# Mimetic discretizations of the incompressible Navier-Stokes equations for polyhedral meshes 

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# Mimetic Discretizations of the Incompressible Navier-Stokes Equations for Polyhedral Meshes 

René Beltman

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# Mimetic Discretizations of the Incompressible Navier-Stokes Equations for Polyhedral Meshes 

## PROEFSCHRIFT

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## Introduction

### 1.1 Motivation

To combat the climate crisis the European Commission presented the European Green Deal which aims for no net emissions of greenhouse gases in 2050. The energy production accounts for $75 \%$ of the EU's emissions. The commission states that to reach the aims of the deal increasing offshore wind energy production will be essential and that between 240 and 450 GW of offshore wind power is needed by 2050 to keep temperature rises below $1.5^{\circ} \mathrm{C}[1]$. This amounts to a capacity increase by a factor up to 20 .

To bring down the cost of energy, wind turbines have become increasingly larger. The maximum tip height of commercially available turbines increased from just over 100 meters ( 3 MW turbine) in 2010 to more than 200 meters in 2016 ( 8 MW turbine). The industry is even targeting $15-20 \mathrm{MW}$ turbines for 2030 with a tip height up to 250 meters [2].

With increasing turbine size, the loads acting on the blades and tower become more extreme. The turbines extract a huge amount of energy from the flow causing it to slow down. Moreover, the turbines produce a highly turbulent wake which affects the loads on, and efficiency of, downstream turbines. Accurate quantifications of these effects (and their uncertainties) are essential to predict the lifetime and output of turbines in a wind farm design. Based on these predictions the turbines and wind farm lay-out can be optimized to further bring down the cost of energy.

Numerical simulation plays a crucial role in this. The air flow through the wind farm can be accurately modeled by the incompressible Navier-Stokes equations ${ }^{1}$. Analytic solutions to these equations can only be found in highly idealized settings; Navier-Stokes solutions for wind-farm flows are computed by numerical approximation methods. These numerical methods are the topic of this thesis.

The size of the wind turbines, the high wind speeds ( $5-25 \mathrm{~m} / \mathrm{s}$ ) and the resulting turbulence make it very challenging to find accurate approximations. The turbulent nature of the flow implies that it admits a wide range of length and time scales. The largest turbulent scales are in the order of kilometers, while the smallest vortices are smaller than a tenth of a millimeter. ${ }^{2}$ Numerical simulation of this wide range of scales demands enormous computational resources and very efficient numerical methods.

[^0]Efficient numerical techniques and high computing power are not only of importance for wind farm analysis. Computational methods are in general an important tool for solid mechanics, fluid mechanics, electromagnetics, optics and more; for science and engineering in general. The numerical techniques discussed in this thesis can also be applied in other domains.

### 1.2 The incompressible Navier-Stokes equations and the energy balance

We now introduce the Navier-Stokes equations, the continuum equations that together with initial and boundary conditions completely describe the fluid flow in our computational domain $\Omega$.

For a fluid described by a density $\rho$ and a velocity field $\underline{u}$ Newton's second law is given by

$$
\begin{equation*}
\frac{\partial \pi}{\partial t}+\nabla \cdot(\underline{u} \otimes \pi)=\nabla \cdot \underline{\underline{\sigma}}+\underline{f} \tag{1.1}
\end{equation*}
$$

where $\pi:=\rho \underline{u}$ is the momentum density, $\underline{\underline{\sigma}}$ is the stress tensor of the fluid and $\underline{f}$ a force field acting on the fluid [4]. In components the tensor product $\underline{u} \otimes(\rho \underline{u})$ is given by

$$
\underline{u} \otimes \pi=\rho\left[\begin{array}{lll}
u_{1} u_{1} & u_{1} u_{2} & u_{1} u_{3} \\
u_{2} u_{1} & u_{2} u_{2} & u_{2} u_{3} \\
u_{3} u_{1} & u_{3} u_{2} & u_{3} u_{3}
\end{array}\right] .
$$

The stress tensor is given by $\underline{\overline{\underline{\sigma}}}=-p \underline{\underline{\underline{I}}}+\underline{\underline{\tau}}$, where $p$ is the pressure, $\underline{\underline{I}}$ is the identity tensor and for an incompressible $\overline{\overline{\text { N }}}$ ewtonian $\overline{=}_{\text {fluid the deviatoric stress tensor }}^{\underline{\tau}}$ is given by

$$
\underline{\underline{\tau}}:=\mu\left(\nabla \underline{u}+(\nabla \underline{u})^{T}\right)
$$

where $\mu$ is the dynamic viscosity of the fluid. Substituting $\underline{\underline{\tau}}$ in (1.1) and expressing the fluid's incompressibility as $\nabla \cdot \underline{u}=0$, the incompressible Navier-Stokes equations take the form

$$
\begin{align*}
\frac{\partial \rho \underline{u}}{\partial t}+\nabla \cdot(\underline{u} \otimes(\rho \underline{u})) & =\mu \Delta \underline{u}-\nabla p+\underline{f},  \tag{1.2a}\\
\nabla \cdot \underline{u} & =0 . \tag{1.2b}
\end{align*}
$$

These differential equations, supplemented with initial and boundary conditions provide a complete model for the description of turbulent flows.

The Reynolds number is given by $\operatorname{Re}=\rho U L / \mu$, with $U$ a characteristic speed and $L$ a characteristic length for the problem at hand. The Reynolds number is a dimensionless number that indicates the ratio between the convective terms and viscous terms. As this number is very large $\left(\operatorname{Re}>10^{8}\right)$ for a wind farm, the flow is dominated by the nonlinear convective term. It is unfeasible to solve all length scales by direct numerical simulation (DNS). To make numerical simulation possible the effect of the small scales on the flow
is modeled. The state-of-the-art turbulence model is provided by Large Eddy Simulation (LES), which allows for unsteady anisotropic turbulent flows [5].

Flow simulations by DNS or LES require a careful approach when discretizing the Navier-Stokes equations. Upwind discretizations of the convective term often result in a stable numerical method but also introduce numerical dissipation. The numerical energy dissipation can have a detrimental effect on the simulation as it can interfere with the modeled turbulent dissipation or even overwhelm it [6, 7]. This influences the energy cascade from large to small scales and can result in premature turbulence decay. For wind farm simulations this might, for example, lead to an under-prediction of the velocity deficit in the turbine wake [8, 9].

The kinetic energy of incompressible flow is defined as

$$
\begin{equation*}
\mathcal{K}:=\frac{1}{2} \int_{\Omega} \rho \underline{u} \cdot \underline{u} d V \tag{1.3}
\end{equation*}
$$

where $\Omega$ is the flow domain. This energy can change only in a specific way.
Theorem 1.1. The global energy only changes due to viscous dissipation, fluxes over the boundary, and work done by forces:

$$
\begin{equation*}
\frac{\partial \mathcal{K}}{\partial t}=\int_{\Omega}-\mu \underline{\omega} \cdot \underline{\omega}+\underline{u} \cdot \underline{f} d V-\int_{\partial \Omega}\left(\frac{1}{2} \rho \underline{u} \cdot \underline{u}+p\right) \underline{u} \cdot \underline{n}+\mu(\underline{\omega} \times \underline{u}) \cdot \underline{n} d A \tag{1.4}
\end{equation*}
$$

where $\underline{\omega}:=\nabla \times \underline{u}$ is the vorticity and $\underline{n}$ is the unit outward normal on $\partial \Omega$.
Proof. From the momentum equation (1.2a), it follows

$$
\frac{\partial \mathcal{K}}{\partial t}=\int_{\Omega} \underline{u} \cdot \frac{\partial \rho \underline{u}}{\partial t} d V=-\int_{\Omega} \underline{u} \cdot(\nabla \cdot(\underline{u} \otimes(\rho \underline{u}))+\mu \nabla \times \underline{\omega}+\nabla p-\underline{f}) d V
$$

where we rewrote the viscous term according to $\mu \Delta \underline{u}=-\mu \nabla \times \underline{\omega}$ using the incompressibility of the fluid. We first notice the fact that ${ }^{3}$

$$
\underline{u} \cdot(\nabla \cdot(\underline{u} \otimes(\rho \underline{u})))=\nabla \cdot\left(\left(\frac{1}{2} \rho \underline{u} \cdot \underline{u}\right) \underline{u}\right) .
$$

Using corollaries of the Gauss Divergence Theorem we find

$$
\begin{aligned}
& \int_{\Omega} \underline{u} \cdot(\nabla \cdot(\underline{u} \otimes(\rho \underline{u}))+\mu \nabla \times \underline{\omega}+\nabla p-\underline{f}) d V \\
& \quad=\int_{\partial \Omega}\left(\frac{1}{2} \rho \underline{u} \cdot \underline{u}\right) \underline{u} \cdot \underline{n}+\mu(\underline{\omega} \times \underline{u}) \cdot \underline{n}+p \underline{p} \cdot \underline{n} d A-\int_{\Omega}-\mu \underline{\omega} \cdot \underline{\omega}+(\nabla \cdot \underline{u}) p-\underline{u} \cdot \underline{f} d V
\end{aligned}
$$

From this follows (1.4), by the incompressibility of the fluid.
Theorem 1.1 shows that, in the absence of external forces and boundary fluxes, the energy of the flow can only diminish by viscous dissipation:

$$
\begin{equation*}
\frac{d \mathcal{K}}{d t}=-\int_{\Omega} \mu \underline{\omega} \cdot \underline{\omega} d V . \tag{1.5}
\end{equation*}
$$

[^1]It is generally agreed upon that for DNS or LES of turbulent flow it is very important that (in absence of forces and boundary contributions) the numerical scheme used mimics the balance equation (1.5) [5, 7,10-15]. This means that one can define an approximation of $\mathcal{K}$ using the discrete variables, that changes only due to the viscous term in the discretized Navier-Stokes equations. Such a discretization is also called energy-conserving, because energy is conserved in the inviscid limit.

Besides the absence of numerical dissipation, energy-conserving methods have the added benefit that the discrete version of (1.5) provides a nonlinear stability bound on the discrete solution. If the spatial discretization is combined with an energy-conserving time-integration method, then the time step can be solely chosen based on accuracy requirements instead of on stability requirements [16].

### 1.3 Starting point: the MAC method

The first energy-conserving numerical scheme for the incompressible Navier-Stokes equations was proposed in 1965 and is known as the MAC (Marker-and-Cell) method [17]. The distinguishing feature of the MAC method is the positioning of the variables in the mesh. The velocity field is discretized on the mesh as the normal components on the cell faces and the pressure variables are located in the cell centers (Figure 1.1). The staggered position of the velocity variables allows for a straightforward conservative discretization of the continuity equation (1.2b) as

$$
\begin{equation*}
0=u_{i, j} \Delta y-u_{i-1, j} \Delta y+v_{i, j} \Delta x-v_{i, j-1} \Delta x . \tag{1.6}
\end{equation*}
$$

The positioning of the pressure variables in the center of the cells makes that the MAC method has a pressure variable corresponding to each discrete continuity equation.

The momentum equation is discretized componentwise using simple, second order accurate, central averages. For example, the $x$-momentum equation of (1.2a) corresponding to the velocity variable $u_{i, j}$ is discretized in space as (Figure 1.1)

$$
\begin{align*}
& \rho \frac{\partial u_{i, j}}{\partial t} \Delta x \Delta y \\
& =-\frac{\rho \Delta y}{4}\left(u_{i+1, j}+u_{i, j}\right)^{2}+\frac{\rho \Delta y}{4}\left(u_{i, j}+u_{i-1, j}\right)^{2}-\frac{\rho \Delta x}{4}\left(u_{i, j+1}+u_{i, j}\right)\left(v_{i, j}+v_{i+1, j}\right) \\
& \quad+\frac{\rho \Delta x}{4}\left(u_{i, j}+u_{i, j-1}\right)\left(v_{i+1, j-1}+v_{i, j-1}\right)  \tag{1.7}\\
& \quad+\mu \Delta y\left(\frac{u_{i+1, j}-u_{i, j}}{\Delta x}\right)-\mu \Delta y\left(\frac{u_{i, j}-u_{i-1, j}}{\Delta x}\right) \\
& \quad+\mu \Delta x\left(\frac{u_{i, j+1}-u_{i, j}}{\Delta y}\right)+\mu \Delta x\left(\frac{u_{i, j}-u_{i, j-1}}{\Delta y}\right) \\
& \quad-p_{i+1, j} \Delta y+p_{i, j} \Delta y
\end{align*}
$$

where the discrete convection term, discrete diffusion term and discrete pressure term, are on the second, third and fourth line respectively. The $y$-momentum equation is


Figure 1.1: On the left: the stencil for the discrete continuity equation (1.6) corresponding to $p_{i, j}$ with the control volume shown in red. In the middle: the stencil for the discrete $x$-momentum equation (1.7) corresponding to $u_{i, j}$ with the control volume shown in green. On the right: the stencil for the discrete $y$-momentum equation (1.8) corresponding to $v_{i, j}$ with the control volume shown in yellow.
discretized analogously:

$$
\begin{align*}
& \rho \frac{\partial v_{i, j}}{\partial t} \Delta x \Delta y \\
& =-\frac{\rho \Delta x}{4}\left(v_{i, j+1}+v_{i, j}\right)^{2}+\frac{\rho \Delta x}{4}\left(v_{i, j}+v_{i, j-1}\right)^{2}-\frac{\rho \Delta y}{4}\left(v_{i+1, j}+v_{i, j}\right)\left(u_{i, j}+u_{i, j+1}\right) \\
& \quad+\frac{\rho \Delta y}{4}\left(v_{i, j}+v_{i-1, j}\right)\left(u_{i-1, j+1}+u_{i-1, j}\right)  \tag{1.8}\\
& \quad+\mu \Delta x\left(\frac{v_{i, j+1}-v_{i, j}}{\Delta y}\right)-\mu \Delta x\left(\frac{v_{i, j}-v_{i, j-1}}{\Delta y}\right) \\
& \quad+\mu \Delta y\left(\frac{v_{i+1, j}-v_{i, j}}{\Delta x}\right)+\mu \Delta y\left(\frac{v_{i, j}-v_{i-1, j}}{\Delta x}\right) \\
& \quad-p_{i, j+1} \Delta x+p_{i, j} \Delta x
\end{align*}
$$

The MAC discretization can be viewed as a finite volume method where the control volumes are centered around the relevant variable as depicted in Figure 1.1. This implies that the method is mass and momentum conserving in a finite volume sense. Besides these primary quantities, it can be shown that the method is energy-conserving (in the inviscid limit).

The conservation of energy can be viewed as a consequence of the symmetry properties of the discretization matrices of the method [18]. It can be shown that the convection matrix of the method is skew-symmetric and as a result does not dissipate the kinetic energy of the flow. Similarly, it can be shown that the gradient matrix which acts on the pressure variables is exactly minus the transpose of the divergence matrix of the discrete continuity equation. This can be used to show that also the discretized pressure gradient in the MAC method only influences the energy balance in a physical (non-dissipative) way. The energy decays only due to the symmetric negative definite matrix representing the viscous term.

### 1.4 The goal: an efficient energy-conserving method for complicated geometries

Although the original MAC method is a very successful and still much used method, the necessity of a Cartesian mesh is very restrictive. To make it wider applicable the method has been generalized to triangular meshes [19-21]. However, these unstructured MAC methods are only first order accurate for unstructured triangulations [21,22] and second order accurate for structured triangulations, where they are sometimes (depending on the discretization of the convection term) equivalent to the original MAC method. The MAC method was also extended to curvilinear meshes [23], allowing for curvilinear domains. This widely extends the applicability of the method but is still too restrictive for a situation where the geometry contains multiple independent moving components.

For a wind farm the difficult time-dependent geometry is often not modeled directly. Instead the presence of the turbine blades and their effect on the flow are modeled through an extra force term in momentum equation (1.2a). These methods differ in terms of how accurately the force is prescribed ranging from actuator disks to the more sophisticated actuator surface method, see [3] for an overview of these methods and further references. The more sophisticated actuator methods require additional information to prescribe the force for given flow conditions. This can come from experimental data, a separate database of reference simulations or a coupled 2D airfoil calculation. Another more accurate approach is to use body-fitted meshes for the moving components. However, this is computationally very expensive and such calculations are limited to a single turbine and wake.

In this thesis we propose to use a cut-cell approach. The Cartesian mesh is locally adapted to model the presence of objects. This allows the use of a single mesh and avoids the indirect modeling through a force term. A successful energy-conserving extension of the MAC method to 2D cut-cell meshes has been proposed in [24]. The difficulty lies in the fact that the polytopal cut-cells can have a wide range of shapes each requiring a different discretization stencil [25-27]. This issue becomes especially pressing because this discretization is derived using staggered control volumes (for the momentum equation) and therefore leads to combinations of two cut-cells that have to be considered. This makes the implementation of the method tedious and inhibits a widespread use.

The aim of this thesis is to find an extension of the MAC method to cut-cell meshes that

- is energy-conserving,
- simplifies to the original MAC method in the Cartesian regions of the mesh,
- uses general discretization formulae and is easily implementable,
- allows for local mesh refinement.

We will make use of and contribute to the rapid development of compatible polytopal discretization methods [28-31].

### 1.5 Outline

In Chapter 2 we start by introducing exterior calculus, an alternative to vector calculus that is better suited for the mimetic discretizations studied in this thesis. In this chapter we review concepts that will be used in later chapters. We introduce differential forms, the exterior derivative and Stokes' theorem, that form the starting point for the mimetic discretizations of Chapters 3 and 4 . We define the Helmholtz-Hodge decomposition that describes the decomposition of the velocity field in terms of a vector potential, scalar potential and a harmonic part and whose discretization is the topic of Chapter 5. Furthermore, we define $k$-vectors that allow us to derive the new discretization methods of Chapter 6.

Next, in Chapter 3, we review the fundamental concepts of a mimetic discretization: The primal mesh, the dual mesh and the discrete Hodge matrices. We review the different possible choices for the discrete Hodge matrices and present them in a common framework. We discuss in Section 3.3 the interpolation from the primal to the dual mesh. Using the exterior calculus we can represent this using a single simple interpolation formula that covers all space and mesh element dimensions at once. This generic geometric formula will be the starting point for new mimetic discretization methods discussed in Chapter 6.

In Chapter 4 we will address the main aim of this thesis. We will use the mimetic discretization techniques discussed in earlier chapters to extend the MAC scheme to non-Cartesian meshes. We discuss two ways of doing this and discuss three different convection discretizations. We end the chapter by applying one of the methods as a cut-cell method to calculate the flow around a cylinder and an airfoil.

In Chapters 5 and 6 we discuss two separate issues closely related to the mimetic discretizations of the earlier chapters, but not directly concerned with the cut-cell methods of Chapter 4. In Chapter 5 it is shown how the velocity field can be found from the vorticity field through the solution of non-singular linear systems. At the same time it is shown how a discrete Helmholtz-Hodge decomposition can be calculated, when either normal or tangential boundary conditions apply for the velocity field. The results of this chapter directly apply to the mimetic discretization methods in earlier chapters. The Helmholtz-Hodge decomposition plays an important role in projection methods for the time integration of the Navier-Stokes equations.

Chapter 6 is concerned with mimetic methods based on explicit interpolation from the dual mesh to the primal mesh. It is known that explicit interpolation is possible when the dual mesh is a circumcentric dual mesh and when the discrete Hodge matrix, which interpolates from the primal to the dual mesh, is diagonal. In this chapter we show that it is also possible when the primal mesh is a simplicial mesh and the dual mesh is a barycentric dual mesh. This relieves the stringent requirements on the regularity of the simplicial mesh that need to be met when the circumcentric dual mesh is used.

The thesis is concluded in Chapter 7. In this final chapter we also show some directions for follow-up research. In particular, we discuss the extension of the mimetic cut-cell method to moving immersed objects and the development of a new mimetic method that conserves, besides energy, also the important physical quantity helicity in the absence of viscosity.

## Exterior Calculus

In Chapter 1 we saw that additional physical principles can be derived from the incompressible Navier-Stokes equations. As an example we considered in Theorem 1.1 the energy of the flow. Conservation of energy was derived by a careful utilization of the relations between the differential operators involved in the equations.

The equations were thus far expressed in terms of vector calculus. In this chapter we change to a different mathematical formalism: exterior calculus. For a number of reasons this will better serve our purpose, we expound some of these first.

A conservation law states that the amount of a certain quantity that leaves a subdomain $\Omega \subset \mathbb{R}^{3}$ by crossing its boundary $\partial \Omega$ is exactly matched by the amount of that quantity that enters $\Omega$ and the amount that is produced within $\Omega$. The simplest example for an incompressible fluid is conservation of mass:

$$
\int_{\partial \Omega} \underline{u} \cdot d \underline{A}=0 .
$$

The fact that above integral, for an incompressible fluid, is zero hinges only on one thing, namely, the Gauss Divergence Theorem. This theorem, which equates an integral over $\partial \Omega$ with an integral over $\Omega$, holds irrespectively of the shape and size of $\Omega$. The set $\Omega$ can have any metrical realization, i.e., any shape and size. Why then does the metric enter the equation in the form of the inner product between $\underline{u}$ and $d \underline{A}$ ? Contrary to vector calculus, exterior calculus allows to express the Gauss Divergence Theorem (and also the Kelvin-Stokes Theorem and the Fundamental Theorem for Line Integrals) in a metrical independent way (known as Stokes' theorem).

This is important because we will see that this implies that, after an appropriate (mimetic) discretization, it is possible to express the conservation laws in terms of matrices only depending on the mesh connectivity and independent of the size and shape of the cells. Moreover, the separation of the topological and metrical operators allows for a discretization of the differential operators that preserves their symmetries and properties. For example, the adjointness between the divergence and (minus) the gradient and the fact that the curl of a gradient is always zero will be preserved in the discretization. These issues are crucial for an energy conserving and stable numerical scheme.

Already in Chapter 1 we saw that one of crucial points for the MAC discretization by Harlow and Welch is the staggered positioning of the variables. The natural location for the velocity variables are the mesh faces, where only their normal component is located, and not the cell centers. The discretizations that will be introduced in the next Chapter embroiders on this viewpoint. All discrete variables will find their natural place in the mesh depending on their place in (discretized) differential equations.

The discrete variables in these discretization methods are so-called $k$-cochains. They can be thought of as an assignment of a real number to each mesh element of dimension $k$. The discrete velocity variables in the MAC scheme are a 2-cochain, an assignment of a real number to each 2-dimensional mesh element, the mesh faces. We will introduce cochains in the next chapter.

Instead of vector fields and scalar fields, the fundamental element in exterior calculus is the differential $k$-form. In exterior calculus a $k$-form can be integrated over a $k$ dimensional manifold to produce a real number. Examples of these are the line integral (integral of 1-form over a 1-dimensional manifold), the integral of a flux through a surface (integral of a 2 -form over a 2 -dimensional manifold) and the integral of a density over a volume (integral of a 3 -form over a 3 -dimensional manifold).

In exterior calculus the relation between the differential form and the manifold is more symmetric than the relation between the argument and domain of integration in vector calculus. To stress this the integral of a $k$-form $a^{(k)}$ over a $k$-dimensional manifold $M_{(k)}$ can be expressed as

$$
\left\langle M_{(k)}, a^{(k)}\right\rangle:=\int_{M_{(k)}} a^{(k)} .
$$

The $k$-form $a^{(k)}$ and the $k$-dimensional manifold $M_{(k)}$ together produce the real number $\left\langle M_{(k)}, a^{(k)}\right\rangle$.

The most appealing feature of exterior calculus is that it unites the fundamental calculus theorems, i.e., the fundamental theorem of line integrals, the Kelvin-Stokes theorem and the Gauss divergence theorem, in one elegant theorem, the general Stokes theorem:

$$
\left\langle M_{(k+1)}, d^{(k)} a^{(k)}\right\rangle=\left\langle\partial_{(k)} M_{(k+1)}, a^{(k)}\right\rangle .
$$

In this equation $d^{(k)}$ replaces, the gradient $(k=0)$, the curl $(k=1)$ or the divergence $(k=2)$ and $\partial_{(k)}$ simply returns the $k$-dimensional boundary of a $(k+1)$-dimensional manifold.

The Stokes theorem above will actually be used to discretize the differential operator $d^{(k)}$ in the next chapter. The mesh will provide us with a set of $0-, 1-, 2-$ and 3 -dimensional manifolds which are related to each other by the boundary operator, e.g., for a 3 -dimensional mesh cell $\boldsymbol{c}_{(3)}, \partial_{(2)} \boldsymbol{c}_{(3)}$ is the set of its 2 -dimensional faces. This discrete representation of the boundary operator will let us define a discretized version of the differential operator $d^{(k)}$ such that Stokes theorem holds for all instances provided by the mesh.

The duality relation between manifolds and forms is the recurring theme in this chapter. We first consider this duality in linear space and focus on the algebra. Then we study the continuum limit and introduce the necessary concepts from exterior calculus.

### 2.1 Exterior algebra: vectors and forms

In this thesis mesh edges and faces will be affine, i.e, edges are straight line segments and faces are planar. Moreover, the mesh elements will have an orientation. This orientation indicates, for example, the direction of flux through a face.


Figure 2.1: Switching the order in the exterior product, or changing the orientation of one of the vectors changes the orientation of the 2 -vector.

Such a linear oriented edge can be represented by a vector, which has the length and orientation of the edge. In this section we generalize the vector to objects called $k$ vectors. A $k$-vector can be thought of as a $k$-dimensional object that has an orientation and a $k$-dimensional "volume". For example, a 2 -vector has an area and an orientation and a 3 -vector has a volume and an orientation, just like a vector has a length and an orientation. We can use 2 -vectors to represent the area and orientation of the mesh faces and 3 -vectors to represent the volume and orientation of the mesh cells.

### 2.1.1 The exterior product

To introduce $k$-vectors we need the exterior product [32]. We first introduce the exterior product of two vectors, giving us 2 -vectors, and then define $k$-vectors and exterior products between them.

Definition 2.1. Let $V$ be a $d$-dimensional vector space over $\mathbb{R}$. We define the space of 1-vectors as $\Lambda_{(1)} V=V$. We can form the exterior product $a_{(1)} \wedge b_{(1)}$ of two vectors $a_{(1)}, b_{(1)} \in \Lambda_{(1)} V$. The exterior product has the following properties. It is anti-symmetric

$$
a_{(1)} \wedge b_{(1)}=-b_{(1)} \wedge a_{(1)}
$$

and it is bilinear

$$
\begin{array}{rlrl}
\alpha\left(a_{(1)} \wedge b_{(1)}\right)=\left(\alpha a_{(1)}\right) \wedge b_{(1)} & =a_{(1)} \wedge\left(\alpha b_{(1)}\right) & & \forall \alpha \in \mathbb{R}, a_{(1)}, b_{(1)} \in V \\
\left(a_{(1)}+c_{(1)}\right) \wedge b_{(1)}=a_{(1)} \wedge b_{(1)}+c_{(1)} \wedge b_{(1)}, & & \forall a_{(1)}, b_{(1)}, c_{(1)} \in V \\
a_{(1)} \wedge\left(b_{(1)}+c_{(1)}\right) & =a_{(1)} \wedge b_{(1)}+a_{(1)} \wedge c_{(1)}, & & \forall a_{(1)}, b_{(1)}, c_{(1)} \in V .
\end{array}
$$

We define the space of 2-vectors $\Lambda_{(2)} V$ to be the vector space generated by the set $\left\{a_{(1)} \wedge b_{(1)} \mid a_{(1)}, b_{(1)} \in V\right\}$.

The 2 -vector $a_{(1)} \wedge b_{(1)}$ can be respresented graphically as a parallelogram spanned by $a_{(1)}$ and $b_{(1)}$. The orientation of the 2 -vector (i.e., $a_{(1)} \wedge b_{(1)}$ vs. $\left.b_{(1)} \wedge a_{(1)}\right)$ can then be represented graphically by a curved vector from $a_{(1)}$ to $b_{(1)}$. This is shown in Figure 2.1. The 2 -vector is here represented by a parallelogram but this is just a choice. The 2 -vector could equally well have been drawn as, for example, an ellipse with the same area and orientation. In Figure 2.2 linearity of the exterior product is exemplified.


Figure 2.2: The exterior product is linear.

Calculations of exterior products are most conveniently done in terms of a basis. Let $V=\mathbb{R}^{3}$ and let $\left\{e_{(1), 1}, e_{(1), 2}, e_{(1), 3}\right\}$ be a basis for $V$. The exterior product of $a_{(1)}=a^{1} e_{1,(1)}+a^{2} e_{(1), 2}+a^{3} e_{(1), 3}$ and $b_{(1)}=b^{1} e_{(1), 1}+b^{2} e_{(1), 2}+b^{3} e_{(1), 3}$ is given by

$$
\begin{aligned}
a_{(1)} \wedge b_{(1)}= & \left(a^{1} e_{(1), 1}+a^{2} e_{(1), 2}+a^{3} e_{(1), 3}\right) \wedge\left(b^{1} e_{(1), 1}+b^{2} e_{(1), 2}+b^{3} e_{(1), 3}\right) \\
= & \left(a^{1} b^{2}-a^{2} b^{1}\right) e_{(1), 1} \wedge e_{(1), 2}+\left(a^{3} b^{1}-a^{1} b^{3}\right) e_{(1), 3} \wedge e_{(1), 1} \\
& +\left(a^{2} b^{3}-a^{3} b^{2}\right) e_{(1), 2} \wedge e_{(1), 3},
\end{aligned}
$$

where terms of the form $e_{(1), i} \wedge e_{(1), i}$ cancel because of anti-symmetry. From this it becomes apparent that $\left\{e_{(1), 1} \wedge e_{(1), 2}, e_{(1), 3} \wedge e_{(1), 1}, e_{(1), 2} \wedge e_{(1), 3}\right\}$ forms a basis for $\Lambda_{(2)} V$.

We see that the exterior product between two vectors is very similar to the cross product. However, no sense of orthogonality is used in the definition of the exterior product: it is metric-independent.
Definition 2.2. Let $V \cong \mathbb{R}^{d}$ and $0 \leq k \leq d$. We define the space of $k$-vectors $\Lambda_{(k)} V$ to be the vector space generated by the set $\left\{a_{(1), 1} \wedge \cdots \wedge a_{(1), k} \mid a_{(1), 1}, \ldots, a_{(1), k} \in V\right\}$, where we extend the definition of the exterior product to more than two terms by demanding it to be associative, i.e.,

$$
a_{(k)} \wedge\left(b_{(l)} \wedge c_{(m)}\right)=a_{(k)} \wedge b_{(l)} \wedge c_{(m)}=\left(a_{(k)} \wedge b_{(l)}\right) \wedge c_{(m)}
$$

for all $a_{(k)} \in \Lambda_{(k)} V, b_{(l)} \in \Lambda_{(l)} V$ and $c_{(m)} \in \Lambda_{(m)} V, 0 \leq k, l, m \leq d$, and also multilinear. Moreover, we denote $\Lambda_{(1)} V:=V$ and $\Lambda_{(0)} V:=\mathbb{R}$.

As a result, for the general exterior product between a $k$-vector and an $l$-vector the anti-symmetry takes the form

$$
a_{(k)} \wedge b_{(l)}=(-1)^{k l} b_{(l)} \wedge a_{(k)} \quad \forall a_{(k)} \in \Lambda_{(k)} V, b_{(l)} \in \Lambda_{(l)} V
$$

To make the definition complete we have defined 0 -vectors just to be real numbers. Graphically, we can represent 0 -vectors as points. A positive real number we represent as a source (outgoing arrows) and a negative real number we represent as a sink (ingoing arrows). These graphical representations of 0 -vectors will make more sense in the context of the next chapter.

If $\left\{e_{(1), 1}, \ldots, e_{(1), d}\right\}$ is a basis for $V$, then a basis for $\Lambda_{(k)} V$ is given by

$$
\begin{equation*}
\left\{e_{(1), i_{1}} \wedge \cdots \wedge e_{(1), i_{k}} \mid \forall i_{1}, \ldots i_{k}: 1 \leq i_{1}<\cdots<i_{k} \leq d\right\} \tag{2.1}
\end{equation*}
$$



Figure 2.3: From left to right: the 0 -vector +1 , the 1 -vector $a_{(1)}$, the 2 -vector $a_{(1)} \wedge b_{(1)}$ and the 3 -vector $a_{(1)} \wedge b_{(1)} \wedge c_{(1)}$. We draw outgoing arrows for the 0 -vector to indicate that it has positive sign.
and a general $k$-vector can be written as

$$
\begin{equation*}
a_{(k)}=\sum_{1 \leq i_{1}<\cdots i_{k} \leq d} a^{i_{1}, \ldots, i_{k}} e_{(1), i_{1}} \wedge \cdots \wedge e_{(1), i_{k}}, \tag{2.2}
\end{equation*}
$$

where the $a^{i_{1}, \ldots, i_{k}}$ are real coefficients. The dimension of $\Lambda_{(k)} V$ is thus equal to $\binom{d}{k}$.
A $k$-vector is called decomposable if it can be written as an exterior product of $k$ 1 -vectors. One quickly checks that for $d \leq 3$ every $k$-vector is decomposable. However, this is not true in general. For $d \geq 4$ the 2 -vector $e_{(1), 1} \wedge e_{(1), 2}+e_{(1), 3} \wedge e_{(1), 4}$ is for example not decomposable.

If we only want to denote the orientation defined by a $k$-vector $a_{(k)}$, we write $\left\{a_{(k)}\right\}$ for the equivalence class with the same orientation. We have $\left\{-a_{(k)}\right\}=-\left\{a_{(k)}\right\}$ and $\left\{c a_{(k)}\right\}=\left\{a_{(k)}\right\}$ for $c>0$. Thus $\left\{a_{(k)}\right\}$ is an orientation for the subspace in which $a_{(k)}$ lies and hence $\left\{a_{(d)}\right\}$ is an orientation for the space $V$.

Definition 2.3. An orientation for a $d$-dimensional vector space $V$ is a choice for one of the two equivalence classes $\left\{o_{(d)}\right\}$ or $-\left\{o_{(d)}\right\}$, where $o_{(d)}$ is some $d$-vector in $\Lambda_{(d)} V$.

### 2.1.2 Inner- and outer-oriented $k$-vectors

In Figure 2.3 examples of a 0 -, 1 -, 2 - and 3 -vector are given for $V=\mathbb{R}^{3}$. The orientation of the 3 -vector is indicated by a right-handed screw, because $\left\{a_{(1)}, b_{(1)}, c_{(1)}\right\}$ form a right-handed basis for $V$.

The $k$-vectors give an orientation for the $k$-dimensional space in which they lie. For many physical quantities this is useful. If we want to calculate a fluid circulation, we might need to calculate the path integral of the velocity and then we need an orientation along this path. Similarly, if we want to calculate the rotation of a fluid in some surface we need to calculate the integral of the vorticity over this surface and for this we need a direction of rotation in the surface.

However, for other physical quantities this "inner" orientation is not appropriate and we actually need an orientation "transverse" to the $k$-dimensional space in which the $k$-vector lies. An example of this is when we want to calculate the flux of some quantity through a surface. In this situation we need to have an orientation of the dimension transverse to the surface to properly determine the sign of the flux. Similarly, if we


Figure 2.4: Let the vectors $a_{(1)}, b_{(1)}$ and $c_{(1)}$ be as in Figure 2.3. The orientation of outer $k$-vectors is given in gray and is independent of, although expressed in terms of, the orientation of the 1 -vectors. From left to right: the outer 0-vector $+\left\{a_{(1)} \wedge b_{(1)} \wedge c_{(1)}\right\}$, the outer 1-vector $-\left\{a_{(1)} \wedge b_{(1)} \wedge c_{(1)}\right\} a_{(1)}$, the outer 2-vector $\left\{a_{(1)} \wedge b_{(1)} \wedge c_{(1)}\right\} a_{(1)} \wedge b_{(1)}$ and the outer 3-vector $\left\{a_{(1)} \wedge b_{(1)} \wedge c_{(1)}\right\} a_{(1)} \wedge b_{(1)} \wedge c_{(1)}$. (We again draw ingoing arrows for a positive orientation of a 0 -dimensional space.)
want to calculate the rotation of a fluid around a line, we calculate the line integral of the vorticity along the line and we need a direction of rotation along the line, i.e., an orientation of the two dimensions transverse to the line.

The first type of physical quantities described above need an orientation as the $k$ vectors we introduced above possess. To accommodate the second type of physical quantities we introduce $k$-vectors that have an orientation that is outer to them. We will call the type of $k$-vectors we introduced so far inner $k$-vectors (short for inner-oriented $k$-vectors) to contrast them to the outer $k$-vectors (short for outer-oriented $k$-vectors) ${ }^{1}$ that we introduce now.

Definition 2.4. An outer $k$-vector [33] is given by a pair $\left\{o_{(d)}\right\} a_{(k)}$, where $a_{(k)}$ is an inner $k$-vector and $\left\{o_{(d)}\right\}$ is an orientation for the $d$-dimensional vector space $V$. We denote the set of outer $k$-vectors by $\tilde{\Lambda}_{(k)} V$.

The inner orientation of $a_{(k)}$ and the orientation $\left\{o_{(d)}\right\}$ together define an outer orientation, namely let $a_{(d-k)}^{\perp}$ be the $(d-k)$-vector such that $a_{(d-k)}^{\perp} \wedge a_{(k)}=o_{(d)}$, then the inner orientation of $a_{(d-k)}^{( }$is the outer orientation of $\left\{o_{(d)}\right\} a_{(k)}$. Graphical representations of the outer $k$-vectors for $k$ equal 0 up to and including 3 are shown in Figure 2.4.

Just like inner $k$-vectors, outer $k$-vectors are independent of the coordinate system used. This is shown in Figure 2.5 for an outer 2-vector. We can take exterior products between outer $k$-vectors according to

$$
\left(\left\{o_{(d)}\right\} a_{(k)}\right) \wedge\left(\left\{o_{(d)}\right\} b_{(l)}\right)=\left\{o_{(d)}\right\} a_{(k)} \wedge b_{(l)}
$$

where the exterior product on the right-hand side is the usual exterior product between two inner $k$-vectors. The exterior product between an inner $k$-vector and an outer $l$-vector results in an outer $(k+l)$-vector according to

$$
a_{(k)} \wedge\left(\left\{o_{(d)}\right\} b_{(l)}\right)=\left\{o_{(d)}\right\}\left(a_{(k)} \wedge b_{(l)}\right)=\left(\left\{o_{(d)}\right\} a_{(k)}\right) \wedge b_{(l)} .
$$

[^2]

Figure 2.5: The figure shows for the example of an outer 2-vector oriented in the $e_{x}$ direction that it is independent of the orientation and basis used for $\mathbb{R}^{3}$. From left to right we use three coordinate systems and bases: $(x, y, z)=(-\hat{x}, \hat{y}, \hat{z})=(-\tilde{x}, \tilde{y}, \tilde{z})$ and $e_{(1), x}=-e_{(1), \hat{x}}=-e_{(1), \tilde{x}}, e_{(1), y}=e_{(1), \hat{y}}=e_{(1), \tilde{y}}$, $e_{(1), z}=e_{(1), \hat{z}}=-e_{(1), \tilde{z}}$. In these different coordinate systems the outer 2-vector is given by $\left\{e_{(1), x} \wedge\right.$ $\left.e_{(1), y} \wedge e_{(1), z}\right\} e_{(1), y} \wedge e_{(1), z}=\left\{e_{(1), \hat{y}} \wedge e_{(1), \hat{x}} \wedge e_{(1), \hat{z}}\right\} e_{(1), \hat{y}} \wedge e_{(1), \hat{z}}=\left\{e_{(1), \tilde{x}} \wedge e_{(1), \tilde{y}} \wedge e_{(1), \tilde{z}}\right\} e_{(1), \tilde{z}} \wedge e_{(1), \tilde{y}}$ and the outer 2 -vector indeed remains unchanged.

$a_{(1)}$
Figure 2.6: Three wedge products that result in the same outer 2 -vector. The inner 1-vectors are given by $a_{(1)}=e_{(1), x}$ and $b_{(1)}=e_{(1), y}$, and the outer 1-vectors are given by $\tilde{a}_{(1)}=\left\{e_{(1), x} \wedge e_{(1), y} \wedge e_{(1), z}\right\} e_{(1), x}$ and $\tilde{b}_{(1)}=\left\{e_{(1), x} \wedge e_{(1), y} \wedge e_{(1), z}\right\} e_{(1), y}$. We see, from left to right, that $\tilde{a}_{(1)} \wedge \tilde{b}_{(1)}=\tilde{a}_{(1)} \wedge b_{(1)}=$ $a_{(1)} \wedge \tilde{b}_{(1)}=\left\{e_{(1), x} \wedge e_{(1), y} \wedge e_{(1), z}\right\} a_{(1)} \wedge b_{(1)}$.

In Figure 2.6 some examples of exterior products involving outer vectors are shown.

### 2.1.3 The Hodge operator

Through the introduction of an inner product we can assign a size to the $k$-vectors. For every inner $k$-vector there exists then an outer $(d-k)$-vector with the same orientation and the same size. For example, in $\mathbb{R}^{3}$, for every vector there is an outer 2-vector that has an outer-orientation coinciding with the inner-orientation of the vector and the size of the 2 -vector is given by the same scalar as the length of the vector. Of course, vice versa, there is for every outer $k$-vector an inner $(d-k)$-vector with the same orientation and size. We first define the inner product on the spaces $\Lambda^{(k)} V$.

Definition 2.5. On the spaces $\Lambda_{(k)} V$ we define the inner product $(\cdot, \cdot): \Lambda_{(k)} V \times \Lambda_{(k)} V \rightarrow$ $\mathbb{R}$ by linear extension on simple $k$-vectors ${ }^{2}$ as

$$
\left(u_{(1), 1} \wedge \cdots \wedge u_{(1), k}, v_{(1), 1} \wedge \cdots \wedge v_{(1), k}\right)=\operatorname{det}\left(u_{(1), i}, v_{(1), j}\right)
$$

where $\left(u_{(1), i}, v_{(1), j}\right)$ is the usual inner product on $V$ (Euclidean inner product if $V=\mathbb{R}^{k}$ ). Furthermore, we define the inner product $(\cdot, \cdot): \tilde{\Lambda}_{(k)} V \times \tilde{\Lambda}_{(k)} V \rightarrow \mathbb{R}$ by linear extension

[^3]on the simple $k$-vectors as
$$
\left(\left\{o_{(d)}\right\} u_{(1), 1} \wedge \cdots \wedge u_{(1), k},\left\{o_{(d)}\right\} v_{(1), 1} \wedge \cdots \wedge v_{(1), k}\right)=\operatorname{det}\left(u_{(1), i}, v_{(1), j}\right) .
$$

The bijection mentioned above, between $\Lambda_{(k)} V$ and $\tilde{\Lambda}_{(d-k)} V$, is called Hodge duality. The bijection is denoted by ${ }_{(k)}: \Lambda_{(k)} V \rightarrow \tilde{\Lambda}_{(d-k)} V$. The operator is linear and therefore defined by its action on the orthonormal basis elements of $\Lambda_{(k)} V(k=0,1, \ldots, d)$. Take some orientation $\left\{e_{(1), 1} \wedge \cdots \wedge e_{(1), d}\right\}$ for $V$. Then we have

$$
\begin{equation*}
\star_{(k)}\left(e_{(1), i_{1}} \wedge \cdots \wedge e_{(1), i_{k}}\right)=\left\{e_{(1), 1} \wedge \cdots \wedge e_{(1), d}\right\} e_{(1), i_{k+1}} \wedge \cdots \wedge e_{(1), i_{d}}, \tag{2.3}
\end{equation*}
$$

for an even permutation $\left(i_{1}, \ldots, i_{d}\right)$ of $(1, \ldots, d)$.
Similarly, we can define a map $\tilde{\star}_{(k)}: \tilde{\Lambda}_{(k)} V \rightarrow \Lambda_{(d-k)} V$ that returns for each outer $k$-vector an inner $k$-vector. The operator is commonly defined by [33]

$$
\begin{equation*}
\tilde{\star}_{(k)}\left(\left\{e_{(1), 1} \wedge \cdots \wedge e_{(1), d}\right\} e_{(1), i_{1}} \wedge \cdots \wedge e_{(1), i_{k}}\right)=e_{(1), i_{k+1}} \wedge \cdots \wedge e_{(1), i_{d}}, \tag{2.4}
\end{equation*}
$$

for an even permutation $\left(i_{1}, \ldots, i_{d}\right)$ of $(1, \ldots, d)$. From the definition it might seem that $\star$ and $\tilde{\star}$ depend on a choice of orientation for $V$, but this is not the case.

In Figure $2.7, \star_{(k)}$ is applied to a basis $0-, 1-, 2-$, and 3 -vector and the resulting outer $k$-vectors are shown. Note that $\tilde{\star}_{(d-k)}{ }_{(k)} a_{(k)}=(-1)^{k(d-k)} a_{(k)}$, which implies that for $d=3, \tilde{\star}_{(d-k)}=\star_{(k)}^{-1}$. We sometimes omit the super- and subscripts on the operators. However, often we do denote it because the discretizations of $\star^{(k)}$ will result in different matrices for different $k$.

In Chapter 3 we will see that the discretization of the Hodge operator is a crucial element for the numerical method. The discrete Hodge operator contains the metrical properties of the mesh, i.e., the sizes and shapes of the mesh elements. As a result, different types of meshes, e.g. simplicial, Cartesian, or polytopal, demand different discrete Hodge operators.

### 2.1.4 Inner- and outer-oriented $k$-covectors

In the introduction to this chapter we explained that by taking a view to integration that better represents the symmetry between where we integrate (manifold) and what we integrate (differential form) leads to a reformulation of the fundamental calculus theorems, as the general Stokes theorem, that is a better starting point for a discretization that preserves the fundamental properties of the differential operators. In the preceding subsection we discussed $k$-vectors. They can be used to represent linear $k$-dimensional manifolds or to locally approximate general $k$-dimensional manifolds. We now continue by introducing the duality between $k$-vectors and exterior $k$-covectors that is the linear/local version of the relation that exists between manifolds and differential forms which we subsequently describe.

We discuss the linear case not only to be able to later on introduce the exterior calculus, but also because it will be used frequently to derive the discretization methods. The linear approximation will be the basis for interpolation there.

$+1$

$\left\{o_{(3)}\right\} e_{(1), x} \wedge e_{(1), y} \wedge e_{(1), z}\left\{o_{(3)}\right\} e_{(1), y} \wedge e_{(1), z}$


$$
e_{(1), x} \wedge e_{(1), y}
$$


$\left\{o_{(3)}\right\} e_{(1), z}$
$e_{(1), x} \wedge e_{(1), y} \wedge e_{(1), z}$

$\left\{o_{(3)}\right\}$

Figure 2.7: The action of $*$ and $\tilde{\not}$ are depicted. Here $o_{(3)}=e_{(1), x} \wedge e_{(1), y} \wedge e_{(1), z}$.

Definition 2.6. Let $V^{*}$ be the dual space to $V$, i.e., $V^{*}$ is the space of linear functionals on $V$. An element of $V^{*}$ is called a 1 -covector. Given a basis $\left\{e_{(1), 1}, \ldots, e_{(1), d}\right\}$ for $V$, the dual basis $\left\{e^{(1), 1}, \ldots, e^{(1), d}\right\}$ for $V^{*}$ is the basis defined by $e^{(1), i}\left(e_{(1), j}\right)=\delta_{j}^{i}$.

The fact that the dual basis is indeed a basis for $V^{*}$ follows from the linearity of the functionals. This shows that the dimension of $V^{*}$ is the same as the dimension of $V$.

A 1-covector applied to a vector produces a real number. However, (for finite dimensional $V$ ) the situation is more symmetric, because the dual space to $V^{*}$, i.e., $V^{* *}$ can be identified with $V$ again. We can equally well consider the elements of $V$ to be linear functionals on $V^{*}$. We therefore write $\left\langle a_{(1)}, b^{(1)}\right\rangle$ for the real number produced by a 1-covector and a 1 -vector.

Taking exterior products of vectors we formed $k$-vectors. Completely analogous we can introduce the exterior product of covectors and form $k$-covectors, we therefore omit the defintion. We denote by $\Lambda^{(k)} V$ the space of $k$-covectors. The space $\Lambda^{(1)} V$ is just $V^{*}$ and $\Lambda^{(0)} V=\Lambda_{(0)} V=\mathbb{R}$. If $\left\{e^{(1), 1}, \ldots, e^{(1), d}\right\}$ is a basis for $V^{*}$, then analogous to (2.1) and (2.2) we have that a basis for $\Lambda^{(k)} V$ is given by

$$
\left\{e^{(1), i_{1}} \wedge \cdots \wedge e^{(1), i_{k}} \mid \forall i_{1}, \ldots, i_{k}: 1 \leq i_{1}<\cdots<i_{k} \leq d\right\}
$$



Figure 2.8: On the left: a covector and a vector. The corresponding real number is about 2. On the right: 2 -covector and a 2 -vector. The corresponding real number is about -4 .
and a general $k$-covector can be written as

$$
a^{(k)}=\sum_{1 \leq i_{1}<\cdots i_{k} \leq d} a_{i_{1}, \ldots, i_{k}} e^{(1), i_{1}} \wedge \cdots \wedge e^{(1), i_{k}},
$$

where the $a_{i_{1}, \ldots, i_{k}}$ are real coefficients. The dimension of $\Lambda^{(k)} V$ is also equal to $\binom{d}{k}$.
Just like vectors, covectors have an orientation and a size. Furthermore, they can also be represented graphically. A 1-covector is represented by two parallel ( $d-1$ )dimensional planes. The orientation of the planes corresponds to the orientation of the 1 -covector and the distance between the planes corresponds to its "size". Note that the size of the 1 -covector increases as the planes are moved closer together. The real number corresponding to a pair of 1-covector and 1 -vector is given by the scalar needed to rescale the 1-covector such that the vector precisely fits between the two planes of the 1 -covector without changing the orientation of the vector (with exception of the sign of the orientation). See Figure 2.8 for an example.

Similarly, a 2 -covector can be represented by a cylinder. The area of the cross-section of the cylinder corresponds to the size of the 2-covector and the orientation of the cylinder corresponds to the orientation of the 2 -covector. Applying the 2 -covector to a 2 -vector gives a real number. The real number is given by a scalar by which one has to rescale the 2 -covector such that the 2 -vector exactly fits in the cylinder representing the 2 -covector without changing its orientation (with exception of the sign of the orientation). An example is again shown in Figure 2.8. More examples can be found in [33].

Outer $k$-covectors $\tilde{\Lambda}^{(k)} V$ can be defined analogously to outer $k$-vectors, as a product of an orientation and a $k$-covector, i.e. $\tilde{a}^{(k)} \in \tilde{\Lambda}^{(k)} V$ is given by $\tilde{a}^{(k)}=\left\{o_{(d)}\right\} a^{(k)}$, where $o_{(d)} \in \Lambda_{(d)} V$ and $a^{(k)} \in \Lambda^{(k)} V$. We denote the space of outer-oriented $k$-covectors by $\tilde{\Lambda}^{(k)} V$.

Just as on the space of $k$-vectors, we can define an inner product on the space of $k$-covectors.

Definition 2.7. On the spaces $\Lambda^{(k)} V$ we define the inner product $(\cdot, \cdot): \Lambda^{(k)} V \times \lambda^{(k)} V \rightarrow$ $\mathbb{R}$ by linear extension on simple $k$-covectors as

$$
\left(u^{(1), 1} \wedge \cdots \wedge u^{(1), k}, v^{(1), 1} \wedge \cdots \wedge v^{(1), k}\right)=\operatorname{det}\left(u^{(1), i}, v^{(1), j}\right),
$$

where $\left(u^{(1), i}, v^{(1), j}\right)$ is the inner product on $V^{*}$ induced by the inner product on $V$. Furthermore, we define the inner product $(\cdot, \cdot): \tilde{\Lambda}^{(k)} V \times \tilde{\Lambda}^{(k)} V \rightarrow \mathbb{R}$ by linear extension


Figure 2.9: On the left: an outer covector and an outer vector, producing approximately the real number 2 . On the right: an outer 2 -covector and outer 2 -vector, producing approximately the real number 4 .
on the simple $k$-covectors as

$$
\left(\left\{o^{(d)}\right\} u^{(1), 1} \wedge \cdots \wedge u^{(1), k},\left\{o^{(d)}\right\} v^{(1), 1} \wedge \cdots \wedge v^{(1), k}\right)=\operatorname{det}\left(u^{(1), i}, v^{(1), j}\right)
$$

The Hodge operators $\star^{(k)}: \Lambda^{(k)} V \rightarrow \tilde{\Lambda}^{(d-k)} V$ and $\tilde{\star}^{(k)}: \tilde{\Lambda}^{(k)} V \rightarrow \Lambda^{(d-k)} V$, which are defined completely analogous to (2.3) and (2.4).

Just as we can take the duality product between $k$-covectors and $k$-vectors, we can take the duality product between outer $k$-covectors and outer $k$-vectors. In Figure 2.9 we have shown an outer-oriented version of Figure 2.8.

### 2.1.5 The musical isomorphisms $b$ and $\sharp$

We saw that the Hodge operator defines a bijection between inner $k$-vectors and outer $(d-k)$-vectors. We now introduce a bijection between inner $k$-vectors and inner $k$ covectors and between outer $k$-vectors and outer $k$-covectors.

Definition 2.8. The musical isomorphisms, flat $b_{(k)}: \Lambda_{(k)} V \rightarrow \Lambda^{(k)} V$ and sharp $\sharp^{(k)}$ : $\Lambda^{(k)} V \rightarrow \Lambda_{(k)} V$ are defined by

$$
b_{(k)} a_{(k)}=\left(\cdot, a_{(k)}\right) \quad \text { and } \quad \not \forall^{(k)} a^{(k)}=\left(\cdot, a^{(k)}\right),
$$

for all $a_{(k)} \in \Lambda_{(k)} V$ and $a^{(k)} \in \Lambda^{(k)} V$, respectively. When the inner product on $V$ and $V^{\star}$ is just the usual Euclidean inner product, we have in terms of coefficients

$$
\begin{aligned}
& b(k)\left(\sum_{i_{1}<\cdots<i_{k}} a^{i_{1}, \ldots, i_{k}} e_{(1), i_{1}} \wedge \cdots \wedge e_{(1), i_{k}}\right)=\sum_{i_{1}<\cdots<i_{k}} a_{i_{1}, \ldots, i_{k}} e^{(1), i_{1}} \wedge \cdots \wedge e^{(1), i_{k}}, \\
& \sharp^{(k)}\left(\sum_{i_{1}<\cdots<i_{k}} b_{i_{1}, \ldots, i_{k}} e^{(1), i_{1}} \wedge \cdots \wedge e^{(1), i_{k}}\right)=\sum_{i_{1}<\cdots<i_{k}} b^{i_{1}, \ldots, i_{k}} e_{(1), i_{1}} \wedge \cdots \wedge e_{(1), i_{k}},
\end{aligned}
$$

where $a^{i_{1}, \ldots, i_{k}}=a_{i_{1}, \ldots, i_{k}}$ and $b_{i_{1}, \ldots, i_{k}}=b^{i_{1}, \ldots, i_{k}}$. We define the musical isomorphisms, flat $b_{(k)}: \tilde{\Lambda}_{(k)} V \rightarrow \tilde{\Lambda}^{(k)} V$ and sharp $\sharp^{(k)}: \tilde{\Lambda}^{(k)} V \rightarrow \tilde{\Lambda}_{(k)} V$ analogously.


Figure 2.10: The bijections: Hodge operators and musical isomorphisms.

### 2.1.6 The volume covector and Hodge duality

For a vector space with an orientation and an inner product we can define a unit volume.
Definition 2.9. Let $V$ be an oriented $d$-dimensional vector space with inner product $(\cdot, \cdot): V \times V \rightarrow \mathbb{R}$. Furthermore, let $e_{(1), 1}, \ldots, e_{(1), d}$ be an orthonormal basis for $V$, such that $\left\{e_{(1), 1} \wedge \cdots \wedge e_{(1), d}\right\}$ is the orientation of $V$ and let $e^{(1), 1}, \ldots, e^{(1), d}$ be the corresponding dual basis. The volume covector on $V$ is the outer $d$-covector $\tilde{\mu}^{(d)} \in \tilde{\Lambda}^{(d)} V$ given by

$$
\tilde{\mu}^{(d)}=\left\{e_{(1), 1} \wedge \cdots \wedge e_{(1), d}\right\} e^{(1), 1} \wedge \cdots \wedge e^{(1), d} .
$$

The volume covector is an outer-oriented covector, because it should not change sign if the choice for orientation of $V$ is changed. This property of outer-oriented vectors was exemplified in Figure 2.5 for a 2 -vector.

We can use the volume covector to give a second equivalent definition of the Hodge star operator. To do this, we need the inner product to the spaces of $k$-covectors $\Lambda^{(k)} V$ and $\tilde{\Lambda}^{(k)} V$. It is defined completely analogous

Definition 2.10. Analogously, we can define the inner product on the spaces $\Lambda^{(k)} V$ and $\tilde{\Lambda}^{(k)} V$.

In equations (2.3) and (2.4) we defined the Hodge star operator by describing its action on a basis. It is also possible to define it without explicit mentioning of a basis.
Definition 2.11. The Hodge star operator $\star^{(k)}: \Lambda^{(k)} V \rightarrow \tilde{\Lambda}^{(d-k)} V$ is the linear operator defined by the relation

$$
a^{(k)} \wedge \star^{(k)} b^{(k)}=\left(a^{(k)}, b^{(k)}\right) \tilde{\mu}^{(d)},
$$

where $a^{(k)}, b^{(k)} \in \Lambda^{(k)} V$ and $\tilde{\mu}^{(d)}$ is the volume covector on $V$.
Similarly, the dual Hodge star operator $\tilde{\star}^{(k)}: \tilde{\Lambda}^{(k)} V \rightarrow \Lambda^{(d-k)} V$ is the linear operator defined by the relation

$$
\tilde{a}^{(k)} \wedge \tilde{\star}^{(k)} \tilde{b}^{(k)}=\left(\tilde{a}^{(k)}, \tilde{b}^{(k)}\right) \tilde{\mu}^{(d)},
$$

where $\tilde{a}^{(k)}, \tilde{b}^{(k)} \in \tilde{\Lambda}^{(k)} V$.
The Hodge star operator is defined analogously on the space of inner and outer $k$ vectors.

It can be verified that the above definition and the earlier definition coincide by taking basis $k$-covectors in above definition.

Often the Hodge star operator also incorporates material properties. An example is given by the relation between the electric field and the electric displacement field. In an anisotropic medium they are related by the permittivity tensor. In the language of exterior calculus the electric field and displacement field are (at a given point in space) represented by an inner 1 -covector $e^{(1)}$ and outer 2 -covector $\tilde{d}^{(2)}$, respectively. They are related by a weighted Hodge operator: $\tilde{d}^{(2)}=\star_{\epsilon}^{(1)} e^{(1)}$. The operator $\star_{\epsilon}^{(1)}$ is defined analogous to $\star^{(1)}$ only using an inner product on $V$ weighted by the permittivity tensor. In general it is defined as follows.

Definition 2.12. Let $\chi: V \rightarrow V^{*}$ be a symmetric positive definite tensor. The $\chi$ weighted Hodge star operator $\star_{\chi}^{(k)}: \Lambda^{(k)} V \rightarrow \tilde{\Lambda}^{(d-k)} V$ is the linear operator defined by the relation

$$
a^{(k)} \wedge \star_{\chi}^{(k)} b^{(k)}=\left(a^{(k)}, b^{(k)}\right)_{\chi} \tilde{\mu}^{(d)}
$$

where $a^{(k)}, b^{(k)} \in \Lambda^{(k)} V$ and $\tilde{\mu}^{(d)}$ is the volume covector on $V$. Furthermore, $(\cdot, \cdot)_{\chi}$ : $\Lambda_{(k)} V \times \Lambda_{(k)} V \rightarrow \mathbb{R}$ is the $\chi$-weighted inner product.

The $\chi$-weighted dual Hodge star operator $\tilde{\star}_{\chi}^{(k)}: \tilde{\Lambda}^{(k)} V \rightarrow \Lambda^{(d-k)} V$ is defined analogously.

### 2.2 Exterior calculus: manifolds and differential forms

We come now to the exterior calculus. The fundamental objects in exterior calculus are differential forms, they replace the scalar and vector fields as the objects to be differentiated and integrated. Differential forms live on manifolds. Manifolds are the objects over which the differential forms can be integrated and on which they can be differentiated. A manifold is smooth enough to allow for this

The manifolds that we are interested in are subsets of $\mathbb{R}^{d}$. Examples of these are paths, surfaces and volumes. These may be, respectively, the edges, faces and cells of our mesh. Moreover, the domain of our flow problem is an example of a manifold.

In this section we first introduce manifolds and subsequently differential forms and operations on them.

### 2.2.1 Manifolds

Manifolds are locally like Euclidean space. To express this in a mathematical exact way we use diffeomorphisms. A smooth map $\phi: X \rightarrow Y$ between two subsets $X$ and $Y$ of $\mathbb{R}^{d}$ is a diffeomorphism if it is a bijection and if the inverse map $\phi^{-1}: Y \rightarrow X$ is also smooth. Two subspaces are called diffeomorphic if a diffeomorphism between them exists.

Definition 2.13. $[34,35]$ A subset $M_{(k)}$ of $\mathbb{R}^{d}$ is called a $k$-dimensional manifold (in $\mathbb{R}^{d}$ ) if it is locally diffeomorphic to $\mathbb{R}^{k}$ : each point $x$ possesses a neighborhood $V \subset M_{(k)}$ which is diffeomorphic to an open set $U$ of $\mathbb{R}^{k}$. A diffeomorphism $\phi: U \rightarrow V$ is called a parametrization of the neighborhood $V$. The inverse diffeomorphism $\phi^{-1}: V \rightarrow U$ is
called a coordinate system on $V$. If a manifold $N_{(l)}$ is a subset of another manifold $M_{(m)}$ then a $N_{(l)}$ is called a submanifold of $M_{(k)}$.

Typical examples of 2-dimensional manifolds are the sphere and the torus. The manifolds according to above definition have no boundary. However, most of the objects important to us, i.e., mesh cells, faces and edges, clearly do have a boundary. In these boundary points they are not locally diffeomorphic to $\mathbb{R}^{d}$, but they are locally diffeomorphic to the half-space $\mathbb{R}_{x^{k} \geq 0}^{k}:=\left\{\left(x^{1}, \ldots, x^{k}\right) \in \mathbb{R}^{k} \mid x^{k} \geq 0\right\}$, i.e., points in $\mathbb{R}^{k}$ with nonnegative final coordinate. The boundary of the set $\mathbb{R}_{x^{k} \geq 0}^{k}$ is $\mathbb{R}^{k-1} \cong\left\{\left(x^{1}, \ldots, x^{k}\right) \in \mathbb{R}^{k} \mid x^{k}=0\right\}$.

Definition 2.14. $[34,35]$ A subset $M_{(k)}$ of $\mathbb{R}^{d}$ is called a $k$-dimensional manifold with boundary (in $\mathbb{R}^{d}$ ) if it is locally diffeomorphic to $\mathbb{R}_{x^{k} \geq 0}^{k}$. The boundary of $M_{(k)}$, denoted by $\partial M_{(k)}$, consists of those points that belong to the image of the boundary of $\mathbb{R}_{x^{k} \geq 0}^{k}$ under some local parametrization. ${ }^{3}$ The interior of $M_{(k)}$ is the complement $M_{(k)}-\partial M_{(k)}$.

A manifold as defined in Definition 2.13 is also a manifold with boundary, only the boundary is simply empty. Such a manifold we call a manifold without boundary.

Suppose $M_{(k)}$ is a manifold with boundary. For a point $x \in \partial M_{(k)}$ there exists a neighborhood $U$ and diffeormorphism $\phi: U \rightarrow V \subset \mathbb{R}_{x^{k} \geq 0}^{k}$. The restriction of $\phi$ to $U \cap \partial M_{(k)}$ is a diffeomorphism between a neighborhood $U \cap \partial M_{(k)}$ of $x$ in $\partial M_{(k)}$ and $\mathbb{R}^{k-1}$. As a result, we see that $\partial M_{(k)}$ is locally diffeomorphic to $\mathbb{R}^{k-1}$, hence the boundary of a $k$-dimensional manifold with boundary is a $(k-1)$-dimensional manifold without boundary.

Many of the objects that we need to consider are also not captured by Definition 2.14. For example, think of cubical or simplicial mesh elements. These objects have a boundary that is definitely not smooth, because it contains corners. To allow for such mesh elements also, we once more widen the definition.

Definition 2.15. [36] A subset $M_{(k)}$ of $\mathbb{R}^{d}$ is called a $k$-dimensional manifold with corners (in $\left.\mathbb{R}^{d}\right)$ if it is locally diffeomorphic to $\mathbb{R}_{\geq 0}^{k}:=\left\{\left(x^{1}, \ldots, x^{k}\right) \in \mathbb{R}^{k} \mid x^{1} \geq 0, \ldots, x^{k} \geq\right.$ $0\}$.

The set of points in $\mathbb{R}_{\geq 0}^{k}$ where at least one coordinate vanishes is the boundary of $\mathbb{R}_{\geq 0}^{k}$. The boundary $\partial M_{(k)}$ of a manifold with corners $M_{(k)}$ is again defined as those points of the manifold that belong to the image of the boundary of $\mathbb{R}_{\geq 0}^{k}$ under some local parametrization. We call the points of $\mathbb{R}_{\geq 0}^{k}$ where more than one coordinate vanishes the corner points of $\mathbb{R}_{\geq 0}^{k}$. We define the corner points of $M_{(k)}$ to be the points of the manifold that belong to the image of the corner points of $\mathbb{R}_{\geq 0}^{k}$ under some local parametrization. This is again independent of the parametrization [36, Proposition 16.20]. A manifold with corners is a manifold with boundary if and only if it has no corner points.

Above we saw that the boundary $\partial M_{(k)}$ of a manifold with boundary $M_{(k)}$ is a manifold again. Contrastingly, the boundary of a manifold with corners is not a manifold with corners.

From now on, if we speak of a manifold we mean a manifold with corners, unless specified alternatively.

[^4]

Figure 2.11: From left to right: a 2-dimensional manifold without boundary, a 2-dimensional manifold with boundary and a 2-dimensional manifold with corners.

### 2.2.2 Differential forms

In Section 2.1 we discussed the exterior algebra of $k$-forms. Now, we want to consider fields of these, i.e., an assignment of a $k$-form to every point of the manifold. Such an assignment is called a differential $k$-form. The differential $k$-forms play the role that scalar and vector fields play in vector calculus. For example, the velocity of a fluid can be represented by a differential 1 -form.

To define differential $k$-forms we need the space of exterior $k$-forms $\Lambda^{(k)} V$ at each point of $x \in M_{(l)}(l \geq k)$. The space $V$ in this context is provided by the tangent space to $M_{(l)}$ at $x$. While introducing the tangent space and related concepts we base us on [34], [35] and [37].

Before introducing the tangent space to a general manifold we first reconsider the Euclidean case $\mathbb{R}^{3}$. A vector field can be thought of as an assignment of vector $\underline{v}_{x} \in \mathbb{R}^{3}$ to every point $x$ in $\mathbb{R}^{3}$. Graphically, the vector $\underline{v}_{x}$ at $x$ is represented by an arrow starting at $x \in \mathbb{R}^{3}$ and ending at $x+\underline{v}_{x} \in \mathbb{R}^{3}$. At $x$ there is a copy of $\mathbb{R}^{3}$ in which $\underline{v}_{x}$ lives. We call this copy the tangent space to $\mathbb{R}^{3}$ at $x$ and denote it by $T_{x} \mathbb{R}^{3}$. The vector $\underline{v}_{x} \in T_{x} \mathbb{R}^{3}$ we call a tangent vector to $\mathbb{R}^{3}$ at $x$.

If we have a concept of tangent space $T_{x} M_{(k)}$ to a point $x$ in a general manifold $M_{(k)}$, then we can define a vector field on $M_{(k)}$ as a choice $\underline{v}_{x} \in T_{x} M_{(k)}$ for every $x \in M_{(k)}$. To be able to define the tangent space to a point in a manifold, we first consider the derivative of a function between two Euclidean manifolds, $\mathbb{R}^{n}$ and $\mathbb{R}^{m}$.

For a differentiable function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ the derivative $d f_{x}$ at a point $x \in \mathbb{R}^{n}$ is a linear map from $\mathbb{R}^{n}$ to $\mathbb{R}^{m}$. We can use it to define a linear map from $T_{x} \mathbb{R}^{n}$ to $T_{f(x)} \mathbb{R}^{m}$ according to

$$
\underline{v}_{x} \mapsto\left[d f_{x} \underline{v}\right]_{f(x)},
$$

for every $x$ in $\mathbb{R}^{n}$.
Suppose instead that we have a local parametrization $\phi: U \rightarrow V \subset M_{(k)} \subset \mathbb{R}^{d}$ around $x \in V$, where $U \subset \mathbb{R}^{k}$ and $\phi(a)=x$ for $a \in U$. The derivative $d \phi_{a}$ defines a map from $T_{a} \mathbb{R}^{k}$ to $\mathbb{R}^{d}$ and has, by definition, rank $k$. As such it defines a $k$-dimensional subspace of $\mathbb{R}^{d}$. We define this subspace to be the tangent space to $x$ at $M_{(k)}$.


Figure 2.12: On the left we see a set $U \subset \mathbb{R}^{2}$ centered around a point $a \in U$ (black dot). In light shaded gray the tangent space at $a$ is depicted, which is tangent to $\mathbb{R}^{2}$. On the right we see how this set is mapped onto the torus $M_{(2)}$ by the $\phi$. This is a local parametrization around $x=\phi(a)$ (black dot). The $\operatorname{map} d \phi_{a}$ on $T_{a} \mathbb{R}^{2}$ produces the tangent space $T_{x} M_{(2)}$, which is the plane in $\mathbb{R}^{3}$ that is tangent to $M_{(2)}$ at $x$.

Definition 2.16. Let $M_{(k)}$ be a manifold in $\mathbb{R}^{d}$ and let $\phi: U \rightarrow V \subset M_{(k)} \subset \mathbb{R}^{d}$ be a local parametrization around $x \in V$, where $U \subset \mathbb{R}^{k}$ is an open set and $\phi(a)=x$ with $a \in U$. The tangent space to $M_{(k)}$ at $x$ is defined to be the image of $d \phi_{a}: \mathbb{R}^{k} \rightarrow \mathbb{R}^{d}$. We denote the tangent space by $T_{x} M_{(k)}$.

This definition is independent of the parametrization chosen. Suppose that $\psi: U^{\prime} \rightarrow$ $V^{\prime} \subset M_{(k)}$ is another parametrization around $x$ and $\psi(b)=x$. Assume that we restricted $U$ and $U^{\prime}$ such that $\phi(U)=\psi\left(U^{\prime}\right)$. The map $h:=\psi^{-1} \circ \phi: U \rightarrow U^{\prime}$ is a diffeomorphism. By the chain rule we have $d \phi_{a}=d \psi_{b} \circ d h_{a}$, hence the image of $d \phi_{a}$ is contained in that of $d \psi_{b}$. By interchanging the roles of $\phi$ and $\psi$ we find that also the image of $d \psi_{b}$ is contained in that of $d \phi_{a}$. So every local parametrization around $x$ produces the same tangent space.

It can be easily checked that the tangent space $T_{x} M_{(k)}$ also has dimension $k$ [34]. Furthermore, if we let $\phi$ be as in Definition 2.16 and if $\left\{\underline{e}_{(1), 1}, \ldots, \underline{e}_{(1), k}\right\}$ is a basis for $T_{a} \mathbb{R}^{k}$ then $\left\{d \phi_{a}\left(\underline{e}_{(1), 1}\right), \ldots, d \phi_{a}\left(\underline{e}_{(1), k}\right)\right\}$ is a basis for $T_{x} M_{(k)}$.

In general, given a smooth map between two manifolds ${ }^{4} \phi: M_{(m)} \rightarrow N_{(n)}$, the function $d \phi_{a}: T_{a} M_{(m)} \rightarrow T_{\phi(a)} N_{(n)}$ is called the pushforward and often alternatively denoted by $\phi_{*}$. We can extend the pushforward to $k$-vectors as a function $\phi_{*}: \Lambda_{(k)}\left(T_{a} M_{(m)}\right) \rightarrow$ $\Lambda_{(k)}\left(T_{\phi(a)} N_{(n)}\right)$ by demanding it to be compatible with the exterior product, i.e., $\phi_{*}\left(a_{(k)} \wedge\right.$ $\left.b_{(l)}\right)=\phi_{*}\left(a_{(k)}\right) \wedge \phi_{*}\left(b_{(l)}\right)$.

[^5]Suppose we have a coordinate system on $U \subset M_{(m)}$ given by $x^{1}, \ldots, x^{m}$ with corresponding basis $e_{(1), 1}, \ldots, e_{(1), m}$ and a coordinate system on $V \subset N_{(n)}$ given by $y^{1}, \ldots, y^{n}$ and corresponding basis $f_{(1), 1}, \ldots, f_{(1), n}$. A vector in $u_{(1)} \in \Lambda_{(1)}\left(T_{a} M_{(m)}\right)$ can be written in this coordinate system as $u_{(1)}=\sum_{i} u^{i} e_{(1), i}$. The pushforward $\phi_{*} u_{(1)}$ is then given by

$$
\phi_{*} u_{(1)}=\sum_{j} \sum_{i} u^{i}\left(\frac{\partial y^{j}}{\partial x^{i}}\right) f_{(1), j},
$$

where the $y^{j}$ are a function of the $x^{i}$ via $\phi$. The component vector of $u_{(1)}$ transforms with the Jacobian matrix of $\phi$ into the component vector of $\phi_{*} u_{(1)}$.

Dual to the pushforward of a $k$-vector is the pullback of a $k$-form. For a differential $k$-form $u^{(k)} \in \Lambda^{(k)}\left(T_{\phi(a)} N_{(n)}\right)$ the pullback $\phi^{*}: \Lambda_{(k)}\left(T_{\phi(a)} N_{(n)}\right) \rightarrow \Lambda_{(k)}\left(T_{a} M_{(m)}\right)$ is defined by

$$
\left\langle v_{(k)}(a),\left(\phi^{*} u^{(k)}\right)(a)\right\rangle:=\left\langle\phi_{*} v_{(k)}(a), u^{(k)}(\phi(a))\right\rangle, \quad \forall v_{(k)}(a) \in \Lambda_{(k)}\left(T_{\phi(a)} M_{(n)}\right),
$$

where $\phi_{*}: \Lambda_{(k)}\left(T_{a} M_{(m)}\right) \rightarrow \Lambda_{(k)}\left(T_{\phi(a)} N_{(n)}\right)$ is the pushforward.
Definition 2.17. Let $M_{(m)}$ be a manifold and $0 \leq k \leq m$. A differential $k$-form on $M_{(m)}$ is an assignment of a $k$-form in $\Lambda^{(k)}\left(T_{x} M_{(m)}\right)$ for each $x \in M_{(m)}$. We say that a differential $k$-form $a^{(k)}$ on $M_{(m)}$ is smooth if, when expressed in any local coordinates,

$$
a^{(k)}=\sum_{i_{1}<\cdots<i_{m}} a_{i_{1}, \ldots, i_{m}} e^{(1), i_{1}} \wedge \cdots \wedge e^{(1), i_{m}}
$$

all the coordinate functions $a_{i_{1}, \ldots, i_{m}}$ are smooth. We denote the space of smooth differential $k$-forms on $M_{(m)}$ by $\Lambda^{(k)}\left(M_{(m)}\right)$. We define the space of smooth $k$-vector fields $\Lambda_{(k)}\left(M_{m}\right)$ analogously. Note that $\Lambda^{(0)}\left(M_{(m)}\right)=\Lambda_{(0)}\left(M_{(m)}\right)=C^{\infty}\left(M_{(m)}\right)$, i.e., the space of smooth functions on $M_{(m)}$.

While the pushforward of a vector field does not produce a vector field again when the map between the manifolds is not bijective, the pullback of a differential form always results in a differential form again. Thus, when given a function $\phi: M_{(m)} \rightarrow N_{(n)}$, the pullback $\phi^{*}$ is truly a map from $\Lambda^{(k)}\left(N_{(n)}\right)$ to $\Lambda^{(k)}\left(M_{(m)}\right)$. This makes differential $k$-forms more suitable objects for calculus then $k$-vector fields.

The pullback has the following properties.
Proposition 2.1. [38] Let $L_{(l)}, M_{(m)}$ and $N_{(n)}$ be manifolds and let $\phi: L_{(l)} \rightarrow M_{(m)}$ and $\psi: M_{(m)} \rightarrow N_{(n)}$ be diffeomorphisms. The pullbacks are linear maps
(i) $\phi^{*}: \Lambda^{(k)}\left(M_{(m)}\right) \rightarrow \Lambda^{(k)}\left(L_{(l)}\right)$ and $\psi^{*}: \Lambda^{(k)}\left(N_{(n)}\right) \rightarrow \Lambda^{(k)}\left(M_{(m)}\right)$,
(ii) $(\psi \circ \phi)^{*}=\phi^{*} \circ \psi^{*}$,
(iii) $\phi^{*}\left(a^{(k)} \wedge b^{(p)}\right)=\phi^{*}\left(a^{(k)}\right) \wedge \phi^{*}\left(b^{(p)}\right), \quad \forall a^{(k)} \in \Lambda^{(k)}(\Omega), b^{(p)} \in \Lambda^{(p)}(\Omega)$.

