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ORIGINAL ARTICLE

# On the core of *m*-attribute games

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**Abstract**

We study a special class of cooperative games with transferable utility (TU), called *m-attribute games*. Every player in an *m*-attribute game is endowed with a vector of *m* attributes that can be combined in an additive fashion; that is, if players form a coalition, the attribute vector of this coalition is obtained by adding the attributes of its members. Another fundamental feature of *m*-attribute games is that their characteristic function is defined by a continuous attribute function $\pi$—the value of a coalition depends only on evaluation of $\pi$ on the attribute vector possessed by the coalition, and not on the identity of coalition members. This class of games encompasses many well-known examples, such as queueing games and economic lot-sizing games. We believe that by studying attribute function $\pi$ and its properties, instead of specific examples of games, we are able to develop a common platform for studying different situations and obtain more general results with wider applicability. In this paper, we first show the relationship between nonemptiness of the core and identification of attribute prices that can be used to calculate core allocations. We then derive necessary and sufficient conditions under which every *m*-attribute game embedded in attribute function $\pi$ has a nonempty core, and a set of necessary and sufficient conditions that $\pi$ should satisfy for the embedded game to be convex. We also develop several sufficient conditions for nonemptiness of the core of *m*-attribute games, which are easier to check, and show how to find a core allocation when these conditions hold. Finally, we establish natural connections between TU games and *m*-attribute games.

**KEYWORDS**
convex games, cooperative game theory, the core, totally balanced games

## 1 | INTRODUCTION

In today's global markets, companies are facing growing pressure to increase their efficiencies and reduce costs. Sharing of resources is a well-known approach that simultaneously improves the performance of service and manufacturing systems and reduces costs due to economies of scale and scope. Examples of such collaborative activities are reported in many industries. For instance, the distributors of Okuma America Corporation carry minimum inventory of machine tools and spare parts. If customers order an item that is locally out of stock, the demand is satisfied either by the central stock of Okuma America or the stock of another distributor, thanks to the shared information technology system (Narus & Anderson, 1996). Similar practices are also common among the car dealers and within retail chains. In aviation, airline companies form alliances (e.g., Star Alliance, Sky Team) to improve their operations and customer service through joint mileage programs and hospitality services (e.g., airport lounges), along with spare parts sharing and code sharing. In the public health sector, health agencies share influenza vaccines to alleviate shortages and improve public health (Westerink-Duijzer et al., 2020).

A key issue for stable cooperation is appropriate incentivization of independent parties—anticipated costs and benefits should be allocated among participants in a way conducive to their long-lasting collaboration. Cooperative game theory provides a natural framework to study this type of problem and to identify allocations that encourage stable cooperative relationships among firms. Most notably, allocations that belong to the core of the underlying cooperative game discourage defection, as no subset of participants can generate a higher value on their own. In general, the core can be empty, as it may be impossible to allocate costs and benefits in a way that prevents some players from acting independently and improving their payoffs. Due to its intuitive nature,

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the core is the most commonly used stability concept in cooperative games, and it is defined formally in Section 1.1.

There are an increasing number of papers in the operations management (OM) literature focusing on cooperative games arising from several OM problems (called *OM games* hereafter) and their core. For example, in the area of production management, Owen (1975) considered cooperation among retailers with linear production (LP) function who can pool their resources to produce different products. Van den Heuvel et al. (2007) and Chen and Zhang (2016) studied cooperation in lot-sizing environments. In newsvendor setting, Müller et al. (2002) and Slikker et al. (2001) were the first to show that stable cooperation is possible if retailers share their inventory. Slikker et al. (2005), Özen et al. (2008), and Chen and Zhang (2009) extended this result to more complex newsvendor networks. Cooperation in queueing systems is studied in several more recent papers, for example, Özen et al. (2011), Anily and Haviv (2010), and Karsten et al. (2015).

One common feature seen in OM games is that every player is endowed with a vector of $m$ attributes that can be combined in an additive fashion; that is, if players form a coalition, the attribute vector of this coalition is obtained by adding the attributes of its members. Another fundamental feature is that the value of a coalition depends only on evaluation of a continuous attribute function $\pi$ on the attribute vector possessed by the coalition, and not on the identity of coalition members. For example, in LP games studied by Owen (1975), the attributes of the retailers are $m$ different types of resources owned by each retailer, and they can be pooled in an additive fashion if the retailers cooperate. The value of a coalition of retailers is expressed through the LP function that evaluates the profit achieved through an efficient use of coalition resources. We call this class of cooperative games—that is, the games defined by players with additive attribute sets and an attribute function $\pi$—*m-attribute games*. In this study, we aim to develop a common platform for analyzing $m$-attribute games by studying properties of attribute function $\pi$, and to use this platform to obtain more general results with wider applicability.

A special class of $m$-attribute games closely related to this work, called *single-attribute games*, is introduced by Özen et al. (2011). Each player in a single-attribute game is endowed with a single type of attribute, and a coalition's attribute level is given by the sum of its members' attribute levels. The value of a coalition is determined by a single-variable function, $\pi$, at its attribute level. Özen et al. (2011) showed that elasticity of attribute function $\pi$ plays an important role in single-attribute games. A function $\pi$ is *elastic* if $\frac{\pi(x_1)}{x_1} \le \frac{\pi(x_2)}{x_2}$ for all $x_1, x_2$ such that $x_1 \le x_2$. In other words, elasticity implies increasing average return with increasing attribute levels. Increasing returns can also be interpreted as increasing unit price for the attribute. Therefore, if the attribute function is elastic, compensating each player proportionally to the price and the players' attribute levels increases players' allocations as the coalition grows, making the largest coalition the most desirable one. The allocation mechanisms with this property are called population monotonic allocation schemes (PMAS), and they constitute core allocations of the underlying game and its subgames. It is then natural to study elasticity as a sufficient condition for showing nonemptiness of the core. More interestingly, Özen et al. (2011) showed that all single-attribute games that are defined by an attribute function, $\pi$, have a nonempty core if and only if $\pi$ is elastic. This result reveals some interesting characteristics of single-attribute games. First, one only needs to focus on elasticity of $\pi$ to study the core of single-attribute games defined by $\pi$. This approach was employed by Özen et al. (2011) to study the core of several games of Erlang B and C queueing systems, by Karsten et al. (2015) for $OPT^N$-M/M/s and $OPT^{\mathbb{R}}$-M/M/s games, and by Karsten and Basten (2014) for spare parts games. Next, if all single-attribute games defined by $\pi$ have a nonempty core, then the attribute function $\pi$ is elastic and, as discussed above, there exists an attribute price that can be used to create a core allocation of payoffs. This price reflects the worth of a unit attribute level in a stable coalition regardless of the identity of its owner. Hence, it can be calculated a priori, without knowing exact players or their attribute levels. The existence of increasing prices (due to elasticity), which also implies the existence of a PMAS, encourages formation of larger coalitions as everyone benefits from increased prices.

In this paper, we extend the analysis by Özen et al. (2011) to $m$-attribute games, in which each player is endowed with a vector of $m$ attributes. Our results indicate that superhomogeneity of degree one (which is the extension of elasticity to functions of $m$ variables and is defined in Section 3) of the attribute function $\pi$ is not a sufficient and necessary condition for core nonemptiness in itself. That is, we require additional conditions (modified concavity) to establish that the core is nonempty. To gain additional insights into $m$-attribute games, we derive several additional sufficient conditions for core nonemptiness of $m$-attribute games, and sufficient and necessary conditions for their convexity. Unlike single-attribute games, $m$-attribute games do not always possess a PMAS. However, we show existence of attribute prices (which we refer to as *core prices*) that can be used to derive core allocations for $m$-attribute games with a nonempty core. We also show how $m$-attribute games can be seen as a generalization of several well-known OM games, and we illustrate how core allocations obtained for those games can be seen as special cases of our results. Finally, we establish natural connections between transferable utility (TU) games and $m$-attribute games. We provide a more detailed discussion of our results at the end of this section.

## 1.1 | Game theory preliminaries

Before we explain our findings in more detail, we need to introduce some concepts and results from game theory. A *TU game* is a pair $(N, v)$, where $N$ is a finite set of agents, also called the *grand coalition*, while $v : 2^N \to \mathbb{R}$, which is called the characteristic function, is a map assigning to each subset of agents $S \subseteq N$ a real number, $v(S)$, with $v(\emptyset) = 0$. The set $S$ is called a *coalition*, and $v(S)$ can be interpreted as the profit that coalition $S$ can achieve by cooperation. A TU game $(N, v)$ is called *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for every $S, T \subset N$ such that $S \cap T = \emptyset$. If a game is superadditive, two separate coalitions can increase their total profit by cooperating. However, superadditivity does not guarantee stability of the grand coalition.

One of the common stability concepts is the core. Let $z = (z_i)_{i \in N} \in \mathbb{R}^N$ be a *payoff vector* specifying the payoff $z_i$ for each player $i$. The *core* of the game $(N, v)$ is

$$\text{Core}(N, v) = \left\{ z \in \mathbb{R}^N | \sum_{i \in N} z_i = v(N) \text{ and } \sum_{i \in S} z_i \geq v(S) \text{ for each } S \subseteq N \right\}. \quad (1)$$

The core consists of all efficient payoff vectors such that no group of players benefits by a defection from the grand coalition. We say that the grand coalition is *stable* if the core of the corresponding game is nonempty. The two central questions in the analysis of the grand coalition stability are as follows: Is the core empty, and if it is nonempty, how can we find allocations in the core? Answering these questions can be a difficult task; even checking if a specific allocation belongs to the core can be NP-hard due to the exponential number of constraints that define the core (Chen & Zhang, 2009).

Convexity gives us sufficient conditions for nonemptiness of the core. A TU game $(N, v)$ is called *convex* if for all $i \in N$ and all $S \subset T \subseteq N \setminus \{i\}$, $v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$; that is, a player's marginal contribution weakly increases as it joins a bigger coalition. Convex games have a nonempty core (Shapley, 1971).

Bondareva (1963) and Shapley (1967) studied nonemptiness of the core and independently identified a set of necessary and sufficient conditions. For every $S \subseteq N$, let us define vector $e^S \in \{0, 1\}^N$ by

$$e_i^S = \begin{cases} 1, & i \in S; \\ 0, & i \in N \setminus S. \end{cases} \quad (2)$$

A map $\kappa : 2^N \setminus \{\emptyset\} \to [0, 1]$ is called a *balanced map* if $\sum_{S \in 2^N \setminus \{\emptyset\}} \kappa(S) e^S = e^N$, and a game $(N, v)$ is called a *balanced game* if for every balanced map $\kappa : 2^N \setminus \{\emptyset\} \to [0, 1]$, it holds that $\sum_{S \in 2^N \setminus \{\emptyset\}} \kappa(S) v(S) \leq v(N)$. Bondareva (1963) and Shapley (1967) proved that a TU game has a nonempty core if and only if it is balanced. Finding a core element can be an intricate task even when the game is shown to be balanced. A TU game $(N, v)$ is *totally balanced* if it is balanced and, for each coalition $T \subset N$, the subgame $(T, v_{|T})$, defined by $v_{|T}(S) = v(S)$ for all $S \subseteq T$, is balanced as well.

It follows from the previous discussion that there are three common approaches to study core nonemptiness: (i) identifying a core allocation, (ii) checking if the game is convex,[1] and (iii) checking if the game is balanced. We illustrate each of these approaches with an example from OM games. First, Chen and Zhang (2016) studied economic lot-sizing (ELS) games with general concave ordering cost and showed how to find a core allocation using linear programming duality. This technique was first applied by Owen (1975) to find a core allocation in LP games. Second, Anily and Haviv (2010) studied cooperation in $M/M/1$ queueing systems and showed that the resulting auxiliary game is concave. Lastly, Müller et al. (2002) and Slikker et al. (2001) studied newsvendor games and showed that such games satisfy balancedness conditions regardless of the distribution of demand faced by the players. We refer to Borm et al. (2001), Fiestras-Janeiro et al. (2011), Dror and Hartman (2011), and Nagarajan and Sošić (2008) for comprehensive reviews of OM games. Next, we review the OM games related to $m$-attribute games.

