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Stability criteria of two-port networks, application to thermo-acoustic systems

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Abstract
System theory methods are developed and applied to introduce a new analysis methodology based on the stability criteria of active two-ports, to the problem of thermo-acoustic instability in a combustion appliance. The analogy between thermo-acoustics of combustion and small-signal operation of microwave amplifiers is utilized. Notions of unconditional and conditional stabilities of an (active) two-port, representing a burner with flame, are introduced and analyzed. Unconditional stability of two-port means the absence of autonomous oscillation at any embedding of the given two-port by any passive network at the system's upstream (source) and downstream (load) sides. It has been shown that for velocity-sensitive compact burners in the limit of zero Mach number, the criteria of unconditional stability cannot be fulfilled. The analysis is performed in the spirit of a known criterion in microwave network theory, the so-called Edwards-Sinsky's criterion. Therefore, two methods have been applied to elucidate the necessary and sufficient conditions of a linear active two-port system to be conditionally stable. The first method is a new algebraic technique to prove and derive the conditional and unconditional stability criteria, and the second method is based on certain properties of Mobius (bilinear) transformations for combinations of reflection coefficients and scattering matrix of (active) two-ports. The developed framework allows formulating design requirements for the stabilization of operation of a combustion appliance via purposeful modifications of either the burner properties or/and of its acoustic embeddings. The analytical derivations have been examined in a case study to show the power of the methodology in the thermo-acoustics system application.

Introduction

Thermo-acoustic combustion instability manifests itself as a high level of tonal noise, vibration, and may cause the performance deterioration or even structural damage of combustion appliance. The ability to eliminate and/or control combustion instability at the appliance design phase is one of the main goals of combustion-acoustics research. The low-order (acoustic network) modeling approach is one of the intensively developing tools which has proven its efficiency in performing problem analysis, synthesis, and eventually the appliances design tasks.

Various acoustic network models have been developed[1–3] that are used to analyze combustion thermo-acoustic instabilities and the design of combustion equipment in recent studies.[4–7] This modeling allows treating combustion appliance components as acoustic two-ports.[4,8,9] Accordingly, the availability of a purely acoustic characterization of the burner with flame is the prerequisite of the model. This is achievable within the concept of the transfer matrix (T) or scattering matrix (S).[7] Then, a network model of the combustion system is obtained when all two-port components are combined.

The methodological similarity of approaches to and the network models equivalence of the electrical circuits and

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combustion acoustic systems have been shown in various papers since 1957.[10] However, the stability analysis and design methods of the two-port networks have not been developed/applied in the combustion field as much as in microwave theory. The linear two-port network theory has been an intensively developing research subject and the results have been applied ubiquitously in the practice of microwave devices' design. The closest analogy can be established between the combustion thermo-acoustic instability problem and the problem of stability of operation of microwave amplifiers. Here, the burner with flame and the amplifier (e.g., transistor) both represent a so-called, "dependent source" or active element. Furthermore, the acoustics of the burner upstream and downstream parts in a combustion appliance are an analogy of the "source" and "load" passive network embeddings of the microwave amplifier. One of the extremely useful and well-developed concepts of the microwave amplifier's design process is the notion of unconditional stability.

In microwave theory, this means that there is no passive source and passive load combination that can cause the circuit with the given amplifier to oscillate. Correspondingly, the unconditional stability in a thermo-acoustic context means operation stability regardless of the (passive) acoustics at upstream and downstream sides of the burner/flame. The pioneering work on this subject was done by Rollett in 1962.[11] He introduced a quantity (criterion) to characterize the degree of stability. Later, it was shown that the combination of validity of certain inequality requested from the Rollett factor together with only one other auxiliary condition are necessary and sufficient to provide unconditional stability.[12–15]

In 1992, Edwards and Sinsky proposed a single parameter, instead of two of Rollett's conditions, to determine the necessary and sufficient unconditional stability requirements.[16] The arguments and analysis were based on a geometrical approach. Various applications and design tools based on the Edwards-Sinsky criteria were developed and discussed.[17–19] Particularly, Balsi et al. extended the geometrical approach and derived the necessary and sufficient conditions for a linear active two-port to be conditionally stable. A recent work of Lombardi and Neri[20] presented the existence of a duality mapping between the input and the output of the two-port network; then by using certain properties of Mobius Transformation (MT), they demonstrated all possible cases of mapping between the input and the output of the system. MT is the bilinear rational transformation[21] as one of the mathematical concepts named after A.F. Mobius. It is well-known that the MT maps a line or circle into another line or circle.[22] Çakmak et al. derived explicit formulas relating the centers and the radii of the mapped circles.[23]

On the other hand, the beginning of active development of the acoustic network modeling approach to the problem of combustion instability falls in the period after ~1990.

The main focus was on formulating thermo-acoustic network models, predicting the instability, searching for unstable frequencies, calculating/measuring the growth rate, searching methods for stabilizing systems, etc. References to most of the performed research can be found in the review paper.[24] The conventional methodology for analyzing the stability of thermo-acoustic systems consists of measuring/modeling the Flame Transfer Function (FTF). Then, a wave-based 1D linear two-port network approach is applied to provide the system matrix. The eigenfrequencies of the system (zeros of the matrix' determinant) determine the (in)-stability frequencies and growth/decay rates. Therefore, the common practice is to create a system matrix each time when the system is altered to check the corresponding effect on the eigenfrequencies. This procedure allows resolving the dilemma of the (in-)stability of operation of a particular system and gives a reasonably accurate prediction of frequencies of oscillation. This approach has been successfully applied in numerous studies before.[7,25,26] However, the conventional modeling approach does not provide a good overview of conditions and guidelines for designing the upstream and downstream sides of the flame such that the system would be stable. The crux is in the absence of specific parameters or criteria to determine the system stability and lack of tools (rules) on how to manipulate the system design as it is done in microwave theory.

A new impetus in the development of system-level analysis of thermo-acoustic network models was given by the discovery of the phenomenon of the burner intrinsic thermo-acoustic mode of instability.[27,28] This and further research on the subject use system theory. Particularly, the derived system instability conditions are based on the gain and phase of the TFT for only ITA modes.[29] A review of literature on this subject can be found in the recent publication.[30]

Another research direction was introduced by Kornilov and de Goey who showed the analogy between the thermo-acoustic and microwave circuits linear two-port networks[31] and use it to investigate two unconditional stability criteria, of 'Rollett' and 'Edwards-Sinsky', for the purpose of evaluation of a burner/flame figure of merit.[32] In turn, this work gave the inspiration to develop a prospective method to assess thermo-acoustic instabilities based on reflection coefficients measured only from the upstream side of the burner (cold side) by Kojourimanesh et al..[33] In this approach, two reflection coefficients, R_{up} and R_{in}, at the cold side of the flame are measured. As displayed in Figure 1, R_{up} is the reflection coefficient of the upstream side of the burner and R_{in} is the input reflection coefficient of the burner terminated by R_{dn}. In other words, if we disconnect the network in Figure 1 from the R_{up}, send in wave f_1 and measure reflected wave g_1, then the corresponding reflection coefficient would be: $R_{in} \triangleq g_1 / f_1$.[33]
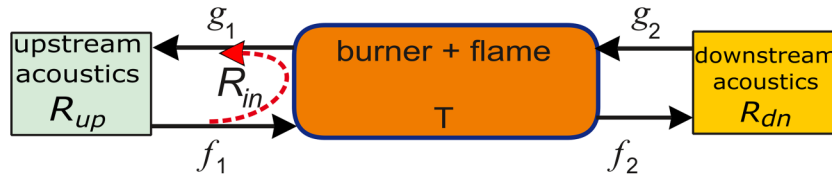
Figure 1. Thermo-acoustic model of a combustion system.[33]

In this method, the stability of the system can be determined by inspection of the Nyquist plot of the measured R_{up} times R_{in}. They showed that the condition applied to the magnitude of $R_{up}R_{in}(i\omega)$ being less than 1 for all frequencies from 0 to infinity is sufficient (but not necessary) to result in the thermo-acoustic stability of the system. Accordingly, in another study, they applied the MT properties to provide the necessary conditions of R_{dn} to ensure that the magnitude of R_{in} becomes less than 1.[34]

The present paper contributes to the further development of the research on the framework of the system-level analysis of thermo-acoustic instability of combustion and utilizes the close analogy with the theory of microwave networks. The particular goal of the present contribution is to introduce a new analysis methodology that is based on the stability criteria of active two-ports. The criteria will be derived using the original approach based on properties of a MT in combination with some algebraic transformations. The more general goal is to illustrate the power of the system analysis method in the application to thermo-acoustic network modeling and introduce an approach that allows designing optimal terminations at the upstream/downstream sides of the flame.

The model of a thermo-acoustic system is first written in the form of the network of a scattering matrix for power waves to show the analogy with microwave theory and derive system stability conditions. Then, new algebraic proofs of unconditional stability, namely the Edwards-Sinsky criterion, and conditional stability are proposed. Besides, an alternative approach using MT is introduced to determine the stability condition. Next, by combining the outcome of the aforementioned methods, conditional stability criteria for the thermo-acoustic systems are provided. Furthermore, the necessary condition is derived which if it is satisfied by R_{dn} ensuring that the magnitude of R_{in} becomes less than 1 in a frequency range. This condition results in passive thermoacoustics stability of the system's operation. In addition, an optimum value of R_{dn} is obtained which provides that the value of R_{in} becomes minimum. In its turn, this promises a higher potential of stability of the system.