The pullback of a $k$-form to a submanifold will be used frequently and therefore we have a name for it.

Definition 2.18. Let $N_{(n)}$ be a submanifold of $M_{(m)}$. We define the inclusion map to be the map $i_{N_{(n)}}: N_{(n)} \rightarrow M_{(m)}$ with $i(x)=x$ for all $x \in N_{(n)}$. The pullback $i_{N_{(n)}}^{*}$ of the inclusion map defines a map from $\Lambda^{(k)}\left(M_{(m)}\right)$ to $\Lambda^{(k)}\left(N_{(n)}\right)$. We denote this map by $t_{N_{(n)}}^{(k)}$ and call it the trace on $N_{(n)}$.

An example of a much used trace is the trace on the boundary $t_{\partial \Omega_{(d)}}^{(k)}: \Lambda^{(k)}\left(\Omega_{(d)}\right) \rightarrow$ $\Lambda^{(k)}\left(\partial \Omega_{(d)}\right)$ for a manifold with smooth boundary. It can for example be used to set boundary conditions for partial differential equations.

We can define an inner product on a manifold $M_{(m)}$ by defining an inner product to each of the tangent spaces in a smooth way. Note that an inner product on a tangent space $T_{x} M_{(m)}$ is a map $(\cdot, \cdot): T_{x} M_{(m)} \times T_{x} M_{(m)} \rightarrow \mathbb{R}$. However, it is symmetric instead of antisymmetric and therefore not a 2 -form, but a symmetric covariant 2-tensor [38].

We only deal with manifolds in $\mathbb{R}^{d}$ and therefore the tangent space is always a subset of $\mathbb{R}^{d}$. We will always use as inner product the usual Euclidean inner product induced by the Euclidean inner product of $\mathbb{R}^{d}$.

Using the inner product on $M_{(m)}$ we can define the musical isomorphisms flat, $b_{(k)}$ : $\Lambda_{(k)}\left(M_{(m)}\right) \rightarrow \Lambda^{(k)}\left(M_{(m)}\right)$, and sharp, $\sharp^{(k)}: \Lambda^{(k)}\left(M_{(m)}\right) \rightarrow \Lambda_{(k)}\left(M_{(m)}\right)$, on $M_{(m)}$ by

$$
b_{(k)} a_{(k)}=\left(\cdot, a_{(k)}\right) \quad \text { and } \quad \sharp^{(k)} a^{(k)}=\left(\cdot, a^{(k)}\right) .
$$

### 2.2.3 The exterior derivative

The main attractive feature of exterior calculus is that differentiation and integration are defined in a uniform manner for all dimensions. The derivative operators from vector calculus, the gradient, curl and divergence, are replaced by a single derivative operator: the exterior derivative.

Definition 2.19. Let $M_{(m)}$ be an $m$-dimensional manifold in $\mathbb{R}^{d}$. The exterior derivative $d^{(k)}: \Lambda^{(k)}\left(M_{(m)}\right) \rightarrow \Lambda^{(k+1)}\left(M_{(m)}\right)$ is the linear operator that, in local coordinates $x^{1}, \ldots, x^{m}$ and corresponding dual basis $e^{(1), 1}, \ldots, e^{(1), m}$, is defined as follows. For a 0 -form $f \in \Lambda^{(0)}\left(M_{(m)}\right), d f$ is the 1 -form given by

$$
d^{(0)} f=\sum_{i=1}^{m}\left(\partial_{i} f\right) e^{(1), i}
$$

and for a monomial $k$-form $a^{(k)}=a e^{(1), i_{1}} \wedge \cdots \wedge e^{(1), i_{m}}$ the exterior derivative is given by

$$
d^{(k)} a^{(k)}=d^{(0)} a \wedge \wedge^{(1), i_{1}} \wedge \cdots \wedge e^{(1), i_{m}} .
$$

This together with the linearity completely determine $d^{(k)}$. Sometimes we omit the superscript and simply write $d$.

Note that the exterior product of an $m$-form in $\Lambda^{(m)}\left(M_{(m)}\right)$ involves the wedge product of $m+11$-forms and as a result is necessarily 0 .

If we apply the exterior derivative to the coordinate 0 -form $x^{i}$ we find

$$
d x^{i}=\sum_{j=i}^{m}\left(\partial_{j} x^{i}\right) e^{(1), j}=\sum_{j=i}^{m} \delta_{j}^{i} e^{(1), j}=e^{(1), i} .
$$

Therefore, we can write $d x^{i}$ for $e^{(1), i}$. This is actually more common and we will do so from now on.

That this notation is also more intuitive becomes apparent when we consider the pullback in terms of local coordinates. Suppose we have a smooth map $\phi: M_{(m)} \rightarrow N_{(n)}$ and local coordinates $x^{1}, \ldots, x^{m}$ and $y^{1}, \ldots, y^{n}$ on $M_{(m)}$ and $N_{(n)}$ respectively. The pullback of a 1 -form $a^{(1)} \in \Lambda^{(1)}\left(N_{(n)}\right)$ by $\phi$ is given by

$$
\phi^{*} a^{(1)}=\phi^{*}\left(\sum_{i=1}^{n} a_{i}\left(y^{1}, \ldots, y^{n}\right) d y^{i}\right)=\sum_{i=1}^{n} a_{i}\left(\phi\left(x^{1}, \ldots, x^{m}\right)\right) \sum_{j=1}^{m}\left(\frac{\partial y^{i}}{\partial x^{j}}\right) d x^{j} .
$$

Thus the pullback is just given by applying the chain rule to $y^{i} \circ \phi$. This generalizes to $k$-forms.

The exterior derivative replaces the fundamental differential operators from vector calculus, the gradient, curl and divergence. To see this we consider the exterior derivative on $\mathbb{R}^{3}$ using Cartesian coordinates. Taking the exterior derivative of a scalar function $f^{(0)} \in \Lambda^{(0)}\left(\mathbb{R}^{3}\right)$ we obtain

$$
d^{(0)} f^{(0)}=\frac{\partial f}{\partial x^{1}} d x^{1}+\frac{\partial f}{\partial x^{2}} d x^{2}+\frac{\partial f}{\partial x^{3}} d x^{3}
$$

The vector corresponding to this differential by the inner product is the usual gradient of $f^{(0)}$, given by $\nabla f^{(0)}=\sharp d f^{(0)}$.

Suppose now that we have a vector field $u_{(1)} \in \Lambda_{(1)}\left(\mathbb{R}^{3}\right)$ and let the corresponding 1 -form be given by $u^{(1)}=b u_{(1)}=u_{1} d x^{1}+u_{2} d x^{2}+u_{3} d x^{3}$. Applying the exterior derivative gives

$$
\begin{aligned}
d^{(1)} u^{(1)} & =\sum_{i=1}^{3} \sum_{j=1}^{3} \frac{\partial u_{j}}{\partial x^{i}} d x^{i} \wedge d x^{j} \\
& =\left(\frac{\partial u_{3}}{\partial x^{2}}-\frac{\partial u_{2}}{\partial x^{3}}\right) d x^{2} \wedge d x^{3}+\left(\frac{\partial u_{1}}{\partial x^{3}}-\frac{\partial u_{3}}{\partial x^{1}}\right) d x^{3} \wedge d x^{1}+\left(\frac{\partial u_{2}}{\partial x^{1}}-\frac{\partial u_{1}}{\partial x^{2}}\right) d x^{1} \wedge d x^{2} .
\end{aligned}
$$

This 2 -form can be related to the vector $\nabla \times u_{(1)}$ by the Hodge map (which we soon define for differential forms) and the sharp operator.

Instead, relating the vector field $u_{(1)}$ to a 2 -form by the flat operator and the Hodge duality we obtain the 2 -form $u^{(2)}=u_{1} d x^{2} \wedge d x^{3}+u_{2} d x^{3} \wedge d x^{1} u_{3} d x^{1} \wedge d x^{2}$. If we apply the exterior derivative to this 2 -form we find

$$
\begin{aligned}
d^{(2)} u^{(2)} & =\frac{\partial u_{1}}{\partial x^{1}} d x^{1} \wedge d x^{2} \wedge d x^{3}+\frac{\partial u_{2}}{\partial x^{2}} d x^{2} \wedge d x^{3} \wedge d x^{1}+\frac{\partial u_{3}}{\partial x^{3}} d x^{3} \wedge d x^{1} \wedge d x^{2} \\
& =\left(\frac{\partial u_{1}}{\partial x^{1}}+\frac{\partial u_{2}}{\partial x^{2}}+\frac{\partial u_{3}}{\partial x^{3}}\right) d x^{1} \wedge d x^{2} \wedge d x^{3} .
\end{aligned}
$$

Thus, we see that the $d^{(0)}, d^{(1)}$ and $d^{(2)}$ are closely related to the gradient, curl and divergence operators, respectively.

However, there is an important difference as well. To exactly write out the vector calculus operators we need besides the exterior derivative also the Hodge isomorphism and the musical isomorphisms. In contrast to the exterior derivative, these depend on the metric of our space. Changing the inner product will change the vector calculus operators, but the exterior derivative remains unchanged: the exterior derivative is a metric independent operator.
Theorem 2.1. The exterior derivative has the following properties [35, 38].
(i) Leibniz rule: $d\left(a^{(k)} \wedge b^{(l)}\right)=d a^{(k)} \wedge b^{(l)}+(-1)^{k} a^{(k)} \wedge d b^{(l)}$.
(ii) $d\left(d a^{(k)}\right)=0$, for all differential forms, i.e. $d \circ d \equiv 0$.
(iii) For smooth $\phi: M_{(m)} \rightarrow N_{(n)}$ and $a^{(k)} \in \Lambda^{(k)}\left(N_{(n)}\right)$ we have $\phi^{*}\left(d a^{(k)}\right)=d\left(\phi^{*} a^{(k)}\right)$.

The first property in above theorem produces familiar product rules known from vector calculus. For $k=0$ and $l=0$ it corresponds to the product rule for the gradient: $\nabla(a b)=a \nabla b+b \nabla a$. Next, consider $k=0$ and $l=1$, with $a^{(0)}=a$ and $b^{(1)}=b_{1} d x^{1}+b_{2} d x^{2}+$ $b_{3} d x^{3}$. The left-hand side of $(i)$ gives

$$
\begin{aligned}
d\left(a^{(0} \wedge b^{(1)}\right)= & \left(\frac{\partial a b_{3}}{\partial x^{2}}-\frac{\partial a b_{2}}{\partial x^{3}}\right) d x^{2} \wedge d x^{3}+\left(\frac{\partial a b_{1}}{\partial x^{3}}-\frac{\partial a b_{3}}{\partial x^{1}}\right) d x^{3} \wedge d x^{1} \\
& +\left(\frac{\partial a b_{2}}{\partial x^{1}}-\frac{\partial a b_{1}}{\partial x^{2}}\right) d x^{1} \wedge d x^{2},
\end{aligned}
$$

and we see that this corresponds to $\nabla \times\left(a b_{(1)}\right)$, where $b_{(1)}=\sharp b^{(1)}$. For the right-hand side of $(i)$ we get

$$
d a^{(0)} \wedge b^{(1)}=\left(\frac{\partial a}{\partial x^{1}} d x^{1}+\frac{\partial a}{\partial x^{2}} d x^{2}+\frac{\partial a}{\partial x^{3}} d x^{3}\right) \wedge\left(b_{1} d x^{1}+b_{2} d x^{2}+b_{3} d x^{3}\right),
$$

and

$$
\begin{aligned}
a^{(0)} \wedge d b^{(1)}= & a\left(\frac{\partial b_{3}}{\partial x^{2}}-\frac{\partial b_{2}}{\partial x^{3}}\right) d x^{2} \wedge d x^{3}+a\left(\frac{\partial b_{1}}{\partial x^{3}}-\frac{\partial b_{3}}{\partial x^{1}}\right) d x^{3} \wedge d x^{1} \\
& +a\left(\frac{\partial b_{2}}{\partial x^{1}}-\frac{\partial b_{1}}{\partial x^{2}}\right) d x^{1} \wedge d x^{2},
\end{aligned}
$$

which correspond to $\nabla a \times b_{(1)}$ and $a \nabla \times b_{(1)}$, respectively. So, we see that $(i)$, for $k=0$ and $l=1$ corresponds to the vector calculus identity $\nabla \times\left(a b_{(1)}\right)=\nabla a \times b_{(1)}+a \nabla \times b_{(1)}$.

Similarly, it can be shown that we find the vector identities

$$
\begin{aligned}
\nabla \cdot\left(a_{(1)} \times b_{(1)}\right) & =\left(\nabla \times a_{(1)}\right) \cdot b_{(1)}-a_{(1)} \cdot\left(\nabla \times b_{(1)}\right), & & \text { for } k=1 \text { and } l=1, \\
\nabla \cdot\left(a b_{(1)}\right) & =\nabla a \cdot b_{(1)}+a \nabla \cdot b_{(1)}, & & \text { for } k=0 \text { and } l=2 .
\end{aligned}
$$

Property (ii) in Theorem 2.1 corresponds to the identities $\nabla \times \nabla=0$ and $\nabla \cdot \nabla \equiv 0$.
Corollary 2.1. Let $N_{(n)}$ be a submanifold of $M_{(m)}$ and let $t^{(k)}: \Lambda^{(k)}\left(M_{(m)}\right) \rightarrow \Lambda^{(k)}\left(N_{(n)}\right)$, with $0 \leq k \leq n$, be the trace maps. From property (iii) of Theorem 2.1 it follows that the trace and exterior derivative commute:

$$
t^{(k+1)}\left(d^{(k)} a^{(k)}\right)=d^{(k)}\left(t^{(k)} a^{(k)}\right), \quad \forall a^{(k)} \in \Lambda^{(k)}\left(M_{(m)}\right)
$$

### 2.2.4 Orientation and outer-oriented differential forms

The differential forms introduced so far are fields of inner $k$-forms, i.e., they are inneroriented differential forms. Besides these, also outer-oriented differential forms, fields of outer $k$-forms, play a fundamental role. To be able to introduce these we first need to discuss the orientation of a manifold.

The concept of orientation for vector spaces (Definition 2.3) carries over to manifolds at least locally, because locally every manifold can be approximated by a vector space, namely the tangent space. To come to a global definition of orientation for a manifold $M_{(m)}$ we need a nonvanishing $m$-vector field $o_{(m)} \in \Lambda_{(m)}\left(M_{(m)}\right)$. The $m$-vector field smoothly assigns an orientation to each tangent space $T_{x} M_{(m)}$.

An $m$-dimensional manifold for which such an $m$-vector field exists is called orientable. Not every manifold is orientable. A well-known example of a nonorientable manifold is the Möbius band. All the manifolds relevant for this work are orientable.

Definition 2.20. Let $M_{(m)}$ be an orientable manifold. For a nonvanishing $m$-vector field $o_{(m)} \in \Lambda_{(m)}\left(M_{(m)}\right)$ we define the equivalence class $\left\{o_{(m)}\right\}$ by the equivalence relation where two nonvanishing $m$-vector fields $o_{(m)}$ and $o_{(m)}^{\prime}$ are equivalent if there exists a strictly positive $f \in \Lambda_{(0)}\left(M_{(m)}\right)$ such that $o_{(m)}=f o_{(m)}^{\prime}$. An inner orientation for $M_{(m)}$ is a choice for one of two equivalence classes $\left\{o_{(m)}\right\}$ and $-\left\{o_{(m)}\right\}$.

With the concept of inner orientation for a manifold we are in the position to define the space of outer-oriented differential forms.

Definition 2.21. Let $M_{(m)}$ be a manifold and $0 \leq k \leq m$. An outer-oriented differential $k$-form on $M_{(m)}$ is a pair $\tilde{a}^{(k)}=\left\{o_{(m)}\right\} a_{(k)}$ consisting of an inner-oriented differential $k$-form $a_{(k)} \in \Lambda^{(k)}\left(M_{(m)}\right)$ and an orientation $\left\{o_{(m)}\right\}$ for $M_{(m)}$. The outer-oriented differential form $\tilde{a}^{(k)}$ is called smooth if $a^{(k)}$ is smooth. We denote the space of smooth outer-oriented differential $k$-forms by $\tilde{\Lambda}^{(k)}\left(M_{(m)}\right)$. We define the space of smooth outeroriented $k$-vector fields $\tilde{\Lambda}_{(k)}\left(M_{(m)}\right)$ analogously.

An example of an outer differential form is given by the volume form.
Definition 2.22. Let $M_{(m)}$ be an inner-oriented manifold with an inner product. In each tangent space $T_{x} M_{(m)}$ the orientation and inner product define a volume form $\tilde{\mu}^{(m)}(x) \in \tilde{\Lambda}^{(m)}\left(T_{x} M_{(m)}\right)$ (cf. Definition 2.9). The inner product on $M_{(m)}$ is by definition smooth, hence $\tilde{\mu}^{(m)}(x)$, for each $x \in M_{(m)}$ defines a smooth outer differential form $\tilde{\mu}^{(m)} \in \tilde{\Lambda}^{(m)}\left(M_{(m)}\right)$. This form is called the volume form on $M_{(m)}$.

The spaces $\tilde{\Lambda}^{(k)}\left(M_{(m)}\right)$ and $\tilde{\Lambda}_{(k)}\left(M_{(m)}\right)$ can be related by the musical isomorphisms and the spaces of inner and outer differential forms are related by the Hodge maps $\star^{(k)}: \Lambda^{(k)}\left(M_{(m)}\right) \rightarrow \tilde{\Lambda}^{(m-k)}\left(M_{(m)}\right)$ and $\tilde{\star}^{(k)}: \tilde{\Lambda}^{(k)}\left(M_{(m)}\right) \rightarrow \Lambda^{(m-k)}\left(M_{(m)}\right)$. These maps are defined locally on the tangent space algebra as in Section 2.1.

The exterior derivative $\tilde{d}^{(k)}: \tilde{\Lambda}^{(k)}\left(M_{(m)}\right) \rightarrow \tilde{\Lambda}^{(k+1)}\left(M_{(m)}\right)$ applied to an outer differential $k$-form $\tilde{a}^{(k)}=\left\{o_{(m)}\right\} a^{(k)}$ is defined according $\tilde{d}^{(k)} \tilde{a}^{(k)}=\left\{o_{(d)}\right\} d^{(k)} a^{(k)}$, where $d^{(k)}$
is the exterior derivative for inner differential $k$-forms. The exterior derivative $\tilde{d}^{(k)}$ satisfies the outer-oriented analogue of Theorem 2.1. Moreover, it follows from the Leibniz rule for inner-oriented forms and the definition of $\tilde{d}$ that we have

$$
\tilde{d}\left(a^{(k)} \wedge \star b^{(l)}\right)=d a^{(k)} \wedge \star b^{(l)}+(-1)^{k} a^{(k)} \wedge \tilde{d} \star b^{(l)}
$$

for $a^{(k)} \in \Lambda^{(k)}\left(M_{(m)}\right)$ and $b^{(l)} \in \Lambda^{(l)}\left(M_{(m)}\right)$ with $l-m \leq k \leq l$.

### 2.2.5 Integration

An inner-oriented differential $k$-form can be integrated over an inner-oriented $k$-dimensional manifold.

Definition 2.23. Let $M_{(k)}$ be an inner-oriented manifold and $a^{(k)} \in \Lambda^{(k)}\left(M_{(k)}\right)$. Furthermore, let $\phi: U \rightarrow M_{(k)}$ be a parametrization. Then the integral of $a^{(k)}$ over $M_{(k)}$ is given by ${ }^{5}$

$$
\int_{M_{(k)}} a^{(k)}:=\int_{U} \phi^{*} a^{(k)}
$$

Here the integral on the right-hand side should be interpreted as a Lebesgue integral on the subset of $\mathbb{R}^{k}$. So if in the local coordinates $\phi^{*} a^{(k)}=\sum_{i_{1}<\cdots<i_{k}} a_{i_{1}, \ldots, i_{k}} d x^{i_{1}} \wedge \cdots \wedge d x^{i_{k}}$, then the integral should be interpreted as

$$
\int_{U} \phi^{*} a^{(k)}=\int_{U} \sum_{i_{1}<\cdots<i_{k}} a_{i_{1}, \ldots, i_{k}} d x^{i_{1}} \wedge \cdots \wedge d x^{i_{k}}=\int_{U} \sum_{i_{1}<\cdots<i_{k}} a_{i_{1}, \ldots, i_{k}} d x^{i_{1}} \cdots d x^{i_{k}} .
$$

In Section 2.1 we saw that the outer orientation for an outer-oriented $m$-vector $\tilde{a}_{(m)}=$ $\left\{o_{(d)}\right\} a_{(m)} \in \tilde{\Lambda}_{(m)} V$ was provided by the inner orientations of $a_{(m)}$ and $V$. The $m$-vector $a_{(m)}$ defines an orientation for the $m$-dimensional subspace of $V$ in which it lies and together with the orientation $\left\{o_{(d)}\right\}$ for the $d$-dimensional vector space $V$, this defines an orientation $\left\{o_{(d-m)}\right\}$ for the transverse dimensions to $a_{(k)}$ in $V$ by demanding $\left\{o_{(d-m)}\right\}$ to be such that $\left\{o_{(d-m)} \wedge a_{(m)}\right\}=\left\{o_{(d)}\right\}$.

In the current situation we have, instead of an $m$-vector $a_{(m)}$ providing an innerorientation for a subset of $V$, a nonvanishing $m$-vector field $o_{(m)}$ assigning an innerorientation to the tangent spaces $T_{x} M_{(m)} \subset \mathbb{R}^{d}, x \in M_{(m)}$. The tangent space to $\mathbb{R}^{d}$ can be seen as $T_{x} \mathbb{R}^{(d)} \cong T_{x} M_{(m)} \oplus\left(T_{x} M_{(m)}\right)^{t}$, where $\left(T_{x} M_{(m)}\right)^{t} \subset T_{x} \mathbb{R}^{(d)}$ is a $(d-m)$ dimensional subspace. This is a subspace of transverse directions, it is not necessarily orthogonal (it exists independent of a metric on $\mathbb{R}^{d}$ ) and it is also not unique. An orientation $\left\{o_{(d)}\right\}$ for $\mathbb{R}^{d}$ defines together with the orientation $\left\{o_{(m)}\right\}$ for $M_{(m)}$ an orientation $\left\{o_{(d-m)}\right\}$ for $\left(T_{x} M_{(m)}\right)^{t}$, for each $x \in M_{(m)}$, by demanding $\left\{o_{(d-m)} \wedge o_{(m)}\right\}=\left\{o_{(d)}\right\}$.

[^6]Definition 2.24. An outer-oriented ${ }^{6}$ manifold $\tilde{M}_{(m)}$ in $\mathbb{R}^{d}$ is given by a pair $\tilde{M}_{(m)}=$ $\left\{o_{(d)}\right\} M_{(m)}$, where $M_{(m)}$ is an oriented manifold in $\mathbb{R}^{d}$ and $\left\{o_{(d)}\right\}$ is an orientation for $\mathbb{R}^{d}$. Suppose that $\left\{o_{(m)}\right\}$ is the inner-orientation of $M_{m}$, then we denote the outer-orientation of $\left\{o_{(d)}\right\} M_{(m)}$ by $\left\{\left\{o_{(d)}\right\} o_{(m)}\right\}$.

Outer-oriented differential forms can be integrated over outer-oriented manifolds. Suppose $\tilde{M}_{(k)}$ is an outer-oriented manifold in $\mathbb{R}^{d}$ and $\tilde{a}^{(k)}$ is an outer-oriented $k$ form on $\tilde{M}_{(k)}$, then it is possible to define the integral of $\tilde{a}^{(k)}$ over $\tilde{M}_{(k)}$. We use $\tilde{\Lambda}^{(k)}\left(M_{(m)} \subset \mathbb{R}^{d}\right)$ to denote the space of outer-oriented differential $k$-forms on $M_{(m)}$ that have their outer-orientation in $\mathbb{R}^{d}$, i.e., $\tilde{a}^{(k)} \in \tilde{\Lambda}^{(k)}\left(M_{(m)} \subset \mathbb{R}^{d}\right)$ is of the form $\left\{o_{(d)}\right\} a^{(k)}$, where $a^{(k)} \in \Lambda^{(k)}\left(M_{(m)}\right)$ and $\left\{o_{(d)}\right\}$ is an orientation for $\mathbb{R}^{d}$.

For a general map between outer-oriented manifolds the pullback does not make sense. However, if these outer-oriented manifolds live in the same ambient manifold then we can define the pullback. Especially, the trace of outer-oriented differential forms will be useful.

Definition 2.25. Let $\tilde{N}_{(n)}$ be a submanifold of $\tilde{M}_{(m)}$ in $\mathbb{R}^{d}$. We define the trace, $\tilde{t}_{N_{(n)}}^{(k)}: \tilde{\Lambda}^{(k)}\left(M_{(m)} \subset \mathbb{R}^{d}\right) \rightarrow \tilde{\Lambda}^{(k)}\left(N_{(n)} \subset \mathbb{R}^{d}\right)$, of an outer-oriented differential $k$-form $\tilde{a}^{(k)}=\left\{o_{(d)}\right\} a^{(k)}$ according to $\tilde{t}_{\tilde{N}_{(n)}}^{(k)} \tilde{a}^{(k)}=\left\{o_{(d)}\right\} t_{N_{(n)}}^{(k)} a^{(k)}$, where $t_{N_{(n)}}^{(k)} a^{(k)} \in \Lambda^{(k)}\left(N_{(n)}\right)$ is the trace of the inner-oriented differential form $a^{(k)}$.

Definition 2.26. Let $\tilde{M}_{(k)}:=\left\{o_{(d)}\right\} M_{(k)}$ be an outer-oriented manifold in $\mathbb{R}^{d}$ and let $\tilde{a}^{(k)}:=\left\{o_{(d)}^{\prime}\right\} a^{(k)} \in \tilde{\Lambda}^{(k)}\left(M_{(k)} \subset \mathbb{R}^{d}\right)$ be an outer-oriented $k$-form on $\tilde{M}_{(k)} \subset \mathbb{R}^{d}$. We define the integral of $\tilde{a}^{(k)}$ over $\tilde{M}_{(k)}$ by

$$
\int_{\tilde{M}_{(k)}} \tilde{a}^{(k)}:=\left\{\begin{aligned}
\int_{M_{(k)}} a^{(k)} & \text { if }\left\{o_{(d)}\right\}=\left\{o_{(d)}^{\prime}\right\}, \\
-\int_{M_{(k)}} a^{(k)} & \text { if }\left\{o_{(d)}\right\}=-\left\{o_{(d)}^{\prime}\right\},
\end{aligned}\right.
$$

where the integrals on the right-hand side are defined as in Definition 2.23.
In Section 2.1 we saw that a $k$-covector and a $k$-vector together produce a real number, expressed by the duality product. Similarly, a $k$-dimensional manifold and a differential $k$-form produce together a real number. By definition of the integral, the real number produced by the integral of a differential $k$-form over a $k$-dimensional manifold can be viewed as the limit of a sum of duality products.

Consider again $a^{(m)} \in \Lambda^{(m)}\left(M_{(m)}\right)$ and the global parametrization $\phi: U \rightarrow M_{(m)}$. We can subdivide $U$ in a set of small $m$-vectors $b_{(m)}^{i}, i \in \mathcal{I}_{h}$, where all $m$-vectors are smaller (in length, area, volume, etc.) than some small $h>0$. Let $x_{i} \in U$ be the initial points of the $m$-vectors $b_{(m)}^{i}$ and let $a_{i}^{(m)}:=a^{(m)}\left(\phi\left(x_{i}\right)\right)$, for all $i \in \mathcal{I}_{h}$. We can then pull the $m$-covectors $a_{i}^{(m)}$ back and take the duality product with the $m$-vectors to find an

[^7]
$$
U \subset \mathbb{R}^{2}
$$


Figure 2.13: In this 2-dimensional example $U \subset \mathbb{R}^{2}$ is subdivided into the set $\left\{b_{(2)}^{i} \in \Lambda_{(2)}\left(T_{x_{i}} U\right) \mid i \in \mathcal{I}_{h}\right\}$ of 36 oriented 2 -vectors. The push-forward $\phi_{*} b_{(2)}^{i}$ of these 2 -vectors form an approximation of $M_{(2)}$ (as shown on the right-hand side). Suppose we have a differential 2-form $a^{(2)} \in \Lambda^{(2)}\left(M_{(2)}\right)$, then we can probe this field in the points $\phi\left(x_{i}\right)$ giving us $\left\{a_{i}^{(2)} \in \Lambda^{(2)}\left(T_{\phi\left(x_{i}\right)} M_{(2)}\right) \mid i \in \mathcal{I}_{h}\right\}$. We can either pull-back the 2 -forms $a_{i}^{(2)}$ or push-forward the 2 -vectors $b_{(2)}^{i}$ and subsequently take the duality product. The sums $\sum_{i \in \mathcal{I}_{h}}\left\langle b_{(2)}^{i}, \phi^{*} a_{i}^{(2)}\right\rangle=\sum_{i \in \mathcal{I}_{h}}\left\langle\phi_{*} b_{(2)}^{i}, a_{i}^{(2)}\right\rangle$ approximate the integral of $a^{(2)}$ over $M_{(2)}$. Increasingly finer partitions of $U$ give increasingly more accurate approximations of the integral, i.e., $\lim _{h \rightarrow 0} \sum_{i \in \mathcal{I}_{h}}\left\langle b_{(2)}^{i}, \phi^{*} a_{i}^{(2)}\right\rangle=\lim _{h \rightarrow 0} \sum_{i \in \mathcal{I}_{h}}\left\langle\phi_{*} b_{(2)}^{i}, a_{i}^{(2)}\right\rangle=\int_{M_{(2)}} a^{(2)}$.
approximation of the integral:

$$
\int_{M_{(m)}} a^{(m)}=\lim _{h \rightarrow 0} \sum_{i \in \mathcal{I}_{h}}\left\langle b_{(m)}^{i}, \phi^{*} a_{i}^{(m)}\right\rangle .
$$

Alternatively, we can also take the dual view point, namely push forward the $k$-vectors to the spaces $\Lambda_{(m)}\left(T_{x_{i}} M_{(m)}\right), i \in \mathcal{I}_{h}$. The integral is then given by the approximation

$$
\int_{M_{(m)}} a^{(m)}=\lim _{h \rightarrow 0} \sum_{i \in \mathcal{I}_{h}}\left\langle\phi_{*} b_{(m)}^{i}, a_{i}^{(m)}\right\rangle
$$

These two dual viewpoints are depicted in Figure 2.13 for the 2-dimensional case.
To express this viewpoint of a manifold and a differential form together producing a real number we will also use the bracket notation for the integral, i.e.,

$$
\left\langle M_{(m)}, a^{(m)}\right\rangle:=\int_{M_{(m)}} a^{(m)}
$$

We extend the duality product to the 0-dimensional case. The duality product between a 0 -form $a^{(0)}$ and a 0 -dimensional manifold $x^{(0)}$, which is just a point in space, is defined according to

$$
\left\langle x_{(0)}, a^{(0)}\right\rangle=a^{(0)}\left(x_{(0)}\right)
$$

i.e., the evaluation of $a^{(0)}$ in the point $x_{(0)}$.


Figure 2.14: Examples of inner- and outer-oriented manifolds. We represent the velocity field either by an inner-oriented 1 -form $u^{(1)}$ or by an outer-oriented 2 -form $\tilde{u}^{(2)}$, which we can integrate over, respectively, $M_{(1)}$ and $\tilde{M}_{(2)}$. Similarly, we can represent the vorticity field either by an outer-oriented 1-form $\tilde{\omega}^{(1)}$ or an inner-oriented 2-form $\omega^{(2)}$, which we can integrate over, respectively, $\tilde{M}_{(1)}$ and $M_{(2)}$.

Above we have defined eight different types of integrals in $\mathbb{R}^{3}$, namely integrals of inner- and outer-oriented $k$-forms, where $k$ equals $0,1,2$ or 3 . Let us give some examples, relevant for this work, of integrals and the role that orientation plays.

We earlier saw that we can represent the velocity field by a 1-form $u^{(1)} \in \Lambda^{(1)}\left(\Omega_{(3)}\right)$. The velocity 1 -form can be integrated over an inner-oriented curve $M_{(1)} \subset \Omega_{(3)}$. If the curve is a closed loop, then the real number $\left\langle M_{(1)}, u^{(1)}\right\rangle$ is known as the circulation around $M_{(1)} .{ }^{7}$

Alternatively, we might want to quantify the rotation of the fluid around the curve $M_{(1)}$. The rotation of the fluid is given by the vorticity 1-form $\tilde{\omega}^{(1)}=\star d u^{(1)} \in \tilde{\Lambda}^{(1)}\left(\Omega_{(3)}\right)$. If we give the curve an outer orientation instead, as shown in Figure 2.14, we can integrate $\tilde{\omega}^{(1)}$ along it. The outer orientation of the curve defines a direction of positive rotation.

[^8]Similarly, we can represent the velocity field by the outer-oriented 2 -form $\tilde{u}^{(2)}=$ $\star u^{(1)} \in \tilde{\Lambda}^{(2)}\left(\Omega_{(3)}\right)$. Integrating this over an outer-oriented surface $\tilde{M}_{(2)}$ gives us the flux $\left\langle\tilde{M}_{(2)}, \tilde{u}^{(2)}\right\rangle$ through the surface. The outer orientation of the surface defines a direction of positive flux.

Finally, instead of determining the rotation of the fluid around a curve, we can calculate the rotation of the fluid in a surface $M_{(2)}$ by integrating the inner-oriented vorticity 2 -form $\omega^{(2)}=d u^{(1)} \in \Lambda^{(2)}\left(\Omega_{(3)}\right)$.

The four different integrals introduced above will all play an important role in the discretization methods in this thesis. In Chapter 3 we will see that the oriented manifolds $M_{(1)}, \tilde{M}_{(2)}, M_{(2)}$ and $\tilde{M}_{(1)}$ will be oriented mesh elemens, i.e., mesh edges or faces. The real numbers $\left\langle M_{(1)}, u^{(1)}\right\rangle,\left\langle\tilde{M}_{(2)}, \tilde{u}^{(2)}\right\rangle,\left\langle M_{(2)}, \omega^{(2)}\right\rangle$ and $\left\langle\tilde{M}_{(1)}, \tilde{\omega}^{(1)}\right\rangle$ will be the discrete variables of the discretization method.

Now that since we have defined the integral inner and outer differential forms we can also define the $L^{2}$-inner product

Definition 2.27. Let $\Omega_{(d)} \subset \mathbb{R}^{d}$ be an inner-oriented manifold. The $L^{2}$-inner product on $\Omega_{(d)},(\cdot, \cdot)_{\Omega_{(d)}}: \Lambda^{(k)}\left(\Omega_{(d)}\right) \times \Lambda^{(k)}\left(\Omega_{(d)}\right) \rightarrow \mathbb{R}$ is given by

$$
\left(a^{(k)}, b^{(k)}\right)_{\Omega_{(d)}}:=\int_{\Omega_{(d)}} a^{(k)} \wedge \star b^{(k)}=\int_{\Omega}\left(a^{(k)}, b^{(k)}\right) \tilde{\mu}^{(d)}
$$

where $\tilde{\mu}^{(d)}$ is the volume form on $\Omega_{(d)}$.
The $L^{2}$-inner product on $\tilde{\Lambda}^{(k)}\left(\Omega_{(d)}\right)$ is defined analogously. We denote by $\|\cdot\|_{\Omega_{(d)}}$ the corresponding norm. When it is not confusing we will sometimes simply omit the subscripts and write $(\cdot, \cdot)$ and $\|\cdot\|$.

### 2.2.6 Stokes' theorem

Stokes' theorem is the basis for the mimetic discretizations in this thesis. It generalizes the fundamental theorem of line integrals, the Kelvin-Stokes theorem and the Gauss divergence theorem to arbitrary dimension. It is stated by the very simple and elegant expression

$$
\left\langle M_{(k+1)}, d a^{(k)}\right\rangle=\left\langle\partial M_{(k+1)}, a^{(k)}\right\rangle
$$

where $M_{(k+1)}$ is a manifold with boundary $\partial M_{(k+1)}$ and $a^{(k)}$ a differential $k$-form on $M_{(k+1)}$.

In this subsection we first introduce Stokes' theorem for inner-oriented forms and manifolds. We define the orientation which a manifold induces on its boundary and then state Stokes' theorem. Subsequently, we do the same for the outer-oriented case, and, finally, we show how different cases of Stokes' theorem relate to the familiar theorems from vector calculus.

Stokes' theorem equates the integral of an exterior derivative of a differential form over a manifold to the integral of the differential form over the boundary of the manifold. In Definition 2.14 we already introduced the boundary of manifold as a subset of that manifold. However, to integrate over the boundary it also needs an orientation. This orientation is induced as follows by the orientation of the manifold.


Figure 2.15: In black we see the inner orientation of $M_{(k)}$ and in gray the induced inner orientation on $\partial M_{(k)}$, for, from left to right, $k=1, k=2$ and $k=3$.

Definition 2.28. Let $M_{(m)}$ be a manifold with boundary and let $\left\{o_{(k)}\right\}$ be its orientation. The induced inner orientation on $\partial M_{(m)}$ is given by $\left\{o_{(k-1)}\right\}$, where

$$
\left\{v_{(1)} \wedge o_{(k-1)}\right\}=\left\{o_{(k)}\right\},
$$

where $v_{(1)} \in \Lambda_{(1)}\left(M_{(m)}\right)$ is a vector field which points out off $M_{(m)}$ at $\partial M_{(m)}$. One can take for example $v_{(1)}=\sharp d \chi$, where $\chi \in \Lambda^{(0)}\left(M_{(m)}\right)$ is increasing to the outside of $M_{(m)}$. From now on, when we write $\partial M_{(m)}$ we mean the manifold $\partial M_{(m)}$ with the orientation induced by $M_{(m)}$.

Although the definition of the induced inner orientation is slightly abstract, the idea is quite intuitive, especially in three dimensions. This is shown in Figure 2.15.

With the induced orientation defined we are almost ready to state Stokes' theorem accurately. There is one thing we still need to deal with. The boundary of a manifold with boundary (Definition 2.14) is a manifold without boundary. Many of the objects that we need to deal with are actually manifolds with corners, like, for example the triangle and tetrahedron in Figure 2.15. The boundary of a manifold with corners is not a manifold itself, because it is not smooth in the corner points.

A manifold with corners $M_{(m)}$ (Definition 2.15) is locally diffeomorphic to $\mathbb{R}_{\geq 0}^{m}$. Also the boundary of $\mathbb{R}_{\geq 0}^{m}$ is not a manifold with corners. It is, nonetheless, a finite union of manifolds, namely, $\partial \mathbb{R}_{\geq 0}^{m}=\sum_{i=1}^{m} H_{i}$, where $H_{i}:=\left\{\left(x^{1}, \ldots, x^{m}\right) \in \mathbb{R}_{\geq 0}^{m} \mid x^{i}=0\right\}$.

Suppose now that we have a manifold with corners $M_{(m)}$ and an ( $m-1$ )-form $a^{(m-1)}$ on $\partial M_{(m)}$. Furthermore, let $\phi: U \rightarrow \mathbb{R}_{\geq 0}^{m}$ be a coordinate chart on $U \subset M_{(m)}$ and assume that $a^{(m-1)}$ is supported in $U . .^{8}$ Then we define the integral of $a^{(m-1)}$ by [36]

$$
\int_{\partial M_{(m)}} a^{(m-1)}=\sum_{i=1}^{m} \int_{H_{i}}\left(\phi^{-1}\right)^{*} a^{(m-1)} .
$$

Theorem 2.2. (Stokes' theorem for inner-oriented forms) [36] Let $M_{(k+1)}$ be an inner-oriented manifold with corners and let $a^{(k)} \in \Lambda^{(k)}\left(M_{(k+1)}\right)$ be an inner-oriented differential form on $M_{(k+1)}$. Then

$$
\left\langle M_{(k+1)}, d a^{(k)}\right\rangle=\left\langle\partial M_{(k+1)}, a^{(k)}\right\rangle
$$

[^9]

Figure 2.16: In black the outer orientation of $\Omega_{(k)}$ and in gray the induced outer orientation on $\partial \Omega_{(k)}$, for, from left to right, $k=3, k=2$ and $k=1$.

Here $a^{(k)}$ on the right-hand side is to be interpreted as $t_{\partial M_{(k+1)}} a^{(k)}$.
To state Stokes' theorem for outer-oriented forms we only need to define the outerorientation induced by a manifold on its boundary. The rest is completely analogous to the inner-oriented case.

Definition 2.29. Let $\tilde{M}_{(m)}$ be an outer-oriented manifold in $\mathbb{R}^{d}$ and let $\left\{\left\{o_{(d)}\right\} o_{(m)}\right\}$ be the outer-orientation of $\tilde{M}_{(m)}$. The induced outer orientation on $\partial \tilde{M}_{(m)}$ is given by $\left\{\left\{o_{(d)}\right\} o_{(m-1)}\right\}$, where $\left\{o_{(m-1)}\right\}$ is the inner orientation induced on $\partial M_{(m)}$ by the orientation $\left\{o_{(m)}\right\}$ for $M_{(m)}$.

Pictorial examples of the induced outer orientation are shown in Figure 2.16.
Theorem 2.3. (Stokes' theorem for outer-oriented forms) [33, 36, 39] Let $\tilde{M}_{(k+1)}$ be an outer-oriented manifold with corners and let $\tilde{a}^{(k)} \in \tilde{\Lambda}^{(k)}\left(M_{(k+1)}\right)$ be an outeroriented differential form on $\tilde{M}_{(k+1)}$. Then

$$
\left\langle\tilde{M}_{(k+1)}, \tilde{d} \tilde{a}^{(k)}\right\rangle=\left\langle\partial \tilde{M}_{(k+1)}, \tilde{a}^{(k)}\right\rangle
$$

Here $\tilde{a}^{(k)}$ on the right-hand side is to be interpreted as $\tilde{t}_{\partial \tilde{M}_{(k+1)}} \tilde{a}^{(k)}$.
We see that the exterior derivative and the boundary operator are adjoint to each other. In Chapter 3 this adjoint relation will form the basis for the discretization of the exterior derivative.

We consider a few examples to illustrate Stokes' theorem. To accommodate the incompressibility constraint of the fluid the gradient of the pressure is present in the momentum equation. We will see in Section 2.4 that it is possible to write this term as the exterior derivative of a pressure 0 -form, i.e., as $d p^{(0)}$. In the mimetic discretizations presented further on in this thesis, we often encounter the integral of this term over a path $M_{(1)}$ from a point $x_{a} \in \Omega_{(3)}$ to a point $x_{b} \in \Omega_{(3)}$, e.g. a mesh edge running from $x_{a}$ to $x_{b}$. Stokes' theorem then tells us that

$$
\int_{M_{(1)}} d p^{(0)}=\left\langle M_{(1)}, d p^{(0)}\right\rangle=\left\langle\partial M_{(1)}, p^{(0)}\right\rangle=p^{(0)}\left(x_{b}\right)-p^{(0)}\left(x_{a}\right),
$$

where we assumed in the last equality that $M_{(1)}$ is oriented from $x_{a}$ to $x_{b}$ and both end points are oriented as sinks. Not surprisingly the integral only depends on the value of $p^{(0)}$ in the end points of $M_{(1)}$. This is simply the fundamental theorem of line integrals.

Another relevant example is the integral of the inner-oriented vorticity 2 -form $\omega^{(2)}=$ $d u^{(1)}$ over a surface $M_{(2)}$. By Stokes' theorem it follows that

$$
\int_{M_{(2)}} \omega^{(2)}=\left\langle M_{(2)}, d u^{(1)}\right\rangle=\left\langle\partial M_{(1)}, u^{(1)}\right\rangle=\int_{\partial M_{(1)}} u^{(1)} .
$$

Again, the integral depends only on the boundary $\partial M_{(2)}$, i.e., for a second surface $M_{(2)}^{\prime}$ such that $\partial M_{(2)}^{\prime}=\partial M_{(2)}$ the integral of $\omega^{(2)}$ would also be equal to the circulation around $\partial M_{(2)}$.

As a final example we consider the outer-oriented velocity 2 -form $\tilde{u}^{(2)}$. The incompressibility constraint can be written as $0=\tilde{d} \tilde{u}^{(2)}$. For any outer-oriented volume $\tilde{M}_{(3)}$ we have

$$
\int_{\partial \tilde{M}_{(3)}} \tilde{u}^{(2)}=\left\langle\partial \tilde{M}_{(3)}, \tilde{u}^{(2)}\right\rangle=\left\langle\tilde{M}_{(3)}, \tilde{d}^{(2)}\right\rangle=0
$$

hence the incompressibility constraint implies that the net influx in $\tilde{M}_{(3)}$ is zero.
The combination of Stokes' theorem and the Leibniz rule for the exterior derivative (Theorem 2.1) leads to the following result.
Corollary 2.2. For $M_{(m)}$ an inner-oriented manifold, $a^{(k)} \in \Lambda^{(k)}\left(M_{(m)}\right)$ and $b^{(m-k-1)} \epsilon$ $\Lambda^{(m-k-1)}\left(M_{(m)}\right)$, with $0 \leq k \leq m$, we have the integration by parts formula

$$
\int_{M_{(m)}} d a^{(k)} \wedge b^{(m-k-1)}=(-1)^{k-1} \int_{M_{(m)}} a^{(k)} \wedge d b^{(m-k-1)}+\int_{\partial M_{(m)}} t_{\partial M_{(m)}}\left(a^{(k)} \wedge b^{(m-k-1)}\right)
$$

An analogous formula holds for outer-oriented forms.
Before we saw that the boundary $\partial M_{(m)}$ of a manifold with corners $M_{(m)}$ is not itself a manifold with corners, because in the corner points $\partial M_{(m)}$ is not smooth. If we consider, for example a cube in $\mathbb{R}^{3}$, which we denote by $M_{(3)}$, then the boundary of $\partial M_{(2)}$ is not smooth in the edges and vertices of the cube. However, each of the faces by itself is a smooth manifold with corners again.

Let us denote by $F^{1} M_{(m)}$ the points $x$ in $\partial M_{(m)}$ that under a chart $\phi: U \rightarrow \mathbb{R}_{\geq 0}^{m}$ map to a point $\phi(x) \in \mathbb{R}_{\geq 0}^{m}$ for which exactly only one of the coordinates is zero and all the others are greater than zero. This definition is independent of the chart used [36]. Analogously, we can define $F^{n} M_{(m)}$ as the points $x$ in $\partial M_{(m)}$ that map under charts to points in $\mathbb{R}_{\geq 0}^{m}$ for which $n$ of the coordinates are zero.

For the cube $M_{(3)}, F^{1} M_{(3)}$ are the points interior in the faces of the cube, $F^{2} M_{(3)}$ are the points interior in the edges and $F^{3} M_{(3)}$ are the vertices of the cube. The boundary points $\partial M_{(3)}$ are the union of these different type of boundary points, i.e., $\partial M_{(3)}=$ $F^{1} M_{(3)} \cup F^{2} M_{(3)} \cup F^{3} M_{(3)}$.

It can be shown (see for example [40]) that $F^{n} M_{(m)}$ is a union of connected components and that for each of these components its closure within $\cup_{i \geq n}^{m} F^{i} M_{(3)}$ is again a manifold with corners. In the case of the cube, the connected components of $F^{1} M_{(3)}$ are the


Figure 2.17: Left: the oriented cube $M_{(3)}$. Center: the faces $\partial^{1} M_{(3)}$ with induced orientation. Right: the set $\cup_{M_{(2)} \epsilon \partial^{1} M_{(3)}} \partial^{1} M_{(2)}$. Every edge $M_{(1)}$ of the cube occurs twice in this set, because it is part of $\partial^{1} M_{(2)}$ for two faces $M_{(2)} \in \partial^{1} M_{(3)}$. However, the two induced orientations are opposite.
interiors of the faces. The closure of one of these faces within $F^{1} M_{(3)} \cup F^{2} M_{(3)} \cup F^{3} M_{(3)}$ is the face including the four edges and vertices in the boundary of this face. This is clearly a 2 -dimensional manifold with corners. Similarly, the connected components in $F^{2} M_{(3)}$ are the interiors of the edges. The closure of one of these within $F^{2} M_{(3)} \cup F^{3} M_{(3)}$ is simply an edge of the cube including the vertices in its endpoints. This is obviously a 1-dimensional manifold with corners.

From now on we denote by $\partial^{n} M_{(m)}$ the set of $(m-n)$-dimensional manifolds with corners found by taking the closures of the connected components of $F^{n} M_{(m)}$. For the elements of $\partial^{1} M_{(m)}$ we assume that they have the same orientation as $\partial M_{(m)}$. It follows from the definition of the integral on a manifold with corners [36] that we can write Stokes' theorem as

$$
\begin{equation*}
\left\langle M_{(k+1)}, d a^{(k)}\right\rangle=\sum_{M_{(k)} \in \partial^{1} M_{(k+1)}}\left\langle M_{(k)}, a^{(k)}\right\rangle . \tag{2.5}
\end{equation*}
$$

The analogous relation holds for the outer-oriented case.
Stokes' theorem as in (2.5) also allows us to view the fact that $d d a^{(k)}=0$ in a more geometric ${ }^{9}$ way. Applying Stokes' theorem twice successively gives

$$
\begin{aligned}
\left\langle M_{(k+1)}, d d a^{(k-1)}\right\rangle & =\sum_{M_{(k)} \in \partial^{1} M_{(k+1)}}\left\langle M_{(k)}, d a^{(k-1)}\right\rangle \\
& =\sum_{M_{(k)} \in \partial^{1} M_{(k+1)}}\left(\sum_{M_{(k-1)} \in \partial^{1} M_{(k)}}\left\langle M_{(k-1)}, a^{(k-1)}\right\rangle\right) .
\end{aligned}
$$

In this double sum every $M_{(k-1)} \in \partial^{2} M_{(k+1)}$ occurs twice but with opposite orientation and therefore $\left\langle M_{(k+1)}, d d a^{(k-1)}\right\rangle=0$. This is illustrated for the cube $M_{(3)}$ in Figure 2.17.

[^10]
### 2.3 De Rham complexes

Given a divergence-free velocity field $u_{(1)}$, does there always exist a vector potential $a_{(1)}$ such that $u_{(1)}=\nabla \times a_{(1)}$ ? If it exists, is such an $a_{(1)}$ unique? What if the flow domain has a non-trivial topology? It turns out that questions of this type are best answered by analyzing the sequence

$$
\begin{equation*}
\Lambda^{(0)}\left(\Omega_{(d)}\right) \xrightarrow{d^{(0)}} \Lambda^{(1)}\left(\Omega_{(d)}\right) \xrightarrow{d^{(1)}} \quad \cdots \quad \xrightarrow{d^{(d-1)}} \Lambda^{(d)}\left(\Omega_{(d)}\right), \tag{2.6}
\end{equation*}
$$

which is known as a de Rham complex.
The spaces in the sequence (2.6) contain smooth differential forms. In practice this is rather restrictive and differential forms with less regularity need to be considered. With this aim we will introduce $L^{2}$-spaces of differential forms and corresponding complexes. We will consider these spaces and complexes for different boundary conditions.

Associated to each complex is an orthogonal decomposition of the spaces of differential forms in the complex. These decompositions are known as Helmholtz-Hodge decompositions, which are a differential form version of the Helmholtz decomposition that says that under certain circumstances a vector field can be written as the sum of a gradient and a curl.

Later on, in Chapter 5, we show how the Helmholtz-Hodge decompositions can be mimicked in the discrete setting. These discrete decompositions can then be used to find the velocity field for an incompressible flow when only its vorticity is known.

### 2.3.1 Smooth de Rham complexes

Before discussing complexes for differential forms we first define a general complex.
Definition 2.30. [41,42] A chain complex is a sequence of groups and homomorphisms $\left(V_{(k)}, A_{(k)}\right)$ :

$$
\cdots \longleftarrow V_{(k-1)} \stackrel{A_{(k)}}{\leftrightarrows} V_{(k)} \stackrel{A_{(k+1)}}{\leftrightarrows} V_{(k+1)} \longleftarrow \cdots,
$$

such that $A_{(k+1)} \circ A_{(k)}=0$.
A cochain complex is a sequence of groups and homomorphisms $\left(V^{(k)}, A^{(k)}\right)$ :

$$
\cdots \longrightarrow V^{(k-1)} \xrightarrow{A^{(k-1)}} V^{(k)} \xrightarrow{A^{(k)}} V^{(k+1)} \longrightarrow \cdots,
$$

such that $A_{(k+1)} \circ A_{(k)}=0$.
The chain and cochain complexes that we encounter are finite, i.e., the sequence of groups and homomorphisms is finite. Furthermore, the groups will often be (infinitedimensional) vector spaces and the homomorphisms will in this chapter be (denselydefined) linear operators.

The sequence ${ }^{10}$

$$
\Lambda^{(0)}(\Omega) \xrightarrow{d^{(0)}} \Lambda^{(1)}(\Omega) \xrightarrow{d^{(1)}} \quad \cdots \quad \xrightarrow{d^{(d-1)}} \Lambda^{(d)}(\Omega),
$$

[^11]\[

$$
\begin{aligned}
& \Lambda^{(0)}(\Omega) \xrightarrow{d^{(0)}} \Lambda^{(1)}(\Omega) \xrightarrow{d^{(1)}} \Lambda^{(2)}(\Omega) \xrightarrow{d^{(2)}} \Lambda^{(3)}(\Omega) \\
& \star^{(0)}\left|\uparrow \tilde{\star}^{(3)} \quad \star^{(1)}\right| \uparrow \tilde{\star}^{(2)} \quad \star^{(2)} \downarrow\left|\tilde{\star}^{(1)} \quad \star^{(3)}\right| \mid \tilde{\star}^{(0)} \\
& \tilde{\Lambda}^{(3)}(\Omega) \longleftarrow \tilde{d}^{(2)} \tilde{\Lambda}^{(2)}(\Omega) \longleftarrow \tilde{d}^{(1)} \tilde{\Lambda}^{(1)}(\Omega) \longleftarrow \tilde{d}^{(0)} \tilde{\Lambda}^{(0)}(\Omega)
\end{aligned}
$$
\]

Figure 2.18: The exterior derivative and Hodge star operator connect the different spaces. The top row and bottom row of the diagram constitute a complex, because we have $d \circ d \equiv 0$ and $\tilde{d} \circ \tilde{d} \equiv 0$.
is obviously a cochain complex by Theorem 2.1. This cochain complex is known as the smooth de Rham complex. The space of outer-oriented smooth differential forms form a cochain complex as well. The complexes of inner- and outer-oriented forms are related by the Hodge duality. This double complex is illustrated in Figure 2.18 for $d=3$.

From Figure 2.18, we see that we can form compositions $\tilde{\star}^{(d-k+1)} \circ \tilde{d}^{(d-k)} \circ \star^{(k)}$ : $\Lambda^{(k)}(\Omega) \rightarrow \Lambda^{(k-1)}(\Omega)$ which together with the spaces $\Lambda^{(k)}(\Omega)$, for $k=0, \ldots, d$ form a sequence of spaces. This composition (modulo its sign) is called the coderivative operator.
Definition 2.31. [38, 43] The coderivative operator $d^{*(k)}: \Lambda^{(k)}(\Omega) \rightarrow \Lambda^{(k-1)}(\Omega)$ is defined according to

$$
\star^{(k-1)} d^{*(k)} a^{(k)}=(-1)^{k} \tilde{d}^{(d-k)} \star^{(k)} a^{(k)}, \quad a^{(k)} \in \Lambda^{(k)}(\Omega) .
$$

Multiplying both sides by $\tilde{\star}^{(d-k+1)}$ and using $\tilde{\star}^{(d-k+1)}{ }_{\star}{ }^{(k-1)}=(-1)^{(k-1)(d-k+1)}$ we obtain $d^{\star(k)} a^{(k)}=(-1)^{d(k-1)+1} \tilde{\star} \tilde{d} \star a^{(k)}$.

Proposition 2.2. The sequence

$$
\begin{equation*}
\Lambda^{(0)}(\Omega) \stackrel{d^{*(1)}}{\leftrightarrows} \Lambda^{(1)}(\Omega) \stackrel{d^{*(2)}}{\leftrightarrows} \cdots \quad \stackrel{d^{*(d)}}{\longleftarrow} \Lambda^{(d)}(\Omega), \tag{2.7}
\end{equation*}
$$

forms a chain complex.
Proof. From (ii) of Theorem 2.1 (which of course also applies to $\tilde{d}$ ) it follows that for all $a^{(k+1)} \in \Lambda^{(k+1)}(\Omega)$,

$$
d^{\star(k)} d^{*(k+1)} a^{(k+1)}=-\left(\star^{(k-1)}\right)^{-1} \tilde{d}^{(d-k)} \tilde{d}^{(d-k-1)} \star^{(k+1)} a^{(k+1)}=0 .
$$

The sign in Definition 2.31 is chosen such that $d^{*}$ is the formal adjoint to $d$ with respect to the $L^{2}$-inner product (Definition 2.27).

Lemma 2.1. For any $a^{(k)} \in \Lambda^{(k)}(\Omega)$ and $b^{(k+1)} \in \Lambda^{(k+1)}(\Omega)$ we have

$$
\left(d^{(k)} a^{(k)}, b^{(k+1)}\right)_{\Omega}=\left(a^{(k)}, d^{*(k+1)} b^{(k+1)}\right)_{\Omega}+\int_{\partial \Omega} t a^{(k)} \wedge \tilde{t} \star b^{(k+1)} .
$$

Proof. From the definition of the coderivative and by the Leibniz rule (Theorem 2.1) we find

$$
\begin{aligned}
\left(a^{(k)}, d^{*} b^{(k+1)}\right)_{\Omega} & =\int_{\Omega} a^{(k)} \wedge(-1)^{k+1} \tilde{d} \star b^{(k+1)} \\
& =-\int_{\Omega} \tilde{d}\left(a^{(k)} \wedge \star b^{(k+1)}\right)+\int_{\Omega} d a^{(k)} \wedge \star b^{(k+1)}
\end{aligned}
$$

Applying Stokes' theorem subsequently gives the result.
This shows that $d$ and $d^{*}$ are each others adjoint in absence of a boundary contribution. To exemplify this, let us define the spaces of smooth differential forms with zero trace,

$$
\Lambda^{(k)}(\Omega):=\left\{a^{(k)} \in \Lambda^{(k)}(\Omega) \mid t_{\partial \Omega} a^{(k)}=0\right\} .
$$

For all $a^{(k)} \in \AA^{(k)}(\Omega)$ and $b^{(k+1)} \in \Lambda^{(k+1)}(\Omega)$ we have

$$
\left(d^{(k)} a^{(k)}, b^{(k+1)}\right)_{\Omega}=\left(a^{(k)}, d^{*(k+1)} b^{(k+1)}\right)_{\Omega} .
$$

When the operators in a chain complex and cochain complex satisfy this adjointness relation, then the chain and cochain complex are called adjoint. Thus, sequence (2.7) and the sequence given by

$$
\AA^{(0)}(\Omega) \xrightarrow{d^{(0)}} \AA^{(1)}(\Omega) \xrightarrow{d^{(1)}} \quad \cdots \quad \xrightarrow{d^{(d-1)}} \AA^{(d)}(\Omega),
$$

are adjoint. In Section 2.3.3 we will see many more examples of this.

### 2.3.2 Sobolev spaces of differential forms

In this section we mostly rely on [44] and [45]. We define the space $L^{2} \Lambda^{(k)}(\Omega)$ of differential forms with $L^{2}$-regularity as the completion of the spaces $\Lambda^{(k)}(\Omega)$ with respect to the $L^{2}$-norm $\|\cdot\|_{\Omega}$ (Definition 2.27).

We define the weak generalizations of the exterior derivative and coderivative, and, the subspaces of $L^{2} \Lambda^{(k)}(\Omega)$ for which these weak derivatives exist.
Definition 2.32. Given $a^{(k)} \in L^{2} \Lambda^{(k)}(\Omega)$ we call $d^{(k)} a^{(k)}$ the weak exterior derivative of $a^{(k)}$ if

$$
\left(d^{(k)} a^{(k)}, \phi^{(k+1)}\right)_{\Omega}=\left(a^{(k)}, d^{*(k+1)} \phi^{(k+1)}\right)_{\Omega}, \quad \forall \phi^{(k+1)} \in \AA^{(k+1)}(\Omega)
$$

Analogously, we call $d^{*(k)} a^{(k)}$ the weak coderivative of $a^{(k)}$ if

$$
\left(d^{*(k)} a^{(k)}, \phi^{(k-1)}\right)_{\Omega}=\left(a^{(k)}, d^{(k-1)} \phi^{(k-1)}\right)_{\Omega}, \quad \forall \phi^{(k-1)} \in \AA^{(k-1)}(\Omega)
$$

The weak exterior (co)derivative coincides with the earlier defined exterior (co)derivative on $\Lambda^{(k)}(\Omega)$. Using the weak derivatives we can define the following Sobolev spaces

$$
\begin{aligned}
H \Lambda^{(k)}(d, \Omega) & :=\left\{a^{(k)} \in L^{2} \Lambda^{(k)}(\Omega) \mid d^{(k)} a^{(k)} \in L^{2} \Lambda^{(k+1)}(\Omega)\right\} \\
H \Lambda^{(k)}\left(d^{*}, \Omega\right) & :=\left\{a^{(k)} \in L^{2} \Lambda^{(k)}(\Omega) \mid d^{*(k)} a^{(k)} \in L^{2} \Lambda^{(k-1)}(\Omega)\right\},
\end{aligned}
$$



Figure 2.19: The exterior derivative and star operator connect the different spaces. The top row and bottom row of the diagram constitute a complex, because we have $d \circ d \equiv 0$ and $\tilde{d} \circ \tilde{d} \equiv 0$. Note that they are only each others adjoint when for either the top row or bottom row additionally holds that the trace of forms on the boundary is zero.
with norms $\left\|a^{(k)}\right\|_{d}:=\left\|a^{(k)}\right\|+\left\|d a^{(k)}\right\|$ and $\left\|a^{(k)}\right\|_{d^{*}}:=\left\|a^{(k)}\right\|+\left\|d^{*} a^{(k)}\right\|$, respectively. Furthermore we need $H \Lambda^{(k)}\left(d, d^{*}, \Omega\right):=H \Lambda^{(k)}(d, \Omega) \cap H \Lambda^{(k)}\left(d^{*}, \Omega\right)$ with norm $\left\|a^{(k)}\right\|_{d, d^{*}}:=$ $\left\|a^{(k)}\right\|+\left\|d a^{(k)}\right\|+\left\|d^{*} a^{(k)}\right\|$. We denote by $H \tilde{\Lambda}^{(k)}(\Omega)$ the outer-oriented version of $H \Lambda^{(k)}(\Omega)$, i.e.,

$$
H \tilde{\Lambda}^{(k)}(\tilde{d}, \Omega):=\left\{\tilde{a}^{(k)} \in L^{2} \tilde{\Lambda}^{(k+1)}(\Omega) \mid \tilde{d}^{(k)} \tilde{a}^{(k)} \in L^{2} \tilde{\Lambda}^{(k+1)}(\Omega)\right\}
$$

The sequences of spaces and derivatives $\left(H \Lambda^{(k)}(d, \Omega), d^{(k)}\right)$ and $\left(H \tilde{\Lambda}^{(k)}(\tilde{d}, \Omega), \tilde{d}^{(k)}\right)$ with increasing $k$ are obviously cochain complexes. These complexes are linked by the Hodge star operator ${ }^{11}$. These linked complexes are depicted for $d=3$ in Figure 2.19.

Note that by definition of the coderivative it follows that $a^{(k)}$ is in $H \Lambda^{(k)}\left(d^{*}, \Omega\right)$ if and only if $\star^{(k)} a^{(k)} \in H \tilde{\Lambda}^{(d-k)}(\tilde{d}, \Omega)$. Furthermore, the Hodge star map is an isomorphism. Thus we have $H \tilde{\Lambda}^{(d-k)}(\tilde{d}, \Omega)=\star^{(k)} H \Lambda^{(k)}\left(d^{*}, \Omega\right)$.

We assume that the domain $\Omega$ has a Lipschitz boundary $\partial \Omega$. On $\partial \Omega$ we can almost everywhere introduce a tangent space and the induced Riemannian metric allows for the definition of a boundary Hodge star operator ${ }_{{ }_{\mathrm{b}}}$. Also on $\partial \Omega$ we can define the space of $L^{2}$ functions $L^{2} \Lambda^{(k)}(\partial \Omega)$ together with the inner product [45]

$$
\begin{equation*}
\left(a^{(k)}, b^{(k)}\right)_{\mathrm{b}}:=\int_{\partial \Omega} a^{(k)} \wedge \star_{\mathrm{b}} b^{(k)}, \tag{2.8}
\end{equation*}
$$

for $a^{(k)}, b^{(k)} \in L^{(2)} \Lambda^{(k)}(\partial \Omega)$.
Assume for a moment that $\partial \Omega$ is smooth. Let $H^{1} \Lambda^{(k)}(\Omega)$ denote the subspace of $L^{2} \Lambda^{(k)}(\Omega)$ for which the (scalar) component functions in any coordinate system are in the Sobolev space $H^{1}(\Omega)=H \Lambda^{(0)}(d, \Omega)$. The trace operator can be extended from the map $t^{(k)}: \Lambda^{(k)}(\Omega) \rightarrow \Lambda^{(k)}(\partial \Omega)$ to a map $t^{(k)}: H^{1} \Lambda^{(k)}(\Omega) \rightarrow L^{2} \Lambda^{(k)}(\partial \Omega)[44,45]$. Similarly, we have a map $\tilde{t}^{(k)}: H^{1} \tilde{\Lambda}^{(k)}(\Omega) \rightarrow L^{2} \tilde{\Lambda}^{(k)}(\partial \Omega)$. We define the following trace

[^12]spaces: ${ }^{12}$
\[

$$
\begin{aligned}
& H_{\pi}^{\frac{1}{2}} \Lambda^{(k)}(\partial \Omega):=t^{(k)}\left(H^{1} \Lambda^{(k)}(\Omega)\right) \\
& H_{\rho}^{\frac{1}{2}} \Lambda^{(k)}(\partial \Omega):=\tilde{\star}_{\mathrm{b}} \tilde{t}^{(d-1-k)}\left(H^{1} \tilde{\Lambda}^{(d-1-k)}(\Omega)\right)
\end{aligned}
$$
\]

For a smooth boundary $\partial \Omega$ these spaces are both equal to the usual trace space $H^{\frac{1}{2}} \Lambda^{(k)}(\partial \Omega)$. However, even for a piecewise smooth boundary they are no longer equal [46]. Analogously, we can define the trace spaces of outer forms on $\partial \Omega$ :

$$
\begin{aligned}
& H_{\pi}^{\frac{1}{2}} \tilde{\Lambda}^{(k)}(\partial \Omega):=\tilde{t}^{(k)}\left(H^{1} \tilde{\Lambda}^{(k)}(\Omega)\right) \\
& H_{\rho}^{\frac{1}{2}} \tilde{\Lambda}^{(k)}(\partial \Omega):=\star_{\mathrm{b}} t^{(d-1-k)}\left(H^{1} \Lambda^{(d-1-k)}(\Omega)\right)
\end{aligned}
$$

We assume again a Lipschitz boundary $\partial \Omega$. In [45] it is shown that weak exterior derivatives $d_{\mathrm{b}}$ and $\tilde{d}_{\mathrm{b}}$ on the boundary can be introduced on the spaces $H_{\rho}^{-\frac{1}{2}} \Lambda^{(k)}(\partial \Omega)$ and $H_{\rho}^{-\frac{1}{2}} \tilde{\Lambda}^{(k)}(\partial \Omega)$, which are the topological dual spaces to $H_{\rho}^{\frac{1}{2}} \Lambda^{(k)}(\partial \Omega)$ and $H_{\rho}^{\frac{1}{2}} \tilde{\Lambda}^{(k)}(\partial \Omega)$ with respect to $L^{2} \Lambda^{(k)}(\partial \Omega)$. Note that

$$
\begin{aligned}
& H_{\rho}^{\frac{1}{2}} \Lambda^{(k)}(\partial \Omega) \subset L^{2} \Lambda^{(k)}(\partial \Omega) \subset H_{\rho}^{-\frac{1}{2}} \Lambda^{(k)}(\partial \Omega), \\
& H_{\rho}^{\frac{1}{2}} \tilde{\Lambda}^{(k)}(\partial \Omega) \subset L^{2} \Lambda^{(k)}(\partial \Omega) \subset H_{\rho}^{-\frac{1}{2}} \tilde{\Lambda}^{(k)}(\partial \Omega) .
\end{aligned}
$$

The definition of $d_{\mathrm{b}}$ and $\tilde{d}_{\mathrm{b}}$ can be found in [45]. They are of course generalizations of $d_{\mathrm{b}}: \Lambda^{(k)}(\partial \Omega) \rightarrow \Lambda^{(k+1)}(\partial \Omega)$ and $\tilde{d}_{\mathrm{b}}: \tilde{\Lambda}^{(k)}(\partial \Omega) \rightarrow \tilde{\Lambda}^{(k+1)}(\partial \Omega)$ when $\Omega$ is a manifold with smooth boundary.

It is shown in [45] that it is possible to further extend the trace operators. Let us define

$$
\begin{aligned}
& H \Lambda^{(k)}\left(d_{\mathrm{b}}, \partial \Omega\right):=\left\{a^{(k)} \in H_{\rho}^{-\frac{1}{2}} \Lambda^{(k)}(\partial \Omega) \left\lvert\, d_{\mathrm{b}} a^{(k)} \in H_{\rho}^{-\frac{1}{2}} \Lambda^{(k+1)}(\partial \Omega)\right.\right\}, \\
& H \tilde{\Lambda}^{(k)}\left(\tilde{d}_{\mathrm{b}}, \partial \Omega\right):=\left\{a^{(k)} \in H_{\rho}^{-\frac{1}{2}} \tilde{\Lambda}^{(k)}(\partial \Omega) \left\lvert\, \tilde{d}_{\mathrm{b}} a^{(k)} \in H_{\rho}^{-\frac{1}{2}} \tilde{\Lambda}^{(k+1)}(\partial \Omega)\right.\right\} .
\end{aligned}
$$

The trace operators $t$ and $\tilde{t}$ can be extended to operators

$$
\begin{aligned}
& t^{(k)}: H \Lambda^{(k)}(d, \Omega) \rightarrow H \Lambda^{(k)}\left(d_{\mathrm{b}}, \partial \Omega\right), \\
& \tilde{t}^{(k)}: H \tilde{\Lambda}^{(k)}(\tilde{d}, \Omega) \rightarrow H \tilde{\Lambda}^{(k)}\left(\tilde{d}_{\mathrm{b}}, \partial \Omega\right)
\end{aligned}
$$

It is shown that these operators are linear, continuous and surjective and hence admit a continuous right inverse. Furthermore, the operators satisfy the commutation relations $t \circ d=d_{\mathrm{b}} \circ t$ and $\tilde{t}^{(k+1)} \circ \tilde{d}^{(k)}=\tilde{d}_{\mathrm{b}}^{(k)} \circ \tilde{t}^{(k)}[44,45]$.

Earlier we saw that $H \tilde{\Lambda}^{(d-k)}(\tilde{d}, \Omega)=\star H \Lambda^{(k)}\left(d^{*}, \Omega\right)$. Similarly, we can define the trace space $H^{-\frac{1}{2}} \Lambda^{(k)}\left(d_{\mathrm{b}}^{*}, \partial \Omega\right):=\tilde{\star}_{\mathrm{b}} H^{-\frac{1}{2}} \tilde{\Lambda}^{(d-1-k)}\left(\tilde{d}_{\mathrm{b}}, \partial \Omega\right)$, where $d_{\mathrm{b}}^{*(k)}:=(-1)^{k} \star_{\mathrm{b}}^{-1} \tilde{d}_{\mathrm{b}} \star_{\mathrm{b}}$. These isometries allow us to define a trace on $H \Lambda^{(k)}\left(d^{*}, \Omega\right)$ via $\tilde{t}$. We define $t^{*(k)}$ :

[^13]$H \Lambda^{(k)}\left(d^{*}, \Omega\right) \rightarrow H^{-\frac{1}{2}} \Lambda^{(k-1)}\left(d^{*}, \partial \Omega\right)$ as $t^{*}=\star_{b}^{-1} \tilde{t} \star$. It is shown in [45] that $t^{*}$ is a surjective, linear and continuous map from $H \Lambda^{(k)}\left(d^{*}, \Omega\right)$ to $H^{-\frac{1}{2}} \Lambda^{(k-1)}\left(d_{\mathrm{b}}^{*}, \partial \Omega\right)$. From their definitions it immediately follows that the operators $d_{\mathrm{b}}^{*}$ and $t^{*}$ satisfy the anticommutation relation $d_{\mathrm{b}}^{*(k-1)} \circ t^{*(k)}=-t^{*(k-1)} \circ d_{\mathrm{b}}^{*(k)}$.

From the previous definitions, Stokes' theorem and the Leibniz rule it follows that for all pairs $a^{(k)} \in H^{1} \Lambda^{(k)}(\Omega)$ and $b^{(k+1)} \in H^{1} \Lambda^{(k+1)}(\Omega)$ the integration by parts formula is given by

$$
\left(d a^{(k)}, b^{(k+1)}\right)_{L^{2}(\Omega)}-\left(a^{(k)}, d^{*} b^{(k+1)}\right)_{L^{2}(\Omega)}=\left(t a^{(k)}, t^{*} b^{(k+1)}\right)_{L^{2}(\partial \Omega)} .
$$

The product $(\cdot, \cdot)_{L^{2}(\partial \Omega)}$ can actually be extended to a continuous bilinear form on $H^{-\frac{1}{2}} \Lambda^{(k)}(d, \partial \Omega) \times H^{-\frac{1}{2}} \Lambda^{(k)}\left(d^{*}, \partial \Omega\right)$ by using the left-hand side of the above equation as definition and by the fact that $t$ and $t^{*}$ admit a right inverse that is in $H^{1} \Lambda^{(k)}(\Omega)$. We will denote this extension by $(\cdot, \cdot)_{\mathrm{b}}$. Thus for all $a^{(k)} \in H^{-\frac{1}{2}} \Lambda^{(k)}(d, \partial \Omega), b^{(k+1)} \in$ $H^{-\frac{1}{2}} \Lambda^{(k)}\left(d^{*}, \partial \Omega\right)$ we have

$$
\begin{equation*}
\left(d a^{(k)}, b^{(k+1)}\right)_{\Omega}-\left(a^{(k)}, d^{*} b^{(k+1)}\right)_{\Omega}=\left(t a^{(k)}, t^{*} b^{(k+1)}\right)_{\mathrm{b}} \tag{2.9}
\end{equation*}
$$

The trace operators that we just defined can be used to define Sobolev spaces with zero trace. These spaces are needed in the next section.

Definition 2.33. The Sobolev spaces with zero traces are defined as

$$
\begin{align*}
\stackrel{\circ}{H} \Lambda^{(k)}(d, \Omega) & :=\left\{a^{(k)} \in H \Lambda^{(k)}(d, \Omega) \mid t a^{(k)}=0\right\}  \tag{2.10}\\
\stackrel{\circ}{H} \Lambda^{(k)}\left(d^{*}, \Omega\right) & :=\left\{a^{(k)} \in H \Lambda^{(k)}\left(d^{*}, \Omega\right) \mid t^{*} a^{(k)}=0\right\},  \tag{2.11}\\
\stackrel{\circ}{\Lambda^{(k)}}(\tilde{d}, \Omega) & :=\left\{a^{(k)} \in H \Lambda^{(k)}(\tilde{d}, \Omega) \mid \tilde{t}_{\tilde{a}}^{(k)}=0\right\} . \tag{2.12}
\end{align*}
$$

and

$$
\begin{gathered}
\stackrel{\circ}{H}_{t} \Lambda^{(k)}\left(d, d^{*}, \Omega\right):=\left\{a^{(k)} \in H \Lambda^{(k)}\left(d, d^{*}, \Omega\right) \mid t a^{(k)}=0\right\}, \\
\stackrel{\circ}{H}_{t^{*}} \Lambda^{(k)}\left(d, d^{*}, \Omega\right):=\left\{a^{(k)} \in H \Lambda^{(k)}\left(d, d^{*}, \Omega\right) \mid t^{*} a^{(k)}=0\right\} .
\end{gathered}
$$

### 2.3.3 Hilbert complexes

In a previous section we already briefly introduced the concept of a complex. Complexes play an important role in this thesis, because we will, for example, need them to formulate Helmholtz-Hodge decompositions. Therefore we introduce them in more detail. We mostly follow [43] and [47].

The type of complex we need is called a Hilbert complex.
Definition 2.34. A Hilbert complex is a sequence of Hilbert spaces $W^{(k)}$ and closed ${ }^{13}$, densely-defined ${ }^{14}$ linear operators $A^{(k)}: V^{(k)} \subset W^{(k)} \rightarrow W^{(k+1)}\left(V^{(k)}\right.$ is the domain of $A^{(k)}$ ) such that the range of $A^{(k)}\left(V^{(k)}\right) \subset V^{(k+1)}$ and $A^{(k+1)} \circ A^{(k)} \equiv 0$. A Hilbert complex is called a closed Hilbert complex if the range $A^{(k)}\left(V^{(k)}\right)$ is closed in $W^{(k)}$ for each $k$.