## 1.2 | OM games and regular games

The coalition value in a TU game can depend on the identities of its members—coalitions with the same vector of attributes but different players can receive different payoffs. In a typical OM game, every coalition is usually identified by a vector of attributes, and the value of its characteristic function is given by evaluating a function of these attributes—an *attribute function*—for given attributes' values, regardless of the identities of coalition members. We say that such a game is *embedded* in the corresponding attribute function. When the attribute vectors of the players are additive—that is, when the attribute vector of a coalition is given by the sum of the attribute vectors of its players—these OM games fall into the class of $m$-attribute games (for a formal definition, see Definition 1 in Section 2). Anily and Haviv (2014) introduced a new class of games, which they called *regular games*. In a regular game, every player is identified by a vector of attributes, and the characteristic function assigns a value to any possible profile of vectors of which coalitions might be endowed. This setting is more conducive for analysis of the OM games. Anily and Haviv (2014) showed that a regular subadditive game that is homogeneous of degree one has a nonempty core. A game is said to be homogeneous of degree *one* if for any positive integer $k$, the characteristic function value obtained by cloning $k$ times a collection of players equals $k$ times the value of the original collection of players. However, many OM games, including the ones we focus on ($m$-attribute games), exhibit economies of scale and scope, as cooperation creates additional benefits. Therefore, many OM games are not homogeneous of degree one, and this result does not help in their analysis.

In a more recent paper, Anily (2018) studied centralizing aggregation games, a special class of regular games. An aggregation game is a regular game in which an aggregation function aggregates any profile of vectors into a vector of attributes. The game is centralizing if the characteristic function value for the aggregated vector lays between the lowest and the highest value vectors within the profile. In other words, the characteristic function acts like a measure of centrality under the aggregation function. Anily (2018) fully characterized the nonnegative core of centralizing aggregation games under a decreasing variation condition. Because adding the players' attribute levels acts as an aggregation function, the $m$-attribute games studied in this paper are a subclass of aggregation games; however, they are not centralizing. We refer to Anily and Haviv (2014) and Anily (2018) for formal definitions of regular and centralizing aggregation games.

## 1.3 | Related economics work

One paper closely related to our $m$-attribute games comes from the economics literature. Sharkey and Telser (1978) studied supportable cost functions. A cost function $c$ defined as $\mathbb{R}_+^m$ is called *supportable* if for every $\bar{x} > \mathbf{0} = (0, \dots, 0)$ there exists a price vector $p \in \mathbb{R}^m$ such that $\sum_{j=1}^m p_j x_j \le c(x)$ for all $x \in \mathbb{R}_+^m$ with $x \le \bar{x}$ (i.e., for all $\mathbf{0} \le x \le \bar{x}$) and $\sum_{j=1}^m p_j \bar{x}_j = c(\bar{x})$. Sharkey and Telser (1978) derived the necessary and sufficient conditions for a nonnegative and nondecreasing cost function $c$ with $c(\mathbf{0}) = 0$ to be supportable. One important implication of supportability is that if the characteristic function of a cooperative cost game is embedded in a supportable cost function (as it is the case in $m$-attribute games), these price vectors can be used to calculate core allocations. In this paper, we show that these conditions are also necessary and sufficient for core nonemptiness of $m$-attribute games. Note that we consider profit function $\pi$ and profit games instead of cost function $c$ and cost games. However, price vectors that can be used to derive core allocations for profit games exist if the corresponding cost function $c = -\pi$ is supportable; nonnegativity is not required for a function to be supportable. Interestingly, these results have not been considered in the OM literature, which has not used the supportability concept to date.

While deriving the sufficient and necessary conditions for supportability, Sharkey and Telser (1978) implicitly refer to a result in Telser (1978) similar to our Theorem 1. Telser (1978) studied the core of profit games with a continuum of players and showed that the core of such games is nonempty if and only if their characteristic function $V$ is kind, that is, if $-V$ is supportable. Sharkey and Telser (1978) assumed that the cost function is nondecreasing. Therefore, their model did not cover some of the OM games, such as the queueing games in Anily and Haviv (2010) and the FIX-M/M/s games in Karsten et al. (2015), which we would like to study under the $m$-attribute games framework. Moreover, Sharkey and Telser (1978) never attempted to identify the abovementioned price vectors but focused mainly on the conditions under which a function is supportable. Telser (1978) focused on a class of games with a continuum of players that is different from our setting with a finite number of players; while it is more challenging to prove results with a finite number of players, such a setting is more natural in the analysis of OM games. In this paper, we revisit results from Sharkey and Telser (1978) and provide a framework that is helpful in the analysis of OM games, including the queueing and FIX-M/M/s games.

In a recent working paper, Cao (2019) built on the results from Sharkey and Telser (1978) and studied superadditive market games. Unlike our paper, Cao (2019) assumed that cooperative functions (corresponding to our attribute functions) are nonnegative, superadditive, and, hence, nondecreasing; these conditions are violated by both the queueing and FIX-M/M/s games. He independently showed a result similar to our Theorem 1 and derived the conditions for convexity of the $m$-attribute games. However, the mathematical techniques used in Cao (2019) are significantly different from those used in our paper, hence our results are complementary to each other. Although Cao (2019) assumed increasing cooperative functions, we use continuity of attribute functions to derive our results. The new sufficiency conditions and core price vectors for the queueing games and FIX-M/M/s games are the novel results unique to our paper.

A setting closely related to our work is considered by the seminal work of Shapley and Shubik (1969). Shapley and Shubik (1969) studied a class of games called market games, which result from an exchange economy—that is, a market in which every player with a continuous and convex utility function owns a vector of divisible and transferable resources. In a market game, players can form coalitions by reallocating their resources to maximize the utility they can jointly achieve. Shapley and Shubik (1969) showed that all market games are totally balanced, and that there exists a price vector that can be used to calculate a payoff vector that is in the core. As discussed by Cao (2019), $m$-attribute games can be derived from superadditive market games in which the players' utility functions are identical and superadditive (but not necessarily convex), and hence, are not equivalent to market games. For example, unlike market games, $m$-attribute games can have an empty core. Although these games are fundamentally very different, in Theorem 1 we show a result similar to the one from Shapley and Shubik (1969); that is, the existence of core prices when the core of $m$-attribute games is nonempty. Cao (2019) referred to the set of the core prices as the Walrasian core and showed that these prices are equivalent to Walrasian equilibrium prices of an exchange economy.

Another famous result of Shapley and Shubik (1969) is that every totally balanced TU game can be reformulated as a market game. Although market games and $m$-attribute games are very different, in the final part of this paper we show that a similar relationship holds for TU games and $m$-attribute games.

## 1.4 | Results

Let us now review our main results. We start by providing a feasibility problem that can be used to check for core nonemptiness of an underlying cooperative game. More precisely, by finding a feasible solution of a semi-infinite linear program, we can find prices for all attributes (which we refer to as *attribute prices*) that can be used to calculate a core allocation of the corresponding $m$-attribute game. More interestingly, our findings also imply that establishing nonemptiness of the core is analogous to checking whether such prices exist. This result is a good starting point as it has a nice intuitive interpretation, achieved by linking attribute prices with core allocations. It can be useful for studying the core of some special classes of $m$-attribute games, as illustrated in the queueing games in Section 3.

We then derive two sets of necessary and sufficient conditions for attribute function $\pi$ that lead to nonemptiness of the core of the embedded cooperative $m$-attribute game. Both these conditions require a type of modified concavity of the attribute function $\pi$. We show that once we move from single-attribute to $m$-attribute games, an extended version of elasticity is no longer enough, and we require additional conditions (modified concavity) for establishing that the core is nonempty. We then develop conditions for convexity of all $m$-attribute games embedded in attribute function $\pi$—additivity and strongly increasing differences (for the definition, see Section 4). It is interesting to observe that strongly increasing differences imply a convex-type behavior, while our first two conditions imply a concave-type behavior. Thus, our results indicate that nonemptiness of the core is not a feature of only convex-behaving or of only concave-behaving attribute functions. In other words, attribute functions that exhibit either convex or concave type of behavior can have embedded balanced $m$-attribute games. We illustrate in Section 4 how our results can be applied to several problems from the OM literature, such as ELS games and queueing games.

Next, as it may be difficult to verify necessary and sufficient conditions for core nonemptiness, we study several sets of sufficient conditions, which might be easier to check. A common theme for these conditions is a type of generalized elasticity coupled with a type of (relaxed) concavity. Finally, we show that concavity of function $\pi$ can be relaxed to concavity in $m-1$ variables, and use this to describe how to find attribute prices that can be used to generate a core allocation for balanced $m$-attribute games. In Section 5, we illustrate the application of these sufficient conditions with some examples from OM games.

In the final part of this paper, we show that for each TU game there exists an equivalent $m$-attribute game embedded in attribute function $\pi$ such that the TU game is totally balanced if and only if all $m$-attribute games embedded in $\pi$ have a nonempty core.

The remainder of the paper is organized as follows: We introduce $m$-attribute games and provide examples of OM games that can be formulated as $m$-attribute games in Section 2. In Section 3, we provide a formulation of the optimization problem that calculates core allocations for $m$-attribute games and use this formulation to calculate a core allocation for special classes of $m$-attribute games. In Section 4, we present the necessary and sufficient conditions for core nonemptiness and convexity of $m$-attribute games. This is followed by Section 5, which considers three sets of sufficient conditions for core nonemptiness. We conclude the paper by establishing connections between TU games and $m$-attribute games in Section 6.

## 2 | *m*-ATTRIBUTE GAMES

In this section, we formally introduce $m$-attribute games and then provide some examples of OM games that fall into this class.

**Definition 1.** Let $m_1, m_2 \in \mathbb{Z}_+$. Consider a domain $\mathcal{D}^m = \mathbb{R}_+^{m_1} \times \mathbb{R}_{++}^{m_2} \cup \{\mathbf{0}\}$, where $\mathbb{R}_{++} = \{y \in \mathbb{R}, y > 0\}$, and let $\pi : \mathcal{D}^m \to \mathbb{R}$ with $\pi(\mathbf{0}) = 0$. Assume that each player $i \in N$ comes with an input vector $\bar{x}^i \in \mathcal{D}^m$. When players form a coalition, $S$, they pool their resources and are endowed with an input vector $\bar{x}^S = \sum_{i\in S} \bar{x}^i$. An *m-attribute game embedded in* $\pi$ is defined by $(N, v)$ such that

$$v(S) = \pi(\bar{x}^S) \quad \text{for all } S \subseteq N\backslash\{\emptyset\} \text{ and } v(\emptyset) = 0. \tag{3}$$

We refer to $\mathcal{D}^m$ as the *set of attribute vectors*, and we call $\pi$ the *attribute function*.

Note that $m$-attribute games can have players with zero input vector, $\mathbf{0}$. Because such players do not have any attributes, they cannot create positive return or contribute to any coalition they join. It is then natural to assume $\pi(\mathbf{0}) = 0$, which is in line with the assumption $v(\emptyset) = 0$ in TU games. Thus, players with zero input vector are dummy players and are irrelevant for the analysis.[2] We also note that $\mathcal{D}^m$ is not necessarily a closed set (when $m_2 > 0$). These features are essential for the analysis of some important OM games (e.g., queueing games) whose domains are not closed.

Next, we provide some examples of OM games that fall into this class.

### 2.1 | Economic lot-sizing games

Our first example comes from the inventory/production environment—ELS games, which were first introduced by Van den Heuvel et al. (2007) and Chen and Zhang (2016). Denote by $N$ the set of $n$ retailers, each selling the same product of a single manufacturer and solving an ELS problem to determine their ordering schedules. For each retailer $i \in N$ we assume identical revenues and costs. Let $t \in \{1, 2, \ldots, m\}$ be the index for periods in the planning horizon, with $m$ being the last period. Let $\bar{x}_t^i$ denote the demand faced by retailer $i \in N$ in period $t$. The products are sold at a price $r_t$ in period $t$. At the beginning of period $t$, the cost of ordering $q_t$ units is given by a concave function $K_t(q_t)$, with $K_t(0) = 0$. If there is inventory on hand at the end of period $t$, a holding cost of $h_t$ per unit is incurred. Backorders are not allowed. If the players form a coalition $S$, coalition $S$ chooses an ordering plan $(q_1, \ldots, q_m)$ to satisfy the demand vector $\bar{x}^S = \sum_{i\in S} \bar{x}^i$. Thus, coalition $S$ is facing the following optimization problem:

$$\pi(\bar{x}^S) = \max_{(q_1,\ldots,q_m)} \sum_{t=1}^{m} r_t \bar{x}_t^S - K_t(q_t) - h_t I_t \tag{4a}$$

$$\text{s.t. } I_t + \bar{x}_t^S = I_{t-1} + q_t, \quad t = 1, \ldots, m, \tag{4b}$$

$$q_t, I_t \geq 0, \quad t = 1, \ldots, m, \tag{4c}$$

$$I_0 = 0. \tag{4d}$$

The ELS game $(N, v)$ is then defined by $v(S) = \pi(\bar{x}^S)$ for all $S \subseteq N$. This is an $m$-attribute game in which demand in each period is an attribute, $m_1 = m$, $m_2 = 0$, and $\bar{x}^i$ is the input vector for player $i$.

