Arithmetical structures on dominated polynomials

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SPECIAL ISSUE IN HONOR OF RAFAEL H. VILLARREAL
ON THE OCCASION OF HIS 70TH BIRTHDAY

Arithmetical structures on dominated polynomials

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Abstract

Arithmetical structures on matrices were introduced in Corrales H, Valencia CE (Arithmetical structures on graphs. Linear Algebra Appl, 536:120–151, 2018), which are finite whenever the matrix is irreducible. We generalize the algorithm that computes arithmetical structures on matrices given in Valencia CE, Villagrán RR (Algorithmic aspects of arithmetical structures. Linear Algeb Appl, 640:191–208, 2022), to an algorithm that computes arithmetical structures on dominated polynomials. A dominated polynomial is an integer multivariate polynomial, such that it contains a monomial, which is divided by all of its monomials. We give an example of a dominated polynomial which is not the determinant of an integer matrix and show how the algorithm works on it.

Keywords Dominated polynomials · Arithmetical structures · Diophantine equation

Mathematical Subject Classification Primary · 11D72 · 11Y50 · Secondary · 11C20 · 15B48

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Dedicated to Professor Rafael H. Villarreal on the occasion of his seventieth birthday.

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1 Introduction

Arithmetical structures on matrices were introduced in 2018 by Corrales and Valencia in [3], where it was proven that arithmetical structures on irreducible matrices are finite. Recently arithmetical structures aroused some interest, see for instance [1, 2, 4, 5, 7]. Some algorithmic aspects of arithmetical structures on matrices were discussed in [6] where an algorithm that computes them was given.

The main goal of this article is to generalize some of the ideas contained in [6] for matrices to dominated polynomials and to get an algorithm that computes arithmetical structures on dominated polynomials. A dominated polynomial is an integer multivariate polynomial with a monomial which is divided by all its monomials. We recall that not every dominated polynomial is the determinant of an integer matrix, see for instance Example 2.13.

Now, we recall the definition of an arithmetic structure on a matrix. Given a non-negative integer matrix L with zero diagonal (for instance the adjacency matrix of a graph), a pair $(\mathbf{d}, \mathbf{r}) \in \mathbb{N}_+^n \times \mathbb{N}_+^n$ is called an arithmetical structure on L whenever

$$(\mathrm{Diag}(\mathbf{d}) - L)\mathbf{r}^t = \mathbf{0}^t \text{ and } \gcd(r_1, \dots, r_n) = 1.$$

It is not difficult to check that the vector $\mathbf{d}$ is a solution of the polynomial Diophantine equation

$$f_L(X) := \det(\mathrm{Diag}(\mathbf{X}) - L) = 0.$$

Therefore, computing arithmetical structures on matrices consists on computing a subset of the solutions of the Diophantine equation defined by the determinant of a matrix with variables in the diagonal.

Throughout this article we use the usual partial order over $\mathbb{R}^n$ given by $\mathbf{a} \leq \mathbf{b}$ if and only if $a_i \leq b_i$ for all $i = 1, \dots, n$ and $\mathbf{a},\mathbf{b} \in \mathbb{R}^n$. In a similar way, $\mathbf{a} < \mathbf{b}$ whenever $\mathbf{a} \leq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$. It is well known that $\leq$ is a well partial order over $\mathbb{N}^n$.

2 Arithmetical structures on dominated polynomials.

We generalize Algorithms [6, 3.2 and 3.4] given for the determinant of a matrix to dominated polynomials. Some concepts are preserved in this new setting and others are not. For instance, the concept of d-arithmetical structure is generalized easily. However, this does not happen in the case of the r-arithmetical structure.

2.1 Dominated polynomials

Given a polynomial f, let $\mathcal{M}_f$ be its set of monomials. A monomial $p \in \mathcal{M}_f$ is called dominant whenever it is divided by every monomial in $\mathcal{M}_f$.

Definition 2.1 A polynomial f is called dominated whenever $\mathcal{M}_f$ has a dominant monomial.