[^14]We have already encountered the Hilbert complex given by

$$
\begin{equation*}
W^{(k)}=L^{2} \Lambda^{(k)}(\Omega), \quad V^{(k)}=H \Lambda^{(k)}(d, \Omega), \quad A^{(k)}=d^{(k)} . \tag{2.13}
\end{equation*}
$$

It can be shown this is a closed Hilbert complex [43]. The second Hilbert complex that we need corresponds to essential boundary conditions and is given by

$$
\begin{equation*}
W^{(k)}=L^{2} \Lambda^{(k)}(\Omega), \quad V^{(k)}=\dot{H} \Lambda^{(k)}(d, \Omega), \quad A^{(k)}=d^{(k)} . \tag{2.14}
\end{equation*}
$$

Note that this is indeed a Hilbert complex, because the trace and the exterior derivative commute. Moreover, it is also a closed Hilbert complex [43].

The complexes above are cochain complexes. We consider the following two chain complexes:

$$
\begin{equation*}
W^{(k)}=L^{2} \Lambda^{(k)}(\Omega), \quad V^{(k)}=H \Lambda^{(k)}\left(d^{*}, \Omega\right), \quad A^{(k)}=d^{*(k)}, \tag{2.15}
\end{equation*}
$$

and

$$
\begin{equation*}
W^{(k)}=L^{2} \Lambda^{(k)}(\Omega), \quad V^{(k)}=\stackrel{\circ}{H} \Lambda^{(k)}\left(d^{*}, \Omega\right), \quad A^{(k)}=d^{*(k)} . \tag{2.16}
\end{equation*}
$$

Note that the fact that $t^{*}$ and $d^{*}$ commute implies that $t^{*} d^{*} a^{(k)}=0$ if $t^{*} a^{(k)}=0$. Therefore, also the second sequence is a chain complex. Both are actually closed Hilbert complexes [43].

From (2.9) it follows that

$$
\begin{align*}
\forall a^{(k)} \in H \Lambda^{(k)}(d, \Omega), & b^{(k+1)} \in \stackrel{\circ}{H} \Lambda^{(k)}\left(d^{*}, \Omega\right): \\
& \left(d a^{(k)}, b^{(k+1)}\right)_{\Omega}=\left(a^{(k)}, d^{*} b^{(k+1)}\right)_{\Omega}  \tag{2.17}\\
\forall a^{(k)} \in \stackrel{\circ}{H} \Lambda^{(k)}(d, \Omega), & b^{(k+1)} \in H \Lambda^{(k)}\left(d^{*}, \Omega\right): \\
& \left(d a^{(k)}, b^{(k+1)}\right)_{\Omega}=\left(a^{(k)}, d^{*} b^{(k+1)}\right)_{\Omega} \tag{2.18}
\end{align*}
$$

From this it follows that the cochain complex (2.13) and the chain complex (2.16) are each others adjoint, and, similarly, the cochain complex (2.14) and the chain complex (2.15) are each others adjoint. Both pairs of adjoint complexes lead to a Helmholtz-Hodge decomposition. These two decompositions will have different boundary conditions.

We derive these in the next section, but first define two important subspaces of the spaces in a complex.

Definition 2.35. Let $\left(V_{(k)}, A_{(k)}\right)$ be a chain complex. The elements of the kernel $Z_{(k)}:=\operatorname{ker} A_{(k)}$ are called the $k$-cycles of the chain complex. The elements in the set $B_{(k)}:=\operatorname{im} A_{(k+1)}$ are called the $k$-boundaries.

Analogously, given a cochain complex $\left(V^{(k)}, A^{(k)}\right)$, the elements of $Z^{(k)}:=\operatorname{ker} A^{(k)}$ are called the $k$-cocycles and the elements of $B^{(k)}:=\operatorname{im} A^{(k-1)}$ are called the $k$-coboundaries.

When the spaces $V^{(k)}$ are spaces of differential form and $A^{(k)}=d^{(k)}$ is the exterior derivative, the differential $k$-forms in $B^{(k)}$ are also called exact and the $k$-forms in $Z^{(k)}$ are also known as closed.

Note that by definition of a complex we always have $B_{(k)} \subset Z_{(k)}$ and $B^{(k)} \subset Z^{(k)}$.

### 2.3.4 Helmholtz-Hodge decompositions

Now, we consider the Helmholtz-Hodge decompositions. These are decompositions of an arbitrary differential form $u^{(k)} \in L^{2} \Lambda^{(k)}(\Omega)$ in three terms:

$$
u^{(k)}=d a^{(k-1)}+d^{*} b^{(k+1)}+h^{(k)},
$$

where $h^{(k)}$ is known as a harmonic form. Harmonic forms are absent when $\Omega$ is smoothly contractible to a point.

The boundary conditions that $a^{(k-1)}, b^{(k+1)}$ and $h^{(k)}$ satisfy are the point at which the two Helmholtz-Hodge decompositions that we will introduce differ.

These decompositions have important consequences, for example, because it says something about the existence of a, possibly unique, "vector potential" for a divergencefree velocity field. Furthermore, it can be used to calculate the velocity field from a corresponding vorticity field when the flow is incompressible.

This subsection is again mostly based on [43]. However, we present some things differently, because it better fits with the discretization approach in Chapter 3.

To start we set the notation. For the cochain complex $\left(H \Lambda^{(k)}(d, \Omega), d^{(k)}\right)$, we denote the subspaces of exact and closed forms as

$$
\begin{aligned}
& \mathcal{B}^{(k)}:=\operatorname{im}\left(d^{(k-1)}: H \Lambda^{(k-1)}(d, \Omega) \rightarrow H \Lambda^{(k)}(d, \Omega)\right), \\
& \mathcal{Z}^{(k)}:=\operatorname{ker}\left(d^{(k)}: H \Lambda^{(k)}(d, \Omega) \rightarrow H \Lambda^{(k+1)}(d, \Omega)\right)
\end{aligned}
$$

and for the cochain complex $\left({ }_{\circ} \Lambda^{(k)}(d, \Omega), d^{(k)}\right)$ we denote the subspaces of exact and closed forms by $\dot{\mathcal{B}}^{(k)}$ and $\dot{\mathcal{Z}}^{(k)}$. Similarly for the chain complex $\left(H \Lambda^{(k)}\left(d^{*}, \Omega\right), d^{*(k)}\right)$ we denote the spaces of $k$-cycles and $k$-boundaries by $\mathcal{B}^{*(k)}$ and $\mathcal{Z}^{*(k)}$, and, for the chain complex $\left(\dot{H} \Lambda^{(k)}\left(d^{*}, \Omega\right), d^{*(k)}\right)$ by $\dot{\mathcal{B}}^{(*(k)}$ and $\dot{\mathcal{Z}}^{*(k)}$.

The inclusions $\mathcal{B}_{(k)} \subset \mathcal{Z}_{(k)}$ and $\mathcal{B}^{(k)} \subset \mathcal{Z}^{(k)}$ are abstract dimensional-independent reformulations of the familiar facts that the curl of a gradient and the divergence of a curl are always zero.

Although, the curl of a gradient is always zero, it does not necessarily hold that every vector field with zero curl is the gradient of some scalar function. Whether this holds or not depends on the topology of $\Omega$. The Poincaré Lemma expresses when this holds.

Lemma 2.2 (Poincaré). [37, p.225] If $\Omega$ is smoothly contractible to a point $x \in \Omega,{ }^{15}$ then every closed form on $\Omega$ is exact.

Thus, for an incompressible flow, $u^{(2)}$ with $d u^{(2)}=0$, in any smoothly contractible domain, there exists a "vector potential", i.e., a 1 -form $a^{(1)}$ such that $u^{(2)}=d a^{(1)}$. Such a vector potential is not unique, because $d\left(a^{(1)}+d b^{(0)}\right)=d a^{(1)}=u^{(2)}$ for any 0 -form $b^{(0)}$.

Any star-shaped $\Omega$ is smoothly contractible, while any $\Omega$ with holes in it is not. A relevant example in which the flow domain $\Omega$ is not smoothly contractible is the flow around an airfoil in 2D or around an airplane in 3D.

The adjointness of $\left(H \Lambda^{(k)}(d, \Omega), d^{(k)}\right)$ and $\left(\stackrel{\circ}{H} \Lambda^{(k)}\left(d^{*}, \Omega\right), d^{*(k)}\right)$, and the adjointness of ( $\left.\dot{H} \Lambda^{(k)}(d, \Omega), d^{(k)}\right)$ and $\left(H \Lambda^{(k)}\left(d^{*}, \Omega\right), d^{*(k)}\right)$ lead to the following result.

[^15]Lemma 2.3. The subspaces of exact and closed forms for the complex $\left(d, H \Lambda^{(k)}(d, \Omega)\right)$ and its adjoint complex $\left(d^{*}, \stackrel{\circ}{H} \Lambda^{(k)}(d, \Omega)\right)$ satisfy

$$
\mathcal{B}^{(k) \perp}=\dot{\mathcal{Z}}^{*(k)}, \quad \dot{\mathcal{B}}^{*(k) \perp}=\mathcal{Z}^{(k)}
$$

Similarly, the subspaces of exact and closed forms for the complex $\left(d^{*}, H \Lambda^{(k)}(d, \Omega)\right)$ and its adjoint complex $\left(d, \stackrel{\circ}{H} \Lambda^{(k)}(d, \Omega)\right)$ satisfy

$$
\mathcal{B}^{*(k) \perp}=\dot{\mathcal{Z}}^{(k)}, \quad \dot{\mathcal{B}}^{(k) \perp}=\mathcal{Z}^{*(k)}
$$

Proof. This follows immediately from the following standard result (see for example [48, Cor. 2.18]). For a densely defined and closed unbounded linear operator $A: D(A) \subset E \rightarrow$ $F$ and its adjoint $A^{*}: D\left(A^{*}\right) \subset F \rightarrow E$ we have $\operatorname{im}(A)^{\perp}=\operatorname{ker}\left(A^{*}\right)$ and $\operatorname{im}\left(A^{*}\right)^{\perp}=\operatorname{ker}(A)$.

Applying it to $\left(H \Lambda^{(k)}(d, \Omega), d^{(k)}\right)$ and its adjoint $\left({ }_{\circ} \Lambda^{(k)}\left(d^{*}, \Omega\right), d^{*(k)}\right)$ gives the first result. Applying it to $\left(\stackrel{\circ}{H} \Lambda^{(k)}(d, \Omega), d^{(k)}\right)$ and its adjoint $\left(H \Lambda^{(k)}\left(d^{*}, \Omega\right), d^{*(k)}\right.$ ) gives the second result.

From these orthogonality relations the Helmholtz-Hodge decompositions follow.
Theorem 2.4 (Helmholtz-Hodge decompositions). Define the following spaces of harmonic forms

$$
\begin{aligned}
& \mathcal{H}_{t^{*}}^{(k)}:=\left\{a^{(k)} \in H \Lambda^{(k)}\left(d, d^{*}, \Omega\right) \mid d a^{(k)}=0, d^{*} a^{(k)}=0, t^{*} a^{(k)}=0\right\}, \\
& \mathcal{H}_{t}^{(k)}:=\left\{a^{(k)} \in H \Lambda^{(k)}\left(d, d^{*}, \Omega\right) \mid d a^{(k)}=0, d^{*} a^{(k)}=0, t a^{(k)}=0\right\} .
\end{aligned}
$$

We have the following two orthogonal decompositions of $L^{2} \Lambda^{(k)}(\Omega)$ [43]:

$$
\begin{equation*}
L^{2} \Lambda^{(k)}(\Omega)=\mathcal{B}^{(k)} \oplus \dot{\mathcal{B}}^{*(k)} \oplus \mathcal{H}_{t^{*}}^{(k)}=\mathcal{B}^{(k)} \oplus \star^{-1} \dot{\mathcal{B}}^{(d-k)} \oplus \mathcal{H}_{t^{*}}^{(k)} \tag{2.19}
\end{equation*}
$$

and

$$
\begin{equation*}
L^{2} \Lambda^{(k)}(\Omega)=\mathcal{\mathcal { B }}^{(k)} \oplus \mathcal{B}^{\star(k)} \oplus \mathcal{H}_{t}^{(k)}=\dot{\mathcal{B}}^{(k)} \oplus \star^{-1} \tilde{\mathcal{B}}^{(d-k)} \oplus \mathcal{H}_{t}^{(k)} \tag{2.20}
\end{equation*}
$$

where $\dot{\mathcal{B}}^{(d-k)}:=\tilde{d}\left(\dot{H}^{(k)}(\tilde{d}, \Omega)\right)$ and $\tilde{\mathcal{B}}^{(d-k)}:=\tilde{d}\left(H \tilde{\Lambda}^{(k)}(\tilde{d}, \Omega)\right)$.
Proof. We briefly repeat from [43]. By Lemma 2.3 we have $L^{2} \Lambda^{(k)}(\Omega)=\mathcal{B}^{(k)} \oplus \dot{\mathcal{Z}}^{*(k)}$. Similarly $\dot{\mathcal{B}}^{*(k)}$ is a closed subspace of $\dot{\mathcal{Z}}^{*(k)}$, therefore, by Lemma 2.3 again, the orthogonal complement to $\dot{\mathcal{B}}^{*(k)}$ within $\dot{\mathcal{Z}}^{*(k)}$ is given by $\dot{\mathcal{Z}}^{*(k)} \cap \mathcal{Z}^{(k)}=\mathcal{H}_{t^{*}}^{(k)}$. This implies that we have the orthogonal decomposition $\dot{\mathcal{Z}}^{*(k)}=\dot{\mathcal{B}}^{*(k)} \oplus \mathcal{H}_{t^{*}}^{(k)}$ and therefore (2.19). The fact that $\dot{\mathcal{B}}^{*(k)}=*^{-1} \dot{\mathcal{B}}^{(d-k)}$ directly follows from the fact that $\star \stackrel{\circ}{H} \Lambda^{(k)}\left(d^{*}, \Omega\right)=\stackrel{\circ}{H} \tilde{\Lambda}^{(d-k)}(\tilde{d}, \Omega)$ and the definition of $d^{*}$. The decomposition (2.20) follows analogously.

We write (2.19) and (2.20) in the second form, because this is the form of the Helmholtz-Hodge decompositions which we will mimic in a discrete setting in Chapter 5.

We denote the dimensions of the spaces of harmonic forms $\mathcal{H}_{t^{*}}^{(k)}$ and $\mathcal{H}_{t}^{(k)}$ by $\beta_{t^{*}}^{(k)}$ and $\beta_{t}^{(k)}$, respectively. These numbers are called Betti numbers. Note that $\star$ is an isometry
between $\mathcal{H}_{t^{*}}^{(k)}$ and $\mathcal{H}_{t}^{(d-k)}$. Therefore we have $\beta_{t^{*}}^{(k)}=\beta_{t}^{(d-k)}$. The Betti numbers are topological invariants of the domain [39]. In 3D $\beta_{t^{*}}^{(0)}$ is equal to the number of connected components of $\Omega, \beta_{t^{*}}^{(1)}$ is equal to the number of handles, $\beta_{t^{*}}^{(2)}$ is equal to the number of voids and $\beta_{t^{*}}^{(3)}$ is always zero. In 2D $\beta_{t^{*}}^{(0)}$ is again equal to the number of connected components, $\beta_{t^{*}}^{(1)}$ is equal to the number of holes and $\beta_{t^{*}}^{(2)}$ is always zero.

### 2.4 Incompressible Navier-Stokes equations

In the last part of this chapter we will rewrite the incompressible Navier-Stokes equations (1.2) in terms of differential forms. The alternative formulations that we introduce will be the form in which we will discretize the equations in Chapter 4.

Using the incompressibility constraint we can rewrite the convective term in (1.2a) as

$$
\begin{aligned}
\nabla \cdot\left(u_{(1)} \otimes\left(\rho u_{(1)}\right)\right) & =\rho\left(u_{(1)} \cdot \nabla\right) u_{(1)} \\
& =\rho \omega_{(1)} \times u_{(1)}+\nabla\left(\frac{\rho u_{(1)} \cdot u_{(1)}}{2}\right),
\end{aligned}
$$

where we adopted the new notation for vector fields and $\omega_{(1)}=\nabla \times u_{(1)}$ is the vorticity vector. If we define the total pressure as $q:=p+u_{(1)} \cdot u_{(1)} / 2$ and use the relation

$$
\nabla \cdot\left(\nabla u_{(1)}\right)=\Delta u_{(1)}=\nabla\left(\nabla \cdot u_{(1)}\right)-\nabla \times \omega_{(1)}=-\nabla \times \omega_{(1)}
$$

we obtain

$$
\frac{\partial \rho u_{(1)}}{\partial t}+\rho \omega_{(1)} \times u_{(1)}=-\mu \nabla \times \omega_{(1)}+f_{(1)} .
$$

In this form the momentum equation only involves curl operators as spatial differential operators, which we can express in terms of the exterior derivative. Furthermore, the cross product can be expressed using the exterior product.

We identify the vector field $u_{(1)}$ with the outer 2-form $\tilde{u}^{(2)}:=\star b u_{(1)}$ and define the total pressure form as $q^{(0)}=p+\rho u_{(1)} \cdot u_{(1)} / 2$. In terms of these forms the 3-dimensional incompressible Navier-Stokes equations are given by

$$
\begin{align*}
\frac{\partial \rho \tilde{u}^{(2)}}{\partial t}+\rho \tilde{\omega}^{(1)} \wedge \tilde{\star} \tilde{u}^{(2)}+\mu \tilde{d} \tilde{\omega}^{(1)}+\star d q^{(0)} & =\tilde{f}^{(2)},  \tag{2.21a}\\
\tilde{\omega}^{(1)}-\star d \tilde{\star} \tilde{u}^{(2)} & =0,  \tag{2.21b}\\
\tilde{d} \tilde{u}^{(2)} & =0 . \tag{2.21c}
\end{align*}
$$

Here $\tilde{\omega}^{(1)}$ is the outer vorticity form.
Alternatively, we could also represent the velocity field as inner 1-form $u^{(1)}:=\tilde{\star} \tilde{u}^{(2)}$ and the vorticity as inner 2 -form $\omega^{(2)}:=\tilde{\star} \tilde{\omega}^{(1)}$. In terms of these variables the Navier-

Stokes equations are given by

$$
\begin{align*}
\frac{\partial \rho u^{(1)}}{\partial t}+\rho \tilde{\star}\left(\star \omega^{(2)} \wedge u^{(1)}\right)+\mu \tilde{\star} \tilde{d} \star \omega^{(2)}+d q^{(0)} & =f^{(1)},  \tag{2.22a}\\
\omega^{(2)}-d u^{(1)} & =0,  \tag{2.22b}\\
\tilde{d} \star u^{(1)} & =0 . \tag{2.22c}
\end{align*}
$$

When we discretize the Navier-Stokes equations, formulations (2.21) and (2.22) will result in different numerical methods. It is therefore important to consider both. This is discussed at length in Chapter 4.

We will also consider discretizations of the 2-dimensional Navier-Stokes equations. The 2-dimensional versions of formulations (2.21) and (2.22) are given by

$$
\begin{align*}
& \frac{\partial \rho \tilde{u}^{(1)}}{\partial t}+\rho \tilde{\omega}^{(0)} \wedge \tilde{\star} \tilde{u}^{(1)}+\mu \tilde{d}_{\tilde{\omega}^{(0)}+\star d q^{(0)}}=\tilde{f}^{(1)},  \tag{2.23a}\\
& \tilde{\omega}^{(0)}+\star d \tilde{\star} \tilde{u}^{(1)}=0,  \tag{2.23b}\\
& \tilde{d} \tilde{u}^{(1)}=0 . \tag{2.23c}
\end{align*}
$$

and

$$
\begin{align*}
\frac{\partial \rho u^{(1)}}{\partial t}+\rho \tilde{\star}\left(\star \omega^{(2)} \wedge u^{(1)}\right)-\mu \tilde{\star} \tilde{d} \star \omega^{(2)}+d q^{(0)} & =f^{(1)},  \tag{2.24a}\\
\omega^{(2)}-d u^{(1)} & =0,  \tag{2.24b}\\
\tilde{d} \star u^{(1)} & =0 . \tag{2.24c}
\end{align*}
$$

The inner and outer forms are related by $\underline{u}^{(1)}=\star u^{(1)}, \tilde{\omega}^{(0)}=\star \omega^{(0)}$ and $\tilde{f}^{(1)}=\star f^{(1)}$. Note that the difference in some minus signs between the 3-dimensional and 2-dimensional formulations result from the fact that $\tilde{\star}^{(1)} \star^{(1)} a^{(1)}=-a^{(1)}$ for all $a^{(1)}$ in 2D, while in 3D we always have $\tilde{\star}=\star^{-1}$.

## Mimetic Discretization

### 3.1 The computational mesh, chains and cochains

In the previous chapter we saw that a differential $k$-form and a $k$-dimensional manifold together produce a real number. Stokes' theorem can be written as a relation between real numbers produced by the duality pairing of differential forms and manifolds. For example, let $\tilde{u}^{(2)}$ be the velocity 2 -form and let $\tilde{M}_{(3)}$ be any 3 -dimensional volume, then Stokes' theorem says

$$
\begin{equation*}
\left\langle\tilde{M}_{(3)}, \tilde{d}^{(2)} \tilde{u}^{(2)}\right\rangle=\sum_{\tilde{M}_{(2)} \in \mathscr{\partial}^{1} \tilde{M}_{(3)}}\left\langle\tilde{M}_{(2)}, u^{(2)}\right\rangle, \tag{3.1}
\end{equation*}
$$

where $\partial^{1} \tilde{M}_{(3)}$, the set of sides of $\tilde{M}_{(3)}$, is defined in Section 2.2.6.
We can partition our flow domain $\Omega$ in volumes like this and then for each of these volumes (3.1) holds. The 2-dimensional sides $\tilde{M}_{(2)}$ are then the interfaces between the different volumes. We obtain a set of equations, one equation for each volume.

Let us order the real numbers $\left\langle\tilde{M}_{(2)}, u^{(2)}\right\rangle$ in a vector $\boldsymbol{u}^{(2)}$. We know that the fluid is incompressible, hence $\tilde{0}^{(3)}=\tilde{d} \tilde{u}^{(2)}$. The equations are linear, therefore we can write them in a matrix system as $\mathbf{0}^{(3)}=\mathbb{D}^{(2)} \boldsymbol{u}^{(2)}$. This is equivalent to the discretization of the incompressibility constraint in the MAC method.

More generally, we see that when we discretize differential $k$-forms by integrating them over $k$-manifolds in some mesh, then by Stokes' theorem we can express the discrete variables $\boldsymbol{b}^{(k+1)}$ corresponding to a $(k+1)$-form $b^{(k+1)}=d a^{(k)}$ as $\boldsymbol{b}^{(k+1)}=\mathbb{D}^{(k)} \boldsymbol{a}^{(k)}$ where $\boldsymbol{a}^{(k)}$ is the vector of discrete variables corresponding to the $k$-form $a^{(k)}$. As a result not only the 3 -dimensional cells and 2 -dimensional mesh faces play an important role in a mimetic discretization, but also the mesh edges and the mesh vertices.

To make this more precise and clear, we start with a definition of the computational mesh.

### 3.1.1 The computational mesh

Definition 3.1. A $k$-cell $\sigma_{(k)}$ is a $k$-dimensional manifold with corners in $\mathbb{R}^{d}$ that is homeomorphic to the closed $k$-dimensional ball and for which $\partial^{1} \sigma_{(k)}$ is a finite set of $(k-1)$-cells. A 0 -cell $\sigma_{(0)}$ is simply a point in $\mathbb{R}^{d}$.

A $k$-cell with an inner orientation, we call an inner-oriented $k$-cell. Similarly, a $k$-cell in $\mathbb{R}^{d}$, with an outer orientation in $\mathbb{R}^{d}$, we call an outer-oriented $k$-cell.


Figure 3.1: The cube $M_{(3)}$ is an inner-oriented 3-cell. Here it is implied that the elements of $\partial^{1} M_{(3)}$ are inner-oriented 2-cells again.

The definition implies that for a $k$-cell $\sigma_{(k)}$ the sets $\partial^{l} \sigma_{(k)}$ with $l \leq k$ (as defined in Section 2.2.6) are all finite as well. Note that the just defined $k$-cells can even be curved. However, the $k$-cells are often demanded to be flat.

Definition 3.2. A polytopal $k$-cell is a $k$-cell which is contained in a $k$-dimensional hyperplane of $\mathbb{R}^{d}$ and for which $\partial^{1} \sigma_{(k)}$ is a collection of polytopal $k$-cells.

The computational mesh is a collection of $k$-cells with $k=0,1, \ldots, d$. For example, a 3 -dimensional mesh is collection of 3 -cells (volumes), 2 -cells (faces), 1-cells (edges) and 0 -cells (vertices). We demand this collection of cells to have some sensible properties, for example, we demand that lower dimensional cells lie in the boundary of higher dimensional meshes. More precisely we use the following definition.

Definition 3.3. A mesh $\mathcal{M}$ on $\Omega$ is a finite collection of $k$-cells, with $k=0,1, \ldots, d$, submanifolds of $\Omega$ such that the following properties hold:
(i) The $d$-cells cover $\Omega$.
(ii) For distinct $k$-cells $\sigma_{(k)}$ and $\tau_{(k)}, \sigma_{(k)} \cap \tau_{(k)}$ is either empty or a union of lower dimensional cells.
(iii) For distinct $\sigma_{(k)}$ and $\tau_{(l)}$ with $l \leq k$ and $\sigma_{(k)} \cap \tau_{(l)} \neq \varnothing$, we have $\sigma_{(k)} \cap \tau_{(l)} \subset \partial \sigma_{(k)}$.


Figure 3.2: Examples of implications of items (ii) and (iii) of Definition 3.3. We assume $\sigma_{(3)}, \tau_{(3)} \in$ $\mathcal{M}_{(3)}$. For the situation on the left (ii) implies that $\eta_{(1)}$ must be part of the mesh. For the situation on the right ( $i i i$ ) implies that such $\eta_{(1)}$ cannot be part of the mesh

We denote the mesh as $\mathcal{M}:=\left\{\mathcal{M}_{(0)}, \ldots, \mathcal{M}_{(d)}\right\}$, with $\mathcal{M}_{(k)}$ the collection of $k$-cells $(k=0,1, \ldots, d)$ and we denote the number of $k$-cells by $N_{(k)}$.

A mesh for which all its cells have an inner (outer) orientation is called an inneroriented (outer-oriented) mesh.

In Figure 3.2 we present two examples to clarify Definition 3.3. Examples, of meshes that satisfy these properties, are the usual Cartesian and simplicial meshes, but also curved versions of these, and polytopal meshes.

A polytopal $k$-cell $\sigma_{(k)} \in \mathcal{M}_{(k)}(\Omega)$ has a size and orientation and as a result we can represent it by a $k$-vector. This $k$-vector is most conveniently calculated using a simplicial submesh for $\sigma_{(k)}$. The following index sets of tuples of cells will be helpful in this calculation and later on:

$$
\begin{gathered}
I^{\sigma_{(k)}:=\left\{\left(\tau_{(0)}, \ldots, \tau_{(k-1)}\right) \in \mathcal{M}_{(0)}(\Omega) \times \cdots \times \mathcal{M}_{(k-1)}(\Omega)\right.} \begin{array}{r}
\left.\mid \tau_{(0)} \in \partial \tau_{(1)}, \ldots, \tau_{(k-1)} \in \partial \sigma_{(k)}\right\}, \\
I_{\sigma_{(l)}}:=\left\{\left(\tau_{(l+1)}, \ldots, \tau_{(d)}\right) \in \mathcal{M}_{(l+1)}(\Omega) \times \cdots \times \mathcal{M}_{(d)}(\Omega)\right. \\
\left.\mid \sigma_{(l)} \in \partial \tau_{(l+1)}, \ldots, \tau_{(d-1)} \in \partial \tau_{(d)}\right\}, \\
I_{\sigma_{(l)}}^{\sigma_{(k)}}:=\left\{\left(\tau_{(l+1)}, \ldots, \tau_{(k-1)}\right) \in \mathcal{M}_{(l+1)}(\Omega) \times \cdots \times \mathcal{M}_{(k-1)}(\Omega)\right. \\
\left.\mid \sigma_{(l)} \in \partial \tau_{(l+1)}, \ldots, \tau_{(k-1)} \in \partial \sigma_{(k)}\right\} .
\end{array}
\end{gathered}
$$

Lemma 3.1. Let $\mathcal{M}$ be a mesh of polytopal cells. An inner-oriented $k$-cell $\sigma_{(k)} \in \mathcal{M}_{(k)}$ is represented by the inner $k$-vector

$$
\begin{equation*}
\sigma_{(k)}:=\sum_{\substack{\left(\tau_{(k-1)}, \ldots, \tau_{(0)}\right) \\ \epsilon I^{\prime}(k)}} \frac{o_{\sigma_{(k)}, \tau_{(k-1)}, \ldots, \tau_{(0)}}}{k!}\left(x_{(1)}^{\tau_{(k-1)}}-x_{(1)}^{\sigma_{(k)}}\right) \wedge \cdots \wedge\left(x_{(1)}^{\tau_{(0)}}-x_{(1)}^{\tau_{(1)}}\right), \tag{3.2}
\end{equation*}
$$

with $o_{\sigma_{(k)}, \tau_{(k-1)}, \ldots, \tau_{(0)}}:=o_{\sigma_{(k)} \tau_{(k-1)}} \cdots o_{\tau_{(1)} \tau_{(0)}} o_{\tau_{(0)}}$, and, where $x_{(1)}^{\tau_{(l)}}$ is the center ${ }^{1}$ of $\tau_{(l)}$ and hence $x_{(1)}^{\tau_{(k-1)}}-x_{(1)}^{\sigma_{(k)}}$ is the vector that points from the center of $\sigma_{(k)}$ to the center of $\tau_{(k-1)}$. Furthermore, $o_{\tau_{(0)}}$ is the orientation of $\tau_{(0)}$, i.e., $o_{\tau_{(0)}}=-1$ if $\tau_{(0)}$ is a sink (ingoing arrows, see Figure 2.8) and $o_{\tau_{(0)}}=+1$ is a source (outgoing arrows). Note that we use $\sigma_{(k)}$ both for the cell as an element of $\mathcal{M}_{(k)}(\Omega)$ and for the corresponding $k$-vector.

Proof. The term in the sum corresponding to $\left(\tau_{(k-1)}, \ldots, \tau_{(0)}\right) \in I^{\sigma_{(k)}}$ is the $k$-vector with as size the size of the simplex and with as vertices the barycenters of $\tau_{(k-1)}, \ldots, \tau_{(0)}$. The simplices in the sum constitute the barycentric subdivision of $\sigma_{(k)}$. So, it is clear that formula (3.2) gives the correct size for $\sigma_{(k)}$.

Taking into account the definition of the induced orientation (Definition 2.28) we see that the following recursive relation should hold:

$$
\sigma_{(l)}=\sum_{\tau_{(l-1)} \in \partial \sigma_{(l)}} \frac{1}{l} o_{\sigma_{(l)} \tau_{(l-1)}}\left(x_{(1)}^{\tau_{(l-1)}}-x_{(1)}^{\sigma_{(l)}}\right) \wedge \tau_{(l-1)}
$$

because $x_{(1)}^{\tau_{(l-1)}}-x_{(1)}^{\sigma_{(l)}}$ points out of $\sigma_{(l)}$ and hence $\tau_{(l-1)}$ has the orientation induced by $\sigma_{(l)}$ if $o_{\sigma_{(l)} \tau_{(l-1)}}=1$ and the other orientation otherwise. This recursive relation implies (3.2).

Examples for simplicial subcells of different dimensions are shown in Figure 3.3. We will call a $k$-cell $\sigma_{(k)}$ well-centered if the center $x_{(1)}^{\sigma_{(k)}}$ is contained in $\sigma_{(1)}$. (Note that this depends on both the shape of the cell and the choice for the center.) Formula (3.2) holds for any choice of cell center as long as the cell center lies in the same $k$-dimensional hyperplane in $\mathbb{R}^{d}$. However, if all the cells in the sum (3.2) are well-centered then all the terms in the sum have the same orientation and the sum in fact defines a simplicial subdivision of $\sigma_{(k)}$.

### 3.1.2 The chain complex

The $k$-cells in the mesh provide a finite number of integration domains for the $k$-forms to be integrated on. We want Stokes' theorem to apply in all these cases. We saw in Section 2.2 that Stokes' theorem can alternatively be stated as the fact that in the duality pairing between $k$-forms and $k$-dimensional manifolds the exterior derivative and the boundary operator are each others adjoint. To discretize the exterior derivative we first introduce a boundary operator for the mesh and a duality pairing. The discrete exterior derivative will then be implied because it is adjoint to the boundary operator in this duality pairing.

The boundary of a $k$-cell $\sigma_{(k)}$ is the finite set $\partial^{1} \sigma_{(k)}$ of $(k-1)$-cells. To be able to express this boundary in terms of the mesh elements in $\mathcal{M}$, the $(k-1)$-cells in $\partial^{1} \sigma_{(k)}$ should be elements of $\mathcal{M}_{(k-1)}$.

[^16]

Figure 3.3: Here one of the terms in the cell $k$-vector of Lemma 3.1 is given for a 1 -cell, 2 -cell and 3 -cell.

Definition 3.4. A cell complex on $\Omega$ is a mesh $\mathcal{M}$ on $\Omega$ such that for each cell $\sigma_{(k)} \in \mathcal{M}_{(k)}$ the set $\partial^{1} \sigma_{(k)}$ consists of $(k-1)$-cells in $\mathcal{M}_{(k-1)}$, for $k=0,1, \ldots, d$. (These $(k-1)$-cells need not to have the orientation induced by $\sigma_{(k)}$.)

For the integral of $d^{(k-1)} a^{(k-1)}$ over a $k$-cell $\sigma_{(k)}$ in a cell complex $\mathcal{M}$ Stokes' theorem says that $\left\langle\sigma_{(k)}, d^{(k-1)} a^{(k-1)}\right\rangle$ is equal to the integral of $a^{(k-1)}$ over the elements of $\partial^{1} \sigma_{(k)}$, with the orientation induced by $\sigma_{(k)}$. If we write $-\sigma_{(k-1)}$ for the cell $\sigma_{(k-1)}$ but with opposite orientation, then we can write

$$
\partial \sigma_{(k)}=\sum_{\sigma_{(k-1)} \in \partial^{1} \sigma_{(k)}} o_{\sigma_{(k)} \sigma_{(k-1)}} \sigma_{(k-1)},
$$

where $o_{\sigma_{(k)}, \sigma_{(k-1)}}$ equals +1 if $\sigma_{(k-1)} \in \mathcal{M}_{(k-1)}$ has the orientation induced by that of $\sigma_{(k)}$ and -1 if it has the opposite orientation. We see that the boundary of a $k$-cell is, with respect to integration, a linear combination of $(k-1)$-cells.

Definition 3.5. Let $\mathcal{M}$ be a mesh on $\Omega$. A $k$-chain $\boldsymbol{a}_{(k)}$ is a formal linear combination of $k$-cells:

$$
\boldsymbol{a}_{(k)}=\sum_{i} a^{i} \sigma_{(k), i}
$$

where the $\sigma_{(k), i}$ are $k$-cells in $\mathcal{M}_{(k)}$ and the $a^{i}$ are real coefficients. The space of $k$-chains is the free ${ }^{2}$ abelian group generated by the $k$-cells in $\mathcal{M}_{(k)}$, which we denote by $C_{(k)}(\Omega)$. Every $k$-cell is a $k$-chain also, we will therefore from now on write the $k$-cells in this context in bold as $\boldsymbol{\sigma}_{(k), i}$ as well. The $k$-cells form a basis for $C_{(k)}(\Omega)$. We call this basis the canonical basis for $C_{(k)}(\Omega)$.

We see that for a $k$-chain $\boldsymbol{a}_{(k)}$ in a cell complex on $\Omega$, the boundary $\partial \boldsymbol{a}_{(k)}$ is an element of $C_{(k-1)}(\Omega)$. We define the boundary operator now by linear extension to the space of chains.

[^17]Definition 3.6. Let $\mathcal{M}$ be a cell complex on $\Omega$. The boundary operator $\partial_{(k)}: C_{(k)}(\Omega) \rightarrow$ $C_{(k-1)}(\Omega)$ is defined by

$$
\partial_{(k)} \boldsymbol{a}_{(k)}:=\partial_{(k)}\left(\sum_{i} a^{i} \boldsymbol{\sigma}_{(k), i}\right)=\sum_{i} a^{i} \partial_{(k)} \boldsymbol{\sigma}_{(k), i},
$$

where $\boldsymbol{a}_{(k)}=\sum_{i} a^{i} \boldsymbol{\sigma}_{(k), i}$ and the boundary of a $k$-cell is defined according to

$$
\partial_{(k)} \boldsymbol{\sigma}_{(k)}=\sum_{\sigma_{(k-1)} \in \partial^{1} \sigma_{(k)}} o_{\sigma_{(k)} \sigma_{(k-1)}} \boldsymbol{\sigma}_{(k-1)},
$$

where $o_{\sigma_{(k)} \sigma_{(k-1)}}$ equals +1 if $\sigma_{(k-1)} \in \mathcal{M}_{(k-1)}$ has the orientation induced by that of $\sigma_{(k)}$ and -1 if it has the opposite orientation.

If a mesh $\mathcal{M}$ is not a cell complex, then there exists a cell $\sigma_{(k)}$ for which $\partial^{1} \sigma_{(k)}$ contains a $(k-1)$-cell that is not an element of $\mathcal{M}_{(k-1)}$. As a result it is not possible to define the boundary operator $\partial_{(k)}$ from $C_{(k)}(\Omega)$ to $C_{(k-1)}(\Omega)$, because $\partial_{(k)} \sigma_{(k)}$ is not an element of $C_{(k-1)}(\Omega)$. Later on in Section 3.2, we will introduce the dual mesh. The dual mesh is an example of a mesh that is not a cell complex.

Already in Section 2.2 we saw that taking the boundary of a boundary gives the empty set. Similarly, applying the boundary operator successively gives a zero chain.

Proposition 3.1. Let $\mathcal{M}$ be a cell complex on $\Omega$. The sequence

$$
C_{(0)}(\Omega) \stackrel{\partial_{(1)}}{\longleftarrow} C_{(1)}(\Omega) \stackrel{\partial_{(2)}}{\longleftrightarrow} \cdots \quad \stackrel{\partial_{(d)}}{\longleftrightarrow} C_{(d)}(\Omega),
$$

is a chain complex.
Proof. We need to show that $\partial_{(k-1)} \circ \partial_{(k)} \boldsymbol{a}_{(k)}=\mathbf{0}_{(k-2)}$ for every $k$-chain in $\boldsymbol{a}_{(k)} \in C_{(k)}(\Omega)$. By linearity we only need to show it for the $k$-cells $\boldsymbol{\sigma}_{(k)}$. Applying the boundary operator twice gives

$$
\partial_{(k-1)} \partial_{(k)} \sigma_{(k)}=\sum_{\sigma_{(k-1)} \in \partial^{1} \sigma_{(k)} \sigma_{(k-2)} \in \partial^{1} \sigma_{(k-1)}} o_{\sigma_{(k)} \sigma_{(k-1)}} o_{\sigma_{(k-1)} \sigma_{(k-2)}} \boldsymbol{\sigma}_{(k-2)},
$$

where $o_{\sigma_{(k)} \sigma_{(k-1)}}$ and $o_{\sigma_{(k-1)} \sigma_{(k-2)}}$ are equal to +1 if the orientation of the cells in subscript agrees and -1 otherwise.

Consider a coordinate chart $\phi: U \rightarrow \mathbb{R}_{\geq 0}^{k}$ for $\sigma_{(k)}$ around a point $x \in \sigma_{(k-2)}$ for one of the $(k-2)$-cells in the summation, with $x \in \sigma_{(k-2)} \subset U \subset \sigma_{(k)}$. By Definition 2.15 and the discussion in Section 2.2.6, $\phi\left(\sigma_{(k-2)}\right) \subset\left\{\left(x^{1}, \ldots, x^{k}\right) \in \mathbb{R}_{\geq 0}^{k} \mid x^{i}=0, x^{j}=0\right\}$ for some $1 \leq i<j \leq k$. Now let

$$
\begin{aligned}
U_{i} & :=\phi^{-1}\left(\left\{\left(x^{1}, \ldots, x^{k}\right) \in \phi\left(\sigma_{(k)}\right) \mid x^{i}=0\right\}\right), \\
U_{j} & :=\phi^{-1}\left(\left\{\left(x^{1}, \ldots, x^{k}\right) \in \phi\left(\sigma_{(k)}\right) \mid x^{j}=0\right\}\right) .
\end{aligned}
$$

$\mathcal{M}$ is a cell complex, hence there are $\sigma_{(k-1), i}$ and $\sigma_{(k-1), j}$ such that $x \in U_{i} \subset \sigma_{(k-1), i}$ and $x \in U_{j} \subset \sigma_{(k-1), j}$. Thus for every $\sigma_{(k-2)}$ in above sum there are exactly two ( $k-1$ )cells in $\partial^{1} \sigma_{(k)} \cap \partial^{-1} \sigma_{(k-2)}\left(\partial^{-1} \sigma_{(k-2)}\right.$ denotes the set of $(k-1)$-cells $\sigma_{(k-1)}$ such that
$\left.\sigma_{(k-2)} \in \partial^{1} \sigma_{(k-1)}\right)$. We can rewrite the summation as

$$
\partial_{(k-1)} \partial_{(k)} \boldsymbol{\sigma}_{(k)}=\sum_{\sigma_{(k-2)} \in \partial^{2} \sigma_{(k)}}\left(\sum_{\sigma_{(k-1)} \in \partial^{1} \sigma_{(k)} \cap \partial^{-1} \sigma_{(k-2)}} o_{\sigma_{(k)}, \sigma_{(k-1)}} o_{\sigma_{(k-1)}, \sigma_{(k-2)}}\right) \boldsymbol{\sigma}_{(k-2)}
$$

where $\partial^{2} \sigma_{(k)}$ is the set of $(k-2)$-cells for which there is a $\sigma_{(k-1)}$ such that $\sigma_{(k-2)} \in \partial^{1} \sigma_{(k-1)}$ and $\sigma_{(k-1)} \in \partial \sigma_{(k)}$. So, because the orientation induced on $\sigma_{(k-2)}$ by $\sigma_{(k-1), i}$ and the orientation induced on $\sigma_{(k-2)}$ by $\sigma_{(k-1), j}$ are opposite we find that the sum between parentheses is zero for every $\sigma_{(k-2)} \in \partial^{2} \sigma_{(k)}$.

We show now that these induced orientations are indeed opposite. Assume that $\left\{o^{(k)}=d x^{1} \wedge \cdots \wedge d x^{k}\right\}$ is the orientation on $\sigma_{(k)}$. Then the induced orientations on $\sigma_{(k-1), i}$ and $\sigma_{(k-1), j}$ are, respectively,

$$
\begin{aligned}
\left\{o_{i}^{(k-1)}\right\} & =\left\{(-1)^{i+1} d x^{1} \wedge \cdots \wedge \widehat{d x^{i}} \wedge \cdots \wedge d x^{k}\right\} \\
\left\{o_{j}^{(k-1)}\right\} & =\left\{(-1)^{j+1} d x^{1} \wedge \cdots \wedge \widehat{d x^{j}} \wedge \cdots \wedge d x^{k}\right\}
\end{aligned}
$$

where the hat above a term indicates that it is omitted. From this it follows that the induced orientations on $\sigma_{(k-2)}$ by $\sigma_{(k-1), i}$ and $\sigma_{(k-1), j}$ are, respectively,

$$
\begin{aligned}
& \left\{o_{i}^{(k-2)}\right\}=\left\{(-1)^{i+1}(-1)^{j} d x^{1} \wedge \cdots \wedge \widehat{d x^{i}} \wedge \cdots \wedge \widehat{d x^{j}} \wedge \cdots \wedge d x^{k}\right\} \\
& \left\{o_{j}^{(k-2)}\right\}=\left\{(-1)^{j+1}(-1)^{i+1} d x^{1} \wedge \cdots \wedge \widehat{d x^{i}} \wedge \cdots \wedge \widehat{d x^{j}} \wedge \cdots \wedge d x^{k}\right\}
\end{aligned}
$$

because $i<j$. As a result we have that the induced orientations on $\sigma_{(k-2)}$ are opposite and therefore $o_{\sigma_{(k)} \sigma_{(k-1), i}} o_{\sigma_{(k-1), i}} \sigma_{(k-2)}=-o_{\sigma_{(k)} \sigma_{(k-1), j}} o_{\sigma_{(k-1), j} \sigma_{(k-2)}}$.

### 3.1.3 The cochain complex

In Chapter 2 we considered duality products between $k$-dimensional manifolds and differential $k$-forms which together produce a real number by integrating the differential form over the manifold. The $k$-dimensional manifolds are replaced in the discrete setting by a finite dimensional space of $k$-chains $C_{(k)}(\Omega)$. The role of $k$-forms will be replaced by $k$-cochains. A $k$-cochain and a $k$-chain together produce a real number and therefore the $k$-cochains are elements of the dual space of $C_{(k)}(\Omega)$.
Definition 3.7. Let $\mathcal{M}$ be a mesh on $\Omega$ and $C_{(k)}(\Omega)$ the corresponding space of $k$-chains. We define the space of $k$-cochains $C^{(k)}(\Omega)$ to be the dual space of linear functionals $\boldsymbol{a}^{(k)}: C_{(k)}(\Omega) \rightarrow \mathbb{R}$. We denote the real number produced by a $k$-chain $\boldsymbol{a}_{(k)}$ and $k$ cochain $\boldsymbol{b}^{(k)}$ as $\left\langle\boldsymbol{a}_{(k)}, \boldsymbol{b}^{(k)}\right\rangle$.

Given a basis $\left\{\boldsymbol{\sigma}_{(k), i} \mid 1 \leq i \leq N_{(k)}\right\}$ for $C_{(k)}(\Omega)$, the corresponding dual basis $\left\{\boldsymbol{\sigma}^{(k), i} \mid 1 \leq i \leq N_{(k)}\right\}$ is defined according to

$$
\left\langle\boldsymbol{\sigma}_{(k), j}, \boldsymbol{\sigma}^{(k), i}\right\rangle=\delta_{j}^{i}
$$

and any $k$-cochain $\boldsymbol{a} \in C^{(k)}(\Omega)$ can be written as

$$
\boldsymbol{a}^{(k)}=\sum_{i=1}^{N_{(k)}} a_{i} \boldsymbol{\sigma}^{(k), i}
$$

We can represent $k$-chains and $k$-cochains by vectors of size $N_{(k)}$ that contain the real numbers with respect to the numbered basis. A $k$-chain $\boldsymbol{a}_{(k)}=\sum_{i=1}^{N_{(k)}} a^{i} \boldsymbol{\sigma}_{(k), i}$ we represent by the row vector $\boldsymbol{a}_{(k)}=\left[a^{1} \cdots a^{N_{(k)}}\right]$ and a $k$-cochain $\boldsymbol{b}^{(k)}=\sum_{i=1}^{N_{(k)}} b_{i} \boldsymbol{\sigma}^{(k), i}$ we represent by the column vector $\boldsymbol{b}^{(k)}=\left[\begin{array}{lll}b_{1} & \cdots & b_{N_{(k)}}\end{array}\right]^{T}$. Their duality product is then simply given by

$$
\left\langle\boldsymbol{a}_{(k)}, \boldsymbol{b}^{(k)}\right\rangle=\left[a^{1} \cdots a^{N_{(k)}}\right]\left[\begin{array}{c}
b_{1} \\
\vdots \\
b_{N_{(k)}}
\end{array}\right]=a^{1} b_{1}+\cdots+a^{N_{(k)}} b_{N_{(k)}} .
$$

In similar vein, we can represent the boundary operator $\partial_{(k)}$ by a matrix $\mathbb{D}_{(k)}$ because it is a linear operator from $C_{(k)}(\Omega)$ to $C_{(k-1)}(\Omega)$. Multiplying a row vector $\boldsymbol{a}_{(k)}$ of length $N_{(k)}$ from the right by a $\left(N_{(k)} \times N_{(k-1)}\right)$-matrix $\mathbb{D}_{(k)}$ results in the row vector $\boldsymbol{a}_{(k)} \mathbb{D}_{(k)}$ of length $N_{(k-1)}$. From the Definition 3.6 it is clear that the entries of $\mathbb{D}_{(k)}$ are given by

$$
\left[\mathbb{D}_{(k)}\right]_{i, j}=o_{\sigma_{(k), i} \sigma_{(k-1), j}}:= \begin{cases}+1 & \text { if } \sigma_{(k-1), j} \text { has the orientation induced by } \sigma_{(k), j}, \\ -1 & \text { if } \sigma_{(k-1), j} \text { has the opposite orientation, } \\ 0 & \text { otherwise. }\end{cases}
$$

In the duality product between manifolds and differential forms the exterior derivative is the adjoint of the boundary operator. In the finite dimensional setting provided by the mesh the adjoint to the boundary operator is called the coboundary operator.
Definition 3.8. [41, p. 251] The coboundary operator $\delta^{(k)}: C^{(k)}(\Omega) \rightarrow C^{(k+1)}(\Omega)$ is defined to be the dual of the boundary operator, i.e., it is defined by

$$
\left\langle\boldsymbol{a}_{(k+1)}, \delta^{(k)} \boldsymbol{b}^{(k)}\right\rangle=\left\langle\partial_{(k+1)} \boldsymbol{a}_{(k+1)}, \boldsymbol{b}^{(k)}\right\rangle, \quad \forall \boldsymbol{a}_{(k+1)} \in C_{(k+1)}(\Omega), \boldsymbol{b}^{(k)} \in C^{(k)}(\Omega)
$$

where $k=0,1, \ldots, d-1$.
The operator $\delta^{(k)}$ can be represented with respect to the canonical basis as a matrix $\mathbb{D}^{(k)}$ multiplied from the left. By definition we need to have

$$
\boldsymbol{a}_{(k+1)} \mathbb{D}^{(k)} \boldsymbol{b}^{(k)}=\boldsymbol{a}_{(k+1)} \mathbb{D}_{(k+1)} \boldsymbol{b}^{(k)}
$$

Thus, $\mathbb{D}^{(k)}=\mathbb{D}_{(k+1)}$ for $k=0,1, \ldots, d-1$.
Proposition 3.2. Let $\mathcal{M}$ be a cell complex on $\Omega$. The sequence

$$
C^{(0)}(\Omega) \xrightarrow{\delta^{(0)}} C^{(1)}(\Omega) \xrightarrow{\delta^{(1)}} \quad \cdots \quad \xrightarrow{\delta^{(d-1)}} C^{(d)}(\Omega),
$$

is a cochain complex.
Proof. By definition, for all $\boldsymbol{a}_{(k+1)} \in C_{(k+1)}(\Omega)$ and $\boldsymbol{b}^{(k)} \in C^{(k)}(\Omega)$,

$$
\left\langle\boldsymbol{a}_{(k+1)}, \delta^{(k)} \delta^{(k-1)} \boldsymbol{b}^{(k-1)}\right\rangle=\left\langle\partial_{(k)} \partial_{(k+1)} \boldsymbol{a}_{(k+1)}, \boldsymbol{b}^{(k-1)}\right\rangle=0
$$

hence $\delta^{(k)} \delta^{(k-1)} \boldsymbol{b}^{(k-1)}=\mathbf{0}^{(k+1)}$ for all $\boldsymbol{b}^{(k)} \in C^{(k)}(\Omega)$, which implies $\delta^{(k)} \circ \delta^{(k-1)}=0$.


Figure 3.4: The de Rham maps map from the continuous de Rham cochain complex to the discrete cochain complex. The diagram is commutative, i.e., $R^{(k+1)} \circ d^{(k)}=\delta^{(k)} \circ R^{(k)}$.

A continuous differential $k$-form is discretized by integrating it over the $k$-cells in the mesh.

Definition 3.9. The de Rham maps $R^{(k)}: \Lambda^{(k)}(\Omega) \rightarrow C^{(k)}(\Omega)^{3}$ are defined by

$$
\left\langle\boldsymbol{\sigma}_{(k), i}, R^{(k)}\left(a^{(k)}\right)\right\rangle=\int_{\sigma_{(k), i}} a^{(k)},
$$

for any $a^{(k)} \in \Lambda^{(k)}(\Omega), 1 \leq i \leq N_{(k)}$ and $0 \leq k \leq d$, with $\sigma_{(k), i} \in \mathcal{M}_{(k)}$ the $i$-th $k$-cell.
Proposition 3.3. The set of de Rham maps form a cochain map from the cochain complex $\left(\Lambda^{(k)}(\Omega), d^{(k)}\right)$ to the cochain complex $\left(C^{(k)}(\Omega), \delta^{(k)}\right)$, which means that $R^{(k+1)}$ 。 $d^{(k)}=\delta^{(k)} \circ R^{(k)}$, for $k=0,1, \ldots, d-1$, i.e., the diagram in Figure 3.4 commutes.

Proof. This follows directly from the definition of the coboundary operator and Stokes' theorem:

$$
\int_{\sigma_{(k+1), i}} d^{(k)} a^{(k)}=\int_{\partial \sigma_{(k+1), i}} a^{(k)}=\left\langle\partial_{(k+1)} \boldsymbol{\sigma}_{(k+1), i}, R^{(k)} a^{(k)}\right\rangle=\left\langle\boldsymbol{\sigma}_{(k+1), i}, \delta^{(k)} R^{(k)} a^{(k)}\right\rangle .
$$

So we find $\left\langle\boldsymbol{\sigma}_{(k+1), i}, R^{(k+1)} d^{(k)} a^{(k)}\right\rangle=\left\langle\boldsymbol{\sigma}_{(k+1), i}, \delta^{(k)} R^{(k)} a^{(k)}\right\rangle$ for the any $k$-cell $\boldsymbol{\sigma}_{(k+1), i}$ and hence $R^{(k+1)} \circ d^{(k)}=\delta^{(k)} \circ R^{(k)}$.

### 3.1.4 The boundary complex

Boundary conditions are enforced on the boundary of the cell complex by discrete trace operators. Before we introduce these operators we first prove a useful lemma.

Lemma 3.2. Let $\mathcal{M}=\left\{\mathcal{M}_{(0)}, \ldots, \mathcal{M}_{(d)}\right\}$ be a cell complex on $\Omega$. Let $\mathcal{N}_{(n)}$ be a subset of $\mathcal{M}_{(n)}$ and define, for $k=n-1, n-2, \ldots, 0$, successively

$$
\mathcal{N}_{(k)}:=\bigcup_{\sigma_{(k+1)} \in \mathcal{N}_{(k+1)}} \partial^{1} \sigma_{(k+1)} .
$$

The collection of cells $\mathcal{N}:=\left\{\mathcal{N}_{(0)}, \ldots, \mathcal{N}_{(n)}\right\}$ is a cell (sub)complex on $\left.\Omega\right|_{\mathcal{N}}:=\underset{\sigma_{(n)} \in \mathcal{N}_{(n)}}{ } \sigma_{(n)}$.

[^18]

Figure 3.5: Left: An example of a cell complex: a simplicial complex. The edges and vertices are shown. Right: The corresponding boundary cell complex.

Proof. Note that if $\mathcal{N}$ is a mesh on $\left.\Omega\right|_{\mathcal{N}}$, then by construction of the sets $\mathcal{N}_{(k)}, 0 \leq k \leq n-1$ it is also a cell complex. Furthermore, by definition of $\left.\Omega\right|_{\mathcal{N}}$, the $n$-cells in $\mathcal{N}$ cover it. Point (iii) of Definition 3.3 holds for $\mathcal{N}$ because it holds for $\mathcal{M}$. Rests us to show (ii) of the definition.

Because $k$-cells $\sigma_{(k)}$ and $\tau_{(k)}$ in $\mathcal{N}$ are also in the mesh $\mathcal{M}$ it follows that $\sigma_{(k)} \cap \tau_{(k)}$ is either empty or a union of $l$-cells in $\mathcal{M}_{(l)}$. Moreover, by item (iii) of the definition we have that for any $l$-cell $\sigma_{(l)}$ in this union $\sigma_{(l)} \subset \partial \sigma_{(k)}$ and $\sigma_{(l)} \subset \partial \tau_{(k)}$. This implies that $\sigma_{(l)}$ has to be in $\mathcal{N}_{(l)}$ as well.

If a $(d-1)$-cell $\sigma_{(d-1)}$ in $\mathcal{M}$ has an intersection with $\partial \Omega$ such that $\partial \sigma_{(d-1)} \cap \partial \Omega \neq$ $\sigma_{(d-1)} \cap \partial \Omega$, i.e., its $(d-1)$-dimensional interior intersects with $\partial \Omega$, then $\sigma_{(d-1)} \subset \partial \Omega$. Suppose $\sigma_{(d-1)} \notin \partial \Omega$, then necessarily there exist two $d$-cells "between" which $\sigma_{(d-1)}$ lies. However, by definition their intersection is a union of ( $d-1$ )-cells which would imply that $\sigma_{(d-1)}$ lies completely in this intersection and therefore leads to a contradiction with $\partial \sigma_{(d-1)} \cap \partial \Omega \neq \sigma_{(d-1)} \cap \partial \Omega$.

From this we can conclude that for each $d$-cell $\sigma_{(d)}$ in a cell complex $\mathcal{M}$ that has a $(d-1)$-dimensional intersection with the boundary, this intersection, $\sigma_{(d)} \cap \partial \Omega$, is a union of $(d-1)$-cells that completely lies within $\partial \Omega$. From this it follows that the set $\mathcal{M}_{(d-1)}^{b}:=\left\{\sigma_{(d-1)} \in \mathcal{M} \mid \sigma_{(d-1)} \subset \partial \Omega\right\}$ covers $\partial \Omega$.

Definition 3.10. Let $\mathcal{M}$ be a cell complex on $\Omega$. The set $\mathcal{M}^{b}=\left\{\mathcal{M}_{(0)}^{b}, \ldots, \mathcal{M}_{(d-1)}^{b}\right\}$, which is defined by $\mathcal{M}_{(d-1)}^{b}:=\left\{\sigma_{(d-1)} \in \mathcal{M} \mid \sigma_{(d-1)} \subset \partial \Omega\right\}$ and, for $k=d-2, \ldots, 0$, by

$$
\mathcal{M}_{(k)}^{b}:=\bigcup_{\sigma_{(k+1)} \in \mathcal{M}_{(k+1)}^{b}} \partial^{1} \sigma_{(k+1)}
$$

is called the boundary cell complex. It is a cell subcomplex of $\mathcal{M}$ on $\partial \Omega$ by Lemma 3.2 and the fact that the $(d-1)$-cells cover $\partial \Omega$. We denote the number of boundary $k$-cells in the set $\mathcal{M}_{(k)}^{b}$ by $N_{(k)}^{b}$.

An example of a cell complex and corresponding boundary cell complex is shown in Figure 3.5.

Analogous to the continuous trace operator $t: \Lambda^{(k)}(\Omega) \rightarrow \Lambda^{(k)}(\partial \Omega)$, see Definition 2.18, we can define a discrete trace operator.

Definition 3.11. The discrete trace operator $\mathbb{T}^{(k)}: C^{(k)}(\Omega) \rightarrow C^{(k)}(\partial \Omega)$ is the linear operator that restricts a $k$-cochain to the boundary cell complex. It is defined by

$$
\left\langle\boldsymbol{\sigma}_{(k), i}, \mathbb{T}^{(k)} a^{(k)}\right\rangle_{\partial \Omega}=\left\langle\boldsymbol{\sigma}_{(k), i}, a^{(k)}\right\rangle_{\Omega}
$$

for any of the $k$-cells $\boldsymbol{\sigma}_{(k), i} \in \mathcal{M}_{(k)}^{b} \subset \mathcal{M}_{(k)}, 1 \leq i \leq N_{(k)}^{b}$, and $\boldsymbol{a}^{(k)} \in C^{(k)}(\Omega)$.
Related useful operators are the projections that restrict a cochain to the interior of the mesh or to the boundary of the mesh, respectively, and set the cochain to zero in the respective complement.

Definition 3.12. We define the projection operators $\mathbb{P}_{\mathcal{M}_{(k)}^{b}}^{(k)}: C^{(k)}(\Omega) \rightarrow C^{(k)}(\Omega)$ and $\mathbb{P}_{\mathcal{M}_{(k)} \mathcal{U}_{(k)}^{b}}^{(k)}: C^{(k)}(\Omega) \rightarrow C^{(k)}(\Omega)$, for all $\boldsymbol{a}^{(k)} \in C^{(k)}(\Omega)$ and $\boldsymbol{\sigma}_{(k)} \in \mathcal{M}_{(k)}$, by, respectively,

$$
\left\langle\boldsymbol{\sigma}_{(k)}, \mathbb{P}_{\partial \Omega}^{(k)} \boldsymbol{a}^{(k)}\right\rangle= \begin{cases}\left\langle\boldsymbol{\sigma}_{(k)}, \boldsymbol{a}^{(k)}\right\rangle & \text { if } \boldsymbol{\sigma}_{(k)} \in \mathcal{M}_{(k)}^{b} \\ 0 & \text { otherwise }\end{cases}
$$

and

$$
\left\langle\boldsymbol{\sigma}_{(k)}, \mathbb{P}_{\Omega \backslash \partial \Omega}^{(k)} a^{(k)}\right\rangle= \begin{cases}\left\langle\boldsymbol{\sigma}_{(k)}, \boldsymbol{a}^{(k)}\right\rangle & \text { if } \boldsymbol{\sigma}_{(k)} \notin \mathcal{M}_{(k)}^{b}, \\ 0 & \text { otherwise } .\end{cases}
$$

These projection matrices can be expressed in terms of the trace operator.
Lemma 3.3. Let $\mathbb{I}^{(k)}: C^{(k)}(\Omega) \rightarrow C^{(k)}(\Omega)$ be the identity. The projection operators are given by

$$
\mathbb{P}_{\partial \Omega}^{(k)}=\left(\mathbb{T}^{(k)}\right)^{T} \mathbb{T}^{(k)}, \quad \text { and } \quad \mathbb{P}_{\Omega \backslash \partial \Omega}^{(k)}=\mathbb{I}^{(k)}-\left(\mathbb{T}^{(k)}\right)^{T} \mathbb{T}^{(k)}
$$

Proof. The trace operator $\mathbb{T}^{(k)}: C^{(k)}(\Omega) \rightarrow C^{(k)}(\partial \Omega)$ restricts a cochain to the boundary mesh. The transpose of the trace matrix $\left(\mathbb{T}^{(k)}\right)^{T}: C^{(k)}(\partial \Omega) \rightarrow C^{(k)}(\Omega)$ is simply the canonical embedding of a boundary $k$-cochain into $C^{(k)}(\Omega)$.

For a discrete implementation of the boundary conditions the boundary de Rham maps come in handy.

Proposition 3.4. Let $R_{b}^{(k)}: \Lambda^{(k)}(\partial \Omega) \rightarrow C^{(k)}(\partial \Omega)$ be the de Rham maps on the boundary cell complex defined by

$$
\left\langle\boldsymbol{\sigma}_{(k), i}, R_{b}^{(k)}\left(a^{(k)}\right)\right\rangle=\int_{\sigma_{(k), i}} a^{(k)}
$$

for any $a^{(k)} \in \Lambda^{(k)}(\partial \Omega), 1 \leq i \leq N_{(k)}^{b}$ and $0 \leq k<d$, with $\sigma_{(k), i} \in \mathcal{M}_{(k)}^{b}$ the $i$-th $k$ cell. The de Rham maps and trace operators commute, i.e., $R_{b}^{(k)} \circ t^{(k)}=\mathbb{T}^{(k)} \circ R^{(k)}$, for $k=0,1, \ldots, d-1$.

Proof. For any of the $k$-cells $\sigma_{(k), i} \in \mathcal{M}_{(k)}^{b}, 1 \leq i \leq N_{(k)}^{b}$ we have

$$
\begin{aligned}
& \left\langle\boldsymbol{\sigma}_{(k), i}, R_{b}^{(k)} t^{(k)} a^{(k)}\right\rangle_{\partial \Omega} \\
& \quad=\int_{\sigma_{(k), i}} a^{(k)}=\int_{\sigma_{(k), i}} t^{(k)} a^{(k)}=\left\langle\boldsymbol{\sigma}_{(k), i}, R^{(k)} a^{(k)}\right\rangle_{\Omega}=\left\langle\boldsymbol{\sigma}_{(k), i}, \mathbb{T}^{(k)} R^{(k)} a^{(k)}\right\rangle_{\partial \Omega} .
\end{aligned}
$$

We denote the boundary and coboundary operators on the boundary cell complex by $\partial_{(k)}^{b}$ and $\delta_{b}^{(k)}$, respectively. These operators are simply the restrictions of $\partial_{(k)}$ and $\delta^{(k)}$ to the boundary cell complex. As a result we have that the matrix representation $\mathbb{D}_{b}^{(k)}$ of $\delta_{b}^{(k)}$ on the canonical basis is given by

$$
\mathbb{D}_{b}^{(k)}=\mathbb{T}^{(k+1)} \mathbb{D}^{(k)}\left(\mathbb{T}^{(k)}\right)^{T}
$$

Proposition 3.5. The discrete trace and coboundary operators commute, i.e., we have

$$
\mathbb{T}^{(k)} \mathbb{D}^{(k-1)}=\mathbb{D}_{b}^{(k-1)} \mathbb{T}^{(k-1)}
$$

Proof. For a $k$-cell $\boldsymbol{\sigma}_{(k)} \in \mathcal{M}_{(k)}^{b}$ its boundary also lies in $\partial \Omega$. This implies that the values of $\mathbb{T}^{(k)} \mathbb{D}^{(k-1)} \boldsymbol{a}^{(k-1)}$, for any $\boldsymbol{a}^{(k-1)} \in C^{(k)}(\Omega)$, depend only on the values of $\boldsymbol{a}^{(k-1)}$ on the boundary complex, hence $\mathbb{T}^{(k)} \mathbb{D}^{(k-1)} \boldsymbol{a}^{(k-1)}=\mathbb{T}^{(k)} \mathbb{D}^{(k-1)} \mathbb{P}_{\partial \Omega}^{(k-1)} \boldsymbol{a}^{(k-1)}$. Thus we have

$$
\mathbb{T}^{(k)} \mathbb{D}^{(k-1)}=\mathbb{T}^{(k)} \mathbb{D}^{(k-1)} \mathbb{P}_{\partial \Omega}^{(k-1)}=\mathbb{T}^{(k)} \mathbb{D}^{(k-1)}\left(\mathbb{T}^{(k-1)}\right)^{T} \mathbb{T}^{(k-1)}=\mathbb{D}_{b}^{(k-1)} \mathbb{T}^{(k-1)}
$$

Proposition 3.6. Let $\mathcal{M}$ be a cell complex on $\Omega$ and let $\mathcal{M}_{b}$ be the corresponding boundary cell complex. The sequence

$$
C^{(0)}(\partial \Omega) \xrightarrow{\delta_{b}^{(0)}} C^{(1)}(\partial \Omega) \xrightarrow{\delta_{b}^{(1)}} \quad \cdots \quad \xrightarrow{\delta_{b}^{(d-2)}} C^{(d-1)}(\partial \Omega),
$$

is a cochain complex.
Proof. Using the matrix representation we find that

$$
\mathbb{D}_{b}^{(k)} \mathbb{D}_{b}^{(k-1)}=\mathbb{T}^{(k+1)} \mathbb{D}^{(k)} \mathbb{P}_{\partial \Omega}^{(k)} \mathbb{D}^{(k-1)}\left(\mathbb{T}^{(k-1)}\right)^{T}=\mathbb{T}^{(k+1)} \mathbb{D}^{(k)} \mathbb{D}^{(k-1)}\left(\mathbb{T}^{(k-1)}\right)^{T}=0,
$$

where we used that $\mathbb{P}_{\partial \Omega}^{(k)}$ is the identity on $C^{(k)}(\partial \Omega)$ and $\mathbb{D}^{(k)} \mathbb{D}^{(k-1)}=0$.
The de Rham maps for the boundary complexes form a cochain map again, just like the de Rham maps for the original complexes.
Proposition 3.7. The set of de Rham maps $R_{b}^{(k)}: \Lambda^{(k)}(\partial \Omega) \rightarrow C^{(k)}(\partial \Omega)$ form a cochain map from the continuous cochain complex $\left(\Lambda^{(k)}(\partial \Omega), d^{(k)}\right)_{\mathrm{b}}$ to the discrete cochain complex $\left(C^{(k)}(\partial \Omega), \delta_{b}^{(k)}\right)$, i.e., $R_{b}^{(k+1)} \circ d_{\mathrm{b}}^{(k)}=\delta_{b}^{(k)} \circ R_{b}^{(k)}$, for $k=0,1, \ldots, d-2$.
Proof. The proof is completely analogous to the proof of Proposition 3.3.
The commutation relations from Propositions 3.3, 3.4, 3.5 and 3.7 are shown as a commuting diagram in Figure 3.6 for $d=3$.


Figure 3.6: Five commutation relations are shown: $t^{(k+1)} \circ d^{(k)}=d_{\mathrm{b}}^{(k)} \circ t^{(k)}($ Section 2.3.2 $), R^{(k+1)} \circ d^{(k)}=$ $\delta^{(k)} \circ R^{(k)}$ (Proposition 3.3), $R_{\mathrm{b}}^{(k)} \circ t^{(k)}=\mathbb{T}^{(k)} \circ R^{(k)}$ (Proposition 3.4), $\mathbb{T}^{(k+1)} \circ \mathbb{D}^{(k)}=\mathbb{D}_{\mathrm{b}}^{(k)} \circ \mathbb{T}^{(k)}$ (Proposition 3.5) and $\mathbb{D}_{\mathrm{b}}^{(k)} \circ R_{\mathrm{b}}^{(k)}=R_{\mathrm{b}}^{(k+1)} \circ d_{\mathrm{b}}^{(k)}$ (Proposition 3.7).

### 3.2 The dual complex

So far we have shown how to discretize inner differential forms on an inner-oriented primal cell complex. We have essentially discretized the top half of Figure 2.18. We now come to the bottom half of this sequence. To represent the Hodge duality we introduce a dual mesh.

### 3.2.1 The dual mesh

The Hodge star assigns to every inner $k$-form an outer $(d-k)$-form. In the discrete setting the Hodge star operator will be discretized as a matrix that maps a $k$-cochain to a ( $d-k$ )-cochain. The $k$-cochain is an element of the dual space to the chain complex formed by the inner-oriented mesh. The $(d-k)$-cochain will be an element of the dual space to a second mesh: the outer-oriented dual mesh.

The dual mesh is a second mesh on $\Omega$. For each inner-oriented $k$-cell in the original mesh (which we will call the primal mesh from now on) there is an outer-oriented dual $(d-k)$-cell in the dual mesh. We define two types of dual meshes. They are both defined using the simplicial subdivision from (3.2). One definition uses as cell centers in this subdivision the barycenters ${ }^{4}$ and the other uses the circumcenters ${ }^{5}$.

[^19]

Figure 3.7: Different types of dual meshes are shown for 2D primal meshes.

Definition 3.13. Let $\mathcal{M}:=\left\{\mathcal{M}_{(0)}, \ldots, \mathcal{M}_{(d)}\right\}$ be an inner-oriented mesh on $\Omega$. For every $k$-cell $\sigma_{(k)} \in \mathcal{M}_{(k)}(\Omega)$, the dual cell to $\sigma_{(k)}$ is given by the set

$$
\begin{equation*}
\sharp \sigma_{(k)}:=\bigcup_{\left(\tau_{(d)}, \ldots, \tau_{(k+1)}\right) \in I_{\sigma_{(k)}}}\left[x_{(1)}^{\tau_{(d)}}, \ldots, x_{(1)}^{\tau_{(k+1)}}, x_{(1)}^{\sigma_{(k)}}\right], \tag{3.3}
\end{equation*}
$$

where $\left[x_{(1)}^{\tau_{(d)}}, \ldots, x_{(1)}^{\tau_{(k+1)}}, x_{(1)}^{\sigma_{(k)}}\right]$ is the simplex with $x_{(1)}^{\tau_{(d)}}, \ldots, x_{(1)}^{\tau_{(k+1)}}, x_{(1)}^{\sigma_{(k)}}$ as vertices, $x_{(1)}^{\tau_{(l)}}$ is the center of $\tau_{(l)}$, and $\approx \sigma_{(k)}$ has as outer orientation the inner orientation of $\sigma_{(k)}$. The dual mesh on $\Omega$ is the mesh given by $\tilde{\mathcal{M}}:=\left\{\tilde{\mathcal{M}}_{(d)}, \ldots, \tilde{\mathcal{M}}_{(0)}\right\}$, where $\tilde{\mathcal{M}}_{(d-k)}$ is the set of dual cells to the cells in $\mathcal{M}_{(k)}$.

If as cell centers of the primal cells the barycenters are used then the dual cells are called barycentric dual cells and the dual mesh is called the barycentric dual mesh. Alternatively, if as centers of the primal cells the circumcenters are used and the primal mesh is well-centered ${ }^{6}$ then the dual cells are called circumcentric dual cells and the dual mesh is called the circumcentric dual mesh.

In Figure 3.7 we show three examples of a dual mesh. We show the dual mesh for a Cartesian, simplicial and quadrangular primal mesh. For the simplicial mesh we show the circumcentric dual mesh, for the quadrangular mesh we show the barycentric dual mesh. For the Cartesian mesh the barycentric and circumcentric dual meshes coincide.

Before we show that the barycentric and circumcentric dual meshes are indeed meshes on $\Omega$ according to Definition 3.3 we introduce the dual cell vectors analogue to (3.2).

[^20]Note that the boundary of the dual cell $\sharp \sigma_{(k)}$ is given by the set of points

$$
\partial \sharp \sigma_{(k)}=\sum_{\tau_{(k+1)} \in \partial^{-1} \sigma_{(k)}\left(\tau_{(d)} \in \ldots, \tau_{(k+2)} \in I_{\tau_{(k+1)}}\right.}\left[x_{(1)}^{\tau_{(d)}}, \ldots, x_{(1)}^{\tau_{(k+2)}}, x_{(1)}^{\tau_{(k+1)}}\right]=\sum_{\tau_{(k+1)} \in \partial^{-1} \sigma_{(k)}} \sharp \tau_{(k+1)},
$$

where we write $\partial^{-l} \sigma_{(k)}, k+l \leq d$, for the set of cells $\tau_{(k+l)} \in \mathcal{M}_{(k+l)}$ such that $\sigma_{(k)} \in \partial^{l} \tau_{(k+l)}$. Thus, we see that $\sharp \tau_{(k+1)} \in \partial \sharp \sigma_{(k)}$ if and only if $\sigma_{(k)} \in \partial \tau_{(k+1)}$. Furthermore, observe that the definition of the dual cells implies that the dual $k$-cells lie in the boundary of the dual $(k+1)$-cells for $0 \leq k<d$.

Now, because the cells $\approx \sigma_{(k)}$ and $\sharp \sigma_{(k-1)}$ with $\sharp \sigma_{(k)} \in \partial \sharp \sigma_{(k-1)}$ have an outer orientation given by their respective primal cells the outer orientation induced by $\approx \sigma_{(k-1)}$ on $\sharp \sigma_{(k)}$ can either agree or disagree. We denote this by $o_{\star \sigma_{(k-1)} \star \sigma_{(k)}}$ which we define analogously to how we did for the primal mesh:

$$
o_{\sharp \sigma_{(k-1)} \star \sigma_{(k)}}=\left\{\begin{aligned}
+1 & \text { if } \approx \sigma_{(k)} \in \partial \star \sigma_{(k-1)} \text { and has the orientation induced by } \approx \sigma_{(k-1)}, \\
-1 & \text { if } \approx \sigma_{(k)} \in \partial \sharp \sigma_{(k-1)} \text { and has the opposite orientation, } \\
0 & \text { if } \approx \sigma_{(k)} \notin \partial \star \sigma_{(k-1)} .
\end{aligned}\right.
$$

Lemma 3.4. The relative orientations $o_{\star \sigma_{(k-1)}{ }^{\star \sigma_{(k)}}}$ on the dual mesh are related to the relative orientations $o_{\sigma_{(k)} \sigma_{(k-1)}}$ on the primal mesh according to

$$
\begin{equation*}
o_{\sharp \sigma_{(k-1)} \star \sigma_{(k)}}=(-1)^{k} o_{\sigma_{(k)} \sigma_{(k-1)}} . \tag{3.4}
\end{equation*}
$$

Proof. Suppose that $\sigma_{(k-1)}$ has orientation $\left\{o_{(k-1)}\right\}$. Then the orientation of $\sigma_{(k)}$ is, according to Definition 2.28, given by

$$
\left\{\sigma_{(k)}\right\}=o_{\sigma_{(k)} \sigma_{(k-1)}}\left\{v_{(1)} \wedge o_{(k-1)}\right\}
$$

where $v_{(1)}$ points out of $\sigma_{(k)}$ at $\sigma_{(k-1)}$. This implies that the orientations of the corresponding dual cells are given by

$$
\begin{aligned}
\left\{\sharp \sigma_{(k-1)}\right\} & =\left\{\left\{o_{(k-1)} \wedge o_{(d-k+1)}\right\} o_{(d-k+1)}\right\} \\
\left\{\sharp \sigma_{(k)}\right\} & =\left\{\left\{v_{(1)} \wedge o_{(k-1)} \wedge o_{(d-k)}\right\} o_{(d-k)}\right\} .
\end{aligned}
$$

However, from the definition of the outer orientation (Definition 2.29) and the definition of $o_{\star \sigma_{(k-1)} * \sigma_{(k)}}$ it follows that the outer orientation of $\approx \sigma_{(k)}$ is given by $o_{\sharp \sigma_{(k-1)} \star \sigma_{(k)}}\left\{\left\{o_{(k-1)} \wedge\right.\right.$ $\left.\left.v_{(1)}^{\prime} \wedge o_{(d-k)}\right\} o_{(d-k)}\right\}$, where $v_{(1)}^{\prime}$ points out of $\sharp \sigma_{(k-1)}$ at $\sharp \sigma_{(k)}$. For simplicity we can assume that $v_{(1)}$ and $v_{(1)}^{\prime}$ both lie in the 1-dimensional orthogonal complement of $o_{(k-1)} \wedge$ $o_{(d-k)}$. We can take $v_{(1)}=x_{(1)}^{\sigma_{(k-1)}}-x_{(1)}^{\sigma_{(k)}}$ as vector pointing out of $\sigma_{(k)}$ and $v_{(1)}^{\prime}=$ $x_{(1)}^{\sigma_{(k)}}-x_{(1)}^{\sigma_{(k-1)}}$ as vector pointing out of $\leadsto \sigma_{(k-1)}$. This shows that we have

$$
\begin{aligned}
\left\{\sharp \sigma_{(k)}\right\} & =o_{\sharp \sigma_{(k-1)} \star \sigma_{(k)}}\left\{\left\{o_{(k-1)} \wedge v_{(1)}^{\prime} \wedge o_{(d-k)}\right\} o_{(d-k)}\right\} \\
& =(-1)^{k} o_{\sharp \sigma_{(k-1)} \star \sigma_{(k)}}\left\{\left\{v_{(1)} \wedge o_{(k-1)} \wedge o_{(d-k)}\right\} o_{(d-k)}\right\},
\end{aligned}
$$

which has to equal the orientation $o_{\sigma_{(k)} \sigma_{(k-1)}}\left\{\left\{v_{(1)} \wedge o_{(k-1)} \wedge O_{(d-k)}\right\} o_{(d-k)}\right\}$ which is implied by the primal mesh cell $\sigma_{(k)}$. This can only be the case if (3.4) holds.