## 2.2 | Queueing games

Our second example is taken from queueing games studied by Anily and Haviv (2010). Note that Anily and Haviv (2010) studied cost games, whereas we study profit games in this paper. Nevertheless, their cost game can be reformulated as a profit game and vice versa. Therefore, they are equivalent problems.

Each player $i$ in the set $N = \{1, \ldots, n\}$ manages an $M/M/1$-queueing system with arrival rate $\lambda_i$ and service rate $\mu_i$, such that $\mu_i > \lambda_i$. The players can form coalitions to take advantage of reduced congestion in the pooled system. Once a coalition $S$ is formed, it manages an $M/M/1$-queueing system with arrival rate $\lambda^S = \sum_{i \in S} \lambda_i$ and service rate $\mu^S = \sum_{i \in S} \mu_i$. Each coalition $S \subseteq N$, $S \neq \emptyset$, serves a customer stream $\lambda^S$, earns $r\lambda^S$, and pays congestion cost $p\frac{\lambda^S}{\mu^S - \lambda^S}$. If we let $\rho^S = \mu^S - \lambda^S$, the queueing game $(N, v)$ is then defined by

$$v(S) = \pi(\lambda^S, \rho^S) = r\lambda^S - p\frac{\lambda^S}{\rho^S} \quad \text{for all } S \subseteq N \text{ with } S \neq \emptyset,$$
$$\text{and } v(\emptyset) = \pi(\mathbf{0}) = 0. \tag{5}$$

The queueing game is an $m$-attribute game in which arrival rate $\lambda$ and $\rho$ are the two attributes, $m_1 = 1$, and $m_2 = 1$ because $\rho_i > 0$ for all $i \in N$. Note that the cost component $p\frac{\lambda}{\rho}$ (and hence the corresponding cost function $c = -\pi$ when $r = 0$) is decreasing with $\rho$, so the model of Sharkey and Telser (1978) does not cover these games.

## 2.3 | FIX-M/M/s games

Our third example considers a queueing game studied by Karsten et al. (2015), in which each player has multiple servers. More precisely, each player $i \in N = \{1, \ldots, n\}$ manages an $M/M/l$-queueing system with arrival rate $\lambda_i$ and $l_i$ servers, each with a service rate $\mu$, such that $l_i\mu > \lambda_i$. The players can form coalitions to reduce congestion and the waiting times of customers in the pooled system. Once a coalition $S$ is formed, it manages an $M/M/l$-queueing system with arrival rate $\lambda^S = \sum_{i \in S} \lambda_i$ and number of servers $l^S = \sum_{i \in S} l_i$. For coalition $S$ with $\lambda^S > 0$, $\mu > 0$, and $l^S \in \mathbb{R}$ with $l^S > \frac{\lambda^S}{\mu}$, the expected sojourn time of a customer is given by

$$W(l^S, \lambda^S) = \frac{C\left(l^S, \frac{\lambda^S}{\mu}\right)}{l^S\mu - \lambda} + \frac{1}{\mu}, \tag{6}$$

where

$$C(l, a) = \left( \int_0^\infty a\, e^{-ax}(1+x)^{l-1} x\, dx \right)^{-1} \text{ for each a} > 0 \text{ and}$$
$$l \in \mathbb{R} \text{ with } l > a. \tag{7}$$

Each coalition $S \subseteq N$, $S \neq \emptyset$, serves a customer stream $\lambda^S$, earns $r\lambda^S$, and pays the congestion cost $p\lambda^S W(l^S, \lambda^S)$. If we let $\rho^S = l^S - \frac{\lambda^S}{\mu}$, then FIX-M/M/s game $(N, v)$ is defined by

$$v(S) = \pi(\lambda^S, \rho^S) = r\lambda^S - p\lambda^S W\left(\rho^S + \frac{\lambda^S}{\mu}, \lambda^S\right)$$
$$\text{for all } S \subseteq N \text{ with } S \neq \emptyset, \text{ and } v(\emptyset) = \pi(\mathbf{0}) = 0. \tag{8}$$

Similar to queueing games, the FIX-M/M/s game is an $m$-attribute game in which arrival rate $\lambda$ and $\rho$ are the two attributes, $m_1 = 1$ and $m_2 = 1$ because $\rho_i = l_i - \frac{\lambda_i}{\mu} > 0$ for all $i \in N$. Note that for this class of games, the cost component $p\lambda W(\rho + \frac{\lambda}{\mu}, \lambda)$ is decreasing with $\rho$ (due to Theorem 2.2 of Karsten et al., 2015), so the model of Sharkey and Telser (1978) does not cover these games.

## 2.4 | Linear production games

Next, we consider the LP game studied in Owen (1975). Let $N = \{1, \ldots, n\}$ denote the retailers. Each retailer $i$ possesses $b_k^i$ level of resource $k \in \{1, 2, \ldots, m\}$ and can produce $p$ different products. Each unit of product $j \in \{1, 2, \ldots, p\}$ requires $a_{jk}$ units of resource $k$ and sells at a price of $c_j$. A retailer, $i$, endowed with the resource vector $b^i = \{b_1^i, \ldots, b_m^i\}$, has to determine how many units of each product to produce to maximize its profit. If several retailers form a coalition $S$, it will be endowed with $b^S = \sum_{i \in S} b^i$. The problem faced by coalition $S$ can be formulated as

$$\pi(b^S) = \max_{(x_1,\ldots,x_p)} \sum_{j=1}^{p} c_j x_j$$

$$\text{s.t.} \quad \sum_{j=1}^{p} a_{jk}x_j \leq b_k^S \qquad k = 1, \ldots, m,$$

$$x_j \geq 0 \qquad j = 1, \ldots, p. \tag{9}$$

Note that the LP game is an $m$-attribute game in which each resource type is an attribute, $m_1 = m$ and $m_2 = 0$.

### 2.5 | Newsvendor games

In the standard newsvendor game, each player $i \in \{1, \ldots, n\}$ buys the same product at a cost $c$ and sells it at a price $p$ to satisfy its discrete random demand, $X^i$, with finite expectation and finite sample space. The goal of each player, $i$, is to determine its order quantity, $Q_i$, which maximizes its expected profit. Let $\Omega$ denote the finite sample space. For a realization $\omega \in \Omega$ and order quantity $Q$, the profit of player $i$ is given by $\Pi(Q, X^i(\omega)) = p\min\{Q, X^i(\omega)\} - cQ$, hence its maximum expected profit is

$$\pi(X^i) = \max_{Q \geq 0} E[\Pi(Q, X^i(\omega))]. \tag{10}$$

If the players form a coalition $S$, the coalition faces demand $X^S$ such that $X^S(\omega) = \sum_{i\in S} X^i(\omega)$ for each $\omega \in \Omega$. The newsvendor game can then be seen as an $m$-attribute game in which $\Omega$ is the set of attributes, $m_1 = |\Omega|$, $m_2 = 0$, and each player $i$ possesses input vector $(X^i(\omega))_{\omega\in\Omega}$.

As can be seen, $m$-attribute games provide a more general setting than those used in Sharkey and Telser (1978) for studying diverse classes of OM games, and our results can be applied to different special cases. We will provide some additional examples in subsequent sections.

## 3 | DETERMINING A CORE ALLOCATION AND CORE PRICES

Determining a core allocation of a TU game might be a hard problem. In this section, we will investigate this problem for $m$-attribute games and we will look for existence of (attribute) price vectors (i.e., core prices) that can be used to calculate core allocations for each $m$-attribute game embedded in $\pi$. We will first introduce and discuss the price vectors of interest for our results. After presenting several novel concepts that will be used in the analysis, we will state our first main result in Theorem 1, showing the existence of core prices that can be used to calculate core allocations for $m$-attribute games that possess a nonempty core. In this theorem, we will introduce a semi-infinite linear problem formulation that is used to determine core prices. At the end of the section, we will illustrate how this formulation can be used to study the core of some special classes of $m$-attribute games.

Consider a function $\pi : \mathcal{D}^m \to \mathbb{R}$, $\bar{x} \in \mathcal{D}^m$ and an $m$-attribute game embedded in $\pi$ with $\sum_{i\in N} \bar{x}^i = \bar{x}$. Let $(\eta_j)_{j=1,\ldots,m} \in \mathbb{R}^m$ be a price vector, with $\eta_j$ being the unit price for attribute $j$. Suppose that $\eta$ is used to compensate each player $i \in N$ proportionally with its attribute levels, that is, player $i$ receives the payoff $z_i = \sum_{j=1}^m \bar{x}_j^i \eta_j$. The payoff vector $(z_i)_{i\in N}$ is a core allocation if $\sum_{i\in S} z_i = \sum_{j=1}^m \bar{x}_j^S \eta_j \geq \pi(\bar{x}^S) = v(S)$ for all $S \subset N$ and $\sum_{i\in N} z_i = \sum_{j=1}^m \bar{x}_j^N \eta_j = \pi(\bar{x}) = v(N)$. We are interested in price vectors that can be used to calculate core allocations proportional with players' attribute levels regardless of the specific profile of players' input vectors, that is, for all $m$-attribute games embedded in $\pi$ with $\sum_{i\in N} \bar{x}^i = \bar{x}$. As any vector $x \in \mathcal{D}^m$ with $x \leq \bar{x}$ (i.e., $x_j \leq \bar{x}_j$ for all $j = 1, \ldots, m$) can be an input vector of a coalition in one of these $m$-attribute games, a price vector $\eta$ can be used to calculate core allocations for all of these $m$-attribute games if

$$\sum_{j=1}^m x_j\eta_j \geq \pi(x) \quad \text{for all } x \in \mathcal{D}^m \text{ with } x \leq \bar{x}, \tag{11a}$$

$$\sum_{j=1}^m \bar{x}_j\eta_j = \pi(\bar{x}). \tag{11b}$$

We summarize this discussion in the following definition.

**Definition 2.** For an input vector $\bar{x} \in \mathcal{D}^m$, a price vector $(\eta_j)_{j=1,\ldots,m} \in \mathbb{R}^m$ is called a *core price vector (of $\bar{x}$)* if it satisfies (11).

We observe that any core price vector (of $\bar{x}$) can be used to calculate a core allocation for any $m$-attribute game embedded in $\pi$ with $\sum_{i\in N} \bar{x}^i = \bar{x}$.

As mentioned in the introduction, our analysis requires introduction of some novel concepts that represent generalizations of elasticity used in single-attribute games.

**Definition 3.** A function $\pi : \mathcal{D}^m \to \mathbb{R}$ is called *superhomogeneous of degree one* (or SH(1)) if for all $x \in \mathcal{D}^m$ and $0 < \alpha \leq 1$ with $\alpha x \in \mathcal{D}^m$, $\frac{\pi(\alpha x)}{\alpha} \leq \pi(x)$. $\pi$ is called *homogeneous of degree one* (or H(1)) if for all $x \in \mathcal{D}^m$ and $0 < \alpha \leq 1$ with $\alpha x \in \mathcal{D}^m$, $\frac{\pi(\alpha x)}{\alpha} = \pi(x)$. $\pi$ is called *superadditive* if for all $x, y \in \mathcal{D}^m$, $\pi(x) + \pi(y) \leq \pi(x + y)$.

Sharkey and Telser (1978) showed that a cost function is supportable only if it is SH(1). SH(1) is sometimes referred to as increasing return to scale—it implies that if multiple players with the same attribute vector cooperate, the average return increases as more players join the cooperation, which makes cooperation desirable and stable. In a single-attribute game each player is of the same type, and Özen et al. (2011) showed that elasticity (which is SH(1) for functions of one variable) is a sufficient and necessary condition for core nonemptiness. Moreover, when $m = 1$ (as in single-attribute games), the feasibility problem in (11) reduces to $\eta = \pi(\bar{x})/\bar{x}$, and $\eta \geq \pi(x)/x$ for all $0 < x \leq \bar{x}$, which naturally holds if $\pi$ is SH(1). When $m > 1$, problem (11) is more complex and SH(1) does not guarantee existence of feasible prices any more. However, SH(1) can help us to simplify problem (11). In order to develop a tighter formulation of problem (11), we need to introduce the notion of a neighboring face.

**Definition 4.** For an input vector $\bar{x} \in \mathcal{D}^m$, the $k$th *neighboring face of* $\bar{x}$ is defined as

$$F^k(\bar{x}) = \begin{cases} \{x \in \mathcal{D}^m | x_k = \bar{x}_k, x_j \le \bar{x}_j \ \ \forall j \ne k, \} & \text{if } \bar{x}_k > 0; \\ \emptyset & \text{if } \bar{x}_k = 0. \end{cases} \tag{12}$$

$F(\bar{x}) = \cup_{k \in \{1,\dots,m\}} F^k(\bar{x})$ denotes the collection of points in all neighboring faces of $\bar{x}$. A vector $x \in F(\bar{x})$ is called a *neighboring face vector* of $\bar{x}$.

