Stable allocations for choice-based collaborative price setting

Citation for published version (APA):
Schlicher, L., & Lurkin, V. J. C. (2022). Stable allocations for choice-based collaborative price setting. *European Journal of Operational Research*, *302*(3), 1242-1254. https://doi.org/10.1016/j.ejor.2022.01.036

DOI:
10.1016/j.ejor.2022.01.036

Document status and date:
Published: 01/11/2022

Document Version:
Accepted manuscript including changes made at the peer-review stage

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

Download date: 05. Oct. 2023

Stable allocations for choice-based collaborative price setting

Loe Schlicher[a], Virginie Lurkin[b,*]

[a] *Eindhoven University of Technology, Department of Industrial Engineering and Innovation Sciences*
P.O. Box 513, 5600 MB, Eindhoven, The Netherlands
[b] *University of Lausanne, HEC Lausanne Faculty of Business and Economics*
Quartier de Chamberonne CH-1015, Lausanne, Switserland

Abstract

Horizontal agreements can fall within the scope of exemptions to antitrust competition if they are expected to create pro-consumer benefits. Inspired by such horizontal agreements, we introduce a cooperative game in which a set of transport operators can collectively decide at what price to offer sustainable urban mobility services to a pool of travelers. The travelers choose amongst the mobility services according to a multinomial logit model, and the operators aim at maximizing their joint profit under a constant market share constraint. After showing that various well-known allocation rules (i.e., proportional rules and the Shapley value) do not always generate core allocations, we present a core-guaranteeing allocation rule, the market share exchange rule. This rule first allocates to each transport operator the profit he or she generates under collaboration, and then subsequently compensates those transport operators that lose part of their market share, which is paid by the ones that receive some extra market share. This exchange of market share is facilitated by a unique price, which can be expressed as the additional return by cooperating per unit of market share. Finally, we show that, under some natural conditions, the market share exchange rule still sustains the collaboration when the transport operators need to pay back part of the joint profit to society.

Keywords: game theory, choice-based pricing, cooperative game, core, allocation rules

---

1. Introduction

More and more innovative transport solutions, such as e-bikes and e-scooters, are popping up in our urban streets. Together with other shared, small, and light emission-free vehicles, these solutions are known collectively as *micromobility.* Micromobility can help cities and service providers to address unsolved transportation challenges related to urban congestion and pollution (Abduljabbar et al. (2021)). Though micromobility have increased in popularity in major cities, many startups still suffer from profitability issues (Fearnley (2020)). Collaboration between these startups has been acknowledged as an important factor to unlock the sustainable benefits of micromobility (Møller and Simlett (2020)).

---

[*] Corresponding author

Email addresses: `l.p.j.schlicher@tue.nl` (Loe Schlicher), `virginie.lurkin@unil.ch` (Virginie Lurkin)

While various forms of collaboration (e.g., strategic alliances or public private partnerships) are, *in theory*, possible, creating functioning collaborations remains hard *in practice*. Indeed, following Article 101 of the Treaty on the Functioning of the European Union (TFEU), horizontal agreements between competitors (e.g., micromobility startups) are usually perceived as anti-competitive and are heavily fined under the national laws of the EU Members States. Article 101 however permits some exceptions. For instance, horizontal agreements may qualify for exemption if they create sufficient pro-consumer benefits that outweigh the anti-competitive effects (Article 101(3), TFEU). Besides, they should not eliminate the competition in the relevant market, implying that participants should have a small market share and their combined market share should not exceed a specified limit (e.g., 20% in the Netherlands). A recent example of such an exemption stems from the Webtaxi case in Luxembourg, where the competition authority allowed various taxi companies to use a pricing algorithm to determine their taxi prices. While it was acknowledged that the joint use of such an algorithm constituted a situation of price fixing (i.e, a price agreement between competitors), it was decided that the agreement could be exempted since they expected huge pro-consumer benefits, mainly less waiting time, and lower prices, as well as more rides for drivers (Bostoen (2018)). At the same time, the combined market shares of the taxi operators would remain far below the threshold set by the Luxembourg authorities.

Similar to the horizontal agreement in the taxi sector, micromobility startups in major cities such as Paris, London or Amsterdam could also decide to bundle their forces and set prices of their mobility services collectively. Such an initiative could also qualify for an antitrust exemption, since it helps mobility startups with their profitability issues, which is so crucial for the continuation and embedding of environmental friendly mobility solutions in our society. However, according to Article 101 (TFEU), such an agreement is only allowed if the combined market share does not exceed a specified limit. To fall within this requirement, the mobility startups could decide to set new prices, while keeping the combined market share stable. Basically, this boils down to a setting where some startups are losing market share, some others are winning market where, but where the overall joint profit (i.e., the sum of the profits of all startups) is increasing. To sustain such form of collaboration, it is important that the losing startups are financially compensated by the winning startups. This can be facilitated by (the development of) an allocation rule, which identifies how to properly distribute the overall joint profit amongst the mobility startups.

Inspired by this joint profit (re)allocation problem, we study a setting in which a set of transport operators (e.g., micromobility startups) can collaborate and decide at what price to offer sustainable urban mobility solutions (e.g., electric scooters or bikes) to a pool of travelers. To better reflect the decisions of these travelers, we assume that they choose among the services offered according to a multinomial logit model, one of the most widely-used disaggregate demand model (Ben-Akiva and Bierlaire (2003)). By considering a choice model on the demand side, our setting involves pricing decisions that better capture the supply-demand interactions between the operators objective of maximizing their expected revenue and the travelers objective of maximizing the expected utility (Sumida et al. (2019)). To be in line with the conditions associated with the horizontal agreement exemptions, we assume that the transport operators set their prices in such a way that the total joint profit is maximized and their total market share remains constant, and as such remains below the authorized limit. We then formulate a cooperative game for this setting, the transport choice (TC) game, and introduce various intuitive allocation rules. We study to which extent these allocation rules produce core allocations, i.e., allocations that sustain the collaboration because they give no reason for any transport operator, nor group of operators, to break from the collaboration. We show that intuitive proportional rules as well as the well-known Shapley value produce allocations that do not always belong to the core (i.e., these allocations can not sustain the collaboration). We also study a market share exchange allocation rule. This rule first allocates to each transport operator the profit it generates under collaboration. Subsequently, the rule compensates those transport operators that lose part of their market share, which is paid by the ones that receive some extra market share. This exchange of market share is facilitated by a unique price, which can be expressed as the additional return by cooperating per unit of market share. We prove that the allocations of this allocation rule always belong to the core. Finally, we study a setting where the transport operators need to pay back part of the joint profit to society. We show that, under some natural conditions, the allocations of the market share exchange allocation rule are still in the core.

The rest of the paper is organized as follows. Section 2 provides an overview of the main advancements in the two main disciplines related to this paper: choice-based pricing and cooperative game theory. In Section 3, we introduce preliminaries on discrete choice theory and cooperative game theory. Transport choice situations are introduced in Section 4. The associated transport choice game is introduced in Section 5. We study allocation rules for our game in Section 6. Then in Section 7 we study an extended TC game, where players need to pay back part of the joint profit to society. We conclude this paper with final remarks in Section 8. While complete proofs of lemmas and theorems are relegated to Appendix A, a sketch of proof is given in the main text for the theorems that constitute our main results.

2. Related literature

The determination of optimal prices for different services (or products) is an essential component of operations management (Sun et al., 2021). This is also a delicate task in many organization, as it affects corporate profitability and market competitiveness. The higher the service price, the better the company can cover its costs and generate a profit, but the higher the service price the least attractive the service becomes for the customers (Tawfik and Limbourg, 2018). This is especially true when newly competitive markets are emerging, which has happened in the transport sector in recent years. As a result of their practical significance, pricing problems have attracted a lot of attention in many fields, including transportation (Azadian and Murat (2018), Arbib et al. (2020), Zhong et al. (2021)).

Most (if not all) pricing problems require demand data as input. In many pricing problems, aggregate representations of demand is used. This aggregate modeling approach is not able to capture the causal relationship between the pricing decisions and the individual customer purchase decisions. To better represent the supply-demand interactions, dis-aggregate demand models have been integrating within pricing problems. The state-of-the-art for the modeling of dis-aggregate demand relies on Discrete-choice modeling (DCM) (Bierlaire and Lurkin, 2017). Pricing models are usually referred to as *choice-based pricing* if customer's choice behavior is modelled using DCM. These models are mathematically complex since they are nonlinear and non-convex in prices. Still, as discussed in the next section, the operations research community put remarkable efforts in solving these models because they better reflect the trade-off between the business objective of maximizing the expected revenue and the customer objective of maximizing the expected utility (Sumida et al., 2019).

2.1. Choice-based pricing

Hanson and Martin (1996) pioneer this research by showing that the expected revenue function is not concave in prices, even for the simple *multinomial logit* (MNL) model. Subsequent authors have showed that, under uniform price sensitivities across all products, the expected revenue function is concave in the choice probability vector (Song and Xue, 2007, Dong et al., 2009, Zhang and Lu, 2013). Li and Huh (2011) proved that the concavity remains for asymmetric price-sensitivities, for both the MNL model and the *nested logit* (NL) model that generalizes the MNL model by grouping alternatives into different nests based on their degree of substitution.

Parallel works have used first-order conditions to show that, under restrictive assumptions on the degree of asymmetry in the price sensitivity parameters, there exist unique price solutions for some logit models. This was demonstrated for the MNL model (e.g., (Aydin and Ryan, 2000, Hopp and Xu, 2005, Maddah and Bish, 2007, Aydin and Porteus, 2008, Akçay et al., 2010)), the NL model (e.g., Aydin and Ryan (2000), Hopp and Xu (2005), Maddah and Bish (2007), Aydin and Porteus (2008), Akçay et al. (2010), Gallego and Wang (2014), Huh and Li (2015)), and the *paired combinatorial logit* (PCL) model (Li and Webster, 2017). Lately Zhang et al. (2018) showed that this result actually holds for the entire family of *generalized extreme value* (GEV) models. This stream of research also includes studies in which pricing decisions are optimized jointly with other decisions such as assortment or scheduling decisions (e.g., Du et al. (2016), Jalali et al. (2019), Bertsimas et al. (2020)).

Both Gilbert et al. (2014) and Li et al. (2019) consider a pricing problem under a *mixed logit model* (ML), a popular choice model that allows the price sensitivity parameter to vary across individuals. Under ML assumption, the concavity property with respect to the choice probabilities breaks down (even for entirely symmetric price sensitivities). The theoretical results obtained for the other logit models therefore do not apply to ML-based pricing problems. Li et al. (2019) assume a finite number of market segments, with product demand in each segment governed by the MNL model. To solve this problem, the authors propose an algorithm that converges to a local optimum by solving two concave maximization problems, which work as lower and upper bounds for the objective value of the revenue function. Gilbert et al. (2014) consider a ML demand model within a revenue-maximizing network pricing problem. Unlike Li et al. (2019), the price sensitivity parameter is distributed across the population according to a continuous random variable. To solve this complex problem, the authors rely on a tractable approximation of the ML-pricing problem.

Apart from Li and Huh (2011), all above studies consider a monopoly setting. However, in practice multiple groups of decision-makers are simultaneously involved within transport markets. As such, game theory is a suitable framework to analyze these choice-based problems (Adler et al., 2020). Choice-based pricing problems have therefore also been studied from a non-cooperative game theory perspective, as highlighted in the next section.

2.2. Non-cooperative game theory

In non-cooperative games based on choice-based pricing, important research efforts are made on showing conditions for existence and uniqueness of Nash equilibria. Existence conditions for Nash equilibria for non-cooperative games under MNL and NL models are provided by Milgrom and Roberts (1990), Bernstein and Federgruen (2004) and Li and Huh (2011), among others. Aksoy-Pierson et al. (2013) identify conditions on price bounds and segment market shares that guarantee the existence and uniqueness of equilibrium for a logit-based game involving differentiated firms offering a single-product at a unique price to groups of homogeneous customers. Following several empirical evidence, and using the adjusted markup as a single decision variable, Gallego and Wang (2014) show that under mild conditions a unique Nash equilibrium exists for a market with homogeneous demand and nested logit models.

Lin and Sibdari (2009) show the existence of an equilibrium for a dynamic logit-based price competition game between firms selling a single product in a market of substitutable products. The authors propose policies to find the equilibrium in case of full and partial information. In a similar spirit, Levin et al. (2009) consider a stochastic dynamic game based on a generalized choice model of demand where customers are subdivided into market segments. Under certain assumptions with respect to information and competition, the authors show the existence of subgame equilibrium solution for each period of this game.

