Γ_b satisfies this requirement.

Let ε be a suitably small value such that for each vertex v in Γ_h, a 6ε-sized square box centered at v contains only v and a 3ε-part of each outgoing segment from v. A *rotation* for edge e leaving v rightwards, is a redrawing of e within the 6ε-box centered at v using the coordinates $(v, v+(0,-\varepsilon), v+(2\varepsilon,-\varepsilon), v+(2\varepsilon,0), v+(3\varepsilon,0))$ (see Figure 16). Analogous rerouting can be done for edges leaving v in other directions. For an edge crossed by a negative spirality link invert the left and right turns. An *edge rotation* is a rotation at both endpoints of the edge. We use the following technical lemma to prove that this redraw step can safely be done without introducing planarity violations.

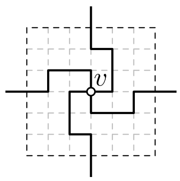


Figure 16: A 6ε-box surrounding a vertex v (dashed) with four redrawn edges.

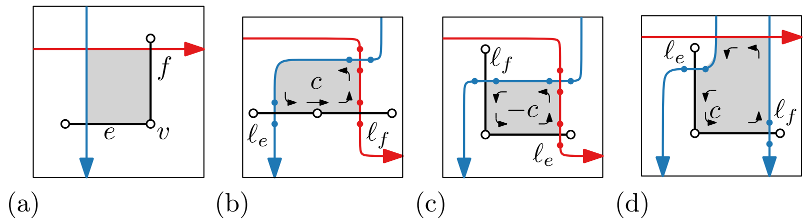


Figure 17: (a) Configuration in Γ_O with e left-oriented and f up-oriented. (b) A configuration in Γ_a when e is counter-clockwise adjacent to f in Γ_O. (c) A configuration when e is clockwise adjacent to f in Γ_O. (d) A configuration when e and f are horizontal in Γ_O.

Lemma 15. *Let v be a vertex with at least two outgoing edges e, f and let c be the turn made at v going from e to f, where $c = -1$ for a right turn, $c = 1$ for a left turn, and $c = 0$ otherwise. Let ℓ_e be a link crossing e and ℓ_f be a link crossing f. We have $s(\ell_e) + c - 1 \leq s(\ell_f) \leq s(\ell_e) + c + 1$.*

Proof. In Γ_O edges e and f are either both horizontal (vertical) or they have different orientations. For the case where the edges have different orientations, w.l.o.g. assume that in Γ_O e is a left-edge for v and f a top-edge for v. By construction e and f are intersected by a pair of wires $w \in W_\rightarrow$ and $w' \in W_\downarrow$, and they cross before crossing e respectively f (see Figure 17(a)). Wires w and w' together with edges e and f enclose a simple region in Γ_O. As the wires in Γ_a form an equivalent set this simple region also exists in Γ_a, though its shape and the orientation of the outgoing edges at v may be different.

By construction the border of this region contains a left turn at the intersection between the wires and two left turns at the intersection of the wires with the respective edges e and f (Figure 17(b)). The turn at v depends on the configuration of e and f in Γ_a. Let ℓ_e, ℓ_f be the links of w, w' crossing e and f. Furthermore, let k be the spirality of the links of w and w' at the crossing between w and w'. As the number of left turns is four larger than the number of right turns when traversing a cycle counter-clockwise we have $(k - s(\ell_e)) + (s(\ell_f) - k) + 3 + c = 4$, simplified $s(\ell_f) = s(\ell_e) + c - 1$. When, in Γ_O, e is clockwise adjacent to f at v then we get $s(\ell_f) = s(\ell_e) + c + 1$ (Figure 17(c)).

In the case where both edges are horizontal (vertical) a similar argument holds, but now the cycle is formed by two wires from $W_\downarrow$ and one wire from $W_\rightarrow$ resulting in one more left turn. We find $s(\ell_f) = s(\ell_e) + c$ (Figure 17(d)). Combining all bounds gives the result. ☐

Lemma 16. *We can rotate all edges in Γ_h that were crossed by a maximal spirality link in Γ_a, without causing planarity violations and without changing the cyclic order.*

Proof. Trivially planarity violations occur only inside the 6ε-boxes at the endpoints. There are two possible cases causing a planarity violation when redrawing Γ_h. First, two perpendicular edges leaving v coincide internally after the redraw step. This occurs if one of the edges is crossed by links of absolute spirality s and the other is not. Second, two edges leaving v in

CC BY

opposing direction coincide internally after the redraw step. This occurs if one of the edges is crossed by links of spirality s and the other by links of spirality $-s$.