It is not difficult to check that if $\mathcal{M}_f$ has a dominant monomial, then it is unique. For simplicity we can assume that f is square-free and $x_1 \cdots x_n$ is its dominant monomial.

Note that if L is a nonnegative integer square matrix with zeros in the diagonal, then $f_L(X)$ is a square-free dominated polynomial, whose dominant monomial is the product of the variables. It is not difficult to check that $f_L(X)$ is irreducible if and only if the matrix L is irreducible. Moreover, a vector $\mathbf{d} \in \mathbb{N}_+^n$ is an arithmetical structure on L whenever $f_L(\mathbf{d}) = 0$ and the non-constant coefficients of $f_{L,\mathbf{d}}(X) := f_L(X + \mathbf{d})$ are positive, see [3, Remark 2.7]. Now, we define the concept of a d-arithmetical structure on an irreducible square-free dominated polynomial.

Definition 2.2 Given a polynomial f with its leading coefficient positive, an arithmetical structure on f is a vector $\mathbf{d} \in \mathbb{N}_+^n$ such that $f(\mathbf{d}) = 0$ and all the non-constant coefficients of $f_{\mathbf{d}}(X) := f(X + \mathbf{d})$ are positive.

Note that if f does not have its leading coefficient positive, then it does not have any arithmetical structures. However, since either f or $-f$ has its leading coefficient positive, then we can assume that f has positive leading coefficient. From now on, let us assume that the leading coefficient is always positive unless the contrary is stated. We recall that not every polynomial is the determinant of a matrix with variables in the diagonal, see for instance Example 2.13.

If a dominated square-free polynomial f is reducible, that is, $f = \prod_{i=1}^{s} f_i$ for some irreducible square-free polynomial f_i, then each f_i is a dominated polynomial. Moreover, if $\mathbf{d}(f_i)$ is the vector with the entries of $\mathbf{d}$ that corresponds to the variables of f_i, then $\mathbf{d}$ is an arithmetical structure on f if and only if $\mathbf{d}(f_i)$ is an arithmetical structure on at least one of the f_i, and the non-constant coefficients of $f_{i,\mathbf{d}(f_i)}(X)$ are positive and the constant coefficient is non-negative for all i. Thus, if f is a reducible square-free polynomial, then it has an infinite number of arithmetical structures.

Definition 2.3 Given a square-free dominated polynomial f on n variables, let

$$\mathcal{D}(f) = \{\mathbf{d} \in \mathbb{N}_+^n \mid \mathbf{d} \text{ is an arithmetical structure on } f\}.$$

This definition generalizes the one given in [6, Section 2]. More precisely, if L is a non-negative matrix with zero diagonal, then $\mathcal{D}(L) = \mathcal{D}(f_L)$ where $f_L = \det(\mathrm{Diag}(\mathbf{x}) - L)$.

Now, let $\mathcal{D}_{\geq 0}(f) = \left\{\mathbf{d} \in \mathbb{N}_+^n \,\middle|\, \text{all non-constant coefficients of } f_{\mathbf{d}}(X) \text{ are positive and } f(\mathbf{d}) \geq 0\right\}$.

2.2 The algorithm in the dominated polynomial case

Here we give an algorithm that finds the arithmetical structures on a square-free irreducible dominated polynomial.

If f is an integer multivariate polynomial and all nonconstant coefficients of f are positive, let

$$\mathcal{C}(f) = \{\mathbf{d} \in \mathbb{N}_+^n \mid f(X + \mathbf{d}) \in \mathcal{D}_{\geq 0}(f)\}.$$

Now, let $\min \mathcal{D}_{\geq 0}(f)$ be the set of all minimal elements of $\mathcal{D}_{\geq 0}(f)$. It is not difficult to check that $\min \mathcal{C}(f)$ exists and it is finite by Dickson's Lemma. Also, for any $\mathbf{d} \in \mathbb{Z}^{n-1}$ and $1 \leq s \leq n$, let $\mathbf{d}^{(s)} \in \mathbb{Z}^n$ be given by

$$(\mathbf{d}^{(s)})_i = \begin{cases} \mathbf{d}_i & \text{if } 1 \leq i < s, \\ 1 & \text{if } i = s, \\ \mathbf{d}_{i-1} & \text{if } s < i \leq n. \end{cases} \tag{2.1}$$

Algorithm 2.4 **Input**: An integer irreducible square-free dominated polynomial f.
Output: $\min \mathcal{D}_{\geq 0}(f)$ and $\mathcal{D}(f)$.