The results obtained and presented below can be in principle generalized to the case of an arbitrary burner with flame for which the purely acoustic representation in the form of a two-port is known and given, e.g., by the burner transfer matrix. However, here we limit the consideration to one particular type of burner, namely, an acoustic velocity-sensitive dependent source of acoustic velocity (analogy of current sensitive current source in microwave theory). In this case, we may use some internal symmetries of the transfer and scattering matrices. Physically, this type of thermo-acoustic property is appropriate to a wide class of perfectly premixed gaseous fuel burners operating in the limit of low Mach numbers for the mean flow when the heat release zone is compact with respect to the acoustic wavelength under consideration. In addition, the conditions presented below should be satisfied for all frequencies from 0 to infinity. For brevity, we may discuss all relations for a fixed frequency point but finally, all criteria and conditions should be satisfied for the whole frequency range.

Furthermore, we will work in the frequency domain, and consider only plane longitudinal waves, 1-D acoustics. The network model variables will be represented by the forward and backward traveling waves f and g and the convention for the time dependence is e^{st} where s is the complex frequency $s = i\omega + \sigma$, where σ is the growth rate.

Stability criteria of thermo-acoustic systems

For a compact velocity-sensitive flame in the limit of zero mean Mach number the transfer matrix takes the form of[6]

$$T = 0.5 \begin{bmatrix} \varepsilon + 1 + \theta\,FTF & \varepsilon - 1 - \theta\,FTF \\ \varepsilon - 1 - \theta\,FTF & \varepsilon + 1 + \theta\,FTF \end{bmatrix}. \quad (1)$$

In this notation, $\theta = \frac{T_2}{T_1} - 1$ is the temperature jump ratio; $T_{1,2}$ being the temperature at upstream and downstream sides of the flame; $\varepsilon = \frac{\rho_1 c_1}{\rho_2 c_2}$ is the jump in characteristic acoustic impedance across the flame; and FTF is the flame transfer function which relates the oscillation of heat release rate of the flame to the oscillation of acoustic velocity at upstream of the burner and scaled to mean values of heat release and unburned gas velocity.

The corresponding scattering matrix of the thermo-acoustic two-port can be defined, if one rearranges

equations of the transfer matrix (T) such that the ingoing waves appear as inputs to the matrix, and the outgoing waves as outputs. In that case, the scattering matrix (S) of the thermo-acoustic system would be[6]

$$S = \frac{1}{2T_{11}} \begin{bmatrix} -2T_{21} & 4 \\ T_{11}^2 - T_{21}^2 & 2T_{21} \end{bmatrix},$$ (2)

with the determinant of $\Delta = -1$, where T_{ij} are transfer matrix elements (See Appendix A.0).

The scattering matrix representation also includes the ITA mode as a special case when $T_{11} = 0$. However, in the present study, we will not focus particularly on pure ITA mode analysis.

Unconditional stability in thermo-acoustic systems

Unconditional stability of a given burner with flame in a thermo-acoustic context means that regardless of the acoustic reflection coefficients of passive upstream and downstream sides of the burner/flame, the combined system would be always thermo-acoustically stable. To establish whether the unconditional stability can be ensured for the thermo-acoustic two-port defined in equations (1) and (2), the evaluation of the Rollett stability condition or the Edwards-Sinsky parameter can be performed.

Lemma 1. The defined thermo-acoustic system in equation (1), cannot be unconditionally stable.

Proof. The Rollett stability condition says that the combination of Rollett stability factor $K > 1$ where

$$K = \frac{1 - |S_{11}|^2 - |S_{22}|^2 + |\Delta|^2}{2|S_{12}S_{21}|},$$ (3)

together with any one of the following auxiliary conditions given in equation (4) are necessary and sufficient for unconditional stability of an (active) two-port described by the scattering matrix S,[16]

$$\begin{cases} B_1 = 1 + |S_{11}|^2 - |S_{22}|^2 - |\Delta|^2 > 0, & or \\ B_2 = 1 - |S_{11}|^2 + |S_{22}|^2 - |\Delta|^2 > 0, & or \\ |\Delta| = |S_{11}S_{22} - S_{12}S_{21}| < 1, & or \\ \quad 1 - |S_{11}|^2 > |S_{12}S_{21}|, & or \\ \quad 1 - |S_{22}|^2 > |S_{12}S_{21}|. \end{cases}$$ (4)

Therefore, one can simplify the Rollett stability factor for the Thermo-Acoustic system i.e., K_{TA} as

$$K_{TA} = \frac{1 - \left|-\frac{T_{21}}{T_{11}}\right|^2 - \left|\frac{T_{21}}{T_{11}}\right|^2 + |-1|^2}{2\left|\frac{2}{T_{11}}\frac{T_{11}^2 - T_{21}^2}{2T_{11}}\right|} = \frac{1 - \left|\frac{T_{21}}{T_{11}}\right|^2}{\left|1 - \left(\frac{T_{21}}{T_{11}}\right)^2\right|} \leq 1.$$ (5)

In Appendix A.1, it is made clear that for any complex number z the function $\frac{1-|z|^2}{|1-z^2|} \leq 1$. Accordingly, based on the Rollett criteria, the considered thermo-acoustic system cannot be unconditionally stable because one of the necessary conditions of unconditional stability is not satisfied, namely, $K_{TA} > 1$.

Alternately, it is also possible to prove *Lemma 1* by analyzing the Edwards and Sinsky parameter, μ. They proved that the necessary and sufficient condition to qualify a two-port as an unconditionally stable element is $\mu > 1$ where[16]

$$\mu = \frac{1 - |S_{11}|^2}{|S_{22} - \overline{S_{11}}\Delta| + |S_{12}S_{21}|}.$$ (6)

In this notation, the bar symbol is used to denote the conjugate of a complex number.

By substituting parameters, like in Appendix A.2, one can also derive $\frac{1-|z|^2}{|2Im(z)|+|1-z^2|} \leq 1$. Hence, for the thermo-acoustic system which obeys equation (2), $\mu_{TA} \leq 1$. Therefore, the thermo-acoustic two-port (a burner with flame) for which the transfer (scattering) matrix has symmetry properties, as in equation (1), cannot be unconditionally stable. For brevity of notation, the index of *TA* is omitted from K_{TA} and μ_{TA} in the rest of the paper.

Conditional stability in thermo-acoustic systems

When considering the notion of conditional stability of a thermo-acoustic system, the upstream and downstream acoustic boundary conditions are playing a role. One needs to search for the range of R_{up} (or R_{dn}) values in the complex domain such that the system would be always stable. It is worth mentioning that outside of that range, the system may be stable or unstable. As expressed in paper,[33] the stability of a combustion appliance can be determined by measuring two reflection coefficients at the cold side of the burner, i.e., R_{up} and R_{in}. In that paper, the expression of R_{in} is written in terms of the transfer matrix's elements, i.e., $R_{in} = \frac{T_{11}R_{dn} - T_{21}}{-T_{12}R_{dn} + T_{22}}$. This expression can be rewritten with scattering matrix's entries with help of equations (1) and (2). The result would be the same expression used in microwave theory, e.g., equation 7.4.1 in book[35]:

$$R_{in} = \frac{-\Delta R_{dn} + S_{11}}{-S_{22}R_{dn} + 1}.$$ (7)

The main idea of the Edwards-Sinsky criterion regarding the stability of a system expressed by equation (7) is that the system is unconditionally stable if the unit disk in the R_{dn} plane (note that the interior of the unit circle represents all possible reflection coefficients of a passive system/termination) is mapped by the equation (7) to the interior of the unit disk in the R_{in} plane.[16] In other words, the system

with passive upstream and downstream terminations is unconditionally stable if and only if $|R_{in}| < 1$. Besides, Kojourimanesh et al. have shown that if $|R_{up}R_{in}| < 1$ then the system is conditionally stable.[33,36] The case of pure ITA modes requires special consideration. For pure ITA modes, (i.e., $T_{11} = 0$, $R_{up} = 0$, $R_{dn} = 0$) the expression written in equation (2) will be undefined due to division by zero. Also, the conditional stability criterion such as $|R_{up}R_{in}| < 1$ is not applicable because both R_{up} and R_{dn} are already zeros.

Accordingly, *Lemma 2* which involves requirements to the magnitudes of R_{up} and R_{dn} is introduced. It provides conditions imposed on the upstream or downstream sides of the flame/burner sufficient for the system to become thermo-acoustically stable.

Lemma 2. For the thermo-acoustic system with the active two-port defined in equation (2), conditions for the downstream and upstream terminations which are sufficient to qualify the system to be conditionally stable are

$$|R_{up}| < \frac{1}{|R_{in}|}, \qquad (8.a)$$

or

$$|R_{up}| < \frac{-|S_{11} - |R_{dn}|^2\overline{S_{22}}\Delta| + |R_{dn}||S_{12}S_{21}|}{(|R_{dn}|^2|\Delta|^2 - |S_{11}|^2)} \qquad (8.b)$$

$$|R_{dn}| < \frac{-|S_{22} - |R_{up}|^2\overline{S_{11}}\Delta| + |R_{up}||S_{12}S_{21}|}{(|R_{up}|^2|\Delta|^2 - |S_{22}|^2)} \qquad (8.c)$$

Proof. As mentioned before, the system is conditionally stable if $|R_{up}R_{in}| < 1$. Therefore, one can say that a sufficient condition for a conditionally stable system could be $|R_{up}R_{in}|^2 < 1$.