Lemma 3.5. An outer-oriented $(d-k)$-cell $\hbar \sigma_{(k)} \in \mathcal{M}_{(d-k)}$ is represented by the outer ( $d-k$ )-vector
with $o_{\sigma_{(k)}, \tau_{(k+1)}, \ldots, \tau_{(d)}}:=o_{\sharp \sigma_{(k)} \sharp \tau_{(k+1)}} \cdots o_{\sharp \tau_{(d-1)} \sharp \tau_{(d)}} o_{\sharp \tau_{(d)}}$, where $o_{\tau_{(d)}}:=\left\{\tau_{(d)}\right\}$, i.e., the orientation of $\tau_{(d)}$.

Proof. The size of the simplices in the sum in Definition 3.13 is given by

$$
\left|\left[x_{(1)}^{\tau_{(d)}}, \ldots, x_{(1)}^{\tau_{(k+1)}}, x_{(1)}^{\sigma_{(k)}}\right]\right|=\frac{1}{(d-k)!}\left|\left(x_{(1)}^{\tau_{(k+1)}}-x_{(1)}^{\sigma_{(k)}}\right) \wedge \cdots \wedge\left(x_{(1)}^{\tau_{(d)}}-x_{(1)}^{\tau_{(d-1)}}\right)\right| .
$$

Rests us to check that the terms in sum (3.5) have the correct orientation. Dual cell $\sharp \tau_{(d)}$ has outer orientation $o_{\sharp} \tau_{(d)}=\left\{\tau_{(d)}\right\}$. We will now argue that the definition of outer orientation implies that the following recursive relation should hold:

$$
\sharp \sigma_{(k)}=\sum_{\tau_{(k+1)} \partial^{-1} \sigma_{(k)}} \frac{1}{(d-k)} o_{\sharp \sigma_{(k)} \star \tau_{(k+1)}}\left(x_{(1)}^{\tau_{(k+1)}}-x_{(1)}^{\sigma_{(k)}}\right) \wedge \sharp \tau_{(k+1)} .
$$

We have $\left\{\stackrel{\downarrow}{d} \tau_{(k+1)}\right\}=\left\{\left\{\tau_{(k+1)} \wedge o_{(d-k-1)}\right\} o_{(d-k-1)}\right\}$. Note that $v_{(1)}:=x_{(1)}^{\tau_{(k+1)}}-x_{(1)}^{\sigma_{(k)}}$ points out of $\approx \sigma_{(k)}$ and into $\tau_{(k+1)}$. Hence, by the definition of induced inner orientation (Definition 2.28),

$$
\begin{aligned}
\left\{\tau_{(k+1)}\right\} & =-o_{\tau_{(k+1)} \sigma_{(k)}}\left\{v_{(1)} \wedge \sigma_{(k)}\right\} \\
& =(-1)^{k+1} o_{\tau_{(k+1)} \sigma_{(k)}}\left\{\sigma_{(k)} \wedge v_{(1)}\right\} \\
& =o_{* \sigma_{(k)} *} \tau_{(k+1)}\left\{\sigma_{(k)} \wedge v_{(1)}\right\} .
\end{aligned}
$$

This shows, after substitution, that terms in the recursive sum above have orientation

$$
\begin{aligned}
o_{\star \sigma_{(k)} \star \tau_{(k+1)}} & \left\{\left\{\tau_{(k+1)} \wedge o_{(d-k-1)}\right\} v_{(1)} \wedge o_{(d-k-1)}\right\} \\
& =\left\{\left\{\sigma_{(k)} \wedge v_{(1)} \wedge o_{(d-k-1)}\right\} v_{(1)} \wedge o_{(d-k-1)}\right\},
\end{aligned}
$$

which is indeed the outer orientation of $\sharp \sigma_{(k)}$.
Formula (3.5) is actually more general than the set of points defining $\downarrow \sigma_{(k)}$ in Definition 3.13. Formula (3.5) namely takes into account that cell centers may lie out of the cell itself. As a result some of the subsimplices in Definition 3.13 get a negative sign and cancel parts of other simplices to get the actual dual cell. This is especially relevant for the circumcentric dual cells, because only for a completely well-centered mesh (i.e., a mesh for which every $k$-cell contains its circumcenter) all the simplices are positive. This is a very stringent requirement. In [49] a sign convention regarding the aforementioned signs of the subsimplices in Definition 3.13 is presented. This sign convention is actually contained in formula (3.5) which therefore always gives the correct dual cell size and orientation. A primal simplicial mesh may actually be such that even some of the cell
sizes become "negative" and the dual mesh overlaps itself. In our definition of orientation introduced in the previous chapter this merely means that the orientation changes.

Another situation, where the dual cell as defined in Definition 3.13 is not adequate and only formula (3.5) applies, is when the primal mesh cells are not star-shaped with respect to their centers. A $k$-cell $\sigma_{(k)}$ is said to be star-shaped if every point in $\sigma_{(k)}$ can be reached from $x_{(1)}^{\sigma_{(k)}}$ by a straight line that is contained in $\sigma_{(k)}$. When the barycenters are used as cell centers, then this is satisfied for any mesh with convex $d$-cells. When the circumcenters are used as cell centers the $d$-cells are likely to be simplices anyway and then if the cells are well-centered they are also star-shaped.

Proposition 3.8. Let $\mathcal{M}:=\left\{\mathcal{M}_{(0)}, \ldots, \mathcal{M}_{(d)}\right\}$ be a mesh of polytopal cells on $\Omega$ such that the cells are well-centered and star-shaped. The dual mesh $\tilde{\mathcal{M}}:=\left\{\tilde{\mathcal{M}}_{(d)}, \ldots, \tilde{\mathcal{M}}_{(0)}\right\}$ as defined in Definition 3.13 is indeed a mesh on $\Omega$.

Proof. We verify the three properties of Definition 3.3, starting with the first point. From (3.2) it follows that, for well-centered cells, the union of the primal $d$-cells (which by definition cover $\Omega$ ) is given by

$$
\Omega=\bigcup_{\substack{\tau_{(d)} \in \mathcal{M}_{(d)}\left(\tau_{(d-1)}, \ldots, \tau_{(0)}\right) \\ \in I^{\tau}(d)}}\left[x_{(1)}^{\tau_{(d)}}, \ldots, x_{(1)}^{\tau_{(0)}}\right]=\bigcup_{\substack{\tau_{(0)} \in \mathcal{M}_{(0)}\left(\tau_{(d)}, \ldots, \tau_{(1)}\right) \\ \in I_{\tau_{(0)}}}}\left[x_{(1)}^{\tau_{(d)}}, \ldots, x_{(1)}^{\tau_{(0)}}\right]
$$

where we reorder the union in the second equality. The last expression is the union of cells $\stackrel{\tau_{(0)}}{ }$, i.e., the union of $d$-dimensional dual cells. So the $d$-cells in $\tilde{\mathcal{M}}$ cover $\Omega$.

We continue with the second item. Suppose we have two distinct dual $k$-cells $\tilde{\sigma}_{(k)}, \tilde{\tau}_{(k)} \epsilon$ $\tilde{\mathcal{M}}_{(k)}$. Note that the simplicial subdivision discussed above is both a subdivision of the primal and dual mesh. From this it follows that if the distinct cells intersect it must be in the boundary of one of their subsimplices, i.e., there must be some vertex $x_{(1)}^{\eta_{(m)}} \in \tilde{\sigma}_{(k)} \cap \tilde{\tau}_{(k)}$ for some $\eta_{(m)} \in \mathcal{M}_{(m)}$ with $m>k$. From the definition of the dual cells $\tilde{\sigma}_{(k)}$ and $\tilde{\tau}_{(k)}$ it follows that $\sqcap \eta_{(m)} \subset \tilde{\sigma}_{(k)} \cap \tilde{\tau}_{(k)}$. So, every intersection between $\tilde{\sigma}_{(k)}$ and $\tilde{\tau}_{(k)}$ is contained in a lower dimensional cell $\forall \eta_{(m)}$ which is contained in $\tilde{\sigma}_{(k)} \cap \tilde{\tau}_{(k)}$, hence the union of all such cells is $\tilde{\sigma}_{(k)} \cap \tilde{\tau}_{(k)}$.

Finally, the third item follows from the fact that the definition of the dual cells implies that the dual $k$-cells lie in the boundary of the dual $(k+1)$-cells for $0 \leq k<d$. As a consequence an intersection of distinct $\tilde{\tau}_{(l)} \in \tilde{\mathcal{M}}_{(l)}$ and $\tilde{\sigma}_{(k)} \in \tilde{\mathcal{M}}_{(k)}$ with $l \leq k$ with $\tilde{\sigma}_{(k)} \cap \tilde{\tau}_{(l)} \neq \varnothing$ must clearly satisfy $\tilde{\sigma}_{(k)} \cap \tilde{\tau}_{(l)} \subset \partial \sigma_{(k)}$.

Proposition 3.8 states that for a mesh $\Omega$ with well-centered and star-shaped cells its corresponding dual mesh is again a mesh on $\Omega$ according to Definition 3.3. When the primal mesh has cells that are not well-centered and star-shaped, then the dual mesh might become self-overlapping. This happens when the primal mesh is not Delaunay, which means that in the circumscribed $(d-1)$-sphere of some primal $d$-cell the circumcenter of another $d$-cell can be found. ${ }^{7}$ This is shown in Figure 3.8.

[^21]

Figure 3.8: On the left we simply plot the points in the union (3.3). For dual vertices (cell circumcenters) that lie out of their simplex this gives the wrong result. If orientation as in formula (3.5) is taken into account we get the correct result shown in the center. However, if we have a primal mesh which is not Delaunay the dual mesh might still become self-overlapping as shown on the right.

### 3.2.2 The dual cell complex

We define chains and cochains on the dual mesh, just as we did on the primal mesh. We assume for now that the primal mesh is such that the dual mesh is again a mesh on $\Omega$.

Definition 3.14. Let $\mathcal{M}$ be a cell complex on $\Omega$ and let $\tilde{\mathcal{M}}$ be the corresponding dual mesh on $\Omega$. We denote the spaces of dual $k$-chains by $\tilde{C}_{(k)}(\Omega)$ and the space of dual $k$-cochains by $\tilde{C}^{(k)}(\Omega)$. We denote the canonical basis for $\tilde{C}_{(k)}(\Omega)$ by $\left\{\tilde{\boldsymbol{\sigma}}_{(k), i}=\right.$ $\left.\approx \boldsymbol{\sigma}_{(d-k), i} \mid 1 \leq k \leq N_{(d-k)}\right\}$, where $\left\{\boldsymbol{\sigma}_{(d-k), i} \mid 1 \leq k \leq N_{(d-k)}\right\}$ is the canonical basis for $C_{(d-k)}(\Omega)$, and the corresponding dual basis for $\tilde{C}^{(k)}(\Omega)$ by $\left\{\tilde{\boldsymbol{\sigma}}^{(k), i} \mid 1 \leq k \leq N_{(d-k)}\right\}$.

We want to introduce a dual mesh boundary operator $\tilde{\partial}_{(k)}: \tilde{C}_{(k)}(\Omega) \rightarrow \tilde{C}_{(k-1)}(\Omega)$ analogous to $\partial_{(k)}$. For such an operator to make sense the dual mesh needs to be a cell complex. However, the dual mesh is not a cell complex. This is clear from Figure 3.7 and Figure 3.8. At the domain boundary $\partial \Omega$ cells are missing. More specifically, for any $k$-cell $\tilde{\sigma}_{(k)} \in \tilde{\mathcal{M}}_{(k)}$ with $\tilde{\sigma}_{(k)} \cap \partial \Omega \neq \varnothing$, there is no $\tilde{\sigma}_{(l)}(l<k)$ such that $\partial \tilde{\sigma}_{(k)} \cap \partial \Omega=\tilde{\sigma}_{(l)}$.

Fortunately, the dual mesh can be completed in a simple way to a dual cell complex by adding the dual mesh to the boundary complex $\mathcal{M}^{\mathrm{b}}$.

Proposition 3.9. Let $\mathcal{M}$ be a cell complex on $\Omega$ and let $\tilde{\mathcal{M}}$ be the corresponding dual mesh on $\Omega$. Furthermore, let $\tilde{\mathcal{M}}^{\mathrm{b}}$ be the dual mesh to the boundary complex $\mathcal{M}^{\mathrm{b}}$ on $\partial \Omega$. The collection of cells $\tilde{\mathcal{M}} \cup \tilde{\mathcal{M}}^{\mathrm{b}}$ is a cell complex on $\Omega$.

Proof. A $(d-k)$-cell $\sharp \sigma_{(k)}$ dual to the primal cell $\sigma_{(k)}$ is given by

$$
\approx \sigma_{(k)}=\bigcup_{\left(\tau_{(d)}, \ldots, \tau_{(k+1)}\right) \in I_{\sigma_{(k)}}}\left[x_{(1)}^{\tau_{(d)}}, \ldots, x_{(1)}^{\tau_{(k+1)}}, x_{(1)}^{\sigma_{(k)}}\right]
$$

To determine the boundary of $\leadsto \sigma_{(k)}$ we consider the boundaries of the $(d-k)$-simplices in the above union. A $(d-k)$-simplex in this union has $(d-k+1)$ faces:


Figure 3.9: The construction of the dual cell complex in 2D.
(a) $\left[x_{(1)}^{\tau_{(d-1)}}, \ldots, x_{(1)}^{\tau_{(k+1)}}, x_{(1)}^{\sigma_{(k)}}\right]$,
(b) $\left[x_{(1)}^{\tau_{(d)}}, \ldots, x_{(1)}^{\tau_{(d-l)}}, x_{(1)}^{\tau_{(d-l-1)}} \ldots, x_{(1)}^{\tau_{(k+1)}}, x_{(1)}^{\sigma_{(k)}}\right], \quad$ for $l=1, \ldots, d-k-1$,
(c) $\left[x_{(1)}^{\tau_{(d)}}, \ldots, x_{(1)}^{\tau_{(k+1)}}\right]$.

It can be easily shown that the faces of type (b) are interior simplices lying in $\star \sigma_{(k)}$. The union of faces of type (c) constitute the cell $\sharp \tau_{(k+1)} \in \partial^{1} \sharp \sigma_{(k)}$. Rest us the faces of type (a).

A face of type (a) also lies in the interior of $\tilde{\sigma}_{(d-k)}$ when there are two $d$-cells, $\tau_{(d)}$ and $\eta_{(d)}$, such that it lies in $\tau_{(d)} \cap \eta_{(d)}$. This is not the case when the face lies in $\partial \Omega$. Note that $\tau_{(d-1)} \subset \partial \Omega$ and $\tau_{(d-1)} \in \partial^{k-d+1} \sigma_{(k)}$ implies that $\sigma_{(k} \in \partial \Omega$. So if $\sigma_{(k)} \notin \mathcal{M}_{(k)}^{\mathrm{b}}$ then the faces of type (a) also lie in the interior of $\star \sigma_{(k)}$, but if $\sigma_{(k)} \in \mathcal{M}_{(k)}^{\mathrm{b}}$ we have

$$
\begin{aligned}
\partial \approx \sigma_{(k)} & =\left(\bigcup_{\left(\tau_{(d-1)}, \ldots, \tau_{(k+1)}\right) \in I_{\sigma_{(k)}^{\mathrm{b}}}}\left[x_{(1)}^{\tau_{(d-1)}}, \ldots, x_{(1)}^{\tau_{(k+1)}}, x_{(1)}^{\sigma_{(k)}}\right]\right) \cup\left(\bigcup_{\tau_{(k+1)} \in \mathcal{D}^{-1} \sigma_{(k)}} \approx \tau_{(k)}\right) \\
& =\star_{\mathrm{b}} \sigma_{(k)} \cup(\bigcup_{\tau_{(k+1)} \in \mathcal{O}^{-1} \sigma_{(k)}} \overbrace{(k)}),
\end{aligned}
$$

where the index set $I_{\sigma_{(k)}}^{\mathrm{b}}$ is the boundary cell complex analogue of $I_{\sigma_{(k)}}$ and $\star_{\mathrm{b}} \sigma_{(k)} \in$ $\tilde{\mathcal{M}}_{(d-1-k)}^{\text {b }}$ denotes the dual cell to $\sigma_{(k)}$ within the boundary.

The construction of the dual cell complex is exemplified in 2D in Figure 3.9 and in 3D in Figure 3.10.

The boundary dual mesh $\tilde{\mathcal{M}}^{\mathrm{b}}$ is besides a mesh on $\partial \Omega$ also a cell complex on $\partial \Omega$. This follows directly from the fact that $\partial \partial \Omega=\varnothing$.

To be able to define the boundary operator on the dual mesh we also need to give an outer orientation to the dual boundary cells in $\tilde{\mathcal{M}}^{\mathrm{b}}$. The inner orientation of the


Figure 3.10: The construction of the dual cell complex in 3D.
cells in $\mathcal{M}$ provides an outer orientation for the cells in $\tilde{\mathcal{M}}^{\mathrm{b}}$. However, this is an outer orientation with respect to $\partial \Omega$. We define the outer orientation on the dual mesh as follows.

Definition 3.15. Let $\mathcal{M}$ be a cell complex on $\Omega, \mathcal{M}^{\text {b }}$ the boundary complex on $\partial \Omega$ and $\tilde{\mathcal{M}}^{\text {b }}$ the boundary dual mesh. We define the outer orientation of a boundary dual cell ${ }^{{ }^{\mathrm{b}}}{ }^{\mathrm{b}} \sigma_{(k)} \in \tilde{\mathcal{M}}_{(d-1-k)}^{\mathrm{b}}$ to be given by

$$
\left\{\approx_{\mathrm{b}} \sigma_{(k)}\right\}:=\left\{\left\{v_{(1)}^{\partial \Omega} \wedge \sigma_{(k)} \wedge o_{(d-k-1)}\right\} o_{(d-k-1)}\right\}
$$

where $o_{(d-k-1)}$ is an element of $\Lambda^{(d-k-1)}\left({ }_{\star} \sigma_{(k)}\right)$ and $v_{(1)}^{\partial \Omega}$ is the outward normal ${ }^{8}$ vector on $\partial \Omega$.

Now we can define the boundary operator on the dual mesh.
Definition 3.16. Let $\mathcal{M}$ be a cell complex on $\Omega, \mathcal{M}^{\text {b }}$ the boundary complex on $\partial \Omega$ and let $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}^{\text {b }}$ be the respective dual meshes. Furthermore, let the spaces of boundary dual $k$-chains be denoted by $\tilde{C}_{(k)}(\partial \Omega)$, with $0 \leq k \leq d-1$. We define the dual boundary operator $\tilde{\partial}_{(k)}: \tilde{C}_{(k)}(\Omega) \rightarrow \tilde{C}_{(k-1)}(\Omega) \oplus \tilde{C}_{(k-1)}(\partial \Omega)$ according to
for any dual $k$-cell $\curvearrowleft \boldsymbol{\sigma}_{(d-k)}$ and the operator extends linearly to all $k$-chains in $\tilde{C}_{(k)}(\Omega)$.
 with the outer orientation of ${ }_{\star} \sigma_{(d-k)}$ and $o_{\star \sigma_{(d-k)}{ }^{\star} \mathrm{b} \sigma_{(d-k)}}=-1$ otherwise.

We define the dual coboundary operator $\tilde{\delta}^{(k)}: \tilde{C}^{(k)}(\Omega) \oplus \tilde{C}^{(k)}(\partial \Omega) \rightarrow \tilde{C}^{(k+1)}(\Omega)$ by

$$
\left\langle\tilde{\boldsymbol{a}}_{(k+1)}, \tilde{\delta}^{(k)}\left[\begin{array}{c}
\tilde{\boldsymbol{b}}^{(k)} \\
\tilde{\boldsymbol{b}}_{\mathrm{b}}^{(k)}
\end{array}\right]\right\rangle=\left\langle\tilde{\partial}_{(k+1)} \tilde{\boldsymbol{a}}_{(k+1)},\left[\begin{array}{c}
\tilde{\boldsymbol{b}}^{(k)} \\
\tilde{\boldsymbol{b}}_{\mathrm{b}}^{(k)}
\end{array}\right]\right\rangle,
$$

for all $\tilde{\boldsymbol{a}}_{(k+1)} \in \tilde{C}_{(k+1)}(\Omega), \tilde{\boldsymbol{b}}^{(k)} \in \tilde{C}^{(k)}(\Omega), \tilde{\boldsymbol{b}}_{\mathrm{b}}^{(k)} \in \tilde{C}^{(k)}(\partial \Omega)$.

[^22]Proposition 3.10. The matrix representation of the dual boundary operator $\tilde{\partial}_{(k)}$ : $\tilde{C}_{(k)}(\Omega) \rightarrow \tilde{C}_{(k-1)}(\Omega) \oplus \tilde{C}_{(k-1)}(\partial \Omega)$ is given by

$$
\tilde{\mathbb{D}}_{(k)}=(-1)^{d-k+1}\left[\mathbb{D}_{(d-k)}^{T} \quad-\mathbb{T}^{(d-k) T}\right]=(-1)^{d-k+1}\left[\begin{array}{ll}
\mathbb{D}^{(d-k) T} & -\mathbb{T}^{(d-k) T}
\end{array}\right]
$$

and the matrix representation of the coboundary operator $\tilde{\delta}^{(k)}: \tilde{C}^{(k)}(\Omega) \oplus \tilde{C}^{(k)}(\partial \Omega) \rightarrow$ $\tilde{C}^{(k+1)}(\Omega)$ is given by

$$
\tilde{\mathbb{D}}^{(k)}=(-1)^{d-k}\left[\begin{array}{ll}
\mathbb{D}^{(d-k-1) T} & -\mathbb{T}^{(d-k-1) T}
\end{array}\right] .
$$

Proof. The matrix representation of the dual boundary operator can be decomposed in two blocks as

$$
\tilde{\mathbb{D}}_{(k)}=\left[\begin{array}{ll}
\tilde{\mathbb{D}}_{(k)}^{\mathrm{i}} & \tilde{\mathbb{D}}_{(k)}^{\mathrm{b}}
\end{array}\right],
$$

with $\tilde{\mathbb{D}}_{(k)}^{\mathrm{i}}: \tilde{C}_{(k)}(\Omega) \rightarrow \tilde{C}_{(k-1)}(\Omega)$ and $\tilde{\mathbb{D}}_{(k)}^{\mathrm{b}}: \tilde{C}_{(k)}(\Omega) \rightarrow \tilde{C}_{(k-1)}(\partial \Omega)$. By Lemma 3.4 we have $o_{\star \sigma_{(d-k) *}^{*} \sigma_{(d-k+1)}}=(-1)^{d-k+1} o_{\sigma_{(d-k+1)} \sigma_{(d-k)}}$. This implies that $\tilde{\mathbb{D}}_{(k)}^{\mathrm{i}}=(-1)^{d-k+1} \mathbb{D}_{(d-k+1)}=$ $(-1)^{d-k+1} \mathbb{D}^{(d-k)}$. By definition of the dual cell complex, we have that the entries of $\tilde{\mathbb{D}}_{(k)}^{\mathrm{b}}$ are given by $o_{\sharp \sigma_{(d-k)}{ }^{\sharp} \mathrm{b} \tau_{(d-k)}}$, which is only nonzero when $\tau_{(d-k)}=\sigma_{(d-k)}$. Furthermore, from Definition 3.15 and the definition of the induced outer orientation (Definition 2.29) it follows that $o_{\sharp \sigma_{(d-k)}{ }^{\star} \mathrm{b} \sigma_{(d-k)}}=(-1)^{d-k}$, because

$$
\begin{aligned}
\left\{\star \sigma_{(d-k-1)}\right\} & =\left\{\left\{\sigma_{(d-k-1)} \wedge o_{(k+1)}\right\} o_{(k+1)}\right\}, \\
\left\{\stackrel{\star}{\mathrm{b}} \sigma_{(d-k-1)}\right\} & =\left\{\left\{v_{(1)}^{\partial \Omega} \wedge \sigma_{(d-k-1)} \wedge o_{(k)}\right\} o_{(k)}\right\},
\end{aligned}
$$

where $v_{(1)}^{\partial \Omega}$ obviously points out of $\sharp \sigma_{(d-k)}$. This shows that $\tilde{\mathbb{D}}_{(k)}^{\mathrm{b}}=(-1)^{d-k} \mathbb{T}^{(d-k)}$.
Applying the dual boundary operator to a $(k+1)$-chain $\tilde{\boldsymbol{a}}_{(k+1)} \in \tilde{C}_{(k+1)}(\Omega)$ corresponds to multiplying the row vector $\tilde{\boldsymbol{a}}_{(k+1)}$ by $\tilde{\mathbb{D}}_{(k+1)}$ from the right. Applying the dual coboundary operator to a $k$-cochain $\left[\tilde{\boldsymbol{b}}^{(k) T} \quad \tilde{\boldsymbol{b}}_{\mathrm{b}}^{(k) T}\right]^{T} \in \tilde{C}^{(k)}(\Omega) \oplus \tilde{C}^{(k)}(\partial \Omega)$ corresponds to multiplying the column vector $\left[\tilde{\boldsymbol{b}}^{(k) T} \quad \tilde{\boldsymbol{b}}_{\mathrm{b}}^{(k) T}\right]^{T}$ by $\tilde{\mathbb{D}}^{(k)}$ from the right. Therefore, by definition of the coboundary operator we have

$$
\tilde{\boldsymbol{a}}_{(k+1)} \tilde{\mathbb{D}}_{(k+1)}\left[\begin{array}{l}
\tilde{\boldsymbol{b}}^{(k)} \\
\tilde{\boldsymbol{b}}_{\mathrm{b}}^{(k)}
\end{array}\right]=\tilde{\boldsymbol{a}}_{(k+1)} \tilde{\mathbb{D}}^{(k)}\left[\begin{array}{l}
\tilde{\boldsymbol{b}}^{(k)} \\
\tilde{\boldsymbol{b}}_{\mathrm{b}}^{(k)}
\end{array}\right],
$$

and hence $\tilde{\mathbb{D}}^{(k)}=\tilde{\mathbb{D}}_{(k+1)}$.
In Propositions 3.3 and 3.6 we see that the coboundary operator on the primal cell complex is defined such that the de Rham maps form a cochain map from the continuous cochain complex to the discrete cochain complex, i.e., $R^{(k+1)} \circ d^{(k)}=\delta^{(k)} \circ R^{(k)}$ and $R_{b}^{(k+1)} \circ d_{\mathrm{b}}^{(k)}=\delta_{b}^{(k)} \circ R_{b}^{(k)}$. A similar kind of commutation relation is satisfied by their analogues on the dual cell complex.

Definition 3.17. The dual mesh de Rham maps $\tilde{R}^{(k)}: \tilde{\Lambda}^{(k)}(\Omega) \rightarrow \tilde{C}^{(k)}(\Omega)$ are defined by

$$
\left\langle\tilde{\boldsymbol{\sigma}}_{(k), i}, \tilde{R}^{(k)}\left(\tilde{a}^{(k)}\right)\right\rangle=\int_{\tilde{\sigma}_{(k), i}} \tilde{a}^{(k)}
$$

for any $\tilde{a}^{(k)} \in \tilde{\Lambda}^{(k)}(\Omega), 1 \leq i \leq N_{(d-k)}$ and $0 \leq k \leq d$, with $\tilde{\sigma}_{(k), i} \in \tilde{\mathcal{M}}_{(k)}$ the $i$-th dual $k$-cell.

Analogously, we define the boundary dual mesh de Rham maps $\tilde{R}_{\mathrm{b}}^{(k)}: \tilde{\Lambda}^{(k)}(\partial \Omega) \rightarrow$ $\tilde{C}^{(k)}(\partial \Omega)$ according to

$$
\left\langle\tilde{\boldsymbol{\sigma}}_{(k), i}, \tilde{R}_{\mathrm{b}}^{(k)}\left(\tilde{a}^{(k)}\right)\right\rangle=\int_{\tilde{\sigma}_{(k), i}} \tilde{a}^{(k)}
$$

for any $\tilde{a}^{(k)} \in \tilde{\Lambda}^{(k)}(\partial \Omega), 1 \leq i \leq N_{(d-1-k)}^{\mathrm{b}}$ and $0 \leq k \leq d-1$, with $\tilde{\sigma}_{(k), i} \in \tilde{\mathcal{M}}_{(k)}^{\mathrm{b}}$ the $i$-th boundary dual $k$-cell.
Proposition 3.11. The set of dual mesh de Rham maps $\tilde{R}^{(k)}$, boundary dual mesh de Rham maps $\tilde{R}_{\mathrm{b}}^{(k)}$, exterior derivatives $\tilde{d}^{(k)}$ and dual coboundary operators $\tilde{\delta}^{(k)}$ satisfy the following commutation relation, for $0 \leq k \leq d-1$,

$$
\tilde{R}^{(k+1)}\left(\tilde{d}^{(k)} \tilde{a}^{(k)}\right)=\tilde{\delta}^{(k)}\left(\left[\begin{array}{c}
\tilde{R}^{(k)}\left(\tilde{a}^{(k)}\right) \\
\tilde{R}_{\mathrm{b}}^{(k)}\left(\tilde{t}^{(k)} \tilde{a}^{(k)}\right)
\end{array}\right]\right) \quad \forall \tilde{a}^{(k)} \in \tilde{\Lambda}^{(k)}(\Omega)
$$

Proof. Take any $\tilde{a}^{(k)} \in \tilde{\Lambda}^{(k)}(\Omega)$. By Stokes' theorem for outer-oriented forms (Theorem 2.3) we have, for any dual cell $\tilde{\sigma}_{(k+1)}=\sharp \sigma_{(d-k-1)} \in \tilde{\mathcal{M}}_{(k)}(\Omega)$,

$$
\begin{aligned}
& {\left[\tilde{R}^{(k+1)}\left(\tilde{d}^{(k)} \tilde{a}^{(k)}\right)\right]_{\tilde{\sigma}_{(k+1)}}=\int_{\tilde{\sigma}_{(k+1)}} \tilde{d}^{(k)} \tilde{a}^{(k)}} \\
& =\sum_{\sharp \sigma_{(d-k)} \in \partial \star \sigma_{(d-k-1)}} o_{\sharp \sigma_{(d-k-1)} \star \sigma_{(d-k)}} \int_{\sharp \sigma_{(d-k)}} \tilde{a}^{(k)} \\
& +o_{\sharp \sigma_{(d-k-1)}{ }^{\mathrm{b}_{\mathrm{b}}} \sigma_{(d-k-1)}} \int_{\star_{\mathrm{b}} \sigma_{(d-k-1)}} \tilde{t}^{(k)} \tilde{a}^{(k)} \\
& =(-1)^{d-k}\left[\mathbb{D}^{(d-k-1) T} \tilde{R}^{(k)}\left(\tilde{a}^{(k)}\right)\right]_{\tilde{\sigma}_{(k+1)}} \\
& +o_{\text {ャ } \sigma_{(d-k-1)}{ }^{\star} \mathrm{b} \sigma_{(d-k-1)}}\left[\tilde{R}_{\mathrm{b}}^{(k)}\left(\tilde{t}^{(k)}\left(\tilde{a}^{(k)}\right)\right)\right]_{\mathfrak{*}_{\mathrm{b}} \sigma_{(d-k-1)}} \\
& =\left[\tilde{\delta}^{(k)}\left(\left[\begin{array}{c}
\tilde{R}^{(k)}\left(\tilde{a}^{(k)}\right) \\
\tilde{R}_{\mathrm{b}}^{(k)}\left(\tilde{t}^{(k)} \tilde{a}^{(k)}\right)
\end{array}\right]\right)\right]_{\tilde{\sigma}_{(k+1)}},
\end{aligned}
$$

where the boundary terms are absent if $\sigma_{(d-k-1)} \notin \mathcal{M}_{(d-k-1)}^{\mathrm{b}}$.
Analogously to the boundary mesh coboundary operator $\delta_{\mathrm{b}}^{(k)}: C^{(k)}(\partial \Omega) \rightarrow C^{(k+1)}(\partial \Omega)$ we define the boundary mesh dual coboundary operator $\tilde{\delta}_{\mathrm{b}}^{(k)}: \tilde{C}^{(k)}(\partial \Omega) \rightarrow \tilde{C}^{(k+1)}(\partial \Omega)$. The boundary dual mesh $\tilde{\mathcal{M}}^{\text {b }}$ is a cell complex, hence the sequence $\left(\tilde{\delta}_{\mathrm{b}}^{(k)}, \tilde{C}^{(k)}(\partial \Omega)\right)$ is a cochain complex. We denote the matrix representation of $\tilde{\delta}_{\mathrm{b}}^{(k)}$ by $\tilde{\mathbb{D}}_{\mathrm{b}}^{(k)}$.


Figure 3.11: Three commutation relations are shown: $\tilde{t}^{(k+1)} \circ \tilde{d}^{(k)}=\tilde{d}_{\mathrm{b}}^{(k)} \circ \tilde{t}^{(k)}$ (Section 2.2.6), the commutation relation given in Proposition 3.11, and $\tilde{R}_{\mathrm{b}}^{(k+1)} \circ \tilde{d}_{\mathrm{b}}^{(k)}=\tilde{\delta}_{\mathrm{b}}^{(k)} \circ \tilde{R}_{\mathrm{b}}^{(k)}$ (Proposition 3.12).

Proposition 3.12. The matrix representation of $\tilde{\delta}_{\mathrm{b}}^{(k)}$ is given by $\tilde{\mathbb{D}}_{\mathrm{b}}^{(k)}=(-1) \mathbb{D}_{\mathrm{b}}^{(d-2-k)}$. Furthermore, we have $\tilde{R}_{\mathrm{b}}^{(k+1)} \circ \tilde{d}_{\mathrm{b}}^{(k)}=\tilde{\delta}_{\mathrm{b}}^{(k)} \circ \tilde{R}_{\mathrm{b}}^{(k)}$, for $0 \leq k \leq d-2$.

Proof. By a similar argumentation as used in the proof of Lemma 3.4 we have that $o_{\dot{\sharp}_{\mathrm{b}} \sigma_{(k-1)^{\star} \mathrm{b}} \sigma_{(k)}}=(-1)^{k} o_{\sigma_{(k)} \sigma_{(k-1)}}$. From this it follows that $\tilde{\mathbb{D}}_{\mathrm{b}}^{(d-1-k)}=(-1)^{k} \mathbb{D}_{\mathrm{b}}^{(k-1) T}$ for $1 \leq k \leq d-1$. The commutation relation is just a special case of Proposition 3.11.

The commutation relations from Propositions 3.11 and 3.12 are shown in Figure 3.11 for $d=3$.

### 3.3 Discrete Hodge operators

We now come to the next element of the mimetic discretization: the discrete Hodge operators. We have seen that the discretization of the exterior derivative as incidence matrices uniquely follows from Stokes' theorem and the mesh connectivity. The discretization of the Hodge operators allows for more variety. It is in the discrete Hodge matrices where different mimetic methods vary. For example, on a given Cartesian mesh, the incidence and trace matrices for the Discrete Exterior Calculus (DEC) [50], Mimetic Finite Difference (MFD) [28], Mimetic Spectral Element (MSE) [51] methods, and for the Discrete Geometric Approach (DGA) [52] are all the same. It is only in the discrete Hodge matrices that they vary.

In this section we consider the discretization of the final pieces of the diagram in Figure 2.18: the Hodge operators $\star^{(k)}: \Lambda^{(k)}(\Omega) \rightarrow \tilde{\Lambda}^{(d-k)}(\Omega)$ and $\tilde{\star}^{(k)}: \tilde{\Lambda}^{(k)}(\Omega) \rightarrow$ $\Lambda^{(d-k)}(\Omega)$. They will be discretized as square matrices $\mathbb{H}^{(k)}: C^{(k)}(\Omega) \rightarrow \tilde{C}^{(d-k)}(\Omega)$ and $\tilde{\mathbb{H}}^{(k)}: \tilde{C}^{(k)}(\Omega) \rightarrow C^{(d-k)}(\Omega)$. This will make the discretization of the double cochain complex of Figure 2.18 complete. The discrete version, including trace operators, is


Figure 3.12: The discrete operators and spaces.
given in Figure 3.12 in terms of four types of matrices only: the incidence matrices $\mathbb{D}^{(k)}$, the trace matrices $\mathbb{T}^{(k)}$, the primal discrete Hodge matrices $\mathbb{H}^{(k)}$ and the dual discrete Hodge matrices $\tilde{\mathbb{H}}^{(k)}$.

Most mimetic methods are based on the primal discrete Hodge matrices $\mathbb{H}^{(k)}$. We saw already in Section 2.1.3 that $\tilde{\star}^{(k)}=(-1)^{k(d-k)}\left(\star^{(d-k)}\right)^{-1}$. Similarly, with the primal discrete Hodge matrices given we immediately have the dual discrete Hodge matrices given by

$$
\begin{equation*}
\tilde{\mathbb{H}}^{(k)}=(-1)^{k(d-k)}\left(\mathbb{H}^{(d-k)}\right)^{-1} . \tag{3.6}
\end{equation*}
$$

However, the inverse on the right-hand side is almost never explicitly known and to apply the dual discrete Hodge matrices a linear system needs to be solved. When a circumcentric dual mesh is used a diagonal primal discrete Hodge matrix can be used and the inverses are given explicitly. This is the only situation where the inverses of the primal discrete Hodge matrices are given.

In Chapter 6 for a barycentric dual mesh, we will define new dual discrete Hodge matrices $\tilde{\mathbb{H}}^{(k)}$ in explicit form. In this section we first briefly recall the construction of the diagonal discrete Hodge matrices for a circumcentric dual mesh. Subsequently, we discuss the construction of the primal discrete Hodge matrices for a barycentric dual mesh. This derivation is not really new (see for example [53]), but we present it in a new geometric way which gives a hint of how dual discrete Hodge matrices might be formed.

### 3.3.1 Circumcentric diagonal Hodge matrices

The circumcentric dual mesh has the important feature that any primal cell $\sigma_{(l)} \in \mathcal{M}_{(l)}$ and its corresponding dual cell $\approx \sigma_{(l)} \in \tilde{\mathcal{M}}_{(d-l)}$ are mutually orthogonal. This means that, if we consider $\sigma_{(l)}$ and $\sharp \sigma_{(l)}$ as inner $l$-vector and outer $(d-l)$-vector (recall (3.2) and
(3.5)), then there exists a $c_{\sigma_{(l)}} \in \mathbb{R}$ such that

$$
\begin{equation*}
\star \sigma_{(l)}=c_{\sigma_{(l)}} \sharp \sigma_{(l)}, \tag{3.7}
\end{equation*}
$$

where on the left-hand side we have $\star: \Lambda_{(l)} V \rightarrow \tilde{\Lambda}_{(d-l)} V$. Taking the inner product with $\star \sigma_{(l)}$ on both sides we find

$$
c_{\sigma_{(l)}}=\frac{\left(\sigma_{(l)}, \sigma_{(l)}\right)}{\left(\star \sigma_{(l)}, \star \sigma_{(l)}\right)},
$$

where we used that the Hodge star operator is an isometry and hence $\left(\star \sigma_{(l)}, \star \sigma_{(l)}\right)=$ $\left(\sigma_{(l)}, \sigma_{(l)}\right)$. Note that $\left|c_{\sigma_{(l)}}\right|=\left|\sigma_{(l)}\right| /\left|\hbar \sigma_{(l)}\right|$. The sign is important because the dual cell might be reversed if the primal mesh is not well-centered.

For any $a^{(l)} \in \Lambda^{(l)}(V),(3.7)$ implies that

$$
\left\langle a^{(l)}, \sigma_{(l)}\right\rangle=\left\langle\star a^{(l)}, \star \sigma_{(l)}\right\rangle=\left(\frac{\left(\sigma_{(l)}, \sigma_{(l)}\right)}{\left(\star \sigma_{(l)}, \star \sigma_{(l)}\right)}\right)\left\langle\star a^{(l)}, \star \sigma_{(l)}\right\rangle .
$$

Thus, given the value $a_{\sigma_{(l)}}^{(l)}$ of a cochain on $\boldsymbol{a}^{(l)}:=R^{(l)}\left(a^{(l)}\right)$ on the cell $\sigma_{(l)}$, we can approximate the value $\tilde{a}_{\star \sigma \sigma_{(l)}}^{(d-l)}$ of the cochain $\tilde{\boldsymbol{a}}^{(d-l)}:=\tilde{R}^{(d-l)}\left(\star a^{(l)}\right)$ on $\star \sigma_{(l)}$ by

$$
\tilde{a}_{\sharp \sigma_{(l)}}^{(d-l)} \approx\left(\frac{\left(\star \sigma_{(l)}, \star \sigma_{(l)}\right)}{\left(\sigma_{(l)}, \sigma_{(l)}\right)}\right) a_{\sigma_{(l)}}^{(l)}
$$

Similarly, $\tilde{\star} \approx \sigma_{(l)}=\tilde{c}_{\sigma_{(l)}} \sigma_{(l)}$, with

$$
\tilde{c}_{\sigma_{(l)}}=\frac{\left(\star \tilde{\star} \star \sigma_{(l)}, \star \sigma_{(l)}\right)}{\left(\star \sigma_{(l)}, \star \sigma_{(l)}\right)}=(-1)^{l(d-l)} \frac{\left(\star \sigma_{(l)}, \star \sigma_{(l)}\right)}{\left(\sigma_{(l)}, \sigma_{(l)}\right)} .
$$

This implies that for $\tilde{b}^{(d-l)} \in \tilde{\Lambda}^{(d-l)} V$ we have

$$
\left\langle\tilde{b}^{(d-l)}, \star \sigma_{(l)}\right\rangle=\left\langle\tilde{\tilde{b}}^{(d-l)}, \tilde{\star} \approx \sigma_{(l)}\right\rangle=(-1)^{l(d-l)}\left(\frac{\left(\stackrel{\wedge}{ } \sigma_{(l)}, \star \sigma_{(l)}\right)}{\left(\sigma_{(l)}, \sigma_{(l)}\right)}\right)\left\langle\tilde{\tilde{b}}^{(d-l)}, \sigma_{(l)}\right\rangle .
$$

So, given the value $\tilde{b}_{\sharp<\sigma_{(l)}}^{(d-l)}$ of a cochain on $\tilde{\boldsymbol{b}}^{(d-l)}:=\tilde{R}^{(d-l)}\left(\tilde{b}^{(d-l)}\right)$ on the cell $\sharp \sigma_{(l)}$, we can approximate the value $b_{\sigma_{(l)}}^{(l)}$ of the cochain $\boldsymbol{b}^{(l)}:=R^{(l)}\left(\tilde{\varkappa}^{(d-l)}\right)$ on $\sigma_{(l)}$ by

$$
b_{\sigma_{(l)}}^{(l)} \approx(-1)^{l(d-l)}\left(\frac{\left(\star \sigma_{(l)}, \star \sigma_{(l)}\right)}{\left(\sigma_{(l)}, \sigma_{(l)}\right)}\right) \tilde{b}_{\mathfrak{k} \sigma_{(l)}}^{(d-l)} .
$$

This suggests the following definition of Hodge matrices.

Definition 3.18. Let $\mathcal{M}$ be a cell complex on $\Omega$ with circumcentric dual mesh $\tilde{\mathcal{M}}$. The circumcentric primal Hodge matrices $\mathbb{H}_{c}^{(l)}: C^{(l)}(\Omega) \rightarrow \tilde{C}^{(d-l)}(\Omega)$, with $0 \leq l \leq d$, are given by

$$
\left[\mathbb{H}_{c}^{(l)}\right]_{\star \tau_{(l)}, \sigma_{(l)}}:=\delta_{\sigma_{(l)}, \tau_{(l)}} \frac{\left(\star \tau_{(l)}, \star \sigma_{(l)}\right)}{\left(\sigma_{(l)}, \sigma_{(l)}\right)},
$$

where $\delta_{\sigma_{(l)}, \tau_{(l)}}=1$ if $\sigma_{(l)}=\tau_{(l)}$ and $\delta_{\sigma_{(l)}, \tau_{(l)}}=0$ if $\sigma_{(l)} \neq \tau_{(l)}$.
The circumcentric dual Hodge matrices $\tilde{\mathbb{H}}^{(d-l)}: \tilde{C}^{(d-l)}(\Omega) \rightarrow C^{(l)}(\Omega)$, with $0 \leq l \leq d$, are given by

$$
\left[\tilde{\mathbb{H}}_{c}^{(l)}\right]_{\star \tau_{(l)}, \sigma_{(l)}}:=(-1)^{l(d-l)} \delta_{\sigma_{(l)}, \tau_{(l)}} \frac{\left(\sigma_{(l)}, \sigma_{(l)}\right)}{\left(\star \sigma_{(l)}, \star \sigma_{(l)}\right)} .
$$

These Hodge matrices satisfy (3.6). This implies that $\tilde{H}^{(d-l)} \mathbb{H}^{(l)} \boldsymbol{a}^{(l)}=(-1)^{l(d-l)} \boldsymbol{a}^{(l)}$ for all $\boldsymbol{a}^{(l)} \in C^{(l)}(\Omega)$, mimicking $\tilde{\star}^{(d-l)} \star^{(l)} a^{(l)}=(-1)^{l(d-l)} a^{(l)}$ for all $a^{(l)} \in \Lambda^{(l)}(\Omega)$.

### 3.3.2 Barycentric primal Hodge matrices: Consistency

The orthogonality of the circumcentric dual mesh with respect to the primal mesh results in diagonal Hodge matrices. The barycentric dual mesh is in general not orthogonal to the primal mesh. Therefore consistent diagonal Hodge matrices are not possible.

To derive a consistent interpolation formula from the primal mesh to the barycentric dual mesh, multiple cochain values are combined. Suppose we want to interpolate a cochain $\boldsymbol{a}^{(l)}:=R^{(l)}\left(a^{(l)}\right) \in C^{(l)}(\Omega)$ to a consistent approximation of $\tilde{a}^{(d-l)}:=$ $\tilde{R}^{(d-l)}\left(\star a^{(l)}\right) \in \tilde{C}^{(d-l)}(\Omega)$, for some $l$-form $a^{(l)} \in \Lambda^{(l)}(\Omega)$. The interpolation from $C^{(l)}(\Omega)$ to $\tilde{C}^{(d-l)}(\Omega)$ is performed in two steps. First a consistent approximation of $a^{(l)}\left(x_{(1)}^{\sigma_{(d)}}\right)$, the value of $a^{(l)}$ in the barycenter of $\sigma_{(d)}$, is determined using the values $a_{\sigma_{(l)}}^{(l)}:=\left\langle\boldsymbol{a}^{(l)}, \boldsymbol{\sigma}_{(l)}\right\rangle$, for $\sigma_{(l)} \in \partial^{d-l} \sigma_{(d)}$. This is done for every $d$-cell $\sigma_{(d)} \in \mathcal{M}_{(d)}$. Then in the second step these approximate values of $a^{(l)}$ in the barycentra of the $d$-cells are used to determine an approximation of $\tilde{\boldsymbol{a}}^{(d-l)}$.

This interpolation procedure has a number of appealing aspects. The operations in the two steps are each others transpose when viewed as matrix operations. This implies that the resulting matrix is symmetric. Furthermore, the interpolation is local leading to sparse discrete Hodge matrices.

However, the discrete Hodge matrices found using the interpolation procedure sketched above not always result in a positive definite matrix. To define discrete Hodge matrices that are sparse, symmetric and positive definite a second matrix is added to each consistent Hodge matrix, which does not impair its sparsity, consistency and symmetry, but makes sure that the resulting matrices are positive definite.

In this section we will derive the consistent part. This derivation can also be found in [53] and [54]. Here we present it using exterior calculus, which allows us to portray the full symmetry of the derivation and give a clearer geometric interpretation of the formulas. This will be helpful also when we derive new barycentric dual Hodge matrices in Chapter 6.

An essential ingredient for the cell-wise reconstruction formulas are local dual cell vectors.

Definition 3.19. The $(k-l)$-cell ${ }_{\wedge_{(k)}} \sigma_{(l)}$ that is dual to the $l$-cell $\sigma_{(l)}$ with respect to the $k$-cell $\sigma_{(k)} \in \partial^{l-k} \sigma_{(l)}$ with $l \leq k$, is defined by

$$
\sharp_{\sigma_{(k)}} \sigma_{(l)}:=\sum_{\substack{\left(\tau_{(k-1)}, \ldots, \tau_{(l+1)}\right) \\ \epsilon I_{\sigma_{(l)}}^{\sigma}(k)}} \frac{o_{\sigma_{(l)}, \tau_{(l+1)}, \ldots, \tau_{(k-1)}, \sigma_{(k)}}}{(k-l)!}\left(x_{(1)}^{\tau_{(l+1)}}-x_{(1)}^{\sigma_{(l)}}\right) \wedge \cdots \wedge\left(x_{(1)}^{\sigma_{(k)}}-x_{(1)}^{\tau_{(k-1)}}\right),
$$

with $o_{\sigma_{(l)}}^{*} \tau_{(l+1)}, \ldots, \tau_{(k-1)}, \sigma_{(k)}:=o_{\sharp \sigma_{(l)} \sharp \tau_{(l+1)}} \cdots o_{\sharp} \tau_{(k-1)} \sharp \sigma_{(k)} o_{\sharp} \sigma_{(k)}$, when $l-1<k$, $\sigma_{(k)} \sigma_{(k-1)}=$ $o_{\star \sigma_{(k-1)} \sigma_{(k)}} o_{\sharp \sigma_{(k)}}\left(x_{(1)}^{\sigma_{(k)}}-x_{(1)}^{\sigma_{(k-1)}}\right)$, and ${ }^{\star} \sigma_{(k)} \sigma_{(k)}=o_{\sharp \sigma_{(k)}}:=\left\{\sigma_{(k)}\right\}$.

Furthermore, we define ${ }^{\wedge_{\sigma_{(k)}}} \sigma^{(l)}:=b{ }^{\star} \sigma_{(k)} \sharp \sigma^{(l)}$. This implies that

$$
{\stackrel{»}{\sigma_{(k)}}} \sigma^{(l)}:=\sum_{\substack{\left(\tau_{(k-1)}, \ldots, \tau_{(l+1)}\right) \\ \epsilon I_{\sigma_{(l)}}^{\sigma(k)}}} \frac{o_{\sigma_{(l)}, \tau_{(l+1)}, \ldots, \tau_{(k-1)}, \sigma_{(k)}}^{\sigma_{(k)}}}{(k-l)!}\left(x_{\tau_{(l+1)}}^{(1)}-x_{\sigma_{(l)}}^{(1)}\right) \wedge \cdots \wedge\left(x_{\sigma_{(k)}}^{(1)}-x_{\tau_{(k-1)}}^{(1)}\right),
$$

where $x_{\tau_{(m)}}^{(1)}:=b x_{(1)}^{\tau_{(m)}}$.
Note that, just like $\sharp \sigma_{(l)},{ }^{\star} \sigma_{(k)} \sigma_{(l)}$ is an outer vector, however the outer-orientation is now an orientation of the $l$-dimensional complement to ${ }^{{ }^{\sigma_{(k)}}} \sigma_{(l)}$ within $\sigma_{(k)}$. This is illustrated in Figure 3.13.

The dual cell forms satisfy the recursive relation

$$
\sharp_{\sigma_{(k)}} \sigma^{(l)}=\sum_{\tau_{(k-1)} \in \partial \sigma_{(k)} \cap \partial^{l-k+1} \sigma_{(l)}}\left(\frac{o_{\star<\sigma_{(k)} \star \tau_{(k-1)}}}{k-l}\right)\left(\AA_{\tau_{(k-1)}} \sigma^{(l)}\right) \wedge\left(x_{\sigma_{(k)}}^{(1)}-x_{\tau_{(k-1)}}^{(1)}\right),
$$

where it should be noted that the outer orientation on the left-hand side is with respect to $\sigma_{(k)}$ and on the right-hand side with respect to $\tau_{(k-1)}$. However, these orientations can be translated into each other because an orientation for $\sigma_{(k)}$ implies an orientation for $\tau_{(k-1)} \in \partial \sigma_{(k)}$ and vice versa. Using Lemma 3.4 and reordering the terms we can also write this recursive relation as

$$
\begin{equation*}
{ }^{\star} \sigma_{(k)} \sigma^{(l)}=\sum_{\tau_{(k-1)} \in \partial \sigma_{(k)} \cap \partial^{l-k+1} \sigma_{(l)}}(-1)^{l}\left(\frac{o_{\sigma_{(k)} \tau_{(k-1)}}}{k-l}\right)\left(x_{\tau_{(k-1)}}^{(1)}-x_{\sigma_{(k)}}^{(1)}\right) \wedge\left(\AA_{\tau_{(k-1)}} \sigma^{(l)}\right) . \tag{3.8}
\end{equation*}
$$

| $k$ | 0 |  | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0 |  |  |  |  |  |

Figure 3.13: This figure represents the different realizations of the operation $\sigma_{(l)} \rightarrow \stackrel{\star}{ } \sigma_{(k)} \sigma_{(l)}$. In blue we show $\sigma_{(l)}$ and in red $\boldsymbol{\sharp}_{\sigma_{(k)}} \sigma_{(l)}$.

Suppose we have an $l$-form $a^{(l)} \in \Lambda^{(k)}(\Omega)$ which is discretized on the $l$-cells of the mesh as $\boldsymbol{a}^{(l)}=R^{(l)}\left(a^{(l)}\right)$. Let us only consider a single $d$-dimensional mesh cell $\sigma_{(d)}$ and let $a_{\sigma_{(l)}}^{(l)}:=\left\langle\boldsymbol{a}^{(l)}, \boldsymbol{\sigma}_{(l)}\right\rangle$ be the value of $\boldsymbol{a}^{(l)}$ on cell $\sigma_{(l)}$. The dual cell $(d-l)$-forms ${ }_{\boldsymbol{\wedge}_{\sigma_{(d)}}} \sigma^{(l)}$ can be used to calculate a consistent approximation of the $(d-l)$-form $\star a^{(l)}$ in the cell $\sigma_{(d)}$. If $a^{(l)}$ is constant in $\sigma_{(d)}$, then

$$
\begin{equation*}
\star a^{(l)}=\frac{1}{\left|\sigma_{(d)}\right|} \sum_{\sigma_{(l)} \in \partial^{d-l} \sigma_{(d)}} a_{\sigma_{(l)}}^{(l)}{ }^{\sharp} \sigma_{(d)} \sigma^{(l)} . \tag{3.9}
\end{equation*}
$$

This formula can be used to derive a cell-wise symmetric interpolation formula from the primal mesh to the dual mesh.

Let us prove reconstruction formula (3.9). The proof is recursive, as such also the lower-dimensional cases will be proved at once. So this also shows immediately how to interpolate, for example, from the boundary primal mesh to the boundary dual mesh. In the proof we use relative Hodge-star operators like $\star_{\sigma_{(k)}}$, which is the Hodge-star operator on the spaces $\Lambda^{(l)}\left(\sigma_{(k)}\right)$ of differential $l$-forms, with $0 \leq l \leq k$. An expression like $\star_{\sigma_{(k)}} a^{(l)}$ for $a \in \Lambda^{(l)}(\Omega)$ should be read like ${ }^{\sigma_{(k)}} t_{\sigma_{(k)}} a^{(l)}$, where $t_{\sigma_{(k)}}: \Lambda^{(l)}(\Omega) \rightarrow \Lambda^{(l)}\left(\sigma_{(k)}\right)$ gives the trace of $a^{(l)}$ on $\sigma_{(k)}$. Thus $\star_{\sigma_{(k)}} a^{(l)}$ is an outer-oriented $(k-l)$-form, where the outer orientation is with respect to $\sigma_{(k)}$.
Theorem 3.1. (Primal mesh interpolation property.) Let $\mathcal{M}$ be a cell complex. For all $l$-forms $a^{(l)} \in \Lambda^{(l)} V$ and $\sigma_{(k)} \in \mathcal{M}_{(k)}$, with $l \leq k \leq d$, we have

$$
\begin{equation*}
\left|\sigma_{(k)}\right| \star_{\sigma_{(k)}} a^{(l)}=\sum_{\sigma_{(l)} \in \partial^{k-l} \sigma_{(k)}}\left\langle a^{(l)}, \sigma_{(l)}\right\rangle{ }^{\star} \sigma_{(k)} \sigma^{(l)} . \tag{3.10}
\end{equation*}
$$

Proof. We prove (3.10) by showing that the left- and right-hand side satisfy the same recursion formula, namely

$$
A_{\sigma_{(k)}}^{(l)}=\sum_{\tau_{(k-1)} \in \partial \sigma_{(k)}}(-1)^{l}\left(\frac{o_{\sigma_{(k)} \tau_{(k-1)}}}{k-l}\right)\left(x_{\tau_{(k-1)}}^{(1)}-x_{\sigma_{(k)}}^{(1)}\right) \wedge A_{\tau_{(k-1)}}^{(l)},
$$

together with $A_{\sigma_{(l)}}^{(l)}=\left\{\sigma_{(l)}\right\}\left\langle a^{(l)}, \sigma_{(l)}\right\rangle$.
We start with the right-hand side. From (3.8) we find that the right-hand side indeed follows the recursion formula:

$$
\begin{aligned}
& \sum_{\sigma_{(l)} \in \partial^{k-l} \sigma_{(k)}}\left\langle a^{(l)}, \sigma_{(l)}\right\rangle{ }^{\sharp} \sigma_{(k)} \sigma^{(l)} \\
& =\sum_{\sigma_{(l)} \in \partial^{k-l} \sigma_{(k)}} \sum_{\substack{\tau_{(k-1)} \\
\cap \partial \sigma_{(k)} \cap \partial^{l-(k-1)} \sigma_{(l)}}}\left\langle a^{(l)}, \sigma_{(l)}\right\rangle(-1)^{l}\left(\frac{o_{\sigma_{(k)} \tau_{(k-1)}}}{k-l}\right)\left(x_{\tau_{(k-1)}}^{(1)}-x_{\sigma_{(k)}}^{(1)}\right) \wedge{ }^{\star_{\tau_{(k-1)}}} \sigma^{(l)} \\
& =\sum_{\tau_{(k-1)} \in \partial \sigma_{(k)}} \sum_{\substack{\sigma_{(l)} \\
\epsilon \partial^{(k-1)-l} \tau_{(k-1)}}}\left\langle a^{(l)}, \sigma_{(l)}\right\rangle(-1)^{l}\left(\frac{o_{\sigma_{(k)} \tau_{(k-1)}}}{k-l}\right)\left(x_{\tau_{(k-1)}}^{(1)}-x_{\sigma_{(k)}}^{(1)}\right) \wedge{ }^{\wedge_{\tau_{(k-1)}}} \sigma^{(l)} \\
& =\sum_{\tau_{(k-1)} \in \partial \sigma_{(k)}}(-1)^{l}\left(\frac{o_{\sigma_{(k)} \tau_{(k-1)}}}{k-l}\right)\left(x_{\tau_{(k-1)}}^{(1)}-x_{\sigma_{(k)}}^{(1)}\right) \wedge\left(\sum_{\sigma_{(l)} \in \partial^{(k-1)-l} \tau_{(k-1)}}\left\langle a^{(l)}, \sigma_{(l)}\right\rangle{ }_{\wedge^{\prime} \tau_{(k-1)}} \sigma^{(l)}\right) .
\end{aligned}
$$

To show that the left hand side satisfies the same recursion formula requires somewhat more work. The $(k-l)$-form ${ }_{\sigma_{(k)}} a^{(l)}$ is a constant element of the space $\tilde{\Lambda}^{(k-l)}\left(\sigma_{(k)}\right)$. Let us define

$$
\left(\tilde{a}^{(p)}, \tilde{b}^{(p)}\right)_{\sigma_{(k)}}:=\int_{\sigma_{(k)}} \tilde{a}^{(p)} \wedge \tilde{\star}_{\sigma_{(k)}} \tilde{b}^{(p)}, \quad \forall \tilde{a}^{(p)}, \tilde{b}^{(p)} \in \tilde{\Lambda}^{(p)}\left(\sigma_{(k)}\right) .
$$

Let $x^{i}, 1 \leq i \leq k$, be some Cartesian coordinates on $\sigma_{(k)}$ such that for the corresponding basis $e_{(1)}^{i}, 1 \leq i \leq k$, we have $\left\{e_{(1)}^{1} \wedge \cdots \wedge e_{(1)}^{k}\right\}=\left\{\sigma_{(k)}\right\}$. A basis for the space of constant elements of $\tilde{\Lambda}^{(k-l)}\left(\sigma_{(k)}\right)$ is given by

$$
\left\{\left\{\sigma_{(k)}\right\} d x^{i_{1}} \wedge \cdots \wedge d x^{i_{k-l}} \mid 1 \leq i_{1}<\cdots<i_{k-l} \leq k\right\} .
$$

Thus, we can rewrite the left hand side as

$$
\begin{aligned}
\star_{\sigma_{(k)}} a^{(l)}\left|\sigma_{(k)}\right| & =\sum_{i_{1}<\cdots<i_{k-l}}\left(\left\{\sigma_{(k)}\right\} d x^{i_{1}} \wedge \cdots \wedge d x^{i_{k-l}}, \star_{\sigma_{(k)}} a^{(l)}\right)_{\sigma_{(k)}}\left\{\sigma_{(k)}\right\} d x^{i_{1}} \wedge \cdots \wedge d x^{i_{k-l}}, \\
& =(-1)^{(k-l) l} \sum_{i_{1}<\cdots<i_{k-l}}\left(\int_{\sigma_{(k)}} d x^{i_{1}} \wedge \cdots \wedge d x^{i_{k-l}} \wedge a^{(l)}\right) d x^{i_{1}} \wedge \cdots \wedge d x^{i_{k-l}},
\end{aligned}
$$

where we used $\tilde{\star}_{\sigma_{(k)}} \star_{\sigma_{(k)}} a^{(l)}=(-1)^{(k-l) l} a^{(l)}$.
By definition of the exterior derivative and $d a^{(l)}=0$ we have

$$
\begin{aligned}
d x^{i_{1}} \wedge \cdots \wedge & d x^{i_{k-l}} \wedge a^{(l)} \\
& =\frac{1}{k-l} d\left(\sum_{t=1}^{k-l}(-1)^{t+1}\left(x_{i_{t}}-x_{i_{t}}^{\sigma_{(k)}}\right) d x^{i_{1}} \wedge \cdots \wedge \widehat{d x^{i_{t}}} \wedge \cdots \wedge d x^{i_{k-l}}\right) \wedge a^{(l)} \\
& =\frac{1}{k-l} d\left(\sum_{t=1}^{k-l}(-1)^{t+1}\left(x_{i_{t}}-x_{i_{t}}^{\sigma_{(k)}}\right) d x^{i_{1}} \wedge \cdots \wedge \widehat{d x^{i_{t}}} \wedge \cdots \wedge d x^{i_{k-l}} \wedge a^{(l)}\right),
\end{aligned}
$$

where $x_{i_{t}}^{\sigma_{(k)}}$ is the $i_{t}$-th coordinate of $x_{\sigma_{(k)}}^{(1)}$. From now on we often omit $\wedge$ in the repeated products. By Stokes' Theorem we obtain

$$
\begin{aligned}
\int_{\sigma_{(k)}} & d x^{i_{1}} \wedge \cdots \wedge d x^{i_{k-l}} \wedge a^{(l)} \\
& =\sum_{\tau_{(k-1)} \in \partial \sigma_{(k)}}\left(\frac{o_{\sigma_{(k)} \tau_{(k-1)}}}{k-l}\right)_{\tau_{(k-1)}} \sum_{t=1}^{k-l}(-1)^{t+1}\left(x_{i_{t}}-x_{i_{t}}^{\sigma_{(k)}}\right) d x^{i_{1}} \cdots \widehat{d x^{i_{t}} \cdots d x^{i_{k-l}} \wedge a^{(l)},}
\end{aligned}
$$

where, for convenience, we do not explicitly write the trace which is implied in the right-hand side.

We consider the integral for one of the $\tau_{(k-1)}$, rewrite this by adding and subtracting the face barycenter coordinate $x_{i_{t}}^{\tau_{(k-1)}}$ and apply the midpoint rule, which is exact for the
linear integrand:

$$
\begin{aligned}
\int_{\tau_{(k-1)}}\left(x_{i_{t}}-x_{i_{t}}^{\sigma_{(k)}}\right) d x^{i_{1}} \cdots \widehat{d x^{i_{t}} \cdots d x^{i_{k-l}} \wedge a^{(l)}}= & \int_{\tau_{(k-1)}}\left(x_{i_{t}}-x_{i_{t}}^{\tau_{(k-1)}}\right) d x^{i_{1}} \cdots \widehat{d x^{i_{t}} \cdots d x^{i_{k-l}} \wedge a^{(l)}} \\
& +\left(x_{i_{t}}^{\tau_{(k-1)}}-x_{i_{t}}^{\sigma_{(k)}}\right) \int_{\tau_{(k-1)}} d x^{i_{1} \cdots \widehat{d x^{i_{t}} \cdots} d x^{i_{k-l}} \wedge a^{(l)}} \\
= & \left(x_{i_{t}}^{\tau_{(k-1)}}-x_{i_{t}}^{\sigma_{(k)}}\right) \int_{\tau_{(k-1)}} d x^{i_{1} \cdots \widehat{d x^{i_{t}} \cdots} d x^{i_{k-l}} \wedge a^{(l)} .} \text {. }
\end{aligned}
$$

We continue now with one of the faces and consider all the components of the ( $k-l$ )vector (we omit the prefactor $(-1)^{(k-l) l}$ to save some space) and find

For convenience we denote the last quantity by $I_{\tau_{(k-1)}}$. We change the summation order:

$$
\begin{aligned}
& I_{\tau_{(k-1)}}
\end{aligned}
$$

$$
\begin{aligned}
& =\left(x_{\tau_{(k-1)}^{(1)}}^{(1)}-x_{\sigma_{(k)}}^{(1)}\right) \wedge \sum_{i_{1}<\cdots<i_{k-1-l}}\left(\int_{\tau_{(k-1)}} d x^{\left.i_{1} \cdots d x^{i_{k-1-l}} \wedge a^{(l)}\right) d x^{i_{1}} \cdots d x^{i_{k-1-l}} . ~ . ~ . ~ . ~ . ~}\right.
\end{aligned}
$$

Note that the indices in the sum run from 1 up to $k$. Let $y^{j}, 1 \leq j \leq k$ be another set of Cartesian coordinates on $\sigma_{(k)}$ with the same orientation, but such that $y^{j}, 1 \leq j \leq k-1$ are Cartesian coordinates on $\tau_{(k-1)}$. Let $x^{i}=\sum_{j=1}^{k} \alpha_{j}^{i} y^{j}$ and also $d x^{i}=\sum_{j=1}^{k} \alpha_{j}^{i} d y^{j}$ because the transformation matrix $\left(\alpha_{j}^{i}\right)$ is constant. We rewrite the integral as follows

$$
\begin{aligned}
& =\sum_{i_{1}, \ldots, i_{k}=1}^{k}(-1)^{(l+1)(k-l-1)}\left(\frac{\operatorname{sgn}\left\{i_{1}, \ldots, i_{k}\right\}}{(l+1)!(k-l-1)!}\right)
\end{aligned}
$$

where we used that

$$
\begin{aligned}
d x^{i_{1} \cdots d x^{i_{k-1-l}}} & =\operatorname{sgn}\left\{i_{k-l}, \ldots, i_{k}, i_{1}, \ldots, i_{k-l-1}\right\} \tilde{\star}_{\sigma_{(k)}}\left(\left\{\sigma_{(k)}\right\} d x^{\left.i_{k-l} \cdots d x^{i_{k}}\right)}\right. \\
& =(-1)^{(l+1)(k-l-1)} \operatorname{sgn}\left\{i_{1}, \ldots, i_{k}\right\} \tilde{\star}_{\sigma_{(k)}}\left(\left\{\sigma_{(k)}\right\} d x^{i_{k-l} \cdots d x^{i_{k}}}\right) .
\end{aligned}
$$

The fact that $\left(\alpha_{j}^{i}\right)$ is a transformation matrix between two Cartesian coordinate systems with the same orientation implies $\operatorname{det}\left(\alpha_{j}^{i}\right)=1$. Furthermore, by definition of the determinant we have

$$
\sum_{i_{1}, \ldots, i_{k}=1}^{k} \operatorname{sgn}\left\{i_{1}, \ldots, i_{k}\right\} \alpha_{j_{1}}^{i_{1}} \cdots \alpha_{j_{k}}^{i_{k}}=\operatorname{sgn}\left\{j_{1}, \ldots, j_{k}\right\} \operatorname{det}\left(\alpha_{j}^{i}\right)=\operatorname{sgn}\left\{j_{1}, \ldots, j_{k}\right\} .
$$

Using this after substituting the new coordinates we find

$$
\begin{aligned}
& =\sum_{j_{1}, \ldots, j_{k}=1}^{k} \sum_{i_{1}, \ldots, i_{k}=1}^{k}(-1)^{(l+1)(k-l-1)}\left(\frac{\operatorname{sgn}\left\{i_{1}, \ldots, i_{k}\right\} \alpha_{j_{1}}^{i_{1}} \cdots \alpha_{j_{k}}^{i_{k}}}{(l+1)!(k-l-1)!}\right)
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{j_{1}, \ldots, j_{k}=1}^{k}(-1)^{(l+1)(k-l-1)}\left(\frac{\operatorname{sgn}\left\{j_{1}, \ldots, j_{k}\right\}}{(l+1)!(k-l-1)!}\right) \\
& \times\left(\int _ { \tau _ { ( k - 1 ) } } \tilde { \star } _ { \sigma _ { ( k ) } } \left(\left\{\sigma_{(k)}\right\} d y^{\left.\left.j_{k-l} \cdots d y^{j_{k}}\right) \wedge a^{(l)}\right) d y^{j_{1} \ldots d y^{j_{k-1-l}}}, ~(l)}\right.\right.
\end{aligned}
$$

with $1 \leq j_{1}<\cdots<j_{k-1-l} \leq k-1$, because the trace of $d y^{k}$ on $\tau_{(k-1)}$ is zero. So, we find
where $1 \leq j_{1}<\cdots<j_{k-1-l} \leq k-1$ and $y^{j}, 1 \leq j \leq k-1$, are Cartesian coordinates on $\tau_{(k-1)}$.
Finally, we see that $(-1)^{(k-1-l) l}(-1)^{(k-l) l}=(-1)^{l}$ and the complete expression gives

$$
\begin{aligned}
& \left|\sigma_{(k)}\right| \star_{\sigma_{(k)}} a^{(l)} \\
& =\sum_{\tau_{(k-1)} \in \partial \sigma_{(k)}}(-1)^{l}\left(\frac{o_{\sigma_{(k)} \tau_{(k-1)}}}{k-l}\right)\left(x_{\tau_{(k-1)}}^{(1)}-x_{\sigma_{(k)}}^{(1)}\right)
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{\tau_{(k-1)} \in \partial \sigma_{(k)}}(-1)^{l}\left(\frac{o_{\sigma_{(k)} \tau_{(k-1)}}}{k-l}\right)\left(x_{\tau_{(k-1)}}^{(1)}-x_{\sigma_{(k)}}^{(1)}\right) \wedge\left(\left|\tau_{(k-1)}\right| \star_{\tau_{(k-1)}} a^{(l)}\right) \text {. }
\end{aligned}
$$

The global discrete Hodge matrices $\mathbb{H}^{(l)}: C^{(l)}(\Omega) \rightarrow \tilde{C}^{(d-l)}(\Omega)$ are constructed by adding the contributions of local cell-wise matrices. The local cell-wise Hodge matrices $\mathbb{H}_{\sigma_{(d)}}^{(l)}$ interpolate from the $l$-cells $\sigma_{(l)} \in \partial^{d-l} \sigma_{(d)}$ to the dual mesh in $\sigma_{(d)}$, i.e., to the dual cells ${ }^{\star} \sigma_{(d)} \sigma_{(l)}$. To discuss the local Hodge matrices we can act as if the primal mesh is given by the single cell $\sigma_{(d)}$ (and its lower dimensional faces $\sigma_{(k)} \in \partial^{d-k} \sigma_{(k)}$ for $0 \leq k<d$ ).

We denote the cochain spaces corresponding to this single cell primal mesh by $C^{(l)}\left(\sigma_{(d)}\right)$ and the space of dual cochains by $\tilde{C}^{(l)}\left(\tilde{\sigma}_{(d)}\right)$. Note that for a cell $\sigma_{(l)} \in \partial^{d-k} \sigma_{(d)}$ only the part $\star_{\sigma_{(d)}} \sigma_{(l)}$ of $\stackrel{\sigma_{(l)}}{ }$ lies within $\sigma_{(d)}$. The local Hodge matrices are square matrices

$$
\mathbb{H}_{\sigma_{(d)}}^{(l)}: C^{(l)}\left(\sigma_{(d)}\right) \rightarrow \tilde{C}^{(d-l)}\left(\sigma_{(d)}\right), \quad \forall \sigma_{(d)} \in \mathcal{M}_{(k)}
$$

To build the global Hodge matrices from the local ones we use the trace operator

$$
\mathbb{T}_{\sigma_{(d)}}^{(l)}: C^{(l)}(\Omega) \rightarrow C^{(l)}\left(\sigma_{(d)}\right), \quad \forall \sigma_{(d)} \in \mathcal{M}_{(k)}
$$

which simply restricts an $l$-cochain to the $l$-cells in $\sigma_{(d)}$. The global Hodge matrices can be determined by adding the contributions of the local Hodge matrices:

$$
\begin{equation*}
\mathbb{H}^{(l)}:=\sum_{\sigma_{(d)} \in \mathcal{M}_{(d)}} \mathbb{T}_{\sigma_{(d)}}^{(l) T} \mathbb{H}_{\sigma_{(d)}}^{(l)} \mathbb{T}_{\sigma_{(d)}}^{(l)} \tag{3.11}
\end{equation*}
$$

To give an example of this cell-wise assembly process we again consider the circumcentric Hodge matrices. We define the local circumcentric Hodge matrices as

$$
\left[\mathbb{H}_{c}^{(l) \sigma_{(d)}}\right]_{\AA_{\sigma_{(d)}} \tau_{(l)}, \sigma_{(l)}}:=\delta_{\sigma_{(l)}, \tau_{(l)}} \frac{\left(\star \sigma_{(l)},{ }_{\star} \sigma_{(d)} \sigma_{(l)}\right)}{\left(\sigma_{(l)}, \sigma_{(l)}\right)} .
$$

Note that we now have the circumcentric dual cell $\boldsymbol{\hbar}_{\sigma_{(d)}} \sigma_{(l)}$ in the numerator instead of $\hbar \sigma_{(l)}$ as in Definition 3.18. The global circumcentric Hodge matrix $\mathbb{H}_{c}^{(l)}$ is found by (3.11).

The consistency of the global discrete Hodge matrices follows from the consistency of the local discrete Hodge matrices. Let us denote by $R_{\sigma_{(d)}}^{(l)}: \Lambda^{(l)}\left(\sigma_{(d)}\right) \rightarrow C^{(l)}\left(\sigma_{(d)}\right)$ and by $\tilde{R}^{(l)}: \tilde{\Lambda}^{(l)}\left(\sigma_{(d)}\right) \rightarrow \tilde{C}^{(l)}\left(\sigma_{(d)}\right)$ the local versions of the de Rham maps.