(1) Let $\partial_s f = \dfrac{\partial f}{\partial x_s}$ for all $1 \leq s \leq n$.
(2) Compute $\tilde{A}_s = \min \mathcal{D}_{\geq 0}(\partial_s f)$ for all $1 \leq s \leq n$.
(3) Let $A_s = \{\tilde{\mathbf{d}}^{(s)} \mid \tilde{\mathbf{d}} \in \tilde{A}_s\}$.
(4) **For** $\boldsymbol{\delta}$ in $\prod := \prod_{s=1}^n A_s$:
(5) $\quad \mathbf{d} = \sup\{\boldsymbol{\delta}_1, \boldsymbol{\delta}_2, \dots, \boldsymbol{\delta}_n\}$.
(6) $\quad$ Let $S = \{s \mid \mathrm{coef}_{\mathbf{d}}(x_s) = 0\}$
(7) $\quad$ **If** $|S| = 0$:
(8) $\quad\quad$ **For** $\mathbf{d}^* \in \min \mathcal{C}(\mathrm{Diag}(f(X + \mathbf{d}))$:
(9) $\quad\quad\quad$ "Add" $\mathbf{d}^* + \mathbf{d}$ to $\min \mathcal{D}_{\geq 0}(f)$.
(10) $\quad$ **If** $|S| \geq 1$:
(11) $\quad\quad$ **For** $t \notin S$:
(12) $\quad\quad\quad$ Make $\mathbf{d}'_t = \mathbf{d}_t + 1, \mathbf{d}'_r = \mathbf{d}_r$ for all $r \in [n] \setminus \{t\}$
(13) $\quad\quad\quad$ **For** $\mathbf{d}^* \in \min \mathcal{C}(\mathrm{Diag}(f(X + \mathbf{d}'))$:
(14) $\quad\quad\quad\quad$ "Add" $\mathbf{d}^* + \mathbf{d}'$ to $\min \mathcal{D}_{\geq 0}(f)$.
(15) $\quad$ **If** $|S| \geq 2$:
(16) $\quad\quad$ **For** $s_1, s_2 \in S$ $(s_1 \neq s_2)$:
(17) $\quad\quad\quad$ Make $\mathbf{d}'_{s_1} = \mathbf{d}_{s_1} + 1, \mathbf{d}'_{s_2} = \mathbf{d}_{s_2} + 1, \mathbf{d}'_r = \mathbf{d}_r$ for all $r \in [n] \setminus \{s_1, s_2\}$
(18) $\quad\quad\quad$ **For** $\mathbf{d}^* \in \min \mathcal{C}(\mathrm{Diag}(f(X + \mathbf{d}'))$:
(19) $\quad\quad\quad\quad$ "Add" $\mathbf{d}^* + \mathbf{d}'$ to $\min \mathcal{D}_{\geq 0}(f)$.
(20) Return $\min \mathcal{D}_{\geq 0}(f)$ and $\mathcal{D}(f) = \{\mathbf{d} \in \min \mathcal{D}_{\geq 0}(f) \mid f(\mathbf{d}) = 0\}$.

The vector at step (5) is the supremum or maximum of the set of vectors $\{\boldsymbol{\delta}_1, \dots, \boldsymbol{\delta}_n\}$ under the usual (entry by entry) order. The function "add" at steps (9), (14) and (19) means that we add the corresponding vector to the set $\min \mathcal{D}_{\geq 0}(L)$ whenever it is not greater than other vector already in the set. Afterwards, by eliminating from the set any vector greater than that vector, then the minimality of the set is ensured. The proof of the correctness of Algorithm 2.4 will be similar to the one given for [6, Algorithm 3.2]. Thus we begin by extending [6, Lemma 3.1] for the polynomial case.