Morrow and Skerlos (2011) present necessary stationarity conditions and analyze numerical methods to compute equilibrium prices for a market with multi-product offer and homogeneous prices under a general ML model of demand. The authors then acknowledge that determining existence or uniqueness of equilibrium prices with general discrete choice models, heterogeneous multi-product firms and heterogeneous consumers is an open problem. In Bortolomiol et al. (2021), a simulation-based heuristic framework is presented to solve pricing problems based on advanced logit models such as the ML, with heterogeneous population, multi-product offer by suppliers and price differentiation. The flexibility of the methodology regarding the choice of the demand model however comes at the expense of pure equilibrium conditions.

In our paper, we also consider a choice-based pricing problem involving multiple firms. Our study therefore fits in the established literature on price optimization under logit choices. However, inspired by horizontal agreement exemptions (as discussed in the introduction), we assume that the firms can collaborate and collectively decide at what price to offer their services. Cooperative game theory is then the most appropriate methodology to adopt to allocate the associated joint profit between the firms, as explained in the next section.

2.3. Cooperative game theory

Cooperative game theory was successfully applied to various types of real-life collaborative settings. For instance, it has been used to identify fair prices for vaccine exchange between countries in times of pandemics (Westerink-Duijzer et al. (2020)), to help museums to decide how to share the profits arising from a museum pass (Ginsburgh and Zang (2003)), to identify fair prices to share railway equipment amongst railway contractors (Schlicher et al. (2017, 2018, 2020)), and to help service operations in factories to divide cost savings when they decide to optimally re-balance their production lines. (Anily and Haviv (2017)).

The transportation industry also offers many situations suitable for the application of game theory. One can, for example, think of various transport operators that decide to team up to perform (parts of) their logistics operations jointly. By exchanging transport requests among each other, logistics operations can potentially take place in a more efficient and sustainable way. Examples of such application can be found in Lozano et al. (2013), Engevall et al. (2004), Hezarkhani et al. (2016) and Kimms and Kozeletskyi (2016).

Our paper is enrolled in the line of this existing literature on cooperative games inspired by real-life settings in transport. In particular, it tackles a sensitive topic: price collaboration. To the best of our knowledge, we are the first in this stream of research to focus on collaborative price setting between transport operators. This is not surprising since price fixing is an obvious horizontal agreement and is therefore perceived as anti-competitive and heavily fined under the national laws of the EU Members States. As such, the industry has very little or no motivation to study profit allocation aspects under collaborative price setting. However, as pointed out in our introduction, under some strict conditions, antitrust rules can be repealed, allowing competitors to collaborate on prices. These exemptions have been our source of inspiration for the development of a cooperative game in which a set of transport operators can collectively decide at what price to offer sustainable urban mobility services to a pool of travelers whose purchase decisions are characterized by the MNL model.

In line with the recent literature on cooperative games inspired by real-life settings in transport, we also investigate the non-emptiness of the core of our game and study fairness properties of various allocation rules (e.g., proportional rules or the Shapley value).

There are some evident similarities between our TC game and the recent contributions of Lardon (2019, 2020) that study cooperative Bertrand oligopoly games. Like us the authors are interested in a set of firms that need to set prices, and have an associated demand function that describes how many customers will opt for that firm. Under the assumption that their demand function is linear in price, they show that their game has a non-empty core, meaning that there are incentives for the firms to cooperate on prices. Our TC game significantly departs from these works by using a non-linear demand function: the MNL model that represents the customers' purchase decisions. Besides, we assume that the sum of the demand functions is stable in price, while it is not in Lardon (2019, 2020).

Finally, it is worth nothing that in a more theoretical context, our TC game also has some similarities with market games (Shapley and Shubik (1969). In these games, each player is associated with a set of resources and a concave profit function, identifying the amount of profit realized for a given set of resources. Players collaborate by reallocating their resources to maximize the sum of the utility functions. In our TC game, players are also reallocating resources, namely market share, and are equipped with an implicitly defined utility function that describes to profit per player, for a given amount of market share. However, opposed to market games, in our game the market share is strictly positive for any combination of prices. This due to the logit-form of the MNL model. As such, it is not possible to translate our TC game into a market game, directly. We would like to mention that, although this translation cannot be made directly, we don't rule out that it is still possible. However this transformation may be a difficult as studying the game itself (Anily and Haviv (2017))

3. Preliminaries on Discrete Choice and Cooperative Game Theory

3.1. Discrete Choice Theory

Rooted in microeconomics, DCM are powerful operational tools that aim at capturing the causality between a set of explanatory variables and the behavioral choice of economic actors. The set of alternatives considered as a potential choice is assumed to be finite and discrete, and is referred to as the choice set, denoted C (with $C \neq \emptyset$). For example, in the context of the choice of a transport mode to commute to work, the choice set of an individual could include the car, train, bus, walking, and biking options (i.e., $C = \{\text{car, train, bus, walking, biking}\}$).

A fundamental assumption behind these models is that each individual is a rational utility maximizer. It means that, when making a choice among a set of available alternatives, the individual, or the decision maker, is choosing the alternative that maximizes a utility function. The exact specification of this utility function is unknown and therefore typically modeled as a continuous random variable. In this paper, we assume a homogeneous population and that there is no discrimination among individuals. The utility associated with a specific alternative $i \in C$ is thus the same for each individual and given by

$$U_i = V_i(x_i; \beta) + \varepsilon_i,$$

where $V_i(x_i; \beta)$ is a deterministic function of the attributes of the alternatives $x_i \in \mathbb{R}$ (e.g., the price of the transport mode), $\beta \in \mathbb{R}$ is a vector of estimable parameters for alternative i (e.g., the willingness to pay), and ε_{in} is a continuous error term, capturing the specification and measurement errors. The choice model that predicts the probability for an individual to choose alternative $i \in C$ is therefore probabilistic and defined as

$$\mathbb{P}(i) = \mathbb{P}(U_i \geq U_j, \forall j \in C).$$

In the context of discrete choice, it is custom to assume that all error terms are independent, identically, and extreme value distributed with location parameter 0 and scale parameter 1 (i.e., $\varepsilon_i \sim \text{EV}(0,1)$). These assumptions lead to a widely used choice model in practice, the logit model whose choice probability is given by

$$\mathbb{P}(i) = \frac{e^{V_i(x_i;\beta)}}{\sum_{j \in C} e^{V_j(x_j;\beta)}} \text{ for all } i \in C.$$

DCM are commonly used in the scientific literature and in practice to understand and predict individual human behavior. The problem then consists in finding the parameters values, i.e., the coefficients of the variables in the utility functions, that maximize the probability that the choice model correctly predicts all observed choices (called the likelihood). In the last decade these choice models have also been more and more used within optimization models to represent more realistically the complexity of human behavior.

3.2. Cooperative Game Theory

Cooperative game theory primarily deals with the modelling and analysis of situations in which a group of players can benefit from coordinating their actions. In this paper, we model and analyze a specific type of cooperative game: a cooperative game with transferable utility. In what follows, we formally introduce this type of game and discuss desirable properties they may satisfy. We conclude with a description of an allocation rule.

A cooperative game with transferable utility, shortly called a (TU) game, is a pair (N, v) where N is a non-empty, finite player set and $v : 2^N \to \mathbb{R}$ a characteristic function with $v(\emptyset) = 0$. A subset $M \subseteq N$ is a coalition and $v(M)$ is the worth coalition M can achieve by itself. This worth can be transferred freely amongst the players. The set N is called the grand coalition. A game (N, v) is monotonic if the value of every coalition is at least the value of any of its subcoalitions, i.e., $v(M) \leq v(K)$ for all $M, K \subseteq N$ with $M \subseteq K$. When the value of the union of any two disjoint coalitions is larger than or equal to the sum of the values of these disjoint coalitions, a game (N, v) is superadditive, i.e., $v(M) + v(K) \leq v(M \cup M)$ for all $M, K \subseteq N$ with $M \cap K = \emptyset$. A game (N, v) is convex if the marginal contribution of any players to any coalition is less than his marginal contribution to a larger coalition, i.e., $v(K \cup \{i\} - V(K)) \geq v(M \cup \{i\}) - v(M))$ for all $M \subseteq K \subseteq N \backslash \{i\})$ and all $i \in N$.

An allocation for a game (N, v) is an N-dimensional vector $x \in \mathbb{R}^N$ describing the payoffs to the players, where player $i \in N$ receives x_i. An allocation is called efficient if $\sum_{i \in N} x_i = v(N)$. This implies that all worth is divided amongst the players of the grand coalition N. An allocation is individual rational if $x_i \geq v(\{i\})$ for all $i \in N$ and stable if no group of players has an incentive to leave the grand coalition N, i.e. $\sum_{i \in M} x_i \geq v(M)$ for all $M \subseteq N$. The set of efficient and stable allocations, called the core of (N, v), is denoted by

$$\mathscr{C}(N, v) := \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in M} x_i \geq v(M) \text{ for all } M \subseteq N \text{ and } \sum_{i \in N} x_i = v(N) \right\}.$$

An allocation rule is a function f that assigns to any game (N, v) in a class of cooperative games a vector $f(N, v) \in \mathbb{R}^N$ satisfying $\sum_{i \in N} f_i(N, v) = v(N)$. A well-known allocation rule defined on the set of all games is the Shapley value (Shapley (1953)). This allocation rule assigns to each player a weighed average over all marginal contributions (s)he can make to any possibly coalition. Formally, for any game (N, v) the Shapley value can be defined by:

$$SV_i = \sum_{M \subseteq N \backslash \{i\}} \frac{|M|!(|N| - 1 - |M|)!}{|N|!} (v(M \cup \{i\}) - v(M)) \text{ for all } i \in N.$$

4. Transport Choice situations

We consider a setting in which a group of homogeneous travelers is buying mobility services from a set of $N \subseteq \mathbb{N}$ transport operators. Each operator $i \in N$ offers one micromobility service (e.g., a e-bike or a segway) against price $p_i \in \mathbb{R}_+$ and cost price $c_i \in \mathbb{R}_+$. The mobility choices of travelers are represented using the logit model described in Section 3.1. In doing so, we let, per mobility service $i \in N$, function V_i be defined as:

$$V_i = \alpha_i - \beta p_i,$$

where $\alpha_i \in \mathbb{R}_+$ is an alternative-specific constant and $\beta \in \mathbb{R}_+$ a price sensitivity parameter. Moreover, the choice set C of the logit model consists of (i) the transport operators, where each operator refers to the operator-specific mobility service, and (ii) the choice to not buy any service, for which the deterministic utility is normalized to zero. We denote the choice of not buying service by 0, and consequently $C = N \cup \{0\}$ and $V_0 = 0$.

By using the choice probabilities of the logit model, we can define the share of travelers that opts for mobility service $i \in N$ as follows

$$\frac{e^{\alpha_i - \beta p_i}}{\sum_{j \in C} e^{\alpha_j - \beta p_j}} = \frac{e^{\alpha_i - \beta p_i}}{1 + \sum_{j \in N} e^{\alpha_j - \beta p_j}}. \tag{1}$$

From now on, we refer to this share of travelers as the market share of transport operator i. Given this market share, the profit of transport operator $i \in N$ is defined by:

$$(p_i - c_i) \cdot \frac{e^{\alpha_i - \beta p_i}}{1 + \sum_{j \in N} e^{\alpha_j - \beta p_j}}. \tag{2}$$

We summarize this setting by tuple $\theta = (N, p, c, \alpha, \beta)$ with N the set of transport operators, $p = (p_i)_{i \in N}$ the vector of prices, $c = (c_i)_{i \in N}$ the vector of cost prices, $\alpha = (\alpha_i)_{i \in N}$ the vector of alternative-specific constants, and β the price sensitivity parameter. We refer to θ as the transport choice situation and let Θ be the set of all possible transport choice situations.

We now illustrate our TC situation with a fictitious example.