For the first case, w.l.o.g. assume e is a right-edge of v and f a bottom-edge. Let ℓ_e be a link crossing e and ℓ_f a link crossing f in Γ_a. Assume w.l.o.g. $s(\ell_e) = s$ and $s(\ell_f) < s$. By Lemma 15, using $c = 1$, in Γ_a we have $s(\ell_e) \leq s(\ell_f) \leq s(\ell_e) + 2$. As $s(\ell_e) = s$ and no larger spiralities exists in Γ_a, we must have $s(\ell_f) = s(\ell_e)$. Contradiction.

For the second case, w.l.o.g. assume e is a right-edge of v and f a left-edge. Let ℓ_e be a link crossing e and ℓ_f a link crossing f, where $s(\ell_e) = s$ and $s(\ell_f) = -s$. Using Lemma 15, with $c = 0$, we get $-s < 0 \leq s(\ell_e) - 1 \leq s(\ell_f)$. Contradiction.

The cyclic order can only change when there are two perpendicular edges e, f in Γ_a that are redrawn in opposing directions and hence are crossed by a link of positive respective negative spirality. W.l.o.g. let the turn from e to f at v be a left-turn. To change in the cyclic order e must be crossed by a link of spirality s and f must be crossed by a link of spirality $-s$. However, by Lemma 15 $s(\ell_e) \leq s(\ell_f)$. Contradiction. □

Lemma 17. *There exists a straight-line drawing Γ'_h with spirality equal to Γ_h.*

Proof. Consider an edge e that in Γ_a is crossed by links of maximum absolute spirality s. Without loss of generality let $s > 0$. For the case where $s < 0$ exchange left and right turns in the following argument. All links crossing e have positive (equivalent) spirality (Lemma 2), therefore a linear slide along any such link ℓ_s introduces a right bend followed by a left bend in e. Thus e has an odd number of segments in Γ_h and the turns along e are alternating right and left turns, starting with a right turn (see Figure 18(b)). Moreover, at least one linear slide was performed on a link crossing e and hence in Γ_h at least one link of spirality $s - 1$ crosses e and it crosses e in a segment started by a right bend and followed by a left bend.

Rotate e within the 6ε-boxes near the endpoints to create two left turns and a right turn at the start of the edge, and one left turn and two right turns at the end of the edge (see Figure 18(c)). We can do this while maintaining planarity of the drawing (Lemma 16). Thus the bends in e in Γ_h can be encoded as $LLR\ (RL)^+\ LRR$, where L encodes a left turn, R a right turn, and $(RL)^+$ is the alternating sequence of turns starting with a right turn and ending with a left turn. Split differently we have $LLRR\ (LR)^*\ LLRR$, where $(LR)^*$ is the possibly empty alternating sequence of left and right turns starting with a left turn and ending with a right turn. Two turns in this sequence enclose a segment of e.

We can remove a pair of consecutive bends LR by performing a zigzag-removing slide on the segment σ between the bends. As every link with spirality $s - 1$ crosses a segment started by a right bend, any link crossing σ must have spirality s. If multiple wires intersect σ, then we can first split σ using a bend-introducing slide. Thus, assume σ is crossed by at most one link. Removing σ with a zigzag-removing slide introduces a single new link in the crossing wire and it crosses e. After removing every pair of consecutive LR bends, the edge forms a single straight-line segment (see Figure 18(d)). This segment is crossed by at least one link of spirality $s - 1$, and by all newly introduced links. As all crossing links must have the same spirality (by Lemma 2) all newly introduced links have spirality $s - 1$. Thus this operation does not increase the spirality. The result is a straight-line version Γ'_h of Γ_h with equal spirality. □

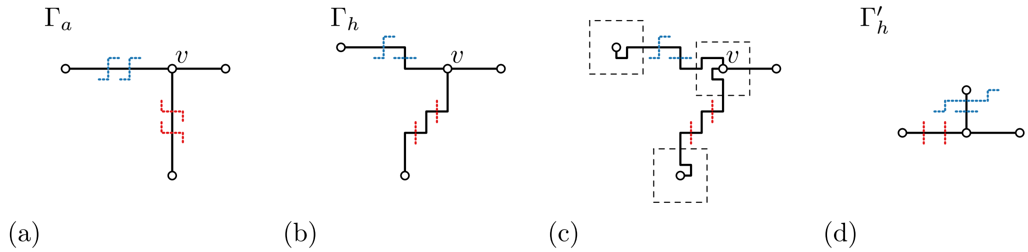


Figure 18: (a) Drawing Γ_a (black) and the maximum absolute spirality links (red, blue) crossing the edges of Γ_a. (b) Drawing Γ_h that results by performing linear slides along all maximum absolute spirality links except the top-left link. (c) The drawing after locally redrawing incoming edges that are not a straight-line segment. (d) Straightening the drawing removes all bends from the edges. Additional segments are introduced in remaining maximum spirality links, however this does not increase the maximum absolute spirality.