Lemma 2.5 *If* $f = ax_1x_2 + b_1x_1 + b_2x_2 + c$ *with* $a, b_1, b_2, c \in \mathbb{Z}$ *and* $a \geq 1$, *then*

$$\min \mathcal{D}_{\geq 0}(f) = \min \left\{ \left(d, \max\left(d_2^+, \left\lceil \frac{-(c+b_1d)}{ad+b_2} \right\rceil \right) \right) \Big| \, d \in \mathbb{N}_+,\, d_1^+ \leq d \leq \max\left(d_1^+, \left\lceil \frac{-(c+b_2d_2^+)}{ad_2^+ + b_1} \right\rceil \right) \right\},$$

where $d_1^+ = \max(1, \lceil \frac{1-b_2}{a} \rceil)$ *and* $d_2^+ = \max(1, \lceil \frac{1-b_1}{a} \rceil)$.

Proof A vector $\mathbf{d} = (d_1, d_2) \in \mathbb{Z}^2$ is in $\mathcal{D}_{\geq 0}(f)$ if and only if $d_1, d_2 \geq 1$, $ad_1 + b_2, ad_2 + b_1 \geq 1$ and

$$ad_1d_2 + b_1d_1 + b_2d_2 + c \geq 0. \tag{2.2}$$

We set $d_1^+ = \max(1, \lceil \frac{1-b_2}{a} \rceil)$ and $d_2^+ = \max(1, \lceil \frac{1-b_1}{a} \rceil)$. It is clear that if $\mathbf{d} \in \mathcal{D}_{\geq 0}(f)$, then $\mathbf{d} \geq (d_1^+, d_2^+)$. On the other hand, if $(d_1, d_2) \geq (d_1^+, d_2^+)$, then the only condition left for $\mathbf{d}$ to be in $\mathcal{D}_{\geq 0}(f)$ is 2.2. Therefore, if $ad_1^+d_2^+ + b_1d_1^+ + b_2d_2^+ + c \geq 0$, then $\min \mathcal{D}_{\geq 0}(f) = \{(d_1^+, d_2^+)\}$. Henceforth, let us assume that

$$ad_1^+d_2^+ + b_1d_1^+ + b_2d_2^+ + c < 0 \ (\leq -1) \tag{2.3}$$

and

$$ad_1d_2^+ + b_1d_1 + b_2d_2^+ + c < 0. \tag{2.4}$$

Thus $d_1^+ \leq d_1 < \frac{-(c+b_2d_2^+)}{ad_2^+ + b_1}$ and in order to fulfill condition (2.2), we have that $d_2 \geq \frac{-(c+b_1d_1)}{ad+b_2}$. Also note that $\max(d_2^+, \frac{-(c+b_1d_1)}{ad_1+b_2}) = \frac{-(c+b_1d_1)}{ad_1+b_2}$ by (2.4). Then

$$\min \left\{ (d_1, \lceil \frac{-(c+b_1d_1)}{ad_1+b_2} \rceil) |\, d_1^+ \leq d_1 \leq \lfloor \frac{-(c+b_2d_2^+)}{ad_2^+ + b_1} \rfloor \right\} \subseteq \min \mathcal{D}_{\geq 0}(f).$$

Finally, if

$$ad_1d_2^+ + b_1d_1 + b_2d_2^+ + c \geq 0, \tag{2.5}$$

then we have that $\max(d_2^+, \frac{-(c+b_1d_1)}{ad_1+b_2}) = d_2^+$ and $d_1 \geq \frac{-(c+b_2d_2^+)}{ad_2^+ + b_1}$. Thus

$$\min\{\mathbf{d} \in \mathcal{D}_{\geq 0}(f) \mid 2.3 \text{ and} 2.5 \text{ holds}\} = \{(\lceil \frac{-(c+b_2d_2^+)}{ad_2^+ + b_1} \rceil, d_2^+)\}.$$