Example 1. *Let $\theta \in \Theta$ with $N = \{1, 2, 3\}$, $p = (6, 8, 15)$, $c = (8, 4, 1)$, $\alpha = (1, 0.5, 1.5)$ and $\beta = 0.36$. The prices, corresponding market shares and associated profits of the transport operators are presented in Table 1.* ◇

i	1	2	3
Price i	6.0	8.0	15.0
market share i	0.220	0.065	0.014
profit i	-0.440	0.260	0.199

Table 1: Prices, market shares and profits of the transport operators of situation θ

Remark 1. *If the aim of the transport operators is to maximize their profits, it is reasonable to consider only those $\theta \in \Theta$ for which p is a Nash equilibrium (i.e., vector p is such that no transport operator $i \in N$ would unilaterally deviate his or her p_i, for the given N, c, α and β). It can be shown (see Appendix A), that for those situations, p satisfies*

$$p_i = \frac{1 + W\left(\frac{e^{\alpha_i - 1 - \beta c_i}}{1 + \sum_{j \in N \setminus i} e^{\alpha_j - p_j \beta}}\right)}{\beta} + c_i \quad \text{for all } i \in N, \tag{3}$$

where W denotes the Lambert W function.

Rather than operating individually, the transport operators can decide to cooperate. If the transport operators decide to do so, they will set new prices, which maximize the sum of the operator-specific profits. By doing so, they have to take into account that the total market share remains stable[1]. Formally, if the transport operators in N decide to collaborate,

---

[1] With this assumption, we guarantee that the operators cannot start dominating the market, which is one of the governmental rules that needs to be satisfied in order to allow for an antitrust law exception.

they are facing the following non-linear, constrained, optimization problem:

$$\begin{aligned}\mathscr{P} := \quad & \max_{x\in\mathbb{R}^N} \sum_{i\in N}(x_i - c_i)\frac{e^{\alpha_i-\beta x_i}}{1+\sum_{j\in N} e^{\alpha_i-\beta x_j}} \\ & \text{s.t.} \sum_{i\in N}\left(\frac{e^{\alpha_i-\beta p_i}}{1+\sum_{j\in N} e^{\alpha_j-\beta p_j}}\right) = \sum_{i\in N}\left(\frac{e^{\alpha_i-\beta x_i}}{1+\sum_{j\in N} e^{\alpha_j-\beta x_j}}\right)\end{aligned} \tag{4}$$

We refer to an optimal solution of $\mathscr{P}$ as an optimal price vector p^* and call the optimal value of $\mathscr{P}$ the optimal joint profit $\mathscr{P}^*$. There exists a closed-form expression for p^* and $\mathscr{P}^*$. Before presenting them, we first introduce some new notation. For each TC situation $\theta \in \Theta$, we define $D(x) = \sum_{i\in N} e^{\alpha_i-\beta x_i}$ for all $x \in \mathbb{R}^N$. Note that this new notation can be used to describe the total market share compactly (i.e., $\frac{\sum_{i\in N} e^{\alpha_i-\beta p_i}}{\sum_{j\in N} e^{\alpha_i-\beta p_j}+1} = \frac{D(p)}{D(p)+1}$). Now, we are ready to present an optimal price vector p^* and the associated optimal joint profit $\mathscr{P}^*$.

Theorem 1. *For each TC situation $\theta \in \Theta$ an optimal price vector p^* is given by*

$$p_i^* = c_i + \frac{1}{\beta}\ln\left(\frac{D(c)}{D(p)}\right) \text{ for all } i \in N,$$

and the associated optimal joint profit equals $\mathscr{P}^* = \frac{D(p)}{\beta(D(p)+1)}\ln\left(\frac{D(c)}{D(p)}\right)$.

The proof of Theorem 1 consists of three steps. First we identify how our optimization problem $\mathscr{P}$ relates to another optimization problem. This optimization problem has a much simpler form of constraint. Then, we identify an optimal price vector and the associated optimal value for this optimization problem, by using a Lagrangian type of optimality result from Bazaraa et al. (2013). Finally, we relate back these outcomes to $\mathscr{P}$.

We now make some remarks regarding Theorem 1.

Remark 2. *Observe that vector p^* is only player-specific in the cost price. So, if $c_i = 0$ for all $i \in N$, every transport operator will select the same price, under collaboration.*

Remark 3. *The gain of collaboration is always non-negative, i.e.,*

$$\mathscr{P}^* - \sum_{i\in N}(p_i - c_i)\cdot\frac{e^{\alpha_i-\beta p_i}}{D(p)+1} \geq 0,$$

which is due to the fact that price vector p is a feasible solution of $\mathscr{P}$.

Remark 4. *If $p_i - c_i = p_j - c_j$ for all $i, j \in N$, the gain of collaboration equals zero, i.e.,*

$$\mathscr{P}^* - \sum_{i\in N}(p_i - c_i)\cdot\frac{e^{\alpha_i-\beta p_i}}{D(p)+1} = \frac{(p_j - c_j)D(p)}{D(p)+1} - \frac{(p_j - c_j)D(p)}{D(p)+1} = 0 \text{ for all } j \in N.$$

Note, this condition describes a TC situation with a constant marginal profit per operator and a total market share of $D(p)/(D(p)+1)$. As such, it can be used as a benchmark for situations with the same total market share, but where operators gain from collaboration.

We now illustrate how Theorem 1 applies to our TC situation of Example 1.

Example 2. *Reconsider the situation of Example 1. In Table 2, we present an optimal price vector and the corresponding market share*[2] *and profit per transport operator.* ◇

i	1	2	3
optimal price i	13.980	9.980	6.980
market share i	0.012	0.032	0.255
profit i	0.074	0.190	1.523

Table 2: Prices, market share and profits of the transport operators of situation θ

From Table 1 of Example 1 and Table 2 of Example 2, we learn that the joint profit, which is 1.787, exceeds the sum of individual profits without collaboration, namely $-0.440 + 0.260 + 0.199 = 0.019$. However, at the same time, we also observe that the individual profit of transport operator 2 decreases (from 0.260 to 0.190). So, in case of collaboration among the three transport operators, it would be natural that operator 1 and operator 3 would compensate operator 2 in some way. But, by how much? In the upcoming section, we address this question by making use of cooperative game theory.

5. A Cooperative Transport Choice Game

In this section, we introduce a cooperative game, associated to our transport choice situation. Formally, for each TC situation $\theta \in \Theta$, we introduce a cooperative game (N, v^θ), where N represents the set of players (i.e., transport operators) and v^θ represents the characteristic value function. In this game, $v^\theta(M)$ reflects the joint profit coalition $M \subseteq N\backslash\{\emptyset\}$ can realize. This joint profit is obtained by taking into account that (i) the sum of the market shares of the players in M remains stable (i.e., the new vector of prices should be such that the sum of their market shares remains the same) and (ii) all players outside coalition M (i.e., players in $N\backslash M$) keep their initially set prices. This game, which we refer to as a cooperative transport choice game, is formally defined as follows.

Definition 1. *For every TC situation $\theta \in \Theta$, the associated cooperative transport choice (TC) game (N, v^θ) is defined by*

$$\begin{aligned} v^\theta(M) &= \max_{x \in \mathbb{R}^M} \sum_{i \in M} (x_i - c_i) \frac{e^{\alpha_i - \beta x_i}}{1 + \sum_{j \in M} e^{\alpha_j - \beta x_j} + \sum_{j \in N\backslash M} e^{\alpha_j - \beta p_j}} \\ s.t. \sum_{i \in M} \left(\frac{e^{\alpha_i - \beta p_i}}{1 + \sum_{j \in N} e^{\alpha_i - \beta p_j}} \right) &= \sum_{i \in M} \left(\frac{e^{\alpha_i - \beta x_i}}{1 + \sum_{j \in M} e^{\alpha_i - \beta x_j} + \sum_{j \in N\backslash M} e^{\alpha_i - \beta p_j}} \right) \end{aligned} \tag{5}$$

for all $M \subseteq N\backslash\{\emptyset\}$ and $v^\theta(\emptyset) = 0$.

---

[2] Note that because of rounding the total market share (0.008 + 0.024 + 0.214) seems to have changed, but this is not the case. The total market share still remains 0.245.

Similar to the optimization problem in Section 4, we present a closed-form expression for the optimal joint profit of any coalition $M \subseteq N \backslash \{\emptyset\}$. Before doing so, we need to introduce some coalition-specific notation. For each TC situation $\theta \in \Theta$ and each $M \subseteq N$, we let $D^M(x) = \sum_{i \in M} e^{\alpha_i - \beta x_i}$. Please, note that we have $D^N(x) = D(x)$ for all $x \in \mathbb{R}$. Now, we are ready to present the closed-form expression for any coalition.

Theorem 2. *For every TC situation $\theta \in \Theta$ it holds, for all $M \subseteq N \backslash \{\emptyset\}$, that*

$$v^\theta(M) = \frac{D^M(p)}{\beta(D^N(p)+1)} \ln\left(\frac{D^M(c)}{D^M(p)}\right)$$

The structure of the proof of Theorem 2 is similar to the structure of the proof of Theorem 1. We now present an example of a TC game.

Example 3. *Reconsider the setting of Example 1. The coalitional values of TC game (N, v^θ) are represented in Table 3 below.*

M	{∅}	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$v^\theta(M)$	0	-0.440	0.260	0.199	0.230	1.485	0.756	1.787

Table 3: Coalitional values of game (N, v^θ)

Please, observe that the coalitional values of the individual coalitions (-0.440, 0.260 and 0.199) match with the profits of Table 1, and that the coalitional value of the grand coalition (1.787) matches with the sum of the profits of Table 2. ◇

Remark 5. *Some readers may see similarities between our TC game and a market game (see e.g., Anily and Haviv (2017)). We want to emphasize that it is not straightforward to recognize our TC game as a market game. In a market game, players can freely reallocate their resources, implying that some of the players may end up with no resources at all. However, in our game, where market shares could be recognized as resources, it is not possible to assign a player with no market share (note that $D^{\{i\}}(p)/(D^N(p)+1) > 0$ for all $p \in \mathbb{R}^N$). As such, our game does not fall in the framework of a market game, directly.*

Remark 6. *As discussed in the preliminaries (Section 3), it is natural to study a cooperative game on properties like monotonicity, superadditivity and convexity. As a side result, we would like to share that our TC game is superadditive, but not monotonic nor convex.*

6. Allocation rules

The central question is this section is how players of our TC game should distribute the joint profit, in order to sustain the collaboration. In the game theory literature, it is common to address this question by introducing several allocation rules and by investigating whether their allocations belong to the core (see, e.g., Westerink-Duijzer et al. (2020)). In this paper, we also follow this approach. In particular, we introduce four intuitive allocation rules and study whether their allocations belong the core. Recall (from Section 3.2) that the core is the set of allocations that makes no group of players break from the grand coalition, i.e., for any TC situation $\theta \in \Theta$ and associated (N, v^θ) the core is given by

$$\mathscr{C}(N, v^\theta) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in M} x_i \geq v^\theta(M) \text{ for all } M \subseteq N \text{ and } \sum_{i \in N} x_i = v^\theta(N) \right\}.$$

For each of the four allocation rules, we generate 10,000 random TC situations $\theta \in \Theta$ for $N = 3, 4$, and 5 players, respectively. For each $\theta \in \Theta$, we generate random numbers for the vector of costs (c) as well as for the utility parameters (α and β). More specifically, for each $i \in N$, cost c_i and parameter α_i are drawn for a uniform distribution in the interval [0.5, 15.0] with a 0.5 step. The price sensitivity parameter β is drawn from a uniform distribution in the interval (0,1] with step 0.1. The price vector p is computed based on the cost and utility parameters, as shown in equation (3), i.e., vector p is a Nash equilibrium. Per generation of 10,000 TC situations, and per allocation rule, we are then interested in the outcome

$$\frac{\#\text{ of TC situations for which allocation is in the core}}{\text{total }\#\text{ of TC situations (10,000)}}.$$

In the upcoming paragraphs, we introduce our four allocation rules and present and discuss the associated outcomes per allocation rule.

Allocation rules 1 and 2: Proportional. Allocation rules that have a long tradition when costs, profits, or savings have to be shared among different agents, are proportional rules (see, e.g., Moulin (1987)). As the name suggests, these rules allocate the worth to the players in a proportional way. A common proportional rule is to divide the worth proportional to the value of the individual coalitions. Formally, for any $\theta \in \Theta$ and associated TC game, this individual-proportional allocation rule is given by

$$\text{I-PROP}_i = \frac{v^\theta(\{i\})}{\sum_{j \in N} v^\theta(\{j\})} \cdot v^\theta(N) \text{ for all } i \in N.$$

As an alternative, one can also decide to divide the worth proportional to the initial market shares of the players. In that case, for every $\theta \in \Theta$ and associated TC game, the allocation rule, which we call the market share-proportional rule, is given by

$$\text{M-PROP}_i = \frac{S_i}{\sum_{j \in N} S_j} \cdot v^\theta(N) \text{ for all } i \in N.$$

The results of the experiments with respect to the above proportional rules are presented in Table 4. We can see that almost all M-PROP allocations don't belong to the core. We also see that the I-PROP allocation rule performs much better, with around 95% of allocations belonging to the core for a 3 player TC game. Still it does not always lead to core allocations. In particular, the number of allocations not belonging to the core increases as the size of the game grows.