Lemma 18. *Drawing Γ_a has spirality one relative to Γ'_h.*

Proof. Consider a straight-line wire grid in Γ'_h. We revert Γ'_h to Γ_h using bend-introducing slides to re-insert the additional bends in the edges. During this process we move the wires from Γ'_h along. As bend-introducing slides do not create any additional bends in the straight-line wires, there exists a set of equivalent wires in Γ_h, to the straight-line wire-grid in Γ'_h, that are also straight lines. This set of equivalent wires in Γ_h must be a subset of the complete straight-line wire-grid in Γ_h, and thus by Lemma 14 there must exist an equivalent set of wires in Γ_a that is monotone. But then Γ_a has spirality one relative to Γ'_h. □

As trivially Γ_b can be created from Γ_a by performing linear slides along a suitable subset of the maximum spirality links (viz. the complete set), we conclude that a straight-line drawing Γ'_b exists such that Γ_a has spirality one relative to Γ'_b, and Γ'_b has spirality $s-1$.

5.4 Single linear morph

We have shown that given a straight-line drawing Γ_a with spirality s, we can find a straight-line drawing Γ'_b with spirality $s-1$ such that Γ_a has spirality one relative to Γ'_b. We now show that any two planar (unified) orthogonal drawings Γ_i and Γ_j, where Γ_i has spirality one relative to Γ_j, can be morphed using a single linear morph while maintaining planarity.

Two drawings are *shape-equivalent* if for each edge the sequence of left and right turns is identical and the initial segment is horizontal (vertical) in both drawings. We say two drawings are *degenerate shape-equivalent* if edges may contain zero-length segments but an assignment of orientations to the segments exists that is consistent with both drawings. Two (degenerate) shape-equivalent drawings are per definition also unified. We discuss how to make Γ_i degenerate shape-equivalent to Γ_j such that the linear interpolation from Γ_i to Γ_j is planar. To achieve this we use the intersections of the drawing with the wires (with spirality one). Note that we need not concern ourselves with planarity of the wires during the morph as this is not required for planarity of the final morph (of the drawing).

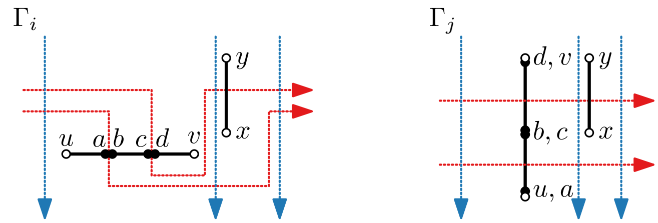


Figure 19: Drawing Γ_i with spirality one to Γ_j made degenerate shape-equivalent by adding segments $\overline{ab}$ and $\overline{cd}$. (Vertices offset slightly for visualization purposes.) All bends and vertices in Γ_O that are between a pair of consecutive vertical (horizontal) wires have the same x- (y-)coordinate.

Lemma 19. *Let Γ_i and Γ_j be two unified planar orthogonal drawings, where Γ_i has spirality one to Γ_j. There exists a single linear morph from Γ_i to Γ_j that maintains planarity and orthogonality.*

Proof. We say two *points* p and q on a drawing are *split* by a wire when p and q lie on different sides of the wire. The partition of the drawing by all wires defines *cells*; regions of the plane not split by any wire.

We first make Γ_i degenerate shape-equivalent to Γ_j. Consider all intersections in Γ_i between a maximum-spirality link and a segment of the drawing. For each intersection between an edge e and a maximum-spirality link ℓ_s we add a zero-length segment in Γ_i at the intersection of e and ℓ_s. We symbolically perturb the endpoints of the zero-length segment such that each endpoint is in a different cell. To ensure that Γ_j remains unified we also add an additional segment in Γ_j. However here we place the endpoints strictly inside the cells of the drawing while ensuring that in Γ_j: (1) the endpoints of the segment are in the same cells as the (symbolically perturbed) endpoints in Γ_i, (2) all bends and vertices enclosed by two consecutive horizontal wires have the same y-coordinate, (3) all bends and vertices enclosed by two consecutive vertical wires have the same x-coordinate (see Figure 19).

For each cell containing at least one bend or vertex, linearly interpolate all vertices and bends in Γ_i to the unique vertex or bend location in Γ_j. This directly defines a linear interpolation for each point (not necessarily a vertex or bend) between Γ_i and Γ_j.

First, we prove that the described linear morph maintains orthogonality. The endpoints of all (zero-length) segments crossing a tb-wire have the same y- coordinates in Γ_i and Γ_j, hence they remain horizontal. Symmetrically all segments crossing a lr-wire remain vertical. All other segments morph to a single point and, as the x-coordinates (y-coordinates) of the endpoints are equivalent at both the start and end of the linear morph, they remain equivalent throughout the interpolation as well.