We conclude that

$$\min \mathcal{D}_{\geq 0}(f) = \begin{cases} \min\left\{\{(d, \lceil \frac{-(c+b_1d)}{ad+b_2} \rceil) \mid d_1^+ \leq d \leq \lfloor \frac{-(c+b_2d_2^+)}{ad_2^++b_1} \rfloor\} \cup \{(\lceil \frac{-(c+b_2d_2^+)}{ad_2^++b_1} \rceil, d_2^+)\}\right\} & \text{if } 2.3 \text{ holds}, \\ \{(d_1^+, d_2^+)\} & \text{otherwise.} \end{cases}$$

Clearly, this can be restated so that we have the result. □

Remark 2.6 Note that $\mathcal{D}_{\geq 0}(f)$ is an infinite set, but by Dickson's Lemma $\min \mathcal{D}_{\geq 0}(f)$ is finite. Also f is monotone, that is, if $g(x_1, x_2) = f(x_1 + d_1^+, x_2 + d_2^+)$ has positive non-constant coefficients and therefore $g(x_1 + \epsilon_1', x_2 + \epsilon_2') > g(x_1 + \epsilon_1, x_2 + \epsilon_2) > g(x_1, x_2)$ for every $(\epsilon_1', \epsilon_2') > (\epsilon_1, \epsilon_2) > 0$.

Next example illustrates how to calculate $\min \mathcal{D}_{\geq 0}(f)$ in the two variable case.

Example 2.7 Let $f = f(x_1, x_2) = 2x_1x_2 - 7x_1 - 10x_2 + 16$ and let d_1^+ and d_2^+ be as in Lemma 2.5. It is not difficult to check that $(d_1^+, d_2^+) = (6, 4)$ and

$$\begin{aligned} \min \mathcal{D}_{\geq 0}(f) &= \min\left\{\left(d, \max\left(4, \left\lceil \frac{-(16-7d)}{2d-10} \right\rceil\right)\right) \middle| d \in \mathbb{N}_+,\ 6 \leq d \leq 24\right\} \\ &= \min\left\{ \begin{array}{c} (6,13),(7,9),(8,7),(9,6),(10,6),(11,6),(12,5),(13,5),(14,5),(15,5), \\ (16,5),(17,5),(18,5),(19,5),(20,5),(21,5),(22,5),(23,5),(24,4) \end{array} \right\} \\ &= \left\{ (6,13),(7,9),(8,7),(9,6),(12,5),(24,4) \right\}. \end{aligned}$$

And therefore $\mathcal{D}(f) = \{(6, 13), (24, 4)\}$.

Now we proceed to prove that Algorithm 2.4 is correct.

Theorem 2.8 *Algorithm 2.4 computes the sets* $\min \mathcal{D}_{\geq 0}(f)$ *and* $\mathcal{D}(f)$ *for any integer multivariate irreducible square-free dominated polynomial* f.

Proof First, without loss of generality we can assume that every variable in X appears in some monomial of f and that $|X| = n$. In the case of a matrix L, induction on the size of L and the $n-1$ minors of $(\mathrm{Diag}(X + \mathbf{d}) - L)$ corresponds to induction on the degree of f and its first partial derivatives, respectively. Thus, we will proceed by induction on the number of variables in X, which is the degree of f.

If $f = f(X)$ is a square-free dominated polynomial with $|X| = 2$ and positive leading coefficient, we have that $X = \{x_1, x_2\}$ and $f = ax_1x_2 + b_1x_1 + b_2x_2 + c$ and therefore the result follows by Lemma 2.5.

Now, assume that the algorithm is correct for every number of variables up to $n-1$ and let $X = \{x_1, \ldots, x_n\}$ and f be an integer multivariate irreducible square-free dominated polynomial of degree n with positive leading coefficient. It is not difficult to check that steps (1) to (5) of Algorithm 2.4 create a set of vectors Δ, such that if $\mathbf{d} \in \Delta$, then the nonconstant coefficients of any monomial of degree at least 2 in $f(x_1 + d_1, \dots, x_n + d_n)$ are positive. Moreover, the nonconstant coefficients of any term of degree one of $f(X + \mathbf{d})$ are non-negative whereas the constant term may be negative.