Considering equation (7) and $\Delta = -1$, Appendix B.1 shows how $|R_{up}R_{in}|^2 < 1$ can be extended to get a form of a quadratic inequality as

$$A|R_{dn}|^2 + B|R_{dn}| + C < 0 \qquad (9)$$

where $A = |R_{up}|^2|\Delta|^2 - |S_{22}|^2$,
$B = 2(|S_{22} - |R_{up}|^2\overline{S_{11}}\Delta|)$, and $C = |R_{up}|^2|S_{11}|^2 - 1$.

As shown in Appendix B.2, the discriminant, $B^2 - 4AC$, of equation (9) would be

$$B^2 - 4AC = 4|R_{up}|^2|S_{12}S_{21}|^2 \qquad (10)$$

Because of $B^2 - 4AC \geq 0$, the only way that equation (9) becomes always negative is $C < 0$ & $|R_{dn}| < \lambda_1$, where λ_1 is the first root of the quadratic equation, i.e.,

$$\lambda_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}. \qquad (11)$$

Appendix B.3 shows that by substituting A, B, C into λ_1, the right side of equation (8.c) would be the same as λ_1. Therefore, if $C < 0$ & $|R_{dn}| < \lambda_1$ then the considered thermo-acoustic system is stable.

Equations (8.b) and (8.c) are almost the same as the conditions suggested by Balsi et al. Appendix C.1 shows how their conditions can be derived using this algebraic method. Moreover, Appendix C.2 provides a new proof of the Edwards-Sinsky criterion from the mentioned algebraic technique.

Mobius transformation between R_{in} and R_{dn}

As stated before, the relation between R_{in} and R_{dn} has the form of so-called bilinear (Mobius) transformation. The general (so-called, normalized) form of this transformation is $H = \frac{aZ+b}{cZ+d}$, where $ad - bc = 1$.

Among multiple specific properties of this transformation, one which will be used below is that the MT maps a line or a circle in the complex plane of its input Z into another line or circle in the plane of its output H. Particularly, if the mapped contour is the unit circle, then the resulting circle of the unit circle has a specified center, M, and radius, r,[22,23] namely,

$$M = \frac{b \cdot \overline{d} - a \cdot \overline{c}}{|d|^2 - |c|^2}, \qquad r = \frac{1}{||d|^2 - |c|^2|} \qquad (12)$$

For a fixed value of the frequency, the entries of the scattering matrix, S, are fixed complex numbers. By looking at the expression for $R_{in} = \frac{-\Delta R_{dn} + S_{11}}{-S_{22}R_{dn}+1}$, it is clear that it has the form of MT. However, it needs to be normalized first i.e., divided by $\sqrt{ad - bc} = \sqrt{S_{12}S_{21}}$. The downstream side of the burner/flame is an acoustically passive termination, i.e., $|R_{dn}| \leq 1$. Consequently, MT transforms the unit circle of R_{dn} into a circle in the plane of R_{in} with a center and radius as.[34]

$$M = \frac{S_{11} + \overline{S_{22}}}{1 - |S_{22}|^2} = \frac{-2Im(S_{22})}{1 - |S_{22}|^2}i, r = \frac{|S_{12}S_{21}|}{|1 - |S_{22}|^2|}. \qquad (13)$$

Furthermore, the center of the R_{dn} unit circle transforms into point $O = \frac{b}{d}$ in the R_{in} plane which indicates to where the inside area of the unit circle of R_{dn} is mapped to. It can be either the inside or outside area of the circle in R_{in} plane. Figure 2 shows four possible cases for this transformation between R_{dn} and R_{in} planes.[34] This kind of plot provides new insight into the design strategy, one may follow to ensure system stability. For instance, considering Case a in Figure 2 one may conclude that for most reflection coefficients of the downstream side of the flame, R_{dn}, the gain of R_{in} is higher than 1. Accordingly, to stabilize the system for this particular case the design of the stabilizing downstream acoustics is more demanding because there is a quite limited range of R_{dn} which provides $|R_{in}| \leq 1$. Therefore, tuning of the upstream side of the burner becomes a more appropriate approach. Contrarily, when facing a situation similar to the Cases b, and c in Figure 2, one may conclude that there is wide freedom to select R_{dn} such that $|R_{in}| \leq 1$
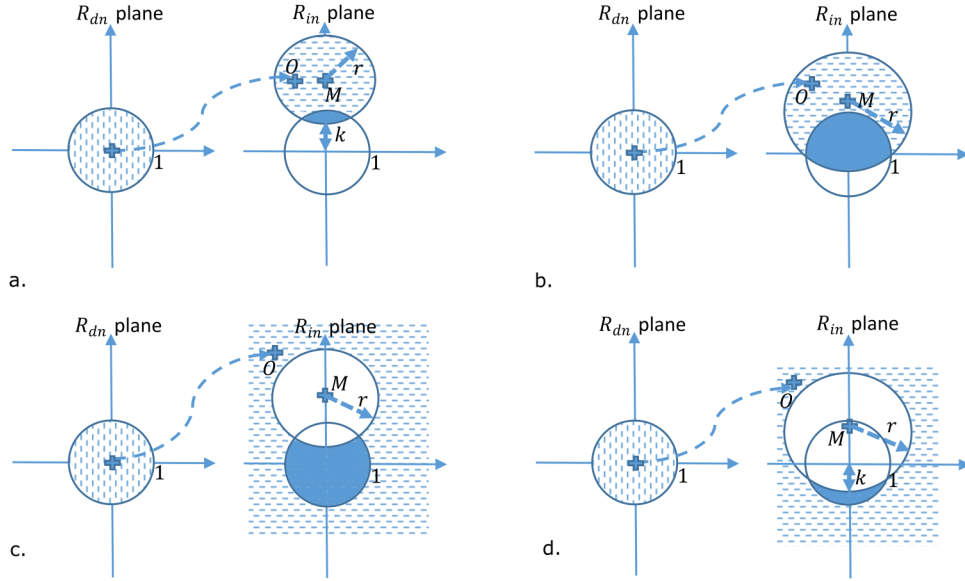
Figure 2. Four possible cases for the Mobius transformation of the unit circle of R_{dn} into R_{in} plane.[34]
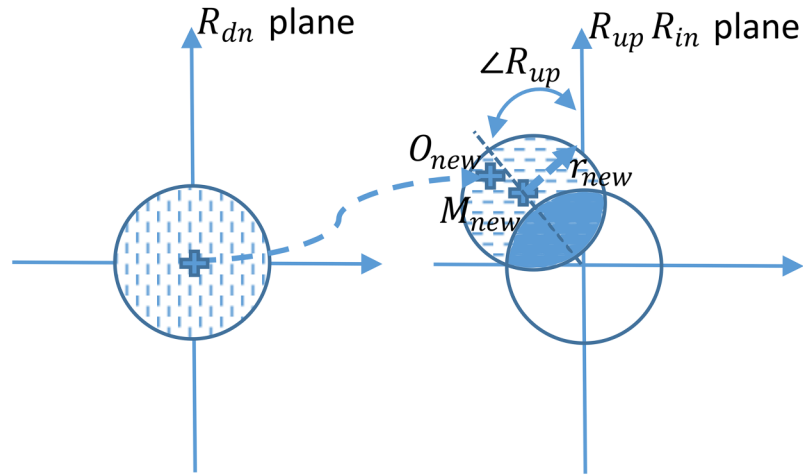

Figure 3. Mobius transformation of R_{dn} unit circle into $R_{up}R_{in}$.

and even it is possible to design a proper R_{dn} where $|R_{in}| = 0$. Elaboration of these design guidance ideas is the subject of the next sections. It should be noted that the shaded rectangular areas in Cases c, and d in Figure 2 are outside domains of the mapped circle.

For the case $|R_{up}| \neq 1$, one can apply the same strategy as in equation (7) but for MT of

$$R_{up}R_{in} = \frac{(-R_{up}\Delta)R_{dn} + (R_{up}S_{11})}{-S_{22}R_{dn} + 1}. \quad (14)$$

Then, the unit circle in R_{dn} plane maps into a circle in $R_{up}R_{in}$ plane. Considering the formula in equation (13),

one can easily show that the location of the center, radius and point O in the $R_{up}R_{in}$ plane would be the scaled one in the R_{in} plane. Equation (15) and Figure 3 show the location of the center, radius and point O of the circle in the $R_{up}R_{in}$ plane which is mapped from the R_{dn} plane.

$$M_{new} = R_{up} M \ \& \ O_{new} = R_{up}\frac{b}{d} \ \& \ r_{new} = |R_{up}|r \quad (15)$$

Figure 3 depicts the MT of the unit circle in the R_{dn} plane into the $R_{up}R_{in}$ plane. Comparing Figure 2 with Figure 3, and also equations (13) and (15), one can conclude that the properties of the disk in the $R_{up}R_{in}$ plane, i.e.,
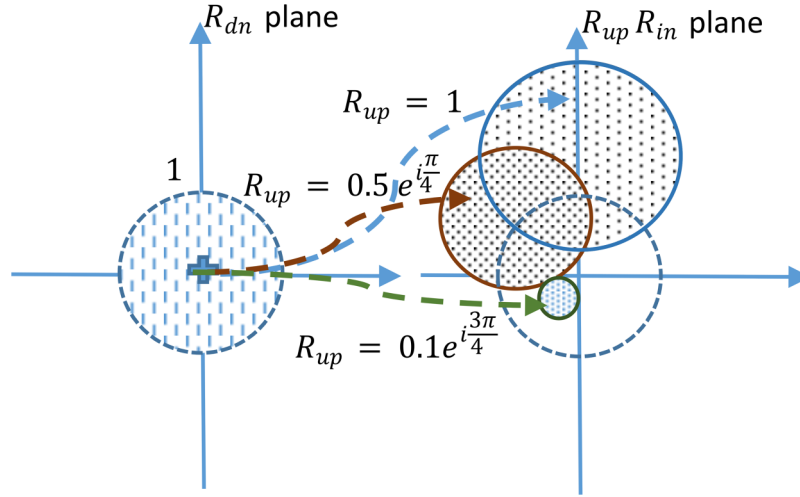
Figure 4. MT of R_{dn} unit circle into $R_{up}R_{in}$ for 3 different R_{up}.