Second, we prove that during the described linear morph the drawing remains planar. Assume for contradiction there exist two distinct points p, q on an edge or vertex of the drawing that coincide during the linear interpolation (excluding Γ_i, Γ_j). By linear motion the x-coordinates and y-coordinates of p and q change linearly. To be identical at a time $0 < t < 1$ during the morph we need that either p and q have identical x-coordinates in

CC BY

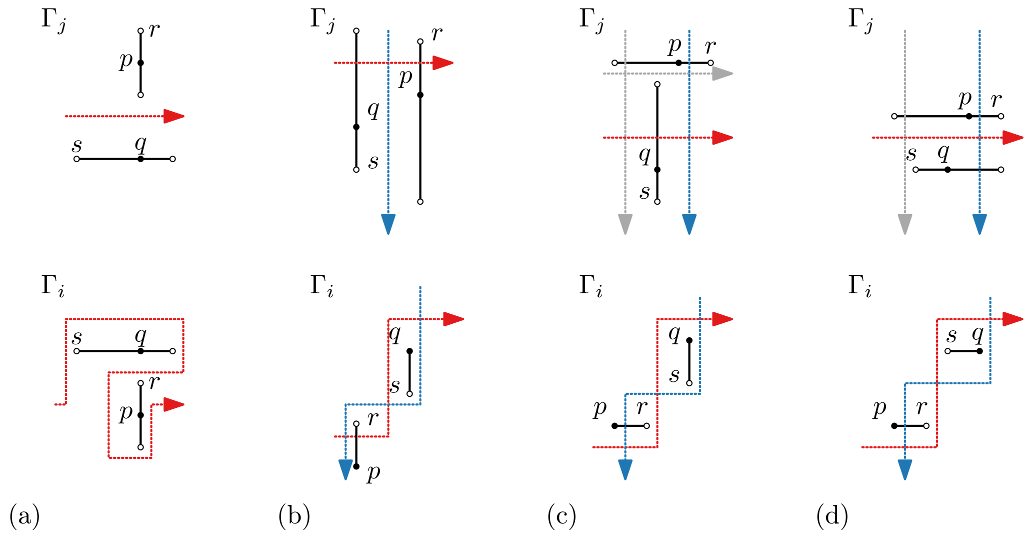


Figure 20: (a) Two points p and q on the same vertical line in Γ_i and Γ_j require spirality at least two. (b) Two points p and q on vertical segments of the drawing that are inverted along both axes imply wires in Γ_i that are not equivalent to Γ_j. (c) Points p and q on a horizontal and vertical segment. (d) Points p and q on horizontal segments.

Γ_i and Γ_j but are y-order inverted, p and q have identical y-coordinates but are x-order inverted, or p and q are inverted along both axes.

Case 1: (p and q have identical x-coordinates but are y-order inverted)
We prove by contradiction that this cannot be the case. Assume w.l.o.g. that $p.y < q.y$ in Γ_i and $p.y > q.y$ in Γ_j. Note that if $p.y = q.y$ in either Γ_i or Γ_j, then by linear motion there can not be an intersection during the linear morph itself as there already is an intersection at one of the endpoints of the morph.

Points p and q cannot be on the same segment as the orientation of each segment is (degenerately) maintained. Thus p and q must be on different segments. Let s be a lowest endpoint of the segment containing q and r a highest endpoint of the segment containing p (see Figure 20(a)). We must have either $r.y < s.y$ in Γ_i and $r.y > s.y$ in Γ_j or $r.y = s.y$. The assumption $r = s$ results in a contradiction as in Γ_j we need that $r.y \geq p.y > q.y \geq s.y$. Thus we must have that $r.y < s.y$ in Γ_i and $r.y > s.y$ in Γ_j. But if $r.y > s.y$ in Γ_j then r and s (and thus p and q) are split by a horizontal straight-line wire. This wire passes below p and above q. However, there exists no x-monotone wire in Γ_i that passes below p and above q. Contradiction as Γ_i has spirality one.

Case 2: (p and q have identical y-coordinates but are x-order inverted)
This case is symmetrical to case 1.

Case 3: (p and q are inverted along both axes)
Assume w.l.o.g. that $p.x < q.x$ and $p.y < q.y$ in Γ_i. We distinguish whether p and q are on a horizontal or vertical segment. We will work out the first case in detail and indicate the setup for the other cases, which are analogous.