We now illustrate these proportional allocation rules to our TC situation of Example 1 and investigate whether their allocations do belong to the core or not.

	N		
	3	4	5
I-PROP	0.9460	0.8959	0.8538
M-PROP	0.0001	0.0000	0.0000

Table 4: Outcome for the two proportional rules (based on 10,000 TC situations)

Example 4. *Reconsider the TC situation $\theta \in \Theta$ and game (N, v^θ) of Example 1. The allocations of the proportional allocation rules for (N, v^θ) are reported in Table 5.* ◇

		$\sum_{i \in M} x_i$	
M	$v^\theta(M)$	I-PROP	M-PROP
$\{1\}$	-0.440	-42.101	1.314
$\{2\}$	0.260	24.859	0.388
$\{3\}$	0.199	19.029	0.085
$\{1,2\}$	0.230	-17.242	1.702
$\{1,3\}$	1.485	-23.072	1.399
$\{2,3\}$	0.756	43.889	0.473
$\{1,2,3\}$	1.787	1.787	1.787

Table 5: Illustration of proportional allocation rules

Recall that an allocation $x \in \mathbb{R}^3$ is in the core if

- $x_1 + x_2 + x_3 = v^\theta(N)$
- $x_1 \geq v^\theta(\{1\})$, $x_2 \geq v^\theta(\{2\})$, $x_3 \geq v^\theta(\{3\})$,
- $x_1 + x_2 \geq v^\theta(\{1,2\})$, $x_1 + x_3 \geq v^\theta(\{1,3\})$, *and* $x_2 + x_3 \geq v^\theta(\{2,3\})$.

I-PROP is not in the core, since $\textit{I-PROP}_1 + \textit{I-PROP}_2 = -17.242 < 0.230 = v^\theta(\{1,2\})$. *That means, players 1 and 2 together can earn more by breaking up and forming a new coalition together. Similarly, M-PROP is not in the core, since* $\textit{M-PROP}_3 = 0.085 < 0.199 = v^\theta(\{3\})$. *That means, player 3 is better of by working individually.*

Allocation rule 3: Shapley value. Another well-known allocation rule is the Shapley value. This allocation rule is introduced in Shapley (1953), and has shown to be applicable in various settings, such as cost sharing in horizontal cooperation among shippers (Lozano et al. (2013)), carpool problems (Naor (2005)), and data sharing settings (Dehez and Tellone (2013)). In words, the Shapley value assigns to each player a weighted average over all marginal contributions a player can make to any possible coalition. Formally, for any TC situation $\theta \in \Theta$ and associated TC game, the Shapley value is defined as:

$$SV_i = \sum_{M \subseteq N \backslash \{i\}} \frac{|M|!(|N| - 1 - |M|)!}{|N|!}(v^\theta(M \cup \{i\}) - v^\theta(M)) \text{ for all } i \in N.$$

Our numerical results in Table 6 demonstrate that also the shapley value does not always belong the core. In terms of performance, the Shapley value is slightly better than the I-PROP rule (see Table 5). The Shapley value does belong to the core in 96% of the instances for a 3 player TC game, and in 90% for a 5 player TC game.

	N		
	3	4	5
SV	0.9620	0.9303	0.8991

Table 6: Outcome for the Shapley value (based on 10,000 TC situations)

We provide below an example, illustrating the calculation of SV.

Example 5. *Reconsider the TC situation* $\theta \in \Theta$ *and game* (N, v^{θ}) *of Example 1. Then,* $SV = (0.407, 0.392, 0.989)$. *One can check that the Shapley does not belong to the core.*

Allocation rule 4: Market Share Exchange. From the previous paragraphs, we learned that the allocations of the proportional rules and the Shapley value do not always belong to the core. One reason could be that these allocation rules do not explicitly compensate for the exchange of market share between players. Therefore, in this paragraph, we study an allocation rule that does explicitly compensate for this exchange of market share. In particular, we study an allocation rule that first allocates to each player the profit he/she generates under full collaboration, i.e., player $i \in N$ receives $(p_i^* - c_i)\frac{D^{\{i\}}(p^*)}{D^N(p^*)+1}$. Thereafter, we identify for each player $i \in N$ the increase (or decrease) in the market share, which is $\left(\frac{D^{\{i\}}(p^*)}{D^N(p^*)+1} - \frac{D^{\{i\}}(p)}{D^N(p)+1}\right)$. Player i then receives a price ϕ for each exchanged unit of market share, and pays the same price for each extra unit of market share. Formally, for each $\theta \in \Theta$ and associated TC game, the market share exchange (MSE) rule is given by

$$MSE_i = (p_i^* - c_i)\frac{D^{\{i\}}(p^*)}{D^N(p^*)+1} - \phi\left(\frac{D^{\{i\}}(p^*)}{D^N(p^*)+1} - \frac{D^{\{i\}}(p)}{D^N(p)+1}\right),$$

where the price ϕ is given by

$$\phi = \frac{v^{\theta}(N) - v^{\widehat{\theta}}(N)}{\left(\frac{D^N(p)}{D^N(p)+1}\right)},$$

with $\widehat{\theta} = (N, p, (p_i - 1/\beta)_{i \in N}, \alpha, \beta)$, i.e., $\widehat{\theta}$ is a TC situation with a constant marginal profit for all operators $(1/\beta)$ and with the same total market share as θ $(D(p)/(D(p)+1))$. Recall from Remark 4 that players cannot gain from such a TC situation (because $p_i - c_i = \frac{1}{\beta}$ for all $i, j \in N$). Hence, the numerator of ϕ represents the total additional return that is gained compared to a TC situation with the same total market share and where collaborating is not beneficial at all. This total additional return is then divided by the total market share. Indeed, ϕ can be recognized as the additional return per unit of market share.

From Table 7, we learn that, for the given TC situations, the allocations of the MSE rule always belong to the core. This does not turn out be a coincidence. Actually, the allocations of the MSE rule are core guaranteed, which will be formalized next.

	N		
	3	4	5
MSE	1.0000	1.0000	1.0000

Table 7: Outcome for the Market rule (based on 10,000 TC situations)

Theorem 3. *For each TC situation $\theta \in \Theta$ and associated TC game (N, v^θ), it holds that $MSE \in \mathscr{C}(N, v^\theta)$.*

The proof of Theorem 3 consists of two steps. First, we show the MSE satisfies efficiency, which follows by construction of MSE. Thereafter, we show that MSE satisfies stability. Here we make use of an elementary property of the e-function (see Lemma 1 in Appendix A).

Remark 7. *We could also have formulated other $\widehat{\theta}$'s for which the marginal profit is constant (e.g., we could have selected $\widehat{\theta}$ with $p_i - c_i = \frac{2}{\beta}$ for all $i, j, \in N$). However, for such settings, core non-emptiness cannot be guaranteed.*

We conclude this section with an example, illustrating the calculation of MSE.

Example 6. *Reconsider the TC situation $\theta \in \Theta$ and game (N, v^θ) of Example 1. Then, $\phi = 3.202$ and the allocation of the MSE rule is given by $MSE = (0.738, 0.296, 0.753)$. One can check that MSE does belong to the core.*

7. An extension of the TC game

In the introduction, we described that horizontal agreements may qualify for an exemption if they create sufficient pro-consumer benefits that outweigh the anti-competitive effects. Besides, they should not eliminate the competition in the relevant market, implying that participants should have a small market share and their combined market share should not exceed a specified limit. For some forms of collaboration, it is also necessary to pay back part of the joint profit to society (Article 101(3), TFEU). In this section, we investigate an extended TC game where part of the joint profit can be reallocated to society. In particular, we study which fraction can be reallocated to society, such that the MSE rule, which has proven to be very successful for a stable collaboration, still produces core allocations.

Consider a TC situation $\theta \in \Theta$ and associated game (N, v^θ). Now, let $\delta \in (0, 1)$ be the fraction of the joint profit $(v^\theta(N))$ that players want to reallocate to society. We assume that this applies to any coalition $M \subseteq N$ with $|M| \geq 2$, i.e., any M with $|M| \geq 2$ will reallocate $\delta \cdot v^\theta(M)$ to society. So, the remaining joint profit of M equals $(1 - \delta) \cdot v^\theta(M)$. We formalize this new setting in a game $(N, v^{\theta,\delta})$, which we call the TC-δ game, and it reads as follows

$$v^{\theta,\delta}(M) = \begin{cases} (1-\delta) \cdot v^\theta(M) & \text{if } |M| \geq 2 \\ \\ v^\theta(M) & \text{if } |M| \leq 1. \end{cases}$$

First of all, observe that there is no reason to collaborate (i.e., the core is empty) if $v^{\theta,\delta}(N) < \sum_{i \in N} v^{\theta,\delta}(\{i\})$. In other words, the fraction that can be paid back at most is

$$\delta \leq 1 - \frac{\sum_{i\in N} v^\theta(\{i\})}{v^\theta(N)} = 1 - \beta \frac{\sum_{i\in N}(p_i - c_i)D^{\{i\}}(N)}{D^N(p^*)\ln\left(\frac{D^N(c)}{D^N(p)}\right)}. \tag{6}$$

Naturally, we restrict our attention to TC-δ games for which (6) holds true. For these games, we provide a sufficient condition for core non-emptiness. Along with this result, we also provide an allocation rule that produces allocations that belong to the core.

Theorem 4. *If* $\delta \leq 1 - \max_{i\in N} \frac{v^\theta(\{i\})}{MSE_i}$, *then* $(1-\delta)MSE \in \mathscr{C}(N, v^\delta) \neq \emptyset$.

The proof of Theorem 4 consists of two steps. First we show that $(1-\delta)MSE$ is efficient, which follows by its construction. Thereafter, we show that $(1-\delta)MSE$ is stable. In doing so, we use that allocations of MSE belong to the core of game (N, v^θ).

We conclude this section with an example.

Example 7. *Reconsider the TC situation of Example 1 with* $\delta = 0.08$. *In Table 8 below, we present the coalitional values for game* $(N, v^{\theta,\delta})$.

M	{∅}	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$v^\theta(M)$	0	-0.440	0.260	0.199	0.212	1.366	0.695	1.644

Table 8: Coalitional values of game $(N, v^{\theta,\delta})$

We have $MSE = (0.738, 0.296, 0.753)$ *and so* $(1-\delta)MSE = (0.679, 0.272, 0.693)$. *It is easy to check that* $(1-\delta)MSE$ *is a core allocation. We could also have concluded this from Theorem 4, since* $\delta = 0.08 < 1 - \max\{\frac{-0.440}{0.738}, \frac{0.260}{0.296}, \frac{0.199}{0.753}\} = 0.124$.

8. Conclusions

In this paper, we introduced a cooperative transport choice (TC) game in which a set of transport operators can collaborate and decide at what prices to offer sustainable urban mobility solutions. To better reflect the decision making process of the travelers, we assume that they chose among the services offered according to the most widely-used disaggregate demand model, the multinomial logit model. To be in line with the conditions associated with horizontal agreement exemptions, our TC game assumes that the transport operators optimize their prices, while keeping their total market share constant.

We presented various intuitive allocation rules for our TC game and studied to which extent these allocation rules produce allocations that belong to the core. We showed that two intuitive proportional allocation rules, as well as the well-known Shapley value do not always generate core allocations and therefore cannot sustain the collaboration. We then introduced a market share exchange allocation rule that first allocates to each transport operator the profit he or she generates under collaboration and subsequently compensates those transport operators that lost market share, with additional profit earned by the ones that gained some extra market share. This exchange of market share is facilitated by a unique price, which can be expressed as the additional return by cooperating per unit of market share. We proved that this allocation rule sustains the collaboration (i.e., the allocations of the market share exchange rule always belong to the core). Finally, we studied a setting where the transport operators need to pay back part of the joint profit to society. We showed that, under some natural conditions, the market share exchange rule still sustains the collaboration. We would like to emphasize that most of our results are stable against some deviations in the modelling of players outside a coalition. For instance, if we would use the pessimistic approach of Lardon (2019) (i.e., players outside a coalition select prices that minimize the coalitional profit), the allocations of the market share exchange rule are still core allocations, implying that the core of our TC game is still non-empty.