M_{new}, O_{new}, and r_{new}, can be deduced from the properties of the disk in R_{in} plane, i.e., M, O, and r, by scaling them with the factor $|R_{up}|$ and rotating it with the phase of R_{up}.

Besides, equation (15) suggests that by decreasing the magnitude of R_{up}, the center of the corresponding disk in the $R_{up}R_{in}$ plane and the point O_{new} would converge to the origin and the radius of the disk goes to zero as illustrated in Figure 4. In other words, the whole disk of $R_{up}R_{in}$ will be inside the unit disk, $|R_{up}R_{in}| < 1$. Therefore, decreasing the magnitude of R_{up} causes a high chance of stable system operation (note, that the possibility of the intrinsic thermoacoustic instability still remains).

Comparing algebraic and MT methods' results

In this section, we are aiming to search for correspondences between the results derived from the algebraic and the MT methods. In the aforementioned thermo-acoustic two-port system, i.e., obeying equation (2), the symmetry implies that $S_{11} = -S_{22}$ and therefore $\Delta = -1$, accordingly $S_{12}S_{21} = 1 - S_{22}^2$. As explained before for any complex z function, $\frac{1-|z|^2}{|1-z^2|} \leq 1$. Then one can readily conclude that the radius of the MT circle in the R_{in} plane would be always bigger than 1, i.e.,

$$r = \frac{|1 - S_{22}^2|}{|1 - |S_{22}|^2|} \geq 1. \tag{16}$$

In addition, for the case $|R_{up}| = 1$, the expression $\frac{\sqrt{B^2-4AC}}{2A} = \frac{2|R_{up}||S_{12}S_{21}|}{2(|R_{up}|^2|\Delta|^2-|S_{22}|^2)}$ would be the same as r, i.e.,

$$r = \frac{\sqrt{B^2 - 4AC}}{2A}.$$

The same procedure confirms that $\frac{-B}{2A} = \frac{-|S_{22}+\overline{S_{11}}|}{1-|S_{22}|^2}$.

By comparing the expression of $\frac{-B}{2A}$ with $M = \frac{S_{11}+\overline{S_{22}}}{1-|S_{22}|^2}$ and considering $S_{11} = -S_{22}$ and $C < 0$, it is obvious that

$$\frac{B}{2A} = |M|.$$

Therefore, one can relate the first root of the quadratic equation to the MT circle parameters as

$$\lambda_1 = \frac{-B}{2A} + \frac{\sqrt{B^2 - 4AC}}{2A} = -|M| + r \tag{17}$$

For the case $|R_{up}| = 1$, $|R_{dn}| = 1$, Appendix C.2 provides the prove that the Edwards-Sinsky factor is indeed λ_1. Hence,

$$\mu = -|M| + r \tag{18}$$

The other point is that the values for the center and radius depend on each other for the system defined in equation (2) when $|R_{up}| = 1$. Due to the equality $A = -C$, one can write the relation between them as

$$r = \sqrt{\frac{B^2 - 4A(-A)}{4A^2}} = \sqrt{\left(\frac{B}{2A}\right)^2 + 1} = \sqrt{M^2 + 1} \tag{19}$$

Equations (17) and (18) reveal the relation between Edwards-Sinsky factors, μ & μ^*, and MT parameters which is shown in Figure 5. As can be seen, the Edwards-Sinsky factor, μ, is the closest point of the R_{in} disk from the origin i.e., $|-|M| + r|$. Moreover, the second root of the quadratic equation (9) is the farthest point of the R_{in} disk from the origin. Thus, both approaches provide the same results regarding the elaboration of criteria for the analysis of (in)-stability of the system operation.

Optimal value of R_{dn} to obtain minimum value of R_{in}

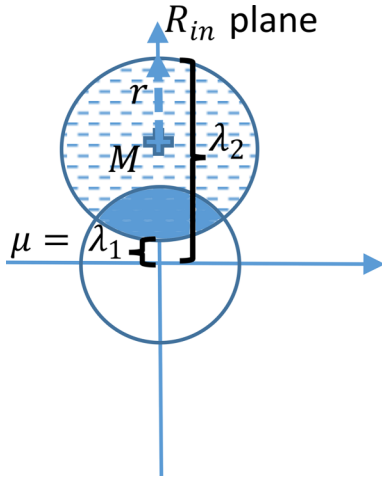In this subsection, an optimum value of R_{dn} is derived which provides a minimum value of R_{in} at given entries

Figure 5. Relation of *MT*, algebraic equation and μ.

of the burner's scattering matrix, *S*. As shown in Figure 2, a minimum value of R_{in} is equal to either *zero* for cases *b* and *c* or it is equal to *k* for the cases *a* and *d*.[34] It has been shown that $|k|$ is actually the Edwards-Sinsky factor, μ. Therefore, the value of R_{dn} can be restored which causes the minimum value of R_{in} by inverse mapping at point *k* back to R_{dn} plane. Hence,

$$R_{dn_{opt.}} = \frac{-d\,k + b}{c\,k - a}. \tag{20}$$

It is obvious that for the cases *b* and *c* the optimum value of R_{dn} is $R_{dn_{opt.}} = -\frac{b}{a}$.

Possible range of R_{dn} to obtain a passive stable system

The practical motivation of the considerations presented below stems from the fact that many producers of burners for industrial or domestic applications tend to combine the burner with the downstream part of an appliance as a unified product. A typical example is the combination of a burner and heat-exchanger as the "power module". Accordingly, only the upstream side (for instance, the blower/fan, venturi, snorkel, etc.) is designed by the boiler manufacturer. In this situation, designers of the power module would like to optimize their product in terms of ensuring the thermo-acoustic stability for the module in combination with any/arbitrary acoustics of the upstream part of an appliance.

In this subsection, the conditions are investigated how to select a passive termination R_{dn} in such a way that for the given burner transfer matrix the corresponding magnitude of R_{in} becomes less than 1 over a certain frequency range. The idea to ensure $|R_{in}| < 1$ is motivated by the fact that in this case the burner with downstream acoustics

behaves as a passive subsystem and therefore it will be stable in combination with any passive upstream acoustics of the considered appliance.

The region of R_{dn}, which we are looking for, belongs to a section of the unit circle in the plane of R_{dn} which maps these R_{dn} to the double shaded (stable) regions in the plane of R_{in}, see Figure 2.[34] To find the corresponding area in the R_{dn} plane, one may use the inverse transformation of equation (7). For this case, the inverse transformation is also of Mobius kind, and it is given by the expression

$$R_{dn} = \frac{-d\,R_{in} + b}{c\,R_{in} - a}. \tag{21}$$

We are interested where the unit circle of R_{in} maps into the R_{dn} plane. The mapping is similar to the one presented in Figure 2 with the only replacement of R_{dn} and R_{in} planes. The described procedure provides the area inside the unit circle of R_{dn} which is considered as a preferable value of R_{dn} to obtain a passive stable system without considering the upstream acoustic's condition.

Case study

To show the power of this methodology, a test setup, shown in Figure 6, is prepared to measure the *FTF*, R_{dn}, and R_{in} in the frequency domain. A constant temperature anemometer (CTA) and a photomultiplier tube (PMT) with OH filter are used to measure the acoustic velocity fluctuation before the flame and the varying heat release signal, respectively. Two Bronckhorst mass flow controllers control the methane and airflow rates. A water-cooled burner deck holder is used to keep the outer perimeter of the burner deck at a fixed temperature to obtain a steady flame and *FTF* independent from the time of measurement. A set of National instruments DAQ systems connected to a PC running LabVIEW is used to transfer the data with a sampling rate of 10 kHz. A loudspeaker is installed at the end part of the upstream side to provide a forced excitation supplied by the sequence of pure tone signals. Software written in LabVIEW is implemented to control the amplitude of the excitation force, sampling rate, measuring time duration, as well as setting the desired range, steps of frequencies, data preprocessing, etc.