Case 3a: (p and q are both on a vertical segment in Γ_j – Figure 20(b))
Let r be the top endpoint of the segment containing p and s the bottom endpoint of the segment containing q. In Γ_j we have $r.y > s.y$ and $r.x > s.x$. As r and s have distinct x- and y-coordinates they are split by at least one tb-wire and one lr-wire in Γ_j. As $\overline{pr}$ and $\overline{qs}$ are also vertical segments in Γ_i we must have $r.x < s.x$. The matching (monotone) tb-wire in Γ_i splits p and q identically and, similarly to Γ_j does not cross the segments $\overline{pr}$ and $\overline{qs}$. Due to the relative position of p and q, the monotonicity of the tb-wire, and $r.x < s.x$, it must also be that $r.y < s.y$ in Γ_i. But then the lr-wire splitting r and s must cross the tb-wire at least three times in Γ_i. Contradiction.

Case 3b: (p is on a horizontal segment and q on a vertical segment – Figure 20(c))
Let r be the right endpoint of the segment containing p and s be the bottom endpoint of the segment containing q. Once again we reach the contradiction that the tb-wire and the lr-wire splitting r and s must cross at least three times in Γ_i.

Case 3c: (p and q are both on a horizontal segment – Figure 20(d))
Let r be the right endpoint of the segment containing p and s be the left-endpoint of the segment containing q.

We conclude there do not exist two distinct points p, q on the edges (vertices) of the drawing that coincide during the linear morph. □

To reduce the need for a final linear morph from a drawing similar to Γ_O to Γ_O itself we make one more observation.

Corollary 1. *If a morph from a drawing Γ_a with spirality one to a drawing Γ_b is planar, then the morph from Γ_a to any drawing similar to Γ_b is planar.*

Proof. Two similar drawings have the same x- and y-order. The planarity of a linear morph remains intact if the x- and y-order of the vertices and bends does not change. □

Theorem 4. *Let Γ_I and Γ_O be two unified planar orthogonal drawings of a (potentially disconnected) graph G, where Γ_I has spirality s to Γ_O. We can morph Γ_I into Γ_O using exactly s linear morphs while maintaining planarity and orthogonality, and keeping the intermediate complexity of the drawing reduced to $O(n^2)$.*

Proof. By Lemma 14 for every iteration the initial drawing Γ_s has spirality one relative to the first drawing Γ_{s-1} of the next iteration. By Lemma 17 and 18 there also exists a straight-line drawing Γ'_{s-1} such that Γ_s has spirality one to Γ'_{s-1} and Γ'_{s-1} has spirality $s-1$ (to Γ_O). By Lemma 19 we can morph Γ_s to Γ'_{s-1} with a single linear morph, reducing the spirality by one. After repeating this process s times the spirality is reduced to zero and the resulting drawing must be similar to Γ_O. A final single linear morph simplifies the drawing to Γ_O. We can slightly improve this as by Corollary 1 we can merge the last two morphs without affecting the planarity of the morph.

As each single linear morph can increase the intermediate complexity by at most $O(n^2)$ and after each single linear morph we have a straight-line drawing again (of complexity $O(n)$), the maximum intermediate complexity during the morph is $O(n^2)$. □

6 Maintaining linear complexity

The approach from Section 5 maintains planarity and orthogonality and reduces complexity of the drawing to $O(n)$ after each linear morph. However, during a single linear morph complexity may increase to $\Theta(n^2)$ as each edge may be crossed by $O(n)$ wires each introducing two additional bends. We refine the approach to ensure that the drawing also maintains $O(n)$ complexity during the morph. For each edge intersected by links of maximum absolute spirality, we perform a linear slide along only one of the intersecting links. Thus at most two bends are introduced in each edge, directly ensuring the linear complexity of the drawing is maintained.

The remaining wires are *rerouted* to cross the newly introduced segment of the edge. In general rerouting the wires cannot be done without increasing their absolute spirality, which is problematic as we aim to reduce the maximum absolute spirality with each single linear morph. We show that we can perform an initial step locally inserting *windmills* in the wires adjacent to the crossed edges. Windmills introduce additional initial complexity to the wires to prevent the need to add extra complexity to the drawing or extra spirality to the wires during the morph itself. This initial rerouting of the wires in Γ_I increases the maximum absolute spirality by one.Thus, using Theorem 4, $s+1$ morphs are sufficient to morph two equivalent drawings into each other while maintaining planarity, orthogonality, and linear complexity.

Windmills. The initial rerouting step reroutes all wires in $W_\downarrow$ and $W_\rightarrow$ locally at each crossed edge if the wires satisfy the following criteria: 1) the absolute spirality of the crossing links is greater than zero and 2) at least two links cross the edge. W.l.o.g. consider a horizontal edge e that is crossed by at least two wires in Γ_I. By Lemma 2 all crossing links have the same spirality. Assume w.l.o.g. that this spirality is positive, otherwise mirror the rotations and replace right by left. Let ε be a small distance such that the ε-band above and below e is empty except for the links crossing e (see Figure 21(a)). We reroute the wires within the ε-band around e. First disconnect all crossing links within the ε-band. Then reroute all wires below e in a parallel bundle to the left, past the left-most wire w_l crossing e.