While the inspiration for our TC game came from the field of urban mobility, results also hold for other applications, for example in city logistics where horizontal collaboration between courier, express and parcel carriers has been recognized as a possible solution to tackle the 'last-mile' issue. Accordingly, a first natural direction for future research therefore lies in applying our TC game and its properties to real-world problems (and data). Another direction could be to investigate cooperative games based on more advanced discrete choice models allowing even more complex and precise representations of individual behavior.

Finally we want to conclude by saying that within the transport community, there is a growing interest in exploiting multidisciplinary methods. By investigating a choice-based cooperative game, we hope that we successfully contributed to bridge the gap between cooperative game theory and discrete choice modelling and that our study can encourage researchers to combine the strengths of these two fields.

Acknowledgements

The authors would like to thank dr. Ahmadreza Marandi for the fruitful discussion on optimality conditions for non-linear, constraint, optimization problems.

References

R. L. Abduljabbar, S. Liyanage, and H. Dia. The role of micro-mobility in shaping sustainable cities: A systematic literature review. *Transportation research part D: transport and environment*, 92:102734, 2021.

N. Adler, A. Brudner, and S. Proost. A review of transport market modeling using game-theoretic principles. *European Journal of Operational Research*, 2020.

Y. Akçay, H. P. Natarajan, and S. H. Xu. Joint dynamic pricing of multiple perishable products under consumer choice. *Management Science*, 56(8):1345–1361, 2010.

M. Aksoy-Pierson, G. Allon, and A. Federgruen. Price competition under mixed multinomial logit demand functions. *Management Science*, 59(8):1817–1835, 2013.

S. Anily and M. Haviv. Line balancing in parallel M/M/1 lines and loss systems as cooperative games. *Production and Operations Management*, 26(8):1568–1584, 2017.

C. Arbib, M. Ç. Pınar, and M. Tonelli. Competitive location and pricing on a line with metric transportation costs. *European journal of operational research*, 282(1):188–200, 2020.

G. Aydin and E. L. Porteus. Joint inventory and pricing decisions for an assortment. *Operations Research*, 56(5):1247–1255, 2008.

G. Aydin and J. K. Ryan. Product line selection and pricing under the multinomial logit choice model. In *Proceedings of the 2000 MSOM conference*. Citeseer, 2000.

F. Azadian and A. Murat. Service location grouping and pricing in transportation: Application in air cargo. *European Journal of Operational Research*, 267(3):933–943, 2018.

M. S. Bazaraa, H. D. Sherali, and C. M. Shetty. *Nonlinear programming: theory and algorithms*. John Wiley & Sons, 2013.

M. Ben-Akiva and M. Bierlaire. Discrete choice models with applications to departure time and route choice. In *Handbook of transportation science*, pages 7–37. Springer, 2003.

F. Bernstein and A. Federgruen. Dynamic inventory and pricing models for competing retailers. *Naval Research Logistics (NRL)*, 51(2):258–274, 2004.

D. Bertsimas, Y. Sian Ng, and J. Yan. Joint frequency-setting and pricing optimization on multimodal transit networks at scale. *Transportation Science*, 54(3):839–853, 2020.

M. Bierlaire and V. Lurkin. Introduction to disaggregate demand models. In *Leading Developments from INFORMS Communities*, pages 48–67. INFORMS, 2017.

S. Bortolomiol, V. Lurkin, and M. Bierlaire. A simulation-based heuristic to find approximate equilibria with disaggregate demand models. *Transportation Science*, 2021.

F. Bostoen. Competition law in the peer-to-peer economy. *The Platform Economy (Intersentia, 2019)*, 2018.

P. Dehez and D. Tellone. Data games: Sharing public goods with exclusion. *Journal of Public Economic Theory*, 15(4):654–673, 2013.

L. Dong, P. Kouvelis, and Z. Tian. Dynamic pricing and inventory control of substitute products. *Manufacturing & Service Operations Management*, 11(2):317–339, 2009.

C. Du, W. L. Cooper, and Z. Wang. Optimal pricing for a multinomial logit choice model with network effects. *Operations Research*, 64(2):441–455, 2016.

S. Engevall, M. Göthe-Lundgren, and P. Värbrand. The heterogeneous vehicle-routing game. *Transportation Science*, 38(1):71–85, 2004.

N. Fearnley. Micromobility–regulatory challenges and opportunities. In *Shaping Smart Mobility Futures: Governance and Policy Instruments in times of Sustainability Transitions*. Emerald Publishing Limited, 2020.

G. Gallego and R. Wang. Multiproduct price optimization and competition under the nested logit model with product-differentiated price sensitivities. *Operations Research*, 62(2):450–461, 2014.

F. Gilbert, P. Marcotte, and G. Savard. Mixed-logit network pricing. *Computational Optimization and Applications*, 57(1):105–127, 2014.

V. Ginsburgh and I. Zang. The museum pass game and its value. *Games and economic behavior*, 43(2):322–325, 2003.

W. Hanson and K. Martin. Optimizing multinomial logit profit functions. *Management Science*, 42(7):992–1003, 1996.

B. Hezarkhani, M. Slikker, and T. Van Woensel. A competitive solution for cooperative truckload delivery. *Or Spectrum*, 38(1):51–80, 2016.

W. J. Hopp and X. Xu. Product line selection and pricing with modularity in design. *Manufacturing & Service Operations Management*, 7(3):172–187, 2005.

W. T. Huh and H. Li. Pricing under the nested attraction model with a multistage choice structure. *Operations Research*, 63(4):840–850, 2015.

H. Jalali, R. Carmen, I. Van Nieuwenhuyse, and R. Boute. Quality and pricing decisions in production/inventory systems. *European Journal of Operational Research*, 272(1):195–206, 2019.

A. Kimms and I. Kozeletskyi. Core-based cost allocation in the cooperative traveling salesman problem. *European Journal of Operational Research*, 248(3):910–916, 2016.

A. Lardon. On the coalitional stability of monopoly power in differentiated bertrand and cournot oligopolies. *Theory and Decision*, 87(4):421–449, 2019.

A. Lardon. The core in bertrand oligopoly tu-games with transferable technologies. *The BE Journal of Theoretical Economics*, 20(1), 2020.

Y. Levin, J. McGill, and M. Nediak. Dynamic pricing in the presence of strategic consumers and oligopolistic competition. *Management science*, 55(1):32–46, 2009.

H. Li and W. T. Huh. Pricing multiple products with the multinomial logit and nested logit models: Concavity and implications. *Manufacturing & Service Operations Management*, 13(4):549–563, 2011.

H. Li and S. Webster. Optimal pricing of correlated product options under the paired combinatorial logit model. *Operations Research*, 65(5):1215–1230, 2017.

H. Li, S. Webster, N. Mason, and K. Kempf. Product-line pricing under discrete mixed multinomial logit demand: Winner—2017 M&SOM practice-based research competition. *Manufacturing & Service Operations Management*, 21(1):14–28, 2019.

K. Y. Lin and S. Y. Sibdari. Dynamic price competition with discrete customer choices. *European Journal of Operational Research*, 197(3):969–980, 2009.

S. Lozano, P. Moreno, B. Adenso-Díaz, and E. Algaba. Cooperative game theory approach to allocating benefits of horizontal cooperation. *European Journal of Operational Research*, 229(2):444–452, 2013.

B. Maddah and E. K. Bish. Joint pricing, assortment, and inventory decisions for a retailer's product line. *Naval Research Logistics (NRL)*, 54(3):315–330, 2007.

P. Milgrom and J. Roberts. Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica: Journal of the Econometric Society*, pages 1255–1277, 1990.

T. Møller and J. Simlett. Micromobility: moving cities into a sustainable future. ey, 2020.

W. R. Morrow and S. J. Skerlos. Fixed-point approaches to computing Bertrand-Nash equilibrium prices under mixed-logit demand. *Operations research*, 59(2):328–345, 2011.

H. Moulin. Equal or proportional division of a surplus, and other methods. *International Journal of Game Theory*, 16(3):161–186, 1987.

M. Naor. On fairness in the carpool problem. *Journal of Algorithms*, 55(1):93–98, 2005.

L. Schlicher, M. Slikker, and G.-J. van Houtum. Probabilistic resource pooling games. *Naval Research Logistics (NRL)*, 64(7):531–546, 2017.

L. Schlicher, M. Slikker, and G.-J. van Houtum. Pooling of critical, low-utilization resources with unavailability. *OR Spectrum*, 40(1):233–263, 2018.

L. Schlicher, M. Slikker, W. van Jaarsveld, and G.-J. van Houtum. Core nonemptiness of stratified pooling games: a structured markov decision process approach. *Mathematics of Operations Research*, 45(4):1445–1465, 2020.

L. S. Shapley. A value for n-person games. *Contributions to the Theory of Games*, 2(28): 307–317, 1953.

L. S. Shapley and M. Shubik. On market games. *Journal of Economic Theory*, 1(1):9–25, 1969.

J.-S. Song and Z. Xue. Demand management and inventory control for substitutable products. *Working paper*, 2007.

M. Sumida, G. Gallego, P. Rusmevichientong, H. Topaloglu, and J. Davis. Revenue-utility tradeoff in assortment optimization under the multinomial logit model with totally unimodular constraints. Technical report, Cornell University, Ithaca, NY, 2019.

Z. Sun, A. C. Hupman, and A. E. Abbas. The value of information for price dependent demand. *European Journal of Operational Research*, 288(2):511–522, 2021.

C. Tawfik and S. Limbourg. Pricing problems in intermodal freight transport: Research overview and prospects. *Sustainability*, 10(9):3341, 2018.

L. E. Westerink-Duijzer, L. Schlicher, and M. Musegaas. Core allocations for cooperation problems in vaccination. *Production and Operations Management*, 29(7):1720–1737, 2020.

D. Zhang and Z. Lu. Assessing the value of dynamic pricing in network revenue management. *INFORMS Journal on Computing*, 25(1):102–115, 2013.

H. Zhang, P. Rusmevichientong, and H. Topaloglu. Multiproduct pricing under the generalized extreme value models with homogeneous price sensitivity parameters. *Operations Research*, 66(6):1559–1570, 2018.

S. Zhong, Y. Jiang, and O. A. Nielsen. Lexicographic multi-objective road pricing optimization considering land use and transportation effects. *European Journal of Operational Research*, 2021.

Appendix A

In this section, we present the proofs of all theorems and lemmas. For proofs of remarks (Remark 1 and 6) we refer to the document "Supplementary material Appendix A".

Proof of Theorem 1

This proof consists of three steps. First we identify how our maximization optimization problem relates to another, minimization optimization problem. Then, we identify an optimal price vector and the associated optimal value for this minimization optimization problem. Finally, we relate these outcomes to our original optimization problem.

Step 1. An equivalent optimization problem

Let $\theta \in \Theta$. First, we replace the constraint of optimization problem $\mathscr{P}$ by another constraint. Recall that the constraint of $\mathscr{P}$ is given by

$$\frac{\sum_{i\in N} e^{\alpha_i-\beta p_i}}{1+\sum_{j\in N} e^{\alpha_j-\beta p_j}} = \frac{\sum_{i\in N} e^{\alpha_i-\beta x_i}}{1+\sum_{j\in N} e^{\alpha_j-\beta x_j}}. \tag{7}$$

Since an e-function is always strictly positive, the above constraint can be rewritten as

$$\left(\sum_{i\in N} e^{\alpha_i-\beta p_i}\right)\left(1+\sum_{i\in N} e^{\alpha_i-\beta x_i}\right) = \left(\sum_{i\in N} e^{\alpha_i-\beta x_i}\right)\left(1+\sum_{i\in N} e^{\alpha_i-\beta p_i}\right),$$

which is equal to

$$\sum_{i\in N} e^{\alpha_i-\beta p_i} + \left(\sum_{i\in N} e^{\alpha_i-\beta p_i}\right)\left(\sum_{i\in N} e^{\alpha_i-\beta x_i}\right) = \sum_{i\in N} e^{\alpha_i-\beta x_i} + \left(\sum_{i\in N} e^{\alpha_i-\beta x_i}\right)\left(\sum_{i\in N} e^{\alpha_i-\beta p_i}\right).$$

By subtracting $\left(\sum_{i\in N} e^{\alpha_i-\beta p_i}\right)\left(\sum_{i\in N} e^{\alpha_i-\beta x_i}\right)$ on both sides, we obtain:

$$\sum_{i\in N} e^{\alpha_i-\beta p_i} = \sum_{i\in N} e^{\alpha_i-\beta x_i}, \tag{8}$$

Hence, we can replace constraint (7) by constraint (8).