The upstream side has a broadband damper with $|R_{up}|$ less than 0.4 for the frequency range of 15–310 Hz. More details of the design of the upstream damper, the setup, and the measurement procedures can be found in paper.[33] The downstream part of the burner is a 110 mm long quartz tube with an open end. The reflection coefficient of the downstream side is measured immediately after turning off the flame to obtain a reasonable estimation of R_{dn} value, i.e., close to the hot condition. Figure 7 shows
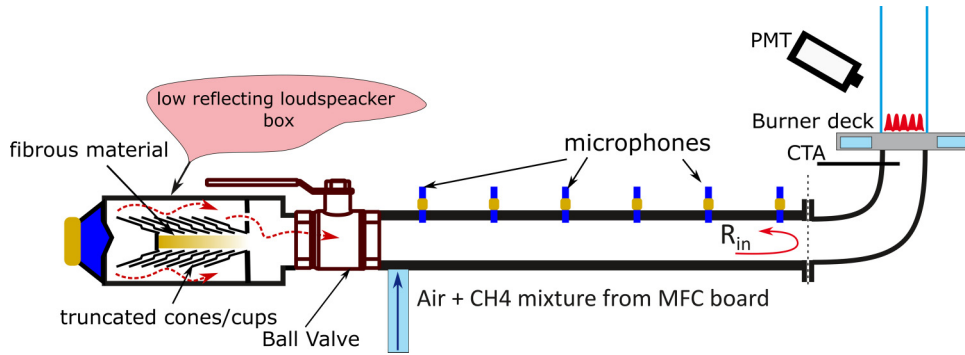
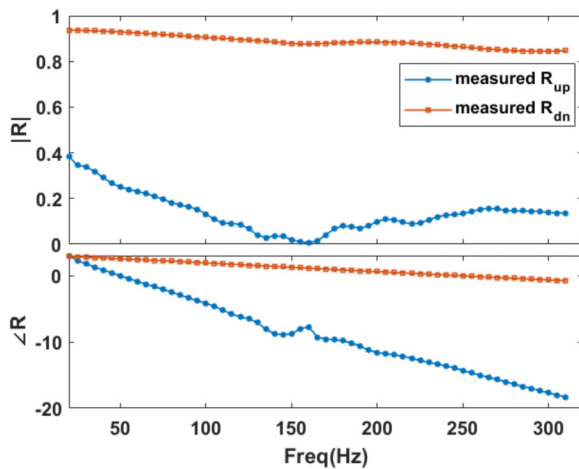**Figure 6.** A test setup to measure the *FTF*, R_{dn}, and R_{in}.

Figure 7. Measured R_{up}, and R_{dn} of the test setup.

the reflection coefficients of the upstream and downstream sides of the burner in the frequency domain.

As an example, the flame transfer function for a brass plate burner with premixed burner-stabilized Bunsen-type flame is measured. The burner deck is a disk that has a thickness of 1 mm, and a diameter of 5 cm. The hexagonal pattern of round holes with a diameter of holes of 2 mm and the pitch between the holes of 4.5 mm is used. For brevity, the burner is called "D2P4.5". The total open area of the burner is 399 mm.[2] The measured flame transfer function is used to calculate coefficients *a, b, c, d* of the MT. Figure 8 shows the flame transfer function of burner D2P4.5, at the mean velocity of the mixture through the burner holes of 70 cm/s and equivalence ratio of $\phi = 0.7$ in the frequency range 15–310 Hz.

The impedance tube with 6 microphones shown in Figure 6, including the low reflecting loudspeaker box (damper) at the upstream side, is used to measure the reflection coefficient R_{in} from the combination of a bended supply tube, the burner with flame, and the downstream subsystem. The measured data of R_{in} in the frequency domain is plotted in Figure 9. The measurement procedure is explained in detail in papers.[33,36]

Results and Discussions

In this section, the analytical results derived in section 2 are examined for the case study expressed in section 3.

Stability analysis using Flame Transfer Function

For the case study as defined in the section 3, the Rollett stability factor *K*, and Edwards-Sinsky parameter *μ* are calculated based on equations (3) and (6), respectively. The results are presented in Figure 10.

As mentioned before, due to symmetry features of the transfer matrix for thermo-acoustic systems, both Rollett factor and Edwards-Sinsky parameter should be less than 1, $K_{Th} \leq 1$, $\mu_{TA} \leq 1$. Therefore, it is not surprising that this fact is reflected in Figure 10 for the case study. By increasing the frequency from 15 to 220 Hz, *K* and *μ* decrease. In the practice of microwave circuits design, the value of *μ* is also used as the qualitative measure/indicator of the active two-port potential of (in)-stability. Accordingly, it also reflects the qualitative measure for the potential of (in)-stability when the considered two-port is embedded between arbitrary passive up and downstream terminations. The argument which substantiates such the role of *μ* factor is the following. If to estimate the potential of stability at each frequency as the ratio of the area of the double shaded region and the area of the unit circle in R_{in} (shown in Figures 2 and 5), then exists the monotonic inverse relation between *μ* and the area ratio. Therefore, by decreasing *μ* from 15 to 220 Hz, the ratio of the areas is decreased, also the potential of stability at that frequency is decreased. For the considered test case the minimum value of *μ* happens at the vicinity of the frequency of 220 Hz. It is worth noticing that it is the same frequency where the phase line of the flame transfer function is crossing $-\pi$, see Figure 8, which is the indication of the burner intrinsic mode. The correlation between the frequency ranges of high potential of instability of thermo-acoustic system and frequencies of the burner intrinsic modes was reported earlier.[31,32]
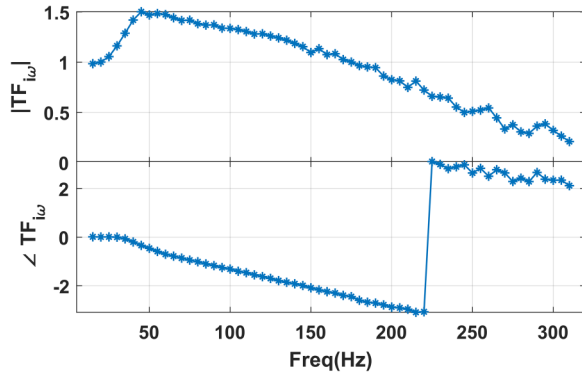
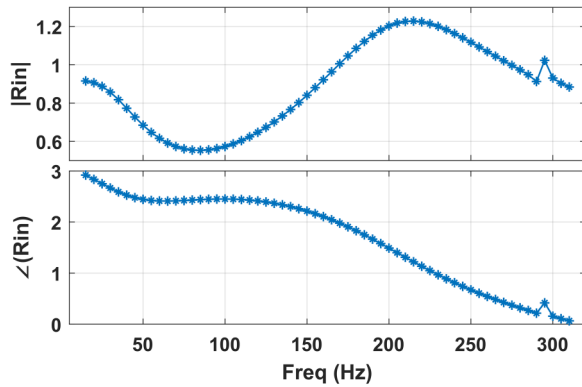**Figure 8.** Measured flame transfer function of the burner D2P4.5 at $v = 70 \frac{cm}{s}$ and $\phi = 0.7$.

Figure 9. Measured R_{in} at $v = 70 \frac{cm}{s}$ and $\phi = 0.7$.

Upstream design to stabilize the system

Equation (8.a) reveals that for a specific R_{in}, it is possible to design R_{up} such that the system becomes stable, i.e., $|R_{up}| < \frac{1}{|R_{in}|}$. For the particular case under consideration, the maximum value of the upstream reflection coefficient is $max(|R_{up}|) = 0.4$ which is two times smaller than $Min\left(\frac{1}{|R_{in}|}\right) = 0.81$. Therefore, the system is stable at this condition. In general, this system with a defined R_{dn} value, i.e., fixed magnitude and fixed phase, would be stable if $max(|R_{up}|) < 0.8$. However, for other values of magnitude and phase of R_{dn}, the value of $|R_{in}|$ could be much higher. When this is the case, an upstream termination with a lower reflection coefficient, for all frequencies, will be needed.

Besides, one can get the advantages of equation (8.b) to design the upstream reflection coefficient. It should be noted that the phase of the reflection coefficient at the downstream side can be easily changed, like by varying the length of the exhaust duct. However, changing the magnitude of the downstream reflection coefficient needs a dedicated gas path design or even external equipment like damper, muffler, etc. Therefore, one of the design strategies is to infer for the fixed value of $|R_{dn}|$ with arbitrary phase of R_{dn}, what will happen with the system. It may suggest what

would be the possible upstream reflection coefficient to stabilize the system. To do that, one can substitute the values of scattering matrix entries and $|R_{dn}|$ for each frequency between 15 and 310 Hz in equation (8.b). If the right-hand side of equation (8.b) becomes negative, then it implies that the system could be unstable at that frequency. Consequently, by changing the upstream part, the system cannot be fully stabilized for all phases of R_{dn}, i.e., any length of the exhaust duct. On the other hand, if the RHS of equation (8.b) is positive then for a fixed value of $|R_{dn}|$ and arbitrary phase of R_{dn}, the region of $|R_{up}|$ where the system remains stable is determined. For the case study under consideration, this region is plotted in Figure 11.

As can be seen in Figure 11, for frequencies higher than 120 Hz there is no value of R_{up} that guarantees the stability of the system for the fixed value of $|R_{dn}|$ with arbitrary phase (length) of the downstream side. Note that this frequency is exactly the same as the frequency where K and μ start to become negative. However, for a frequency less than 120 Hz, the blue area in Figure 11 gives the possible magnitude of $|R_{up}|$ to guarantee the stability of the system at those frequencies.