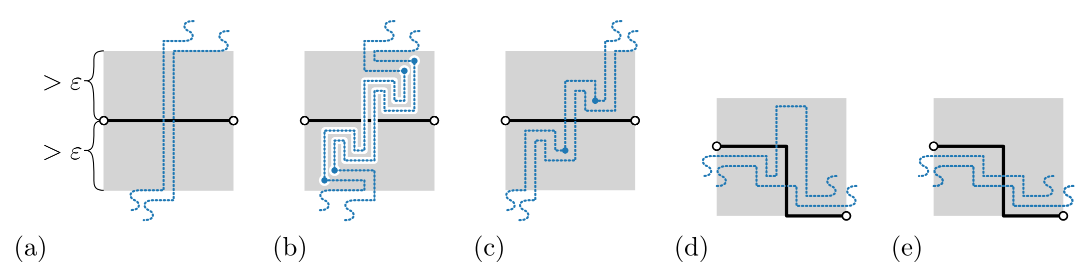


Figure 21: (a) The empty ε-band. (b) Introducing an s-windmill ($s = 1$). The outlined edges form a s-windmill. The s-windmill is connected to the original wire with links of spirality $s+1$. (c) The s-windmill after reducing all links of spirality $s+1$. Marked bends are separated by the right-most crossing link. (d) After a linear slide along the right-most crossing link. (e) After rerouting.

We now insert a so-called *s-windmill* which is defined as follows (see Figure 21(b)). Start by spiralling the bundle using right turns until the spirality of the links reaches minus one. Next we unwind the bundle again within the spiral until we reach links of spirality $s-2$. We then route the wires parallel back along e to ensure each wire crosses e in its original location and with a link of spirality s. We repeat this process above the edge, where we reroute all wires in a parallel bundle above e to the right beyond the right-most wire w_r crossing e. Again we spiral the bundle using right turns until the links have spirality minus one, then unwind the bundle until spirality s. This concludes the s-windmill. Finally we reconnect the wires by routing back parallel to e to maintain the original crossing points.

The creation of the initial s-windmill locally increases the spirality by one. After the first iteration only the s-windmill is left (see Figure 21(c)).

Morphing with rerouting. Once again consider the morph as a sequence of individual linear slides. Specifically consider a single iteration of the resulting morph. The linear slides performed within an iteration are all based on links of the same maximum absolute spirality. As we have not enforced any order on these linear slides we can perform them in any order. We now define a partial ordering on the linear slides that, indirectly, prevents excess complexity from being introduced in the drawing during the individual morphs.

Specifically consider an ordering that first performs all slides caused by maximum-spirality links that do not cross a segment of the drawing. Then, of the remaining maximum-spirality links, the linear slides defined by links meeting the following criteria are performed: For each horizontal edge e crossed by $k > 1$ links of maximum absolute spirality we perform the linear slide caused by the right-most link crossing e. This slide creates a new vertical segment (see Figure 21(d)). For each vertical edge crossed by $k > 1$ links of maximum-spirality we perform the slide defined by the top-most link crossing the edge.

Let Γ_s be the first drawing of a given iteration and let Γ_s have spirality s. Furthermore consider the intermediate drawing Γ^r_{s-1} that results from performing all the slides defined above. Using Lemma 14 we directly derive the following lemma.

Lemma 20. *Drawing Γ_s has spirality one relative to Γ^r_{s-1}.*

We reroute the wires by shortcutting them as a bundle across the new segment (see Figure 21(e)).

Lemma 21. *In Γ^r_{s-1} all remaining maximum-spirality links crossing an edge e can be shortcut across the newly created segment in e.*

Proof. Assume w.l.o.g. that the spirality is positive and that e is horizontal. Let ℓ be the (right-most) link that caused the linear slide introducing a new segment in e. Let w be a wire crossing e that is not the right-most wire. Let ℓ_w be the link from w crossing e. Finally let ℓ_{w+2} be the first link along w after ℓ_w that does not share an endpoint with ℓ_w. In Γ_s the start point s_w of ℓ_w is separated by ℓ from the endpoint t_{w+2} of ℓ_{w+2} (see Figure 21(c)). However, the start point s_{w+2} of ℓ_{w+2} is not separated from s_w. After performing the linear slide along ℓ, endpoint t_{w+2} has a smaller y-coordinate than s_w, whereas s_w still has a larger y-coordinate. But then after performing the linear slide defined by ℓ a rightwards ray from the start point of ℓ_w must cross ℓ_{w+2}. We can horizontally shortcut each wire. As the order of the wires is identical on both sides we can do so without introducing new crossings. □