Note that we set $F^k(\bar{x}) = \emptyset$ if $\bar{x}_k = 0$ because these cases can be analyzed in $m-1$ dimensions. When $x = \mathbf{0}$, constraint (11a) holds for any price vector because $\pi(\mathbf{0}) = 0$; hence, (11a) is redundant. Consider now an $x \in \mathcal{D}^m \backslash \{\mathbf{0}\}$ with $x \le \bar{x}$. If $\pi$ is SH(1), the constraint in (11a) related to $x \notin F(\bar{x})$ is dominated by another constraint because there exists a $\beta > 1$ with $\beta x \in \mathcal{D}^m$ and $\beta x \in F(\bar{x})$. Hence, all such constraints are redundant and it is enough to focus on the constraints for $x \in F(\bar{x})$, as described in our first result.

**Theorem 1.** *Suppose that* $\pi : \mathcal{D}^m \to \mathbb{R}$ *is continuous on* $\mathcal{D}^m \backslash \{\mathbf{0}\}$. *Then, all m-attribute games embedded in* $\pi$ *have a nonempty core if and only if* $\pi$ *is SH*(1) *and for all* $\bar{x} \in \mathcal{D}^m$, *there exists a core price vector* (*of* $\bar{x}$) $\eta$ *that satisfies the following set of constraints:*

$$\sum_{j=1}^{m} x_j \eta_j \ge \pi(x) \quad \textit{for all } x \in F(\bar{x}) \tag{13a}$$

$$\sum_{j=1}^{m} \bar{x}_j \eta_j = \pi(\bar{x}). \tag{13b}$$

In Theorem 1, we introduce feasibility problem (13a)–(13b), which is a tighter formulation of problem (11) under SH(1). Together with SH(1), the solution of problem (13) gives specific core prices. Theorem 1 shows that establishing core nonemptiness is analogous to establishing the existence of core prices. Although the existence of core prices directly implies nonemptiness of the core, our result also shows the opposite implication. This opposite relation is nontrivial and interesting, as the existence of core prices has not been established for some OM games (e.g., queueing games and FIX-M/M/s games) that are known to have a nonempty core. Our result also shows that the feasible set of core prices depends only on the input vector of the grand coalition and that core prices can be used to calculate core allocations regardless of players' specific profile of input vectors.

We will begin the proof by showing that, under SH(1), any core price vector satisfying (13) can be used to calculate a core allocation for any game with the grand coalition input vector $\bar{x}$. This will show the "if" part of the theorem. Showing that all $m$-attribute games embedded in $\pi$ have a nonempty core only if $\pi$ is SH(1) and if, for all $\bar{x} \in \mathcal{D}^m$, there exists a core price vector satisfying (13) is more involved. We will first introduce a semi-infinite LP problem equivalent to the feasibility problem (13) and its dual. Then, we will prove that the strong duality theorem holds for our semi-infinite linear program. Together with SH(1), the dual formulation gives us a set of necessary and sufficient conditions for existence of core prices.[3] In the proof of Theorem 2, we will show that the same set of conditions will also give us sufficient and necessary conditions for nonemptiness of the core. Despite the fact that we do not have all tools required to prove this result at the moment, we choose it as our starting point because it has a nice intuitive interpretation—it links core prices with core nonemptiness.

*Proof of Theorem* 1. In Theorem 2, we show that SH(1) is a necessary condition for all $m$-attribute games embedded in $\pi$ to have a nonempty core. Therefore, in the remainder of the proof, we focus on $\pi$ that is SH(1).

We first prove the "if" part of the theorem. Consider an $\bar{x} \in \mathcal{D}^m$ and suppose that $(\eta_j)_{j=1,\dots,m}$ is a feasible solution for $\bar{x}$. Consider an $m$-attribute game embedded in $\pi$ such that $\bar{x}^N = \bar{x}$, and coalition $S \subseteq N$ with input vector $\bar{x}^S$. Because $\bar{x}^S \le \bar{x}$, there exists an $\alpha_S \ge 1$ such that $\alpha_S \bar{x}^S \in F(\bar{x})$. Then,

$$v(S) = \pi(\bar{x}^S) \le \frac{\pi(\alpha_S \bar{x}^S)}{\alpha_S} \le \frac{\sum_{j=1}^{m} \alpha_S \bar{x}_j^S \eta_j}{\alpha_S} = \sum_{j=1}^{m} \bar{x}_j^S \eta_j = \sum_{i \in S} z_i. \tag{14}$$

The first inequality holds because $\pi$ is SH(1). The second inequality follows because $\eta$ represents a feasible solution for (13) and, hence, it satisfies (13a) if $\alpha_S \bar{x}^S \ne \bar{x}$, and it satisfies (13b) if $\alpha_S \bar{x}^S = \bar{x}$.

It remains to show the "only if" part. Suppose that all $m$-attribute games embedded in $\pi$ have a nonempty core. Due to Lemma 1, which is presented in the Supporting Information Appendix, we can restrict our attention to $\bar{x} \in \mathcal{D}^m$ with $\bar{x}_k > 0$ for all $k = 1, \dots, m$ without loss of generality.

Consider an $\bar{x} \in \mathcal{D}^m$ with $\bar{x}_k > 0$ for all $k = 1, \dots, m$. The "only if" statement will follow from the following observations. The feasibility problem (13) has a feasible solution if and only if the following problem has a solution achieving minimum that is equal to $\pi(\bar{x})$:

$$P: \quad \inf_{(\eta_1,\dots,\eta_m)} \sum_{j=1}^{m} \left(\bar{x}_j \eta_j\right) \tag{15a}$$

$$\text{s.t.} \quad \sum_{j=1}^{m} x_j \eta_j \ge \pi(x) \quad \text{for all } x \in F(\bar{x}). \tag{15b}$$

The dual for problem ($P$) can be written as

$$D: \quad \sup_{(a_x)_{x \in F(\bar{x})}} \sum_{x \in F(\bar{x})} (\pi(x) a_x) \tag{16a}$$

$$\text{s.t.} \sum_{x \in F(\bar{x})} x_j a_x = \bar{x}_j \qquad \text{for all } j = 1, \ldots, m \tag{16b}$$

$$a_x \geq 0 \qquad \text{for all } x \in F(\bar{x}). \tag{16c}$$

We remark that there is a finite number of decision variables and an infinite number of constraints in problem (15). This special type of linear problem is known as a semi-infinite linear problem (Charnes et al., 1963), and the duality result known for its finite counterparts does not directly extend to semi-infinite linear problems (Duffin & Karlovitz, 1965, provided a counterexample[4]). The following conditions guarantee that the primal and dual optimal solutions exist and a strong duality result holds (Corollary 12.3 in Faigle et al., 2002):

- $\pi$ is a continuous function;
- $F(\bar{x})$ is a compact set (i.e., closed and bounded);
- Slater constraint qualification for $P$ (hereafter denoted as $SCQ_P$ for brevity) holds: ($SCQ_P$) there exists a vector $(\eta_j')_{j=1,\ldots,m}$ such that $\sum_{j=1}^{m} \eta_j' x_j > \pi(x)$ for all $x \in F(\bar{x})$;
- dual Slater condition (hereafter denoted as $SC_D$ for brevity) holds: ($SC_D$) $\bar{x} \in int\ cone\{x | x \in F(\bar{x})\}$.

We remark that $F(\bar{x})$ is a compact set when $\mathcal{D}^m = \mathbb{R}_+^m$. However, it is not closed and hence not compact when $\mathcal{D}^m = \mathbb{R}_+^{m_1} \times \mathbb{R}_{++}^{m_2}$ with $m_2 > 0$. Next, we study these two cases separately.

To start, suppose that $m_2 = 0$ and, hence, $\mathcal{D}^m = \mathbb{R}_+^m$. Let $\bar{x} \in \mathcal{D}^m$ with $\bar{x}_k > 0$ for all $k = 1, \ldots, m$. Because $\pi$ is a continuous function, we will show that $F(\bar{x})$ is a compact set, and $SCQ_P$ and $SC_D$ are satisfied by (15) and (16).

First, consider $F(\bar{x})$. Because $\mathcal{D}^m = \mathbb{R}_+^m$ and $\bar{x}_k > 0$ for all $k = 1, \ldots, m$, the $k$th neighboring face can be restated as $F^k(\bar{x}) = \{x \in \mathbb{R}^m | x_k = \bar{x}_k, 0 \leq x_j \leq \bar{x}_j \ \forall j \neq k\}$, which is closed and bounded. Therefore, $F(\bar{x}) = \cup_{k \in \{1,\ldots,m\}} F^k(\bar{x})$ is closed and bounded as well.

Next, we consider $SCQ_P$. Because $F(\bar{x})$ is closed and bounded, and $\pi$ is continuous and finite in $\mathcal{D}^m$, $\max_{x \in F(\bar{x})} \pi(x)$ is well defined. Note that for each $x \in F(\bar{x})$ there exists a $j$ such that $x_j = \bar{x}_j$. As a result, for any $(\eta_j')_{j=1,\ldots,m}$ such that $\eta_j' > \max_{x \in F(\bar{x})}(\pi(x))/\bar{x}_j$, it holds that $\sum_{j=1}^{m} \eta_j' x_j > \pi(x)$ for all $x \in F(\bar{x})$. Therefore, $SCQ_P$ is satisfied. Finally, because $F(\bar{x})$ contains points other than $\bar{x}$, $SC_D$ is satisfied and the strong duality theorem holds for our semi-infinite linear program.

To conclude, when $\mathcal{D}^m = \mathbb{R}_+^m$, problem (16) (and problem (15), due to the strong duality theorem) has a feasible solution with an objective function value equal to $\pi(\bar{x})$ if and only if attribute function $\pi$ has the following property: $\sum_{x \in CF(\bar{x})} a_x \pi(x) \leq \pi(\bar{x})$ for each $(a_x)_{x \in F(\bar{x})}$ satisfying constraints (16b) and (16c). We refer to this property as constructive face concavity (see Section 4 for a formal definition). In Theorem 2 we will show that SH(1) and constructive face concavity are necessary and sufficient conditions for nonemptiness of the core.

The proof for the case $m_2 > 0$ follows similar steps; we present it in the Supporting Information Appendix. □

The main difficulty in solving problem (13) is that it contains an infinite number of constraints. However, this formulation can be useful for studying the core of some special classes of $m$-attribute games, as illustrated in the following example.

## 3.1 | Queueing games

Consider the queueing game $(N, v)$ defined by $v(S) = \pi(\lambda^S, \rho^S)$ for all $S \subseteq N$, with attribute function $\pi$ satisfying SH(1). When $\lambda^N = 0$, it is a null game with $v(S) = 0$ for all $S \subseteq N$, and each player receives zero payoff. When $\lambda^N > 0$, a price vector $\eta \in \mathbb{R}^2$, which describes a core allocation, should satisfy the following set of constraints:

$$\lambda\eta_1 + \rho^N\eta_2 \geq \pi(\lambda, \rho^N) \quad \text{for all } 0 \leq \lambda < \lambda^N, \tag{17a}$$

$$\lambda^N\eta_1 + \rho\eta_2 \geq \pi(\lambda^N, \rho) \quad \text{for all } 0 < \rho < \rho^N, \tag{17b}$$

$$\lambda^N\eta_1 + \rho^N\eta_2 = \pi(\lambda^N, \rho^N). \tag{17c}$$

It follows from (17c) that $\lambda\eta_1 + \rho^N\eta_2 = \pi(\lambda^N, \rho^N) - \eta_1(\lambda^N - \lambda)$. Together with (17a), this implies that for all $0 \leq \lambda \leq \lambda^N$,

$$\begin{aligned} &\pi(\lambda^N, \rho^N) - \eta_1(\lambda^N - \lambda) \geq \pi(\lambda, \rho^N) \Rightarrow \eta_1 \\ &\leq \frac{\pi(\lambda^N, \rho^N) - \pi(\lambda, \rho^N)}{\lambda^N - \lambda} = \frac{r\lambda^N - p\frac{\lambda^N}{\rho^N} - r\lambda + p\frac{\lambda}{\rho^N}}{\lambda^N - \lambda} \\ &= r - p\frac{1}{\rho^N}, \end{aligned} \tag{18}$$

where the first equality follows from (5). Therefore, constraint (17a) is satisfied if and only if

$$\eta_1 \leq r - p\frac{1}{\rho^N}. \tag{19}$$

Similarly, it follows from (17c) that $\lambda^N\eta_1 + \rho\eta_2 = \pi(\lambda^N, \rho^N) - \eta_2(\rho^N - \rho)$, which together with (17b) implies that for all $0 < \rho \leq \rho^N$,

$$\begin{aligned} &\pi(\lambda^N, \rho^N) - \eta_2(\rho^N - \rho) \geq \pi(\lambda^N, \rho) \Rightarrow \eta_2 \\ &\leq \frac{\pi(\lambda^N, \rho^N) - \pi(\lambda^N, \rho)}{\rho}^N - \rho \leq p\frac{\lambda^N}{\rho}\rho^N, \end{aligned} \tag{20}$$

where the last inequality follows from (5). Because (20) should hold for all $0 < \rho \leq \rho^N$, it holds that $\eta_2 \leq p\frac{\lambda^N}{(\rho^N)^2}$. Now, (17c) and (19) imply that

$$\eta_1 = \frac{\pi(\lambda^N, \rho^N) - \eta_2 \rho^N}{\lambda^N} = \frac{r\lambda^N}{\lambda^N} - p\frac{\lambda^N}{\lambda^N \rho^N} - \frac{\eta_2 \rho^N}{\lambda^N} \Rightarrow$$

$$r - p\frac{1}{\rho^N} - \frac{\eta_2 \rho^N}{\lambda^N} \leq r - p\frac{1}{\rho^N}. \tag{21}$$

As $\lambda^N, \rho^N > 0$, it directly implies that $\eta_2 \geq 0$. Therefore, every price vector $\eta \in \mathbb{R}^2$ such that $0 \leq \eta_2 \leq p\frac{\lambda^N}{(\rho^N)^2}$ and $\eta_1 = \frac{\pi(\lambda^N, \rho^N) - \eta_2 \rho^N}{\lambda^N}$ describes a core allocation. We remark that this range covers all price vectors $\eta \in \mathbb{R}^2$ that can describe a core allocation for all queueing games embedded in $\pi$ with input vector $(\lambda^N, \rho^N)$ assigned to the grand coalition.