Now, let $\gamma = 1/\left(1+\sum_{j\in N} e^{\alpha_j-\beta p_j}\right)$. By using (8), $\mathscr{P}$ can be reformulated as:

$$\begin{aligned} \mathscr{P} = \max_{x\in\mathbb{R}^N} \gamma \sum_{i\in N}(x_i-c_i)e^{\alpha_i-\beta x_i} \\ \text{s.t. } \sum_{i\in N} e^{\alpha_i-\beta p_i} = \sum_{i\in N} e^{\alpha_i-\beta x_i}. \end{aligned} \tag{9}$$

By using the shorthand notation $D(x) = \sum_{i\in N} e^{\alpha_i-\beta x_i}$ for all $x \in \mathbb{R}$, the above optimization problem can be rewritten as:

$$\begin{aligned} \max_{x\in\mathbb{R}^N} \gamma \sum_{i\in N}(x_i-c_i)e^{\alpha_i-\beta x_i} \\ \text{s.t. } D(p) = \sum_{i\in N} e^{\alpha_i-\beta x_i}. \end{aligned}$$

Observe that we could also study the above's optimization problem without constant γ. In that case, the optimal value would be off a factor γ only. Moreover, since maximizing a certain function is the same as minimizing that function times minus one, it is also possible to study the following optimization problem $\mathscr{P}'$, instead of $\mathscr{P}$:

$$\begin{aligned} \mathscr{P}' = \min_{x\in\mathbb{R}^N} -\sum_{i\in N}(x_i-c_i)e^{\alpha_i-\beta x_i} \\ \text{s.t. } D(p) = \sum_{i\in N} e^{\alpha_i-\beta x_i}. \end{aligned}$$

Please, note that the optimal value of optimization problem $\mathscr{P}$ equals the optimal value of optimization problem $\mathscr{P}$ times $-\gamma$. Moreover, any optimal price vector of $\mathscr{P}'$ is also an optimal price vector in $\mathscr{P}$. Hence, we can thus also study optimization problem $\mathscr{P}'$.

Step 2. Optimal price vector and associated optimal value for $\mathscr{P}'$

In this step, we find the minimal value of $\mathscr{P}'$ and an associated optimal price vector. We do so by applying Theorem 5. In terms of Theorem 5, we can recognize our optimization problem $\mathscr{P}'$ as a nonlinear optimization problem with objective function

$$f(x) = -\sum_{i\in N}(x_i-c_i)e^{\alpha_i-\beta x_i} \text{ for all } x \in \mathbb{R}^N,$$

and equality constraint

$$h(x) = \sum_{i\in N} e^{\alpha_i-\beta x_i} - D(p) = 0 \text{ for all } x \in \mathbb{R}^N.$$

Moreover, for any $\lambda \in \mathbb{R}$, function $\mathscr{L}(\lambda)$ of Theorem 5 is given by

$$\mathscr{L}(\lambda) = \min_{x \in \mathbb{R}^N} f(x) + \lambda h(x) = \min_{x \in \mathbb{R}^N} \left[-\sum_{i \in N} (x_i - c_i) e^{\alpha_i - \beta x_i} + \lambda \left(\sum_{i \in N} e^{\alpha_i - \beta x_i} - D(p) \right) \right].$$

In order to apply Theorem 5, we first solve $\mathscr{L}(\lambda)$ analytically for any $\lambda \in \mathbb{R}$. Thereafter, we construct a feasible solution $x^* \in \mathbb{R}^N$ and $\lambda^* \in \mathscr{L}$ such that $f(x^*) = \mathscr{L}(\lambda^*)$. We solve function $\mathscr{L}(\lambda)$ by dividing the minimization problem in $|N|$ subproblems. We can do so, since there is no dependency between the variables in x. Hence, for any $\lambda \in \mathbb{R}$

$$\mathscr{L}(\lambda) = \sum_{i \in N} \min_{x_i \in \mathbb{R}} \left[-(x_i - c_i) e^{\alpha_i - \beta x_i} + \lambda e^{\alpha_i - \beta x_i} \right] - \lambda D(p) \tag{10}$$

We now solve each subproblem of (10). That is, we solve

$$\min_{x_i \in \mathbb{R}} \left[-(x_i - c_i) e^{\alpha_i - \beta x_i} + \lambda e^{\alpha_i - \beta x_i} \right] \text{ for all } i \in N.$$

We do so by studying the derivative of the objective function of each subproblem. Let $i \in N$. The derivative of the objective function, with respect to x_i, is

$$(\beta(x_i - c_i) - 1 - \lambda\beta) e^{\alpha_i - \beta x_i} \tag{11}$$

From 11, we learn that the objective of each subproblem is a decreasing function for $x_i < \frac{1}{\beta} + c_i + \lambda$, constant for $x_i = \frac{1}{\beta} + c_i + \lambda$ and increasing for $x_i > \frac{1}{\beta} + c_i + \lambda$. Hence, the minimum is attained at $x_i = \frac{1}{\beta} + c_i + \lambda$, with objective value

$$-\frac{1}{\beta} e^{a_i - 1 - \beta c_i - \beta\lambda}.$$

By applying the above analysis for each subproblem, we conclude, for each $\lambda \in \mathbb{R}$, that

$$\mathscr{L}(\lambda) = -\frac{1}{\beta} e^{-1-\lambda\beta} \sum_{i \in N} e^{a_i - \beta c_i} - \lambda D(p) = -\frac{1}{\beta} e^{-1-\lambda\beta} D(c) - \lambda D(p).$$

Now, let $\lambda^* = \frac{1}{\beta}\left(\ln\left(\frac{D(c)}{D(p)}\right) - 1\right)$ and $x_i^* = c_i + \frac{1}{\beta}\ln\left(\frac{D(c)}{D(p)}\right)$ for all $i \in N$. By substituting x^* in the objective function of $\mathscr{P}'$, we obtain

$$f(x^*) = -\sum_{i \in N} (x_i^* - c_i) e^{\alpha_i - \beta x_i^*} = -\frac{1}{\beta} \ln\left(\frac{D(c)}{D(p)}\right) \sum_{i \in N} e^{\alpha_i - \beta c_i + \ln\left(\frac{D(p)}{D(c)}\right)} = -\frac{1}{\beta} \ln\left(\frac{D(c)}{D(p)}\right) D(p). \tag{12}$$

Moreover, $\mathscr{L}(\lambda^*)$ gives

$$\begin{aligned}\mathscr{L}(\lambda^*) &= -\frac{1}{\beta}e^{-1-\lambda^*\beta}D(c) - \lambda D(p) = -\frac{1}{\beta}\frac{D(p)}{D(c)}D(c) - \lambda^* D(p) = -D(p)\left(\frac{1}{\beta}+\lambda^*\right) \\ &= -D(p)\frac{1}{\beta}\ln\left(\frac{D(c)}{D(p)}\right)\end{aligned} \tag{13}$$

By combining (12) and (13), we learn that

$$f(x^*) = \mathscr{L}(\lambda^*) = -D(p)\frac{1}{\beta}\ln\left(\frac{D(c)}{D(p)}\right).$$

Hence, by Theorem 5, we can conclude that x^* is an optimal price vector of $\mathscr{P}'$.

Step 3. Back to our original optimization problem

In step 1, we learned that x^* is also an optimal price vector of $\mathscr{P}$. Moreover, in step 1, we learned that the optimal value of $\mathscr{P}$ equals the optimal value of $\mathscr{P}'$ times $-\gamma$. Hence, the optimal value of optimization problem $\mathscr{P}$ is

$$-\gamma \cdot -D(p)\frac{1}{\beta}\ln\left(\frac{D(c)}{D(p)}\right) = \frac{D(p)}{\beta(1+D(p))}\ln\left(\frac{D(c)}{D(p)}\right). \tag{14}$$

This concludes the proof. □

Proof of Theorem 2

Let $\theta \in \Theta$ and $M \subseteq N\backslash\{\emptyset\}$. First, we replace the constraint of the optimization problem of M by another constraint. Recall that the constraint is given by

$$\frac{\sum_{i\in M} e^{\alpha_i-\beta p_i}}{1+\sum_{j\in N} e^{\alpha_i-\beta p_i}} = \frac{\sum_{i\in M} e^{\alpha_i-\beta x_i}}{1+\sum_{i\in M} e^{\alpha_i-\beta x_i}+\sum_{j\in N\backslash M} e^{\alpha_i-\beta p_i}}. \tag{15}$$

Since an e-function is always strictly positive, we also have

$$\left(\sum_{i\in M} e^{\alpha_i-\beta p_i}\right)\left(1+\sum_{i\in M} e^{\alpha_i-\beta x_i}+\sum_{j\in N\backslash M} e^{\alpha_i-\beta p_i}\right) = \left(\sum_{i\in M} e^{\alpha_i-\beta x_i}\right)\left(1+\sum_{j\in N} e^{\alpha_i-\beta p_i}\right).$$

By subtracting $(\sum_{i\in M} e^{\alpha_i-\beta p_i})(\sum_{i\in M} e^{\alpha_i-\beta x_i})$ on both sides, we obtain

$$\left(\sum_{i\in M} e^{\alpha_i-\beta p_i}\right)\left(1+\sum_{j\in N\backslash M} e^{\alpha_i-\beta p_i}\right) = \left(\sum_{i\in M} e^{\alpha_i-\beta x_i}\right)\left(1+\sum_{j\in N\backslash M} e^{\alpha_i-\beta p_i}\right).$$

Observe that $(1+\sum_{i\in N\backslash M} e^{\alpha_i-\beta p_i})$ is a strictly positive constant, and so we can divide both sides of the last equation by this term. This leads to

$$\sum_{i \in M} e^{\alpha_i - \beta p_i} = \sum_{i \in M} e^{\alpha_i - \beta x_i}. \tag{16}$$

Hence, we can replace constraint (15) by constraint (16).

Now, let $\gamma = 1/(1 + \sum_{j \in N} e^{\alpha_i - \beta p_j}) > 0$. By (16), our optimization problem becomes

$$\max_{x \in \mathbb{R}^M} \gamma \sum_{i \in M} (x_i - c_i) e^{\alpha_i - \beta x_i}$$
$$\text{s.t. } \sum_{i \in M} e^{\alpha_i - \beta p_i} = \sum_{i \in M} e^{\alpha_i - \beta x_i}.$$

Above optimization problem is exactly equal to optimization problem (9) of the proof of Theorem 1, except that we consider set M in stead of N. So, we can now use Theorem 1 (see equation (14)), to conclude that the optimal value of our optimization problem equals

$$-\gamma \cdot -D^M(p) \frac{1}{\beta} \ln \left(\frac{D^M(c)}{D^M(p)} \right) = \frac{D^M(p)}{\beta(D^N(p)+1)} \ln \left(\frac{D^M(c)}{D^M(p)} \right).$$

Hence, $v^\theta(M) = \frac{D^M(p)}{\beta(D^N(p)+1)} \ln \left(\frac{D^M(c)}{D^M(p)} \right)$, which concludes the proof.

Lemma 1. *For all $A, B \in \mathbb{R}_{++}$, it holds that*

$$A \geq B \left(\ln \left(\frac{A}{B} \right) + 1 \right)$$

Proof : First, we prove that $e^y - ye \geq 0$ for all $y \in \mathbb{R}$. We do so by showing that the function is convex, and identifying that the minimal value of this function equals 0. The function is convex, since $\frac{d^2}{d^2y}(e^y - ye) = e^y \geq 0$. Moreover, we have $\frac{d}{dy}(e^y - ye) = e^y - e$, implying that the minimal value is attained at $y = 1$ with associated function value $e^1 - 1 \cdot e = 0$.