Corollary 2. *Drawing Γ^r_{s-1} has spirality $s-1$ relative to Γ_O.*

Proof. Rerouting the wires removes all links of maximum-spirality. Thus the resulting wires have maximum-spirality $s-1$, and are equivalent to the straight-line wire grid in Γ_O. But then Γ^r_{s-1} has spirality $s-1$ relative to Γ_O. □

Lemma 22. *At the start of iteration $s-1$ of the final morph including intermediate simplification and rerouting, all wires crossing an edge e with links of spirality $s-1$ form an $(s-1)$-windmill in an empty ε-band next to e.*

Proof. As the ε-band is empty except for the windmill, every link that is not part of the windmill cannot separate two bends that are part of the windmill. Such links by Lemma 10 cannot destroy the structure (the x- and y-order of the bends) of the windmill, so we only concern ourselves with links that are part of the windmill. Performing a linear slide along the right-most link ℓ crossing e maintains all links in the windmill except for ℓ (see Figure 21(d)). Moreover, after the linear slide every wire, after crossing e, will be routed along the newly generated segment past the newly produced intersection of e with the right-most wire. Rerouting the remaining wires (see Figure 21(e)) removes all spirality s links, but maintains all other links. The spirality of the resulting set of wires is at most $s-1$ and they form an $(s-1)$-windmill next to e. □

By Lemma 20 drawing Γ_s has spirality one relative to Γ^r_{s-1}. Thus by Lemma 19 we can reduce the spirality by one using a single linear morph, while increasing the complexity of the drawing by only $O(n)$. By Lemma 22 this process can be repeated without increasing spirality intermittently. All that is left is to reintroduce the intermittent simplification to maintain $O(n)$ complexity.

Lemma 23. *Drawing Γ_s has spirality one relative to drawing Γ'^r_{s-1} and Γ'^r_{s-1} has spirality $s-1$.*

Proof. By Lemma 20 Γ_s has spirality one relative to Γ^r_{s-1}. By Lemma 16 and 17 we can redraw Γ^r_{s-1} to a straight-line drawing Γ'^r_{s-1} while maintaining that Γ_s has spirality one relative to Γ'^r_{s-1}. □

We conclude that given a straight-line orthogonal drawing Γ_s of complexity $O(n)$ and with spirality s to Γ_O, a straight-line orthogonal drawing Γ'^r_{s-1} exists with complexity $O(n)$ and spirality $s-1$ to Γ_O. Moreover, Γ_s has spirality one to Γ'^r_{s-1} and the unified complexity of Γ_s and Γ'^r_{s-1} is $O(n)$. By Lemma 19 we can morph Γ_s to Γ'^r_{s-1} with a single linear morph while maintaining planarity, orthogonality, and linear complexity.

7 Final Algorithm

Including all improvements we obtain the following algorithm (Algorithm 1). Let Γ_I and Γ_O be two equivalent orthogonal drawings of a (potentially disconnected) graph G. We first unify Γ_I and Γ_O to ensure they are equivalent orthogonal *straight-line* drawings. This can be done without increasing the complexity. Find two sets of wires $W_\rightarrow$ and $W_\downarrow$ that are an equivalent set of wires for Γ_I, with respect to the straight-line wire-grid in Γ_O, having maximum spirality $s = O(n)$. By Theorem 2 such a set exists. Introduce windmills locally at each crossed edge if 1) the absolute spirality of the edge is greater than zero, 2) at least two wires cross the edge. This increases the maximum absolute spirality by one to $s + 1$. Consider the drawing $\Gamma_{s+1} = \Gamma_I$ together with the rerouted wire set.

We now repeat the following for each orthogonal straight-line drawing Γ_t with spirality $0 < t \leq s + 1$. Perform linear slides on all maximum-spirality links that do not intersect the drawing. Perform linear slides on all maximum-spirality links that are the right-most link crossing a horizontal edge or the top-most link crossing a vertical edge. Reroute the wires to remove all other maximum-spirality links. Straighten the drawing to the resulting drawing Γ_{t-1}. Drawing Γ_t has spirality one relative to Γ_{t-1} and Γ_{t-1} has spirality $t - 1$ (Lemma 23) and $O(n)$ complexity. By Lemma 19 we can linearly interpolate Γ_t to Γ_{t-1} while maintaining planarity, and orthogonality.