Definition 3.20. A local primal Hodge matrix $\mathbb{H}_{\sigma_{(d)}}^{(l)}: C^{(l)}\left(\sigma_{(d)}\right) \rightarrow \tilde{C}^{(d-l)}\left(\sigma_{(d)}\right)$ is consistent if

$$
\tilde{\boldsymbol{a}}_{\sigma_{(d)}}^{(d-l)}=\mathbb{H}_{\sigma_{(d)}}^{(l)} \boldsymbol{a}_{\sigma_{(d)}}^{(l)},
$$

where $\boldsymbol{a}_{\sigma_{(d)}}^{(l)}:=R_{\sigma_{(d)}}^{(l)}\left(a^{(l)}\right)$ and $\tilde{\boldsymbol{a}}_{\sigma_{(d)}}^{(d-l)}:=\tilde{R}_{\sigma_{(d)}}^{(d-l)}\left(\star a^{(l)}\right)$ for a constant $l$-form $a^{(l)} \in \Lambda^{(l)}\left(\sigma_{(d)}\right)$.
Similarly, a local dual Hodge matrix $\tilde{\mathbb{H}}_{\sigma_{(d)}}^{(d-l)}: \tilde{C}^{(d-l)}\left(\sigma_{(d)}\right) \rightarrow C^{(l)}\left(\sigma_{(d)}\right)$ is consistent if

$$
\tilde{\mathbb{H}}_{\sigma_{(d)}}^{(d-l)} \tilde{\boldsymbol{b}}_{\sigma_{(d)}}^{(d-l)}=\boldsymbol{b}_{\sigma_{(d)}}^{(l)},
$$

where $\tilde{\boldsymbol{b}}_{\sigma_{(d)}}^{(d-l)}:=\tilde{R}_{\sigma_{(d)}}^{(d-l)}\left(\tilde{b}^{(d-l)}\right)$ and $\boldsymbol{a}_{\sigma_{(d)}}^{(l)}:=R_{\sigma_{(d)}}^{(l)}\left(\tilde{\star}^{(d-l)}\right)$ for constant $(d-l)$-form $\tilde{b}^{(d-l)} \epsilon$ $\tilde{\Lambda}^{(d-l)}\left(\sigma_{(d)}\right)$.

Reconstruction formula (3.10) shows that we have

$$
\left\langle\star a^{(l)},{ }_{\star} \sigma_{(d)} \tau_{(l)}\right\rangle=\sum_{\sigma_{(l)} \in \partial^{d-l} \sigma_{(d)}} \frac{\left\langle\star_{\sigma_{(d)}} \sigma^{(l)}, \star_{\sigma_{(d)}} \tau_{(l)}\right\rangle}{\left|\sigma_{(d)}\right|}\left\langle a^{(l)}, \sigma_{(l)}\right\rangle .
$$

This shows that a consistent local Hodge matrix $\mathbb{H}_{\mathrm{b}, \sigma_{(d)}}^{(l)}$ for a barycentric dual mesh is given by the symmetric matrix

$$
\begin{equation*}
\left[\mathbb{H}_{\mathrm{b}, \sigma_{(d)}}^{(l)}\right]_{\sharp \tau_{(l)}, \sigma_{(l)}}=\frac{\left(\stackrel{\star}{\sigma_{(d)}} \sigma_{(l)}, \star_{\sigma_{(d)}} \tau_{(l)}\right)}{\left|\sigma_{(d)}\right|} . \tag{3.12}
\end{equation*}
$$

Although this discrete Hodge matrix is consistent, it is in general not full rank and a discretization based on it might lead to a linear system which is not full rank. It is customary to add a stabilization matrix to the consistency part such that the resulting matrix is symmetric positive definite.

### 3.3.3 Barycentric primal Hodge matrices: Stability

For convenience, we denote by $\Lambda^{(l)} \sigma_{(d)}$ the vector space constant elements of $\Lambda^{(l)}\left(\sigma_{(d)}\right)$. Analogously, we denote by $\tilde{\Lambda}^{(l)} \sigma_{(d)}$ the vector space of constant elements of $\tilde{\Lambda}^{(l)}\left(\sigma_{(d)}\right)$.

To be able to define the stabilization part of the Hodge matrices we first introduce the following matrices.
Definition 3.21. We define the primal reconstruction matrix $\mathbb{R}_{\sigma_{(d)}}^{(l)}: C^{(l)}\left(\sigma_{(d)}\right) \rightarrow$ $\tilde{\Lambda}^{(d-l)} \sigma_{(d)}$ and dual reconstruction matrix $\tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l)}: \tilde{C}^{(d-l)}\left(\sigma_{(d)}\right) \rightarrow \Lambda^{(l)} \sigma_{(d)}$ according to

$$
\begin{aligned}
\mathbb{R}_{\sigma_{(d)}}^{(l)} \boldsymbol{a}_{\sigma_{(d)}}^{(l)} & :=\sum_{\sigma_{(l)} \in \partial^{d-l} \sigma_{(d)}}\left\langle\boldsymbol{a}_{\sigma_{(d)}}^{(l)}, \boldsymbol{\sigma}_{(l)}\right\rangle{\stackrel{\star}{\sigma_{(d)}}} \sigma^{(l)}, \\
\tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l)} \tilde{\boldsymbol{a}}_{\sigma_{(d)}}^{(d-l)} & :=\sum_{\sigma_{(l)} \in \partial^{d-l} \sigma_{(d)}}\left\langle\tilde{\boldsymbol{a}}_{\sigma_{(d)}}^{(d-l)},{ }^{\star} \sigma_{(d)} \boldsymbol{\sigma}_{(l)}\right\rangle \sigma^{(l)} .
\end{aligned}
$$

Thus $\mathbb{R}_{\sigma_{(d)}}^{(l)}$ and $\tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l)}$ are $(d!/(l!(d-l)!)) \times N_{(l)}^{\sigma_{(d)}}$ matrices, where $N_{(l)}^{\sigma_{(d)}}$ is the number of $l$-cells $\sigma_{(l)} \in \partial^{d-l} \sigma_{(d)}$. The columns of these matrices corresponding to $\sigma_{(l)}$ are given by the components of ${ }_{\wedge_{(d)}} \sigma^{(l)}$ and $\sigma^{(l)}$, respectively. In this we use the usual Cartesian basis for $\tilde{\Lambda}^{(d-l)} \sigma_{(d)}$ and $\Lambda^{(l)} \sigma_{(d)}$.

The primal reconstruction formula is the right-hand side of (3.9) for $k=d$ in matrix form. This explains why we call it the primal reconstruction matrix. The name of the dual reconstruction will be explained soon. First observe that the transpose of these reconstruction operators acts like the de Rham maps for constant forms, because the integral of a constant form over a cell is the same as taking the inner product with the form representing the cell. More precisely, we have the following.
Lemma 3.6. For all constant $a^{(l)} \in \Lambda^{(l)}\left(\sigma_{(d)}\right)$ and constant $\tilde{a}^{(d-l)} \in \tilde{\Lambda}^{(d-l)}\left(\sigma_{(d)}\right)$,

$$
\tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} a^{(l)}=R_{\sigma_{(d)}}^{(l)}\left(a^{(l)}\right), \quad \text { and } \quad \mathbb{R}_{\sigma_{(d)}}^{(l) T} \tilde{a}^{(d-l)}=\tilde{R}_{\sigma_{(d)}}^{(d-l)}\left(\tilde{a}^{(d-l)}\right)
$$

Given a constant $l$-form $a^{(l)}$ we can first project it on the $l$-cells using $\tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T}$ and then apply $\mathbb{R}_{\sigma_{(d)}}^{(l)}$ to find an element of $\tilde{\Lambda}^{(d-l)} \sigma_{(d)}$. Similarly, we can project a constant $(d-l)$-form $\tilde{a}^{(d-l)}$ on the local dual cells using $\mathbb{R}_{\sigma_{(d)}}^{(l) T}$ and then apply $\tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l)}$ to find an element of $\Lambda^{(l)} \sigma_{(d)}$. It turns out that these successive operations boil down to multiplying the form by the cell size applying the Hodge star and inverse Hodge star, respectively.

Lemma 3.7. The primal and dual reconstruction operators satisfy

$$
\begin{aligned}
\mathbb{R}_{\sigma_{(d)}^{(l)}}^{()_{\sigma_{(d)}}^{(d-l) T} a^{(l)}}=\left|\sigma_{(d)}\right| \star a^{(l)}, & & \forall a^{(l)} \in \Lambda^{(l)} \sigma_{(d)}, \\
\tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l)} \mathbb{R}_{\sigma_{(d)}}^{(l) T} \tilde{a}^{(d-l)}=\left|\sigma_{(d)}\right| \star^{-1} \tilde{a}^{(d-l)}, & & \forall \tilde{a}^{(d-l)} \in \tilde{\Lambda}^{(d-l)} \sigma_{(d)}
\end{aligned}
$$

Thus, $\mathbb{R}_{\sigma_{(d)}}^{(l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} /\left|\sigma_{(d)}\right|$ is the matrix representation of $\star^{(l)}$ and $(-1)^{l(d-l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l)} \mathbb{R}_{\sigma_{(d)}}^{(l)} /\left|\sigma_{(d)}\right|$ is the matrix representation of $\tilde{\star}^{(d-l)}$.
Proof. For any $a^{(l)} \in \Lambda^{(l)} \sigma_{(d)}$ we have by Lemma 3.6,

$$
\left[\tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} a^{(l)}\right]_{\sigma_{(l)}}=\left\langle a^{(l)}, \sigma_{(l)}\right\rangle
$$

By (3.9) it follows then $\mathbb{R}_{\sigma_{(d)}}^{(l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} a^{(l)}=\left|\sigma_{(d)}\right| \star a^{(l)}$.
The fact that $\tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l)} \mathbb{R}_{\sigma_{(d)}}^{(l) T} /\left|\sigma_{(d)}\right|$ equals $\star^{-1}$ follows from the fact that $*$ is an isometry. Let $(\cdot, \cdot)_{\Lambda^{(l)} \sigma_{(d)}}$ and $(\cdot, \cdot)_{\tilde{\Lambda}^{(d-l)} \sigma_{(d)}}$ be the inner products on $\Lambda^{(l)} \sigma_{(d)}$ and $\tilde{\Lambda}^{(d-l)} \sigma_{(d)}$. We have for all $a^{(l)}, b^{(l)} \in \Lambda^{(l)} \sigma_{(d)}$,

$$
\begin{aligned}
\left(a^{(l)}, \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l)} \mathbb{R}_{\sigma_{(d)}}^{(l) T} \mathbb{R}_{\sigma_{(d)}}^{(l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} b^{(l)}\right)_{\Lambda^{(l)} \sigma_{(d)}} & =\left(\mathbb{R}_{\sigma_{(d)}}^{(l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} a^{(l)}, \mathbb{R}_{\sigma_{(d)}}^{(l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} b^{(l)}\right)_{\tilde{\Lambda}^{(d-l)} \sigma_{(d)}} \\
& =\left|\sigma_{(d)}\right|^{2}\left(\star a^{(l)}, \star b^{(l)}\right)_{\tilde{\Lambda}^{(d-l)} \sigma_{(d)}} \\
& =\left|\sigma_{(d)}\right|^{2}\left(a^{(l)}, b^{(l)}\right)_{\Lambda^{(l)} \sigma_{(d)}},
\end{aligned}
$$

where we used that $\star$ is an isometry in the last equality. This shows that $\tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l)} \mathbb{R}_{\sigma_{(d)}}^{(l) T}$ is a left inverse of $\mathbb{R}_{\sigma_{(d)}}^{(l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T}$. Moreover, these are square matrices so $\tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l)} \mathbb{R}_{\sigma_{(d)}}^{(l) T}$ is actually the inverse of $\mathbb{R}_{\sigma_{(d)}}^{(l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T}$.

From this we get the following result, which clarifies the name, dual reconstruction matrix, we earlier gave to $\tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l)}$.
Lemma 3.8. The primal and dual reconstruction matrices satisfy, for any a ${ }^{(l)} \in \Lambda^{(l)} \sigma_{(d)}$,

$$
\left|\sigma_{(d)}\right| \star a^{(l)}=\mathbb{R}_{\sigma_{(d)}}^{(l)} \boldsymbol{a}_{\sigma_{(d)}}^{(l)}, \quad \text { and } \quad\left|\sigma_{(d)}\right| a^{(l)}=\tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l)} \tilde{\boldsymbol{a}}_{\sigma_{(d)}}^{(d-l)}
$$

where $\boldsymbol{a}_{\sigma_{(d)}}^{(l)}:=R_{\sigma_{(d)}}^{(l)}\left(a^{(l)}\right)$ and $\tilde{\boldsymbol{a}}_{\sigma_{(d)}}^{(d-l)}:=\tilde{R}_{\sigma_{(d)}}^{(d-l)}\left(\star a^{(l)}\right)$.
Proof. The relation for $\mathbb{R}_{\sigma}^{(l)}$ (d) is simply the formula (3.9) restated. The second relation follows immediately from Lemma 3.6 and Lemma 3.7.

Assume that we use the usual Cartesian bases to represent elements in $\Lambda^{(l)} \sigma_{(d)}$ and $\tilde{\Lambda}^{(d-l)} \sigma_{(d)}$. With respect to these bases the Hodge star operator $\star: \Lambda^{(l)} \sigma_{(d)} \rightarrow \tilde{\Lambda}^{(d-l)} \sigma_{(d)}$ and its inverse are then simply given by the identity matrix.

The following result concerning the reconstruction and incidence matrices will be useful in the next chapter, when we derive discrete conservation laws.
Lemma 3.9. For any $\boldsymbol{a}^{(d-2)} \in C^{(d-2)}(\Omega)$ and $\sigma_{(d)} \in C_{(d)}(\Omega)$ we have

$$
\left\{\sigma_{(d)}\right\} \mathbb{R}_{\sigma_{(d)}}^{(d-1)} \mathbb{D}_{\sigma_{(d)}}^{(d-2)} \boldsymbol{a}_{\sigma_{(d)}}^{(d-2)}=\sum_{\sigma_{(d-1)} \in \sigma_{(d)}} o_{\sigma_{(d)} \sigma_{(d-1)}}\left\{\sigma_{(d-1)}\right\} \mathbb{R}_{\sigma_{(d-1)}}^{(d-2)} \boldsymbol{a}_{\sigma_{(d-1)}}^{(d-2)},
$$

where $\boldsymbol{a}_{\sigma_{(d-1)}}^{(d-2)}=\mathbb{T}_{\sigma_{(d-1)}}^{(d-2)} \boldsymbol{a}^{(d-2)}$ and $\boldsymbol{a}_{\sigma_{(d)}}^{(d-2)}=\mathbb{T}_{\sigma_{(d)}}^{(d-2)} \boldsymbol{a}^{(d-2)}$.
Proof. By definition we have

$$
\begin{aligned}
\left\{o_{(d)}\right\} & \mathbb{R}_{\sigma_{(d)}}^{(d-1)} \mathbb{D}_{\sigma_{(d)}}^{(d-2)} \boldsymbol{a}_{\sigma_{(d)}}^{(d-2)} \\
& =\left\{o_{(d)}\right\} \sum_{\sigma_{(d-1)} \in \sigma_{(d)}}{ }^{\sharp} \sigma_{(d)} \sigma^{(d-1)}\left(\sum_{\sigma_{(d-2)} \in \sigma_{(d-1)}} o_{\sigma_{(d-1)} \sigma_{(d-2)}} a_{\sigma_{(d-2)}}^{(d-2)}\right) \\
& =\sum_{\sigma_{(d-1)} \in \sigma_{(d)}}(-1)^{d}\left(\sum_{\sigma_{(d-2)} \in \sigma_{(d-1)}} o_{\sigma_{(d)} \sigma_{(d-1)}} o_{\sigma_{(d-1)} \sigma_{(d-2)}}\left(x_{\sigma_{(d)}}^{(1)}-x_{\sigma_{(d-1)}}^{(1)}\right) a_{\sigma_{(d-2)}}^{(d-2)}\right) \\
& =\sum_{\sigma_{(d-2)} \in \partial^{2} \sigma_{(d)} \sigma_{(d-1)} \in \partial \sigma_{(d)} \cap \partial^{-1} \sigma_{(d-2)}}(-1)^{d} o_{\sigma_{(d)} \sigma_{(d-1)}} o_{\sigma_{(d-1)} \sigma_{(d-2)}}\left(x_{\sigma_{(d)}}^{(1)}-x_{\sigma_{(d-1)}}^{(1)}\right) a_{\left.\sigma_{(d-2)}\right)}^{(d-2)} .
\end{aligned}
$$

The fact that $\partial \sigma_{(d)} \cap \partial^{-1} \sigma_{(d-2)}$ has two elements, say $\sigma_{(d-1)}^{a}$ and $\sigma_{(d-1)}^{b}$, for which $o_{\sigma_{(d)} \sigma_{(d-1)}^{a}} o_{\sigma_{(d-1)}^{a} \sigma_{(d-2)}}=-o_{\sigma_{(d)} \sigma_{(d-1)}^{b}} o_{\sigma_{(d-1)}^{b} \sigma_{(d-2)}}$ implies that we can replace $x_{\sigma_{(d)}}^{(1)}$ by $x_{\sigma_{(d-2)}}^{(1)}$. So, we have

$$
\begin{aligned}
\left\{o_{(d)}\right\} & \mathbb{R}_{\sigma_{(d)}}^{(d-1)} \mathbb{D}_{\sigma_{(d)}}^{(d-2)} \boldsymbol{a}_{\sigma_{(d)}}^{(d-2)} \\
& =\sum_{\sigma_{(d-2)} \in \partial^{2} \sigma_{(d)}} \sum_{\sigma_{(d-1)} \in \partial \sigma_{(d)} \cap \partial^{-1} \sigma_{(d-2)}}(-1)^{d} o_{\sigma_{(d)} \sigma_{(d-1)}} o_{\sigma_{(d-1)} \sigma_{(d-2)}}\left(x_{\sigma_{(d-1)}}^{(1)}-x_{\sigma_{(d-2)}}^{(1)}\right) a_{\sigma_{(d-2)}}^{(d-2)} \\
& =\sum_{\sigma_{(d-1)} \in \sigma_{(d)}} o_{\sigma_{(d)} \sigma_{(d-1)}} \sum_{\sigma_{(d-2)} \in \sigma_{(d-2)}} o_{\sharp \sigma_{(d-2)} \star \sigma_{(d-1)}}\left(x_{\sigma_{(d-1)}}^{(1)}-x_{\sigma_{(d-2)}}^{(1)}\right) a_{\sigma_{(d-2)}^{(d-2)}}^{(d)} \\
& =\sum_{\sigma_{(d-1)} \in \sigma_{(d)}} o_{\sigma_{(d)} \sigma_{(d-1)}}\left\{\sigma_{(d-1)}\right\} \mathbb{R}_{\sigma_{(d-1)}}^{(d-2)} \boldsymbol{a}_{\sigma_{(d-1)}}^{(d-2)} .
\end{aligned}
$$

It should be noted that the consistency of the primal and dual Hodge matrices as defined in Definition 3.20, as a consequence of Lemma 3.6 and 3.7, can also be expressed as follows.
Lemma 3.10. The local primal Hodge matrix $\mathbb{H}_{\sigma_{(d)}^{(l)}}^{(l)}: C^{(l)}\left(\sigma_{(d)}\right) \rightarrow \tilde{C}^{(d-l)}\left(\sigma_{(d)}\right)$ is consistent if and only if, for all $a^{(l)} \in \Lambda^{(l)} \sigma_{(d)}, \mathbb{H}_{\sigma_{(d)}}^{(l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} a^{(l)}=\mathbb{R}_{\sigma_{(d)}}^{(l) T} \star a^{(l)}$, or equivalently, when we assume a Cartesian basis, if the following matrix equation holds:

$$
\begin{equation*}
\mathbb{H}_{\sigma_{(d)}}^{(l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T}=\mathbb{R}_{\sigma_{(d)}}^{(l) T} \tag{3.13}
\end{equation*}
$$

Similarly, the local dual Hodge matrix $\tilde{\mathbb{H}}_{\sigma_{(d)}}^{(d-l)}: \tilde{C}^{(d-l)}\left(\sigma_{(d)}\right) \rightarrow C^{(l)}(\Omega)$ is consistent if and only if, for all $\tilde{a}^{(d-l)} \in \tilde{\Lambda}^{(d-l)} \sigma_{(d)}, \tilde{\mathbb{H}}_{\sigma_{(d)}}^{(d-l)} \mathbb{R}_{\sigma_{(d)}}^{(l) T} \tilde{a}^{(d-l)}=\tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} \tilde{\star} \tilde{a}^{(d-l)}$, or equivalently, when we assume a Cartesian basis, if the following matrix equation holds:

$$
\begin{equation*}
\tilde{\mathbb{H}}_{\sigma_{(d)}}^{(d-l)} \mathbb{R}_{\sigma_{(d)}}^{(l) T}=(-1)^{l(d-l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} . \tag{3.14}
\end{equation*}
$$

The relations (3.13) and (3.14) are in the Mimetic Finite Difference literature known as algebraic consistency conditions [30].

The primal and dual Hodge matrices given by

$$
\begin{aligned}
\mathbb{H}_{\mathrm{b}_{\mathrm{c}}, \sigma_{(d)}}^{(l)} & :=\frac{1}{\left|\sigma_{(d)}\right|} \mathbb{R}_{\sigma_{(d)}}^{(l) T} \mathbb{R}_{\sigma_{(d)}}^{(l)}, \\
\tilde{\mathbb{H}}_{\sigma_{(d)}^{(d)}}^{(d-l)} & :=(-1)^{l(d-l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l)},
\end{aligned}
$$

are obviously consistent. Every consistent Hodge matrix $\mathbb{H}_{\sigma_{(d)}}^{(l)}$ can be written as the sum of this consistent term and a stabilization term:

$$
\begin{equation*}
\mathbb{H}_{\mathrm{b}, \sigma_{(d)}}^{(l)}:=\mathbb{H}_{\mathrm{b}_{\mathrm{c}}, \sigma_{(d)}}^{(l)}+\mathbb{H}_{\mathrm{b}_{\mathbf{s}}, \sigma_{(d)}}^{(l)} \tag{3.15}
\end{equation*}
$$

where the stabilization part $\mathbb{H}_{\mathrm{b}_{\mathrm{s}}, \sigma_{(d)}}^{(l)}$ satisfies

$$
\begin{equation*}
\mathbb{H}_{\mathrm{b}_{\mathbf{s}}, \sigma_{(d)}}^{(l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T}=0 \tag{3.16}
\end{equation*}
$$

This makes sure that $\mathbb{H}_{\sigma_{(d)}}^{(l)}$ still satisfies the consistency condition (3.13). The stabilization term $\mathbb{H}_{\mathrm{b}_{\mathrm{s}}, \sigma_{(d)}}^{(l)}$ should be chosen such that the $\mathbb{H}_{\mathrm{b}, \sigma_{(d)}}^{(l)}$ is symmetric positive definite.

We now define local Hodge matrices found by adding a stabilization term. We will consider two formulations. The stabilization term proposed in the Mimetic Finite Difference literature and the stabilization term proposed in the Compatible Discrete Operator (CDO) framework [54].

Definition 3.22. The local primal CDO Hodge matrices are given, for $0 \leq l \leq d$, by (3.15), where the stabilization term is given by

$$
\mathbb{H}_{\mathrm{b}_{\mathrm{s}}, \sigma_{(d)}}^{(l)}:=\left(\mathbb{I}_{\sigma_{(d)}}^{(l)}-\left|\sigma_{(d)}\right|^{-1} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} \mathbb{R}_{\sigma_{(d)}}^{(l)}\right)^{T} \mathbb{U}_{\beta, \sigma_{(d)}}^{(l)}\left(\mathbb{I}_{\sigma_{(d)}}^{(l)}-\left|\sigma_{(d)}\right|^{-1} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} \mathbb{R}_{\sigma_{(d)}}^{(l)}\right)
$$

with $\mathbb{U}_{\beta, \sigma_{(d)}}^{(l)}$ the square $N_{(l)}^{\sigma_{(d)}} \times N_{(l)}^{\sigma_{(d)}}$ diagonal matrix given by

$$
\left[\mathbb{U}_{\beta, \sigma_{(d)}}^{(l)}\right]_{\sigma_{(l)}, \tau_{(l)}}:=\delta_{\sigma_{(l)}, \tau_{(l)}} \beta^{2} d\left(\frac{\left(\text { 』 }_{\sigma_{(d)}} \sigma_{(l)}, \star_{\sigma_{(d)}} \sigma_{(l)}\right)}{\left(\star \sigma_{(l)},{ }^{\star} \sigma_{(d)} \sigma_{(l)}\right)}\right)
$$

Here $\beta>0$ is a free parameter. ${ }^{9}$

[^23]

Figure 3.14: The set $\Delta_{\sigma_{(d)}}\left(\tau_{(l)}\right)$ is shown for a simplicial cell $\sigma_{(3)}$ and an $l$-cell $\tau_{(l)} \in \partial^{3-l} \sigma_{(3)}$. From left to right $l$ equals $0\left(\tau_{(0)}\right.$ is the most left vertex), $1 \tau_{(1)}$ is the front-bottom edge), and $2\left(\tau_{(2)}\right.$ is the bottom face).

Let $\Delta_{\sigma_{(d)}}\left(\sigma_{(l)}\right)$ be defined as the union of the $d$-dimensional subsimplices in $\sigma_{(d)}$ that contain both $\sigma_{(l)}$ and ${ }^{\boldsymbol{\wedge}_{(d)}} \sigma_{(l)}$. Examples of this are shown in Figure 3.14. When $d=3$ and $l=1$ or $l=2$, the term $d\left(\star_{\sigma_{(d)}} \sigma_{(l)},{ }^{\star} \sigma_{(d)} \sigma_{(l)}\right) /\left(\star \sigma_{(l)},{ }^{\star} \sigma_{(d)} \sigma_{(l)}\right)$ is the (oriented ${ }^{10}$ ) size of $\Delta_{\sigma_{(3)}}\left(\sigma_{(l)}\right)$. In [54] the matrix $\mathbb{U}_{\beta, \sigma_{(d)}}^{(l)}$ is actually defined according to $\left[\mathbb{U}_{\beta, \sigma_{(d)}}^{(l)}\right]_{\sigma_{(l)}, \tau_{(l)}}:=$ $\delta_{\sigma_{(l)}, \tau_{(l)}} \beta^{2}\left|\Delta_{\sigma_{(d)}}\left(\sigma_{(l)}\right)\right|$, which gives a different matrix when $l=0$. We use Definition 3.22, because for this definition, with $\beta=1 / \sqrt{d}$, the matrices $\mathbb{H}_{\mathrm{b}, \sigma_{(d)}}^{(l)}$ simplify to the diagonal circumcentric Hodge matrices $\mathbb{H}_{\mathrm{c}, \sigma_{(d)}}^{(l)}$ when the barycentric and circumcentric dual meshes coincide (e.g. for Cartesian meshes). This fact we will show in Section 3.3.4.

The choice $\beta=1 / d$ corresponds to the primal Hodge matrices used in the Discrete Geometric Approach (DGA) [52] a mimetic approach to discretizing Maxwell's equations. This choice is also considered in [54].

Similar to the local primal CDO Hodge matrices, we can of course form the corresponding local dual versions $\tilde{\mathbb{H}}_{\mathrm{b}, \sigma_{(d)}}^{(d-l)}$, where the stabilization term is given by

$$
\begin{equation*}
\tilde{\mathbb{H}}_{\mathrm{b}_{\mathrm{s}}, \sigma_{(d)}}^{(d-l)}:=\left(\mathbb{I}_{\sigma_{(d)}}^{(l)}-\left|\sigma_{(d)}\right|^{-1} \mathbb{R}^{(l) T} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l)}\right)^{T} \mathbb{U}_{\beta, \sigma_{(d)}}^{(l)}\left(\mathbb{I}_{\sigma_{(d)}}^{(l)}-\left|\sigma_{(d)}\right|^{-1} \mathbb{R}^{(l) T} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l)}\right) \tag{3.17}
\end{equation*}
$$

However, these will not be used in this thesis. They do play a role in some closely related discretization methods, especially those based on hybridization (e.g. [55] for $\beta=1 / \sqrt{d}$ ).

In the Mimetic Finite Difference (MFD) literature [28,30] the Hodge matrices are formed using a different form for the stabilization term.

Definition 3.23. The local MFD Hodge matrices are given, for $0 \leq l \leq d$, by (3.15), where the stabilization term is given by

$$
\mathbb{H}_{\mathrm{b}_{\mathrm{s}}, \sigma_{(d)}}^{(l)}:=\gamma \operatorname{tr}\left(\mathbb{H}_{\mathrm{b}_{\mathrm{c}}, \sigma_{(d)}}^{(l)}\right)\left(\mathbb{I}_{\sigma_{(d)}}^{(l)}-\tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T}\left(\tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T}\right)^{-1} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l)}\right)
$$

Here $\gamma>0$ is a free parameter.
The factor $\operatorname{tr}\left(\mathbb{H}_{\mathrm{b}_{\mathrm{c}}, \sigma_{(d)}}^{(l)}\right)$ is such that the stabilization term gets the proper scaling. (Remember: the trace equals the sum of the eigenvalues.)

[^24]Again, the corresponding dual local Hodge matrices $\tilde{\mathbb{H}}_{\mathrm{b}, \sigma_{(d)}^{(d-l)}}$ can be formed as well. They are given by the stabilization term

$$
\tilde{\mathbb{H}}_{\mathrm{b}_{\mathrm{s}}, \sigma_{(d)}}^{(d-l)}:=\gamma \operatorname{tr}\left(\tilde{\mathbb{H}}_{\mathrm{b}_{\mathrm{c}}, \sigma_{(d)}}^{(d-l)}\right)\left(\mathbb{I}_{\sigma_{(d)}}^{(l)}-\mathbb{R}_{\sigma_{(d)}}^{(l) T}\left(\mathbb{R}_{\sigma_{(d)}}^{(l)} \mathbb{R}_{\sigma_{(d)}}^{(l) T}\right)^{-1} \mathbb{R}_{\sigma_{(d)}}^{(l)}\right)
$$

A common choice is $\gamma=1 / d$ which leads to a diagonal Hodge matrix $\mathbb{H}_{\mathrm{b}, \sigma_{(d)}}^{(l)}$ for $l=1$ and $l=2$, when the mesh is Cartesian and the cells are square (height equals depth equals length). Taking $\gamma=1$ gives a diagonal Hodge for $l=0$ on any Cartesian mesh.

It can be shown that the local Hodge matrices above are, besides symmetric, indeed positive definite. Let us define the following norm $\|\cdot\|_{\sigma_{(d)}}$ on $C^{(l)}\left(\sigma_{(d)}\right)$. For $\boldsymbol{a}_{\sigma_{(d)}}^{(l)} \epsilon$ $C^{(l)}\left(\sigma_{(d)}\right)$ let

$$
\begin{equation*}
\left\|\boldsymbol{a}_{\sigma_{(d)}}^{(l)}\right\|_{C^{(l)}\left(\sigma_{(d)}\right)}^{2}:=\sum_{\sigma_{(l)} \in \partial^{d-l} \sigma_{(d)}}\left|\Delta_{\sigma_{(d)}}\left(\sigma_{(l)}\right)\right|\left(\frac{\left\langle\boldsymbol{a}_{\sigma_{(d)}}^{(l)}, \boldsymbol{\sigma}_{(l)}\right\rangle}{\left|\sigma_{(l)}\right|}\right)^{2} \tag{3.18}
\end{equation*}
$$

Proposition 3.13. Let the mesh $\mathcal{M}$ on $\Omega$ be such that the cells are star-shaped with respect to their barycenter and the simplicial submesh in each cell has a finite number of shape-regular ${ }^{11}$ simplices. Furthermore let $\mathbb{H}_{\mathrm{b}, \sigma_{(d)}}^{(l)}$ be one of the barycentric Hodge matrices defined above. There exists an $\eta>0$ such that for all $\sigma_{(d)} \in \mathcal{M}_{(d)}$,

$$
\eta\left\|\boldsymbol{a}_{\sigma_{(d)}}^{(l)}\right\|_{C^{(l)}\left(\sigma_{(d)}\right)}^{2} \leq \boldsymbol{a}_{\sigma_{(d)}}^{(l) T} \mathbb{H}_{\mathrm{b}, \sigma_{(d)}}^{(l)} \boldsymbol{a}_{\sigma_{(d)}}^{(l)} \leq \eta^{-1}\left\|\boldsymbol{a}_{\sigma_{(d)}}^{(l)}\right\|_{C^{(l)}\left(\sigma_{(d)}\right)}^{2}, \quad \forall \boldsymbol{a}_{\sigma_{(d)}}^{(l)} \in C^{(l)}\left(\sigma_{(d)}\right)
$$

from which it follows that

$$
\eta\left\|\boldsymbol{a}_{\sigma_{(d)}}^{(l)}\right\|_{C^{(l)}(\Omega)}^{2} \leq \boldsymbol{a}^{(l) T} \mathbb{H}_{\mathrm{b}}^{(l)} \boldsymbol{a}^{(l)} \leq \eta^{-1}\left\|\boldsymbol{a}_{\sigma_{(d)}}^{(l)}\right\|_{C^{(l)}(\Omega)}^{2}, \quad \forall \boldsymbol{a}^{(l)} \in C^{(l)}(\Omega)
$$

where $\left\|\boldsymbol{a}_{\sigma_{(d)}}^{(l)}\right\|_{C^{(l)}(\Omega)}^{2}:=\sum_{\sigma_{(d)} \in \mathcal{M}_{(d)}}\left\|\mathbb{T}_{\sigma_{(d)}}^{(l)} \boldsymbol{a}^{(l)}\right\|_{C^{(l)}\left(\sigma_{(d)}\right)}^{2}$ and $\mathbb{H}_{\mathrm{b}}^{(l)}$ is the corresponding global Hodge matrix defined according to (3.11).

Proof. The proof can be found in [54] and [30] for respectively the CDO Hodge matrices and the MFD Hodge matrices.

### 3.3.4 Combining the circumcentric and barycentric Hodge matrices

In most of our domain we want to use a Cartesian mesh. Therefore barycentric Hodge matrices that simplify to the diagonal circumcentric Hodge matrices in the Cartesian regions of the mesh are of particular interest. They lead to smaller discretization stencils and hence sparser linear systems, reducing the cost of the computation.

In Cartesian primal cells $\sigma_{(d)} \in \mathcal{M}_{(d)}$ the circumcentric and barycentric dual meshes coincide. This implies that both $\mathbb{H}_{c}^{(l)} \sigma_{(d)}$ and $\mathbb{H}_{\mathrm{b}_{\mathrm{c}}, \sigma_{(d)}}^{(l)}$ are consistent. Their difference is given by the stabilization term given by the CDO Hodge matrix for $\beta=1 / \sqrt{d}$.

[^25]Proposition 3.14. Let $\mathbb{H}_{\mathrm{b}, \sigma_{(d)}}^{(l)}$ be the CDO Hodge matrix, with $\beta=1 / \sqrt{d}$ in the stabilization term, and $\mathbb{H}_{\mathrm{c}, \sigma_{(d)}}^{(l)}$ the circumcentric Hodge matrix. For cells $\sigma_{(d)} \in \mathcal{M}_{(d)}$ in which the circumcentric and barycentric dual meshes coincide we have $\mathbb{H}_{\mathrm{b}, \sigma_{(d)}}^{(l)}=\mathbb{H}_{\mathrm{c}, \sigma_{(d)}}^{(l)}$.

Proof. For cells $\sigma_{(d)} \in \mathcal{M}_{(d)}$ with coinciding circumcentric and barycentric dual meshes we have, for the matrix $\mathbb{U}_{1 / \sqrt{d}, \sigma_{(d)}}^{(l)}$ in the stabilization term,

$$
\begin{aligned}
{\left[\mathbb{U}_{1 / \sqrt{d}, \sigma_{(d)}}^{(l)}\right]_{\sigma_{(l)}, \tau_{(l)}} } & =\delta_{\sigma_{(l)}, \tau_{(l)}} \frac{\left(\stackrel{\star}{ } \sigma_{(d)} \sigma_{(l)},{ }^{\star} \sigma_{(d)} \sigma_{(l)}\right)}{\left(\star \sigma_{(l)}, \star \sigma_{(d)} \sigma_{(l)}\right)} \\
& =\delta_{\sigma_{(l)}, \tau_{(l)}} \frac{\left(\star \sigma_{(l)},{ }^{\star} \sigma_{(d)} \sigma_{(l)}\right)}{\left(\sigma_{(l)}, \sigma_{(l)}\right)},
\end{aligned}
$$

where we used that we have $\star \sigma_{(l)}=\left(\left(\sigma_{(l)}, \sigma_{(l)}\right) /\left(\star \sigma_{(l)},{ }^{\star} \sigma_{(d)} \sigma_{(l)}\right)\right){ }^{\wedge}{ }_{\sigma_{(d)}} \sigma_{(l)}$ for a circumcentric dual mesh. Thus we have $\mathbb{U}_{1 / \sqrt{d}, \sigma_{(d)}}^{(l)}=\mathbb{H}_{\mathrm{c}, \sigma_{(d)}}^{(l)}$.

After substituting this, using $\mathbb{H}_{c_{c} \sigma_{(d)}}^{(l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T}=\mathbb{R}_{\sigma_{(d)}}^{(l) T}\left(\right.$ Lemma 3.10) and $\mathbb{R}_{\sigma_{(d)}}^{(l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T}=$ $\left|\sigma_{(d)}\right|^{-1} \mathbb{I}_{\sigma_{(d)}}^{(l)}$ (Lemma 3.7), we find

$$
\begin{aligned}
& \mathbb{H}_{\mathrm{b}_{\mathrm{s}}, \sigma_{(d)}}^{(l)}=\left(\mathbb{I}_{\sigma_{(d)}}^{(l)}-\left|\sigma_{(d)}\right|^{-1} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} \mathbb{R}_{\sigma_{(d)}}^{(l)}\right)^{T} \mathbb{H}_{\mathrm{c}, \sigma_{(d)}}^{(l)}\left(\mathbb{I}_{\sigma_{(d)}}^{(l)}-\left|\sigma_{(d)}\right|^{-1} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} \mathbb{R}_{\sigma_{(d)}}^{(l)}\right) \\
& =\mathbb{H}_{\mathbf{c}, \sigma_{(d)}}^{(l)}-\left|\sigma_{(d)}\right|^{-1}\left(\mathbb{H}_{\mathbf{c}, \sigma_{(d)}}^{(l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} \mathbb{R}_{\sigma_{(d)}}^{(l)}\right)^{T}-\left|\sigma_{(d)}\right|^{-1} \mathbb{H}_{\mathbf{c}, \sigma_{(d)}}^{(l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} \mathbb{R}_{\sigma_{(d)}}^{(l)} \\
& +\left|\sigma_{(d)}\right|^{-2}\left(\mathbb{H}_{\mathrm{c}_{-\sigma_{(d)}}^{(l)}} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} \mathbb{R}_{\sigma_{(d)}}^{(l)}\right)^{T} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} \mathbb{R}_{\sigma_{(d)}}^{(l)} \\
& =\mathbb{H}_{\mathrm{c}, \sigma_{(d)}}^{(l)}-2\left|\sigma_{(d)}\right|^{-1} \mathbb{R}_{\sigma_{(d)}}^{(l) T} \mathbb{R}_{\sigma_{(d)}}^{(l)}+\left|\sigma_{(d)}\right|^{-2} \mathbb{R}_{\sigma_{(d)}}^{(l) T} \mathbb{R}_{\sigma_{(d)}}^{(l)} \tilde{\mathbb{R}}_{\sigma_{(d)}}^{(d-l) T} \mathbb{R}_{\sigma_{(d)}}^{(l)} \\
& =\mathbb{H}_{c, \sigma_{(d)}}^{(l)}-\left|\sigma_{(d)}\right|^{-1} \mathbb{R}_{\sigma_{(d)}}^{(l) T} \mathbb{R}_{\sigma_{(d)}}^{(l)} .
\end{aligned}
$$

So we find $\mathbb{H}_{\mathrm{c}, \sigma_{(d)}}^{(l)}=\mathbb{H}_{\mathrm{b}_{\mathrm{c}}, \sigma_{(d)}}^{(l)}+\mathbb{H}_{\mathrm{b}_{\mathrm{s}}, \sigma_{(d)}}^{(l)}$.
This shows that when we use the CDO Hodge matrix with $\beta=1 / \sqrt{d}$ on the whole mesh it automatically simplifies to the diagonal circumcentric Hodge matrix in Cartesian regions of the mesh.

### 3.4 Summary

In this chapter we introduced all the ingredients for a mimetic discretization on polyhedral meshes. We introduced the primal mesh $\mathcal{M}$ and showed that it constitutes a cell complex. This allowed for an exact discretization of the exterior derivative $d$.

Next, we defined the circumcentric and barycentric dual meshes. The dual mesh $\tilde{\mathcal{M}}$ on itself is not a cell complex, but the union $\mathcal{M} \cup \tilde{\mathcal{M}}^{\text {b }}$, of the dual mesh and the boundary dual mesh $\tilde{\mathcal{M}}^{\text {b }}$, is a cell complex. Using the dual cell complex we were able to also exactly discretize $\tilde{d}$.

Furthermore, we derived reconstruction formulas that allowed us to formulate the consistent and stable CDO and MFD discrete Hodge matrices in a generic way.

In the next chapter we will use the tools introduced in this chapter to discretize the incompressible Navier-Stokes equations.

## Mimetic Discretization of Incompressible Flow on General Meshes

One of the most popular numerical methods for simulating viscous incompressible flow is the Marker-And-Cell (MAC) scheme by Harlow and Welch [17]. The MAC scheme, published over 50 years ago, is a staggered mesh method in which the incompressible Navier-Stokes equations are discretized in terms of the normal velocity components at the cell faces and pressure variables in the cell centers of a Cartesian mesh. The staggered positioning of the velocity variables allows for an efficient discretization of the divergence-free condition and leads to discrete conservation of mass. The MAC scheme, conserves not only mass and momentum, but also the secondary quantities vorticity and, in the inviscid case, kinetic energy. Moreover, spurious pressure oscillations, that have to be artificially suppressed for non-staggered colocated schemes, are absent for the staggered MAC scheme. In [57] it is stated that "nobody will dispute, that on Cartesian grids, computation of incompressible flows is best performed with the staggered scheme proposed by Harlow and Welch".

Despite the fact that the MAC scheme is, after 50 years, still the method of choice, much progress has been made. The MAC scheme has been extended to Delaunay triangulations $[19,20]$ and to curvilinear meshes $[23,58]$. Moreover, it was shown how to extend the scheme to non-uniform meshes without affecting the symmetry properties of the discrete operators [18] and how to extend the method to higher order by Richardson extrapolation $[18,59]$. Recently, an extension of the MAC scheme to locally refined meshes was proposed [60]. The conservation properties of the scheme extended to tetrahedral meshes have been studied in $[21,61,62]$. Besides the generalization of the scheme to different types of meshes, also the convergence of the scheme has been studied. Convergence studies of the MAC scheme have been performed for the linearized Navier-Stokes equations [63], the Stokes equations [64-66], the steady incompressible Navier-Stokes equations [67,68] and, recently, the unsteady incompressible Navier-Stokes equations [60].

Besides the generalization of the MAC scheme to tetrahedral and curvilinear meshes, a further generalization to meshes consisting of arbitrary polygons and polyhedrals (polytopes in general dimension) has become within reach as a result of the fruitful developments in the field of polytopal discretization methods. A few examples of discretization methods suitable for general polytopal meshes are the Mimetic Finite Difference (MFD) methods [28, 30, 53,69] and their recent extension, the Virtual Element Method (VEM) [70, 71], Hybrid High-Order method [72, 73], polyhedral finite element methods [74, 75] and more general frameworks encompassing these methods [31, 54, 76]. The polytopal discretization methods allow for more flexibility for modeling in complicated domains. This flexibility is, for example, needed for the meshes in cut-cell MAC meth-
ods [24-26].
The generalization of the MAC method to non-Cartesian meshes can most easily be understood in terms of the mimetic discretization framework reviewed in Chapter 3. If we just focus on the Stokes equations, i.e., omitting the convection term, the generalization is now relatively straightforward. Given the Stokes part of the two formulations (2.21) and (2.22) we can simply replace the exterior derivative and Hodge operators by their discrete versions. The discretization of (2.21) uses velocity flux variables on an outeroriented primal mesh and discretization of (2.22) uses velocity circulation variables on an inner-oriented primal mesh. We will therefore call the discretization of (2.21) the outeroriented scheme and call the discretization of (2.22) the inner-oriented scheme. For the outer-oriented scheme additional vorticity variables are needed. Some of the results presented in this chapter related to the outer-oriented scheme were published in [77].

It is easily checked that if one uses a Cartesian mesh and the circumcentric diagonal Hodge operators, the discretization of (2.21) reduces to the MAC scheme for Stokes equations. Similarly, the discretization of (2.22) simplifies to the MAC scheme for Stokes equations but now with the dual mesh playing the role of the single mesh used in the formulation of the MAC scheme. ${ }^{1}$ Using the circumcentric dual mesh that is available for simplicial meshes leads to the generalization of the MAC method as discussed in [19,20].

To generalize these two discretizations to polyhedral meshes we need to resort to the barycentric dual mesh, because the circumcentric dual mesh does not exist. So, to formulate the MAC method on polyhedral meshes we only need to replace the diagonal circumcentric discrete Hodge matrices for the barycentric discrete Hodge matrices. In this chapter we will use the three different barycentric Hodge matrices discussed in Chapter 3: the CDO-DGA Hodge matrix (Definition 3.22, with $\beta=1 / d$, where $d$ is the spatial dimension), the CDO-SUSHI Hodge matrix ${ }^{2}$ (Definition 3.22, with $\beta=1 / \sqrt{d}$ ) and the MFD Hodge matrix (Definition 3.23, with $\gamma=1 / d$ ). Only the CDO-SUSHI Hodge leads to a true realization of the MAC method in the sense that in the Cartesian parts of the mesh it coincides with the MAC method, because we have shown in Proposition 3.14 that in Cartesian cells the CDO-SUSHI Hodge matrix is simply the diagonal circumcentric Hodge matrix.

The generalization of the convection term to polyhedral meshes is less straightforward. For the outer-oriented scheme we can use a discretization of the convection term in divergence form which was given earlier (for simplicial meshes) in [21, 78]. For both schemes we define a discretization of the convection term in rotational form. All discretizations are energy- and vorticity-conserving. Unfortunately, the rotational-form discretizations of the convection term only conserve momentum on Cartesian meshes. We will show that the rotational convection term for the inner-oriented scheme on Cartesian meshes simplifies to the small stencil of the MAC method in 2D.

[^26]Altogether, we consider three different discrete Hodge matrices, two discretizations of the convection term for the outer-oriented scheme and one convection discretization for the inner-oriented scheme, which amounts to a total of 9 combinations. For these 9 discretizations of the incompressible Navier-Stokes equations we analyze the conservation and convergence properties for a range of different mesh types. Finally, we apply one of these methods as a cut-cell method to compute the vortex shedding of a circular cylinder and the impulsively started flow around a NACA 0012 airfoil.

### 4.1 Mimetic discretizations of the Stokes equations

The formulations (2.21) and (2.22) of the Navier-Stokes equations suggest two different discrete formulations. We start by considering these for the Stokes equations, i.e., we consider (2.21) and (2.22) but without the time derivative and convective terms. We set the dynamic viscosity $\mu=1$. This gives the following two sets of equations:

$$
\begin{align*}
\tilde{d} \tilde{\omega}^{(1)}+\star d p^{(0)} & =\tilde{f}^{(2)},  \tag{4.1a}\\
\tilde{\omega}^{(1)}-\star d \tilde{\star} \tilde{u}^{(2)} & =0,  \tag{4.1b}\\
\tilde{d} \tilde{u}^{(2)} & =0, \tag{4.1c}
\end{align*}
$$

and

$$
\begin{align*}
\tilde{\star} \tilde{d} \star \omega^{(2)}+d p^{(0)} & =f^{(1)},  \tag{4.2a}\\
\omega^{(2)}-d u^{(1)} & =0,  \tag{4.2b}\\
\tilde{d} \star u^{(1)} & =0 . \tag{4.2c}
\end{align*}
$$

Remember that these formulations are very close to the curl-curl formulation of the Stokes equations, which in vector calculus is given by

$$
\begin{array}{r}
\nabla \times \underline{\omega}+\nabla p=\underline{f}, \\
\underline{\omega}-\nabla \times \underline{u}=0, \\
\nabla \cdot \underline{u}=0 .
\end{array}
$$

In Chapter 3 we saw that due to the nature of the barycentric dual mesh, in general, the discrete versions of the dual Hodge $\tilde{\star}$ that maps from the dual to the primal mesh will not be available. This implies that we should discretize above equations such that we avoid the use of these operators.

Furthermore, up to now we have considered the primal mesh to have an inner orientation and the dual mesh to have an outer orientation. This is rather arbitrary and we could equally well assign an outer orientation to the primal mesh. This we will do when we discretize (4.1), because it allows us to avoid the use of dual to primal Hodge matrices.

Up to now, we indicated outer-oriented objects with a tilde~. From now, for discretized objects, we will use the tilde to indicate that they have to do with the dual mesh. For example, we discretize the outer form $\tilde{u}^{(2)}$ on the outer-oriented primal mesh as
$\boldsymbol{u}^{(2)}:=R^{(2)}\left(\tilde{u}^{(2)}\right)$. Similarly, we discretize the inner form $p^{(0)}$ in (4.1) as $\tilde{\boldsymbol{p}}^{(0)}:=\tilde{R}^{(0)}\left(p^{(0)}\right)$ on the inner-oriented dual mesh. Thus, for discrete objects the tilde indicates whether they are located on the primal mesh or on the dual mesh, irrespective of their type of orientation.

### 4.1.1 Discrete formulations

We start with the discretization of (4.1). We use an outer-oriented primal mesh and inner-oriented dual mesh. To avoid the dual to primal Hodge matrices we rewrite (4.1) as

$$
\begin{aligned}
\tilde{\star}^{(2)} \tilde{d}^{(1)} \tilde{\omega}^{(1)}+d^{(0)} p^{(0)} & =\tilde{\star}^{(2)} \tilde{f}^{(2)}, \\
\tilde{\star}^{(1)} \tilde{\omega}^{(1)}-d^{(1)} \tilde{\star}^{(2)} \tilde{u}^{(2)} & =0, \\
\tilde{d}^{(2)} \tilde{u}^{(2)} & =0 .
\end{aligned}
$$

The discrete version of this reads

$$
\begin{align*}
\mathbb{H}^{(2)} \mathbb{D}^{(1)} \boldsymbol{\omega}^{(1)}+\tilde{\mathbb{D}}^{(0)}\left[\begin{array}{c}
\tilde{\boldsymbol{p}}^{(0)} \\
\tilde{\boldsymbol{p}}_{\mathrm{b}}^{(0)}
\end{array}\right] & =\mathbb{H}^{(2)} \boldsymbol{f}^{(2)},  \tag{4.3a}\\
\mathbb{H}^{(1)} \boldsymbol{\omega}^{(1)}-\tilde{\mathbb{D}}^{(1)}\left[\begin{array}{c}
\mathbb{H}^{(2)} \boldsymbol{u}^{(2)} \\
\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}
\end{array}\right] & =\tilde{\boldsymbol{0}}^{(2)},  \tag{4.3b}\\
\mathbb{D}^{(2)} \boldsymbol{u}^{(2)} & =\mathbf{0}^{(3)}, \tag{4.3c}
\end{align*}
$$

where we discretized the force term as $\boldsymbol{f}^{(2)}:=R^{(2)}\left(\tilde{f}^{(2)}\right)$. We see that in this formulation the boundary dual mesh pressure $\tilde{\boldsymbol{p}}_{\mathrm{b}}^{(0)}$ and tangential velocity $\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}$ appear as possible natural boundary conditions. Alternatively, we can choose for essential boundary conditions where either the (normal) trace, i.e., $\mathbb{T}^{(2)} \boldsymbol{u}^{(2)}$, of the velocity is given and/or $\mathbb{T}^{(1)} \boldsymbol{\omega}^{(1)}$, the trace of the vorticity. We will deal with no-slip boundary conditions and therefore the tangential velocity, $\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}$, and the normal velocity, $\mathbb{T}^{(2)} \boldsymbol{u}^{(2)}$, will be given.

Suppose the normal and tangential components of the velocity on the boundary are given as $\tilde{u}_{\mathrm{b}}^{(2)}:=\tilde{t}^{(2)} \tilde{u}^{(2)}$ and $u_{\mathrm{b}}^{(1)}:=t^{(1)} \tilde{\star}^{(2)} \tilde{u}^{(2)}$ on $\partial \Omega$. We define the discrete boundary data as

$$
\boldsymbol{u}_{\mathrm{b}}^{(2)}:=R_{\mathrm{b}}^{(2)}\left(\tilde{u}_{\mathrm{b}}^{(2)}\right) \quad \text { and } \quad \tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}:=\tilde{R}_{\mathrm{b}}^{(1)}\left(u_{\mathrm{b}}^{(1)}\right)
$$

Substituting this in (4.3) and using Proposition 3.10 we find the linear system

$$
\left[\begin{array}{cccc}
-\mathbb{H}^{(1)} & \mathbb{D}^{(1) T} \mathbb{H}^{(2)} & 0 & 0  \tag{4.4}\\
\mathbb{H}^{(2)} \mathbb{D}^{(1)} & 0 & -\mathbb{D}^{(2) T} & \mathbb{T}^{(2) T} \\
0 & -\mathbb{D}^{(2)} & 0 & 0 \\
0 & \mathbb{T}^{(2)} & 0 & 0
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{\omega}^{(1)} \\
\boldsymbol{u}^{(2)} \\
\tilde{\boldsymbol{p}}^{(0)} \\
\tilde{\boldsymbol{p}}_{\mathrm{b}}^{(0)}
\end{array}\right]=\left[\begin{array}{c}
\mathbb{T}^{(1) T} \tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)} \\
\mathbb{H}^{(2)} \boldsymbol{f}^{(2)} \\
\mathbf{0}^{(3)} \\
\boldsymbol{u}_{\mathrm{b}}^{(2)}
\end{array}\right]
$$

The essential boundary condition on $\boldsymbol{u}^{(2)}$ can of course also be incorporated in the definition of the corresponding cochain space and the equations corresponding to the
variables in $C^{(1)}(\partial \Omega)$ and $C^{(2)}(\partial \Omega)$ can be eliminated from the linear system through a proper adjustment of the right-hand side. For simplicity we will write them out explicitly.

Natural boundary conditions are discretized on the dual mesh and they appear in the right-hand side multiplied by a transposed trace operator as is the case for $\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}$. They are the boundary contribution to the discretized exterior derivative calculated on the dual mesh (Proposition 3.10).

Essential boundary conditions are enforced by trace operators applied to cochains on the primal cell complex. The boundary dual mesh contributions to the discretization of the exterior derivative on the dual mesh, which alternatively could have been given as natural boundary conditions, now appear in a symmetric way as unknowns in the linear system. For example, the boundary pressure unknowns $\tilde{\boldsymbol{p}}_{\mathrm{b}}^{(0)}$ in (4.4) (which could have been used as natural boundary conditions) correspond to the essential boundary condition $\mathbb{T}^{(2)} \boldsymbol{u}^{(2)}=\boldsymbol{u}_{\mathrm{b}}^{(2)}$.

For the second discrete formulation we use an inner-oriented primal mesh. We rewrite the system (in order to avoid dual to primal Hodge matrices) as

$$
\begin{aligned}
\tilde{d}^{(1)} \star^{(2)} d^{(1)} u^{(1)}+\star^{(1)} d^{(0)} p^{(0)} & =\star^{(1)} f^{(1)}, \\
\tilde{d}^{(2)} \star{ }^{(1)} u^{(1)} & =0
\end{aligned}
$$

The discretized version of this is given by

$$
\begin{align*}
& \tilde{\mathbb{D}}^{(1)}\left[\begin{array}{c}
\mathbb{H}^{(2)} \mathbb{D}^{(1)} \boldsymbol{u}^{(1)} \\
\tilde{\boldsymbol{\omega}}_{\mathrm{b}}^{(1)}
\end{array}\right]+\mathbb{H}^{(1)} \mathbb{D}^{(0)} \boldsymbol{p}^{(0)}=\mathbb{H}^{(1)} \boldsymbol{f}^{(1)},  \tag{4.5a}\\
& \tilde{\mathbb{D}}^{(2)}\left[\begin{array}{c}
\mathbb{H}^{(1)} \boldsymbol{u}^{(1)} \\
\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(2)}
\end{array}\right]=\tilde{\boldsymbol{0}}^{(3)}, \tag{4.5b}
\end{align*}
$$

where $\boldsymbol{f}^{(1)}:=R^{(1)}\left(f^{(1)}\right)$.
Again we use no-slip boundary conditions. This time the boundary condition for the tangential velocity component is essential and the boundary condition for the normal velocity component is natural. We define the discrete boundary data as

$$
\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(2)}:=\tilde{R}_{\mathrm{b}}^{(2)}\left(\tilde{u}_{\mathrm{b}}^{(2)}\right) \quad \text { and } \quad \boldsymbol{u}_{\mathrm{b}}^{(1)}:=R_{\mathrm{b}}^{(1)}\left(u_{\mathrm{b}}^{(1)}\right)
$$

This results in the linear system

$$
\left[\begin{array}{ccc}
\mathbb{D}^{(1) T} \mathbb{H}^{(2)} \mathbb{D}^{(1)} & \mathbb{H}^{(1)} \mathbb{D}^{(0)} & -\mathbb{T}^{(1) T}  \tag{4.6}\\
\mathbb{D}^{(0) T} \mathbb{H}^{(1)} & 0 & 0 \\
-\mathbb{T}^{(1)} & 0 & 0
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{u}^{(1)} \\
\boldsymbol{p}^{(0)} \\
\tilde{\boldsymbol{\omega}}_{\mathrm{b}}^{(1)}
\end{array}\right]=\left[\begin{array}{c}
\mathbb{H}^{(1)} \boldsymbol{f}^{(1)} \\
\mathbb{T}^{(0) T} \tilde{\boldsymbol{u}}_{\mathrm{b}}^{(2)} \\
-\boldsymbol{u}_{\mathrm{b}}^{(1)}
\end{array}\right]
$$

This time we have extra vorticity variables $\tilde{\boldsymbol{\omega}}_{\mathrm{b}}^{(1)}$ on the boundary to enforce $\mathbb{T}^{(1)} \boldsymbol{u}^{(1)}=$ $\boldsymbol{u}_{\mathrm{b}}^{(1)}$. Again, note that these variables, together with the corresponding boundary velocity variables, can be eliminated from the linear system by incorporating the essential boundary condition in the definition of the cochain space.

The linear systems (4.4) and (4.6) have been considered before in [31]. There it was shown that with natural boundary conditions these discretizations are well-posed.

In the case of the discretization (4.4) it was shown that, under the assumption of mesh regularity ${ }^{3}$, the operators $\mathbb{H}^{(1)} \mathbb{D}^{(0)}$ and $-\mathbb{D}^{(2)}$ satisfy discrete inf-sup conditions. Furthermore, the matrix $-\mathbb{H}^{(1)}$ is self-adjoint and negative definite. These together imply that the discrete system is well-posed [79]. This line of argumentation can now be extended to show that the discrete system (4.4) (with the essential boundary condition on $\boldsymbol{u}^{(2)}$ ) is well-posed, by showing that the operator $\left[-\mathbb{D}^{(2) T} \mathbb{T}^{(2) T}\right]^{T}$ satisfies the inf-sup condition as well.

The linear system (4.6) is more problematic. For natural boundary conditions the discrete system is well-posed because the operator $\mathbb{D}^{(1) T} \mathbb{H}^{(2)} \mathbb{D}^{(1)}$ is coercive and the operator $\mathbb{D}^{(0) T} \mathbb{H}^{(1)}$ satisfies a discrete inf-sup condition. However, we use a combination of essential and natural boundary conditions and as a result have to deal with the operator $\left[\mathbb{H}^{(1)} \mathbb{D}^{(0)}-\mathbb{T}^{(1)}\right]^{T}$. This operator does not satisfy a discrete inf-sup condition for all meshes.

For general meshes the matrix $\left[\mathbb{H}^{(1)} \mathbb{D}^{(0)}-\mathbb{T}^{(1) T}\right]^{T}$ is not full-rank. Nontrivial elements ${ }^{4}$ of the kernel of this matrix may be produced by vertices in the boundary of the mesh that are only connected (via edges) to other vertices that also lie on the boundary of the mesh. This is most easily seen in the case when $\mathbb{H}^{(1)}$ is diagonal. Consider such a vertex $\sigma_{(0)} \in C_{(0)}(\partial \Omega)$ that is only connected to other vertices in the mesh boundary. Let $p_{\sigma_{(0)}}$ be the pressure value in this vertex. An increase of this value can be exactly compensated for by the values of $\tilde{\boldsymbol{\omega}}_{\mathrm{b}}^{(1)}$ on the dual edges (in the boundary mesh) dual to the primal edges going out of $\sigma_{(0)}$. More precisely, the mode with $\boldsymbol{p}^{(0)}$, everywhere zero with exception of $p_{\sigma_{(0)}}$ and $\tilde{\boldsymbol{\omega}}_{\mathrm{b}}^{(1)}$ everywhere zero with the exception of the dual edges ${ }_{\mathrm{b}}^{\mathrm{b}} \sigma_{(1)}$ for $\sigma_{(1)} \in \partial^{-1} \sigma_{(0)}$ where we have $\tilde{\omega}_{\boldsymbol{*}_{\mathrm{b}} \sigma_{(1)}}=-o_{\sigma_{(1)} \sigma_{(0)}} p_{\sigma_{(0)}} / c_{\sigma_{(1)}}$ with $c_{\sigma_{(1)}}$ the diagonal value of $\mathbb{H}^{(1)}$ corresponding to $\sigma_{(1)}$, is an element of the kernel of $\left[\mathbb{H}^{(1)} \mathbb{D}^{(0)}-\mathbb{T}^{(1) T}\right]^{T}$.

Another way of looking at this issue is by considering the zero-divergence constraint for the dual cells corresponding to the vertices in the boundary mesh which are connected to other vertices in the boundary only. For such dual cells, in the case of a diagonal $\mathbb{H}^{(1)}$, the discrete divergence is calculated using only velocity values from the boundary conditions $\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(2)}$ and $\boldsymbol{u}_{\mathrm{b}}^{(1)}$. The pressure variable as Lagrange multiplier to enforce zero divergence does not make sense for this vertex and its corresponding dual cell. We therefore remove these vertices from the linear system and are forced to prescribe them. This leads to a new linear system given by

$$
\left[\begin{array}{ccc}
\mathbb{D}^{(1) T} \mathbb{H}^{(2)} \mathbb{D}^{(1)} & \mathbb{H}^{(1)} \mathbb{D}_{\mathrm{r}}^{(0)} & -\mathbb{T}^{(1) T}  \tag{4.7}\\
\mathbb{D}_{\mathrm{r}}^{(0) T} \mathbb{H}^{(1)} & 0 & 0 \\
-\mathbb{T}^{(1)} & 0 & 0
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{u}^{(1)} \\
\boldsymbol{p}_{\mathrm{r}}^{(0)} \\
\tilde{\boldsymbol{\omega}}_{\mathrm{b}}^{(1)}
\end{array}\right]=\left[\begin{array}{c}
\mathbb{H}^{(1)} \boldsymbol{f}^{(1)}-\mathbb{H}^{(1)} \mathbb{D}_{\mathrm{rc}}^{(0)} \boldsymbol{p}_{\mathrm{rc}}^{(0)} \\
\mathbb{T}_{\mathrm{r}}^{(0) T} \tilde{\boldsymbol{u}}_{\mathrm{b}}^{(2)} \\
-\boldsymbol{u}_{\mathrm{b}}^{(1)}
\end{array}\right]
$$

where the matrices $\mathbb{D}_{\mathrm{r}}^{(0)}$ and $\mathbb{T}_{\mathrm{r}}^{(0)}$ are found from $\mathbb{D}^{(0)}$ and $\mathbb{T}^{(0)}$ by removing the columns corresponding to these special boundary vertices and, similarly, $\boldsymbol{p}_{\mathrm{r}}^{(0)}$ is $\boldsymbol{p}^{(0)}$ with the variables corresponding to these vertices removed. Furthermore, $\mathbb{D}_{\mathrm{rc}}^{(0)}$ is the matrix consisting of the removed columns and $\boldsymbol{p}_{\mathrm{rc}}^{(0)}$ are the pressure variables removed from $\boldsymbol{p}^{(0)}$

[^27]which we now assume to be given. A more complete analysis of these issues is left for future research.

### 4.1.2 Convergence test

To assess the efficiency of these two formulations we test them on a problem taken from the Finite Volumes for Complex Applications conference 8 (FVCA8) benchmark [80]. We take the velocity field and pressure field given by

$$
\underline{u}_{\mathrm{e}}:=\left[\begin{array}{c}
-2 \cos (2 \pi x) \sin (2 \pi y) \sin (2 \pi z) \\
\sin (2 \pi x) \cos (2 \pi y) \sin (2 \pi z) \\
\sin (2 \pi x) \sin (2 \pi y) \cos (2 \pi z)
\end{array}\right], \quad p_{\mathrm{e}}:=-6 \pi \sin (2 \pi x) \sin (2 \pi y) \sin (2 \pi z) .
$$

This is a solution to the Stokes equations when

$$
\underline{f}:=\left[\begin{array}{c}
-36 \pi^{2} \cos (2 \pi x) \sin (2 \pi y) \sin (2 \pi z) \\
0 \\
0
\end{array}\right]
$$

The domain of the problem is $\Omega=[0,1]^{3}$ and the boundary data is calculated from the exact solution.

In the case of the outer-oriented scheme, we discretize the exact velocity, vorticity, pressure and the force term (represented as differential forms as $\tilde{u}_{\mathrm{e}}^{(2)}, \tilde{\omega}_{\mathrm{e}}^{(1)}, p_{\mathrm{e}}^{(0)}$ and $\tilde{f}^{(2)}$ ) on the outer-oriented primal faces, primal edges, dual vertices, and, primal faces, respectively as,

$$
\boldsymbol{u}_{\mathrm{e}}^{(2)}:=R^{(2)}\left(\tilde{u}_{\mathrm{e}}^{(2)}\right), \quad \boldsymbol{\omega}_{\mathrm{e}}^{(1)}:=R^{(1)}\left(\tilde{\omega}_{\mathrm{e}}^{(1)}\right), \quad \tilde{\boldsymbol{p}}_{\mathrm{e}}^{(0)}:=\tilde{R}^{(0)}\left(p_{\mathrm{e}}^{(0)}\right) \quad \boldsymbol{f}^{(2)}:=R^{(2)}\left(\tilde{f}^{(2)}\right)
$$

Similarly, for the inner-oriented scheme, we discretize the exact solution and force (represented as differential forms $u_{\mathrm{e}}^{(1)}, \omega_{\mathrm{e}}^{(2)}, p_{\mathrm{e}}^{(0)}$ and $\left.f^{(1)}\right)$ on the inner-oriented primal edges, primal faces, primal vertices and primal edges, respectively as,

$$
\boldsymbol{u}_{\mathrm{e}}^{(1)}:=R^{(1)}\left(u_{\mathrm{e}}^{(1)}\right), \quad \boldsymbol{\omega}_{\mathrm{e}}^{(2)}:=R^{(2)}\left(\omega_{\mathrm{e}}^{(2)}\right), \quad \boldsymbol{p}_{\mathrm{e}}^{(0)}:=R^{(0)}\left(p_{\mathrm{e}}^{(0)}\right) \quad \boldsymbol{f}^{(1)}:=R^{(1)}\left(f^{(1)}\right)
$$

Using the discrete Hodge matrices we define approximate relative $L^{2}$-norms and the relevant number of variables for the outer-oriented scheme as

$$
\begin{array}{ll}
E_{u}^{\mathrm{o}}:=\sqrt{\frac{\left(\boldsymbol{u}^{(2)}-\boldsymbol{u}_{\mathrm{e}}^{(2)}\right)^{T} \mathbb{H} \mathbb{H}^{(2)}\left(\boldsymbol{u}^{(2)}-\boldsymbol{u}_{\mathrm{e}}^{(2)}\right)}{\boldsymbol{u}_{\mathrm{e}}^{(2) T} \mathbb{H}^{(2)} \boldsymbol{u}_{\mathrm{e}}^{(2)}},} & N_{u}^{\mathrm{o}}:=N^{(2)}, \\
E_{\omega}^{\mathrm{o}}:=\sqrt{\frac{\left(\tilde{\boldsymbol{\omega}}^{(1)}-\tilde{\boldsymbol{\omega}}_{\mathrm{e}}^{(1)}\right)^{T} \mathbb{H}^{(1)}\left(\tilde{\boldsymbol{\omega}}^{(1)}-\tilde{\boldsymbol{\omega}}_{\mathrm{e}}^{(1)}\right)}{\tilde{\boldsymbol{\omega}}_{\mathrm{e}}^{(1) T} \mathbb{H}^{(1)} \tilde{\boldsymbol{\omega}}_{\mathrm{e}}^{(1)}},} & N_{\omega}^{\mathrm{o}}:=N^{(1)},  \tag{4.8}\\
E_{p}^{\mathrm{o}}:=\sqrt{\frac{\left(\tilde{\boldsymbol{p}}^{(0)}-\tilde{\boldsymbol{p}}_{\mathrm{e}}^{(0)}\right)^{T}\left(\mathbb{H}\left(\mathbb{H}^{(3)}\right)^{-1}\left(\tilde{\boldsymbol{p}}^{(0)}-\tilde{\boldsymbol{p}}_{\mathrm{e}}^{(0)}\right)\right.}{\tilde{\boldsymbol{p}}_{\mathrm{e}}^{(0) T}\left(\mathbb{H} H^{(3)}\right)^{-1} \tilde{\boldsymbol{p}}_{\mathrm{e}}^{(0)}}}, & N_{p}^{\mathrm{o}}:=N^{(3)},
\end{array}
$$

and for the inner-oriented scheme as

$$
\begin{array}{ll}
E_{u}^{\mathrm{i}}:=\sqrt{\frac{\left(\boldsymbol{u}^{(1)}-\boldsymbol{u}_{\mathrm{e}}^{(1)}\right)^{T} \mathbb{H}^{(1)}\left(\boldsymbol{u}^{(1)}-\boldsymbol{u}_{\mathrm{e}}^{(1)}\right)}{\boldsymbol{u}_{\mathrm{e}}^{(1) T} \mathbb{H}^{(1)} \boldsymbol{u}_{\mathrm{e}}^{(1)}}}, & N_{u}^{\mathrm{i}}:=N^{(1)}, \\
E_{\omega}^{\mathrm{i}}:=\sqrt{\frac{\left(\tilde{\boldsymbol{\omega}}^{(2)}-\tilde{\boldsymbol{\omega}}_{\mathrm{e}}^{(2)}\right)^{T} \mathbb{H}^{(2)}\left(\tilde{\boldsymbol{\omega}}^{(2)}-\tilde{\boldsymbol{\omega}}_{\mathrm{e}}^{(2)}\right)}{\tilde{\boldsymbol{\omega}}_{\mathrm{e}}^{(2) T} \mathbb{H}^{(2)} \tilde{\boldsymbol{\omega}}_{\mathrm{e}}^{(2)}}}, & N_{\omega}^{\mathrm{i}}:=N^{(2)},  \tag{4.9}\\
E_{p}^{\mathrm{i}}:=\sqrt{\frac{\left(\boldsymbol{p}^{(0)}-\boldsymbol{p}_{\mathrm{e}}^{(0)}\right)^{T} \mathbb{H}^{(0)}\left(\boldsymbol{p}^{(0)}-\boldsymbol{p}_{\mathrm{e}}^{(0)}\right)}{\boldsymbol{p}_{\mathrm{e}}^{(0) T} \mathbb{H}^{(0)} \boldsymbol{p}_{\mathrm{e}}^{(0)}}}, & N_{p}^{\mathrm{i}}:=N^{(0)} .
\end{array}
$$

These error norms depend on the discrete Hodge operator used. Alternatively, we could also use the discrete Hodge-independent norm defined in (3.18), but we have always found the same convergence rates for these norms. This was also observed in [54].


( Convergence behavior for the velocity (blue), the vorticity (green), and the pressure (red). The full ines show the behavior of the outer-oriented scheme and the dashed line that of the inner-oriented scheme. For the outer-oriented scheme $E_{u}$ and $N_{u}$ refer to (4.8) and for the inner-oriented scheme to (4.9), and $E_{\omega}, N_{\omega}, E_{p}$ and $N_{p}$ are defined analogously. The different shades correpond to the three different types of discrete Hodge matrices used.

The convergence results are shown in Figure 4.1. Six different mesh sequences are considered: hexahedral (HE), prismatic-triangular (PT), checkerboard (CB), locallyrefined (LR), prismatic-hexagonal (PH), and, tetrahedral (TE) meshes. Examples of each are shown in Figure 4.1. We define the rate of convergence for the velocity by

$$
\begin{equation*}
R_{u, j}=-d \frac{\log \left(E_{u, j} / E_{u, j-1}\right)}{\log \left(N_{u, j} / N_{u, j-1}\right)} \tag{4.10}
\end{equation*}
$$

where $E_{u, j}$ is the error for the mesh at refinement level $j, N_{u, j}$ is the number of velocity variables for this refinement level and $d$ is the spatial dimension. We define the convergence rates for the vorticity and pressure analogously. In Table 4.1 the convergence rates are given based on the two finest mesh levels for each sequence.

For the HE mesh sequence the hexahedral cells are cubes and as a result both the MFD and CDO-SUSHI Hodge matrices are diagonal. This implies that both the outeroriented and inner-oriented schemes simplify to the MAC scheme for the Stokes equations for these Hodge matrices in the interior of the domain. They only differ in how the noslip boundary conditions are applied. The CDO-DGA Hodge matrices are not diagonal for the HE mesh sequence.

Both schemes show second order convergence rates for the velocity, vorticity and pressure for all discrete Hodge matrices used. The exception is the pressure for the inner-oriented scheme with the MFD- or CDO-SUSHI Hodge matrices, which shows high-order convergence. The pressure is actually captured exactly and we see here the convergence rate of the quadrature rules used to discretize the data. This phenomenon was also observed in [54]. However, there it was also found for the CDO-DGA Hodge matrices. The difference between the current test problem and the problem considered in [54] is the fact that in this study no-slip boundary conditions are used while in [54] essential or natural boundary conditions are used. As mentioned in Section 4.1.1, for the inner-oriented scheme, in some parts of the boundary mesh, the no-slip boundary condition (implemented as a combination of natural and essential boundary conditions) is incompatible and therefore the pressure is prescribed instead in some boundary vertices. We suspect that this combined with the fact the CDO-DGA Hodge matrices are not diagonal destroys the super-convergence of the pressure for the inner-oriented scheme with CDO-DGA Hodge matrices and that therefore second order convergence is found instead.

The convergence of the vorticity and pressure seems to be less robust with respect to the shape of the mesh cells than the convergence of the velocity. The convergence of the velocity is still close to second order for the LR and PT mesh sequences. For the other mesh sequences it is between first and second order. The convergence behavior of the velocity is very similar for the inner-oriented and outer-oriented schemes. The convergence of the outer-oriented scheme with the MFD and CDO-DGA Hodge matrices is somewhat worse for the CB mesh sequence than the other methods. Furthermore, the velocity convergence of the inner-oriented scheme is better than that of the outer-oriented scheme for the TE mesh sequence.

The outer-oriented scheme uses vorticity variables, while for the inner-oriented scheme the vorticity is calculated from the velocity. On average the outer-oriented scheme shows better convergence behavior for the vorticity. Both schemes show second order accuracy
for the HE mesh sequence and a convergence rate close to 1.5 for the PT mesh sequence. However, for the LR and CB mesh sequences the convergence rate for the outer-oriented scheme is around 1.7 and 0.9 , respectively, while only around 0.8 and 0.5 for the inneroriented scheme. The convergence rate of the inner-oriented scheme with CDO-DGA Hodge matrices seems to be closer to that of the outer-oriented scheme than that of the inner-oriented scheme with the other Hodge matrices. Besides this there is again very few influence on the convergence behavior by the choice of Hodge matrix.