Now, let $A, B \in \mathbb{R}_{++}$. Note that $\frac{A}{B}$ exists, because $A, B > 0$. We just learned that $e^y - ye \geq 0$ for all $y \in \mathbb{R}$, and so, we also have

$$e^{\frac{A}{B}} \geq \frac{A}{B} e$$
$$\iff (e^{\frac{A}{B}})^B \geq \left(\frac{A}{B} e \right)^B$$
$$\iff e^A \geq e^{\ln\left(\left(\frac{A}{B} e \right)^B \right)}$$
$$\iff A \geq B \ln \left(\frac{eA}{B} \right)$$
$$\iff A \geq B \left(\ln \left(\frac{A}{B} \right) + 1 \right)$$

where the first implication holds since $e^{\frac{A}{B}} > 0$, $\frac{A}{B}e > 0$ and $B > 0$. The second implication follows from the fact that $e^{\ln x} = x$ for $x \in \mathbb{R}_{++}$ and $(e^x)^y = e^{xy}$ for all $x, y \in \mathbb{R}$. The third implication results from the fact that $e > 0$ and $\ln(x^y) = y\ln(x)$ for all $x, y \in \mathbb{R}_{++}$. The last implication is a result of property $\ln(xy) = \ln(x) + \ln(y)$ for all $x, y \in \mathbb{R}_{++}$ and the fact that $\ln(e) = 1$. This concludes the proof. $\square$

Proof of Theorem 3

Let $\theta \in \Theta$ and consider the associated game (N, v^θ). We show that allocation $MSE \in \mathscr{C}(N, v^\theta)$. We do so by showing that

$$\sum_{i \in N} MSE_i = v^\theta(N),$$

$$\sum_{i \in M} MSE_i \geq v^\theta(M) \text{ for all } M \subseteq N.$$

For the first part, recall that

$$MSE_i = (p^* - c_i)\frac{D^{\{i\}}(p^*)}{D^N(p^*) + 1} - \phi\left(\frac{D^{\{i\}}(p^*)}{D^N(p^*) + 1} - \frac{D^{\{i\}}(p)}{D^N(p) + 1}\right) \text{ for all } i \in N.$$

Now, observe that

$$\begin{aligned}
\sum_{i \in N} MSE_i &= \sum_{i \in N}\left((p_i^* - c_i)\frac{D^{\{i\}}(p^*)}{D^N(p^*) + 1} - \phi\left(\frac{D^{\{i\}}(p^*)}{D^N(p^*) + 1} - \frac{D^{\{i\}}(p)}{D^N(p) + 1}\right)\right) \\
&= \sum_{i \in N}(p_i^* - c_i)\frac{D^{\{i\}}(p^*)}{D^N(p^*) + 1} - \phi\left(\sum_{i \in N}\frac{D^{\{i\}}(p^*)}{D^N(p^*) + 1} - \sum_{i \in N}\frac{D^{\{i\}}(p)}{D^N(p) + 1}\right) \\
&= \frac{1}{\beta}\ln\left(\frac{D^N(c)}{D^N(p)}\right)\frac{D^N(p)}{D^N(p) + 1} \\
&= v^\theta(N).
\end{aligned}$$

In the third equality, we use the definition of p^*, apply that $\sum_{i \in N} D^{\{i\}}(p^*) = D^N(p^*)$ and consequently use that $\frac{D^N(p^*)}{D^N(p^*)+1} = \frac{D^N(p)}{D^N(p)+1}$ (i.e., the total market share remains stable).

Now, it remains to prove $\sum_{i \in M} MSE_i \geq v^\theta(M)$ for all $M \subseteq N$. First, observe that,

$$\phi = \frac{v^\theta(N) - v^{\theta^*}(N)}{\left(\frac{D^N(p)}{D^N(p)+1}\right)} = \frac{1}{\beta}\ln\left(\frac{D^N(c)}{D^N(p)}\right) - \frac{1}{\beta}. \tag{17}$$

Next, by exploiting p^*, we learn that

$$D^N(p^*) = \sum_{i \in N} e^{\alpha_i - \beta p_i^*} = \sum_{i \in N} e^{\alpha_i - \beta\left(c_i + \frac{1}{\beta}\ln\left(\frac{D^N(c)}{D^N(p)}\right)\right)} = D^N(c) \cdot \frac{D^N(p)}{D^N(c)} = D^N(p). \tag{18}$$

By using (17), (18) and exploiting p^*, we can reformulate MSE_i for all $i \in N$ as follows

$$\begin{aligned}
MSE_i &= (p_i^* - c_i)\frac{D^{\{i\}}(p^*)}{D^N(p^*)+1} - \phi\left(\frac{D^{\{i\}}(p^*)}{D^N(p^*)+1} - \frac{D^{\{i\}}(p)}{D^N(p)+1}\right) \\
&= \frac{1}{D^N(p)+1}\left[(p_i^* - c_i)D^{\{i\}}(p^*) - \phi\left(D^{\{i\}}(p^*) - D^{\{i\}}(p)\right)\right] \\
&= \frac{1}{D^N(p)+1}\left[\frac{1}{\beta}\ln\left(\frac{D^N(c)}{D^N(p)}\right)\frac{D^N(p)}{D^N(c)}D^{\{i\}}(c)\right. \\
&\qquad \left. - \left(\frac{1}{\beta}\ln\left(\frac{D^N(c)}{D^N(p)}\right) - \frac{1}{\beta}\right)\left(\frac{D^N(p)}{D^N(c)}D^{\{i\}}(c) - D^{\{i\}}(p)\right)\right] \\
&= \frac{1}{D^N(p)+1}\left[\frac{1}{\beta}\ln\left(\frac{D^N(c)}{D^N(p)}\right)D^{\{i\}}(p) + \frac{1}{\beta}\left(\frac{D^N(p)}{D^N(c)}D^{\{i\}}(c) - D^{\{i\}}(p)\right)\right] \\
&= \frac{1}{\beta(D^N(p)+1)}\left[\ln\left(\frac{D^N(c)}{D^N(p)}\right)D^{\{i\}}(p) + \frac{D^N(p)}{D^N(c)}D^{\{i\}}(c) - D^{\{i\}}(p)\right].
\end{aligned} \tag{19}$$

Moreover, from Lemma 1, we learned that

$$A \geq B\left(\ln\left(\frac{A}{B}\right) + 1\right) \tag{20}$$

Now, fix an $M \subseteq N$. Moreover, let's use the inequality of (20), and set $A = \frac{D^M(c)D^N(p)}{D^N(c)}$ and $B = D^M(p) > 0$. Then,

$$\begin{aligned}
&\frac{D^M(c)D^N(p)}{D^N(c)} \geq D^M(p)\left(\ln\left(\frac{D^M(c)D^N(p)}{D^N(c)D^M(p)}\right) + 1\right) \\
&\iff \frac{D^M(c)D^N(p)}{D^N(c)} \geq D^M(p)\left(\ln\left(\frac{D^M(c)}{D^M(p)}\right) + \ln\left(\frac{D^N(p)}{D^N(c)}\right) + 1\right) \\
&\iff D^M(p)\ln\left(\frac{D^N(c)}{D^N(p)}\right) + \frac{D^M(c)D^N(p)}{D^N(c)} - D^M(p) \geq D^M(p)\ln\left(\frac{D^M(c)}{D^M(p)}\right) \\
&\iff \frac{1}{\beta(D^N(p)+1)}\left(D^M(p)\ln\left(\frac{D^N(c)}{D^N(p)}\right) + \frac{D^N(p)}{D^N(c)}D^M(c) - D^M(p)\right) \\
&\qquad \geq \frac{D^M(p)}{\beta(D^N(p)+1)}\ln\left(\frac{D^M(c)}{D^M(p)}\right) \\
&\iff \frac{1}{\beta(D^N(p)+1)}\left(\ln\left(\frac{D^N(c)}{D^N(p)}\right)D^M(p) + \frac{D^N(p)}{D^N(c)}D^M(c) - D^M(p)\right) \geq v^\theta(M)
\end{aligned} \tag{21}$$

Please, note that in the last inequality we used the definition of $v^\theta(M)$. By using the last equality of (19) and the last inequality of (21), we have

$$\begin{aligned}\sum_{i\in M} MSE_i &= \sum_{i\in M} \frac{1}{\beta(D^N(p)+1)}\left(\ln\left(\frac{D^N(c)}{D^N(p)}\right) D^{\{i\}}(p) + \frac{D^N(p)}{D^N(c)} D^{\{i\}}(c) - D^{\{i\}}(p)\right)\\ &= \frac{1}{\beta(D^N(p)+1)}\left(\ln\left(\frac{D^N(c)}{D^N(p)}\right)\sum_{i\in M} D^{\{i\}}(p) + \frac{D^N(p)}{D^N(c)}\sum_{i\in M} D^{\{i\}}(c) - \sum_{i\in M} D^{\{i\}}(p)\right)\\ &= \frac{1}{\beta(D^N(p)+1)}\left(\ln\left(\frac{D^N(c)}{D^N(p)}\right) D^{M}(p) + \frac{D^N(p)}{D^N(c)} D^{M}(c) - D^{M}(p)\right)\\ &\geq v^{\theta}(M),\end{aligned}$$

which is exactly what we need to show. This concludes the proof. □

<u>Proof of Theorem 4</u>

Let $\delta \leq 1-\max_{i\in N}\frac{v^{\theta}(\{i\})}{MSE_i}$. We will show that allocation rule $(1-\delta)MSE \in \mathscr{C}(N, v^{\theta,\delta}) \neq \emptyset$. We do so by showing that $(1-\delta)MSE$ satisfies

$$\sum_{i\in N}(1-\delta)MSE_i = v^{\theta,\delta}(N),$$

$$\sum_{i\in M}(1-\delta)MSE_i \geq v^{\theta,\delta}(M) \text{ for all } M \subseteq N.$$

For the first part, observe that

$$\sum_{i\in N}(1-\delta)MSE_i = (1-\delta)\sum_{i\in N} MSE_i = (1-\delta)v^{\theta}(N) = v^{\theta,\delta}(N),$$

where the second equality holds since $MSE \in \mathscr{C}(N, v^{\theta})$ (See Theorem 3).

Now, it remains to prove that $\sum_{i\in M}(1-\delta)MSE_i \geq v^{\theta,\delta}(M)$ for all $M \subseteq N$. Since $\delta \leq 1-\max_{i\in N}\frac{v^{\theta}(\{i\})}{MSE_i}$, we also have $\delta \leq 1-\frac{v^{\theta}(\{i\})}{MSE_i}$ for all $i \in N$. From this, we can conclude that $(1-\delta)MSE_i \geq v^{\theta}(\{i\}) = v^{\theta,\delta}(\{i\})$ for all $i \in N$. Hence, it remains to show that $\sum_{i\in M}(1-\delta)MSE_i \geq v^{\theta,\delta}(M)$ for all $M \subseteq N$ with $|M| \geq 2$.

Let $M \subseteq N$ with $|M| \geq 2$. We have

$$\sum_{i\in M}(1-\delta)MSE_i = (1-\delta)\sum_{i\in M} MSE_i \geq (1-\delta)v^{\theta}(M) = v^{\theta,\delta}(M),$$

where the inequality holds since $MSE \in \mathscr{C}(N, v^{\theta})$ (see Theorem 3). This implies that $\sum_{i\in M} MSE_i^{\delta} \geq v^{\theta,\delta}(M)$ for all $M \subseteq N$. This concludes the proof. □

Theorem 5. *Let $f : \mathbb{R}^N \to \mathbb{R}$ and $h(x) : \mathbb{R}^N \to \mathbb{R}$. If $x^* \in \mathbb{R}^N$ is a feasible solution of nonlinear programming problem $\mathscr{G}$:*

$$\begin{aligned}\mathscr{G} = \min f(x)&\\ h(x) = 0&\\ x \in \mathbb{R}^N,&\end{aligned}$$

and $f(x^*) = \mathscr{L}(\lambda)$ *for some* $\lambda \in \mathbb{R}$, *where*

$$\mathscr{L}(\lambda) = \min_{x \in \mathbb{R}^N} \{f(x) + \lambda h(x)\}$$

then x^* *is an optimal solution of* $\mathscr{G}$.

Proof: See Bazaraa et al. (2013), chapter 6, corollary 2. □

Supplementary material Appendix A

Proof of Remark 1

For each TC situation $\theta \in \Theta$ for which p is a Nash equilibrium, we show that

$$p_i = \frac{1 + W\left(\frac{e^{\alpha_i - 1 - \beta c_i}}{1+\sum_{j \neq i} e^{\alpha_j - p_j \beta}}\right)}{\beta} + c_i \quad \text{for all } i \in N. \tag{22}$$

Let $\theta \in \Theta$ and p be a Nash equilibrium. Hence, p is also a solution of the set of first-order conditions, based on the profit functions of the individual transport operators. We will now derive these first-order conditions and show that they coincide with equation (22).