## 4 | NECESSARY AND SUFFICIENT CONDITIONS FOR NONEMPTINESS OF THE CORE

In this section, we will first provide sufficient and necessary conditions for core nonemptiness of every *m*-attribute game embedded in attribute function $\pi$. These conditions coincide with the sufficient and necessary conditions for existence of core prices in the proof of Theorem 1. After presenting alternative sufficient and necessary conditions for core nonemptiness, we will derive sufficient and necessary conditions for convexity of every *m*-attribute game embedded in $\pi$. Before we introduce our first result in this section, we need to introduce the concepts of (minimal) constructive face collection and constructive face concavity, which play an important role in our analysis.

**Definition 5.** A finite set $CF(\bar{x}) \subset F(\bar{x})$ is called a *constructive face collection* of $\bar{x}$ if there exist $a_x > 0$ for all $x \in CF(\bar{x})$ such that $\sum_{x \in CF(\bar{x})} a_x x = \bar{x}$. Vector $(a_x)_{x \in CF(\bar{x})}$ is referred to as the *weights* of constructive face collection $CF(\bar{x})$. A constructive face collection $CF(\bar{x})$ is called *minimal* if there does not exist another constructive face collections $CF'(\bar{x})$ such that $CF'(\bar{x}) \subset CF(\bar{x})$.

**Definition 6.** A function $\pi : \mathcal{D}^m \to \mathbb{R}$ satisfies *constructive face concavity* if for any $\bar{x} \in \mathcal{D}^m$ and any of its constructive face collections[5] $CF(\bar{x})$, $\sum_{x \in CF(\bar{x})} a_x \pi(x) \leq \pi(\bar{x})$.

As for each constructive face collection of $\bar{x}$ we have $\sum_{x \in CF(\bar{x})} a_x x = \bar{x}$, constructive face concavity requires that $\pi$ satisfies a modified concave inequality, $\sum_{x \in CF(\bar{x})} a_x \pi(x) \leq \pi(\sum_{x \in CF(\bar{x})} a_x x)$, when applied to points in any constructive face collection of any point in the domain with their corresponding weights.

**Theorem 2.** *Suppose that $\pi : \mathcal{D}^m \to \mathbb{R}$ is continuous on $\mathcal{D}^m \backslash \{\mathbf{0}\}$. All m-attribute games embedded in $\pi$ have a nonempty core if and only if $\pi$ is SH(1) and satisfies constructive face concavity.*

There are several important messages that we can derive from Theorem 2. First, SH(1) implies an increasing average return along the line of every input vector crossing the origin. In other words, the average return increases with the number of symmetric players joining the cooperation. This condition is an extension of the elasticity condition in single-attribute games. One direct consequence of the elasticity condition for single-attribute games is that the set of totally balanced games and the set of games containing a PMAS coincide. Therefore, all single-attribute games embedded in $\pi$ are (totally) balanced if and only if they all contain a PMAS. When we consider more than one attribute, Theorem 2 reveals that the increasing average return is still a necessary condition for nonemptiness of the core, but it is no longer sufficient—we now require constructive face concavity as well. In addition, for *m*-attribute games the set of totally balanced games and the set of games containing a PMAS do not necessarily coincide (see Example 4.3 in Karsten et al., 2015, for a counterexample). Nonetheless, there still exist core prices for *m*-attribute games with a nonempty core, as we prove in Theorem 1. Finally, Theorem 2 implies that attribute functions defined by (4a) and (5) are SH(1) and that they satisfy constructive face concavity, as ELS games and queueing games are shown to have a nonempty core by Van den Heuvel et al. (2007) and Anily and Haviv (2010), respectively.

One may ask whether constructive face concavity and SH(1) are independent properties. It is easy to see that SH(1) does not imply constructive face concavity; however, the reverse is not clear. This is an open question that requires further analysis.

Although the "if" part of Theorem 2 follows directly from the proof of Theorem 1, proving the "only if" part is more involved. We will first show that if $\pi$ is not SH(1), there exists an *m*-attribute game whose balancedness condition requires SH(1). We will create this game by using symmetric players having the same input vector. We will then show that if the constructive face concavity is violated, there exists an *m*-attribute game whose balancedness condition requires constructive face concavity. To create this game, we will use *m* different types of players, each having only one attribute type.

*Proof of Theorem* 2. The "if" part of the theorem follows directly from the proof of Theorem 1. In this proof, we show that if $\pi$ is SH(1) and satisfies constructive face concavity, there exists a price vector (a feasible solution of problem (13)) for each $\bar{x} \in \mathcal{D}^m$ that can be used to calculate a core allocation, and hence, all *m*-attribute games embedded in $\pi$ have a nonempty core.

Next, we show the "only if" part by identifying *m*-attribute games with an empty core when any of the abovementioned statements are not satisfied.

Suppose first that $\pi$ is not SH(1). Then, there exist $\bar{x} \in \mathcal{D}^m$ and $\alpha < 1$ such that $\frac{\pi(\alpha\bar{x})}{\alpha} > \pi(\bar{x})$. If $\alpha \in \mathbb{Q}_{++} = \mathbb{Q}_+ \setminus \{0\}$ with $p, q \in \mathbf{N}$. Let $y = 1/q$. We can construct an $m$-attribute game $(N, v)$ embedded in $\pi$ with $|N| = q$ wherein each player $i \in N$ has an input vector $z_i = y\bar{x}$. Let $\mathbf{S}^p$ denote the set of all coalitions with $p$ players. For each $S \in \mathbf{S}^p$ it then holds that $\sum_{i \in S} z_i = py\bar{x}$. Note that every coalition $S \in \mathbf{S}^p$ contains $p$ players, there are $\binom{q}{p}$ coalitions in $\mathbf{S}^p$, and every player $i \in N$ appears in $\binom{q-1}{p-1}$ coalitions with $p$ players.

Consider the map $\kappa$ such that

$$\kappa(S) = \begin{cases} \dfrac{1}{\binom{q-1}{p-1}}, & \text{if } |S| = p, \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

As every player $i \in N$ appears in $\binom{q-1}{p-1}$ coalitions with $p$ players, $\kappa$ is a balanced map. The following relation shows that a balancedness condition is not satisfied, and the core of $(N, v)$ is empty:

$$\begin{aligned} \sum_{S \subseteq N} \kappa(S) v(S) &= \sum_{S \in \mathbf{S}^p} \kappa(S) \pi(py\bar{x}) = \sum_{S \in \mathbf{S}^p} \frac{1}{\binom{q-1}{p-1}} \pi(py\bar{x}) \\ &= \frac{\binom{q}{p}}{\binom{q-1}{p-1}} \pi(py\bar{x}) = \frac{q}{p} \pi(py\bar{x}) \\ &= \frac{\pi(\alpha\bar{x})}{\alpha} > \pi(\bar{x}) = v(N). \end{aligned} \tag{23}$$

If $\alpha \in \mathbb{R}_{++} \setminus \mathbb{Q}_{++}$, there exists an $\epsilon > 0$ such that $\alpha + \epsilon \in \mathbb{Q}_{++}$ and $\frac{\pi((\alpha+\epsilon)\bar{x})}{\alpha+\epsilon} > \pi(\bar{x})$. Repeating the abovementioned arguments, there exists an $m$-attribute game embedded in $\pi$ that has an empty core.

Now, suppose that $\pi$ does not satisfy constructive face concavity. Assume that there exist an $\bar{x} \in \mathcal{D}^m$ and a (minimal) constructive face collection $CF(\bar{x})$ such that $\sum_{x \in CF(\bar{x})} a_x \pi(x) > \pi(\bar{x})$.

If $\bar{x} \in \mathbb{Q}_+^m$, there exist $p_j, q_j, p_j^x, q_j^x \in \mathbf{N}$ such that $\bar{x}_j = p_j/q_j$ and $x_j = p_j^x/q_j^x$. Let $y_j = 1/(q_j q_j^x), d_j = p_j q_j^x, d_j^x = p_j^x q_j$ for $j = 1, \ldots, m$ and for all $x \in CF(\bar{x})$.

- Suppose first that $m_2 = 0$ and hence $\mathcal{D}^m = \mathbb{R}_+^m$. We can then construct a game with $m$ different types of players such that there are $d_j$ players of type $j \in \{1, \ldots, m\}$ holding $y_j$ resources of type $j$ only. We remark that when $m_2 > 0$, some player types are not allowed by the definition of $m$-attribute games because $z_i \in \mathcal{D}^m$ for all $i \in N$. We will study this case separately. Note that for each $x \in CF(\bar{x})$ and $j = 1, \ldots, m$, $\frac{d_j^x}{d_j} = \frac{x_j}{\bar{x}_j}$.

Let $\bar{N}$ and $\bar{N}^j$ denote the set of all players and the set of all type $j$ players, respectively. Let $e^k \in \mathbb{R}^m$ denote the vector such that

$$e_j^k = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{otherwise,} \end{cases} \tag{24}$$

and let $z_i$ denote the resource vector of player $i \in \bar{N}$. If $i$ is a type $j$ player, then $z_i = y_j e^j$.
Let $\mathbf{S}^x$ denote the set of coalitions $S$ with $\sum_{i \in S} z_i = x$. Note that every coalition $S \in \mathbf{S}^x$ contains $d_j^x$ type $j$ players and $|\mathbf{S}^x| = \Pi_{j=1}^m \binom{d_j}{d_j^x}$. Consider $(a_x)_{x \in CF(\bar{x})}$ such that $\sum_{x \in CF(\bar{x})} a_x x = \bar{x}$, which exists because $CF(\bar{x})$ is a constructive face collection. Then,

$$\sum_{x \in CF(\bar{x})} a_x \frac{d_j^x}{d_j} = 1 \quad \text{for all } j = 1, \ldots, m. \tag{25}$$

Consider the map $\kappa$ such that

$$\kappa(S) = \begin{cases} \dfrac{a_x}{\Pi_{j=1}^m \binom{d_j}{d_j^x}}, & \text{if } S \in \mathbf{S}^x, \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

We will show that $\kappa$ is a balanced map. Consider a player $\bar{j}$ of type $j$. This player would appear $\Pi_{i \neq j} \binom{d_i}{d_i^x} \times \binom{d_j - 1}{d_j^x - 1}$ times in coalitions in $\mathbf{S}^x$. Then,

$$\begin{aligned} \sum_{x \in CF(\bar{x})} \sum_{S \in \mathbf{S}^x | \bar{j} \in S} \kappa(S) &= \sum_{x \in CF(\bar{x})} \sum_{S \in \mathbf{S}^x | \bar{j} \in S} \frac{a_x}{\Pi_{j=1}^m \binom{d_j}{d_j^x}} \\ &= \sum_{x \in CF(\bar{x})} \Pi_{i \neq j} \binom{d_i}{d_i^x} \times \binom{d_j - 1}{d_j^x - 1} \frac{a_x}{\Pi_{j=1}^m \binom{d_j}{d_j^x}} \\ &= \sum_{x \in CF(\bar{x})} a_x \frac{d_j^x}{d_j} = 1. \end{aligned} \tag{27}$$

The last equality holds by (25), hence $\kappa$ is a balanced map. Then,

$$\begin{aligned} \sum_{x \in CF(\bar{x})} \sum_{S \in \mathbf{S}^x} \kappa(S) v(S) &= \sum_{x \in CF(\bar{x})} \sum_{S \in \mathbf{S}^x} \kappa(S) \pi(x) \\ &= \sum_{x \in CF(\bar{x})} \sum_{S \in \mathbf{S}^x} \frac{a_x}{\Pi_{j=1}^m \binom{d_j}{d_j^x}} \pi(x) \\ &= \sum_{x \in CF(\bar{x})} a_x \pi(x) > \pi(\bar{x}) = v(\bar{N}), \end{aligned} \tag{28}$$

therefore a balancedness condition is not satisfied and the core of the game $(\bar{N}, v)$ is empty.