If $S = \emptyset$ implies that every nonconstant coefficient of $f(X + \mathbf{d})$ is positive, see Step (7). Steps (10) - (12) and steps (15) - (17) handle the other two cases. That is, we have that all nonconstant coefficients of $f(X + \mathbf{d}^{'})$ are positive. Let $\Delta^{'}$ be the set of all of these vectors obtained at steps of Algorithm 2.4. We will prove that in steps (8)-(9), (13)-(14), (18)-(19) and (20), the algorithm increases the vectors in $\Delta^{'}$ further so that we get all the vectors in $\min \mathcal{D}_{\geq 0}(f)$. Note that if $\mathbf{d}^{'} \in \Delta^{'}$, then by the definition of $\mathcal{C}(f)$ every vector $\mathbf{u} \geq \mathbf{d}^{'}$, such that $f(X + \mathbf{u})$ has all of its noncontant coefficients positive and the constant non-negative coefficient can be reached on steps (8)-(9), (13)-(14) and (18)-(19). Therefore we only need to prove that every $\mathbf{u} \in \min \mathcal{D}_{\geq 0}(f)$ is reached by some vector in $\Delta^{'}$.

In order to prove this, for every $\mathbf{u} \in \mathcal{D}_{\geq 0}(f)$, let $\mathbf{u}_{|s}$ be the vector equal to $\mathbf{u}$ without the s-th entry. That is,

$$(\mathbf{u}_{|s})_i = \begin{cases} \mathbf{u}_i, & \text{if } 1 \leq i \leq s-1, \\ \mathbf{u}_{i+1}, & \text{if } s \leq i \leq n-1. \end{cases}$$

Then for every $s \in [n]$, we have that $\mathbf{u}_{|s} \in \mathcal{D}_{\geq 0}(\partial_s f)$ and there exists $\tilde{\mathbf{u}} \in \min \mathcal{D}_{\geq 0}(\partial_s f)$ such that $\tilde{\mathbf{u}} \leq \mathbf{u}_{|s}$. Consequently, we have that

$$\max_{s \in [n]} \left\{ (\tilde{\mathbf{u}}^{(s)})_i \right\} \leq \mathbf{u}_i,$$

where $\mathbf{u}^{(s)}$ is as in equation (2.1). In other words, every $\mathbf{u} \in \min \mathcal{D}_{\geq 0}(f)$ is greater or equal than a vector presented by step (5). Therefore let $\mathbf{u} \in \min \mathcal{D}_{\geq 0}(f)$ and let $\mathbf{d} \leq \mathbf{f}$ be such vector given at step (5). Then, assume that there is no vector $\mathbf{d}^{'} \geq \mathbf{d}$ in $\Delta^{'}$ such that $\mathbf{u} \geq \mathbf{d}^{'}$. Note that $S = \{s \mid \operatorname{coef}_{\mathbf{d}}(x_s) = 0\} \neq \emptyset$ and that any vector in $\Delta^{'}$ can not be greater or equal than $\mathbf{u}$. Thus $\mathbf{u} = \mathbf{d} + a\mathbf{e_s}$ for some $a \in \mathbb{N}_+$ and some $s \in S$, where $\mathbf{e_s} \in \mathbb{N}^n$ is the standard unit vector with its s-th entry equal to 1. Therefore $\operatorname{coef}_{\mathbf{u}}(x_s) = 0$, a contradiction since $\mathbf{u} \in \min \mathcal{D}_{\geq 0}(f)$. Concluding that there is a vector $\mathbf{d}^{'} \geq \mathbf{d}$ in $\Delta^{'}$ such that $\mathbf{u} \geq \mathbf{d}^{'}$ and therefore the algorithm computes $\min \mathcal{D}_{\geq 0}(f)$ and $\mathcal{D}(f)$. □

Next example illustrates how Algorithm 2.4 works on a polynomial which is not the determinant of a matrix with variables in the diagonal.