The derivative of the profit function of transport operator $i \in N$, with respect to price (which we denote by p'_i instead of p_i for notational convenience), equals

$$\begin{aligned}
&\frac{d}{dp'_i}\left((p'_i - c_i) \cdot \left(\frac{e^{\alpha_i - \beta p'_i}}{1 + \sum_{j \in N} e^{\alpha_j - \beta p'_j}}\right)\right) \\
&= \frac{e^{\alpha_i - \beta p'_i}}{1 + \sum_{j \in N} e^{\alpha_j - \beta p'_j}} \left(1 - \beta\left(\frac{1 + \sum_{j \in N \setminus \{i\}} e^{\alpha_j - \beta p'_j}}{1 + \sum_{j \in N} e^{\alpha_j - \beta p'_j}}\right)(p'_i - c_i)\right).
\end{aligned}$$

Since $\frac{e^{\alpha_k - \beta p'_k}}{1+\sum_{j \in N} e^{\alpha_j - \beta p'_j}} > 0$ for all $k \in N$, the first-order conditions (with Nash equilibrium prices p) read as follows

$$1 - \beta\left(1 + \frac{\sum_{j \in N \setminus \{i\}} e^{\alpha_j - \beta p_j}}{1 + \sum_{j \in N} e^{\alpha_j - \beta p_j}}\right)(p_i - c_i) = 0 \text{ for all } i \in N.$$

These first-order conditions can be written in terms of the Nash equilibrium prices p

$$p_i = \frac{1}{\beta \cdot \frac{1+\sum_{j \in N \setminus \{i\}} e^{\alpha_j - \beta p_j}}{1+\sum_{j \in N} e^{\alpha_j - \beta p_j}}} + c_i \text{ for all } i \in N. \tag{23}$$

We continue by rewriting equation (23) towards equation (22). For all $i \in N$, we have

$$\begin{aligned}
p_i &= \frac{1}{\beta \cdot \left(1 - \frac{e^{\alpha_i - \beta p_i}}{1+\sum_{j \in N} e^{\alpha_j - \beta p_j}}\right)} + c_i && (24) \\
&= \frac{1}{\frac{\beta \cdot (1+\sum_{j \in N} e^{\alpha_j - \beta p_j}) - \beta \cdot (e^{\alpha_i - \beta p_i})}{1+\sum_{j \in N} e^{\alpha_j - \beta p_j}}} + c_i && (25) \\
&= \frac{1 + \sum_{j \in N} e^{\alpha_j - \beta p_j}}{\beta \cdot (1 + \sum_{j \in N \setminus \{i\}} e^{\alpha_j - \beta p_j})} + c_i && (26)
\end{aligned}$$

$$= \frac{1 + \sum_{j \in N \setminus \{i\}} e^{\alpha_j - \beta p_j} + e^{\alpha_i - \beta p_i}}{\beta \cdot (1 + \sum_{j \in N \setminus \{i\}} e^{\alpha_j - \beta p_j})} + c_i \tag{27}$$

$$= \frac{1}{\beta} + \frac{e^{\alpha_i - \beta p_i}}{\beta \cdot A} + c_i \tag{28}$$

with $A = 1 + \sum_{j \in N \setminus \{i\}} e^{\alpha_j - \beta p_j}$.

Multiplying equation (28) by β and then subtracting α_i, we obtain

$$\beta p_i - \alpha_i = 1 + \frac{e^{\alpha_i - \beta p_i}}{A} + \beta c_i - \alpha_i \tag{29}$$

$$\iff \frac{e^{\alpha_i - \beta p_i}}{A} - \beta p_i + \alpha_i = \alpha_i - 1 - \beta c_i \tag{30}$$

Taking exponential on both sides of equation (30) and using that $A > 0$, we have

$$e^{\frac{e^{\alpha_i - \beta p_i}}{A} - \beta p_i + \alpha_i} = e^{\alpha_i - 1 - \beta c_i} \tag{31}$$

$$\iff e^{\frac{e^{\alpha_i - \beta p_i}}{A}} \cdot e^{\alpha_i - \beta p_i} = e^{\alpha_i - 1 - \beta c_i} \tag{32}$$

$$\iff e^{\frac{e^{\alpha_i - \beta p_i}}{A}} \cdot \frac{e^{\alpha_i - \beta p_i}}{A} = \frac{e^{\alpha_i - 1 - \beta c_i}}{A}, \tag{33}$$

Now, observe that equation (33) can reformulated in terms of the classic LambertW equation (i..e, as $e^W \cdot W = c$ for some $c \in \mathbb{R}$). In particular, (33) can be reformulated as

$$\frac{e^{\alpha_i - \beta p_i}}{A} = W\left(\frac{e^{\alpha_i - 1 - \beta c_i}}{A}\right) \tag{34}$$

Taking logarithms on both side of (34), which is allowed since $\frac{e^{\alpha_i - \beta p_i}}{A} > 0$, we have

$$\ln(\frac{e^{\alpha_i - \beta p_i}}{A}) = \ln\left(W\left(\frac{e^{\alpha_i - 1 - \beta c_i}}{A}\right)\right) \tag{35}$$

Using the logarithmic property of the LambertW function (i.e., $\ln(W(x)) = \ln(x) - W(x)$ for any $x \in \mathbb{R}_{++}$) and the fact that $e^{\alpha - 1 - \beta c_i}/A, e^{\alpha_i - \beta p_i}/A > 0$, equation (35) becomes:

$$\ln\left(\frac{e^{\alpha_i - \beta p_i}}{A}\right) = \ln\left(\frac{e^{\alpha_i - 1 - \beta c_i}}{A}\right) - W\left(\frac{e^{\alpha_i - 1 - \beta c_i}}{A}\right) \tag{36}$$

$$\iff \alpha_i - \beta p_i - \ln(A) = \alpha_i - 1 - \beta c_i - \ln(A) - W\left(\frac{e^{\alpha_i - 1 - \beta c_i}}{A}\right) \tag{37}$$

$$\iff -\beta p_i = -1 - \beta c_i - W\left(\frac{e^{\alpha_i - 1 - \beta c_i}}{A}\right) \tag{38}$$

$$\iff \beta p_i = 1 + \beta c_i + W\left(\frac{e^{\alpha_i - 1 - \beta c_i}}{A}\right) \tag{39}$$

$$\iff p_i = \frac{1 + W\left(\frac{e^{\alpha_i - 1 - \beta c_i}}{A}\right)}{\beta} + c_i \tag{40}$$

Substituting $A = 1 + \sum_{j \neq i} e^{\alpha_j - \beta p_j}$, we obtain

$$p_i = \frac{1 + W(\frac{e^{\alpha_i - 1 - \beta c_i}}{1 + \sum_{j \neq i} e^{\alpha_j - \beta p_j}})}{\beta} + c_i, \tag{41}$$

which we needed to show. This concludes the proof. □

Proof of Remark 6

First we will show that our game is superadditive, i.e., $v^\theta(M) + v^\theta(K) \leq v^\theta(M \cup K)$ for all $M, K \subseteq N$ with $M \cap K = \emptyset$ and all $\theta \in \Theta$. Let $\theta \in \Theta$ and $M, K \subseteq N$ with $M \cap K = \emptyset$. Moreover, let $x^M = (x_i^M)_{i \in M}$ be an optimal solution of the optimization problem of coalition M. Similarly, let $x^K = (x_i^K)_{i \in K}$ be an optimal solution of the optimization problem of coalition K. Next, let

$$x_i^{M \cup K} = \begin{cases} x_i^M & \text{if } i \in M \\ \\ x_i^K & \text{if } i \in K. \end{cases}$$

We will show that $x^{M \cup K} = (x_i^{M \cup K})_{i \in M \cup K}$ is a feasible solution of the optimization problem of coalition $M \cup K$. That means, we need to show that the market share constraint of the optimization problem of coalition $M \cup K$ is satisfied. Recall that, due to equation (16) of Theorem 2, the total market share constraint of any coalition $T \subseteq N$ reads as

$$\sum_{i \in T} e^{\alpha_i - \beta x_i} = \sum_{i \in T} e^{\alpha_i - \beta p_i}. \tag{42}$$

Hence, for optimal solutions x^M and x^K, we have

$$\begin{aligned} \sum_{i \in M} e^{\alpha_i - \beta x_i^M} &= \sum_{i \in M} e^{\alpha_i - \beta p_i} \\ \sum_{i \in K} e^{\alpha_i - \beta x_i^K} &= \sum_{i \in K} e^{\alpha_i - \beta p_i}. \end{aligned} \tag{43}$$

As a consequence, we have

$$\sum_{i \in M \cup K} e^{\alpha_i - \beta x_i^{M \cup K}} = \sum_{i \in M} e^{\alpha_i - \beta x_i^M} + \sum_{i \in K} e^{\alpha_i - \beta x_i^K} = \sum_{i \in M} e^{\alpha_i - \beta p_i} + \sum_{i \in K} e^{\alpha_i - \beta p_i} = \sum_{i \in M \cup K} e^{\alpha_i - \beta p_i}.$$

From the above equation, we learn that solution $x^{M \cup K}$ is a feasible solution of the optimization problem of coalition $M \cup K$. Next, we will show that the sum of the objective functions of the optimization problems of M and K, evaluated for x^M and x^K coincide with the objective function of the optimization problem of $M \cup K$, evaluated at $x^{M\cup K}$.

The sum of the objective functions of the optimization problems of M and K, evaluated for x^M and x^K reads as follows

$$\frac{\sum_{i\in M}(x_i^M - c_i)e^{\alpha_i - \beta x_i^M}}{1+\sum_{j\in M} e^{\alpha_j-\beta x_j^M} + \sum_{j\in N\setminus M} e^{\alpha_j - \beta p_j}} + \frac{\sum_{i\in K}(x_i^K - c_i)e^{\alpha_i - \beta x_i^K}}{1+\sum_{j\in K} e^{\alpha_j-\beta x_j^K} + \sum_{j\in N\setminus K} e^{\alpha_j - \beta p_j}}$$
$$= \frac{\sum_{i\in M}(x_i^M - c_i)e^{\alpha_i - \beta x_i^M}}{1+\sum_{j\in N} e^{\alpha_j - \beta p_j}} + \frac{\sum_{i\in K}(x_i^K - c_i)e^{\alpha_i - \beta x_i^K}}{1+\sum_{j\in N} e^{\alpha_j - \beta p_j}}$$
$$= \frac{\sum_{i\in M\cup K}(x_i^{M\cup K} - c_i)e^{\alpha_i - \beta x_i^{M\cup K}}}{1+\sum_{j\in N} e^{\alpha_j - \beta p_j}}$$
$$= \frac{\sum_{i\in M\cup K}(x_i^{M\cup K} - c_i)e^{\alpha_i - \beta x_i^{M\cup K}}}{1+\sum_{j\in M\cup K} e^{\alpha_j-\beta x_j^{M\cup K}} + \sum_{j\in N\setminus (M\cup K)} e^{\alpha_j - \beta p_j}}$$

Note that we used equation (42) for coalitions M and K in the first equality and for coalition $M \cup K$ in the last equality. Next, observe that the expression in the last equation coincides with the objective function of optimization problem $M \cup K$, evaluated for $X^{M\cup K}$.

Hence, the sum of the coalitional values $v^\theta(M)$ and $v^\theta(K)$ coincides with the objective value of the optimization problem of $M \cup K$, evaluated as $X^{M\cup K}$. Since solution $x^{M\cup K}$ is as feasible solution, we conclude that $v^\theta(M \cup K)$ is at least this value. Hence,

$$v^\theta(M) + v^\theta(K) \leq v^\theta(M \cup K),$$

which concludes the proof for superadditivty.

Now we will provide an example that illustrates that our TC game is not convex nor monotonic in general. Consider a TC situation $\theta \in \Theta$ with $p = (0.5, 0.5, 2)$, $c = (0.5, 1, 1.5)$, $\alpha = (1, 2, 1.5)$, and $\beta = 0.1$. The coalitional values are represented in Table 9 below.

M	{∅}	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$v^\theta(M)$	0	0	-0.246	0.128	-0.244	0.130	-0.109	-0.109

Table 9: Coalitional values of game (N, v^θ)

Observe that $v^\theta(\{1\}) = 0 > -0.244 = v^\theta(\{1,2\})$, implying that the game is not monotonic. In addition, observe that $v^\theta(\{1,2\}) - v^\theta(\{2\}) = 0.002 > 0 = v^\theta(\{1,2,3\}) - v^\theta(\{2,3\})$, implying that the game is also not convex.