The convergence of the pressure suffers mostly when other meshes than hexahedral meshes are considered. Especially for the PH and TE mesh sequences the pressure seems in some instances not to converge. This is in contrast to the convergence results when natural or essential boundary conditions are used (reported in [54]). We do not have a good explanation for this significant deterioration of the convergence behavior and this should be investigated in a further study.

| $R_{u}$ | Outer-oriented |  |  | scheme | Inner-oriented |  |  | scheme |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Mesh | MFD | DGA | SUS | MFD | DGA | SUS |  |  |
| HE | 2.04 | 1.98 | 2.04 | 2.07 | 2.09 | 2.07 |  |  |
| PT | 1.94 | 1.98 | 1.92 | 1.99 | 2.03 | 2.01 |  |  |
| CB | 0.84 | 0.91 | 0.84 | 1.17 | 1.36 | 1.09 |  |  |
| LR | 1.77 | 1.67 | 1.64 | 1.70 | 2.16 | 1.79 |  |  |
| PH | 1.88 | 1.01 | 0.90 | 1.64 | 1.15 | 1.24 |  |  |
| TE | 0.99 | 0.90 | 1.01 | 2.08 | 1.89 | 2.05 |  |  |


| $R_{\omega}$ | Outer-oriented |  |  |  | scheme | Inner-oriented scheme |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Mesh | MFD | DGA | SUS | MFD | DGA | SUS |  |  |
| HE | 2.04 | 2.04 | 2.04 | 2.04 | 2.06 | 2.04 |  |  |
| PT | 1.64 | 1.47 | 1.57 | 1.62 | 1.39 | 1.54 |  |  |
| CB | 0.99 | 0.78 | 1.05 | 0.37 | 0.73 | 0.44 |  |  |
| LR | 1.74 | 1.46 | 1.80 | 0.78 | 1.47 | 0.88 |  |  |
| PH | 0.83 | 0.44 | 0.34 | 1.13 | 0.55 | 0.52 |  |  |
| TE | 0.50 | 0.41 | 0.52 | 1.97 | 1.97 | 1.97 |  |  |


| $R_{p}$ | Outer-oriented |  |  | scheme | Inner-oriented scheme |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Mesh | MFD | DGA | SUS | MFD | DGA | SUS |  |
| HE | 1.93 | 1.97 | 1.93 | 4.25 | 2.08 | 4.25 |  |
| PT | 1.71 | 1.82 | 1.66 | 1.37 | 1.34 | 1.38 |  |
| CB | 1.00 | 1.14 | 0.95 | 1.63 | 2.10 | 1.57 |  |
| LR | 1.64 | 1.88 | 1.70 | 0.92 | 1.36 | 0.76 |  |
| PH | 1.49 | 0.17 | 0.02 | 1.16 | 0.68 | 0.57 |  |
| TE | 0.20 | 0.12 | 0.24 | 2.54 | 2.54 | 2.40 |  |

Table 4.1: The convergence rates $R_{u}, R_{\omega}$ and $R_{p}$ (defined as in (4.10)) are given for the hexahedral (HE), prismatic-triangular (PT), checkerboard (CB), locally-refined (LR), prismatic-hexagonal (PH) and tetrahedral (TE) mesh sequences, based on the two results found on the two finest meshes for each sequence. The rates are given for the outer-oriented and inner-oriented schemes, using either the Mimetic Finite Difference (MFD), Discrete Geometric Approach (DGA), or SUSHI (SUS) Hodge matrices.

### 4.2 Mimetic discretizations of the Navier-Stokes equations

To extend the two discrete formulations for the Stokes equations to the Navier-Stokes equations we rewrite (2.21) and (2.22) again such that the primal-to-dual discrete Hodge matrices will be avoided in the discretization. The formulation in terms of the outeroriented velocity 2 -form $\tilde{u}^{(2)}$ can be written as

$$
\begin{aligned}
\rho \tilde{\star}^{(2)} \frac{\partial \tilde{u}^{(2)}}{\partial t}+\rho \tilde{\star}^{(2)}\left(\tilde{\omega}^{(1)} \wedge \tilde{\star}^{(2)} \tilde{u}^{(2)}\right)+\mu \tilde{\star}^{(2)} \tilde{d}^{(1)} \tilde{\omega}^{(1)}+d^{(0)} q^{(0)} & =\tilde{\star}^{(2)} \tilde{f}^{(2)}, \\
\tilde{\star}^{(1)} \tilde{\omega}^{(1)}-d^{(1)} \tilde{\star}^{(2)} \tilde{u}^{(2)} & =0, \\
\tilde{d}^{(2)} \tilde{u}^{(2)} & =0,
\end{aligned}
$$

and in terms of the inner-oriented velocity 1-form as

$$
\begin{aligned}
\rho \star^{(1)} \frac{\partial u^{(1)}}{\partial t}+\rho\left(\left(\star^{(2)} \omega^{(2)}\right) \wedge u^{(1)}\right)+\mu \tilde{d}^{(1)} \star^{(2)} d^{(1)} u^{(1)}+\star^{(1)} d^{(0)} q^{(0)} & =\star{ }^{(1)} f^{(1)}, \\
\omega^{(2)}-d^{(1)} u^{(1)} & =0, \\
\tilde{d}^{(2)} \star^{(1)} u^{(1)} & =0 .
\end{aligned}
$$

Note that we assume the density $\rho$ to be constant.

### 4.2.1 Discrete formulations

We extend the discrete formulations for Stokes equations and discretize the two formulations as, respectively,

$$
\begin{align*}
& \rho \mathbb{H}^{(2)} \frac{\partial \boldsymbol{u}^{(2)}}{\partial t}+\mathbb{N}_{\mathrm{o}}\left(\boldsymbol{\omega}^{(1)}, \boldsymbol{u}^{(2)}\right)+ \mu \mathbb{H}^{(2)} \mathbb{D}^{(1)} \boldsymbol{\omega}^{(1)}+\tilde{\mathbb{D}}^{(0)}\left[\begin{array}{c}
\tilde{\boldsymbol{q}}^{(0)} \\
\tilde{\boldsymbol{q}}_{\mathrm{b}}^{(0)}
\end{array}\right]  \tag{4.11a}\\
& \mathbb{H}^{(1)} \boldsymbol{\omega}^{(1)}-\tilde{\mathbb{D}}^{(1)}\left[\begin{array}{c}
\mathbb{H}^{(2)} \boldsymbol{H}^{(2)} \boldsymbol{u}^{(2)} \\
\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}
\end{array}\right]=\tilde{\mathbf{0}}^{(2)},  \tag{4.11b}\\
& \mathbb{D}^{(2)} \boldsymbol{u}^{(2)}=\mathbf{0}^{(3)}, \tag{4.11c}
\end{align*}
$$

$a$ and $^{5}$

$$
\begin{align*}
& \rho \mathbb{H}^{(1)} \frac{\partial \boldsymbol{u}^{(1)}}{\partial t}+\mathbb{N}_{\mathrm{i}}\left(\boldsymbol{\omega}^{(2)}, \boldsymbol{u}^{(1)}\right)+\mu \tilde{\mathbb{D}}^{(1)}\left[\begin{array}{c}
\mathbb{H}^{(2)} \boldsymbol{\omega}^{(2)} \\
\tilde{\boldsymbol{\omega}}_{\mathrm{b}}^{(1)}
\end{array}\right]+\mathbb{H}^{(1)} \mathbb{D}^{(0)} \boldsymbol{q}^{(0)}=\mathbb{H}^{(1)} \boldsymbol{f}^{(1)},  \tag{4.12a}\\
& \boldsymbol{\omega}^{(2)}-\mathbb{D}^{(1)} \boldsymbol{u}^{(1)}=0,  \tag{4.12b}\\
& \tilde{\mathbb{D}}^{(2)}\left[\begin{array}{c}
\mathbb{H}^{(1)} \boldsymbol{u}^{(1)} \\
\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(2)}
\end{array}\right]=\tilde{\boldsymbol{0}}^{(3)}, \tag{4.12c}
\end{align*}
$$

where $\mathbb{N}_{\mathrm{o}}\left(\boldsymbol{\omega}^{(1)}, \boldsymbol{u}^{(2)}\right)$ and $\mathbb{N}_{\mathrm{i}}\left(\mathbb{D}^{(1)} \boldsymbol{u}^{(1)}, \boldsymbol{u}^{(1)}\right)$ are the discretizations of the respective convective terms. We continue by specifying these terms.

[^28]We start with $\mathbb{N}_{\mathrm{o}}\left(\boldsymbol{\omega}^{(1)}, \boldsymbol{u}^{(2)}\right)$ and consider a single cell $\sigma_{(3)}$. Note that for $\tilde{\omega}^{(1)} \in$ $\tilde{\Lambda}^{(1)} \sigma_{(3)}$ and $\tilde{u}^{(2)} \in \tilde{\Lambda}^{(2)} \sigma_{(3)}$, with $\boldsymbol{\omega}_{\sigma_{(3)}}^{(1)}:=R_{\sigma_{(3)}}^{(1)}\left(\tilde{\omega}^{(1)}\right)$ and $\boldsymbol{u}_{\sigma_{(3)}}^{(2)}:=R_{\sigma_{(3)}}^{(2)}\left(\tilde{u}^{(2)}\right)$, we have (by Lemma 3.8)

$$
\tilde{\star}^{(1)} \tilde{\omega}^{(1)}=\frac{1}{\left|\sigma_{(3)}\right|} \mathbb{R}_{\sigma_{(3)}}^{(1)} \boldsymbol{\omega}_{\sigma_{(3)}}^{(1)} \quad \text { and } \quad \tilde{\star}^{(2)} \tilde{u}^{(2)}=\frac{1}{\left|\sigma_{(3)}\right|} \mathbb{R}_{\sigma_{(3)}}^{(2)} \boldsymbol{u}_{\sigma_{(3)}}^{(2)} .
$$

This suggests that a consistent approximation of $\tilde{\omega}^{(1)} \wedge \tilde{\star}^{(2)} \tilde{u}^{(2)}$ in each cell $\sigma_{(3)}$ is given by

$$
\left.\left(\tilde{\omega}^{(1)} \wedge \tilde{\star}^{(2)} \tilde{u}^{(2)}\right)\right|_{x_{(1)}^{\sigma_{(3)}}} \approx\left|\sigma_{(3)}\right|^{-2}\left(\star^{(1)} \mathbb{R}_{\sigma_{(3)}}^{(1)} \boldsymbol{\omega}_{\sigma_{(3)}}^{(1)}\right) \wedge\left(\mathbb{R}_{\sigma_{(3)}}^{(2)} \boldsymbol{u}_{\sigma_{(3)}}^{(2)}\right),
$$

where $\boldsymbol{\omega}_{\sigma_{(3)}}^{(1)}=R_{\sigma_{(3)}}^{(1)}\left(\tilde{\omega}^{(1)}\right)$ and $\boldsymbol{u}_{\sigma_{(3)}}^{(2)}=R_{\sigma_{(3)}}^{(2)}\left(\tilde{u}^{(2)}\right)$. Subsequently, we can use the transpose matrix $\mathbb{R}_{\sigma_{(3)}}^{(2) T}$ to approximate the integral of this term on the dual edges. Thus a consistent approximation of the convective term in the cell $\sigma_{(3)}$ is given by

$$
\mathbb{N}_{\mathrm{o}, \sigma_{(3)}}\left(\boldsymbol{\omega}_{\sigma_{(3)}}^{(1)}, \boldsymbol{u}_{\sigma_{(3)}}^{(2)}\right):=\rho \mathbb{R}_{\sigma_{(3)}}^{(2) T} \tilde{\star}^{(2)}\left(\left|\sigma_{(3)}\right|^{-2}\left(\tilde{\star}^{(1)} \mathbb{R}_{\sigma_{(3)}}^{(1)} \boldsymbol{\omega}_{\sigma_{(3)}}^{(1)}\right) \wedge\left(\mathbb{R}_{\sigma_{(3)}}^{(2)} \boldsymbol{u}_{\sigma_{(3)}}^{(2)}\right)\right)
$$

For implementation we rewrite the exterior product using a skew-symmetric matrix. Suppose that we have the Cartesian representation

$$
\begin{aligned}
\left|\sigma_{(3)}\right|^{-1} \star{ }^{(1)} \mathbb{R}_{\sigma_{(3)}}^{(1)} \boldsymbol{\omega}_{\sigma_{(3)}}^{(1)} & =\{d x \wedge d y \wedge d z\}\left(\omega_{x} d x+\omega_{y} d y+\omega_{z} d z\right), \\
\left|\sigma_{(3)}\right|^{-1} \mathbb{R}_{\sigma_{(3)}}^{(2)} \boldsymbol{u}_{\sigma_{(3)}}^{(2)} & =u_{x} d x+u_{y} d y+u_{z} d z .
\end{aligned}
$$

In terms of these components the exterior product is given by
$\left|\sigma_{(3)}\right|^{-2}\left(\star^{(1)} \mathbb{R}_{\sigma_{(3)}}^{(1)} \boldsymbol{\omega}_{\sigma_{(3)}}^{(1)}\right) \wedge\left(\mathbb{R}_{\sigma_{(3)}}^{(2)} \boldsymbol{u}_{\sigma_{(3)}}^{(2)}\right)$
$=\{d x \wedge d y \wedge d z\}\left(\left(\omega_{y} u_{z}-\omega_{z} u_{y}\right) d y \wedge d z+\left(\omega_{z} u_{x}-\omega_{x} u_{z}\right) d z \wedge d x+\left(\omega_{x} u_{y}-\omega_{y} u_{x}\right) d x \wedge d y\right)$.
Using the Cartesian components we can write ${ }^{6}$

$$
\mathbb{N}_{\mathrm{o}, \sigma_{(3)}}\left(\boldsymbol{\omega}_{\sigma_{(3)}}^{(1)}, \boldsymbol{u}_{\sigma_{(3)}}^{(2)}\right)=\rho \mathbb{R}_{\sigma_{(3)}}^{(2) T} \mathbb{W}\left[\boldsymbol{\omega}_{\sigma_{(3)}}^{(1)}\right] \mathbb{R}_{\sigma_{(3)}}^{(2)} \boldsymbol{u}_{\sigma_{(3)}}^{(2)}
$$

where

$$
\mathbb{W}\left[\boldsymbol{\omega}_{\sigma_{(3)}}^{(1)}\right]=\left[\begin{array}{ccc}
0 & -\omega_{z} & \omega_{y} \\
\omega_{z} & 0 & -\omega_{x} \\
-\omega_{y} & \omega_{x} & 0
\end{array}\right]
$$

We denote the skew-symmetric matrix by $\mathbb{C}_{\mathrm{o}, \sigma_{(3)}}\left[\boldsymbol{\omega}_{\sigma_{(3)}}^{(1)}\right]:=\rho \mathbb{R}_{\sigma_{(3)}}^{(2) T} \mathbb{W}\left[\boldsymbol{\omega}_{\sigma_{(3)}}^{(1)}\right] \mathbb{R}_{\sigma_{(3)}}^{(2)}$ and define the global version according to ${ }^{7}$

$$
\mathbb{C}_{\mathrm{o}}\left[\boldsymbol{\omega}^{(1)}\right]:=\sum_{\sigma_{(3)} \in \mathcal{M}_{(3)}} \mathbb{T}_{\sigma_{(3)}}^{(2) T} \mathbb{C}_{\mathrm{o}, \sigma_{(3)}}\left[\mathbb{T}_{\sigma_{(3)}}^{(1)} \boldsymbol{\omega}^{(1)}\right] \mathbb{T}_{\sigma_{(3)}}^{(2)}
$$

[^29]which, being the sum of skew-symmetric matrices, is skew-symmetric. Using this matrix we can state the global version of the convective term:
$$
\mathbb{N}_{\mathrm{o}}\left(\boldsymbol{\omega}^{(1)}, \boldsymbol{u}^{(2)}\right):=\mathbb{C}_{\mathrm{o}}\left[\boldsymbol{\omega}^{(1)}\right] \boldsymbol{u}^{(2)}
$$

For the inner-oriented scheme we analogously define

$$
\mathbb{N}_{\mathrm{i}, \sigma_{(3)}}\left(\boldsymbol{\omega}_{\sigma_{(3)}}^{(2)}, \boldsymbol{u}_{\sigma_{(3)}}^{(1)}\right):=\rho \mathbb{R}_{\sigma_{(3)}}^{(1) T}\left(\left|\sigma_{(3)}\right|^{-2}\left(\mathbb{R}_{\sigma_{(3)}}^{(2)} \boldsymbol{\omega}_{\sigma_{(3)}}^{(2)}\right) \wedge\left(\tilde{\star}^{(1)} \mathbb{R}_{\sigma_{(1)}}^{(1)} \boldsymbol{u}_{\sigma_{(3)}}^{(1)}\right)\right)
$$

This we can write using Cartesian components as

$$
\mathbb{N}_{\mathrm{i}, \sigma_{(3)}}\left(\boldsymbol{\omega}_{\sigma_{(3)}}^{(2)}, \boldsymbol{u}_{\sigma_{(3)}}^{(1)}\right)=\rho \mathbb{R}_{\sigma_{(3)}}^{(1) T} \mathbb{W}\left[\boldsymbol{\omega}_{\sigma_{(3)}}^{(2)}\right] \mathbb{R}_{\sigma_{(3)}}^{(1)} \boldsymbol{u}_{\sigma_{(3)}}^{(1)},
$$

where also

$$
\mathbb{W}\left[\boldsymbol{\omega}_{\sigma_{(3)}}^{(2)}\right]=\left[\begin{array}{ccc}
0 & -\omega_{z} & \omega_{y} \\
\omega_{z} & 0 & -\omega_{x} \\
-\omega_{y} & \omega_{x} & 0
\end{array}\right]
$$

Similarly, we denote the corresponding skew-symmetric matrix in the inner-oriented case by $\mathbb{C}_{\mathbf{i}, \sigma_{(3)}}\left[\boldsymbol{\omega}_{\sigma_{(3)}}^{(2)}\right]:=\rho \mathbb{R}_{\sigma_{(3)}}^{(1) T} \mathbb{W}\left[\boldsymbol{\omega}_{\sigma_{(3)}}^{(2)}\right] \mathbb{R}_{\sigma_{(3)}}^{(1)}$ and define the global version according to

$$
\mathbb{C}_{\mathrm{i}}\left[\boldsymbol{\omega}^{(2)}\right]:=\sum_{\sigma_{(3)} \in \mathcal{M}_{(3)}} \mathbb{T}_{\sigma_{(3)}}^{(1) T} \mathbb{C}_{\mathrm{i}, \sigma_{(3)}}\left[\mathbb{T}_{\sigma_{(3)}}^{(2)} \boldsymbol{\omega}^{(2)}\right] \mathbb{T}_{\sigma_{(3)}}^{(1)}
$$

Finally, the convective term is for the inner-oriented scheme given by

$$
\mathbb{N}_{\mathrm{i}}\left(\boldsymbol{\omega}^{(2)}, \boldsymbol{u}^{(1)}\right):=\mathbb{C}_{\mathrm{i}}\left[\boldsymbol{\omega}^{(2)}\right] \boldsymbol{u}^{(1)}
$$

This discretization of the convection term for the inner-oriented scheme coincides with the 2 D convection discretization considered in [81], when a triangular mesh and a circumcentric dual mesh is used.

Besides these two discretizations of the convection term based on the rotational form, we introduce a third discretization based on the divergence form of the convection term introduced before in [21,78]. We repeat it here for completeness.

This discretization of the divergence-form of the convection term is given more easily using vector calculus. Let us define $\tilde{u}_{(2)}:=\sharp \tilde{u}^{(2)}$. The convective term, which we denote by $N_{\mathrm{d}}\left(\tilde{u}^{(2)}\right)$, is given by

$$
N_{\mathrm{d}}\left(\tilde{u}^{(2)}\right)=\nabla \cdot\left(\tilde{u}_{(2)} \otimes\left(\rho \tilde{u}_{(2)}\right)\right) .
$$

The term is discretized by integrating it over a primal cell $\sigma_{(3)}$, applying Gauss's divergence theorem and approximating the fluxes. We have

$$
\int_{\sigma_{(3)}} N_{\mathrm{d}}\left(\tilde{u}^{(2)}\right) d V=\int_{\sigma_{(3)}} \nabla \cdot\left(\tilde{u}_{(2)} \otimes\left(\rho \tilde{u}_{(2)}\right)\right) d V=\sum_{\sigma_{(2)} \in \partial \sigma_{(3)}} \int_{\sigma_{(2)}} \rho \tilde{u}_{(2)}\left(\tilde{u}_{(2)}, \tilde{n}_{(2)}\right) d A,
$$

where $\tilde{n}_{(2)}$ is the outward normal at $\sigma_{(2)}$. This we rewrite as

$$
\int_{\sigma_{(3)}} N_{\mathrm{d}}\left(\tilde{u}^{(2)}\right) d V=\underline{\mathbb{D}}_{\sigma_{(3)}}^{(2)}{\underline{\phi_{(3)}}}_{(2)}^{(2)},
$$

where

$$
\underline{D}_{\sigma_{(3)}}^{(2)}:=\left[\begin{array}{ccc}
\mathbb{D}_{\sigma_{(3)}}^{(2)} & 0 & 0 \\
0 & \mathbb{D}_{\sigma_{(3)}}^{(2)} & 0 \\
0 & 0 & \mathbb{D}_{\sigma_{(3)}}^{(2)}
\end{array}\right], \quad \underline{\phi}_{\sigma_{(3)}}^{(2)}:=\left[\begin{array}{l}
\phi_{x, \sigma_{(3)}}^{(2)} \\
\phi_{y, \sigma_{(3)}}^{(2)} \\
\phi_{z, \sigma_{(3)}}^{(2)}
\end{array}\right]
$$

with $\mathbb{D}_{\sigma_{(3)}}^{(2)}:=\mathbb{T}_{\sigma_{(3)}}^{(3)} \mathbb{D}^{(2)} \mathbb{T}_{\sigma_{(2)}}^{(2) T}$ and $\phi_{i, \sigma_{(3)}}^{(2)} \in C^{(2)}\left(\sigma_{(3)}\right)$ the local cochain with $\left[\phi_{i, \sigma_{(3)}}^{(2)}\right]_{\sigma_{(2)}}:=$ $\int_{\sigma_{(2)}} \rho \tilde{u}_{(2), i}\left(\tilde{u}_{(2)}, \tilde{n}_{(2)}\right) d A$ for each $\sigma_{(2)} \in \partial \sigma_{(2)}$ where $\tilde{u}_{(2), i}$ is the $i$-component of $\tilde{u}_{(2)}$ $(i=x, y, z)$.

We replace the exact fluxes, using central difference approximations, by

$$
\left[\phi_{i, \sigma_{(3)}}^{(2)}\right]_{\sigma_{(2)}}=\left(\frac{1}{2} \rho \sum_{\sigma_{(3)} \in \mathcal{Q}^{-1} \sigma_{(2)}} u_{i, \sigma_{(3)}}\right) u_{\sigma_{(2)}}^{(2)},
$$

where $u_{i, \sigma_{(3)}}$ is the $i$-component of the reconstruction of the velocity vector $u_{(1), \sigma_{(3)}}$ given by

$$
\begin{equation*}
u_{(1), \sigma_{(3)}}=\sharp\left(\frac{1}{\left|\sigma_{(3)}\right|} \mathbb{R}_{\sigma_{(3)}}^{(2)} \boldsymbol{u}_{\sigma_{(3)}}^{(2)}\right), \tag{4.13}
\end{equation*}
$$

and

$$
u_{\sigma_{(2)}}^{(2)}:=\int_{\sigma_{(2)}}\left(\tilde{u}_{(2)}, \tilde{n}_{(2)}\right) d A=\int_{\sigma_{(2)}} \tilde{u}^{(2)},
$$

i.e., the entry of $\boldsymbol{u}^{(2)}$ for $\sigma_{(2)}$.

Taking a central difference approximation to calculate the flux is crucial for the discretization being energy-conserving as we will see in Section 4.2.4. When $\sigma_{(2)} \in C_{(2)}(\partial \Omega)$ the flux $\left[\phi_{\sigma_{(3)}}^{(2)}\right]_{\sigma_{(2)}}:=\rho u_{(1), \sigma_{(2)}}^{\mathrm{b}} u_{\sigma_{(2)}}^{(2)}$ will be given by the (no-slip) boundary conditions.

The local discretization of the convection term is now given by

$$
\mathbb{N}_{\mathrm{d}, \sigma_{(3)}}\left(\boldsymbol{u}^{(2)}\right)=\frac{1}{\left|\sigma_{(3)}\right|} \mathbb{R}_{\sigma_{(3)}}^{(2) T} \underline{\underline{\sigma}}_{\sigma_{(3)}}^{(2)} \underline{\phi}_{\sigma_{(3)}}^{(2)},
$$

where $\underline{\phi}_{\sigma_{(3)}}^{(2)}$ now contains the approximate fluxes. Note that to calculate $\mathbb{N}_{\mathrm{d}, \sigma_{(3)}}\left(\boldsymbol{u}^{(2)}\right)$ the velocity variables of neighboring cells are needed. The resulting stencil will therefore be significantly wider than for the two convection discretizations based on the rotational form.

The global discretization is given by

$$
\mathbb{N}_{\mathrm{d}}\left(\boldsymbol{u}^{(2)}\right):=\sum_{\sigma_{(3)} \in C_{(3)}(\Omega)} \mathbb{T}_{\sigma_{(3)}}^{(2) T} \mathbb{N}_{\mathrm{d}, \sigma_{(3)}}\left(\boldsymbol{u}^{(2)}\right)
$$

This adds to (4.11) and (4.12) the third discrete formulation given by

$$
\begin{align*}
\rho \mathbb{H}^{(2)} \frac{\partial \boldsymbol{u}^{(2)}}{\partial t}+\mathbb{N}_{\mathrm{d}}\left(\boldsymbol{u}^{(2)}\right)+\mu \mathbb{H}^{(2)} \mathbb{D}^{(1)} \boldsymbol{\omega}^{(1)}+\tilde{\mathbb{D}}^{(0)}\left[\begin{array}{c}
\tilde{\boldsymbol{p}}^{(0)} \\
\tilde{\boldsymbol{p}}_{\mathrm{b}}^{(0)}
\end{array}\right] & =\mathbb{H}^{(2)} \boldsymbol{f}^{(2)},  \tag{4.14a}\\
\mathbb{H}^{(1)} \boldsymbol{\omega}^{(1)}-\tilde{\mathbb{D}}^{(1)}\left[\begin{array}{c}
\mathbb{H}^{(2)} \boldsymbol{u}^{(2)} \\
\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}
\end{array}\right] & =\tilde{\mathbf{0}}^{(2)},  \tag{4.14b}\\
\mathbb{D}^{(2)} \boldsymbol{u}^{(2)} & =\mathbf{0}^{(3)} . \tag{4.14c}
\end{align*}
$$

Note that $\tilde{\boldsymbol{p}}^{(0)}$ and $\tilde{\boldsymbol{p}}_{\mathrm{b}}^{(0)}$ are now ordinary pressure variables again instead of total pressure variables.

### 4.2.2 Calculating the pressure from the total pressure

In the schemes with the convection term discretized in rotational form we have, instead of pressure variables, total pressure variables: $\tilde{\boldsymbol{q}}^{(0)}$ for the outer-oriented scheme and $\boldsymbol{q}^{(0)}$ for the inner-oriented scheme. Often it is important to know the pressure field itself, for example, when the forces acting on an object present in the flow are of interest.

The pressure is given by $p=q^{(0)}-\left(u_{(1)}, u_{(1)}\right) / 2$. For both the outer- and inner-oriented scheme we calculate the pressure in the cell center. For the outer-oriented scheme the total pressure variables are already located in the cell centers. We approximated the velocity in the cell centers according to Lemma 3.8 as

$$
\tilde{\star}^{(2)} \tilde{u}_{\sigma_{(3)}}^{(2)}=\frac{1}{\left|\sigma_{(3)}\right|} \mathbb{R}_{\sigma_{(3)}}^{(2)} \boldsymbol{u}_{\sigma_{(3)}}^{(2)} .
$$

The pressure in the cell center (dual vertex) is then given by

$$
p_{\star \sigma_{(3)}}=\tilde{q}_{\star \sigma_{(3)}}^{(0)}-\frac{\rho}{2}\left(\tilde{u}_{\sigma_{(3)}}^{(2)}, \tilde{u}_{\sigma_{(3)}}^{(2)}\right) .
$$

For the inner-oriented scheme the total pressure variables are located in the primal vertices and we first use these to find an approximation in the dual vertices, again according to Lemma 3.8, as

$$
q_{\star \sigma_{(3)}}:=\star \star^{(0)} q_{\sigma_{(3)}}^{(0)}=\frac{1}{\left|\sigma_{(3)}\right|} \mathbb{R}_{\sigma_{(3)}}^{(0)} \boldsymbol{q}_{\sigma_{(3)}}^{(0)}
$$

Similarly, we define

$$
\star^{(1)} u_{\sigma_{(3)}}^{(1)}=\frac{1}{\left|\sigma_{(3)}\right|} \mathbb{R}_{\sigma_{(3)}}^{(1)} \boldsymbol{u}_{\sigma_{(3)}}^{(1)}
$$

The pressure we then calculate as

$$
\begin{equation*}
p_{\star \sigma_{(3)}}=q_{\star \sigma_{(3)}}-\frac{\rho}{2}\left(u_{\sigma_{(3)}}^{(1)}, u_{\sigma_{(3)}}^{(1)}\right) . \tag{4.15}
\end{equation*}
$$



Figure 4.2: The inner-oriented primal mesh is shown in blue and the outer-oriented dual mesh is shown in red.

### 4.2.3 The convection discretizations on Cartesian meshes

Our goal is to find a generalization of the MAC scheme to general meshes. Above we formulated three discretizations of the convection term. Although these all use the same discrete variables on a Cartesian mesh, they are not true generalizations of the MAC convection discretization because they have a different stencil. However, we will show that the inner-oriented scheme in 2D does coincide with the MAC discretization on a Cartesian mesh.

It is not immediately obvious that this is the case because the MAC scheme is a discretization of the incompressible Navier-Stokes equations with the convection term in divergence form and it uses the normal pressure, while the inner-oriented scheme defined above is based on the rotational form of the convection and therefore uses the total pressure.

We consider the situation in Figure 4.2. We use here, as in the MAC scheme, point values for the velocity. So, $u_{i, j}:=u_{e_{i, j}}^{(1)} / \Delta x$, where $u_{e_{i, j}}^{(1)}$ is the entry of $\boldsymbol{u}^{(1)}$ corresponding to edges $e_{i, j}$ and $\Delta x$ is the cell width. We denote the energy part of the total pressure on the vertices by $\boldsymbol{k}^{(0)}$, which we calculate from the velocity field as ${ }^{8}$

$$
k_{i, j}^{(0)}=\frac{\rho}{2}\left(\frac{u_{i, j}^{2}+u_{i-1, j}^{2}}{2}\right)+\frac{\rho}{2}\left(\frac{v_{i, j}^{2}+v_{i, j-1}^{2}}{2}\right) .
$$

We substitute for the total pressure $\boldsymbol{q}^{(0)}:=\boldsymbol{p}^{(0)}+\boldsymbol{k}^{(0)}$ and will show that the extra term $\mathbb{H}^{(1)} \mathbb{D}^{(0)} \boldsymbol{k}^{(0)}$ that now appears in the discrete momentum equation, together with the

[^30]discretization of the rotational convection term, will give the MAC convection discretization. This shows that if we find a solution $\left(\boldsymbol{u}^{(1)}, \boldsymbol{q}^{(0)}\right)$ of the inner-oriented scheme, then $\left(\boldsymbol{u}^{(1)}, \boldsymbol{p}^{(0)}\right)$ is the solution of the MAC scheme and vice versa.

The 2D version of the discrete rotational-form convection term for the inner-oriented scheme is given by

$$
\mathbb{N}_{\mathrm{i}, \sigma_{(2)}}\left(\boldsymbol{\omega}_{\sigma_{(2)}}^{(2)}, \boldsymbol{u}_{\sigma_{(2)}}^{(1)}\right):=-\rho \mathbb{R}_{\sigma_{(2)}}^{(1) T}\left(\left|\sigma_{(2)}\right|^{-2}\left(\mathbb{R}_{\sigma_{(2)}}^{(2)} \boldsymbol{\omega}_{\sigma_{(2)}}^{(2)}\right) \wedge\left(\tilde{\star}^{(1)} \mathbb{R}_{\sigma_{(1)}}^{(2)} \boldsymbol{u}_{\sigma_{(2)}}^{(1)}\right)\right)
$$

where the extra minus sign compared to the 3 D version comes from the fact that $\tilde{\star}^{(1)} \star^{(1)}=$ -1 in 2D. Let us denote the cell to the north of $u_{i, j}$ by $\sigma_{(2)}^{\mathrm{n}}$ and the one to the south by $\sigma_{(2)}^{\mathrm{s}}$. We have, using Cartesian coordinates,

$$
\begin{aligned}
\tilde{\star}^{(1)} \mathbb{R}_{\sigma_{(2)}^{(n)}}^{(1)} \boldsymbol{u}_{\sigma_{(2)}^{n}}^{(1)} & =-\frac{\Delta x \Delta y}{2}\left[\begin{array}{l}
u_{i, j}+u_{i, j+1} \\
v_{i, j}+v_{i+1, j}
\end{array}\right], \\
\left(\mathbb{R}_{\sigma_{(2)}^{(2)}}^{(2)} \boldsymbol{\omega}_{\sigma_{(2)}^{(2)}}^{(2)}\right) & =\Delta x u_{i, j}-\Delta x u_{i, j+1}-\Delta y v_{i, j}+\Delta y v_{i+1, j},
\end{aligned}
$$

where $\Delta x$ and $\Delta y$ are respectively, the cell width and cell height. Similarly, we have

$$
\begin{aligned}
\tilde{\star}^{(1)} \mathbb{R}_{\sigma_{(2)}^{s}}^{(1)} \boldsymbol{u}_{\sigma_{(2)}}^{(1)} & =-\frac{\Delta x \Delta y}{2}\left[\begin{array}{c}
u_{i, j-1}+u_{i, j} \\
v_{i, j-1}+v_{i+1, j-1}
\end{array}\right], \\
\left(\mathbb{R}_{\sigma_{(2)}^{(2)}}^{(2)} \boldsymbol{\omega}_{\sigma_{(2)}^{s}}^{(2)}\right) & =\Delta x u_{i, j-1}-\Delta x u_{i, j}-\Delta y v_{i, j-1}+\Delta y v_{i+1, j-1} .
\end{aligned}
$$

We write $\left[\mathbb{N}_{\mathrm{i}, \sigma_{(2)}}\left(\boldsymbol{\omega}_{\sigma_{(2)}}^{(2)}, \boldsymbol{u}_{\sigma_{(2)}}^{(1)}\right)\right]_{u_{i, j}}$ for the value of $\mathbb{N}_{\mathrm{i}, \sigma_{(2)}}\left(\boldsymbol{\omega}_{\sigma_{(2)}}^{(2)}, \boldsymbol{u}_{\sigma_{(2)}}^{(1)}\right)$ on the edge of $u_{i, j}$. We have

$$
\begin{aligned}
& {\left[\mathbb{N}_{\mathbf{i}, \sigma_{(2)}^{\mathrm{n}}}\left(\boldsymbol{\omega}_{\sigma_{(2)}^{\mathrm{n}}}^{(2)}, \boldsymbol{u}_{\sigma_{(2)}^{\mathrm{n}}}^{(1)}\right)\right]_{u_{i, j}}} \\
& \quad=\frac{\rho}{2 \Delta x \Delta y}\left[\frac{\Delta y}{2} \quad 0\right]\left[\begin{array}{cc}
0 & -\omega_{\sigma_{(2)}}^{(2)} \\
\omega_{\sigma_{(2)}^{\mathrm{n}}}^{(2)} & 0
\end{array}\right]\left[\begin{array}{c}
u_{i, j}+u_{i, j+1} \\
v_{i, j}+v_{i+1, j}
\end{array}\right] \\
& \quad=-\frac{\rho}{4 \Delta x} \omega_{\sigma_{(2)}^{(2)}}^{(2)}\left(v_{i, j}+v_{i+1, j}\right)
\end{aligned}
$$

where $\omega_{\sigma_{(2)}^{\mathrm{n}}}^{(2)}:=\Delta x u_{i, j}-\Delta x u_{i, j+1}-\Delta y v_{i, j}+\Delta y v_{i+1, j}$. Thus, we find

$$
\begin{aligned}
{\left[\mathbb{N}_{\mathrm{i}, \sigma_{(2)}^{\mathrm{n}}}\left(\boldsymbol{\omega}_{\sigma_{(2)}^{\mathrm{n}}}^{(2)}, \boldsymbol{u}_{\sigma_{(2)}^{\mathrm{n}}}^{(1)}\right)\right]_{u_{i, j}}=-\frac{\rho}{4 \Delta x} } & \left(\left(u_{i, j}-u_{i, j+1}\right)\left(v_{i, j}+v_{i+1, j}\right) \Delta x\right. \\
& \left.+\left(v_{i+1, j}-v_{i, j}\right)\left(v_{i, j}+v_{i+1, j}\right) \Delta y\right), \\
{\left[\mathbb{N}_{\mathrm{i}, \sigma_{(2)}^{\mathrm{s}}}\left(\boldsymbol{\omega}_{\sigma_{(2)}^{\mathrm{s}}}^{(2)}, \boldsymbol{u}_{\sigma_{(2)}^{\mathrm{s}}}^{(1)}\right)\right]_{u_{i, j}}=-\frac{\rho}{4 \Delta x} } & \left(\left(u_{i, j-1}-u_{i, j}\right)\left(v_{i, j-1}+v_{i+1, j-1}\right) \Delta x\right. \\
& \left.+\left(v_{i+1, j-1}-v_{i, j-1}\right)\left(v_{i, j-1}+v_{i+1, j-1}\right) \Delta y\right)
\end{aligned}
$$

Finally, the entry of the convection term corresponding to $u_{i, j}$ is given by

$$
\begin{aligned}
& {\left[\mathbb{N}_{\mathrm{i}}\left(\omega^{(2)}, \boldsymbol{u}^{(1)}\right)\right]_{u_{i, j}}} \\
& \quad=\frac{\rho}{4 \Delta x}\left(\left(u_{i, j+1}-u_{i, j}\right)\left(v_{i, j}+v_{i+1, j}\right) \Delta x+\left(v_{i, j}-v_{i+1, j}\right)\left(v_{i, j}+v_{i+1, j}\right) \Delta y\right. \\
& \left.\quad+\left(u_{i, j}-u_{i, j-1}\right)\left(v_{i, j-1}+v_{i+1, j-1}\right) \Delta x+\left(v_{i, j-1}-v_{i+1, j-1}\right)\left(v_{i, j-1}+v_{i+1, j-1}\right) \Delta y\right)
\end{aligned}
$$

Similarly, on the Cartesian mesh and for a diagonal Hodge matrix, the extra term resulting from the kinetic energy part of the total pressure is given by

$$
\begin{aligned}
& {\left[\mathbb{H}^{(1)} \mathbb{D}^{(0)} \boldsymbol{k}^{(0)}\right]_{u_{i, j}}} \\
& \quad=\frac{\Delta y}{\Delta x}\left(k_{i+1, j}-k_{i, j}\right) \\
& \quad=\frac{\Delta y}{4 \Delta x}\left(\left(u_{i+1, j}^{2}+u_{i, j}^{2}+v_{i+1, j}^{2}+v_{i+1, j-1}^{2}\right)-\left(u_{i, j}^{2}+u_{i-1, j}^{2}+v_{i, j}^{2}+v_{i, j-1}^{2}\right)\right)
\end{aligned}
$$

Adding the contributions of the rotational convection term and the gradient of the kinetic energy part of the total pressure we obtain

$$
\begin{aligned}
& \Delta x {\left[\mathbb{N}_{\mathbf{i}}\left(\omega^{(2)}, \boldsymbol{u}^{(1)}\right)+\mathbb{H}^{(1)} \mathbb{D}^{(0)} \boldsymbol{k}^{(0)}\right]_{u_{i, j}} } \\
&=\frac{\rho}{4}\left(\left(u_{i, j+1}-u_{i, j}\right)\left(v_{i, j}+v_{i+1, j}\right) \Delta x+\left(v_{i, j}-v_{i+1, j}\right)\left(v_{i, j}+v_{i+1, j}\right) \Delta y\right. \\
&+\left(u_{i, j}-u_{i, j-1}\right)\left(v_{i, j-1}+v_{i+1, j-1}\right) \Delta x+\left(v_{i, j-1}-v_{i+1, j-1}\right)\left(v_{i, j-1}+v_{i+1, j-1}\right) \Delta y \\
&\left.+\left(u_{i+1, j}^{2}+u_{i, j}^{2}+v_{i+1, j}^{2}+v_{i+1, j-1}^{2}\right) \Delta y-\left(u_{i, j}^{2}+u_{i-1, j}^{2}+v_{i, j}^{2}+v_{i, j-1}^{2}\right) \Delta y\right) \\
&=\frac{\rho}{4}\left(\left(u_{i, j+1}-u_{i, j}\right)\left(v_{i, j}+v_{i+1, j}\right) \Delta x+\left(u_{i, j}-u_{i, j-1}\right)\left(v_{i, j-1}+v_{i+1, j-1}\right) \Delta x\right. \\
&\left.+\left(u_{i+1, j}^{2}+u_{i, j}^{2}-u_{i, j}^{2}-u_{i-1, j}^{2}\right) \Delta y\right) .
\end{aligned}
$$

For the inner-oriented scheme, mass conservation holds on the dual cells. For the dual cells to east and west of $u_{i, j}$ we have

$$
\begin{aligned}
& 0=\left(u_{i+1, j}-u_{i, j}\right) \Delta y+\left(v_{i+1, j}-v_{i+1, j-1}\right) \Delta x, \\
& 0=\left(u_{i, j}-u_{i-1, j}\right) \Delta y+\left(v_{i, j}-v_{i, j-1}\right) \Delta x .
\end{aligned}
$$

Adding these and multiplying by $\rho u_{i, j} / 2$ we find

$$
0=\frac{\rho}{4}\left(2 u_{i, j}\left(v_{i+1, j}+v_{i, j}\right) \Delta x-2 u_{i, j}\left(v_{i+1, j-1}+v_{i, j-1}\right) \Delta x\right)+\frac{\rho}{2} u_{i, j}\left(u_{i+1, j}-u_{i-1, j}\right) \Delta y .
$$

Adding this zero term to the expression found earlier we obtain

$$
\begin{aligned}
\Delta x[ & \left.\mathbb{N}_{\mathbf{i}}\left(\omega^{(2)}, \boldsymbol{u}^{(1)}\right)+\mathbb{H}^{(1)} \mathbb{D}^{(0)} \boldsymbol{k}^{(0)}\right]_{u_{i, j}} \\
& =\frac{\rho}{4}\left(\left(\left(u_{i, j+1}+u_{i, j}\right)\left(v_{i, j}+v_{i+1, j}\right) \Delta x-\left(u_{i, j}+u_{i, j-1}\right)\left(v_{i, j-1}+v_{i+1, j-1}\right) \Delta x\right.\right. \\
& \left.\quad+\left(u_{i+1, j}+u_{i, j}\right)^{2} \Delta y-\left(u_{i, j}+u_{i-1, j}\right)^{2} \Delta y\right) .
\end{aligned}
$$

This is exactly the convection discretization of the MAC scheme, given in equation (1.7).
In 3D the inner-oriented scheme has a wider stencil than the MAC convection discretization. While in 2D an edge lies between two 2D cells, in 3D an edge lies between four 3D cells. For the rotational convection stencil of a given edge the velocity variables on all edges of these four cells are needed. The stencil therefore involves 33 velocity variables. This is more than double the 15 velocity variables that are needed for the MAC convection discretization in 3D.

The stencil of the outer-oriented rotational convection discretization needs, for the stencil of a velocity variable located on a face, the velocity and vorticity variables on the two cells between which the face lies. The vorticity variables located on the edges of these cells are again calculated using the velocity variables located on the faces that have that particular edge in its boundary. This leads to a stencil involving 41 velocity variables.

The stencil of the outer-oriented divergence-form discretization is even larger. For the stencil corresponding to a given face the integral of the convection term is approximated in both neighboring cells. For this approximation the velocity vector is used in each cell that shares a face with one of these two cells and the cell velocity vector is calculated using all the velocity variables of that cell. Thus the stencil involves as many velocity variables as there are faces in the union of neighbor cells of the two cells that neighbor a given face. This means that the stencil involves 57 velocity variables in 3D.

### 4.2.4 Conservation properties

In Chapter 1 we saw that besides the primary quantities, momentum and mass, also the secondary derived quanitity energy is conserved (in the inviscid limit). The derivation of these conservation laws is based on the properties and symmetries of the differential operators in the incompressible Navier-Stokes equations.

The discretizations introduced above are based on a mimetic discretization of the differential operators and therefore preserve most of these properties and symmetries. In this section we define the discrete versions of the momentum, vorticity and energy and derive discrete conservation laws for them.

We first consider the global momentum. The $i$-th component of the global momentum is given by

$$
\mathcal{M}_{i}:=\int_{\Omega} \rho \underline{e}_{i} \cdot \underline{u} d V
$$

where $\underline{e}_{i}, i=x, y, z$ is the $x$-, $y$-, $z$-unit vector. Conservation of global momentum is found by integrating the momentum equation (1.2a) over the flow domain $\Omega$ and applying the Gauss Divergence Theorem. This gives

$$
\begin{equation*}
\frac{\partial \mathcal{M}_{i}}{\partial t}=-\int_{\partial \Omega} \rho \underline{u}(\underline{u} \cdot \underline{n})+\mu \underline{e}_{i} \cdot(\underline{n} \times \underline{\omega})+p\left(\underline{e}_{i} \cdot \underline{n}\right) d A+\int_{\Omega} \underline{e}_{i} \cdot \underline{f} d V . \tag{4.16}
\end{equation*}
$$

The convection discretization based on the rotational form does not conserve momentum on general meshes, while the discretization based on the divergence-form is conservative. The definition of discrete momentum depends on the location of the velocity variables.

Definition 4.1. Let the discretizations of the unit basis forms ${ }^{9} e_{i}^{(1)}$, with $i=x, y, z$, be given by $\boldsymbol{e}_{i}^{(1)}:=R^{(1)}\left(e_{i}^{(1)}\right)$ and let $\boldsymbol{e}_{i}^{(2)}:=R^{(2)}\left(\tilde{e}_{i}^{(2)}\right)$, where $\tilde{e}_{i}^{(2)}:=\star e_{i}^{(1)}$. The discrete momentum for the inner-oriented discretization (4.12) is defined as

$$
M_{i}:=\rho \boldsymbol{e}_{i}^{(1) T} \mathbb{H}^{(1)} \boldsymbol{u}^{(1)},
$$

where $i=x, y, z$.
The discrete momentum for the outer-oriented discretizations (4.11) and (4.14) is defined as

$$
M_{i}:=\rho \boldsymbol{e}_{i}^{(2) T} \mathbb{H}^{(2)} \boldsymbol{u}^{(2)},
$$

where $i=x, y, z$.
Note that the definition of the discrete momentum depends on the discrete Hodge matrices used. However, because the discrete Hodge matrices are consistent, the discrete momentum $M_{i}$ is a consistent approximation of the global momentum $\mathcal{M}_{i}$. The discrete momentum is defined analogously in 2D.

Theorem 4.1. The outer-oriented discretization (with convection term in divergenceform) (4.14) is momentum-conserving:

$$
\begin{align*}
\frac{\partial M_{i}}{\partial t}=- & \sum_{\sigma_{(2)} \in C_{(2)}(\partial \Omega)} o_{\sigma_{(2)}} \boldsymbol{\phi}_{i, \sigma_{(2)}}^{(2)}+o_{\sigma_{(2)}} \mu\left(e_{i}^{(1)}, \mathbb{R}_{\sigma_{(2)}}^{(1)} \boldsymbol{\omega}_{\sigma_{(2)}}^{(1)}\right)-\left(\mathbb{T}^{(2)} \boldsymbol{e}_{i}^{(2)}\right)^{T} \tilde{\boldsymbol{p}}_{b}^{(0)}  \tag{4.17}\\
& +\boldsymbol{e}_{i}^{(2) T} \mathbb{H}^{(2)} \boldsymbol{f}^{(2)} .
\end{align*}
$$

Proof. From the discrete momentum equation (4.14a) we find

$$
\begin{aligned}
\frac{\partial M_{i}}{\partial t} & =\boldsymbol{e}_{i}^{(2) T}\left(\rho \mathbb{H}^{(2)} \frac{\partial \boldsymbol{u}^{(2)}}{\partial t}\right) \\
& =\boldsymbol{e}_{i}^{(2) T} \mathbb{H}^{(2)} \boldsymbol{f}^{(2)}-\boldsymbol{e}_{i}^{(2) T} \mathbb{N}_{\mathrm{d}}\left(\boldsymbol{u}^{(2)}\right)-\mu \boldsymbol{e}_{i}^{(2) T} \mathbb{H}^{(2)} \mathbb{D}^{(1)} \boldsymbol{\omega}^{(1)}-\boldsymbol{e}_{i}^{(2) T} \tilde{\mathbb{D}}^{(0)}\left[\begin{array}{c}
\tilde{\boldsymbol{p}}^{(0)} \\
\tilde{\boldsymbol{p}}_{\mathrm{b}}^{(0)}
\end{array}\right]
\end{aligned}
$$

We treat the convection, diffusion and pressure term in turn. For the convection term we get

$$
\boldsymbol{e}_{i}^{(2) T} \mathbb{N}_{\mathrm{d}}\left(\boldsymbol{u}^{(2)}\right)=\sum_{\sigma_{(3)} \in C_{(3)}(\Omega)} \boldsymbol{e}_{i, \sigma_{(3)}}^{(2) T} \mathbb{N}_{\mathrm{d}, \sigma_{(3)}}\left(\boldsymbol{u}^{(2)}\right)
$$

where $\boldsymbol{e}_{i, \sigma_{(3)}}^{(2)}=\mathbb{T}_{\sigma_{(3)}}^{(2)} \boldsymbol{e}_{i}^{(2)}$. Using the definition of $\mathbb{N}_{\mathrm{d}}\left(\boldsymbol{u}^{(2)}\right)$ and Lemma 3.8 we find

$$
\boldsymbol{e}_{i}^{(2) T} \mathbb{N}_{\mathrm{d}}\left(\boldsymbol{u}^{(2)}\right)=\sum_{\sigma_{(3)} \in C_{(3)}(\Omega)} \mathbb{D}_{\sigma_{(3)}}^{(2)} \phi_{i, \sigma_{(3)}}^{(2)}=\sum_{\sigma_{(3)} \in C_{(3)}(\Omega)} \sum_{\sigma_{(2)} \in \partial \sigma_{(3)}} o_{\sigma_{(3)} \sigma_{(2)}} \phi_{i, \sigma_{(2)}}^{(2)}
$$

[^31]The interior fluxes cancel in the summation and hence we finally obtain

$$
\boldsymbol{e}_{i}^{(2) T} \mathbb{N}_{\mathrm{d}}\left(\boldsymbol{u}^{(2)}\right)=\sum_{\sigma_{(2)} \in C_{(2)}(\partial \Omega)} o_{\sigma_{(2)}} \phi_{i, \sigma_{(2)}}^{(2)}
$$

where $o_{\sigma_{(2)}}=o_{\sigma_{(3)} \sigma_{(2)}}$ with $\sigma_{(3)}$ the single element of $\partial^{-1} \sigma_{(2)}$.
We continue with the diffusion term. The consistency of the discrete Hodge matrix implies that

$$
\mu \boldsymbol{e}_{i}^{(2) T} \mathbb{H}^{(2)} \mathbb{D}^{(1)} \boldsymbol{\omega}^{(1)}=\mu \sum_{\sigma_{(3)} \in C_{(3)}(\Omega)}\left(e_{i}^{(1)}, \mathbb{R}_{\sigma_{(3)}}^{(2)} \mathbb{D}_{\sigma_{(3)}}^{(1)} \boldsymbol{\omega}_{\sigma_{(3)}}^{(1)}\right)
$$

Using Lemma 3.9 we get ${ }^{10}$

$$
\begin{aligned}
\mu \boldsymbol{e}_{i}^{(2) T} \mathbb{H}^{(2)} \mathbb{D}^{(1)} \boldsymbol{\omega}^{(1)} & =\mu\left(e_{i}^{(1)}, \sum_{\sigma_{(3)} \in C_{(3)}(\Omega)} \mathbb{R}_{\sigma_{(3)}}^{(2)} \mathbb{D}_{\sigma_{(3)}}^{(1)} \boldsymbol{\omega}_{\sigma_{(3)}}^{(1)}\right) \\
& =\mu \sum_{\sigma_{(3)} \in C_{(3)}(\Omega)} \sum_{\sigma_{(2)} \in \partial \sigma_{(3)}} o_{\sigma_{(3)} \sigma_{(2)}}\left(e_{i}^{(1)}, \mathbb{R}_{\sigma_{(2)}}^{(1)} \boldsymbol{\omega}_{\sigma_{(2)}}^{(1)}\right) \\
& =\mu \sum_{\sigma_{(2)} \in C_{(2)}(\partial \Omega)} o_{\sigma_{(2)}}\left(e_{i}^{(1)}, \mathbb{R}_{\sigma_{(2)}}^{(1)} \boldsymbol{\omega}_{\sigma_{(2)}}^{(1)}\right) .
\end{aligned}
$$

Finally, we come to the pressure term. By Proposition 3.10 we have

$$
\begin{aligned}
\boldsymbol{e}_{i}^{(2) T} \tilde{\mathbb{D}}^{(0)}\left[\begin{array}{c}
\tilde{\boldsymbol{p}}^{(0)} \\
\tilde{\boldsymbol{p}}_{\mathrm{b}}^{(0)}
\end{array}\right] & =-\left(\mathbb{D}^{(2)} \boldsymbol{e}_{i}^{(2)}\right)^{T} \tilde{\boldsymbol{p}}^{(0)}+\left(\mathbb{T}^{(2)} \boldsymbol{e}_{i}^{(2)}\right)^{T} \tilde{\boldsymbol{p}}_{\mathrm{b}}^{(0)} \\
& =\left(\mathbb{T}^{(2)} \boldsymbol{e}_{i}^{(2)}\right)^{T} \tilde{\boldsymbol{p}}_{\mathrm{b}}^{(0)},
\end{aligned}
$$

where we used that $\mathbb{D}^{(2)} \boldsymbol{e}_{i}^{(2)}=\mathbf{0}^{(3)}$.
The terms on the right-hand side of (4.17) are consistent approximations of the terms on the right-hand side of (4.16). To see this for the diffusion term, note that the vector proxy for $\mathbb{R}_{\sigma_{(2)}}^{(1)} \boldsymbol{\omega}_{\sigma_{(2)}}^{(1)}$ is a consistent approximation of the integral of $\underline{n} \times \underline{u}$ over $\sigma_{(2)}$.

Definition 4.2. The discrete energy for the inner-oriented discretization (4.12) is defined as

$$
K:=\frac{1}{2} \rho \boldsymbol{u}^{(1) T} \mathbb{H}^{(1)} \boldsymbol{u}^{(1)},
$$

and the discrete energy for the outer-oriented discretizations (4.11) and (4.14) is defined as

$$
K:=\frac{1}{2} \rho \boldsymbol{u}^{(2) T} \mathbb{H}^{(2)} \boldsymbol{u}^{(2)} .
$$

[^32]Theorem 4.2. The inner-oriented discretization (4.12) is energy-conserving:

$$
\frac{\partial K}{\partial t}=\boldsymbol{u}^{(1) T} \mathbb{H}^{(1)} \boldsymbol{f}^{(1)}-\mu \boldsymbol{\omega}^{(2) T} \mathbb{H}^{(2)} \boldsymbol{\omega}^{(2)}+\mu\left(\mathbb{T}^{(1)} \boldsymbol{u}^{(1)}\right)^{T} \tilde{\boldsymbol{\omega}}_{b}^{(1)}-\tilde{\boldsymbol{u}}_{b}^{(2) T} \mathbb{T}^{(0)} \boldsymbol{q}^{(0)}
$$

The outer-oriented discretizations (4.11) and (4.14) are also energy-conserving. For the outer-oriented discretization (4.11) the energy satisfies

$$
\frac{\partial K}{\partial t}=\boldsymbol{u}^{(2) T} \mathbb{H}^{(2)} \boldsymbol{f}^{(2)}-\mu \boldsymbol{\omega}^{(1) T} \mathbb{H}^{(1)} \boldsymbol{\omega}^{(1)}-\mu \tilde{\boldsymbol{u}}_{b}^{(1) T} \mathbb{T}^{(1)} \boldsymbol{\omega}^{(1)}+\left(\mathbb{T}^{(2)} \tilde{\boldsymbol{u}}^{(2)}\right)^{T} \tilde{\boldsymbol{q}}_{b}^{(0)}
$$

and for the outer-oriented discretization (4.14) the energy satisfies

$$
\begin{aligned}
& \frac{\partial K}{\partial t}=\boldsymbol{u}^{(2) T} \mathbb{H}^{(2)} \boldsymbol{f}^{(2)}-\sum_{\sigma_{(2)} \in \mathcal{M}_{(2)}^{b}} \frac{\rho}{2} o_{\sigma_{(3)} \sigma_{(2)}} u_{\sigma_{(2)}}^{(2)}\left(u_{(1), \sigma_{(3)}}, u_{(1), \sigma_{(3)}}^{\prime}\right) \\
&-\mu \boldsymbol{\omega}^{(1) T} \mathbb{H}^{(1)} \boldsymbol{\omega}^{(1)}-\mu \tilde{\boldsymbol{u}}_{b}^{(1) T} \mathbb{T}^{(1)} \boldsymbol{\omega}^{(1)}+\left(\mathbb{T}^{(2)} \tilde{\boldsymbol{u}}^{(2)}\right)^{T} \tilde{\boldsymbol{p}}_{b}^{(0)}
\end{aligned}
$$

where $u_{(1), \sigma_{(3)}}^{\prime}$ is defined through $u_{(1), \sigma_{(2)}}^{b}=\left(u_{(1), \sigma_{(3)}}^{\prime}+u_{(1), \sigma_{(3)}}\right) / 2$, where $u_{(1), \sigma_{(2)}}^{b}$ is the (given) velocity vector at the boundary face $\sigma_{(2)}$.

Proof. We first consider the inner-oriented discretization. The discrete momentum equation (4.12a) implies that we have

$$
\begin{aligned}
\frac{\partial K}{\partial t}= & \boldsymbol{u}^{(1) T} \mathbb{H}^{(1)} \boldsymbol{f}^{(1)}-\boldsymbol{u}^{(1) T} \mathbb{N}_{\mathrm{i}}\left(\boldsymbol{\omega}^{(2)}, \boldsymbol{u}^{(1)}\right) \\
& \quad-\mu \boldsymbol{u}^{(1) T} \tilde{\mathbb{D}}^{(1)}\left[\begin{array}{c}
\mathbb{H}^{(2)} \boldsymbol{\omega}^{(2)} \\
\tilde{\boldsymbol{\omega}}_{\mathrm{b}}^{(1)}
\end{array}\right]-\boldsymbol{u}^{(1) T} \mathbb{H}^{(1)} \mathbb{D}^{(0)} \boldsymbol{q}^{(0)} .
\end{aligned}
$$

Using $\tilde{\mathbb{D}}^{(1)}=\left[\mathbb{D}^{(1) T}-\mathbb{T}^{(1) T}\right], \tilde{\mathbb{D}}^{(2)}=\left[-\mathbb{D}^{(0) T} \quad \mathbb{T}^{(0) T}\right]$ and the discrete mass equation (4.12c) we find

$$
\begin{aligned}
& \frac{\partial K}{\partial t}= \boldsymbol{u}^{(1) T} \mathbb{H}^{(1)} \boldsymbol{f}^{(1)}-\boldsymbol{u}^{(1) T} \mathbb{N}_{\mathrm{i}}\left(\boldsymbol{\omega}^{(2)}, \boldsymbol{u}^{(1)}\right) \\
& \quad-\mu \boldsymbol{\omega}^{(2) T} \mathbb{H}^{(2)} \boldsymbol{\omega}^{(2)}+\mu\left(\mathbb{T}^{(1)} \boldsymbol{u}^{(1)}\right)^{T} \tilde{\boldsymbol{\omega}}_{\mathrm{b}}^{(1)}-\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(2) T} \mathbb{T}^{(0)} \boldsymbol{q}^{(0)}
\end{aligned}
$$

The convective term gives no contribution because we have

$$
\boldsymbol{u}^{(1) T} \mathbb{N}_{\mathrm{i}}\left(\boldsymbol{\omega}^{(2)}, \boldsymbol{u}^{(1)}\right)=\sum_{\sigma_{(3)} \in \mathcal{M}_{(3)}} \rho \boldsymbol{u}_{\sigma_{(3)}}^{(1) T} \mathbb{R}_{\sigma_{(3)}}^{(1) T} \mathbb{W}\left[\boldsymbol{\omega}_{\sigma_{(3)}}^{(2)}\right] \mathbb{R}_{\sigma_{(3)}}^{(1)} \boldsymbol{u}_{\sigma_{(3)}}^{(1)}=0
$$

as a result of the skew-symmetry of $\mathbb{W}\left[\boldsymbol{\omega}_{\sigma_{(3)}}^{(2)}\right]$ for every $\sigma_{(3)} \in \mathcal{M}_{(3)}$.
For the outer-oriented discretization (4.11) the convection term gives no contribution for the same reason and the derivation for the other terms goes analogously.

For the divergence-form convection term of the outer-oriented discretization (4.14) we use [21]. We have

$$
\boldsymbol{u}^{(2) T} \mathbb{N}_{\mathrm{d}}\left(\boldsymbol{u}^{(2)}\right)=\sum_{\sigma_{(3)} \in \mathcal{M}_{(3)}}\left(\frac{1}{\left|\sigma_{(3)}\right|} \mathbb{R}_{\sigma_{(3)}}^{(2)} \boldsymbol{u}_{\sigma_{(3)}}^{(2)}\right)^{T} \underline{\mathbb{D}}_{\sigma_{(3)}}^{(2)} \underline{\phi}_{(3)}^{(2)}
$$

Using again the notation (4.13) we can write

$$
\begin{aligned}
& \boldsymbol{u}^{(2) T} \mathbb{N}_{\mathrm{d}}\left(\boldsymbol{u}^{(2)}\right) \\
& =\sum_{\sigma_{(3)} \in \mathcal{M}_{(3)}}\left(u_{(1), \sigma_{(3)}}, \mathbb{D}_{\sigma_{(3)}}^{(2)} \boldsymbol{\phi}_{\sigma_{(3)}}^{(2)}\right) \\
& =\sum_{\sigma_{(3)} \in \mathcal{M}_{(3)}} \sum_{\sigma_{(2)} \in \sigma_{(3)}} \frac{1}{2} \rho o_{\sigma_{(3)} \sigma_{(2)}} u_{\sigma_{(2)}}^{(2)}\left(u_{(1), \sigma_{(3)}}, u_{(1), \tau_{(3)}}\right) \\
& +\sum_{\sigma_{(2)} \in \mathcal{M}_{(2)} \backslash \mathcal{M}_{(2)}^{\mathrm{b}}} \frac{1}{2} \rho o_{\sigma_{(3)} \sigma_{(2)}} u_{\sigma_{(2)}}^{(2)}\left(u_{(1), \sigma_{(3)}}, u_{(1), \tau_{(3)}}\right)+\frac{1}{2} \rho o_{\tau_{(3)} \sigma_{(2)}} u_{\sigma_{(2)}}^{(2)}\left(u_{(1), \sigma_{(3)}}, u_{(1), \tau_{(3)}}\right) \\
& +\sum_{\sigma_{(2)} \in \mathcal{M}_{(2)}^{\mathrm{b}}} \rho o_{\sigma_{(3)} \sigma_{(2)}} u_{\sigma_{(2)}}^{(2)}\left(u_{(1), \sigma_{(3)}}, u_{(1), \sigma_{(2)}}^{\mathrm{b}}-\frac{1}{2} u_{(1), \sigma_{(3)}}\right),
\end{aligned}
$$

where in the one before last line $\tau_{(3)}$ is the element in $\partial^{-1} \sigma_{(2)}$ other than $\sigma_{(3)}$. Using the incompressibility constraint (4.14c) and the fact that $o_{\sigma_{(3)} \sigma_{(2)}}=-o_{\tau_{(3)} \sigma_{(2)}}$ we obtain

$$
\boldsymbol{u}^{(2) T} \mathbb{N}_{\mathrm{d}}\left(\boldsymbol{u}^{(2)}\right)=\sum_{\sigma_{(2)} \in \mathcal{M}_{(2)}^{\mathrm{b}}} \rho o_{\sigma_{(3)} \sigma_{(2)}} u_{\sigma_{(2)}}^{(2)}\left(u_{(1), \sigma_{(3)}}, u_{(1), \sigma_{(2)}}^{\mathrm{b}}-\frac{1}{2} u_{(1), \sigma_{(3)}}\right) .
$$

By the definition of $u_{(1), \sigma_{(3)}}^{\prime}$ we have $u_{(1), \sigma_{(3)}}^{\prime} / 2=u_{(1), \sigma_{(2)}}^{\mathrm{b}}-\frac{1}{2} u_{(1), \sigma_{(3)}}$, and we find that

$$
\boldsymbol{u}^{(2) T} \mathbb{N}_{\mathrm{d}}\left(\boldsymbol{u}^{(2)}\right)=\sum_{\sigma_{(2)} \in \mathcal{M}_{(2)}^{\mathrm{b}}} \frac{\rho}{2} o_{\sigma_{(3)} \sigma_{(2)}} u_{\sigma_{(2)}}^{(2)}\left(u_{(1), \sigma_{(3)}}, u_{(1), \sigma_{(3)}}^{\prime}\right)
$$

Note that the convective energy flux, which is explicit for the outer-oriented scheme with convection in divergence form, is contained in the total pressure term for the schemes based on the rotational convection discretization.

Finally we consider the vorticity. The global vorticity is defined as

$$
\mathcal{V}_{i}:=\int_{\Omega} e_{i} \cdot \underline{\omega} d V
$$

In 2 D the vorticity is scalar and the global vorticity is simply given by

$$
\mathcal{V}:=\int_{\Omega} \omega d V .
$$

From the Gauss Divergence Theorem and the definition of the vorticity as curl of the velocity, it follows that the global vorticity is instantly given by the velocity at the boundary.

Theorem 4.3. The global vorticity is given by

$$
\mathcal{V}_{i}=\int_{\partial \Omega} \underline{e}_{i} \cdot(\underline{n} \times \underline{u}) d A .
$$

This implies that in a domain with constant no-slip boundary conditions or a periodic domain the global vorticity is constant (zero for a periodic domain).

The numerical schemes share this feature for the discrete vorticity.

Definition 4.3. Let the discretizations of the unit basis forms $e_{i}^{(2)}$, with $i=x, y, z$, be given by $\boldsymbol{e}_{i}^{(2)}:=R^{(2)}\left(e_{i}^{(2)}\right)$ and let $\boldsymbol{e}_{i}^{(1)}:=R^{(1)}\left(\tilde{e}_{i}^{(1)}\right)$, where $\tilde{e}_{i}^{(1)}:=\star e_{i}^{(2)}$. The discrete vorticity for the inner-oriented discretization (4.12) is defined as

$$
V_{i}:=\boldsymbol{e}_{i}^{(2) T} \mathbb{H}^{(2)} \boldsymbol{\omega}^{(2)},
$$

where $i=x, y, z$.
The discrete vorticity for the outer-oriented discretizations (4.11) and (4.14) is defined as

$$
V_{i}:=\boldsymbol{e}_{i}^{(1) T} \mathbb{H}^{(1)} \boldsymbol{\omega}^{(1)},
$$

where $i=x, y, z$.
In 2D the discrete vorticity is defined as $V:=\boldsymbol{e}^{(2) T} \mathbb{H}^{(2)} \boldsymbol{\omega}^{(2)}$ for the inner-oriented discretization and as $V:=\boldsymbol{e}^{(0) T} \mathbb{H}^{(0)} \boldsymbol{\omega}^{(0)}$ for the outer-oriented discretizations, where, respectively, $e^{(2)}:=R^{(2)}(d x \wedge d y)$ and $e^{(0)}:=R^{(0)}(1)$.
Theorem 4.4. For the inner-oriented discretization (4.12) the global vorticity is given in terms of the boundary velocity as

$$
V_{i}=\sum_{\sigma_{(2)} \in C_{(2)}(\partial \Omega)} o_{\sigma_{(2)}}\left(\star e_{i}^{(2)}, \mathbb{R}_{\sigma_{(2)}}^{(1)} \boldsymbol{u}_{\sigma_{(2)}}^{(1)}\right) .
$$

For the outer-oriented discretizations (4.11) and (4.14) the vorticity is given in terms of the boundary velocity as

$$
V_{i}=-\left(\mathbb{T}^{(1)} \boldsymbol{e}^{(1)}\right)^{T} \tilde{\boldsymbol{u}}_{b}^{(1)}
$$

Both expressions for $V_{i}$ are consistent approximations of $V_{i}=\int_{\partial \Omega} \underline{e}_{i} \cdot(\underline{n} \times \underline{u}) d A$.
Proof. For the inner-oriented discretization the vorticity is given by $\boldsymbol{\omega}^{(2)}=\mathbb{D}^{(1)} \boldsymbol{u}^{(1)}$. Using the fact that $\mathbb{H}^{(2)}$ is consistent and using Lemma 3.9 we find

$$
\begin{aligned}
\boldsymbol{e}_{i}^{(2) T} \mathbb{H}^{(2)} \mathbb{D}^{(1)} \boldsymbol{u}^{(1)} & =\sum_{\sigma_{(3)} \in C_{(3)}(\Omega)}\left(\star e_{i}^{(2)}, \mathbb{R}_{\sigma_{(3)}}^{(2)} \mathbb{D}_{\sigma_{(3)}}^{(1)} \boldsymbol{u}_{\sigma_{(3)}}^{(1)}\right) \\
& =\sum_{\sigma_{(3)} \in C_{(3)}(\Omega)} \sum_{\sigma_{(2)} \in \partial \sigma_{(3)}} o_{\sigma_{(3)} \sigma_{(2)}}\left(\star e_{i}^{(2)}, \mathbb{R}_{\sigma_{(2)}}^{(1)} \boldsymbol{u}_{\sigma_{(2)}}^{(1)}\right) \\
& =\sum_{\sigma_{(2)} \in C_{(2)}(\partial \Omega)} o_{\sigma_{(2)}}\left(\star e_{i}^{(2)}, \mathbb{R}_{\sigma_{(2)}}^{(1)} u_{\sigma_{(2)}}^{(1)}\right) .
\end{aligned}
$$

For the outer-oriented discretizations the vorticity is given by

$$
\mathbb{H}^{(1)} \boldsymbol{\omega}=\tilde{\mathbb{D}}^{(1)}\left[\begin{array}{c}
\mathbb{H}^{(2)} \boldsymbol{u}^{(2)}  \tag{4.18}\\
\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}
\end{array}\right]
$$

This implies that

$$
\begin{aligned}
V_{i} & =\boldsymbol{e}^{(1) T} \tilde{\mathbb{D}}^{(1)}\left[\begin{array}{c}
\mathbb{H}^{(2)} \boldsymbol{u}^{(2)} \\
\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}
\end{array}\right] \\
& =\left(\mathbb{D}^{(1)} \boldsymbol{e}^{(1)}\right)^{T} \mathbb{H}^{(2)} \boldsymbol{u}^{(2)}-\left(\mathbb{T}^{(1)} \boldsymbol{e}^{(1)}\right)^{T} \tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)} .
\end{aligned}
$$

Now the result follows, because $\boldsymbol{e}^{(1)}$ is the discretization of a constant field and therefore $\mathbb{D}^{(1)} \boldsymbol{e}^{(1)}=\mathbf{0}^{(2)}$.

### 4.3 Numerical tests

In this section we perform numerical experiments to verify the discrete conservation properties discussed above and to analyze the convergence behavior for the different numerical discretizations. Some of the results presented in this section have earlier appeared in the publications [82-84].

### 4.3.1 Conservation properties

To test the discrete conservation properties we simulate roll up of a doubly periodic jet. The flow domain is given by $\Omega=[0,2 \pi] \times[0,2 \pi]$ and periodic boundary conditions apply. At $t=0$ the velocity field is given by

$$
u_{x}=\left\{\begin{array}{ll}
\tanh \left(\frac{y-\pi / 2}{\gamma}\right) & \text { if } y \leq \pi, \\
\tanh \left(\frac{3 \pi / 2-y}{\gamma}\right) & \text { if } y>\pi,
\end{array} \quad u_{y}=\delta \sin (x),\right.
$$

with $\gamma=\pi / 15$ and $\delta=0.05$. The flow starts out with two shear layers, one at $y=\pi / 2$ with negative vorticity and one at $y=3 \pi / 2$ with positive vorticity. Due to the small perturbation in $u_{y}$ the shear layers will start to roll up irrespective of the value of $\mu$, which we take equal to zero. We set $\rho=1$. The initial $(t=0)$ and final $(t=8)$ vorticity fields are shown in Figure 4.3.1. This problem has already been considered in [85-87].

We test in this inviscid simulation the conservation of momentum, energy and vorticity. We use the three different discretizations (4.12), (4.11) and (4.14) with the three different choices for Hodge matrices. We test the conservation properties on a $200 \times 200$ Cartesian mesh and a $200 \times 200$ mesh of irregular quadrangular cells.


Figure 4.3: The vorticity field is shown at $t=0$ (left) and at $t=8$ (right). The calculation is performed with the inner-oriented scheme with CDO-DGA Hodge matrices on a Cartesian $200 \times 200$ mesh.

For the time integration we use the implicit midpoint method with time step $\Delta t=$ 0.025 . It can be shown that a spatial discretization that conserves momentum and vorticity also conserves these quantities when combined with any consistent and stable time integrator. For the energy, which is a quadratic quantity, this is a more subtle matter and only some time integrators are conserving. This issue is studied in depth in [16]. Runge-Kutta methods that are energy-conserving are the ones based on Gauss quadrature points. The simplest method in this class is the implicit midpoint method which is second order accurate in time.

The periodic boundary conditions and the absence of viscosity imply that the momentum, energy and vorticity are constant over time. In Figure 4.4 and Figure 4.5 we see the conservation of the momentum components $M_{x}$ and $M_{y}$, respectively. We see that, on the Cartesian mesh, all the numerical schemes considered are momentum-conserving up to machine precision. However, on the quadrangular mesh only the outer-oriented discretization with convection term in divergence form is momentum-conserving. This is in line with Theorem 4.1. Moreover, this experiment shows that the discretizations based on the rotational form of the convection term are indeed not momentum-conserving in general, but as it turns out they are momentum-conserving on Cartesian meshes.

In Figure 4.6 and Figure 4.7 it is shown that all the methods conserve energy and vorticity, both on the Cartesian mesh and the quadrangular mesh. Moreover, it is shown that this holds for all choices of Hodge matrices. This confirms Theorem 4.2 and Theorem 4.3.

### 4.3.2 Convergence behavior in 2D

We test the convergence of the methods in 2D on the familiar Taylor-Green vortex flow. The Taylor-Green vortex is an exact solution to the incompressible Navier-Stokes equations (without force term) given by

$$
\underline{u}_{\mathrm{e}}=e^{-2 \pi^{2} \mu t / \rho}\left[\begin{array}{c}
-\sin (\pi x) \cos (\pi y) \\
\cos (\pi x) \sin (\pi y)
\end{array}\right], \quad p_{\mathrm{e}}=\frac{1}{4} e^{-4 \pi^{2} \mu t / \rho}(\cos (2 \pi x)+\cos (2 \pi y)) .
$$

To focus on the error in the spatial discretization we run the simulation from $t=0$ to $t=0.1$ by performing 100 time steps with a fourth order Runge-Kutta method. We set $\rho=1$ and $\mu=0.001$. The domain is given by $\Omega=[0.25,2.25] \times[0.25,2.25]$ and we prescribe the exact velocity on the boundary.

We discretize the exact solution for the inner-oriented scheme as

$$
u_{\mathrm{e}}^{(1)}:=R^{(1)}\left(u_{\mathrm{e}}^{(1)}\right), \quad \boldsymbol{\omega}_{\mathrm{e}}^{(2)}:=R^{(2)}\left(\omega_{\mathrm{e}}^{(2)}\right), \quad p_{\mathrm{e}}^{(0)}:=R^{(0)}\left(p_{\mathrm{e}}^{(0)}\right),
$$

where $u_{\mathrm{e}}^{(1)}, \omega_{\mathrm{e}}^{(2)}$ and $p_{\mathrm{e}}^{(0)}$ are the inner-oriented differential forms representing $\underline{u}_{\mathrm{e}}, \nabla \times \underline{u}_{\mathrm{e}}$ and $p_{\mathrm{e}}$. Similarly, for the outer-oriented schemes, we discretize the exact solution as

$$
u_{\mathrm{e}}^{(1)}:=R^{(1)}\left(\tilde{u}_{\mathrm{e}}^{(1)}\right), \quad \quad \boldsymbol{\omega}^{(0)}:=R^{(0)}\left(\tilde{\omega}^{(0)}\right), \quad \tilde{p}_{\mathrm{e}}^{(0)}:=\tilde{R}^{(0)}\left(p_{\mathrm{e}}^{(0)}\right)
$$

where $\tilde{u}_{\mathrm{e}}^{(1)}$ and $\tilde{\omega}_{\mathrm{e}}^{(2)}$ are the outer-oriented differential forms representing $\underline{u}_{\mathrm{e}}$ and $\nabla \times \underline{u}_{\mathrm{e}}$.


Figure 4.4: Conservation of $M_{x}$, the discrete $x$-momentum, for the double periodic jet problem. The conservation test results are shown for the outer-oriented scheme with divergence-form convection discretization (4.14) (in black), for the inner-oriented scheme (4.12) (in red) and the outer-oriented scheme with rotational-form convection discretization (4.11) (in blue). The tests have been performed using the MFD (solid line), CDO-DGA (dashed line) and CDO-SUSHI (dashed/dotted line) discrete Hodge matrices on a Cartesian (left column) and Quadrangular (right column) $200 \times 200$ mesh.


Figure 4.5: Conservation of $M_{y}$, the discrete $y$-momentum, for the double periodic jet problem. The conservation test results are shown for the outer-oriented scheme with divergence-form convection discretization (4.14) (in black), for the inner-oriented scheme (4.12) (in red) and the outer-oriented scheme with rotational-form convection discretization (4.11) (in blue). The tests have been performed using the MFD (solid line), CDO-DGA (dashed line) and CDO-SUSHI (dashed/dotted line) discrete Hodge matrices on a Cartesian (left column) and Quadrangular (right column) $200 \times 200$ mesh.