- Now, assume that $m_2 > 0$ and $\mathcal{D}^m = \mathbb{R}_+^{m_1} \times \mathbb{R}_{++}^{m_2} \cup \{\mathbf{0}\}$. Consider $m$ different types of players as defined above and recall that some player types are not allowed by the definition of $m$-attribute games because $m_2 > 0$. We will construct a related game with an empty core. Let $e \in \mathbb{R}^m$ denote the unit vector. Then, there exists an $\epsilon > 0$ such that $x + \epsilon \sum_{j=1}^m d_j^x e \in \mathbb{Q}_{++}$. This statement holds because $\pi$ is continuous. Note that, for all $x \in CF(\bar{x})$, $x + \epsilon \sum_{j=1}^m d_j^x e \leq \bar{x} + \epsilon \sum_{j=1}^m d_j e$ because $d_j^x \leq d_j$ for all $j = 1, \ldots, m$.
We can construct the related game with $m$ different types of players as follows. There are $d_j$ players of type $j \in \{1, \ldots, m\}$ holding resource vector $y_j e^j + \epsilon e$, that is, $z_i = y_j e^j + \epsilon e$ if $i \in N$ is a type $j$ player. Let $\tilde{x} = x + \epsilon \sum_{j=1}^m d_j^x e$ and $\mathbf{S}^{\tilde{x}}$ denote the set of coalitions $S$ with $\sum_{i \in S} z_i = \tilde{x}$. Every coalition $S \in \mathbf{S}^{\tilde{x}}$ contains $d_j^x$ type $j$ players. Replacing $\bar{x}$ and $\{x\}_{x \in CF(\bar{x})}$ as arguments of $\pi$ by $\bar{x} + \epsilon \sum_{j=1}^m d_j e$ and $\{\tilde{x}\}_{x \in CF(\bar{x})}$, and $\{\mathbf{S}^x\}_{x \in CF(\bar{x})}$ by $\{\mathbf{S}^{\tilde{x}}\}_{x \in CF(\bar{x})}$ while leaving $a_x$, $d_j^x$, $\mathbf{S}^x$ and $d_j$ unchanged, the same arguments as above hold, hence a balancedness condition is not satisfied and the core of the related game is empty. This completes the proof of "only if" statement under the assumption that $\bar{x} \in \mathbb{Q}_+^m$. We provide the proof for the real numbers in the Supporting Information Appendix. □

We note that it might be hard to study constructive face concavity for attribute function $\pi$ resulting from an optimization problem (e.g., ELS games). Before we introduce an alternative necessary and sufficient condition for core nonemptiness, we need the following definition.

**Definition 7.** Let $\vee_{j=1,\ldots,\ell} x^j$ denote the component-wise maximum of $x^1, \ldots, x^\ell$, $(\vee_{j=1,\ldots,\ell} x^j)_l = \max\{x_l^1, \ldots, x_l^\ell\}$. A function $\pi : \mathcal{D}^m \to \mathbb{R}$ satisfies *max-concavity* if for all $\ell \geq 1$, $x^1, \ldots, x^\ell \in \mathcal{D}^m$ and $a_1, \ldots, a_\ell \in \mathbb{R}_{++}$ such that $\vee_{j=1,\ldots,\ell} x^j \leq \sum_{j=1}^\ell a_j x^j$, $\sum_{j=1}^\ell a_j \pi(x^j) \leq \pi(\sum_{j=1}^\ell a_j x^j)$.

Max-concavity can be described as follows: For any positive linear combination of points from the domain such that this linear combination has larger elements than any of the original points, the function value at the linear combination is greater than the linear combination of the function values at the original points. The following proposition states compact and equivalent necessary and sufficient conditions.

**Theorem 3.** *Suppose that $\pi : \mathcal{D}^m \to \mathbb{R}$ is continuous on $\mathcal{D}^m \backslash \{\mathbf{0}\}$. All m-attribute games embedded in $\pi$ have a nonempty core if and only if $\pi$ satisfies max-concavity.*

Although max-concavity is not easier to analyze than SH(1) and constructive face concavity, we will use it to derive sufficient conditions for nonemptiness of the core of $m$-attribute games in the next section. Theorem 3 implies that attribute functions defined by (4a) and (5) are max-concave.

*Proof of Theorem* 3. Suppose that $\pi$ satisfies max-concavity and consider the case $\ell = 1$. Then, $a_1 \geq 1$ and $\pi$ is SH(1). Moreover, for all $\bar{x} \in \mathcal{D}^m$ and constructive face collection $CF(\bar{x})$, it holds that $x \leq \bar{x}$ for all $x \in CF(\bar{x})$ and therefore $\vee_{x \in CF(\bar{x})} x \leq \bar{x} = \sum_{x \in CF(\bar{x})} a_x x$, hence constructive face concavity follows from max-concavity. Consequently, if $\pi$ satisfies max-concavity, then $\pi$ is SH(1) and satisfies constructive face concavity.

Suppose that $\pi$ is SH(1) and satisfies constructive face concavity. Consider $\ell \geq 1$, $\bar{x}^1, \ldots, \bar{x}^\ell \in \mathcal{D}^m$ and $a_1, \ldots, a_\ell \in \mathbb{R}_{++}$ such that $\vee_{j=1,\ldots,\ell} \bar{x}^j \leq \sum_{j=1}^\ell a_j \bar{x}^j \in \mathcal{D}^m$. Let $F(\bar{x})$ be the collection of points in all neighboring faces of $\bar{x} = \sum_{j=1}^\ell a_j \bar{x}^j$. There exists $s_j \geq 1$ such that $s_j \bar{x}^j \in F(\bar{x})$ for all $j \in \{1, \ldots, \ell\}$. Hence, $\{s_j \bar{x}^j\}_{j \in \{1,\ldots,\ell\}}$ is a constructive face collection of $\bar{x}$ with weights $a_{s_j \bar{x}^j} = \frac{a_j}{s_j}$. Then, $\sum_{j=1}^\ell a_j \pi(\bar{x}^j) \leq \sum_{j=1}^\ell \frac{a_j}{s_j} \pi(s_j \bar{x}^j) \leq \pi(\sum_{j=1}^\ell \frac{a_j}{s_j} s_j \bar{x}^j)$. The first inequality holds because $\pi$ is SH(1), and the second inequality holds because $\pi$ satisfies constructive face concavity. □

An important subclass of totally balanced games are convex games. Before we present sufficient and necessary conditions for convexity of $m$-attribute games, we need to introduce additional terminology.

**Definition 8.** For a function $\pi : \mathcal{D}^m \to \mathbb{R}$, any pair of indices $j, k \in \{1, \ldots, m\}$ and vector $x_{-jk} = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m) \in \mathcal{D}^{m-2}$, define $\pi_{x_{-jk}}(x_j, x_k) = \pi(x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_m)$.

We say that $\pi : \mathcal{D}^m \to \mathbb{R}$ has *increasing differences*if for any pair of distinct indices $j, k \in \{1, \ldots, m\}$, any $x_{-jk} \in \mathcal{D}^{m-2}$, and any $\varepsilon, \delta > 0$, we have

$$\begin{aligned} &\pi_{x_{-jk}}(x_j + \varepsilon, x_k + \delta) - \pi_{x_{-jk}}(x_j + \varepsilon, x_k) \\ &\quad \geq \pi_{x_{-jk}}(x_j, x_k + \delta) - \pi_{x_{-jk}}(x_j, x_k). \end{aligned} \tag{29}$$

We say that $\pi : \mathcal{D}^m \to \mathbb{R}$ has *strongly increasing differences* if, for any $x, \epsilon^1, \epsilon^2 \in \mathcal{D}^m$,

$$\pi(x + \epsilon^1 + \epsilon^2) - \pi(x + \epsilon^2) \geq \pi(x + \epsilon^1) - \pi(x). \tag{30}$$

Supermodularity of an $m$-variable function requires that the function has increasing differences. The functions with strongly increasing differences are also referred to as ultramodular; see Marinacci and Montrucchio (2005) for a detailed discussion of ultamodular functions. Strongly increasing differences impose stronger conditions than supermodularity, as (29) can be obtained as a special case of (30) in which $\epsilon^1 = \varepsilon e^j, \epsilon^2 = \delta e^k, j < k \in \{1, \ldots, m\}$, $e^j, e^k \in \mathbb{R}^m$

with

$$e_l^j = \begin{cases} 1, & j = l, \\ 0, & \text{otherwise.} \end{cases} \tag{31}$$

A function with strongly increasing differences is convex in all dimensions, but it may not satisfy joint convexity; similarly, a convex function may not have strongly increasing differences. In Proposition 8, Sharkey and Telser (1978) showed that strongly increasing differences are sufficient for supportability. In the following proposition, we show that these conditions are also sufficient and necessary for the embedded games to be convex.

**Proposition 1.** *All m-attribute games embedded in an attribute function $\pi : \mathcal{D}^m \to \mathbb{R}$ are convex if and only if $\pi$ has strongly increasing differences.*

The proposition shows that strongly increasing differences are the necessary and sufficient condition for the resulting $m$-attribute games to be convex. Note that strongly increasing differences imply superadditivity, which is a necessary condition for a TU game to be totally balanced, because $\pi(\mathbf{0}) = 0$. Compared with constructive face concavity, this condition is more in line with convex behavior, and it is easier to check.

*Proof of Proposition* 1. The convexity condition for $m$-attribute games can be restated as follows: An $m$-attribute game is convex if and only if for all $i \in N$ and $S \subseteq T \subseteq N \setminus \{i\}$:

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S) \iff \pi\left(x^{T\cup\{i\}}\right) - \pi\left(x^T\right) \geq \pi\left(x^{S\cup\{i\}}\right) - \pi\left(x^S\right). \tag{32}$$

We first show the "only if" part by identifying a nonconvex $m$-attribute game in a setting in which the statements above are not satisfied. If $\pi$ does not have strongly increasing differences, then there exist $\epsilon^1, \epsilon^2 \in \mathbb{R}_+^m$ such that

$$\pi(x + \epsilon^2 + \epsilon^1) - \pi(x + \epsilon^2) < \pi(x + \epsilon^1) - \pi(x). \tag{33}$$

Consider an $m$-attribute game with three players, $N = \{1, 2, 3\}$. Suppose that the players' attribute levels are $x_1 = x$, $x_2 = \epsilon^1$ and $x_3 = \epsilon^2$. Then, for $T = \{1, 2\}$, $S = \{1\}$ and $i = 3$, it holds that $x^{T\cup\{i\}} = x + \epsilon^1 + \epsilon^2$, $x^T = x + \epsilon^1$, $x^{S\cup\{i\}} = x + \epsilon^2$ and $x^S = x$, hence $v(T \cup \{i\}) - v(T) - v(S \cup \{i\}) + v(S) = \pi(x^{T\cup\{i\}}) - \pi(x^T) - \pi(x^{S\cup\{i\}}) + \pi(x^S) = \pi(x + \epsilon^1 + \epsilon^2) - \pi(x + \epsilon^1) - \pi(x + \epsilon^2) + \pi(x) < 0$, where the inequality follows from (33).

In the second part of the proof, we show the "if" part—that is, all $m$-attribute games embedded in $\pi$ are convex if $\pi$ has strongly increasing differences. Consider an $x \in \mathcal{D}^m$ and $\epsilon^1, \epsilon^2 \in \mathbb{R}_+^m$. Let $\Delta(x, \epsilon^1) = \pi(x + \epsilon^1) - \pi(x)$ and $\Delta^2(x, \epsilon^2, \epsilon^1) = \Delta(x + \epsilon^2, \epsilon^1) - \Delta(x, \epsilon^1)$. Then, $\Delta^2(x, \epsilon^2, \epsilon^1) \geq 0$ because $\pi$ has strongly increasing differences.