Example 2.9 Let $f = x_1x_2x_3 - 19x_1 + 2x_2 + 3x_3 - 23$ be the irreducible polynomial given in Example 2.13. Step (2) of Algorithm 2.4 gives us

$$\partial_1 f = x_2x_3 - 19 \quad \partial_2 f = x_1x_3 + 2 \quad \partial_3 f = x_1x_2 + 3.$$

From step (3) and Lemma 2.5 we get that $\min \mathcal{D}_{\geq 0}(\partial_2 f) = \min \mathcal{D}_{\geq 0}(\partial_3 f) = \{(1, 1)\}$ and

$$\min \mathcal{D}_{\geq 0}(\partial_1 f) = \{(1,19),(19,1),(2,10),(10,2),(3,7),(7,3),(4,5),(5,4)\}.$$

Continuing with Algorithm 2.4 we have the following set of vectors to search,

$$\Pi = \left\{ \begin{matrix} (1,1,19) & (1,2,10) & (1,3,7) & (1,4,5) \\ (1,19,1) & (1,10,2) & (1,7,3) & (1,5,4) \end{matrix} \right\}.$$

Note that $f_{\mathbf{d}}(X)$ has positive constant term for almost every vector $\mathbf{d} \in \Pi$, except for (1, 5, 4). That is, only the vector (1, 5, 4) has the chance to be an arithmetical structure on f. Indeed, since

$$f_{(1,5,4)}(X) = x_1x_2x_3 + 4x_1x_2 + 5x_1x_3 + x_2x_3 + x_1 + 6x_2 + 8x_3 + 0,$$

then $\mathcal{D}(f) = \{(1,5,4)\}$.

Next Figure illustrate the geometry of Lemma 2.5. Let P_G be the green region, which corresponds to $\mathcal{D}_{\geq 0}(f)$ since it is the portion of the $\mathbb{N}_+$-grid "above" (d_1^+, d_2^+) and such that $f \geq 0$. More precisely, $\mathcal{D}_{\geq 0}(f) = P_G \cap \mathbb{N}_+^2$. Furthermore, it is not difficult to see that if g is a polynomial of degree n, then $\mathcal{D}_{\geq 0}(g) = P \cap \mathbb{N}_+^n$ for some unbounded n-dimensional polytope P (Fig. 1).

We recall that if f is a square-free dominated polynomial without any arithmetical structure, then this does not imply (as next example shows) that $f = 0$ has not integer solutions.

Example 2.10 Let $g = x_1x_2 + 17x_1 - 12x_2 + 27$. By Lemma 2.5 we have that

$$\min \mathcal{D}_{\geq 0}(g) = \{(13,1)\}.$$

On the other hand, since $g(13,1) = 249$, then $\mathcal{D}(g) = \emptyset$. Nevertheless $g = 0$ has sixteen different solutions in $\mathbb{Z}^2$. Moreover four of them are solutions in $\mathbb{N}_+^2$, namely

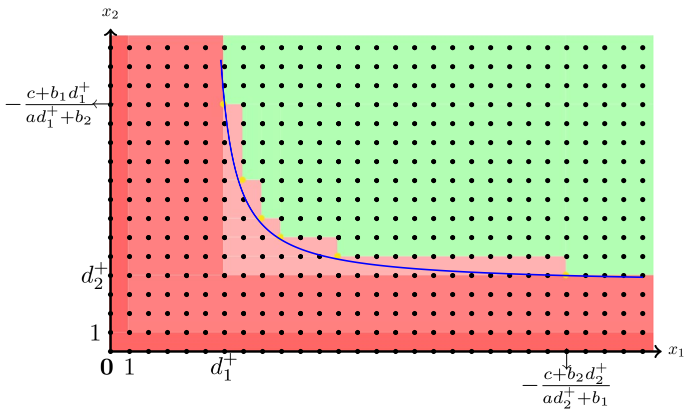

Fig. 1 The blue line represents the curve $f = 2x_1x_2 - 7x_1 - 10x_1 + 16 = 0$ for $x_1 \geq 5.8$ and the yellow points are the elements in $\min \mathcal{D}_{\geq 0}(f)$ (Color figure online)

$$\{(1,4),(5,16),(9,60),(11,214)\}.$$

None of them found by the algorithm, since the condition of having all non-constant coefficients positive is not fulfilled by any of them. For instance $f(x_1+11,x_2+214)=x_1x_2+231x_1-x_2$.