The reflection coefficient of the upstream side can be also designed from the MT strategy. Figure 12 shows the results of mapping the unit circle of R_{dn} first into the plane of R_{in} i.e., equation (7) (interior area of the dark blue circle) and second, into the plane of $R_{up}R_{in}$ plane i.e., equation (14) (interior area the light blue circle), at the frequency of 100 Hz. As can be seen, the size of the circle in $R_{up}R_{in}$ plane is much smaller than the one in the plane of R_{in}, and only a section of mapped R_{in} (purple shading area) is inside the unit circle. However, the whole circle of mapped $R_{up}R_{in}$ (turquoise shadings area) is inside the unit circle. Equation (15) reveals the reason for this property of the mapping that applying the low reflecting termination at the upstream side, i.e., $|R_{up}| = 0.13$ at 100 Hz, the whole disk of $R_{up}R_{in}$ will end up inside the unit disk.

In addition, the system with a high value of the upstream reflection coefficient, $|R_{up}| \sim 1$, cannot be fully stabilized/passivated. The reason, as can be deduced from equation (16), is that the radius of the mapped disk in R_{in} plane is higher than one. Therefore, some particular values of R_{dn} can cause $|R_{in}| > 1$. In general, it can be argued that by providing a low reflecting upstream termination, one may attempt to increase the potential of stability for any R_{in} and/or R_{dn}.

Downstream design for stability

Equation (8.c) could help to design the downstream part for a fixed value of $|R_{up}|$ with arbitrary phase (length) of the upstream side such that the system would be thermo-acoustically stable. Figure 13 shows the possible values of $|R_{dn}|$ that provide the stability of the system studied as the case study, for the fixed value of $|R_{up}|$ with arbitrary
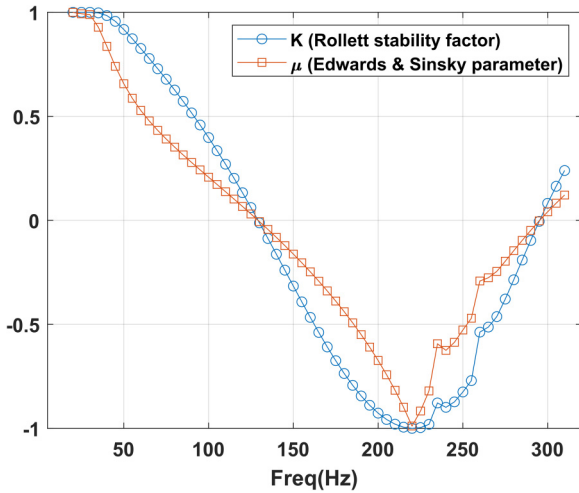
Figure 10. Calculated K and μ for the defined *FTF*.

Figure 12. MT of R_{dn} unit circle into R_{in} and $R_{up}R_{in}$ planes at 100Hz (interior areas of dark-blue and light-blue circles respectively). Purple and turquoise shadings signify overlapping of the circles' interior with the unit circle, respectively.
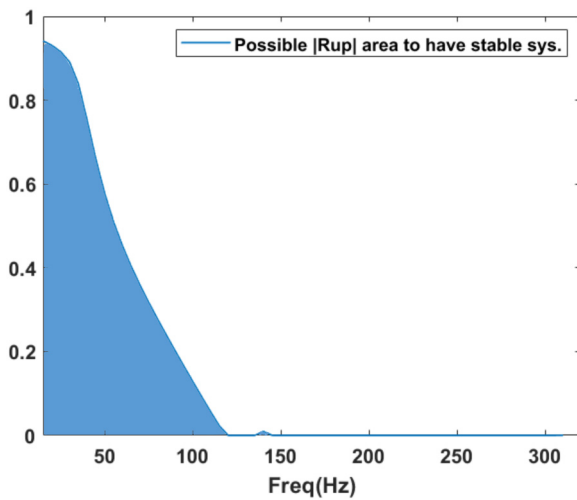

Figure 11. Region of $|R_{up}|$ to have a stable system, for the fixed value of $|R_{dn}|$ and arbitrary phase of R_{dn} of the case study.

Figure 13. Region of $|R_{dn}|$ to have a stable system, for the fixed value of $|R_{up}|$ and arbitrary phase of R_{up} of the case study.

phase (length) of the upstream side at frequencies of 35–310 Hz.

Figure 13 demonstrates that even for the fixed low magnitude of the reflection coefficient of the upstream termination $|R_{up}|$, it is possible that the system becomes unstable in the frequency range between 210 and 225 Hz. It should be noted that this range of frequencies is close to the intrinsic instability of the burner with flame.

As expressed before, some combustion companies prefer to sell their product without the upstream side. Accordingly, the inverse transformation written in equation (21) could provide a method how to ensure the complete system stability for any values of magnitude and phase of R_{up}. This
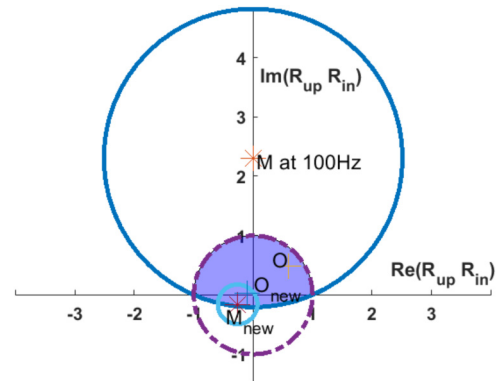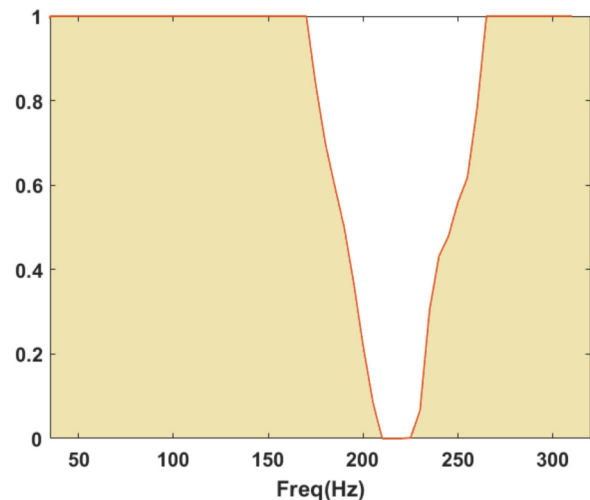
equation allows defining the region of R_{dn} such that $|R_{in}| < 1$, which in its turn means that the system becomes passive and stable for any passive upstream termination. To demonstrate this design option for the case study, R_{dn} circles and the possible ranges of R_{dn} at which $|R_{in}| \leq 1$ are calculated using equation (21). At four arbitrary selected frequencies, the R_{dn} circles and centers' locations of the circles and those preferable R_{dn} areas are shown in Figure 14 with purple color.

By developing these ideas further, one may suggest that the "volume" of possible R_{dn} values (blue region shown in Figure 15) may provide a good indication for the kind of "figure of merit" of the burner.[34] It means that a good burner in terms of the thermo-acoustic
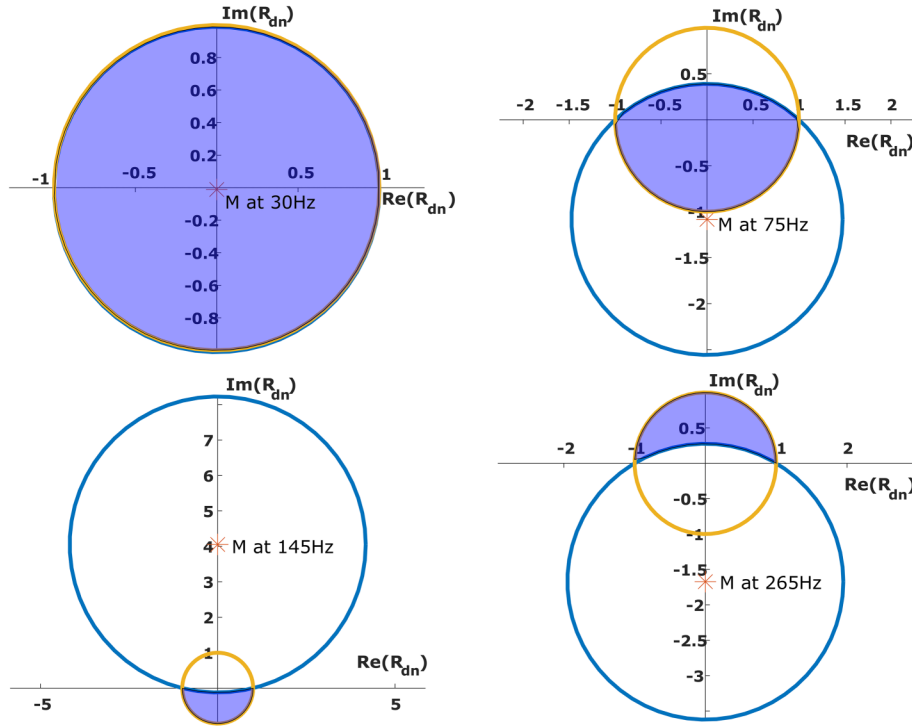
Figure 14. R_{dn} circles, centers' locations, and possible R_{dn} map to the unit circle of R_{in} at 30, 75, 145, 265 Hz.

stability would have a larger choice of possible (stabilizing) R_{dn} values.