Consider an $m$-attribute game embedded in $\pi$. Then, for all $i \in N$ and $S \subseteq T \subseteq N \setminus \{i\}$ it holds that $v(T \cup \{i\}) - v(T) - v(S \cup \{i\}) + v(S) = \pi(x^{T\cup\{i\}}) - \pi(x^T) - \pi(x^{S\cup\{i\}}) + \pi(x^S) = \Delta(x^T, x^{\{i\}}) - \Delta(x^S, x^{\{i\}}) = \Delta^2(x^S, x^T - x^S, x^{\{i\}}) \geq 0$, where the inequality holds due to $\Delta^2(x, \epsilon^2, \epsilon^1) \geq 0$ because $x_j^T \geq x_j^S$ for all $j \in \{1, \ldots, m\}$. Therefore, (32) holds. □

For twice differentiable functions, strongly increasing differences can be restated as in the following corollary.

**Corollary 1.** *If attribute function $\pi$ from Proposition 1 is a twice differentiable function on $\mathbb{R}_+^m$, then all m-attribute games embedded in $\pi$ are convex if and only if for all $x \in \mathbb{R}_+^m$ and $j, k \in \{1, \ldots, m\}$ (including $j = k$),*

$$\frac{\partial^2 \pi(x)}{\partial x_j \partial x_k} \geq 0. \tag{34}$$

Supermodularity conditions for twice-differentiable functions require that (34) holds only for $j \neq k$, while under strongly increasing differences it has to hold for $j = k$ as well. These conditions can be checked more easily than constructive face concavity, as we illustrate with our next example.

Consider queueing games with attribute function $\pi$ defined in (5). We check the second derivatives of $\pi$:

$$\frac{\partial \pi(\lambda, \rho)}{\partial \rho} = p\frac{\lambda}{\rho^2} \geq 0$$

$$\frac{\partial^2 \pi(\lambda, \rho)}{\partial \rho^2} = -2p\frac{\lambda}{\rho^3} \leq 0. \tag{35}$$

Because (35) is negative for $\lambda > 0$, $\pi$ does not have strongly increasing differences, hence queueing games are not convex in general.

## 5 | SUFFICIENT CONDITIONS FOR CORE NONEMPTINESS

In the previous section, we derived the necessary and sufficient conditions for core nonemptiness and for the convexity of all $m$-attribute games embedded in attribute function $\pi$. In this section, we present several sufficient conditions for core nonemptiness that are easier to evaluate.

We start with a special subclass of $m$-attribute games that naturally satisfies max-concavity.

**Proposition 2.** *Suppose that $\pi : \mathcal{D}^m \to \mathbb{R}$ is continuous on $\mathcal{D}^m \setminus \{\mathbf{0}\}$. All m-attribute games embedded in $\pi$ have a nonempty core if $\pi$ is SH(1) and concave on $\mathcal{D}^m \setminus \{\mathbf{0}\}$.*

A natural question that arises is whether we can find any OM games for which the attribute function is concave and SH(1). The answer to this question is positive. For instance, consider a two-period version of the ELS game with no backlogging wherein $\pi$ is defined by (4a) with $\mathcal{D}^2 = \mathbb{R}_+^2$. Let $K_1(q_1) = 15 + q_1$ for all $q_1 > 0$, $K_2(q_2) = 16 + 2q_2$, $K_2(0) = K_1(0) = 0$, and $r_1 = r_2 = 20$. Because it is more expensive to produce in the second period, the optimal solution is to produce enough quantity in the first period to satisfy the total demand. Therefore, for this game, $\pi(x_1, x_2) = \bar{\pi}(x_1 + x_2) = 20(x_1 + x_2) - (15 + x_1 + x_2)$ for all $x_1 + x_2 > 0$. Because $\bar{\pi}$ is linear on $\mathbb{R}_+^2 \backslash \{\mathbf{0}\}$, $\pi$ is concave on $\mathcal{D}^2 \backslash \{\mathbf{0}\}$. Moreover, it is easy to check that

$$\begin{aligned}\frac{\pi(sx_1, sx_2)}{s} &= \frac{19s(x_1 + x_2) - 15}{s} = 19(x_1 + x_2) - \frac{15}{s} \\ &\leq 19(x_1 + x_2) - 15 = \pi(x_1 + x_2) \\ &\quad \text{for all } 0 < s \leq 1 \text{ and } x_1 + x_2 > 0. \end{aligned} \tag{36}$$

Therefore, together with $\pi(\mathbf{0}) = 0$, $\pi$ is SH(1).

In this proposition, we presented an independently derived result analogous to Corollary 1 of Sharkey and Telser (1978). Our proof uses a different approach from theirs, as it is based on max-concavity, while Sharkey and Telser (1978) used subhomogeneity.

*Proof of Proposition* 2. Suppose that $\pi$ is SH(1) and concave on $\mathcal{D}^m \backslash \{\mathbf{0}\}$. Consider an $\ell \geq 1$, $x^1, \ldots, x^\ell \in \mathcal{D}^m$ and $a_1, \ldots, a_\ell \in \mathbb{R}_{++}$ such that $x^1 \vee \ldots \vee x^\ell \leq \sum_{j=1}^{\ell} a_j x^j$. If $x^1, \ldots, x^\ell = \mathbf{0}$, then $\sum_{j=1}^{\ell} a_j \pi(x^j) = \pi(\sum_{j=1}^{\ell} a_j x^j) = 0$ because $\pi(\mathbf{0}) = 0$. Now consider the case where at least one of the vectors $x^1, \ldots, x^\ell$ is not equal to $\mathbf{0}$. Note that $\max\{x_l^1 \ldots, x_l^\ell\} \leq \sum_{j=1}^{\ell} a_j x_l^j$ for all $l = 1, \ldots, m$, hence $\sum_{j=1}^{\ell} a_j \geq 1$. Then,

$$\begin{aligned}\sum_{j=1}^{\ell} a_j \pi(x^j) &= \sum_{l=1}^{\ell} a_l \sum_{j=1}^{\ell} \frac{a_j}{\sum_{k=1}^{\ell} a_k} \pi(x^j) \\ &\leq \sum_{l=1}^{\ell} a_l \pi\left(\sum_{j=1}^{\ell} \frac{a_j}{\sum_{k=1}^{\ell} a_k} x^j\right) \\ &\leq \pi\left(\sum_{l=1}^{\ell} a_l \sum_{j=1}^{\ell} \frac{a_j}{\sum_{k=1}^{\ell} a_k} x^j\right) = \pi\left(\sum_{j=1}^{\ell} a_j x^j\right). \end{aligned} \tag{37}$$

The first inequality holds because $\pi$ is concave, and the second inequality holds because $\pi$ is SH(1). Therefore, $\pi$ that is SH(1) and concave on $\mathcal{D}^m \backslash \{\mathbf{0}\}$ satisfies max-concavity. □

Anily and Haviv (2014) studied regular games and derived sufficient conditions for nonemptiness of their core based on the superadditivity concept and $H(1)$ (homogeneity of degree one) property. As we mentioned earlier, $m$-attribute games are a special class of regular games, and we restate the conditions from Anily and Haviv (2014) for $m$-attribute games. Although Proposition 3 follows directly from Anily and Haviv (2014) and Corollary 2 of Sharkey and Telser (1978) presented analogous results to this proposition, we present it for completeness.

**Proposition 3.** *Suppose that $\pi : \mathcal{D}^m \to \mathbb{R}$ is continuous on $\mathcal{D}^m \backslash \{\mathbf{0}\}$. All m-attribute games embedded in $\pi$ have a nonempty core if $\pi$ is $H(1)$ and superadditive.*

The proof of Proposition 3 indicates that SH(1) and concavity are more general conditions than H(1) and superadditivity. For example, the attribute function for the two-period ELS game introduced above is SH(1) and concave, but not H(1).

*Proof of Proposition* 3. Suppose that $\pi$ is H(1) and superadditive. Then, the homogeneity of degree one implies that $\pi$ is SH(1). Moreover, for all $x, y \in \mathcal{D}^m$ and $a \in (0, 1)$, $a\pi(x) + (1-a)\pi(y) = \pi(ax) + \pi((1-a)y) \leq \pi(ax + (1-a)y)$, where the equality holds because $\pi$ is H(1), and the inequality holds because $\pi$ is superadditive. Therefore, $\pi$ is concave in $x$. Proposition 2 completes the proof. □

The next proposition presents an alternative set of sufficient conditions for nonemptiness of the core. SH(1) and partial concavity of attribute function are identified as sufficient conditions for core nonemptiness for special classes of $m$-attribute games. Consider FIX-M/M/s games with the attribute function described by (5). Karsten et al. (2015) used these conditions to show that FIX-M/M/s games are totally balanced, without identifying a core allocation. In Proposition 4, we not only show that an adaptation of conditions from Karsten et al. (2015) is sufficient for core nonemptiness of $m$-attribute games but we also describe how to find a core allocation. Before we formally state our result, we introduce a special type of partial concavity.

**Definition 9.** For a $j \in \{1, \ldots, m\}$, we say that a function $\pi : \mathcal{D}^m \to \mathbb{R}$ satisfies *j-complement concavity* if $\pi$ is concave in $x_{-j} \in \mathcal{D}^{m-1}$, where $x_{-j} = (x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m)$.

A function satisfies $j$-complement concavity if its projection on $m-1$ dimensional space, which excludes the $j$th component, is concave. This is used in our next set of sufficient conditions.

**Proposition 4.** *Suppose that $\pi : \mathcal{D}^m \to \mathbb{R}$ is continuous on $\mathcal{D}^m \backslash \{\mathbf{0}\}$. All m-attribute games embedded in $\pi$ have a nonempty core if $\pi$ is SH(1) and there exists a $j \in \{1, \ldots, m\}$ such that $\pi$ is j-complement concave.*

*Moreover, consider an m-attribute game embedded in $\pi$, wherein player $i \in N$ has $x^i$ resources. If $\pi$ is SH(1) and j-complement concave, then there exists a vector $a \in \mathbb{R}^m$ with $a_j = 0$ such that $\pi(x^N) + ay \geq \pi(x^N + y)$ for all $y \in \mathbb{R}^m$ with*

*$y_j = 0$ and $x^N + y \in \mathcal{D}^m$. Let*

$$\eta_l = \begin{cases} a_l & \text{for all } l \neq j \\ \dfrac{\pi(x^N) - \sum_{k \neq j} \eta_k x_k^N}{x_j^N}, & l = j. \end{cases} \quad (38)$$

*Then, $z_i = \sum_{l=1}^m \eta_l x_l^i$ for all $i \in N$ is a core allocation.*

Proposition 4 provides more general sufficient conditions than Propositions 2 and 3; that is, any attribute function that is SH(1) and concave in $x$, or that is H(1) and superadditive, is also SH(1) and $j$-complement concave. More importantly, Proposition 4 also describes a set of core allocations for the games with[6] $m \geq 2$ satisfying those conditions; Sharkey and Telser (1978) did not derive any core allocation although they showed that these conditions are sufficient for supportability. The core allocation described in Proposition 4 requires finding the tangent plane to a surface described by $\pi$ at $x^N$; the difficulty of this process depends on the properties of the underlying attribute function $\pi$.

*Proof of Proposition* 4. Suppose that $\pi$ is SH(1) and $j$-complement concave. Due to the supporting hyperplane theorem, $j$-complement concavity directly implies that there exists a vector $a \in \mathbb{R}^m$ with $a_j = 0$ such that $\pi(x^N) + ay \geq \pi(x^N + y)$ for all $y \in \mathbb{R}^m$ with $y_j = 0$ and $x^N + y \in \mathcal{D}^m$. We will show that for all $i \in N$, $z_i = \sum_{l=1}^m \eta_l x_l^i$ is a core allocation, where

$$\eta_l = \begin{cases} a_l, & \text{for all } l \neq j \\ \dfrac{\pi(x^N) - \sum_{k \neq j} \eta_k x_k^N}{x_j^N}, & l = j. \end{cases} \quad (39)$$

Consider coalition $S \subseteq N$ and its input vector $x^S$. Let $\alpha \geq 1$ such that $\alpha x_j^S = x_j^N$, and let $y = \alpha x^S - x^N$. Then,

$$\pi(x^S) \leq \frac{\pi(\alpha x^S)}{\alpha} = \frac{\pi(x^N + y)}{\alpha} \leq \frac{\pi(x^N) + ay}{\alpha}$$

$$= \frac{\pi(x^N) + a(\alpha x^S - x^N)}{\alpha} = \frac{\pi(x^N) + \sum_{l \neq j} \eta_l \left(\alpha x_l^S - x_l^N\right)}{\alpha}$$

$$= \frac{\left(\pi(x^N) - \sum_{l \neq j} \eta_l x_l^N\right) + \sum_{l \neq j} \eta_l \alpha x_l^S}{\alpha}$$

$$= \frac{\left(\frac{\pi(x^N) - \sum_{l \neq j} \eta_l x_l^N}{x_j^N}\right) x_j^N + \sum_{l \neq j} \eta_l \alpha x_l^S}{\alpha} = \frac{\eta_j x_j^N + \sum_{l \neq j} \eta_l \alpha x_l^S}{\alpha}$$

$$= \frac{\eta_j \alpha x_j^S + \sum_{l \neq j} \eta_l \alpha x_l^S}{\alpha} = \sum_{l=1}^m \eta_l x_l^S$$

$$= \sum_{l=1}^m \eta_l \sum_{i \in S} x_l^i = \sum_{i \in S} \sum_{l=1}^m \eta_l x_l^i = \sum_{i \in S} z_i. \quad (40)$$

The first inequality holds because $\pi$ is SH(1), while the second follows from the definition of $\alpha$. □

**Corollary 2.** *If attribute function $\pi$ from Proposition 4 is differentiable at $x^N$, the weights in Proposition 4 can be set as*

$$\eta_l = \begin{cases} \dfrac{\partial \pi(x^N)}{\partial x_l}, & \text{for all } l \neq j \\ \dfrac{\pi(x^N) - \sum_{k \neq j} \eta_k x_k^N}{x_j^N}, & l = j. \end{cases} \quad (41)$$

Next, we illustrate application of sufficient conditions with some examples.