Figure 4.6: Conservation of $K$, the discrete energy, for the double periodic jet problem. The conservation test results are shown for the outer-oriented scheme with divergence-form convection discretization (4.14) (in black), for the inner-oriented scheme (4.12) (in red) and the outer-oriented scheme with rotationalform convection discretization (4.11) (in blue). The tests have been performed using the MFD (solid line), CDO-DGA (dashed line) and CDO-SUSHI (dashed/dotted line) discrete Hodge matrices on a Cartesian (left column) and Quadrangular (right column) $200 \times 200$ mesh.


Figure 4.7: Conservation of $V$, the discrete vorticity, for the double periodic jet problem. The conservation test results are shown for the outer-oriented scheme with divergence-form convection discretization (4.14) (in black), for the inner-oriented scheme (4.12) (in red) and the outer-oriented scheme with rotational-form convection discretization (4.11) (in blue). The tests have been performed using the MFD (solid line), CDO-DGA (dashed line) and CDO-SUSHI (dashed/dotted line) discrete Hodge matrices on a Cartesian (left column) and Quadrangular (right column) $200 \times 200$ mesh.

Using the discrete Hodge matrices we define approximate relative $L^{2}$-norms and the relevant number of variables for the inner-oriented scheme as

$$
\begin{array}{ll}
E_{u}^{\mathrm{i}}:=\sqrt{\frac{\left(\boldsymbol{u}^{(1)}-\boldsymbol{u}_{\mathrm{e}}^{(1)}\right)^{T} \mathbb{H}^{(1)}\left(\boldsymbol{u}^{(1)}-\boldsymbol{u}_{\mathrm{e}}^{(1)}\right)}{\boldsymbol{u}_{\mathrm{e}}^{(1) T} \mathbb{H}^{(1)} \boldsymbol{u}_{\mathrm{e}}^{(1)}}}, & N_{u}^{\mathrm{i}}:=N^{(1)}, \\
E_{\omega}^{\mathrm{i}}:=\sqrt{\frac{\left(\tilde{\boldsymbol{\omega}}^{(2)}-\tilde{\boldsymbol{\omega}}_{\mathrm{e}}^{(2)}\right)^{T} \mathbb{H}^{(2)}\left(\tilde{\boldsymbol{\omega}}^{(2)}-\tilde{\boldsymbol{\omega}}_{\mathrm{e}}^{(2)}\right)}{\tilde{\boldsymbol{\omega}}_{\mathrm{e}}^{(2) T} \mathbb{H}^{(2)} \tilde{\boldsymbol{\omega}}_{\mathrm{e}}^{(2)}}}, & N_{\omega}^{\mathrm{i}}:=N^{(2)},  \tag{4.19}\\
E_{p}^{\mathrm{i}}:=\sqrt{\frac{\left(\boldsymbol{p}^{(0)}-\boldsymbol{p}_{\mathrm{e}}^{(0)}\right)^{T} \mathbb{H}^{(0)}\left(\boldsymbol{p}^{(0)}-\boldsymbol{p}_{\mathrm{e}}^{(0)}\right)}{\boldsymbol{p}_{\mathrm{e}}^{(0) T} \mathbb{H}^{(0)} \boldsymbol{p}_{\mathrm{e}}^{(0)}}}, & N_{p}^{\mathrm{i}}:=N^{(0)} .
\end{array}
$$

and for the outer-oriented scheme as

$$
\begin{array}{ll}
E_{u}^{\mathrm{o}}:=\sqrt{\frac{\left(\boldsymbol{u}^{(1)}-\boldsymbol{u}_{\mathrm{e}}^{(1)}\right)^{T} \mathbb{H}^{(1)}\left(\boldsymbol{u}^{(1)}-\boldsymbol{u}_{\mathrm{e}}^{(1)}\right)}{\boldsymbol{u}_{\mathrm{e}}^{(1) T} \mathbb{H}^{(1)} \boldsymbol{u}_{\mathrm{e}}^{(1)}}}, & N_{u}^{\mathrm{o}}:=N^{(1)}, \\
E_{\omega}^{\mathrm{o}}:=\sqrt{\frac{\left(\tilde{\boldsymbol{\omega}}^{(0)}-\tilde{\boldsymbol{\omega}}_{\mathrm{e}}^{(0)}\right)^{T} \mathbb{H}^{(0)}\left(\tilde{\boldsymbol{\omega}}^{(0)}-\tilde{\boldsymbol{\omega}}_{\mathrm{e}}^{(0)}\right)}{\tilde{\boldsymbol{\omega}}_{\mathrm{e}}^{(0) T} \mathbb{H}^{(0)} \tilde{\boldsymbol{\omega}}_{\mathrm{e}}^{(0)}},} & N_{\omega}^{\mathrm{o}}:=N^{(0)},  \tag{4.20}\\
E_{p}^{\mathrm{o}}:=\sqrt{\frac{\left(\tilde{\boldsymbol{p}}^{(0)}-\tilde{\boldsymbol{p}}_{\mathrm{e}}^{(0)}\right)^{T}\left(\mathbb{H}^{(2)}\right)^{-1}\left(\tilde{\boldsymbol{p}}^{(0)}-\tilde{\boldsymbol{p}}_{\mathrm{e}}^{(0)}\right)}{\tilde{\boldsymbol{p}}_{\mathrm{e}}^{(0) T}\left(\mathbb{H}^{(2)}\right)^{-1} \tilde{\boldsymbol{p}}_{\mathrm{e}}^{(0)}}}, & N_{p}^{\mathrm{o}}:=N^{(2)} .
\end{array}
$$

We define the convergence rate again as in (4.10).
We perform the test on three mesh sequences: Cartesian, quadrangular and triangular meshes. We again compare the three different Hodge matrices for the inner-oriented scheme and for the outer-oriented scheme with the convection term in rotational form and in divergence form. The results are shown in Figure 4.8.









Figure 4.8: These plots show the convergence behavior for the velocity (blue), the vorticity (green), and the pressure (red). The full lines show the behavior of the outer-oriented scheme with divergence-form convection, the dashed line that of the outer-oriented scheme with rotational-form convection, and the dotted/dashed line that of the inner-oriented scheme. For the outer-oriented schemes $E_{u}$ and $N_{u}$ refer to ( 4.20 ) and for the inner-oriented scheme to (4.19), and similarly for $E_{\omega}, N_{\omega}, E_{p}$ and $N_{p}$. The different shades correspond to the three different types of Hodge matrices

We see that the convergence behavior on the Cartesian meshes is second order for the velocity, while on the quadrangular meshes it is close to second and around first order on the triangular meshes. There is not much difference in performance between the different Hodge matrices and schemes with regard to the convergence of the velocity. However, it seems that on the quadrangular and triangular meshes the outer-oriented scheme with divergence-form convection performs worse than the other methods. On the triangular meshes the inner-oriented scheme is to be preferred over the other methods.

The convergence behavior of the vorticity is less uniform across the different methods. On the Cartesian mesh the convergence is roughly first order for the two outer-oriented schemes, while it is second order for the inner-oriented scheme. On the quadrangular meshes the vorticity convergence for the outer-oriented scheme with divergence-form convection is almost absent. The other two schemes show first order convergence. This discrepancy between the outer-oriented scheme with divergence-form convection and the other two schemes is even more explicit on the triangular meshes.

The pressure converges second order on all meshes considered with the sole exception again the outer-oriented scheme with convection in divergence-form. For this scheme the convergence of the pressure is around first order.

### 4.3.3 Convergence behavior in 3D

As a final convergence test case for the methods we solve the incompressible NavierStokes equations in 3D. We consider one of the exact solutions derived in [88]. We consider as domain $\Omega=[-1,1]^{3}$ and the velocity and pressure fields are given by

$$
\underline{u}_{\mathrm{e}}:=-\frac{\pi}{4}\left[\begin{array}{c}
\sin \left(\frac{\pi(y+2 z)}{4}\right) \exp \left(\frac{\pi x}{4}\right)+\cos \left(\frac{\pi(x+2 y)}{4}\right) \exp \left(\frac{\pi z}{4}\right) \\
\sin \left(\frac{\pi(z+2 x)}{4}\right) \exp \left(\frac{\pi y}{4}\right)+\cos \left(\frac{\pi(y+2 z)}{4}\right) \exp \left(\frac{\pi x}{4}\right) \\
\sin \left(\frac{\pi(x+2 y)}{4}\right) \exp \left(\frac{\pi z}{4}\right)+\cos \left(\frac{\pi(z+2 x)}{4}\right) \exp \left(\frac{\pi y}{4}\right)
\end{array}\right],
$$

and

$$
\begin{aligned}
p_{\mathrm{e}}:=-\frac{\pi^{2}}{32} & \left(\exp \left(\frac{\pi x}{2}\right)+2 \sin \left(\frac{\pi(x+2 y)}{4}\right) \cos \left(\frac{\pi(z+2 x)}{4}\right) \exp \left(\frac{\pi(y+z))}{4}\right)\right. \\
& +\exp \left(\frac{\pi y}{2}\right)+2 \sin \left(\frac{\pi(y+2 z)}{4}\right) \cos \left(\frac{\pi(x+2 y)}{4}\right) \exp \left(\frac{\pi(x+z))}{4}\right) \\
& \left.+\exp \left(\frac{\pi z}{2}\right)+2 \sin \left(\frac{\pi(z+2 x)}{4}\right) \cos \left(\frac{\pi(y+2 z)}{4}\right) \exp \left(\frac{\pi(y+x))}{4}\right)\right) .
\end{aligned}
$$

This is a solution to the 3D steady incompressible Navier-Stokes equations if the force term is given by

$$
\underline{f}_{\mathrm{e}}:=-\frac{\mu \pi^{2}}{4} \underline{u}_{\mathrm{e}}
$$



Figure 4.9: The velocity field is shown on the boundary of $\Omega$ as calculated by the inner-oriented scheme with CDO-DGA Hodge matrices on a tetrahedral mesh of 857 vertices, 5206 edges, 8248 faces and 3898 cells.

In this solution the convective term and pressure term balance each other and the diffusive term is balanced by the force term. Furthermore, the total pressure $q_{\mathrm{e}}$ is identically zero for this solution. The velocity field is shown in Figure 4.9.

We discretize the velocity, vorticity and pressure of the inner- and outer-oriented schemes again as we did in Section 4.1.2. We define the numerical $L^{2}$-errors again as in (4.8) and (4.9), for, respectively, the outer-oriented schemes and the inner-oriented scheme. For this test we set $\mu=\rho=1$. The convergence behavior is shown in Figure 4.10 and the rates of convergence can be found in Table 4.2.

The convergence behavior shows many similarities with the test results we obtained before in 2D and for the Stokes test in 3D. On the hexahedral mesh sequence we see again second order for the velocity for all schemes. The vorticity convergence is again first order for the outer-oriented schemes while close to second order for the inner-oriented scheme. The convergence for the pressure is close to second order for the inner-oriented scheme. For the outer-oriented scheme it is around first order, with as exception the case that the CDO-DGA Hodge matrices are used, the convergence is more irregular then.

In general the convergence behavior mostly depends on which of the three schemes is used and not so much on the specific Hodge matrices used. Often the rate of convergence for the different Hodge matrices is the same. A few exceptions can be found. For example, for the outer-oriented schemes on the PH meshes ${ }^{11}$, the MFD Hodge matrices give significantly better results. Furthermore, for the outer-oriented schemes the pressure

[^33]is more accurate and converges faster for the CB meshes when the CDO-DGA Hodge matrices are used.

The differences between the results of the outer-oriented scheme with divergence-form convection and the outer-oriented scheme with rotational-form convection are always very small. The differences between the inner-oriented scheme and the outer-oriented schemes are bigger. The convergence rates for the inner-oriented scheme are often better (on the HE, PH, TE mesh sequences), although the error itself is often larger.



Figure 4.10: These plots show the convergence behavior for the velocity (blue), the vorticity (green), and the pressure (red). The full lines show the behavior of the outer-oriented scheme with divergence-form convection (4.14), the dashed line that of the outer-oriented scheme with rotational-form convection (4.11), and the dotted/dashed line that of the inner-oriented scheme (4.12). For the outer-oriented schemes $E_{u}$ and $N_{u}$ refer to (4.8) and for the inner-oriented scheme to (4.9), and similarly for $E_{\omega}, N_{\omega}, E_{p}$ and $N_{p}$. The different shades correspond to the three different types of Hodge matrices used.

| $R_{u}$ | Outer-oriented (d) |  |  |  | Outer-oriented $(r)$ |  |  |  | Inner-oriented |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Mesh | MFD | DGA | SUS | MFD | DGA | SUS | MFD | DGA | SUS |  |  |
| HE | 1.94 | 2.03 | 1.94 | 1.93 | 2.01 | 1.93 | 1.93 | 2.00 | 1.93 |  |  |
| PT | 1.82 | 1.89 | 1.86 | 1.81 | 1.88 | 1.85 | 1.77 | 1.82 | 1.77 |  |  |
| CB | 0.92 | 0.95 | 0.96 | 0.92 | 0.95 | 0.96 | 1.27 | 1.20 | 1.24 |  |  |
| LR | 1.63 | 1.52 | 1.57 | 1.61 | 1.51 | 1.57 | 1.81 | 1.94 | 1.73 |  |  |
| PH | 1.47 | 0.70 | 0.66 | 1.43 | 0.71 | 0.66 | 1.56 | 1.40 | 1.45 |  |  |
| TE | 0.98 | 0.97 | 0.98 | 0.97 | 0.96 | 0.97 | 1.72 | 1.60 | 1.70 |  |  |


| $R_{\omega}$ | Outer-oriented (d) |  |  |  | Outer-oriented $(r)$ |  |  |  | Inner-oriented |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Mesh | MFD | DGA | SUS | MFD | DGA | SUS | MFD | DGA | SUS |  |  |
| HE | 1.08 | 1.14 | 1.08 | 1.08 | 1.14 | 1.08 | 1.77 | 1.83 | 1.77 |  |  |
| PT | 1.24 | 1.26 | 1.33 | 1.23 | 1.26 | 1.32 | 0.94 | 0.72 | 0.75 |  |  |
| CB | 0.44 | 0.64 | 0.59 | 0.43 | 0.64 | 0.59 | 0.37 | 0.44 | 0.34 |  |  |
| LR | 1.19 | 1.20 | 1.17 | 1.19 | 1.20 | 1.17 | 1.00 | 1.15 | 0.95 |  |  |
| PH | 0.57 | 0.22 | 0.19 | 0.57 | 0.22 | 0.19 | 0.99 | 0.97 | 1.03 |  |  |
| TE | 0.33 | 0.29 | 0.35 | 0.31 | 0.28 | 0.32 | 1.32 | 1.32 | 1.32 |  |  |


| $R_{p}$ | Outer-oriented (d) |  |  |  | Outer-oriented ( $r$ ) |  |  |  | Inner-oriented |  |  |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | MFD | DGA | SUS | MFD | DGA | SUS | MFD | DGA | SUS |  |  |
|  | 1.06 | 0.43 | 1.06 | 1.06 | 0.11 | 1.06 | 1.81 | 1.88 | 1.81 |  |  |
| PT | 1.15 | 1.41 | 1.14 | 1.12 | 1.35 | 1.11 | 1.17 | 0.44 | 1.08 |  |  |
| CB | 2.04 | 1.98 | 2.04 | 0.84 | 0.91 | 0.84 | 1.17 | 1.36 | 1.09 |  |  |
| LR | -1.24 | 4.48 | 0.29 | -2.27 | 4.34 | 0.13 | 1.15 | 1.97 | 1.06 |  |  |
| PH | 0.69 | 0.35 | 0.03 | 0.62 | 0.32 | 0.02 | 1.28 | 1.39 | 1.52 |  |  |
| TE | -0.38 | 0.01 | -0.30 | 0.35 | 0.39 | 0.44 | 2.25 | 2.25 | 2.25 |  |  |

Table 4.2: The convergence rates $R_{u}, R_{\omega}$ and $R_{p}$ (defined as in (4.10)) are given for the Hexahdedral (HE), prismatic-triangular (PT), checkerboard (CB), locally-refined (LR), prismatic-hexagonal (PH) and tetrahedral (TE) mesh sequences, based on the results found on the two finest meshes for each sequence. The convergence rates are shown for the outer-oriented scheme with divergence-form convection (d), the outer-oriented scheme with rotational-form convection (r) and the inner-oriented scheme.

### 4.4 Mimetic cut-cell method

The mimetic discretizations introduced above can be employed as cut-cell methods. In this section we will test the effectiveness of discretization (4.14) as a cut-cell method. We first discuss how the cut-cell mesh is defined and subsequently apply the method to the benchmark flow around a circular cylinder [89] and the impulsively started flow around the NACA 0012 airfoil under a $15^{\circ}$ angle of attack.

### 4.4.1 Construction of the cut-cell mesh

For simplicity, we illustrate here how the cut-cell mesh is calculated in the case of a uniform Cartesian mesh for a square domain containing a cylinder. We start by discretizing the cylinder. We give the cylinder its own discretization independent of the Cartesian mesh, because in a future extension of the method to time-dependent geometries this is needed for conservation of mass. The boundary of the cylinder is discretized by taking equidistantly distributed points on it and connecting these by straight edges. The discretized cylinder is then immersed in the Cartesian mesh and extra vertices are added where the edges of the cylinder cross the edges of the Cartesian mesh. Finally, the cylinder is cut out of the Cartesian mesh.


Figure 4.11: On the left: the Cartesian mesh (blue) with the discretized cylinder (green) and the extra added vertices on the cylinder (blue). In the middle: the resulting primal cell-complex. On the right: the resulting dual cell-complex.

Once the primal mesh has been calculated, the cut-cell primal-dual cell complex is determined just as before. The boundary of the mesh contains both the boundary of the square and the boundary of the cylinder. In Figure 4.11 we depict the construction of the dual cell-complex for the cut-cell mesh.

In the Cartesian regions of the mesh we use the diagonal discrete Hodge matrices and in the non-Cartesian parts of the mesh we use the MFD Hodge matrices for the tests performed in this section. Note that, because of the fact that $\mathbb{H}^{(0)}$ is diagonal in the Cartesian cells, the vorticity variables are only needed in the vertices of the non-Cartesian cells, i.e., in the cut-cells or at an interface between two Cartesian regions with a different refinement level.

If the object almost entirely covers a Cartesian cell a very tiny cut-cell occurs. These very small cells can hamper the stability of the method and demand a very small timestep when an explicit time integrator is used. To avoid this we simply merge these cells with the neighboring cell with which they share the longest edge. We merge cells once they have an area less than a tenth of the smallest uncut Cartesian cell.

### 4.4.2 Unsteady flow around cylinder

We test the mimetic cut-cell method on the benchmark problem of flow around an asymmetrically placed circular cylinder in a channel, taken from [89]. At the top and bottom of the channel no-slip boundary conditions apply. We prescribe a parabolic inflow profile and use a no-stress boundary condition for the outflow. To concentrate mesh cells close to the cylinder we use four different refinement levels. An example mesh is given in Figure 4.12. For time integration we use the explicit midpoint method. We take the time step small enough for the temporal discretization error to be negligible compared to the spatial discretization error.


Figure 4.12: This mesh is constructed by starting with a uniform mesh of $N_{x} \times N_{y}$ cells and successively refining (by dividing cells in four) in regions increasingly closer to the cylinder. The regions of different refinement levels are kept fixed. The mesh shown here corresponds to $N_{x}=50$ and $N_{y}=12$.

The inflow velocity corresponds to $\mathrm{Re}=100$ and after some time an unsteady periodic flow develops. We compute for one period the maximum drag coefficient $c_{\mathrm{dmax}}$, the maximum lift coefficient $c_{\text {lmax }}$, the Strouhal number St and the pressure difference between front and back of the cylinder after half a period, where we take the start of the period to coincide with $t$ such that $c_{\mathrm{L}}(t)=c_{\text {Lmax }}$. We calculate the lift and drag coefficients using the trapezoidal and midpoint approximations using the vorticity and pressure variables, located in, respectively, the boundary vertices and center of the boundary edges of the cylinder.

For the mesh with $N_{x} \times N_{y}=80 \times 20$, the lift and drag coefficients and the pressure difference are shown as a function of time in Figure 4.13 on the left. We see that from approximately $t=5$ the flow becomes periodic. On the right in Figure 4.13 the vorticity field in the first half of the channel is shown. A von Kármán vortex street is formed by the vortex shedding of the cylinder. In Figure 4.14 the magnitude of the velocity field is plotted for the same mesh and time.

The values found for $c_{\mathrm{dmax}}, c_{\mathrm{lmax}}$, St and $\Delta p$ are shown in Table 4.3. We simulated the flow for 6 different meshes. It can be seen that all four values seem to converge to the range where the exact values lie. Even the finest mesh we used here (with $N_{x} \times N_{y}=$


Figure 4.13: On the left $C_{\mathrm{L}}$ (in red), $C_{\mathrm{D}}$ (in blue) and $\Delta p$ are shown as a function of time for the mesh with $N_{x} \times N_{y}=80 \times 20$. On the right the corresponding vorticity field at $t=6$ is shown.


Figure 4.14: The magnitude of the velocity at $t=6$ on the mesh with $N_{x} \times N_{y}=80 \times 20$.

| $N_{x} \times N_{y}$ | $\# \boldsymbol{u}^{(1)}$ | $\# \boldsymbol{\omega}^{(1)}$ | $\# \tilde{\boldsymbol{p}}^{(2)}$ | $c_{\text {Dmax }}$ | $c_{\text {Lmax }}$ | St | $\Delta p$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $50 \times 12$ | 14297 | 814 | 7415 | 3.504 | 1.227 | 0.2950 | 2.599 |
| $60 \times 15$ | 21159 | 981 | 10890 | 3.475 | 1.174 | 0.2950 | 2.578 |
| $70 \times 18$ | 29490 | 1127 | 15099 | 3.419 | 1.171 | 0.2963 | 2.576 |
| $80 \times 20$ | 37171 | 1252 | 18979 | 3.393 | 1.129 | 0.2967 | 2.550 |
| $90 \times 23$ | 48619 | 1418 | 24745 | 3.355 | 1.103 | 0.2985 | 2.540 |
| $100 \times 25$ | 58401 | 1534 | 29676 | 3.266 | 1.074 | 0.2994 | 2.517 |
| [89] (lower bound) | - | - | - | 3.22 | 0.99 | 0.295 | 2.46 |
| [89] (upper bound) | - | - | - | 3.24 | 1.01 | 0.305 | 2.50 |

Table 4.3: Results for the benchmark test. The last two lines give a lower and upper bound for the exact values. These bounds come from [89] and are based on the results for many methods collected there.
$100 \times 25)$ is still quite coarse. However, the results obtained are in good agreement with the most accurate values found in [89].


Figure 4.15: Left: the complete mesh. Right: the mesh close to the airfoil.

### 4.4.3 Impulsively started flow around a NACA 0012 airfoil at a $15^{\circ}$ angle of attack

In a second qualitative test we study the impulsively started flow around a NACA 0012 airfoil at a $15^{\circ}$ angle of attack for $\mathrm{Re}=1200$. From our computation we determine the streamline pattern of the unsteady flow, which we compare with PIV measurements taken from [90].

The cut-cell mesh we use is depicted in Figure 4.15. We again use various levels of refinement to increase the resolution where we expect the wake of the airfoil to be. The resulting mesh consists of 9794 cells, 20244 edges and 10450 vertices. The method uses 20244 velocity variables. We have a pressure variable in every cell center and on every boundary edge, as a result there are 10316 pressure variables. The vorticity variables are only needed in the vertices of the non-Cartesian cells and hence 2596 vorticity variables are used.

In the experiment described in [90] an extrusion of the NACA 0012 airfoil is towed through a watertank. Plastic particles are disseminated in the water tank to scatter laser light to be recorded by a CCD camera comoving with the airfoil as it is towed through the water tank. At the start of the experiment the airfoil is instantaneously set into motion by the towing mechanism. ${ }^{12}$ The dimensions of the experiment correspond to $R e=1200$.

In the numerical experiment we set the computational domain equal to $\Omega=[-10,10] \times$ $[-10,10]$ and locate the front of the airfoil in the origin. The airfoil has a chord length of 1 , hence its trailing edge is located at $(x, y)=(1,0)$. We prescribe Dirichlet boundary conditions on the southern and western sides of $\Omega$ and on the northern and eastern sides we prescribe outflow boundary conditions. To mimic the experiment computationally we start with a zero velocity field in all of the computational domain and at $t=0$ we suddenly set the fluid into motion through the Dirichlet boundary conditions at the southern and western sides of the domain. We prescribe an inflow velocity of magnitude

[^34]

Figure 4.16: Left: the experimental PIV result. Right: the computational result.

1 at an angle of $15^{\circ}$ degrees with the $x$-axis. We set $\mu=1 / 1200$ and use a fourth order explicit Runge-Kutta method with time-step $\Delta t=0.002$. The computational results are shown next to the experimental results in Figure 4.16 and Figure 4.17. ${ }^{13}$ In the experimentally determined streak pictures an exposure time of 0.5 s was used.

At $t=1$ the boundary layer starts to separate from the trailing edge and the separation

[^35]point starts to move upstream. Subsequently a surface vortex grows and rolls downstream again. It reaches the trailing edge around $t=3.131$. (See Figure 4.16.) From this point in time the sub-atmospheric pressure in the large vortex and shear induce a trailing edge vortex (with opposite rotational orientation) which is first seen in both the PIV and computational results at $t=3.914$. The trailing edge vortex enlarges while the first vortex escapes. After this a new vortex upstream of the trailing edge vortex is formed. This is seen in Figure 4.17 at $t=4.958$. Subsequently, both these two vortices are released from the airfoil and the wake flattens as we see at $t=6.524$. It can be seen that the computation clearly captures all of these qualitative flow features but with greater precision and clarity than the experiment, especially in the regions where the flow velocity is small, where the PIV pictures do not show a clear flow direction.


Figure 4.17: Left: the experimental PIV result. Right: the computational result.

### 4.5 Conclusions

In this chapter we investigated various discretizations of the incompressible Navier-Stokes equations. The inner-oriented scheme is a discretization of the Navier-Stokes equations where the convection term is in rotational form. This scheme uses velocity variables on the edges of the mesh and total pressure variables located at the vertices of the mesh.

Besides the inner-oriented scheme we have considered two discretizations that use velocity variables located at the faces of the mesh, vorticity variables located at the edges of the mesh and (total) pressure variables located at the vertices of the dual mesh. The outer-oriented discretizations were defined with two different discretizations of the convection term, one based on the divergence form and another based on the rotational form.

We showed that all three schemes conserve energy and vorticity. Furthermore the outer-oriented scheme with convection term in divergence form also conserves momentum. The other two schemes only conserve momentum on Cartesian meshes. The conservation is independent of the Hodge matrices used in the discretization.

We analyzed the convergence behavior of the methods for a range of mesh sequences. The discrete Hodge matrices are based on interpolation formulas that are, under the assumption of mesh regularity, at least first order accurate. However, for most methods we found better convergence behavior. On Cartesian meshes the velocity converges second order and for most other mesh sequences the convergence of the velocity is between first and second order. It was found that the convergence behavior does not strongly depend on the discrete Hodge matrices used.

We did not include any theoretical a priori error analysis of the methods. It would be good to in future work do such an analysis, for example for the simpler Oseen equations. This would give insight in the dependency of the numerical properties of the method with respect to the Reynolds number.

Finally, we applied the discretization with the convection term in divergence form as a cut-cell method to calculate the vortex shedding of a circular cylinder and the impulsively started flow around a NACA 0012 airfoil at a $15^{\circ}$ angle of attack. We found that the method, despite the use of relatively coarse cut-cell meshes, gives good predictions of the lift and drag, among others. For the flow around the airfoil we obtained an accurate resolution of all the qualitative flow features also found in a physical experiment. The test of the other two methods on cut-cell meshes and an analysis of the relative accuracy of these methods remain topics for future research.

## Div-Curl Problems and Discrete Helmholtz-Hodge Decompositions

We consider the problem of finding a vector field $\underline{u}$ in a 3-dimensional domain $\Omega$ that satisfies

$$
\begin{aligned}
\nabla \times \underline{u} & =\underline{\omega}, & & \text { in } \Omega, \\
\nabla \cdot \underline{u} & =\rho, & & \text { in } \Omega,
\end{aligned}
$$

for given $\underline{\omega}$ and $\rho$, together with normal boundary conditions,

$$
\begin{equation*}
\underline{n} \cdot \underline{u}=r, \quad \text { on } \partial \Omega, \tag{5.1}
\end{equation*}
$$

or, with tangential boundary conditions,

$$
\begin{equation*}
\underline{n} \times \underline{u}=\underline{r}, \quad \text { on } \partial \Omega, \tag{5.2}
\end{equation*}
$$

where $\underline{n}$ is the normal on the domain boundary $\partial \Omega$ and, where $r$ or $\underline{r}$ is given.
This problem occurs in both fluid mechanics and electromagnetism. In fluid dynamics the problem turns up with $\underline{u}$ the velocity field, $\underline{\omega}$ the vorticity and $\rho$ either equal to the density of the fluid or zero. It occurs, for example, in computational algorithms for solving the Navier-Stokes equations in the vorticity-velocity formulation [91]. Moreover, the related Helmholtz-Hodge decomposition of the velocity field plays an important role in, for example, projection methods $[92,93]$ for the numerical solution of the incompressible Navier-Stokes equations, and has many other applications in the physical sciences [94].

In the context of electromagnetism, the problem describes a static electric field, when $\underline{u}$ is the electric field, $\underline{\omega}=\underline{0}$, and $\rho$ is the charge density. Alternatively, with $\underline{\omega}$ equal to the current density and $\rho=0, \underline{u}$ describes a static magnetic field [95]. ${ }^{1}$

In this chapter we show how these problems can be solved using the mimetic discretization methods, introduced in Chapter 3, that result in nonsingular linear systems. This is in contrast to the recent paper [96]. In [96] the problems are discretized by mimetic methods such that singular linear systems resulted and non-standard linear algebra methods are required for solving these systems.

This chapter is structured as follows. In Section 5.1 we reformulate the problem in terms of differential forms. We show that the problem with two types of boundary conditions can be solved by using a corresponding Helmholtz-Hodge decomposition. Solving the problem amounts to solving two Hodge-Laplace problems for a vector and scalar potential field. We use this to establish that the problem has a unique solution.

[^36]In Section 5.2 we derive two discrete Helmholtz-Hodge decompositions. This allows us, in Section 5.3, to repeat the reasoning applied before to show the existence of a unique discrete solution (up to harmonic forms) to discretizations of the div-curl problem with one of the two boundary conditions. Finally, in Section 5.4 we present the numerical performance for the DEC-, MFD- and DGA-discrete Hodge matrices and a wide range of meshes.

### 5.1 Solutions to the div-curl problems through $L^{2}$ Helmholtz-Hodge decompositions

In Section 2.3.2 we introduced Sobolev spaces of differential forms and two types of trace operators were defined: $t$ and $t^{*}$. These trace operators correspond to either the tangential or normal component of the field and can be used to define the div-curl problem with the two types of boundary conditions:

$$
\left(\mathrm{P}_{t^{*}}^{(k)}\right)\left\{\begin{array} { r l } 
{ d ^ { * } u ^ { ( k ) } } & { = f ^ { ( k - 1 ) } , } \\
{ d u ^ { ( k ) } } & { = g ^ { ( k + 1 ) } , } \\
{ t ^ { * } u ^ { ( k ) } } & { = r _ { \mathrm { b } } ^ { ( k - 1 ) } , }
\end{array} \quad ( \mathrm { P } _ { t } ^ { ( k ) } ) \left\{\begin{array}{rl}
d^{*} u^{(k)} & =f^{(k-1)}, \\
d u^{(k)} & =g^{(k+1)}, \\
t u^{(k)} & =r_{\mathrm{b}}^{(k)} .
\end{array}\right.\right.
$$

More precisely, the problems read:
$\left(\mathrm{P}_{t^{*}}^{(k)}\right)$ Given $\left(f^{(k-1)}, g^{(k+1)}, r^{(k-1)} \mathrm{b}\right) \in L^{2}(\Omega) \times L^{2}(\Omega) \times H^{-\frac{1}{2}} \Lambda^{(k-1)}\left(d^{*}, \partial \Omega\right)$ satisfying $f^{(k-1)} \in$ $\operatorname{im}\left(d^{*}\right), g^{(k+1)} \in \operatorname{im}(d)$ and $\left(f^{(k-1)}, z^{(k-1)}\right)+\left(r_{\mathrm{b}}^{(k-1)}, t z^{(k-1)}\right)_{\mathrm{b}}=0$ for all $z^{(k-1)} \epsilon$ $H \Lambda^{(k-1)}(d, \Omega)$ with $d z^{(k-1)}=0$, find $u^{(k)} \in H \Lambda^{(k)}\left(d, d^{*}, \Omega\right)$ that solves $\left(\mathrm{P}_{t^{*}}^{(k)}\right) .{ }^{2}$
$\left(\mathrm{P}_{t}^{(k)}\right)$ Given $\left(f^{(k-1)}, g^{(k+1)}, r^{(k)} \mathrm{b}\right) \in L^{2}(\Omega) \times L^{2}(\Omega) \times H^{-\frac{1}{2}} \Lambda^{(k)}(d, \partial \Omega)$ satisfying $f^{(k-1)} \in \operatorname{im}\left(d^{*}\right), g^{(k+1)} \in \operatorname{im}(d)$ and $\left(g^{(k+1)}, z^{(k+1)}\right)=\left(r_{\mathrm{b}}^{(k)}, t^{*} z^{(k+1)}\right)_{\mathrm{b}}$ for all $z^{(k+1)} \in$ $H \Lambda^{(k+1)}\left(d^{*}, \Omega\right)$ with $d^{*} z^{(k+1)}=0$, find $u^{(k)} \in H \Lambda^{(k)}\left(d, d^{*}, \Omega\right)$ that solves $\left(\mathrm{P}_{t}^{(k)}\right)$.

In three dimensions this leads to four different problems, because a vector field $\underline{u}$ can be represented either as a 1 -form $u^{(1)}$ or a 2 -form $u^{(2)}$. Let the density $\rho$ be represented by a 0 -form $\rho^{(0)}$ or the 3 -form $\rho^{(3)}$ and the vorticity field $\underline{\omega}$ either by $\omega^{(1)}$ or $\omega^{(2)}$. We

[^37]have the following four versions of the 3-dimensional div-curl problem:
\[

$$
\begin{align*}
&\left(\mathrm{P}_{t^{*}}^{(1)}\right)\left\{\begin{array}{rlr}
d^{*} u^{(1)} & =\rho^{(0)}, \\
d u^{(1)} & =\omega^{(2)}, \\
t^{*} u^{(1)} & =r_{\mathrm{b}}^{(0)},
\end{array}\right.\left(\mathrm{P}_{t}^{(1)}\right)\left\{\begin{aligned}
d^{*} u^{(1)} & =\rho^{(0)}, \\
d u^{(1)} & =\omega^{(2)}, \\
t u^{(1)} & =r_{\mathrm{b}}^{(1)},
\end{aligned}\right. \\
&\left(\mathrm{P}_{t^{*}}^{(2)}\right)\left\{\begin{array} { r l } 
{ d ^ { * } u ^ { ( 2 ) } } & { = \omega ^ { ( 1 ) } , } \\
{ d u ^ { ( 2 ) } } & { = \rho ^ { ( 3 ) } , } \\
{ t ^ { * } u ^ { ( 2 ) } } & { = r _ { \mathrm { b } } ^ { ( 1 ) } , }
\end{array} \quad ( \mathrm { P } _ { t } ^ { ( 2 ) } ) \left\{\begin{array}{rl}
d^{*} u^{(2)} & =\omega^{(1)}, \\
d u^{(2)} & =\rho^{(3)}, \\
t u^{(2)} & =r_{\mathrm{b}}^{(2)} .
\end{array}\right.\right. \tag{5.3}
\end{align*}
$$
\]

For the problems $\left(\mathrm{P}_{t}^{(1)}\right)$ and $\left(\mathrm{P}_{t^{*}}^{(2)}\right)$, the trace is a 1-form on $\partial \Omega$, which corresponds to tangential boundary conditions. The cases $\left(\mathrm{P}_{t}^{(2)}\right)$ and $\left(\mathrm{P}_{t^{*}}^{(1)}\right)$ correspond to normal boundary conditions.

### 5.1.1 Hodge-Laplace problems

The Helmholtz-Hodge decompositions, given in Theorem 2.4, will aid us in proving that the problems $\left(\mathrm{P}_{t^{*}}^{(k)}\right)$ and $\left(\mathrm{P}_{t}^{(k)}\right)$ have a solution unique up to harmonic forms. We do this by showing that the potentials $a^{(k-1)}$ and $b^{(k+1)}$ in the Helmholtz-Hodge decomposition, $u^{(k)}=d a^{(k-1)}+d^{*} b^{(k+1)}+h^{(k)}$, of the solution $u^{(k)}$ of the div-curl problem, are themselves solutions of a well-posed problem. Thus we can find $u^{(k)}$ through first finding $a^{(k-1)}$ and $b^{(k+1)}$.

The potentials $a^{(k-1)}$ and $b^{(k+1)}$ satisfy a type of problem that is known as a HodgeLaplace problem. We will introduce it in an abstract form taken from [47], because we will encounter a number of different realizations of this problem.

For convenience we repeat some of the results of Section 2.3. As in Section 2.3.3 we denote a general Hilbert Complex by ( $W^{(k)} \subset V^{(k)}, d^{(k)}$ ) (see also Definition 2.34). In Section 2.3.3 we introduced the two cochain complexes with $W^{(k)}, V^{(k)}$ and $d^{(k)}$ given by, respectively,

$$
\begin{equation*}
W^{(k)}=L^{2} \Lambda^{(k)}(\Omega), \quad V^{(k)}=H \Lambda^{(k)}(d, \Omega), \quad A^{(k)}=d^{(k)}, \tag{5.4}
\end{equation*}
$$

and

$$
\begin{equation*}
W^{(k)}=L^{2} \Lambda^{(k)}(\Omega), \quad V^{(k)}=\stackrel{\circ}{H} \Lambda^{(k)}(d, \Omega), \quad A^{(k)}=d^{(k)} \tag{5.5}
\end{equation*}
$$

We saw that the adjoint chain complexes of (5.4) and (5.5) are given by, respectively,

$$
\begin{equation*}
W^{(k)}=L^{2} \Lambda^{(k)}(\Omega), \quad V^{(k)}=\stackrel{\circ}{H} \Lambda^{(k)}\left(d^{*}, \Omega\right), \quad A^{(k)}=d^{*(k)} \tag{5.6}
\end{equation*}
$$

and

$$
\begin{equation*}
W^{(k)}=L^{2} \Lambda^{(k)}(\Omega), \quad V^{(k)}=H \Lambda^{(k)}\left(d^{*}, \Omega\right), \quad A^{(k)}=d^{*(k)} \tag{5.7}
\end{equation*}
$$

The Helmholtz-Hodge decompositions corresponding to (5.4) and (5.6), and, (5.5) and (5.7), respectively, are given by

$$
\begin{equation*}
L^{2} \Lambda^{(k)}(\Omega)=\mathcal{B}^{(k)} \oplus \dot{\mathcal{B}}^{*(k)} \oplus \mathcal{H}_{t^{*}}^{(k)}=\mathcal{B}^{(k)} \oplus \star^{-1} \stackrel{\tilde{\mathcal{B}}}{ }_{(d-k)}^{\mathcal{H}_{t^{*}}^{(k)} . . . ~ . ~} \tag{5.8}
\end{equation*}
$$

and

$$
\begin{equation*}
L^{2} \Lambda^{(k)}(\Omega)=\dot{\mathcal{B}}^{(k)} \oplus \mathcal{B}^{*(k)} \oplus \mathcal{H}_{t}^{(k)}=\dot{\mathcal{B}}^{(k)} \oplus \star^{-1} \tilde{\mathcal{B}}^{(d-k)} \oplus \mathcal{H}_{t}^{(k)} \tag{5.9}
\end{equation*}
$$

where the spaces of harmonic forms are given by

$$
\begin{aligned}
& \mathcal{H}_{t^{*}}^{(k)}:=\left\{a^{(k)} \in H \Lambda^{(k)}\left(d, d^{*}, \Omega\right) \mid d a^{(k)}=0, d^{*} a^{(k)}=0, t^{*} a^{(k)}=0\right\}, \\
& \mathcal{H}_{t}^{(k)}:=\left\{a^{(k)} \in H \Lambda^{(k)}\left(d, d^{*}, \Omega\right) \mid d a^{(k)}=0, d^{*} a^{(k)}=0, t a^{(k)}=0\right\},
\end{aligned}
$$

and where $\stackrel{\dot{\mathcal{B}}}{ }^{(d-k)}:=\tilde{d}\left(\stackrel{\circ}{H}^{( } \tilde{\Lambda}^{(k)}(\tilde{d}, \Omega)\right)$ and $\tilde{\mathcal{B}}^{(d-k)}:=\tilde{d}\left(H \tilde{\Lambda}^{(k)}(\tilde{d}, \Omega)\right)$.
It can be shown that to any closed Hilbert cochain complex $\left(V^{(k)}, d^{(k)}\right)$ there corresponds an adjoint closed Hilbert chain complex $\left(V^{*(k)}, d^{*(k)}\right)$ and a Helmholtz-Hodge decomposition $u^{(k)}=d a^{(k-1)}+d^{*} b^{(k+1)}+h^{(k)}$, where $a^{(k-1)} \in V^{(k-1)}, b^{(k+1)} \in V^{*(k+1)}$ and $h^{(k)} \in H^{(k)}$, where $H^{(k)}$ is the space of harmonic forms in this general setting [47]. In this context the general Hodge-Laplace problem is to find $u$, given $f$, that solves $\left(d^{*} d+d d^{*}\right) u=f$. This problem will turn out to be most convenient for us in the mixed form. We call it the General Problem, (GP).

Problem (GP). Find $(\sigma, s, r) \in V^{(k-1)} \times V^{(k)} \times H^{(k)}$ satisfying

$$
\begin{aligned}
-(\sigma, \chi)+(s, d \chi) & =0 & & \forall \chi \in V^{(k-1)}, \\
(d s, d v)+(d \sigma, v)+(r, v) & =F(v) & & \forall v \in V^{(k)}, \\
(s, h) & =0 & & \forall h \in H^{(k)},
\end{aligned}
$$

where $F$ is given and an element of the dual space to $V^{(k)} .{ }^{3}$
In [47] it is shown that if $\left(V^{(k)}, d^{(k)}\right)$ is a closed Hilbert complex, then this general Helmholtz-Hodge problem in mixed form has a unique solution. Hereafter, we will use (GP) with for ( $V^{(k)}, d^{(k)}$ ) the Hilbert complexes (5.4-5.7).

### 5.1.2 Characterization of the solution to problem $\left(\mathbf{P}_{t^{*}}^{(k)}\right)$

In this section we will show that the solution to problem $\left(\mathrm{P}_{t^{*}}^{(k)}\right)$ is of the form

$$
\begin{equation*}
u^{(k)}=\underbrace{d a^{(k-1)}}_{\in \mathcal{B}^{(k)}}+\underbrace{d^{*} b^{(k+1)}}_{\epsilon \mathcal{B}^{*(k)}}+\underbrace{h^{(k)}}_{\epsilon \mathcal{H}_{t^{*}}^{(k)}}, \tag{5.10}
\end{equation*}
$$

[^38]where $a^{(k-1)}$ and $b^{(k+1)}$ solve the Hodge-Laplace problems
\[

\left\{$$
\begin{array} { r l } 
{ ( d d ^ { * } + d ^ { * } d ) a ^ { ( k - 1 ) } } & { = f ^ { ( k - 1 ) } , } \\
{ t ^ { * } d a ^ { ( k - 1 ) } } & { = r _ { \mathrm { b } } ^ { ( k - 1 ) } , } \\
{ t ^ { * } a ^ { ( k - 1 ) } } & { = 0 , }
\end{array}
$$ \quad \left\{$$
\begin{array}{r}
\left(d d^{*}+d^{*} d\right) b^{(k+1)}=g^{(k+1)}, \\
t^{*} d b^{(k+1)}=0 \\
t^{*} b^{(k+1)}=0
\end{array}
$$\right.\right.
\]

First, we show that these problems have a unique solution up to harmonic forms. Then we show that they together form the solution $u^{(k)}$ according to (5.10).

### 5.1.2.1 Hodge-Laplace problem for the potential $a^{(k-1)}$

In weak form we can state the Hodge-Laplace problem for $a^{(k-1)}$ as follows.
Find $a \in \stackrel{\circ}{H}_{t^{*}} \Lambda^{(k-1)}\left(d, d^{*}, \Omega\right)$ and $p \in \mathcal{H}_{t^{*}}^{(k-1)}$ such that

$$
\begin{aligned}
(d a, d v)+\left(d^{*} a, d^{*} v\right)+(p, v) & =(f, v)+\left(r_{\mathrm{b}}, t v\right)_{\mathrm{b}} & & \forall v \in{\stackrel{\circ}{{ }_{t}^{*}}}^{*} \Lambda^{(k-1)}\left(d, d^{*}, \Omega\right), \\
& (a, h)=0 & & \forall h \in \mathcal{H}_{t^{*}}^{(k-1)} .
\end{aligned}
$$

The term $(p, v)$ and the second equation appear to make sure that the solution $a^{(k)}$ is orthogonal to $\mathcal{H}_{t^{*}}^{(k-1)}$. Without this restriction the solution would not be unique.

This weak formulation contains terms with the coderivative $d^{*}$ and this involves the inverse Hodge star, which, as we discussed in Chapter 3, is often less suitable for a discretization. We therefore rephrase the Hodge-Laplace problem in the following equivalent mixed form.

Problem ( $\mathrm{P}_{t^{*}}^{a}$. Given $f \in L^{2} \Lambda^{(k-1)}(\Omega)$ and $r_{\mathrm{b}} \in H^{-\frac{1}{2}} \Lambda^{(k-1)}\left(d^{*}, \partial \Omega\right)$ satisfying $d^{*} f=0$, $f \in \mathcal{H}_{t^{*}}^{(k-1) \perp}$ and $(f, z)+\left(r_{\mathrm{b}}, t z\right)_{\mathrm{b}}=0$ for all $z \in \mathcal{Z}^{(k-1)}$. Find $\phi \in H \Lambda^{(k-2)}(d, \Omega), a \in$ $H \Lambda^{(k-1)}(d, \Omega)$ and $p \in \mathcal{H}_{t^{*}}^{(k-1)}$ such that

$$
\begin{align*}
-(\phi, \chi)+(a, d \chi) & =0 & & \forall \chi \in H \Lambda^{(k-2)}(d, \Omega),  \tag{5.11a}\\
(d a, d v)+(d \phi, v)+(p, v) & =(f, v)+\left(r_{\mathrm{b}}, t v\right)_{\mathrm{b}} & & \forall v \in H \Lambda^{(k-1)}(d, \Omega),  \tag{5.11b}\\
(a, h) & =0 & & \forall h \in \mathcal{H}_{t^{*}}^{(k-1)} . \tag{5.11c}
\end{align*}
$$

The following lemma gives some properties of the unique solution to $\left(\mathrm{P}_{t^{*}}^{a}\right)$.
Lemma 5.1. Problem $\left(\mathrm{P}_{t^{*}}^{a}\right)$ has a unique solution $\left(\phi^{(k-2)}, a^{(k-1)}, p^{(k-1)}\right)$. Moreover, we have $a^{(k-1)} \in \stackrel{\circ}{H}_{t^{*}} \Lambda^{(k-1)}\left(d, d^{*}, \Omega\right), d^{*} a^{(k-1)}=\phi^{(k-2)}=0$ and $p^{(k-1)}=0$.

Proof. Recall that $(\cdot, \cdot)_{\mathrm{b}}$ is a bounded linear form on $H^{-\frac{1}{2}} \Lambda^{(k)}\left(d^{*}, \partial \Omega\right) \times H^{-\frac{1}{2}} \Lambda^{(k)}(d, \partial \Omega)$. This implies that $(f, \cdot)+\left(r_{\mathrm{b}}, t(\cdot)\right)_{\mathrm{b}}: H \Lambda^{(k-1)}(d, \Omega) \rightarrow \mathbb{R}$ is an element of the dual space to $H \Lambda^{(k-1)}(d, \Omega)$. Thus this problem is a version of the general Hodge-Laplace problem (GP), with $\left(V^{(k)}, d^{(k)}\right)=\left(H \Lambda^{(k)}(d, \Omega), d^{(k)}\right)$ and $F(v)=(f, v)+\left(r_{\mathrm{b}}, t v\right)_{\mathrm{b}}$, hence it admits a unique solution.

Requirement (5.11a) implies that $a^{(k-1)}$ is in $\stackrel{\circ}{H}_{t^{*}} \Lambda^{(k-1)}\left(d, d^{*}, \Omega\right)$ and $d^{*} a^{(k-1)}=$ $\phi^{(k-2)}$.

Furthermore, to show $\phi^{(k-2)}=0$ we follow an argument similar to one in [97]. We take $v=d \phi$ in (5.11b). Note that $d \phi \in \mathcal{Z}^{(k-1)}$ and therefore $(f, d \phi)+\left(r_{\mathrm{b}}, t d \phi\right)_{\mathrm{b}}=0$. Furthermore, $p \in \mathcal{H}_{t^{*}}^{(k)}$ and hence from the Helmholtz-Hodge decomposition (2.19) we see that $(p, d \phi)=0$, because $d \phi \in \mathcal{B}^{(k-1)}$. Finally, because $d d \phi=0$ we find $(d \phi, d \phi)=0$, which implies that $\phi \in \mathcal{Z}^{(k-2)}$. However, (5.11a) implies that $\phi \in \mathcal{B}^{*(k)}$. Lemma 2.3 now implies that $\phi=0$.

Similarly, taking now $v=p$ in the second line gives $(p, p)=0$.
From (5.11b) one sees that $d^{*} a^{(k-1)}=0$ and $p^{(k-1)}=0$, by integration by parts, imply that $d^{*} d a^{(k-1)}=f^{(k-1)}$ and $t^{*} d a^{(k-1)}=r_{\mathrm{b}}^{(k-1)}$.

### 5.1.2.2 Hodge-Laplace problem for the potential $b^{(k+1)}$

For the potential $b^{(k+1)}$ we have a similar Hodge-Laplace problem in mixed formulation.
Problem $\left(\mathrm{P}_{t^{*}}^{b}\right)$. Given $g \in L^{2} \Lambda^{(k-1)}(\Omega)$ satisfying $d g=0, g \in \mathcal{H}_{t^{*}}^{(k+1) \perp}$. Find $\psi \in$ $H \Lambda^{(k)}(d, \Omega), b \in H \Lambda^{(k+1)}(d, \Omega)$ and $q \in \mathcal{H}_{t^{*}}^{(k+1)}$ such that

$$
\begin{align*}
-(\psi, \chi)+(b, d \chi) & =0 & & \forall \chi \in H \Lambda^{(k)}(d, \Omega), \\
(d b, d v)+(d \psi, v)+(q, v) & =(g, v) & & \forall v \in H \Lambda^{(k+1)}(d, \Omega),  \tag{5.12a}\\
(b, h) & =0 & & \forall h \in \mathcal{H}_{t^{\star}}^{(k+1)} . \tag{5.12b}
\end{align*}
$$

The uniqueness of the solution and some of its other properties are stated in the next lemma.

Lemma 5.2. Problem ( $\mathrm{P}_{t^{*}}^{b}$ ) has a unique solution $\left(\psi^{(k)}, b^{(k+1)}, q^{(k+1)}\right)$. Moreover, we have $b^{(k+1)} \in \dot{H}_{t^{*}} \Lambda^{(k+1)}\left(d, d^{*}, \Omega\right), d b^{(k+1)}=0$ and $q^{(k+1)}=0$.

Proof. This is again a version of problem (GP), now with $\left(V^{(k)}, d^{(k)}\right)=\left(H \Lambda^{(k)}(d, \Omega), d^{(k)}\right)$ and $F(v)=(g, v)$, and hence it is well-posed. To show that $d b=0$ we (again arguing as in [97]) use the orthogonal decomposition $L^{2} \Lambda^{(k+1)}(\Omega)=\dot{\mathcal{B}}^{*(k+1)} \oplus \mathcal{Z}^{(k+1)}$ and the projection $\pi: L^{2} \Lambda^{(k+1)}(\Omega) \rightarrow \dot{\mathcal{B}}^{*(k+1)}$. Applying this to ( 5.12 b ) we obtain, for all $v \in H \Lambda^{(k+1)}(d, \Omega):(d b, d \pi v)=0$, because $d \psi, q$ and $g$ are all in $\mathcal{Z}^{(k+1)}$. However, we have $d \pi v=d v$ for all $v \in H \Lambda^{(k+1)}(d, \Omega)$ by definition of $\pi$ and $\mathcal{Z}^{(k+1)}$, hence $(d b, d b)=0$ holds. Taking $v=q$ in the second line gives, by Helmholtz-Hodge decomposition (2.19), $(q, q)=0$, hence $q=0$.

Lemma 5.2 potential $b^{(k+1)}$ satisfies $d d^{*} b^{(k+1)}=g^{(k+1)}, t^{*} b^{(k+1)}=0$ and $t^{*} d b^{(k+1)}=0$.

### 5.1.2.3 The solution to Problem ( $\mathbf{P}_{t^{*}}^{(k)}$ )

The fact that there is a unique solution to the two Hodge-Laplace problems for the potentials $a^{(k-1)}$ and $b^{(k+1)}$ implies that there is a unique solution to the div-curl problem $\left(\mathrm{P}_{t^{*}}^{(k)}\right)$. In weak form the problem is given as follows.

Problem $\left(\mathrm{P}_{t^{*}}^{(k)}\right)$. Given $f \in L^{2} \Lambda^{(k-1)}(\Omega), g \in L^{2} \Lambda^{(k+1)}(\Omega)$ and $r_{\mathrm{b}} \in H^{-\frac{1}{2}} \Lambda^{(k-1)}\left(d^{*}, \partial \Omega\right)$ satisfying $d^{*} f=0, f \in \mathcal{H}_{t^{*}}^{(k-1) \perp}, d g=0, g \in \mathcal{H}_{t^{*}}^{(k+1) \perp}$ and $(f, z)+\left(r_{\mathrm{b}}, t z\right)_{\mathrm{b}}=0$ for all $z \in \mathcal{Z}^{(k-1)}$. Find $u^{(k)} \in H \Lambda^{(k)}(\Omega)$ such that

$$
\begin{array}{ll}
(u, d v)=(f, v)+\left(r_{\mathrm{b}}, t v\right)_{\mathrm{b}} & \forall v \in H \Lambda^{(k-1)}(d, \Omega), \\
(d u, w)=(g, w) & \forall w \in H \Lambda^{(k+1)}(d, \Omega) . \tag{5.13b}
\end{array}
$$

Next theorem gives the solution of Problem $\left(\mathrm{P}_{t^{*}}^{(k)}\right)$ in terms of the potentials $a^{(k-1)}$ and $b^{(k+1)}$ and shows its uniqueness up to an harmonic form.

Theorem 5.1. The solution of Problem $\left(\mathrm{P}_{t^{*}}^{(k)}\right)$ is given by

$$
u^{(k)}=d a^{(k-1)}+d^{*} b^{(k+1)}+h^{(k)},
$$

where $a^{(k-1)}$ and $b^{(k+1)}$ are the unique solutions of problems $\left(P_{t^{*}}^{a}\right)$ and $\left(P_{t^{*}}^{b}\right)$, respectively, and any $h^{(k)} \in \mathcal{H}_{t^{*}}^{(k)}$. Thus $t^{*} a^{(k-1)}=0, t^{*} d a^{(k-1)}=r_{\mathrm{b}}^{(k-1)}, t^{*} b^{(k+1)}=0, d^{*} a^{(k-1)}=0$ and $d b^{(k+1)}=0$. The solution is unique up to elements of $\mathcal{H}_{t^{*}}^{(k)}$.

Proof. By the Helmholtz-Hodge decomposition the solution can be written as in (5.10), i.e., $u^{(k)}=d a^{(k-1)}+d^{*} b^{(k+1)}+h^{(k)}$. Substitution in (5.13a) gives

$$
(d a, d v)=(f, v)+\left(r_{\mathrm{b}}, t v\right)_{\mathrm{b}} \quad \forall v \in H \Lambda^{(k-1)}(d, \Omega),
$$

which the solution $a^{(k-1)}$ of Problem ( $\mathrm{P}_{t^{*}}^{a}$ ) satisfies. Substitution in (5.13b) gives

$$
\begin{aligned}
-(\psi, \chi)+(b, d \chi) & =0 & & \forall \chi \in H \Lambda^{(k)}(d, \Omega), \\
(d \psi, v) & =(g, v) & & \forall v \in H \Lambda^{(k-1)}(d, \Omega),
\end{aligned}
$$

which the solution $\left(b^{(k+1)}, \psi^{(k)}\right)$ of Problem $\left(\mathrm{P}_{t^{*}}^{b}\right)$ satisfies. So we see that $u^{(k)}$ as given above is indeed a solution.

Suppose there is a second solution of the problem. The difference of the two solutions satisfies the problem with zero right-hand side, which is an alternative way of stating that it is an element of $\mathcal{H}_{t^{*}}^{(k)}$. Therefore any $h^{(k)} \in \mathcal{H}_{t^{*}}^{(k)}$ gives a solution.

### 5.1.3 Characterization of the solution to problem $\left(\mathbf{P}_{t}^{(k)}\right)$

The solution of the problem $\left(\mathrm{P}_{t}^{(k)}\right)$ relies on the second Helmholtz-Hodge decomposition, given in (5.9). We will show that the solution to problem $\left(\mathrm{P}_{t}^{(k)}\right)$ is of the form

$$
\begin{equation*}
u^{(k)}=\underbrace{d a^{(k-1)}}_{\epsilon \mathcal{B}^{(k)}}+\underbrace{d^{*} b^{(k+1)}}_{\epsilon \mathcal{B}^{*(k)}}+\underbrace{h^{(k)}}_{\epsilon \mathcal{H}_{t}^{(k)}}, \tag{5.14}
\end{equation*}
$$

where $a^{(k-1)}$ and $b^{(k+1)}$ solve

$$
\left\{\begin{array} { r l } 
{ ( d d ^ { * } + d ^ { * } d ) a ^ { ( k - 1 ) } } & { = f ^ { ( k - 1 ) } , } \\
{ t d ^ { * } a ^ { ( k - 1 ) } } & { = 0 , } \\
{ t a ^ { ( k - 1 ) } } & { = 0 , }
\end{array} \quad \left\{\begin{array}{rl}
\left(d d^{*}+d^{*} d\right) b^{(k+1)} & =g^{(k+1)}, \\
t d^{*} b^{(k+1)} & =r_{\mathrm{b}}^{(k)}, \\
t b^{(k+1)} & =0 .
\end{array}\right.\right.
$$

In [97] this was shown previously for $k=1$ and $k=2$ in 3 D . The discussion that follows here is quite similar to the one presented there, however, we use a mixed weak form that is more suited for discretization.

### 5.1.3.1 Hodge-Laplace problem for the potential $a^{(k-1)}$

The potential $a^{(k-1)}$ is now the solution of the mixed Hodge-Laplace problem with essential boundary conditions. Given $f \in L^{2} \Lambda^{(k-1)}(\Omega)$ we need to find $\phi \in H^{\prime} \Lambda^{(k-2)}(d, \Omega)$, $a \in \dot{H} \Lambda^{(k-1)}(d, \Omega)$ and $p \in \mathcal{H}_{t}^{(k-1)}$ such that

$$
\begin{aligned}
(\phi, \chi)-(a, d \chi) & =0 & & \forall \chi \in \dot{H} \Lambda^{(k-2)}(d, \Omega), \\
(d a, d v)+(d \phi, v)+(p, v) & =(f, v) & & \forall v \in \circ_{H}^{(k-1)}(d, \Omega), \\
(a, h) & =0 & & \forall h \in \mathcal{H}_{t}^{(k-1)} .
\end{aligned}
$$

However, to reformulate the problem in a form that is easier to discretize, we do not enforce the boundary conditions in the function spaces but impose them via an extra equation.

Problem $\left(\mathrm{P}_{t}^{a}\right)$. Given $f \in L^{2} \Lambda^{(k-1)}(\Omega)$ satisfying $d^{*} f=0, f \in \mathcal{H}_{t}^{(k-1) \perp}$. Find $\phi \in$ $H \Lambda^{(k-2)}(d, \Omega), a \in H \Lambda^{(k-1)}(d, \Omega), \tilde{\mu}_{\mathrm{b}} \in H^{-\frac{1}{2}} \tilde{\Lambda}^{(d-k+1)}(\tilde{d}, \partial \Omega), \tilde{\eta}_{\mathrm{b}} \in H^{-\frac{1}{2}} \tilde{\Lambda}^{(d-k)}(\tilde{d}, \partial \Omega)$ and $p \in \mathcal{H}_{t}^{(k-1)}$ such that

$$
\begin{align*}
(\phi, \chi)-(a, d \chi)+\left[t \chi, \tilde{\mu}_{\mathrm{b}}\right]_{\mathrm{b}} & =0 & & \forall \chi \in H \Lambda^{(k-2)}(d, \Omega),  \tag{5.15a}\\
(d a, d v)+(d \phi, v)+(p, v)-\left[t v, \tilde{\eta}_{\mathrm{b}}\right]_{\mathrm{b}} & =(f, v) & & \forall v \in H \Lambda^{(k-1)}(d, \Omega),  \tag{5.15b}\\
{\left[t a, \tilde{\nu}_{\mathrm{b}}\right]_{\mathrm{b}} } & =0 & & \forall \tilde{\nu}_{\mathrm{b}} \in H^{-\frac{1}{2}} \tilde{\Lambda}^{(d-k)}(\tilde{d}, \partial \Omega),  \tag{5.15c}\\
{\left[t \phi, \tilde{\zeta}_{\mathrm{b}}\right]_{\mathrm{b}} } & =0 & & \forall \tilde{\zeta}_{\mathrm{b}} \in H^{-\frac{1}{2}} \tilde{\Lambda}^{(d-k+1)}(\tilde{d}, \partial \Omega),  \tag{5.15d}\\
(a, h) & =0 & & \forall h \in \mathcal{H}_{t}^{(k-1)}, \tag{5.15e}
\end{align*}
$$

where $[\cdot, \cdot]_{\mathrm{b}}: H^{-\frac{1}{2}} \Lambda^{(k-1)}(d, \partial \Omega) \times H^{-\frac{1}{2}} \tilde{\Lambda}^{(d-k)}(\tilde{d}, \partial \Omega)$, is defined according to

$$
\left[a_{\mathrm{b}}^{(k-1)}, \tilde{b}_{\mathrm{b}}^{(d-k)}\right]_{\mathrm{b}}:=\int_{\partial \Omega} a_{\mathrm{b}}^{(k-1)} \wedge \tilde{b}_{\mathrm{b}}^{(d-k)} .
$$

Lemma 5.3. Problem ( $\mathrm{P}_{t}^{a}$ ) has a unique solution $\left(\phi^{(k-2)}, a^{(k-1)}, \tilde{\mu}_{\mathrm{b}}^{(k-2)}, \tilde{\eta}_{\mathrm{b}}^{(k-1)}, p^{(k-1)}\right)$. Moreover, we have $a^{(k-1)} \in H_{t} \Lambda^{(k-1)}\left(d, d^{*}, \Omega\right), d^{*} a^{(k-1)}=\phi^{(k-2)}=0, \tilde{\mu}_{\mathrm{b}}^{(k-2)}=\star_{\mathrm{b}} t^{*} a^{(k-1)}$, $\tilde{\eta}_{\mathrm{b}}^{(k-1)}=\star_{\mathrm{b}} t^{*} d a^{(k-1)}$ and $p^{(k-1)}=0$.

Proof. The problem is well-posed because it is equivalent to problem (GP) with $\left(V^{(k)}, d^{(k)}\right)=$ $\left(H \Lambda^{(k)}(d, \Omega), d^{(k)}\right)$ and $F(v)=(f, v)$. From (5.15a) it first follows by restricting to $\chi \in \stackrel{\circ}{H} \Lambda^{(k-2)}(d, \Omega)$ that $\phi=d^{*} a$. By (2.9) it then follows that $\left(t \chi, \star_{\mathrm{b}}^{-1} \tilde{\mu}_{\mathrm{b}}\right)_{\mathrm{b}}=\left(t \chi, t^{*} a\right)_{\mathrm{b}}$ for all $\chi \in H \Lambda^{(k-2)}(d, \Omega)$ from which it follows that $\star_{\mathrm{b}} t^{*} a=\tilde{\mu}_{\mathrm{b}}$.

When we take $v=d \phi$ in (5.15b) we obtain $(d \phi, d \phi)=0$, because $d d \phi=0,(p, d \phi)=0$, $t d \phi=0$ and $(f, d \phi)=0$ as $d \phi \in \dot{\mathcal{B}}^{(k-1)}$ and $p, f \in \mathcal{Z}^{*(k-1)}=\dot{\mathcal{B}}^{(k-1) \perp}$. This implies that $\phi \in \dot{\mathcal{Z}}^{(k-2)}$, while by (5.15a) $\phi=d^{*} a \in \mathcal{B}^{*(k-2)}=\dot{\mathcal{Z}}^{(k-2) \perp}$, and hence $\phi=0$.

Finally, if we take $v=p$ in (5.15b), we find because $d p=0, t p=0$ and $f \in \mathcal{H}_{t}^{(k-1) \perp}$, that $(p, p)=0$, i.e., $p=0$.

Subsequently, restricting (5.15b) to $v \in \grave{H}^{\prime} \Lambda^{(k-1)}(d, \Omega)$, we find $(d a, d v)=(f, v)$ for all $v \in \dot{H} \Lambda^{(k-1)}(d, \Omega)$, which shows that $d a \in H \Lambda^{(k)}\left(d^{*}, \Omega\right)$ and $d^{*} d a=f$. Using this and (2.9) we find that $\left(t^{*} d a-\star_{\mathrm{b}}^{-1} \tilde{\eta}_{\mathrm{b}}, t v\right)_{\mathrm{b}}=0$ for all $v \in H \Lambda^{(k-1)}(d, \Omega)$.

### 5.1.3.2 Hodge-Laplace problem for the potential $b^{(k+1)}$

The potential $b^{(k+1)}$ satisfies a similar Hodge-Laplace problem as $a^{(k-1)}$.
Problem $\left(\mathrm{P}_{t}^{b}\right)$. Given $g \in L^{2} \Lambda^{(k+1)}(\Omega)$ and $r_{\mathrm{b}} \in H^{-\frac{1}{2}} \Lambda^{(k+1)}(d, \Omega)$ satisfying $d g=0$, $g \in \mathcal{H}_{t}^{(k+1) \perp}$ and $(g, z)=\left(r_{\mathrm{b}}, t^{*} z\right)_{\mathrm{b}}$ for all $z \in \mathcal{Z}^{*(k+1)}$. Find $\psi \in H \Lambda^{(k)}(d, \Omega), b \in$ $H \Lambda^{(k+1)}(d, \Omega), \tilde{\xi}_{\mathrm{b}} \in H^{-\frac{1}{2}} \tilde{\Lambda}^{(d-k-1)}(\tilde{d}, \partial \Omega), \tilde{\rho}_{\mathrm{b}} \in H^{-\frac{1}{2}} \tilde{\Lambda}^{(d-k-2)}(\tilde{d}, \partial \Omega)$ and $q \in \mathcal{H}_{t}^{(k+1)}$ such that

$$
\begin{align*}
(\psi, \chi)-(b, d \chi)+\left[t \chi, \tilde{\xi}_{\mathrm{b}}\right]_{\mathrm{b}} & =0 & & \forall \chi \in H \Lambda^{(k)}(d, \Omega),  \tag{5.16a}\\
(d b, d v)+(d \psi, v)+(q, v)-\left[t v, \tilde{\rho}_{\mathrm{b}}\right]_{\mathrm{b}} & =(g, v) & & \forall v \in H \Lambda^{(k+1)}(d, \Omega),  \tag{5.16b}\\
{\left[t b, \tilde{\nu}_{\mathrm{b}}\right]_{\mathrm{b}} } & =0 & & \forall \nu_{\mathrm{b}} \in H^{-\frac{1}{2}} \tilde{\Lambda}^{(d-k-2)}(\tilde{d}, \partial \Omega),  \tag{5.16c}\\
{\left[t \psi, \tilde{\zeta}_{\mathrm{b}}\right]_{\mathrm{b}} } & =\left[r_{\mathrm{b}}, \tilde{\zeta}_{\mathrm{b}}\right]_{\mathrm{b}} & & \forall \zeta_{\mathrm{b}} \in H^{-\frac{1}{2}} \tilde{\Lambda}^{(d-k-1)}(\tilde{d}, \partial \Omega),  \tag{5.16d}\\
(b, h) & =0 & & \forall h \in \mathcal{H}_{t}^{(k+1)} . \tag{5.16e}
\end{align*}
$$

Lemma 5.4. Problem $\left(\mathrm{P}_{t}^{b}\right)$ has a unique solution $\left(\psi^{(k)}, b^{(k+1)}, \tilde{\xi}_{\mathrm{b}}^{(k)}, \rho_{\mathrm{b}}^{(k+1)}, q^{(k+1)}\right)$. Moreover, we have $b^{(k+1)} \in \stackrel{\circ}{H}_{t} \Lambda^{(k+1)}\left(d, d^{*}, \Omega\right)$, $d b^{(k+1)}=0$, $\psi^{(k)}=d^{*} b^{(k+1)}$, $t \psi^{(k)}=r_{\mathrm{b}}$, $\tilde{\xi}_{\mathrm{b}}^{(d-k-1)}=\star_{\mathrm{b}} t^{*} b^{(k+1)}, \tilde{\rho}_{\mathrm{b}}^{(d-k-2)}=\star_{\mathrm{b}} t^{*} d b^{(k+1)}=0$ and $q^{(k+1)}=0$.
Proof. We show the well-posedness of this problem in Appendix A.
We first show $d b=0$. Let $\pi: \mathrm{Ł}^{2} \Lambda^{(k+1)}(\Omega) \rightarrow \mathcal{B}^{*(k+1)}$ be the orthogonal projection on $\mathcal{B}^{*(k+1)}$. Applying this to (5.16b) gives, for all $v \in H \Lambda^{(k+1)}(d, \Omega)$,

$$
(d b, d \pi v)+(d \psi, \pi v)+\left(t \pi v, \star_{\mathrm{b}}^{-1} \tilde{\rho}_{\mathrm{b}}\right)_{\mathrm{b}}=(g, \pi v),
$$

where we used that $\pi v \in \mathcal{H}_{t}^{(k+1) \perp}$. Now, because $d^{*} \pi v=0$ we have that $\pi v \in H \Lambda^{(k+1)}\left(d, d^{*}, \Omega\right)$ and hence we can take the trace $t^{*} \pi v \in H^{-\frac{1}{2}} \Lambda^{(k)}\left(d^{*}, \partial \Omega\right)$. Taking in (5.16d) $\tilde{\zeta}_{\mathrm{b}}=\star_{\mathrm{b}} t^{*} \pi v$ and using $(g, \pi v)=\left(r_{\mathrm{b}}, t^{*} \pi v\right)_{\mathrm{b}}$ we obtain

$$
(d b, d \pi v)+(d \psi, \pi v)-\left(t \psi, t^{*} \pi v\right)_{\mathrm{b}}+\left(t \pi v, \star_{\mathrm{b}}^{-1} \tilde{\rho}_{\mathrm{b}}\right)_{\mathrm{b}}=0
$$

for all $v \in H \Lambda^{(k+1)}(d, \Omega)$. By (2.9) we have $(d \psi, \pi v)-\left(t \psi, t^{*} \pi v\right)_{\mathrm{b}}=\left(\psi, d^{*} \pi v\right)=0$ and hence we find $(d b, d \pi v)-\left(\rho_{\mathrm{b}}, t \pi v\right)_{\mathrm{b}}=0$ for all $v \in H \Lambda^{(k+1)}(d, \Omega)$. However, because $\mathcal{B}^{*(k+1) \perp}=\dot{\mathcal{Z}}^{(k+1)}$ we have that $(d b, d \pi v)-\left(\rho_{\mathrm{b}}, t \pi v\right)_{\mathrm{b}}=(d b, d v)-\left(\rho_{\mathrm{b}}, t v\right)_{\mathrm{b}}$. Taking finally $v=b$ and using the fact that by (5.16c) $t b=0$ we obtain $(d b, d b)=0$ implying that $d b=0$.

Taking $v=q$ in (5.16b) gives $(d \psi, q)+(q, q)=0$. Note that because $(g, q)=0$ we necessarily have $\left(r_{\mathrm{b}}, t^{*} q\right)=0$. Taking $\zeta_{\mathrm{b}}=t^{*} q$ in (5.16d) we see that $\left(t \psi, t^{*} q\right)=0$ and therefore we have $(d \psi, q)=\left(\psi, d^{*} q\right)=0$ by (2.9). Thus $(q, q)=0$ and so $q=0$.

Restricting (5.16a) to $H \circ^{(k)}(d, \Omega)$ we find that $b \in H \Lambda^{(k+1)}\left(d^{*}, \Omega\right)$ and $d^{*} b=\psi$. For $b \in H \Lambda^{(k+1)}\left(d^{*}, \Omega\right)$ and $\chi \in \stackrel{\circ}{H} \Lambda^{(k)}(d, \Omega)$ we have $\left(d^{*} b, \chi\right)-(b, d \chi)=\left(t \chi, t^{*} b\right)_{\mathrm{b}}$. Hence, (5.16a) implies that $\left(t \chi, t^{*} b\right)_{\mathrm{b}}=\left(t \chi, \star_{\mathrm{b}}^{-1} \tilde{\xi}_{\mathrm{b}}\right)_{\mathrm{b}}$ for all $\chi \in \stackrel{H}{\circ} \Lambda^{(k)}(d, \Omega)$, which shows that $\tilde{\xi}_{\mathrm{b}}=\star_{\mathrm{b}} t^{*} b$.

Finally, because $d b \in H \Lambda^{(k+2)}\left(d^{*}, \Omega\right)$ and $d^{*} b \in H \Lambda^{(k)}(d, \Omega)$ (see Appendix A), we find by applying integration by parts to (5.16b) that $\left(t v, \star_{\mathrm{b}}^{-1} \tilde{\rho}_{\mathrm{b}}\right)_{\mathrm{b}}=\left(t v, t^{*} d b\right)_{\mathrm{b}}$ for all $v \in H \Lambda^{(k+1)}(d, \Omega)$. This implies that $\tilde{\rho}_{\mathrm{b}}=\star_{\mathrm{b}} t^{*} d b$.

### 5.1.3.3 The solution to Problem ( $\mathbf{P}_{t}^{(k)}$ )

Again the potentials that solve the Problems $\left(\mathrm{P}_{t}^{a}\right)$ and $\left(\mathrm{P}_{t}^{b}\right)$ provide the solution to the div-curl Problem $\left(\mathrm{P}_{t}^{(k)}\right)$.

Problem $\left(\mathrm{P}_{t}^{(k)}\right)$. Given $f \in L^{2} \Lambda^{(k-1)}(\Omega), g \in L^{2} \Lambda^{(k+1)}(\Omega)$ and $r_{\mathrm{b}} \in H^{-\frac{1}{2}} \Lambda^{(k)}(d, \partial \Omega)$ satisfying $d^{*} f=0, f \in \mathcal{H}_{t}^{(k-1) \perp}, d g=0, g \in \mathcal{H}_{t}^{(k+1) \perp}$ and $(g, z)=\left(r_{\mathrm{b}}, t z\right)_{\mathrm{b}}$ for all $z \in$ $\mathcal{Z}^{*(k-1)}$. Find $\left(u^{(k)}, \tilde{\mu}_{\mathrm{b}}^{(d-k-2)}\right) \in H \Lambda^{(k)}(\Omega) \times H^{-\frac{1}{2}} \tilde{\Lambda}^{(d-k-2)}(\tilde{d}, \partial \Omega)$ such that

$$
\begin{align*}
(u, d v)-\left[t v, \tilde{\mu}_{\mathrm{b}}\right]_{\mathrm{b}} & =(f, v) & & \forall v \in H \Lambda^{(k-1)}(d, \Omega),  \tag{5.17a}\\
(d u, w) & =(g, w) & & \forall w \in H \Lambda^{(k+1)}(d, \Omega),  \tag{5.17b}\\
{\left[t u, \tilde{\nu}_{\mathrm{b}}\right]_{\mathrm{b}} } & =\left[r_{\mathrm{b}}, \tilde{\nu}_{\mathrm{b}}\right]_{\mathrm{b}} & & \forall \tilde{\nu}_{\mathrm{b}} \in H^{-\frac{1}{2}} \tilde{\Lambda}^{(d-k-1)}(\tilde{d}, \partial \Omega) . \tag{5.17c}
\end{align*}
$$

Theorem 5.2. Problem $\left(\mathrm{P}_{t}^{(k)}\right)$ has a unique solution given by

$$
u^{(k)}=d a^{(k-1)}+d^{*} b^{(k+1)}+h^{(k)},
$$

where $a^{(k-1)}$ and $b^{(k+1)}$ are the unique solutions of Problem $\left(\mathrm{P}_{t}^{a}\right)$ and Problem $\left(\mathrm{P}_{t}^{b}\right)$, and, $h^{(k)} \in \mathcal{H}_{t}^{(k)}$. Thus $t a^{(k-1)}=0, t d^{*} a^{(k-1)}=0, t b^{(k+1)}=0, t d^{*} b^{(k+1)}=r_{\mathrm{b}}^{(k)}, d^{*} a^{(k-1)}=0$ and $d b^{(k+1)}=0$. Furthermore, $\tilde{\mu}_{\mathrm{b}}^{(d-k-2)}=\star_{\mathrm{b}} t^{*} u^{(k)}$. The solution is unique up to elements of $\mathcal{H}_{t}{ }^{(k)}$.