Algorithm 1 Final morphing algorithm

Require: Two equivalent planar orthogonal drawings Γ_I and Γ_O with maximum complexity $O(n)$. Moreover let s be the spirality of Γ_I with respect to Γ_O.
Ensure: A sequence of $s + 1 = O(n)$ linear morphs that morph Γ_I to Γ_O.
  Unify Γ_I and Γ_O. (Section 2)
  Find an equivalent set of wires for Γ_I compared to Γ_O. (Section 3)
  Let s be the maximum absolute spirality of the wires in Γ_I.
  Add windmills to Γ_I. (Section 6)
  Let Γ_{s+1} be the current drawing and let $t = s + 1$.
  while $t > 0$ **do**
    Based on Γ_t compute an orthogonal straight-line drawing Γ_{t-1}. (Section 6)
    Morph Γ_t to Γ_{t-1} with a single linear morph. (Section 5.4)
    $t \leftarrow t - 1$
  end while

As complexity only increases by $O(n)$ during each linear morph and each drawing is simplified to $O(n)$ complexity again we also maintain linear complexity of the drawing during the morph. Thus using $s + 1$ linear morphs we can morph Γ_I to Γ_O while maintaining planarity, orthogonality, and linear complexity.

Thus we directly obtain the final Theorem:

Theorem 5. *Let Γ_I and Γ_O be two equivalent drawings of a (potentially disconnected) graph G, where Γ_I has spirality s. We can morph Γ_I into Γ_O using $s + 1$ linear morphs while maintaining planarity, orthogonality, and linear complexity of the drawing during the morph.*

8 Conclusion

We have described a morph of two planar orthogonal drawings of a (potentially disconnected) graph G of complexity n, using $O(n)$ linear morphs. To this end we used the spirality s of the drawing Γ of G, which we have shown is $O(n)$ even for disconnected graphs. We further refined the analysis to show that not only are $O(n)$ linear morphs sufficient, but indeed $s+1$ linear morphs are sufficient to maintain planarity, orthogonality, and linear complexity during the complete morph. There remain a number of open questions surrounding morphs of orthogonal drawings.

First, it is clear that s morphs is a lowerbound for our approach. It is unclear whether $s+1$ linear morphs are needed to maintain linear complexity though. Preliminary investigations indicate this may be the case, but as it stands $s+1$ is just one off from a trivial lowerbound. Second, it is not yet clear what the required time complexity is to compute the described morphs. Our proofs are mostly constructive, but the efficiency of computing the morph was not taken into account. Third, from a visualization perspective the reduced number of linear morphs is satisfying, but the drawing may be arbitrarily scaled during the morph. The final morph does not take stability of the drawing into account, something that would be desirable for practical application.

Acknowledgements

The authors would like to thank the anonymous reviewers for their extensive feedback and suggestions to help improve the paper.

References

[1] Soroush Alamdari, Patrizio Angelini, Fidel Barrera-Cruz, Timothy Chan, Giordano Da Lozzo, Giuseppe Di Battista, Fabrizio Frati, Penny Haxell, Anna Lubiw, Maurizio Patrignani, Vincenzo Roselli, Sahil Singla, and Bryan Wilkinson. How to morph planar graph drawings. *SIAM Journal on Computing*, 46(2):824–852, 2017.

[2] Patrizio Angelini, Fabrizio Frati, Maurizio Patrignani, and Vincenzo Roselli. Morphing planar graph drawings efficiently. In *Proc. 21st International Symposium on Graph Drawing*, pages 49–60, 2013.

[3] Therese Biedl, Anna Lubiw, Mark Petrick, and Michael Spriggs. Morphing orthogonal planar graph drawings. *ACM Transactions on Algorithms*, 9(4):29:1–29:24, 2013.

[4] Therese Biedl, Anna Lubiw, and Michael Spriggs. Morphing planar graphs while preserving edge directions. In *Proc. 13th International Symposium on Graph Drawing*, pages 13–24, 2006.

[5] Stewart Cairns. Deformations of plane rectilinear complexes. *The American Mathematical Monthly*, 51(5):247–252, 1944.

[6] Michael Freedman, Joel Hass, and Peter Scott. Closed geodesics on surfaces. *Bulletin of the London Mathematical Society*, 14(5):385–391, 1982.

[7] Linda Kleist, Boris Klemz, Anna Lubiw, Lena Schlipf, Frank Staals, and Darren Strash. Convexity-increasing morphs of planar graphs. *Computational Geometry*, 84:69–88, 2019.

[8] Bettina Speckmann and Kevin Verbeek. Homotopic c-oriented routing with few links and thick edges. *Computational Geometry*, 67:11–28, 2018.

[9] Carsten Thomassen. Deformations of plane graphs. *Journal of Combinatorial Theory, Series B*, 34(3):244–257, 1983.

[10] Arthur van Goethem, Bettina Speckmann, and Kevin Verbeek. Optimal Morphs of Planar Orthogonal Drawings II. *arXiv e-prints*, Aug 2019.

[11] Arthur van Goethem and Kevin Verbeek. Optimal morphs of planar orthogonal drawings. In *Proceedings of the 34th International Symposium on Computational Geometry (SoCG 2018)*, pages 42:1–42:14, 2018.