## 5.1 | Queueing games

Consider queueing games with an attribute function described by (5). Observe that for all $x \in \mathcal{D}^m$ and $0 < \alpha \leq 1$, we have

$$\frac{\pi(\alpha\lambda, \alpha\rho)}{\alpha} = \frac{1}{\alpha}\left(r\alpha\lambda - p\frac{r\alpha\lambda}{\alpha\rho}\right)$$

$$= r\lambda - \frac{1}{\alpha}p\frac{r\lambda}{\rho} \leq r\lambda - p\frac{r\lambda}{\rho} = \pi(\lambda, \rho), \quad (42)$$

hence $\pi$ is SH(1). Moreover, (35) implies that $\pi$ is concave in $\rho$, hence it satisfies $j$-complement concavity. Therefore, $\eta_2 = \frac{\partial \pi(\lambda^N, \rho^N)}{\partial \rho} = p\frac{\lambda^N}{(\rho^N)^2}$ and $\eta_1 = \frac{\pi(\lambda^N, \rho^N) - \eta_2 \rho^N}{\lambda^N}$ describe a core allocation, where each player $i \in N$ gets $z_i = \eta_1 \lambda^i + \eta_2 \rho^i$.

## 5.2 | Economic lot-sizing games

Consider a two-period ELS game with attribute function described by (4a). Suppose that $r_1 = r_2 = 20$, $h_1 = h_2 = 2$, $K_1(1) = K_2(1) = 5$, $K_1(2) = K_2(2) = 9$, $K_1(3) = K_2(3) = 12$, and $K_1(0) = K_2(0) = 0$, hence production cost in each period is concave. Let $x^1 = (1, 0)$, $x^2 = (2, 0)$, $x^3 = (3, 0)$, $y^1 = (0, 1)$, $y^2 = (0, 2)$ and $y^3 = (0, 3)$. It is easy to check that

$$\pi(x^3) - \pi(x^2) = (60 - 12) - (40 - 9) = 17 \geq 16$$

$$= (40 - 9) - (20 - 5) = \pi(x^2) - \pi(x^1),$$

$$\pi(y^3) - \pi(y^2) = (60-12) - (40-9) = 17 \geq 16$$
$$= (40-9) - (20-5) = \pi(y^2) - \pi(y^1). \quad (43)$$

Therefore, $\pi$ is convex in both dimensions, and it is not concave, H(1) or $j$-complement concave, hence it does not satisfy any of the sufficient conditions introduced in this section.

## 5.3 | Linear production games

It is easy to see that previously introduced attribute function $\pi$ for LP games satisfies H(1), and it has been shown in the literature that $\pi$ is a nondecreasing piecewise-linear concave function (Proposition 1 of Ho, 2000). Moreover, its subgradient at point $b^N$, $\Delta(b^N)$, is given by the dual solution (Proposition 2 of Ho, 2000), $\pi(b) = \Delta(b)^T b$. Therefore, the core prices proposed by Owen (1975) that are given by these subgradients are exactly the solution imposed by Proposition 4.

## 5.4 | Newsvendor games

Recall the newsvendor problem under discrete demand with finite expectation and finite sample space. It has been shown by Müller et al. (2002) that $\Pi(Q, X^i(\omega))$ and $\pi(X)$ are H(1) and superadditive. Therefore, the prices imposed by Proposition 4 correspond to those given by the dual solution of the stochastic program (see Chen & Zhang, 2009), which coincides with the solution proposed by Montrucchio and Scarsini (2007).

# 6 | TU GAMES AND *m*-ATTRIBUTE GAMES

$m$-Attribute games studied in this paper are a special class of regular games, and more general, of TU games. The main characteristic of $m$-attribute games is additivity of input vectors that endows players, that is, the input vector of a coalition is obtained by adding input vectors of its members. It can be argued that the additivity assumption of $m$-attribute games is too restrictive and that there are many TU games outside of this class. In this section, we will establish a one-to-one relation between TU games and $m$-attribute games, and show that for each TU game there exists an attribute function $\pi : \mathcal{D}^m \to \mathbb{R}$ and an equivalent $m$-attribute game embedded in $\pi$ such that the TU game is totally balanced if and only if all $m$-attribute games embedded in $\pi$ have a nonempty core. A similar result for the relationship between TU games and market games has been shown by Shapley and Shubik (1969).

Shapley and Shubik (1969) show that every totally balanced TU game can be reformulated as a market game. We use a similar reformulation approach to establish a natural relationship between TU games and $m$-attribute games.

**Theorem 4.** *For every TU game $(M, w)$, there exists an attribute function $\pi : \mathbb{R}_+^M \to \mathbb{R}_+$ such that we can construct an m-attribute game embedded in $\pi$ that coincides with $(M, w)$. If $(M, w)$ is totally balanced, then there exists an attribute function $\pi : \mathbb{R}_+^M \to \mathbb{R}_+$ such that all m-attribute games embedded in $\pi$ have a nonempty core.*

The abovementioned result implies that (i) the class of TU games coincides with the class of $m$-attribute games, and (ii) the class of totally balanced TU games and hence, the class of market games (Shapley & Shubik, 1969), coincides with the class of totally balanced $m$-attribute games. Therefore, in theory, the class of $m$-attribute games is rich enough to study (total balancedness of) TU games. In other words, different classes of games can be shown to be totally balanced if they can be reduced to totally balanced $m$-attribute games (or market games). However, as argued by Anily and Haviv (2014) for market games, this reformulation (or showing that such a reduction is not possible) can be an intricate job and the reformulation presented in the proof is not practical in this sense. As such, if one can find another natural reformulation (as we show for newsvendor games), our results can be used to study the core of these games further.

*Proof of Theorem* 4. Consider a TU game $(M, w)$ with set of players $M = \{1, \ldots, m\}$ and characteristic function $w$. For any $S \subseteq M$, let $\epsilon^S \in \mathbb{R}^m$ denote the vector such that

$$\epsilon_i^S = \begin{cases} 1, & \text{if } i \in S, \\ 0, & \text{otherwise.} \end{cases} \quad (44)$$

For any $x \in \mathbb{R}_+^M$, consider the following optimization problem:

$$\Psi(x) = \max_{(\kappa(S))_{S\subseteq M}} \times \left\{ \sum_{S\subseteq M} \kappa(S) w(S) \middle| \sum_{S\subseteq M} \kappa(S)\epsilon^S = x,\ \kappa(S) \geq 0\ \ \forall S \subseteq M \right\}. \quad (45)$$

Equation (45) always has an optimal solution. It is easy to argue that $\Psi$ is superadditive because the sum of the optimal solutions for $\Psi(x)$ and $\Psi(y)$ is a feasible solution for problem $\Psi(x+y)$ with value $\Psi(x) + \Psi(y)$, hence $\Psi(x+y) \geq \Psi(x) + \Psi(y)$. For any feasible solution $(\kappa(S))_{S\subseteq M}$ of problem $\Psi(x)$, $(s\kappa(S))_{S\subseteq M}$ is a feasible solution for $\Psi(sx)$ with $s > 0$ and $\sum_{S\subseteq M} \kappa(S)w(S) = \sum_{S\subseteq M} s\kappa(S)w(S)/s$, hence $\Psi(x)$ is H(1).

Consider an $m$-attribute game in which each player $i \in N$ has an input vector $x^i \in \mathbb{R}_+^m$. The input vector for coalition $S$

is then $x^S = \sum_{i \in S} x^i$. Define function $\pi$ as follows:

$$\pi(x) = \begin{cases} w(S) & \text{if } x = \epsilon^S \text{ for some } S \subseteq M; \\ \Psi(x) & \text{otherwise.} \end{cases} \quad (46)$$

$\pi$ is not continuous in general. The $m$-attribute game $(M, v)$ embedded in $\pi$ in which each player $i \in M$ has the input vector $x^i = \epsilon^i$ is equivalent to $(M, w)$ by definition. This completes the proof of the first statement of the theorem.

Next, we prove the second statement by showing that if $(M, w)$ is totally balanced, then $\pi(x) = \Psi(x)$ for all $x \in \mathbb{R}_+^M$, and $\pi$ is superadditive and H(1).

Consider an $x \in \mathbb{R}_+^m$. If there is no $S \subseteq M$ such that $x = \epsilon^S$, then by definition $\pi(x) = \Psi(x)$ for all $x \in \mathbb{R}_+^m$. Suppose that there exists an $S \subseteq M$ such that $x = \epsilon^S$. A feasible set of (45) then consists of balanced combinations of $(\epsilon^T)_{T \subseteq S}$. Because $(M, w)$ is totally balanced, its subgame $(S, w_{|S})$ is balanced and $\Psi(x) = w(S)$. Hence, $\pi(x) = \Psi(x)$ for all $x \in \mathbb{R}_+^m$. Because $\Psi$ is H(1) and superadditive, $\pi$ is also H(1) and superadditive. Proposition 3 implies that all $m$-attribute games embedded in $\pi$ have a nonempty core, and Proposition 4 describes a set of core allocations. This completes the proof. □

## 7 | CONCLUSIONS

In this paper, we developed a generalized framework for studying a large class of games in which players are endowed with $m$ attributes, while values generated by coalitions depend on its members' attributes in an additive form expressed through an attribute function, $\pi$. One important feature of this setting is that the value of a specific coalition does not depend on the identity of members of that coalition directly. For the class of games we have described in this paper, $m$-attribute games, we developed a set of conditions that characterize nonemptiness of the core. In other words, we identify sets of conditions that an attribute function has to satisfy in order to assure the existence of a method for apportioning benefits from collaboration that leads to long-term cooperation of companies contemplating pooling their resources. More interestingly, the same conditions also guarantee the existence of core (attribute) prices. Many well-studied games can be shown to fall into the class of $m$-attribute games. We hope that there will be more examples developed in the future, and that our results can help in identifying whether pursuing an all-inclusive collaboration has long-term potential.

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## ENDNOTES

1 Note that this applies to profit games; in the case of cost games, we check for concavity.

2 The games with $v(N) = \pi(\mathbf{0}) = 0$ are zero games where all players are dummy players. Because they are trivial, we will not discuss these cases explicitly in our analysis.

3 In Proposition 4, Sharkey and Telser (1978) derived similar conditions as necessary and sufficient conditions for cost functions to be supportive. Their result follows directly from balancedness conditions for games with an infinite number of players, and a result similar to our Theorem 1 in Telser (1978). As described above, our proof follows a very different approach from theirs. In addition, there are other differences between our results. Telser (1978) studied games with a continuum of players and considered domain $\mathbb{R}_+^m$ throughout—i.e., $m_2 = 0$. Moreover, core nonemptiness is defined in Telser (1978, p. 150) directly in terms of existence of prices, rather than via core nonemptiness of associated cooperative games.

4 The general conditions for the strong duality theorem to hold for semi-infinite linear problems were first derived by Charnes et al. (1963) and then refined by Duffin and Karlovitz (1965). Since then, semi-infinite linear problems have been studied extensively in the literature. As stated in Faigle et al. (2002), another difference to finite linear programming is that the existence of primal and dual feasible solutions need not imply the existence of optimal solutions.

5 By Lemma 3, we could equivalently restrict this definition to *minimal* constructive face collections.

6 Note that when $m = 1$, we already know that SH(1) is a sufficient and necessary condition. Moreover, $\eta = \pi(x)\backslash x$ is a core price.

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