Defining an r-arithmetical structure on an integer square-free dominated polynomial is a more difficult task. On the other hand, the r-arithmetical structures on L and L^t are equal if and only if L is symmetric. Also, $f_L(X)=f_{L^t}(X)$ for any $L\in M_n(\mathbb{Z})$ because the determinant of a matrix is invariant under the transpose, that is, $\det(L)=\det(L^t)$. Moreover, if M is a matrix without rows or columns equal to zero, then $\mathcal{D}(L)=\mathcal{D}(L^t)$. That is, the polynomial $f_L(X)$ does not distinguish between L and L^t. However r-arithmetical structures on L and L^t are not equal when L is not symmetric. Therefore in general we may not try to extract the information of the r-arithmetical structures from $f_L(X)$. Next example illustrates the previous discussion.

Example 2.11 If $L=\begin{pmatrix}0&1\\3&0\end{pmatrix}$, then $f_L(x_1,x_2)=f_{L^t}(x_1,x_2)=x_1x_2-3$ and therefore

$$\mathcal{A}(L)=\{((1,3),(1,1)),((3,1),(1,3))\}\text{ and }\mathcal{A}(L^t)=\{((1,3),(3,1)),((3,1),(1,1))\}.$$

Thus $\mathcal{D}(f_L)=\{(1,3),(3,1)\}=\mathcal{D}(f_{L^t})$ and $\mathcal{R}(f_L)=\{(1,1),(1,3)\}\neq\{(1,1),(3,1)\}=\mathcal{R}(f_{L^t})$.

Remark 2.12 If f is an irreducible polynomial which is the determinant of a matrix with variables in the diagonal irreducible, then it comes from an irreducible matrix.

A symmetric Z-matrix M is an almost non-singular M-matrix with $\det(M)=0$ if and only if there exists a positive vector $\mathbf{r}$ such that

$$Adj(M)=\ |K(M)|\,\mathbf{r}^t\mathbf{r}>\mathbf{0},$$

where $\ker_{\mathbb{Q}}(M)=\langle\mathbf{r}\rangle$ and $K(M)$ is the critical group of M, see [3, Proposition 3.4]. Then it is feasible to define the order of the critical group of a d-arithmetical structure on a polynomial f as

$$|K(f,\mathbf{d})|=\gcd(\mathrm{coef}_{f_{\mathbf{d}}(X)}(x_1),\ldots,\mathrm{coef}_{f_{\mathbf{d}}(X)}(x_n)).$$

Given any non-negative matrix with zero diagonal L such that every of its rows are different from $\mathbf{0}$, then $(L\mathbf{1},\mathbf{1})$ is the canonical arithmetical structure on L. In general for integer multivariate polynomials we can not recover the concept of canonical arithmetical structure. Furthermore, some polynomials are extremal in the sense that they have very few arithmetical structures. We illustrate this idea at the next example.

Example 2.13 If $g=x_1x_2x_3-19x_1+2x_2+3x_3+b$, then

$$b = \frac{-114}{n} - n \text{ where } n \in \mathrm{Div}(114) = \pm\{1, 2, 3, 6, 19, 38, 57, 114\}.$$

Which implies that $b \in \pm\{25, 41, 59, 115\}$. It is not difficult to check by [6, Proposition 3.7] that $f(x_1, x_2, x_3) = x_1x_2x_3 - 19x_1 + 2x_2 + 3x_3 - 23$ is not the determinant of a matrix with variables in the diagonal. Evaluating, it is easy to see that $(d_1, d_2, d_3) \in \mathbb{N}_+^3$ is an arithmetical structure on f if and only if

$$d_2d_3 - 19 \geq 1 \text{ and } (d_2d_3 - 19)d_1 + 2d_2 + 3d_3 = 23.$$

Thus we have that $\mathcal{D}(f) = \{(1, 5, 4)\}$. A follow up problem would be to study this type of polynomials, where we have a single d-arithmetical structure.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author state that there is no conflict of interest.

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