Figure 15 demonstrates a 3D view of the discussed area in a frequency range 15–310 Hz. The four slices (joint areas) for particular frequencies as shown in Figure 14 are also marked in Figure 15 with the red color. If the reflection coefficient of the downstream side of the burner is lying inside this area (looking as a channel), then the system would be at a passive stability condition otherwise, the system will not be passive and the solution of the dilemma of stability-instability is related to the product of upstream and inlet reflection coefficients, i.e., $R_{up}R_{in}$.

Figure 15 could also be a productive tool for further analysis. For instance, one may conclude that at low frequencies, like 15–40 Hz, for almost any downstream termination of the mentioned burner at specified mean velocity and equivalence ratio condition, the system would be passive and, therefore, stable. The frequency ranges around 220 Hz look the most problematic to stabilize because for downstream acoustics the range of values of reflection coefficients needed for stabilizing the system is narrow at these frequencies.

Continuing the case study further, by minimizing R_{in}, the maximum potential of stability could be achieved. Therefore, an optimal R_{dn} should be found that minimizes R_{in}. Equation (20) can be applied to find the optimal R_{dn}

with a minimum value of R_{in} where the system becomes passive and therefore stable for any passive upstream termination of the system. For the considered case, the optimal value of R_{dn} such that the magnitude of R_{in} becomes minimum is derived from equation (20) and results with a corresponding minimum value of R_{in} are demonstrated in Figure 16.

These graphs also highlight the conclusion that for the considered burner/flame the frequencies around 220 Hz may require special measures to ensure system stability.

Conclusions

It is demonstrated that the system-level analysis of a network of two-ports is a very fruitful tool to perform investigations of many aspects of combustion acoustic instability phenomena. Particularly, it provides promising approaches to the task of system design aiming stability of appliance operation. Original algebraic proofs of conditional and unconditional stability criteria of linear two-port network systems are proposed. It has been shown that thermo-acoustic systems cannot be unconditionally stable. Hence, the conditional stability criteria have been investigated based on the algebraic technique. Also, a complementary framework of analysis is proposed which is based on the application of known properties of bilinear MT. The
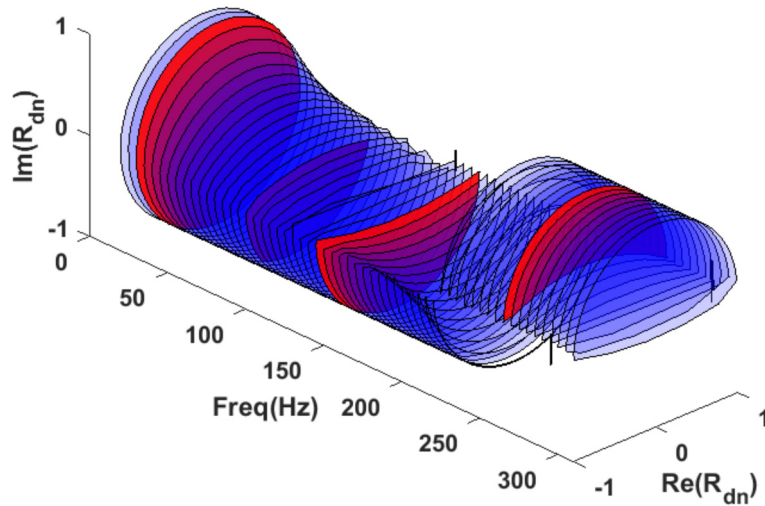
Figure 15. A 3D view of possible R_{dn} maps to the unit circle of R_{in} in a frequency range 15–310 Hz. The four slices shown in Figure 14 are highlighted in red.
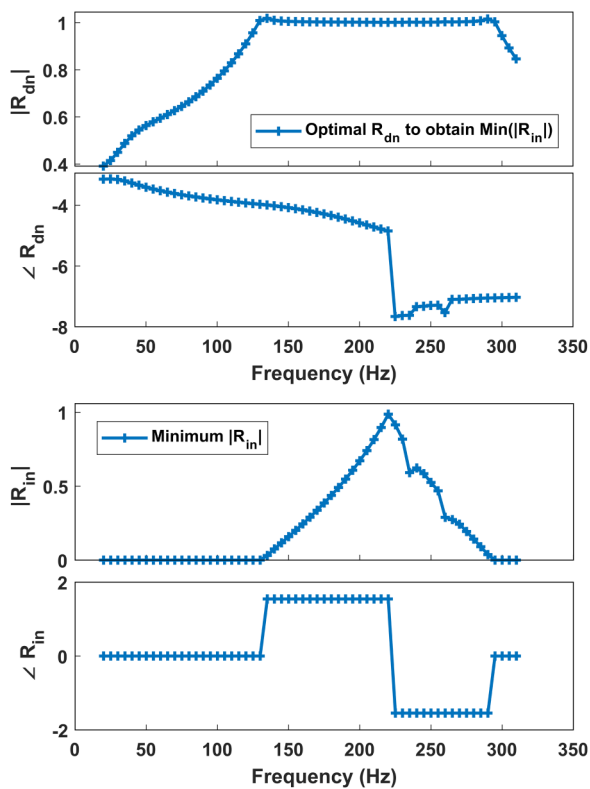

Figure 16. Optimal R_{dn} (top plot) to obtain a minimum amount of R_{in} (bottom plot) at 15–310 Hz.

comparison of different approaches reveals relations between the results of the algebraic derivations, the geometrical approach applied in microwave theory, and the MT technique. Any of these three approaches can be applied to analyze the stability of the thermo-acoustic system. However, the technique based on the MT would provide more insightful information and a better visual and intuitive interpretation of results than the two other techniques.

The elaborated criteria of system stability can be applied for purposeful design of the upstream and downstream sides of the given burner with flame to provide thermo-acoustic system stability.

Furthermore, the elaborated approaches show principal requirements and propose the receipts how to design the acoustics of upstream and downstream sides of the burner such that the system operation would be stable.

Also, for each frequency, the area of possible R_{dn} values to obtain a passive stable system is derived. The area depends on the flame transfer function of the burner. Hypothetically, the size (volume or another measure) of the area can be considered as a possible candidate for an indication for the figure of merit of the burner.

To confirm the validity and usefulness of the proposed ideas related to the combustor design strategies and evaluation of the figure of merit of burners, further theoretical development, and experimental checks are needed. This work is in progress and promises new directions for further research. Particularly, for the experimental verifications of the theoretical results new setup with a wide range of variable reflection coefficients at both up- and downstream terminations is developed. The final goal of future R&D works in this direction would be the elaboration of a convenient designer's toolbox supporting and suggesting design decisions.

Declaration of Conflicting Interests

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Appendix

A.0) Transfer matrix and Scattering matrix of TA

By considering the burner/flame as an acoustically compact lumped element, the relationship between pressure and velocity fluctuations at the upstream and downstream parts of the flame (in the limit of zero mean flow Mach number) can be described via the Transfer matrix as

$$\begin{Bmatrix} p'_{dn} \\ u'_{dn} \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 + \theta \, FTF_s \end{bmatrix} \begin{Bmatrix} p'_{up} \\ u'_{up} \end{Bmatrix}.$$

In terms of Riemann invariants (f, g), pressure and velocity can be written as

$$\begin{cases} p'_{up} = (f_1 + g_1)\rho_{up}c_{up} & p'_{dn} = (f_2 + g_2)\rho_{dn}c_{dn} \\ u'_{up} = f_1 - g_1 & u'_{dn} = f_2 - g_2 \end{cases}$$

Therefore, $\begin{cases} (f_2 + g_2) = \varepsilon(f_1 + g_1) \\ f_2 - g_2 = (1 + \theta \, FTF_s)\,(f_1 - g_1) \end{cases}$.

By calculating g_2 from the second row and put it in the first row then simplify, one can derive

$$\begin{Bmatrix} f_2 \\ g_2 \end{Bmatrix} = 0.5 \begin{bmatrix} \varepsilon + 1 + \theta \, FTF_s & \varepsilon - 1 - \theta \, FTF_s \\ \varepsilon - 1 - \theta \, FTF_s & \varepsilon + 1 + \theta \, FTF_s \end{bmatrix} \begin{Bmatrix} f_1 \\ g_1 \end{Bmatrix},$$

where $\varepsilon = \frac{\rho_1 c_1}{\rho_2 c_2}$. The similar approach should be followed to acquire the scattering matrix in terms of TM elements; therefore, one should calculate f_2 from the second row and put it in the first row to find g_1; then substitute the transfer matrix elements and simplify the equations. Then the below expression will be derived.

$$\begin{Bmatrix} g_1 \\ f_2 \end{Bmatrix} = \frac{1}{2T_{11}} \begin{bmatrix} -2T_{21} & 4 \\ T_{11}^2 - T_{21}^2 & 2T_{21} \end{bmatrix} \begin{Bmatrix} f_1 \\ g_2 \end{Bmatrix}.$$

Hence, the determinant of the scattering matrix would be

$$\Delta = \frac{(-2T_{21})(2T_{21}) - 4(T_{11}^2 - T_{21}^2)}{4T_{11}^2} = -1.$$

A.1) Rollett factor for Unconditional stability of TA

Suppose $z = x + iy$ then $|z|^2 = z\bar{z} = x^2 + y^2$, $|z| = \sqrt{x^2 + y^2}$, and $z^2 = (x^2 - y^2) + 2xyi$.

Rollett factor in thermo-acoustic systems is $K = \frac{1 - |z|^2}{|1 - z^2|}$ where $z = \frac{T_{21}}{T_{11}}$.

Hence, $K = \frac{1 - |z|^2}{|1 - z^2|} = \frac{1 - (x^2 + y^2)}{|(1 - x^2 + y^2) - 2xyi|}$ therefore,

$$K = \frac{1 - x^2 + y^2 - 2y^2}{\sqrt{(1 - x^2 + y^2)^2 + (-2xy)^2}}.$$

Also, it is clear that $\dfrac{1 - x^2 + y^2}{\sqrt{(1 - x^2 + y^2)^2 + (-2xy)^2}} \leq 1$ and

$$\frac{-2y^2}{\sqrt{(1 - x^2 + y^2)^2 + (-2xy)^2}} \leq 0. \text{ Therefore } K \leq 1.$$

A.2) Edwards-Sinsky factor for Thermo-Acoustic

$$\mu = \frac{1 - |S_{11}|^2}{|S_{22} - S_{11}^*| + |S_{12}S_{21}|}$$

$$= \frac{1 - \left| -\dfrac{T_{21}}{T_{11}} \right|^2}{\left| \dfrac{T_{21}}{T_{11}} + \dfrac{T_{21}^*}{T_{11}^*}(-1) \right| + \left| 1 - (\dfrac{T_{21}}{T_{11}})^2 \right|}.$$

$$\rightarrow \mu = \frac{1 - \left| -\dfrac{T_{21}}{T_{11}} \right|^2}{\left| 2Im(\dfrac{T_{21}}{T_{11}}) \right| + \left| 1 - (\dfrac{T_{21}}{T_{11}})^2 \right|}$$

As mentioned before $\frac{1 - |z|^2}{|1 - z^2|} \leq 1$ hence, $\frac{1 - |z|^2}{|2Im(z)| + |1 - z^2|} \leq 1$.

B.1) Quadratic equation for Thermo-Acoustic systems

We know $R_{in} = \frac{S_{11} - \Delta R_2}{1 - S_{22}R_2}$, $|R_{up}R_{in}|^2 < 1$,

$$2Re(z) = z + \bar{z}, \qquad Re(z_1 z_2) \leq |z_1||z_2|$$

and $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\,Re(z_1\overline{z_2})$. Therefore,

$$|R_{up}R_{in}|^2 < 1 \rightarrow |R_{up}(S_{11} - \Delta R_{dn})|^2 < |1 - S_{22}R_{dn}|^2.$$

Expanding the power results in

$$|R_{up}|^2(|S_{11}|^2 + |\Delta R_{dn}|^2 - 2\,Re(S_{11}\overline{\Delta R_{dn}})) < 1 + |S_{22}|^2|R_{dn}|^2 - 2\,Re(\overline{S_{22}R_{dn}}).$$

Applying $2Re(z) = z + \bar{z}$, one writes

$$|R_{up}|^2|S_{11}|^2 + |R_{up}|^2|\Delta|^2|R_{dn}|^2 - [|R_{up}|^2 S_{11}\overline{\Delta R_{dn}} -$$

$$|R_{up}|^2 \overline{S_{11}}\Delta R_{dn}] - 1 - |S_{22}|^2|R_{dn}|^2 +$$

$$[S_{22}R_{dn} + \overline{S_{22}R_{dn}}] < 0.$$

Factorizing $|R_{dn}|^2$, R_{dn}, and $\overline{R_{dn}}$ concludes

$$(|R_{up}|^2|\Delta|^2 - |S_{22}|^2)|R_{dn}|^2 + [R_{dn}(S_{22} - |R_{up}|^2\overline{S_{11}}\Delta)$$

$$+\overline{R_{dn}}(\overline{S_{22}} - |R_{\text{up}}|^2 S_{11}\bar{\Delta})] + |R_{up}|^2|S_{11}|^2 - 1 < 0.$$

Employing $z + \bar{z} = 2Re(z)$, one finds

$$(|R_{up}|^2|\Delta|^2 - |S_{22}|^2)|R_{dn}|^2$$

$$+2\,Re(R_{dn}[S_{22} - |R_{\text{up}}|^2\overline{S_{11}}\Delta]) + |R_{up}|^2|S_{11}|^2 - 1 < 0.$$

Generally speaking, if $A|R_{dn}|^2 + 2|z_1||z_2| + C$ is smaller than zero then for sure $A|R_{dn}|^2 + 2Re(z_1 z_2) + C$ would be less than zero because of $Re(z_1 z_2) \leq |z_1||z_2|$. Therefore, one allows to consider below inequality instead of the last inequality.

$$(|R_{up}|^2|\Delta|^2 - |S_{22}|^2)|R_{dn}|^2 + 2|R_{dn}|\,|S_{22} - |R_{\text{up}}|^2\overline{S_{11}}\Delta|$$

$$+|R_{up}|^2|S_{11}|^2 - 1 < 0.$$

Therefore, if

$$A|R_{dn}|^2 + B|R_{dn}| + C < 0$$

where $A = (|R_{up}|^2|\Delta|^2 - |S_{22}|^2)$;

$$B = \ 2|S_{22} - |R_{\text{up}}|^2\overline{S_{11}}\Delta|;$$

$$C = |R_{up}|^2|S_{11}|^2 - 1,$$

then for sure $|R_{\text{up}}R_{\text{in}}|^2 \ < 1.$

B.2)

$$B^2 - 4AC = 4|S_{22} - |R_{\text{up}}|^2\overline{S_{11}}\Delta|^2$$

$$- 4(|R_{up}|^2|\Delta|^2 - |S_{22}|^2)(|R_{up}|^2|S_{11}|^2 - 1).$$

Expanding the equation and applying $2Re(z) = z + \bar{z}$, one writes

$$4[|S_{22}|^2 + |R_{\text{up}}|^4|\overline{S_{11}}\Delta|^2 -$$

$$S_{22}|R_{\text{up}}|^2 S_{11}\bar{\Delta} - \overline{S_{22}}|R_{\text{up}}|^2\overline{S_{11}}\Delta - |R_{up}|^4|\Delta|^2|S_{11}|^2$$

$$-|R_{up}|^2|\Delta|^2 - |S_{22}|^2|R_{up}|^2|S_{11}|^2 - |S_{22}|^2].$$

Simplifying the equation leads to

$$B^2 - 4AC = 4|R_{\text{up}}|^2|S_{11}S_{22} - \Delta|^2 \rightarrow$$

$$B^2 - 4AC = 4|R_{\text{up}}|^2|S_{12}S_{21}|^2 \geq 0.$$

B.3)

$$\lambda_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

Substituting A, B, C defined in Appendix B.1 leads to

$$\lambda_1 = \frac{-2|S_{22} - |R_{\text{up}}|^2\overline{S_{11}}\Delta| + 2|R_{\text{up}}||S_{12}S_{21}|}{2(|R_{up}|^2|\Delta|^2 - |S_{22}|^2)}.$$

Also, the same procedure can be used to prove equation (8.b) from $|R_{\text{dn}}R_{\text{out}}|^2 < 1$.

C.1) A new proof of Balsi et al. criterion

Balsi et al. showed that necessary and sufficient conditions for conditional stability can be ascertained by means of a single parameter. Their theorem was,[19]

'*Provided that the S-parameters defined for at least one pair of positive constant reference impedances have no RHP poles, the necessary and sufficient condition for a linear active two-port to be stable is*

$$\frac{1 - |S_{11}|^2 R_{\text{up}}^2}{|S_{22} - \overline{S_{11}}\Delta R_{\text{up}}^2|R_{dn} + |S_{12}S_{21}|R_{up}R_{dn}} > 1.$$

Proof. As shown in Edwards-Sinsky paper, one can readily show that [16]

$$|S_{22}|^2 - |R_{up}|^2|\Delta|^2$$

$$= \frac{|S_{22} - |R_{\text{up}}|^2\overline{S_{11}}\Delta|^2 - |R_{\text{up}}|^2|S_{11}S_{22} - \Delta|^2}{1 - |R_{\text{up}}|^2|S_{11}|^2}.$$

By substituting the term into the denominator of equation (8.c) and simplify it, one can derive

$$|R_{dn}| < \frac{1 - |R_{\text{up}}|^2|S_{11}|^2}{|S_{22} - |R_{\text{up}}|^2\overline{S_{11}}\Delta| + |R_{up}||S_{12}S_{21}|}.$$

By moving $|R_{dn}|$ to the right side of the earlier equation, the conditional stability criterion, provided by Balsi et al., is derived.

C.2) A new proof of Edwards-Sinsky criterion

Substituting $|R_{\text{up}}| = 1$ and $|R_{dn}| = 1$ into the aforementioned equation (last equation in C.1), it is easy to realize that the right-hand side of the equation is indeed the Edwards-Sinsky criterion.

$$1 < \frac{1 - |S_{11}|^2}{|S_{22} - \overline{S_{11}}\Delta| + |S_{12}S_{21}|} = \mu.$$