Proof. By the Helmholtz-Hodge decomposition (5.14) the solution can be written as $u^{(k)}=d a^{(k-1)}+d^{*} b^{(k+1)}+h^{(k)}$. Substitution in (5.17a) gives, after integration by parts,

$$
(d a, d v)+\left(t v, t^{*} d^{*} b+t^{*} h-\star_{\mathrm{b}}^{-1} \tilde{\mu}_{\mathrm{b}}\right)_{\mathrm{b}}=(f, v) \quad \forall v \in H \Lambda^{(k-1)}(d, \Omega) .
$$

Substitution in (5.17b) and (5.17c) gives

$$
\begin{aligned}
-(\psi, \chi)+(b, d \chi)+\left[t \chi, \tilde{\xi}_{\mathrm{b}}\right]_{\mathrm{b}} & =0 & & \forall \chi \in H \Lambda^{(k)}(d, \Omega) \\
(d \psi, v) & =(g, v) & & \forall v \in H \Lambda^{(k+1)}(d, \Omega) \\
\left(t \psi, \nu_{\mathrm{b}}\right)_{\mathrm{b}} & =\left(r_{\mathrm{b}}, \nu_{\mathrm{b}}\right)_{\mathrm{b}} & & \forall \nu_{\mathrm{b}} \in H^{-\frac{1}{2}} \Lambda^{(k)}\left(d^{*}, \partial \Omega\right) .
\end{aligned}
$$

We see that if $a^{(k-1)}$ and $b^{(k+1)}$ are the solution to $\left(\mathrm{P}_{t}^{a}\right)$ and $\left(\mathrm{P}_{t}^{b}\right)$ then they satisfy above equations. So we see that $u^{(k)}$ as given above is indeed a solution. Suppose there is a second solution, then the difference of both solutions should satisfy the problem with zero right-hand side, which implies that the difference is an element of $\mathcal{H}_{t}^{(k)}$.

### 5.2 Two discrete Helmholtz-Hodge decompositions

To repeat the discussion from Section 5.1 in a discrete sense and show the solution approach and well-posedness of the discrete problems, we will derive the discrete analogues of Helmholtz-Hodge decompositions (2.19) and (2.20). We do this by formulating first the discrete versions of the closed Hilbert complexes that lead to the Helmholtz-Hodge decompositions.

### 5.2.1 Discrete complexes for natural boundary conditions

The first complex is given by the simple sequence seen before in Proposition 3.2, it is given by

$$
\begin{equation*}
C^{(k-1)}(\Omega) \xrightarrow{\mathbb{D}^{(k-1)}} C^{(k)}(\Omega) \xrightarrow{\mathbb{D}^{(k)}} C^{(k+1)}(\Omega) . \tag{5.18}
\end{equation*}
$$

Lemma 5.5. Sequence (5.18) is a closed Hilbert complex.
Proof. In the notation from Section 2.3.3 we have $W^{(k)}=C^{(k)}(\Omega), V^{(k)}=C^{(k)}(\Omega)$ and $A^{(k)}=\mathbb{D}^{(k)}$. From the fact that $d^{(k)} \circ d^{(k-1)} \equiv 0$ and the commuting diagram in Figure 3.4 it follows that $\mathbb{D}^{(k)} \mathbb{D}^{(k-1)} \equiv 0$, hence the sequence is a complex. We have $W^{(k)}=V^{(k)}$, therefore the operators $\mathbb{D}^{(k)}$ are trivially densely-defined. The spaces have finite dimension, hence the operators are bounded and closed and have a closed range. Moreover, together with the discrete Hodge matrices $\mathbb{H}^{(k)}$ the spaces $C^{(k)}(\Omega)$ are Hilbert spaces.

The adjoint sequence is given by

$$
\begin{equation*}
C^{(k-1)}(\Omega) \stackrel{\mathbb{D}^{*(k)}}{\leftrightarrows} C^{(k)}(\Omega) \stackrel{\mathbb{D}^{*(k+1)}}{\leftrightarrows} C^{(k+1)}(\Omega), \tag{5.19}
\end{equation*}
$$

where $\mathbb{D}^{\star(k)}:=\left(\mathbb{H}^{(k-1)}\right)^{-1} \mathbb{D}^{(k-1) T} \mathbb{H}^{(k)}$. Note that this is a discrete analogue of the sequence $\left(\stackrel{\circ}{H} \Lambda^{(k)}\left(d^{*}, \Omega\right), d^{*(k)}\right)$, because for $\tilde{a}^{(d-k)} \in \check{\Lambda}^{(d-k)}(\tilde{d}, \Omega)$ we have

$$
\tilde{\mathbb{D}}^{(d-k)}\left[\begin{array}{c}
\tilde{R}^{(d-k)}\left(\tilde{a}^{(d-k)}\right) \\
\tilde{R}_{\mathrm{b}}^{(d-k)}\left(\tilde{t}^{(d-k)}\right)
\end{array}\right]=\mathbb{D}^{(k-1) T} \tilde{R}^{(d-k)}\left(\tilde{a}^{(d-k)}\right)=\tilde{R}^{(d-k+1)}\left(\tilde{d}^{(d-k)} \tilde{a}^{(d-k)}\right),
$$

because $\tilde{t} \tilde{a}^{(d-k)}=0$. Thus $\mathbb{D}^{*(k)} R^{(k)}\left(a^{(k)}\right)$ approximates $R^{(k-1)}\left(d^{*(k)} a^{(k)}\right)$ when $t^{*} a^{(k)}=$ 0.

Before we continue, it is important to note that the De Rham maps are not defined for $\check{H} \Lambda^{(k)}\left(d^{*}, \Omega\right)$, hence we apply it in the example above, for simplicity, to the smooth elements. However, the space $\tilde{\Lambda}^{(d-k)}(\tilde{d}, \Omega)$ (or another space of less smooth differential forms for which the De Rham maps are defined) is not a complete and therefore not a convenient space to define the div-curl problems. This poses a serious difficulty if one wants to show the convergence of the discretizations in this chapter to the problems $\left(\mathrm{P}_{t^{*}}^{(k)}\right)$ and $\left(\mathrm{P}_{t}^{(k)}\right)$. We can still in the discrete setting mimic the $L^{2}$ Helmholtz-Hodge decompositions and use this to show the well-posedness of the discretizations.

The adjointness of the two sequences is expressed by

$$
\left(\mathbb{D}^{(k)} \boldsymbol{a}^{(k)}, \boldsymbol{b}^{(k+1)}\right)_{\mathbb{H}}=\left(\boldsymbol{a}^{(k)}, \mathbb{D}^{*(k+1)} \boldsymbol{b}^{(k+1)}\right)_{\mathbb{H}}
$$

which is a discrete version of (2.9) without boundary contribution.
We define the spaces of discrete exact and closed forms for these two sequences according to

$$
\begin{array}{rrr}
B^{(k)}:=\operatorname{im} \mathbb{D}^{(k-1)}, & Z^{(k)}:=\operatorname{ker} \mathbb{D}^{(k)}, \\
\dot{B}^{*(k)}:=\operatorname{im} \mathbb{D}^{*(k+1)}, & \dot{Z}^{*(k)}:=\operatorname{ker} \mathbb{D}^{*(k)} .
\end{array}
$$

They satisfy again the same orthogonality relations as their analogues in Lemma 2.3. By a completely analogous proof we have

Lemma 5.6. The subspaces of discrete exact and closed forms for the complex (5.18) and its adjoint complex (5.19) satisfy

$$
B^{(k) \perp}=\dot{Z}^{*(k)}, \quad \quad \stackrel{\circ}{B}^{*(k) \perp}=Z^{(k)}
$$

Thus we see that the discrete complex (5.18) exactly mimics the complex (5.4).

### 5.2.2 Discrete complexes for essential boundary conditions

To formulate the second pair of discrete complexes we first define projection operators that restrict a cochain to the boundary or interior of the mesh. Let $\mathbb{I}^{(k)}: C^{(k)}(\Omega) \rightarrow$ $C^{(k)}(\Omega)$ be the identity. The boundary and interior projection operators are defined by, respectively,

$$
\mathbb{P}_{\partial \Omega}^{(k)}:=\mathbb{T}^{(k) T} \mathbb{T}^{(k)}, \quad \text { and } \quad \mathbb{P}_{\Omega \backslash \partial \Omega}^{(k)}:=\mathbb{T}^{(k)}-\mathbb{T}^{(k) T} \mathbb{T}^{(k)}
$$

Using the projection and trace operators we can define the second sequence, which will lead to the second discrete Helmholtz-Hodge decomposition of $C^{(k)}(\Omega)$. The sequence is given by

$$
\begin{equation*}
C^{(k-1)}(\Omega) \xrightarrow{\mathbb{E}^{(k-1)}} C^{(k)}(\Omega) \xrightarrow{\mathbb{E}^{(k)}} C^{(k+1)}(\Omega) \oplus C^{(k)}(\partial \Omega), \tag{5.20}
\end{equation*}
$$

with $\mathbb{E}^{(k-1)}: C^{(k-1)}(\Omega) \rightarrow C^{(k)}(\Omega)$ and $\mathbb{E}^{(k)}: C^{(k)}(\Omega) \rightarrow C^{(k+1)}(\Omega) \oplus C^{(k)}(\partial \Omega)$ given by

$$
\mathbb{E}^{(k-1)}:=\mathbb{D}^{(k-1)} \mathbb{P}_{\Omega \backslash \partial \Omega}^{(k-1)} \quad \text { and } \quad \mathbb{E}^{(k)}:=\left[\begin{array}{l}
\mathbb{D}^{(k)} \\
\mathbb{T}^{(k)}
\end{array}\right]
$$

Moreover, these spaces in (5.20) are Hilbert spaces endowed with the inner products $\mathbb{H}^{(k-1)}, \mathbb{H}^{(k)}$ and $\mathbb{H}^{(k+1)} \oplus \mathbb{H}_{\mathrm{b}}^{(k)}$, where $\mathbb{H}_{\mathrm{b}}^{(k)}$ is the boundary discrete Hodge matrix. This matrix is the discrete analogue of $\star_{\mathrm{b}}$. The product $\left(R_{\mathrm{b}}^{(k)}\left(a^{(k)}\right)\right)^{T} \mathbb{H}_{\mathrm{b}}^{(k)} R_{\mathrm{b}}^{(k)}\left(b^{(k)}\right)$ is an approximation of $\int_{\partial \Omega} a^{(k)} \wedge \star_{\mathrm{b}} b^{(k)}$ with $a^{(k)}, b^{(k)}$ smooth. The boundary discrete Hodge matrix can be built just as the interior discrete Hodge matrix. However, in the next section we will see that we do not need it in practice.

Lemma 5.7. Sequence (5.20) is a closed Hilbert complex.
Proof. Using $\mathbb{D}^{(k)} \mathbb{D}^{(k-1)}=0$ we find

$$
\mathbb{E}^{(k)} \mathbb{E}^{(k-1)}=\left[\begin{array}{c}
\mathbb{D}^{(k)} \mathbb{D}^{(k-1)} \mathbb{P}_{\Omega \backslash \Omega \Omega}^{(k-1)} \\
\mathbb{T}^{(k)} \mathbb{D}^{(k-1)} \mathbb{P}_{\Omega \backslash \partial \Omega}^{(k-1)}
\end{array}\right]=\left[\begin{array}{c}
0 \\
\mathbb{T}^{(k)} \mathbb{D}^{(k-1)} \mathbb{P}_{\Omega \backslash \partial \Omega}^{(k-1)}
\end{array}\right]
$$

For a $k$-cell in $\sigma_{(k)} \in C_{(k)}(\partial \Omega)$ its boundary $\partial^{1} \sigma_{(k)}$ is a set of $(k-1)$-cells that also lie in $\partial \Omega$. This implies that the values of $\mathbb{T}^{(k)} \mathbb{D}^{(k-1)} \boldsymbol{a}^{(k-1)}$, for any $\boldsymbol{a}^{(k-1)} \in C^{(k-1)}(\Omega)$, depend only on the values of $\boldsymbol{a}^{(k-1)}$ on the boundary. Thus we have $\mathbb{T}^{(k)} \mathbb{D}^{(k-1)}=$ $\mathbb{T}^{(k)} \mathbb{D}^{(k-1)} \mathbb{P}_{\partial \Omega}^{(k-1)}$.

The fact $\mathbb{P}_{\partial \Omega}^{(k-1)} \mathbb{P}_{\Omega \backslash \partial \Omega}^{(k-1)}=0$, therefore implies $\mathbb{E}^{(k)} \mathbb{E}^{(k-1)}=0$.
Finally, it follows as in Lemma 5.5 that the complex is a closed Hilbert complex.
The adjoint sequence is given by

$$
\begin{equation*}
C^{(k-1)}(\Omega) \stackrel{\mathbb{E}^{*(k)}}{\leftrightarrows} C^{(k)}(\Omega) \stackrel{\mathbb{E}^{*(k+1)}}{\leftrightarrows} C^{(k+1)}(\Omega) \tag{5.21}
\end{equation*}
$$

with

$$
\begin{aligned}
& \mathbb{E}^{*(k)}:=(-1)^{k}\left(\mathbb{H}^{(k-1)}\right)^{-1} \tilde{\mathbb{D}}^{(d-k)}\left[\begin{array}{cc}
\mathbb{H}^{(k)} & 0 \\
0 & 0
\end{array}\right]=\left(\mathbb{H}^{(k-1)}\right)^{-1} \mathbb{P}_{\Omega \lambda \Omega \Omega}^{(k-1)} \mathbb{D}^{(k-1) T} \mathbb{H}^{(k)}, \\
& \mathbb{E}^{*(k+1)}:=(-1)^{k+1}\left(\mathbb{H}^{(k)}\right)^{-1} \tilde{\mathbb{D}}^{(d-k-1)}\left[\begin{array}{cc}
\mathbb{H}^{(k+1)} & 0 \\
0 & \mathbb{H}_{\mathrm{b}}^{(k)}
\end{array}\right] .
\end{aligned}
$$

Note that $\mathbb{E}^{*(k+1)}$ is the discrete analogue of $d^{*(k+1)}=(-1)^{k+1}\left(\star^{(k)}\right)^{-1} \tilde{d}^{(d-k-1)} \star^{(k+1)}$, and, similarly, $\mathbb{E}^{*(k)}$ is the discrete analogue of $d^{*(k))}$ on the space with zero trace.

By definition of $\mathbb{E}^{*(k)}$ and $\mathbb{E}^{*(k+1)}$ they satisfy, for arbitrary cochains,

$$
\left(\mathbb{E}^{*(k)} \boldsymbol{a}^{(k)}, \boldsymbol{b}^{(k-1)}\right)_{\mathbb{H}}=\left(\boldsymbol{a}^{(k)}, \mathbb{D}^{(k-1)} \mathbb{P}_{\Omega \backslash \partial \Omega}^{(k-1)} \boldsymbol{b}^{(k-1)}\right)_{\mathbb{H}}
$$

which shows that $\mathbb{E}^{*(k)}$ is adjoint to $\mathbb{D}^{(k-1)} \mathbb{P}_{\Omega \backslash \Omega}^{(k-1)}$. Furthermore, by Proposition 3.10 , we have

$$
\left(\mathbb{E}^{*(k+1)}\left[\begin{array}{c}
\boldsymbol{a}^{(k+1)} \\
\boldsymbol{a}_{\mathrm{b}}^{(k)}
\end{array}\right], \boldsymbol{b}^{(k)}\right)_{\mathbb{H}}=\left(\boldsymbol{a}^{(k+1)}, \mathbb{D}^{(k)} \boldsymbol{b}^{(k)}\right)_{\mathbb{H}}-\left(\boldsymbol{a}_{\mathrm{b}}^{(k)}, \mathbb{T}^{(k)} \boldsymbol{b}^{(k)}\right)_{\mathbb{H}_{\mathrm{b}}} .
$$

If we take $\boldsymbol{a}^{(k+1)}=R^{(k+1)}\left(a^{(k+1)}\right), \boldsymbol{a}_{\mathrm{b}}^{(k)}=R_{\mathrm{b}}^{(k)}\left(t^{*} a^{(k+1)}\right)$ and $\boldsymbol{b}^{(k)}=R^{(k)}\left(b^{(k)}\right)$ (for smooth $a^{(k+1)}$ and $\left.b^{(k)}\right)$, the relation for $\mathbb{E}^{*(k+1)}$ reads

$$
\begin{aligned}
& \left(R^{(k+1)}\left(a^{(k+1)}\right), R^{(k+1)}\left(d b^{(k)}\right)\right)_{\mathbb{H}}-\left(\mathbb{E}^{*(k+1)}\left[\begin{array}{l}
R^{(k+1)}\left(a^{(k+1)}\right) \\
R_{\mathrm{b}}^{(k)}\left(t^{*} a^{(k+1)}\right)
\end{array}\right], R^{(k)}\left(b^{(k)}\right)\right)_{\mathbb{H}} \\
& \quad=\left(R_{\mathrm{b}}^{(k)}\left(t b^{(k)}\right), R_{\mathrm{b}}^{(k)}\left(t^{*} a^{(k+1)}\right)\right)_{\mathbb{H}_{\mathrm{b}}}
\end{aligned}
$$

which shows that this is the discrete equivalent of the integration by parts formula (2.9).
We also define the spaces of discrete exact and closed forms for these two sequences:

$$
\begin{aligned}
\AA^{(k)} & :=\operatorname{im} \mathbb{E}^{(k-1)}, & \AA^{(k)} & :=\operatorname{ker} \mathbb{E}^{(k)}, \\
B^{*(k)} & :=\operatorname{im} \mathbb{E}^{*(k+1)}, & Z^{*(k)} & :=\operatorname{ker} \mathbb{E}^{*(k)}
\end{aligned}
$$

These satisfy again the usual orthogonality relations as stated in the following lemma.
Lemma 5.8. The subspaces of discrete exact and closed forms for the complex (5.20) and its adjoint complex (5.21) satisfy

$$
\stackrel{\circ}{B}^{(k) \perp}=Z^{*(k)}, \quad B^{*(k) \perp}=\check{Z}^{(k)} .
$$

Thus we see that the discrete complex (5.20) exactly mimics the complex (5.5).

### 5.2.3 The discrete Helmholtz decompositions for both pairs of complexes

We are now in the position to prove the discrete analogue of Theorem 2.4.
Theorem 5.3 (Discrete Helmholtz-Hodge decompositions). Define the following spaces of discrete harmonic forms

$$
\begin{aligned}
H_{t^{*}}^{(k)} & :=\left\{\boldsymbol{c}^{(k)} \in C^{(k)}(\Omega) \mid \mathbb{D}^{(k)} \boldsymbol{c}^{(k)}=0, \mathbb{D}^{*(k)} \boldsymbol{c}^{(k)}=0\right\}, \\
H_{t}^{(k)} & :=\left\{\boldsymbol{c}^{(k)} \in C^{(k)}(\Omega) \mid \mathbb{E}^{(k)} \boldsymbol{c}^{(k)}=0, \mathbb{E}^{*(k)} \boldsymbol{c}^{(k)}=0\right\} .
\end{aligned}
$$

We have the following two orthogonal decompositions (with respect to the inner product defined by $\left.\mathbb{H}^{(k)}\right)$ of $C^{(k)}(\Omega)$ :

$$
\begin{equation*}
C^{(k)}(\Omega)=B^{(k)} \oplus \stackrel{\circ}{B}^{*(k)} \oplus H_{t^{*}}^{(k)}=B^{(k)} \oplus\left(\mathbb{H}^{(k)}\right)^{-1} \tilde{\tilde{B}}^{(d-k)} \oplus H_{t^{*}}^{(k)} . \tag{5.22}
\end{equation*}
$$

and

$$
\begin{equation*}
C^{(k)}(\Omega)=\stackrel{\circ}{B}^{(k)} \oplus B^{*(k)} \oplus H_{t}^{(k)}=\dot{B}^{(k)} \oplus\left(\mathbb{H}^{(k)}\right)^{-1} \tilde{B}^{(d-k)} \oplus H_{t}^{(k)} \tag{5.23}
\end{equation*}
$$

where $\tilde{B}^{(d-k)}:=\mathbb{D}^{(k) T}\left(\tilde{C}^{(d-k-1)}(\Omega)\right)$ and $\tilde{B}^{(d-k)}:=\tilde{\mathbb{D}}^{(d-k-1)}\left(\tilde{C}^{(d-k-1)}(\Omega) \oplus \tilde{C}^{(d-k-1)}(\partial \Omega)\right)$.

### 5.3. Numerical solutions to the div-curl problems through discrete Hodge-Helmholtz decompositions

Proof. The proof is completely analogous to the proof of Theorem 2.4.
The discrete Helmholtz-Hodge decompositions show that a cochain $\boldsymbol{u}^{(k)} \in C^{(k)}(\Omega)$ can be written as ${ }^{4}$

$$
\boldsymbol{u}^{(k)}=\mathbb{D}^{(k-1)} \boldsymbol{a}^{(k-1)}+\left(\mathbb{H}^{(k)}\right)^{-1} \tilde{\mathbb{D}}^{(d-k-1)}\left[\begin{array}{c}
\tilde{\boldsymbol{b}}^{(d-k-1)}  \tag{5.24}\\
\mathbf{0}
\end{array}\right]+\boldsymbol{h}^{(k)},
$$

with $\boldsymbol{a}^{(k-1)} \in C^{(k-1)}(\Omega), \tilde{\boldsymbol{b}}^{(d-k-1)} \in \tilde{C}^{(d-k-1)}(\Omega)$ and $\boldsymbol{h}^{(k)} \in H_{t^{*}}^{(k)}$, or, as

$$
\boldsymbol{u}^{(k)}=\mathbb{D}^{(k-1)} \boldsymbol{a}^{(k-1)}+\left(\mathbb{H}^{(k)}\right)^{-1} \tilde{\mathbb{D}}^{(d-k-1)}\left[\begin{array}{l}
\tilde{\boldsymbol{b}}^{(d-k-1)}  \tag{5.25}\\
\tilde{\boldsymbol{b}}_{\mathrm{b}}^{(d-k-1)}
\end{array}\right]+\boldsymbol{h}^{(k)},
$$

with $\boldsymbol{a}^{(k-1)} \in C^{(k-1)}(\Omega), \mathbb{T}^{(k-1)} \boldsymbol{a}^{(k-1)}=\mathbf{0}, \tilde{\boldsymbol{b}}^{(d-k-1)} \in \tilde{C}^{(d-k-1)}(\Omega), \tilde{\boldsymbol{b}}_{\mathrm{b}}^{(d-k-1)} \in \tilde{C}^{(d-k-1)}(\partial \Omega)$ and $\boldsymbol{h}^{(k)} \in H_{t}^{(k)}$.

Now that we have established the two discrete Helmholtz-Hodge decompositions, we can use a similar reasoning as before to show the well-posedness of the discretized versions of Problem $\left(\mathrm{P}_{t^{*}}^{(k)}\right)$ and Problem $\left(\mathrm{P}_{t}^{(k)}\right)$ that we will introduce in the next section.

### 5.3 Numerical solutions to the div-curl problems through discrete Hodge-Helmholtz decompositions

In this section we will discretize Problems $\left(\mathrm{P}_{t^{*}}^{(k)}\right)$ and $\left(\mathrm{P}_{t}^{(k)}\right)$. We will use a similar reasoning as in Section 5.1 to show that these problems result in non-singular linear systems. Furthermore, we will show that all the properties of the solutions of Problems $\left(\mathrm{P}_{t^{*}}^{(k)}\right)$ and $\left(\mathrm{P}_{t}^{(k)}\right)$ are preserved in the discretization, i.e., the solutions can be written in terms of a discrete Helmholtz-Hodge decomposition, where the discrete potentials satsify the same requirements with respect to their trace and (discretized) exterior derivative as the potentials of Problems $\left(\mathrm{P}_{t^{*}}^{(k)}\right)$ and $\left(\mathrm{P}_{t}^{(k)}\right)$ do (as described in Theorem 5.1 and Theorem 5.2).

### 5.3.1 The problem with natural boundary conditions

We start with the discretization of Problem $\left(\mathrm{P}_{t^{*}}^{(k)}\right)$.
Problem $\left(\mathrm{DP}_{t^{*}}\right)$. Let $f^{(k-1)}, g^{(k+1)}$ and $r_{\mathrm{b}}^{(k-1)}$ be the given data for Problem $\left(\mathrm{P}_{t^{*}}^{(k)}\right)$ as stated in Section 5.1.2. Define the discrete data ${ }^{5}$ as

$$
\begin{aligned}
\tilde{\boldsymbol{f}}^{(d-k+1)} & :=\tilde{R}^{(d-k+1)}\left((-1)^{k} \star f^{(k-1)}\right), \\
\boldsymbol{g}^{(k+1)} & :=R^{(k+1)}\left(g^{(k+1)}\right) \\
\tilde{\boldsymbol{r}}_{\mathrm{b}}^{(d-k)} & :=\tilde{R}_{\mathrm{b}}^{(d-k)}\left(\star_{\mathrm{b}} r_{\mathrm{b}}^{(k-1)}\right)
\end{aligned}
$$

[^39]Find $\boldsymbol{u}^{(k)} \in C^{(k)}(\Omega)$ such that

$$
\tilde{\mathbb{D}}^{(d-k)}\left[\begin{array}{c}
\mathbb{H}^{(k)} \boldsymbol{u}^{(k)} \\
\tilde{\boldsymbol{r}}_{\mathrm{b}}^{(d-k)}
\end{array}\right]=\tilde{\boldsymbol{f}}^{(d-k+1)}, \quad \quad \mathbb{D}^{(k)} \boldsymbol{u}^{(k)}=\boldsymbol{g}^{(k+1)}
$$

The fact that $f^{(k-1)} \in \operatorname{im}\left(d^{*}\right)$ and $g^{(k+1)} \in \operatorname{im}(d)$ implies the following for the discrete data.

Lemma 5.9. The discrete data of Problem $\left(\mathrm{DP}_{t^{*}}\right)$ satisfies

$$
\begin{align*}
\tilde{\mathbb{D}}^{(d-k+1)}\left[\begin{array}{c}
\tilde{\boldsymbol{f}}^{(d-k+1)} \\
\tilde{\boldsymbol{f}}_{\mathrm{b}}^{(d-k+1)}
\end{array}\right] & =\mathbf{0}^{(k-2)},  \tag{5.26a}\\
\mathbb{D}^{(k+1)} \boldsymbol{g}^{(k+1)} & =\mathbf{0}^{(k+2)},  \tag{5.26b}\\
\forall \boldsymbol{z}^{(k-1)} \in Z^{(k-1)}: \quad\left((-1)^{k} \tilde{\boldsymbol{f}}+\mathbb{T}^{(k-1) T} \tilde{\boldsymbol{r}}_{\mathrm{b}}\right)^{T} \boldsymbol{z}^{(k-1)} & =0, \tag{5.26c}
\end{align*}
$$

where $\tilde{\boldsymbol{f}}_{\mathrm{b}}^{(d-k+1)}:=\tilde{\mathbb{D}}_{\mathrm{b}}^{(d-k)} \tilde{\boldsymbol{r}}_{\mathrm{b}}^{(d-k)}$. Moreover, we actually have $\boldsymbol{g}^{(k+1)} \in B^{(k+1)}$.
Proof. Note that $\tilde{\mathbb{D}}_{\mathrm{b}}^{(d-k)} \tilde{\boldsymbol{r}}_{\mathrm{b}}^{(d-k)}=\tilde{R}_{\mathrm{b}}^{(d-k+1)}\left(\tilde{d}_{\mathrm{b}} \star_{\mathrm{b}} r_{\mathrm{b}}^{(k-1)}\right)$. By definition of $\tilde{d}_{\mathrm{b}}$ and $\star_{\mathrm{b}}$ and the fact that $\tilde{d}_{\mathrm{b}} \tilde{t}=\tilde{t} \tilde{d}$ it follows that

$$
\tilde{d}_{\mathrm{b}} \star_{\mathrm{b}} r_{\mathrm{b}}^{(k-1)}=\tilde{d}_{\mathrm{b}} \tilde{t} \star u^{(k)}=\tilde{t} \tilde{d} \star u^{(k)}=(-1)^{k} \tilde{t}\left(\star d^{\star} u^{(k)}\right)=\tilde{t}\left((-1)^{k} \star f^{(k-1)}\right) .
$$

Therefore it follows from Proposition 3.11 that

$$
\begin{aligned}
\tilde{\mathbb{D}}^{(d-k+1)}\left[\begin{array}{c}
\tilde{\boldsymbol{f}}^{(d-k+1)} \\
\tilde{\boldsymbol{f}}_{\mathrm{b}}^{(d-k+1)}
\end{array}\right] & =\tilde{\mathbb{D}}^{(d-k+1)}\left[\begin{array}{c}
\tilde{R}^{(d-k+1)}\left((-1)^{k} \star f^{(k-1)}\right) \\
\tilde{R}_{\mathrm{b}}^{(d-k+1)}\left(\tilde{t}\left((-1)^{k} \star f^{(k-1)}\right)\right)
\end{array}\right] \\
& =\tilde{R}^{(d-k+2)}\left(\tilde{d}^{(d-k+1)}\left((-1)^{k} \star f^{(k-1)}\right)\right) \\
& =0,
\end{aligned}
$$

because $d^{*} f^{(k-1)}=0$.
Relation (5.26c) follows because by Proposition 3.11 we have

$$
\begin{aligned}
\tilde{\boldsymbol{f}}^{(d-k+1)} & =\tilde{R}^{(d-k+1)}\left(\tilde{d} \star u^{(k)}\right) \\
& =\tilde{\mathbb{D}}^{(d-k)}\left[\begin{array}{c}
\tilde{R}^{(d-k)}\left(\star u^{(k)}\right) \\
\tilde{R}_{\mathrm{b}}^{(d-k)}\left(\tilde{t} \star u^{(k)}\right)
\end{array}\right] \\
& =(-1)^{k}\left(\mathbb{D}^{(k-1) T} \tilde{R}^{(d-k)}\left(\star u^{(k)}\right)-\mathbb{T}^{(k-1) T} \tilde{\boldsymbol{r}}_{\mathrm{b}}^{(d-k)}\right)
\end{aligned}
$$

which multiplied from the left by $\boldsymbol{z}^{(k-1) T}$ with $\boldsymbol{z}^{(k-1)} \in Z^{(k-1)}$ gives $(5.26 \mathrm{c})$.
Finally $\boldsymbol{g}^{(k+1)} \in B^{(k+1)}$ because

$$
\boldsymbol{g}^{(k+1)}=R^{(k+1)}\left(g^{(k+1)}\right)=R^{(k+1)}\left(d u^{(k)}\right)=\mathbb{D}^{(k)} R^{(k)}\left(u^{(k)}\right),
$$

which obviously also implies (5.26b).
The above result will be useful in what follows.

### 5.3.1.1 Discrete Hodge-Laplace problem for the potential $\boldsymbol{a}^{(k-1)}$

To formulate the discrete Hodge-Laplace problem for the potential we first need to calculate a basis for the space of discrete harmonic forms. To calculate, for example, a basis for $H_{t}^{(k)}$ we can calculate the kernels of $\mathbb{D}^{(k)}$ and $\mathbb{D}^{*(k)}$ and subsequently determine a basis for their intersection. We denote by $M_{t}^{(k)}$ the matrix with as columns the vectors that comprise such a basis. Analogously, we define $M_{t^{*}}^{(k)}$ to be a similar matrix corresponding to $H_{t^{*}}^{(k)}$. Note that every element of $H_{t}^{(k)}$ can be written as $M_{t}^{(k)} \boldsymbol{p}$, where $\boldsymbol{p} \in \mathbb{R}^{N_{t}^{(k)}}$ with $N_{t}^{(k)}$ equal to the dimension of $H_{t}^{(k)}$, and, similarly, every element of $H_{t^{*}}^{(k)}$ can be written as $M_{t *}^{(k)} \boldsymbol{p}$ where $\boldsymbol{p} \in \mathbb{R}^{N_{t^{*}}^{(k)}}$.

The discrete Hodge-Laplace problem for the discrete potential $\boldsymbol{a}^{(k-1)}$ is given as follows.
Problem $\left(\mathrm{DP}_{t^{*}}^{a}\right)$. Given $\tilde{\boldsymbol{f}}^{(d-k+1)} \in \tilde{C}^{(d-k+1)}(\Omega)$ and $\tilde{\boldsymbol{r}}_{\mathrm{b}}^{(d-k)} \in \tilde{C}^{(d-k)}(\partial \Omega)$ satisfying the requirements (5.26a) and (5.26c). Find $\boldsymbol{\phi}^{(k-2)} \in C^{(k-2)}(\Omega), \boldsymbol{a}^{(k-1)} \in C^{(k-1)}(\Omega)$ and $\boldsymbol{p}_{t^{*}}^{(k-1)} \in \mathbb{R}^{N_{t^{*}}^{(k-1)}}$ such that

$$
\left[\begin{array}{ccc}
-\mathbb{H}^{(k-2)} & \mathbb{D}^{(k-2) T} \mathbb{H}^{(k-1)} & 0 \\
\mathbb{H}^{(k-1)} \mathbb{D}^{(k-2)} & \mathbb{D}^{(k-1) T} \mathbb{H}^{(k)} \mathbb{D}^{(k-1)} & \mathbb{H}^{(k-1)} \mathbb{M}_{t^{*}}^{(k-1)} \\
0 & \mathbb{M}_{t^{*}}^{(k-1) T} \mathbb{H}^{(k-1)} & 0
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{\phi} \\
\boldsymbol{a} \\
\boldsymbol{p}_{t^{*}}
\end{array}\right]=\left[\begin{array}{c}
\tilde{\mathbf{0}} \\
(-1)^{k} \tilde{\boldsymbol{f}}+\mathbb{T}^{(k-1) T} \tilde{\boldsymbol{r}}_{\mathrm{b}} \\
\mathbf{0}
\end{array}\right] .
$$

The following lemma is the discrete version of Lemma 5.1.
Lemma 5.10. Problem $\left(\mathrm{DP}_{t^{*}}^{a}\right)$ has a unique solution $\left(\phi^{(k-2)}, \boldsymbol{a}^{(k-1)}, \boldsymbol{p}_{t^{*}}^{(k-1)}\right)$. Moreover, we have $\boldsymbol{\phi}^{(k-2)}=\mathbf{0}^{(k-2)}$, and hence $\mathbb{D}^{(k-2) T} \mathbb{H}^{(k-1)} \boldsymbol{a}^{(k-1)}=\tilde{\mathbf{0}}^{(d-k+2)}$, and, $\boldsymbol{p}_{t^{*}}^{(k-1)}=\mathbf{0}$.

Proof. Note that we can rewrite the linear system in the following weak from:

$$
\begin{array}{ccc}
-(\boldsymbol{\sigma}, \boldsymbol{\phi})_{\mathbb{H}}+\left(\mathbb{D}^{(k-2)} \boldsymbol{\sigma}, \boldsymbol{a}\right)_{\mathbb{H}}=0 & \forall \boldsymbol{\sigma} \in C^{(k-2)}(\Omega), \\
\left(\boldsymbol{v}, \mathbb{D}^{(k-2)} \boldsymbol{a}\right)_{\mathbb{H}}+\left(\mathbb{D}^{(k-1)} \boldsymbol{v}, \mathbb{D}^{(k-1)} \boldsymbol{a}\right)_{\mathbb{H}} & \\
+\left(\boldsymbol{v}, \mathbb{M}_{t^{*}}^{(k-1)} \boldsymbol{p}_{t^{*}}\right)_{\mathbb{H}}=(-1)^{k} \boldsymbol{v}^{T} \tilde{\boldsymbol{f}}+\left(\mathbb{T}^{(k-1)} \boldsymbol{v}\right)^{T} \tilde{\boldsymbol{r}}_{\mathrm{b}} & \forall \boldsymbol{v} \in C^{(k-1)}(\Omega), \\
\left(\mathbb{M}_{t^{*}}^{(k-1)} \boldsymbol{q}_{t^{*}}, \boldsymbol{a}\right)_{\mathbb{H}}=0 & \forall \boldsymbol{q}_{t^{*}} \in \mathbb{R}^{N_{t^{*}}^{(k-1)}} .
\end{array}
$$

This is a version of Problem (GP) corresponding to the exact sequence (5.18) (with $k$ replaced by $k-1$ ). Therefore, the problem admits a unique solution.

To show $\boldsymbol{p}_{t^{*}}=\mathbf{0}$, we multiply the second equation of the linear system from the left by $\left(\mathbb{M}_{t^{*}}^{(k-1)} \boldsymbol{p}_{t^{*}}\right)^{T}$. Note that $\mathbb{M}_{t^{*}}^{(k-1) T}\left((-1)^{k} \boldsymbol{f}^{(k+1)}+\mathbb{T}^{(k-1) T} \tilde{\boldsymbol{r}}_{\mathrm{b}}^{(d-k)}\right)=0$ by (5.26c). Furthermore we have $\mathbb{M}_{t^{*}}^{(k-1) T} \mathbb{H}^{(k-1)} \mathbb{D}^{(k-2)} \equiv 0$ and $\mathbb{D}^{(k-1)} \mathbb{M}_{t^{*}}^{(k-1)} \equiv 0$ by the discrete Helmholtz-Hodge decomposition (5.22) (with $k$ replaced by $k-1$ ). This shows that $\mathbb{M}_{t^{*}}^{(k-1)} \boldsymbol{p}_{t^{*}}=\mathbf{0}^{(k+1)}$ and hence $\boldsymbol{p}_{t^{*}}^{(k-1)}=\mathbf{0}$, because $\mathbb{M}_{t^{*}}^{(k-1)}$ is full rank.

Multiply the second equation from the left by $\left(\mathbb{D}^{(k-2)} \phi^{(k-2)}\right)^{T}$ to find, using $\mathbb{D}^{(k-2) T} \mathbb{H}^{(k-1)} \mathbb{M}_{t^{*}}^{(k-1)} \equiv 0$ and $\mathbb{D}^{(k-1)} \mathbb{D}^{(k-2)} \equiv 0$, that $\mathbb{D}^{(k-2)} \phi^{(k-2)}=\mathbf{0}^{(k-1)}$. This implies that $\phi^{(k-2)} \in Z^{(k-2)}$. Moreover, by the first equation we have $\phi^{(k-2)} \in \AA^{*(k-2)}=Z^{(k-2) \perp}$, hence we must have $\phi^{(k-2)}=\mathbf{0}^{(k-2)}$ and $\mathbb{D}^{(k-2) T} \mathbb{H}^{(k-1)} \boldsymbol{a}^{(k-1)}=\tilde{\mathbf{0}}^{(d-k+2)}$.

We see that the potential $\boldsymbol{a}^{(k+1)}$ is in all ways the discrete equivalent of $a^{(k+1)}$.

### 5.3.1.2 Discrete Hodge-Laplace problem for the potential $\boldsymbol{b}^{(k+1)}$

The discrete potential $\boldsymbol{b}^{(k+1)}$ solves the following Hodge-Laplace problem.
Problem $\left(\mathrm{DP}_{t^{*}}^{b}\right)$. Given $\boldsymbol{g}^{(k+1)} \in C^{(k+1)}(\Omega)$ satisfying $\boldsymbol{g}^{(k+1)} \in B^{(k+1)}$. Find $\boldsymbol{\psi}^{(k)} \in$ $C^{(k)}(\Omega), \boldsymbol{b}^{(k+1)} \in C^{(k+1)}(\Omega)$ and $\boldsymbol{q}_{t^{*}}^{(k+1)} \in \mathbb{R}^{N_{t^{*}}^{(k+1)}}$ such that

$$
\left[\begin{array}{ccc}
-\mathbb{H}^{(k)} & \mathbb{D}^{(k) T} \mathbb{H}^{(k+1)} & 0 \\
\mathbb{H}^{(k+1)} \mathbb{D}^{(k)} & \mathbb{D}^{(k+1) T} \mathbb{H}^{(k+2)} \mathbb{D}^{(k+1)} & \mathbb{H}^{(k+1)} \mathbb{M}_{t^{*}}^{(k+1)} \\
0 & \mathbb{M}_{t^{*}}^{(k+1) T} \mathbb{H}^{(k+1)} & 0
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{\psi} \\
\boldsymbol{b} \\
\boldsymbol{q}_{t^{*}}
\end{array}\right]=\left[\begin{array}{c}
\tilde{\mathbf{0}} \\
\mathbb{H}^{(k+1)} \boldsymbol{g} \\
\mathbf{0}
\end{array}\right]
$$

The following lemma is the discrete version of Lemma 5.2.
Lemma 5.11. Problem ( $P_{t^{*}}^{b}$ ) has a unique solution $\left(\boldsymbol{\psi}^{(k)}, \boldsymbol{b}^{(k+1)}, \boldsymbol{q}_{t^{*}}^{(k+1)}\right)$. Moreover, we have $\mathbb{D}^{(k+1)} \boldsymbol{b}^{(k+1)}=\mathbf{0}^{(k+2)}$ and $\boldsymbol{q}_{t^{*}}^{(k+1)}=\mathbf{0}$.

Proof. Well-posedness can be shown as it was shown for Problem ( $\mathrm{DP}_{t^{*}}^{a}$ ).
Multiplying the second line from the left by $\left(\left(\mathbb{H}^{(k+1)}\right)^{-1} \mathbb{D}^{(k+1) T} \tilde{\boldsymbol{v}}^{(d-k-2)}\right)^{T}$, with any $\tilde{\boldsymbol{v}}^{(d-k-2)} \in \tilde{C}^{(d-k-2)}(\Omega)$ we find, using $\mathbb{D}^{(k+1)} \mathbb{D}^{(k)} \equiv 0$ and $\mathbb{D}^{(k+1)} \mathbb{M}_{t^{*}}^{(k+1)} \equiv 0$, that

$$
\left(\mathbb{D}^{(k+1)}\left(\mathbb{H}^{(k+1)}\right)^{-1} \mathbb{D}^{(k+1) T} \tilde{\boldsymbol{v}}^{(d-k-2)}, \mathbb{D}^{(k+1)} \boldsymbol{b}^{(k+1)}\right)_{\mathbb{H}}=0 .
$$

Note that $\left(\mathbb{H}^{(k+1)}\right)^{-1} \mathbb{D}^{(k+1) T} \tilde{\boldsymbol{v}}^{(d-k-2)} \in \dot{B}^{*(k+1)}=Z^{(k) \perp}$. From $\mathbb{D}^{(k+1)}\left(Z^{(k)}\right) \equiv 0$ it therefore follows that for any $\boldsymbol{v}^{(k+1)} \in C^{(k+1)}(\Omega)$

$$
\left(\mathbb{D}^{(k+1)} \boldsymbol{v}^{(k+1)}, \mathbb{D}^{(k+1)} \boldsymbol{b}^{(k+1)}\right)_{\mathbb{H}}=0,
$$

and hence $\mathbb{D}^{(k+1)} \boldsymbol{b}^{(k+1)}=\mathbf{0}^{(k+2)}$.
Just as in Lemma 5.10, multiplying the second equation from the left by $\left(\mathbb{M}_{t^{*}}^{(k+1)} \boldsymbol{q}_{t^{*}}\right)^{T}$ shows that $\boldsymbol{q}_{t^{*}}=\mathbf{0}$. This time we use that $\left(\mathbb{M}_{t^{*}}^{(k+1)} \boldsymbol{q}_{t^{\star}}, \boldsymbol{g}\right)_{\mathbb{H}}=0$ because $\boldsymbol{g} \in B^{(k+1)}$.

### 5.3.1.3 The solution to Problem ( $\mathrm{DP}_{t^{*}}$ )

We can now state the solution to Problem $\left(\mathrm{DP}_{t^{*}}\right)$ in terms of the discrete potentials.
Theorem 5.4. The solution to Problem $\left(\mathrm{DP}_{t^{*}}\right)$ is given by

$$
\begin{aligned}
\boldsymbol{u}^{(k)} & =\mathbb{D}^{(k-1)} \boldsymbol{a}^{(k-1)}+\left(\mathbb{H}^{(k)}\right)^{-1} \tilde{\mathbb{D}}^{(d-k-1)}\left[\begin{array}{c}
\mathbb{H}^{(k+1)} \boldsymbol{b}^{(k+1)} \\
\mathbf{0}
\end{array}\right]+\boldsymbol{h}^{(k)} \\
& =\mathbb{D}^{(k-1)} \boldsymbol{a}^{(k-1)}+\boldsymbol{\psi}^{(k)}+\boldsymbol{h}^{(k)},
\end{aligned}
$$

where $\boldsymbol{a}^{(k-1)}$ is the solution of Problem $\left(\mathrm{DP}_{t^{*}}^{a}\right),\left(\boldsymbol{\psi}^{(k)}, \boldsymbol{b}^{(k+1)}\right)$ is the solution of Problem $\left(\mathrm{DP}_{t^{*}}^{b}\right)$ and $\boldsymbol{h}^{(k)} \in H_{t^{*}}^{(k)}$. Thus $\mathbb{D}^{(k-2) T} \mathbb{H}^{(k-1)} \boldsymbol{a}^{(k-1)}=\mathbf{0}, \mathbb{D}^{(k+1)} \boldsymbol{b}^{(k+1)}=\mathbf{0}$ and the solution is unique up to elements of $H_{t^{*}}^{(k)}$.

Proof. By the discrete Helmholtz-Hodge decomposition (5.24) we can write any $\boldsymbol{u}^{(k)} \epsilon$ $C^{(k)}(\Omega)$ as $\boldsymbol{u}^{(k)}=\mathbb{D}^{(k-1)} \boldsymbol{a}^{(k-1)}+\left(\mathbb{H}^{(k)}\right)^{-1} \mathbb{D}^{(k) T} \mathbb{H}^{(k+1)} \boldsymbol{b}^{(k+1)}+\boldsymbol{h}^{(k)}$. Substituting this into $\left(\mathrm{DP}_{t^{*}}\right)$ we obtain

$$
\begin{aligned}
\mathbb{D}^{(k-1) T} \mathbb{H}^{(k)} \mathbb{D}^{(k-1)} \boldsymbol{a}^{(k-1)} & =(-1)^{k} \tilde{\boldsymbol{f}}^{(d-k+1)}+\mathbb{T}^{(k-1) T} \tilde{\boldsymbol{r}}_{\mathrm{b}}^{(d-k)}, \\
\mathbb{D}^{(k)}\left(\mathbb{H}^{(k)}\right)^{-1} \mathbb{D}^{(k) T} \mathbb{H}^{(k+1)} \boldsymbol{b}^{(k+1)} & =\boldsymbol{g}^{(k+1)}
\end{aligned}
$$

These equations are satisfied if $\boldsymbol{a}^{(k-1)}$ and $\boldsymbol{b}^{(k+1)}$ are the solutions of, respectively, Problem $\left(\mathrm{DP}_{t^{*}}^{a}\right)$ and Problem $\left(\mathrm{DP}_{t^{*}}^{b}\right)$.

Furthermore, suppose there is a second solution to $\left(\mathrm{DP}_{t^{*}}\right)$. The difference of the solutions is a solution to $\left(\mathrm{DP}_{t^{*}}\right)$ with vanishing $\tilde{\boldsymbol{f}}^{(d-k+1)}, \boldsymbol{g}^{(k+1)}$ and $\tilde{\boldsymbol{r}}_{\mathrm{b}}^{(d-k)}$, which implies that it is an element of $H_{t^{*}}^{(k)}$.

This shows that we can find $\boldsymbol{u}^{(k)}$ the solution to Problem $\left(\mathrm{DP}_{t^{*}}\right)$ by solving nonsingular linear systems for the two potentials $\boldsymbol{a}^{(k-1)}$ and $\boldsymbol{b}^{(k+1)}$.

### 5.3.2 The problem with essential boundary conditions

Problem $\left(\mathrm{P}_{t}\right)$ with essential boundary conditions is discretized as follows.
Problem $\left(\mathrm{DP}_{t}\right)$. Let $f^{(k-1)}, g^{(k+1)}$ and $r_{\mathrm{b}}^{(k)}$ be the given data for Problem $\left(P_{t^{*}}^{(k)}\right)$ as stated in Section 5.1.3. Define the discrete data as

$$
\begin{aligned}
\tilde{\boldsymbol{f}}^{(d-k+1)} & :=\tilde{R}^{(d-k+1)}\left((-1)^{k} \star f^{(k-1)}\right), \\
\boldsymbol{g}^{(k+1)} & :=R^{(k+1)}\left(g^{(k+1)}\right), \\
\boldsymbol{r}_{\mathrm{b}}^{(k)} & :=R_{\mathrm{b}}^{(k)}\left(r_{\mathrm{b}}^{(k)}\right)
\end{aligned}
$$

Find $\boldsymbol{u}^{(k)} \in C^{(k)}(\Omega), \boldsymbol{\eta}_{\mathrm{b}}^{(d-k)} \in \tilde{C}^{(d-k)}(\partial \Omega)$ such that

$$
\tilde{\mathbb{D}}^{(d-k)}\left[\begin{array}{c}
\mathbb{H}^{(k)} \boldsymbol{u}^{(k)} \\
\tilde{\boldsymbol{\eta}}_{\mathrm{b}}^{(d-k)}
\end{array}\right]=\tilde{\boldsymbol{f}}^{(d-k+1)}, \quad \mathbb{D}^{(k)} \boldsymbol{u}^{(k)}=\boldsymbol{g}^{(k+1)}, \quad \mathbb{T}^{(k)} \boldsymbol{u}^{(k)}=\boldsymbol{r}_{\mathrm{b}}^{(k)}
$$

Note that we can equivalently search only $\boldsymbol{u}^{(k)} \in C^{(k)}(\Omega)$ that satisfies ${ }^{6}$

$$
\mathbb{P}_{\Omega \backslash \Omega}^{(k-1)} \mathbb{D}^{(k-1) T} \mathbb{H}^{(k)} \boldsymbol{u}^{(k)}=(-1)^{k} \mathbb{P}_{\Omega \backslash \partial \Omega}^{(k-1)} \tilde{\boldsymbol{f}}^{(d-k+1)}, \quad \mathbb{D}^{(k)} \boldsymbol{u}^{(k)}=\boldsymbol{g}^{(k+1)}, \quad \mathbb{T}^{(k)} \boldsymbol{u}^{(k)}=\boldsymbol{r}_{\mathrm{b}}^{(k)}
$$

because for each equation corresponding to a boundary cell, there is a single variable in $\tilde{\boldsymbol{\eta}}_{\mathrm{b}}^{(d-k)}$. Thus, the removed equations can, after $\boldsymbol{u}^{(k)}$ has been found, be solved for and they are all decoupled.

From the properties of the original data $f^{(k-1)}, g^{(k+1)}$ and $r_{\mathrm{b}}^{(k)}$ we derive similar properties for the discrete data.

[^40]Lemma 5.12. The discrete data of Problem $\left(\mathrm{DP}_{t}\right)$ satisfies

$$
\begin{align*}
\mathbb{P}_{\Omega \backslash \partial \Omega}^{(k-2)} \mathbb{D}^{(k-2) T} \tilde{\boldsymbol{f}}^{(d-k+1)} & =\mathbf{0}^{(k-2)},  \tag{5.27a}\\
\mathbb{D}^{(k+1)} \boldsymbol{g}^{(k+1)} & =\mathbf{0}^{(k+2)},  \tag{5.27b}\\
\mathbb{T}^{(k+1)} \boldsymbol{g}^{(k+1)} & =\mathbb{D}_{\mathrm{b}}^{(k)} \boldsymbol{r}_{\mathrm{b}}^{(k)},  \tag{5.27c}\\
\forall \boldsymbol{z}^{(k-1)} \in \AA^{(k-1)}: \quad \tilde{\boldsymbol{f}}^{(d-k+1) T} \boldsymbol{z}^{(k-1)} & =0 . \tag{5.27d}
\end{align*}
$$

Moreover, we have $\boldsymbol{g}^{(k+1)} \in B^{(k)}$.
Proof. Multiplying (5.26a) by $\mathbb{P}_{\Omega \backslash \partial \Omega}^{(k-2)}$ gives (5.27a) and (5.27b) was shown before. Moreover, ( 5.27 d ) follows immediately from ( 5.26 c ).

To find (5.27c) we use the mutual commutativity of the de Rham maps, exterior derivative and trace operator:

$$
\mathbb{T}^{(k+1)} \boldsymbol{g}^{(k+1)}=\mathbb{T}^{(k+1)} R^{(k+1)}\left(g^{(k+1)}\right)=R_{\mathrm{b}}^{(k+1)}\left(t d u^{(k)}\right)=R_{\mathrm{b}}^{(k+1)}\left(d_{\mathrm{b}} r^{(k)}\right)=\mathbb{D}_{\mathrm{b}}^{(k)} \boldsymbol{r}_{\mathrm{b}}^{(k)} .
$$

These properties of the discrete data are important for what follows.

### 5.3.2.1 Discrete Hodge-Laplace problem for the potential $\boldsymbol{a}^{(k-1)}$

We formulate discretized Hodge-Laplace problems for the potentials. We start again with the problem for the potential $\boldsymbol{a}^{(k-1)}$.

Problem $\left(\mathrm{DP}_{t}^{a}\right)$. Given $\tilde{\boldsymbol{f}}^{(d-k+1)} \in \tilde{C}^{(d-k+1)}(\Omega)$ satisfying the requirement (5.27a). Find $\phi^{(k-2)} \in C^{(k-2)}(\Omega), \boldsymbol{a}^{(k-1)} \in C^{(k-1)}(\Omega), \tilde{\boldsymbol{\mu}}_{\mathrm{b}}^{(d-k+1)} \in \tilde{C}^{(d-k+1)}(\partial \Omega), \tilde{\boldsymbol{\eta}}_{\mathrm{b}}^{(d-k)} \in \tilde{C}^{(d-k)}(\partial \Omega)$ and $\boldsymbol{p}_{t}^{(k-1)} \in \mathbb{R}^{N_{t}^{(k-1)}}$ such that

$$
\left[\begin{array}{ccccc}
-\mathbb{H}^{(k-2)} & -\mathbb{D}^{(k-2) T} \mathbb{H}^{(k-1)} & \mathbb{T}^{(k-2) T} & 0 & 0 \\
-\mathbb{H}^{(k-1)} \mathbb{D}^{(k-2)} & -\mathbb{D}^{(k-1) T} \mathbb{H}^{(k)} \mathbb{D}^{(k-1)} & 0 & \mathbb{T}^{(k-1) T} & \mathbb{H}^{(k-1)} \mathbb{M}_{t}^{(k-1)} \\
\mathbb{T}^{(k-2)} & 0 & 0 & 0 & 0 \\
0 & \mathbb{T}^{(k-1)} & 0 & 0 & 0 \\
0 & \mathbb{M}_{t}^{(k-1) T} \mathbb{H}^{(k-1)} & 0 & 0 & 0
\end{array}\right]\left[\begin{array}{c}
\phi \\
\boldsymbol{a} \\
\tilde{\boldsymbol{\mu}}_{\mathrm{b}} \\
\tilde{\boldsymbol{\eta}}_{\mathrm{b}} \\
\boldsymbol{p}_{t}
\end{array}\right]=\left[\begin{array}{c}
\tilde{\mathbf{0}} \\
(-1)^{k} \tilde{\boldsymbol{f}} \\
\mathbf{0}_{\mathrm{b}} \\
\mathbf{0}_{\mathrm{b}} \\
\mathbf{0}
\end{array}\right]
$$

Properties of the solution to this problem are stated in the following lemma, which is the discrete version of Lemma 5.3.

Lemma 5.13. Problem ( $\left.\mathrm{P}_{t}^{a}\right)$ has a unique solution $\left(\phi^{(k-2)}, \boldsymbol{a}^{(k-1)}, \tilde{\boldsymbol{\mu}}_{\mathrm{b}}^{(d-k+1)}, \tilde{\boldsymbol{\eta}}_{\mathrm{b}}^{(d-k)}, \boldsymbol{p}_{t}^{(k-1)}\right)$. Moreover, we have $\phi^{(k-2)}=\mathbf{0}^{(k-2)}$, and hence

$$
\tilde{\mathbb{D}}^{(d-k-1)}\left[\begin{array}{c}
\mathbb{H}^{(k-1)} \boldsymbol{a}^{(k-1)} \\
\tilde{\boldsymbol{\mu}}_{\mathrm{b}}^{(d-k+1)}
\end{array}\right]=\tilde{\mathbf{0}}^{(d-k)} .
$$

Furthermore we have again $\boldsymbol{p}_{t}=\mathbf{0}$.

Proof. If we enforce the boundary condition in the discrete spaces, then the problem can be written in weak form as

$$
\begin{array}{rlrl}
-(\boldsymbol{\sigma}, \boldsymbol{\phi})_{\mathbb{H}}+\left(\mathbb{D}^{(k-2)} \boldsymbol{\sigma}, \boldsymbol{a}\right)_{\mathbb{H}} & =0 & \forall \boldsymbol{\sigma} \in \dot{C}^{(k-2)}(\Omega), \\
\left(\boldsymbol{v}, \mathbb{D}^{(k-2)} \boldsymbol{a}\right)_{\mathbb{H}}+\left(\mathbb{D}^{(k-1)} \boldsymbol{v}, \mathbb{D}^{(k-1)} \boldsymbol{a}\right)_{\mathbb{H}} & \\
+\left(\boldsymbol{v}, \mathbb{M}_{t}^{(k-1)} \boldsymbol{p}_{t}\right)_{\mathbb{H}}=(-1)^{k} \boldsymbol{v}^{T} \tilde{\boldsymbol{f}} & \forall \boldsymbol{v} \in \dot{C}^{(k-1)}(\Omega), \\
\left(\mathbb{M}_{t}^{(k-1)} \boldsymbol{q}_{t}, \boldsymbol{a}\right)_{\mathbb{H}}=0 & \forall \boldsymbol{q}_{t} \in \mathbb{R}^{N_{t}^{(k-1)}},
\end{array}
$$

where $\dot{C}^{(k)}(\Omega):=\left\{\boldsymbol{a}^{(k)} \in C^{(k)}(\Omega) \mid \mathbb{T}^{(k)} \boldsymbol{a}^{(k)}=\mathbf{0}_{\mathrm{b}}\right\}$. This is problem (GP) for the Hilbert complex $\left(\dot{C}^{(l)}(\Omega), \mathbb{D}^{(l)}\right)$ with $l=k-2, k-1, k$. Therefore the problem $\left(\mathrm{DP}_{t}^{a}\right)$ is well-posed.

We consider again the matrix formulation. The first line implies that $\phi^{(k-2)} \in B^{*(k-2)}$. Multiplying the second equation from the left by $\left(\mathbb{D}^{(k-2)} \mathbb{P}_{\Omega \text { 人 }}^{(k-2)} \phi^{(k-2)}\right)^{T}$ we obtain, using $\mathbb{T}^{(k-1)} \mathbb{D}^{(k-2)} \mathbb{P}_{\Omega \backslash \partial \Omega}^{(k-2)}=\mathbb{D}_{\mathrm{b}}^{(k-2)} \mathbb{T}^{(k-2)} \mathbb{P}_{\Omega \backslash \partial \Omega}^{(k-2)} \equiv 0,(5.27 \mathrm{a})$ and $\left(\mathbb{D}^{(k-2)}\right)^{T} \mathbb{H}^{(k-1)} \mathbb{M}_{t}^{(k-1)} \equiv 0$,

$$
\left(\mathbb{D}^{(k-2)} \mathbb{P}_{\Omega \backslash \partial \Omega}^{(k-2)} \phi^{(k-2)}, \mathbb{D}^{(k-2)} \phi^{(k-2)}\right)_{\mathbb{H}}=0 .
$$

Finally, because $\mathbb{T}^{(k-2)} \phi^{(k-2)}=\mathbf{0}_{\mathrm{b}}^{(k-2)}$ we have $\mathbb{P}_{\Omega \backslash \Omega \Omega}^{(k-2)} \phi^{(k-2)}=\phi^{(k-2)}$, and therefore $\mathbb{D}^{(k-2)} \boldsymbol{\phi}^{(k-2)}=\mathbf{0}^{(k-1)}$. Thus $\phi^{(k-2)} \in \AA^{(k-2)}=B^{\not(k-2) \perp}$. Together with the fact that $\phi^{(k-2)} \in B^{*(k-2)}$ this implies that $\boldsymbol{\phi}^{(k-2)}=\mathbf{0}^{(k-2)}$.

Multiplying again the second line by $\left(\mathbb{M}_{t}^{(k-1)} \boldsymbol{p}_{t}\right)^{T}$ from the left gives $\boldsymbol{p}_{t}=\mathbf{0}$. This time we additionally need that $\mathbb{T}^{(k-1)} \mathbb{M}_{t}^{(k-1)} \equiv 0$.

### 5.3.2.2 Discrete Hodge-Laplace problem for the potential $\boldsymbol{b}^{(k+1)}$

We continue with the problem for the discrete potential $\boldsymbol{b}^{(k+1)}$.
Problem $\left(\mathrm{DP}_{t}^{b}\right)$. Given $\boldsymbol{g}^{(k+1)} \in C^{(k+1)}(\Omega)$ and $\boldsymbol{r}_{\mathrm{b}}^{(k)} \in C^{(k)}(\partial \Omega)$ satisfying the requirements (5.27b), (5.27c) and (5.27d). Find $\boldsymbol{\psi}^{(k)} \in C^{(k)}(\Omega), \boldsymbol{b}^{(k+1)} \in C^{(k+1)}(\Omega)$, $\tilde{\boldsymbol{\xi}}_{\mathrm{b}}^{(d-k-1)} \in \tilde{C}^{(d-k-1)}(\partial \Omega), \tilde{\boldsymbol{\rho}}_{\mathrm{b}}^{(d-k-2)} \in \tilde{C}^{(d-k-2)}(\partial \Omega)$ and $\boldsymbol{q}_{t}^{(k+1)} \in \mathbb{R}^{N_{t}^{(k-1)}}$ such that
$\left[\begin{array}{ccccc}-\mathbb{H}^{(k)} & -\mathbb{D}^{(k) T} \mathbb{H}^{(k+1)} & \mathbb{T}^{(k) T} & 0 & 0 \\ -\mathbb{H}^{(k+1)} \mathbb{D}^{(k)} & -\mathbb{D}^{(k+1) T} \mathbb{H}^{(k+2)} \mathbb{D}^{(k+1)} & 0 & \mathbb{T}^{(k+1) T} & \mathbb{H}^{(k+1)} \mathbb{M}_{t}^{(k+1)} \\ \mathbb{T}^{(k)} & 0 & 0 & 0 & 0 \\ 0 & \mathbb{T}^{(k+1)} & 0 & 0 & 0 \\ 0 & \mathbb{M}_{t}^{(k+1) T} \mathbb{H}^{(k+1)} & 0 & 0 & 0\end{array}\right]\left[\begin{array}{c}\boldsymbol{\psi} \\ \boldsymbol{b} \\ \tilde{\boldsymbol{\xi}}_{\mathrm{b}} \\ \tilde{\boldsymbol{\rho}}_{\mathrm{b}} \\ \boldsymbol{q}_{t}\end{array}\right]=\left[\begin{array}{c}\tilde{\mathbf{0}} \\ -\mathbb{H}^{(k+1)} \boldsymbol{g} \\ \boldsymbol{r}_{\mathrm{b}} \\ \mathbf{0}_{\mathrm{b}} \\ \mathbf{0}\end{array}\right]$.
The following lemma is the discretized version of Lemma 5.4.
Lemma 5.14. Problem $\left(\mathrm{DP}_{t}^{b}\right)$ has a unique solution $\left(\boldsymbol{\psi}^{(k)}, \boldsymbol{b}^{(k+1)}, \tilde{\boldsymbol{\xi}}_{\mathrm{b}}^{(d-k-1)}, \tilde{\boldsymbol{\rho}}_{\mathrm{b}}^{(d-k-2)}, \boldsymbol{q}_{t}^{(k+1)}\right)$. Moreover, we have $\mathbb{D}^{(k+1)} \boldsymbol{b}^{(k+1)}=\mathbf{0}^{(k+2)}, \tilde{\boldsymbol{\rho}}_{\mathrm{b}}^{(d-k-2)}=\tilde{\mathbf{0}}_{\mathrm{b}}^{(d-k-2)}$ and $\boldsymbol{q}_{t}=\mathbf{0}$.

Proof. The well-posedness follows as before.
Multiplying the second line from the left by an element $\boldsymbol{v}^{(k+1)} \in B^{*(k+1)}$, i.e, by $\left.\left(\boldsymbol{v}^{(k-2)}\right)^{T}=\left(\mathbb{H}^{(k+1)}\right)^{-1}\left(\left(\mathbb{D}^{(k+1)}\right)^{T} \tilde{\boldsymbol{v}}^{(d-k-2)}+\left(\mathbb{T}^{(k+1)}\right)^{T} \tilde{\boldsymbol{v}}_{\mathrm{b}}^{(d-k-2)}\right)\right)^{T}$, for some $\tilde{\boldsymbol{v}}^{(d-k-2)} \epsilon$ $\tilde{C}^{(d-k-2)}(\Omega)$ and $\tilde{\boldsymbol{v}}_{\mathrm{b}}^{(d-k-2)} \in \tilde{C}^{(d-k-2)}(\partial \Omega)$, we find

$$
\begin{aligned}
& \left(\mathbb{D}^{(k+1)}\left(\mathbb{H}^{(k+1)}\right)^{-1}\left(\left(\mathbb{D}^{(k+1)}\right)^{T} \tilde{\boldsymbol{v}}^{(d-k-2)}+\left(\mathbb{T}^{(k+1)}\right)^{T} \tilde{\boldsymbol{v}}_{\mathrm{b}}^{(d-k-2)}\right), \mathbb{D}^{(k+1)} \boldsymbol{b}^{(k+1)}\right)_{\mathbb{H}} \\
& \quad+\left(\mathbb{T}^{(k+1)}\left(\mathbb{H}^{(k+1)}\right)^{-1}\left(\left(\mathbb{D}^{(k+1)}\right)^{T} \tilde{\boldsymbol{v}}^{(d-k-2)}+\left(\mathbb{T}^{(k+1)}\right)^{T} \tilde{\boldsymbol{v}}_{\mathrm{b}}^{(d-k-2)}\right)\right)^{T} \tilde{\boldsymbol{\rho}}_{\mathrm{b}}^{(d-k-2)} \\
& \quad=-\left(\tilde{\boldsymbol{v}}_{\mathrm{b}}^{(d-k-2)}\right)^{T} \mathbb{T}^{(k+1)} \mathbb{D}^{(k)} \boldsymbol{\psi}^{(k)}+\left(\tilde{\boldsymbol{v}}_{\mathrm{b}}^{(d-k-2)}\right)^{T} \mathbb{T}^{(k+1)} \boldsymbol{g}^{(k+1)},
\end{aligned}
$$

where the other terms vanish due to the fact that $\mathbb{D}^{(k+1)} \mathbb{D}^{(k)} \equiv 0, \mathbb{D}^{(k+1)} \mathbb{M}_{t}^{(k+1)} \equiv 0$, $\mathbb{T}^{(k+1)} \mathbb{M}_{t}^{(k+1)} \equiv 0$ and $\mathbb{D}^{(k+1)} \boldsymbol{g}^{(k+1)}=\mathbf{0}^{(k+2)}$.

Using the commutation relation in Proposition 3.5 and the boundary condition we obtain $\mathbb{T}^{(k+1)} \mathbb{D}^{(k)} \boldsymbol{\psi}^{(k)}=\mathbb{D}_{\mathrm{b}}^{(k)} \mathbb{T}^{(k)} \boldsymbol{\psi}^{(k)}=\mathbb{D}_{\mathrm{b}}^{(k)} \boldsymbol{r}_{\mathrm{b}}^{(k)}$. By requirement ( 5.27 d ) we have $\mathbb{T}^{(k+1)} \boldsymbol{g}^{(k+1)}-\mathbb{T}^{(k+1)} \mathbb{D}^{(k)} \boldsymbol{\psi}^{(k)}=\mathbf{0}_{\mathrm{b}}^{(k+1)}$. This shows that

$$
\left(\mathbb{D}^{(k+1)} \boldsymbol{v}^{(k+1)}, \mathbb{D}^{(k+1)} \boldsymbol{b}^{(k+1)}\right)_{\mathbb{H}}+\left(\mathbb{T}^{(k+1)} \boldsymbol{v}^{(k+1)}\right)^{T} \tilde{\boldsymbol{\rho}}_{\mathrm{b}}^{(d-k-2)}=0,
$$

for all $\boldsymbol{v}^{(k+1)} \in B^{*(k+1)}$. However, $B^{*(k+1) \perp}=\dot{Z}^{(k+1)}$ for which $\mathbb{D}^{(k+1)}\left(\dot{Z}^{(k+1)}\right) \equiv 0$ and $\mathbb{T}^{(k-2)}\left(\dot{Z}^{(k+1)}\right) \equiv 0$ and as a result the relation holds actually for all $\boldsymbol{v}^{(k+1)} \epsilon$ $C^{(k+1)}(\Omega)$. Taking $\boldsymbol{v}^{(k+1)}=\boldsymbol{b}^{(k+1)}$ and using the fact that $\mathbb{T}^{(k+1)} \boldsymbol{b}^{(k+1)}=\mathbf{0}_{\mathrm{b}}^{(k+1)}$ we find that $\mathbb{D}^{(k+1)} \boldsymbol{b}^{(k+1)}=\mathbf{0}^{(k+2)}$.

Multiplying the second equation from the left by $\left(\mathbb{M}_{t}^{(k+1)} \boldsymbol{q}_{t}\right)^{T}$ again shows that $\boldsymbol{q}_{t}=\mathbf{0}$.
To see that $\tilde{\boldsymbol{\rho}}_{\mathrm{b}}^{(d-k-2)}=\tilde{\boldsymbol{0}}_{\mathrm{b}}^{(d-k-2)}$ we multiply the second line by $\mathbb{T}^{(k+1)}\left(\mathbb{H}^{(k+1)}\right)^{-1}$ and find

$$
\mathbb{T}^{(k+1)} \mathbb{D}^{(k)} \boldsymbol{\psi}+\mathbb{T}^{(k+1)}\left(\mathbb{H}^{(k+1)}\right)^{-1} \mathbb{T}^{(k+1) T} \tilde{\boldsymbol{\rho}}_{\mathrm{b}}=\mathbb{T}^{(k+1)} \boldsymbol{g}
$$

By (5.27b) we have $\mathbb{T}^{(k+1)} \mathbb{D}^{(k)} \boldsymbol{\psi}=\mathbb{D}_{\mathrm{b}}^{(k)} \boldsymbol{r}_{\mathrm{b}}=\mathbb{T}^{(k+1)} \boldsymbol{g}$. Therefore we see that

$$
\mathbb{T}^{(k+1)}\left(\mathbb{H}^{(k+1)}\right)^{-1} \mathbb{T}^{(k+1) T} \tilde{\boldsymbol{\rho}}_{\mathrm{b}}=\mathbf{0}_{\mathrm{b}}
$$

which implies that $\tilde{\boldsymbol{\rho}}_{\mathrm{b}}=\tilde{\mathbf{0}}_{\mathrm{b}}$ because $\mathbb{T}^{(k+1)}\left(\mathbb{H}^{(k+1)}\right)^{-1} \mathbb{T}^{(k+1) T}$ is full rank.
The discrete potential $\boldsymbol{b}^{(k+1)}$ mimics again all properties of the potential $b^{(k+1)}$ in Problem ( $\mathrm{P}_{t}$ ).

### 5.3.2.3 The solution to Problem ( $\mathrm{DP}_{t}$ )

The following theorem gives the solution to Problem $\left(\mathrm{DP}_{t}\right)$ in terms of the potentials $\boldsymbol{a}^{(k-1)}$ and $\boldsymbol{b}^{(k+1)}$.

Theorem 5.5. The solution to Problem $\left(\mathrm{DP}_{t}\right)$ is given by

$$
\begin{aligned}
\boldsymbol{u}^{(k)} & =\mathbb{D}^{(k-1)} \boldsymbol{a}^{(k-1)}+\left(\mathbb{H}^{(k)}\right)^{-1} \tilde{\mathbb{D}}^{(d-k-1)}\left[\begin{array}{c}
\mathbb{H}^{(k+1)} \boldsymbol{b}^{(k+1)} \\
\tilde{\boldsymbol{\xi}}_{\mathrm{b}}^{(d-k-1)}
\end{array}\right]+\boldsymbol{h}^{(k)} \\
& =\mathbb{D}^{(k-1)} \boldsymbol{a}^{(k-1)}+\boldsymbol{\psi}^{(k)}+\boldsymbol{h}^{(k)},
\end{aligned}
$$

with $\mathbb{T}^{(k-1)} \boldsymbol{a}^{(k-1)}=\mathbf{0}$. Moreover, $\boldsymbol{a}^{(k-1)}$ is the solution to Problem $\left(\mathrm{DP}_{t}^{a}\right)$, $\left(\boldsymbol{\psi}^{(k)}, \boldsymbol{b}^{(k+1)}, \tilde{\boldsymbol{\xi}}_{\mathrm{b}}^{(d-k-1)}\right)$ is the solution to Problem $\left(\mathrm{DP}_{t}^{b}\right)$ and $\boldsymbol{h}^{(k)} \in H_{t}^{(k)}$. Thus

$$
\tilde{\mathbb{D}}^{(d-k-1)}\left[\begin{array}{c}
\mathbb{H}^{(k-1)} \boldsymbol{a}^{(k-1)} \\
\tilde{\boldsymbol{\mu}}_{\mathrm{b}}^{(d-k+1)}
\end{array}\right]=\tilde{\mathbf{0}}^{(d-k)},
$$

where $\tilde{\boldsymbol{\mu}}_{\mathrm{b}}^{(d-k+1)}$ is part of the solution to $\left(\mathrm{DP}_{t}^{a}\right)$, and $\boldsymbol{D}^{(k+1)} \boldsymbol{b}^{(k+1)}=\mathbf{0}$. The solution is again unique up to elements of $H_{t}^{(k)}$.

Proof. We can write $\boldsymbol{u}^{(k)}$ as in the discrete Helmholtz-Hodge decomposition (5.25). By substituting $\boldsymbol{u}^{(k)}$ in this form in $\left(\mathrm{DP}_{t}\right)$ we obtain

$$
\begin{aligned}
\mathbb{D}^{(k)} \boldsymbol{\psi}^{(k)} & =\boldsymbol{g}^{(k+1)}, \\
\mathbb{T}^{(k)} \boldsymbol{\psi}^{(k)} & =\boldsymbol{r}_{\mathrm{b}}^{(k)}, \\
\mathbb{P}_{\Omega \backslash \partial \Omega}^{(k-1)} \mathbb{D}^{(k-1) T} \mathbb{H}^{(k)} \mathbb{D}^{(k-1)} \boldsymbol{a}^{(k-1)} & =(-1)^{k} \mathbb{P}_{\Omega \backslash \partial \Omega}^{(k-1)} \tilde{\boldsymbol{f}}^{(d-k+1)} .
\end{aligned}
$$

These equations are satisfied if $\boldsymbol{a}^{(k-1)}$ and $\psi^{(k)}$ are the solutions of, respectively, Problem $\left(\mathrm{DP}_{t}^{a}\right)$ and Problem $\left(\mathrm{DP}_{t}^{b}\right)$.

Furthermore, suppose there is a second solution to $\left(\mathrm{DP}_{t}\right)$, then the difference of the two solutions is a solution to $\left(\mathrm{DP}_{t}\right)$ with zero right-hand side and hence an element of $H_{t}^{(k)}$.

Finally, we have found that we can calculate the solution $\boldsymbol{u}^{(k)}$ to Problem $\left(\mathrm{DP}_{t}\right)$ by solving non-singular linear systems for the two potentials $\boldsymbol{a}^{(k-1)}$ and $\boldsymbol{b}^{(k+1)}$.

### 5.4 Numerical results

We numerically test the proposed systems of equations by using them to calculate the two different discrete Helmholtz-Hodge decompositions for a given 1-form in 2D. Subsequently we consider a div-curl problem in 3D and we test the method for a number of different discrete Hodge matrices and meshes.

### 5.4.1 Calculation of Helmholtz-Hodge decompositions in 2D

As a first example we show how the discrete formulations of Section 5.3 can be used to calculate the two discrete Helmholtz-Hodge decompositions of a cochain. For simplicity we show this in 2 D .


Figure 5.1: The mesh and 1-cochain on $\Omega$.

### 5.4.1.1 Problem setting

We consider the domain $\Omega$ equal to $[-2,2] \times[-2,2]$ with three disks of radius $2 / 5$ located at $\left(-1, \frac{3}{4}\right),\left(1, \frac{3}{4}\right)$ and $(0,-1)$ removed. The resulting domain has Betti numbers ${ }^{7}$ $\left(\beta_{t^{*}}^{(0)}, \beta_{t^{*}}^{(1)}, \beta_{t^{*}}^{(2)}\right)=(1,3,0)$ and $\left(\beta_{t}^{(0)}, \beta_{t}^{(1)}, \beta_{t}^{(2)}\right)=(0,3,1)$. We will determine the decompositions of $\boldsymbol{u}^{(1)}:=R^{(1)}\left(u^{(1)}\right)$, where the 1 -form $u^{(1)}$ is given by

$$
\begin{align*}
u^{(1)}(x, y)= & -\frac{1}{2} \pi x \sin \left(\frac{\pi(x+2)}{2}\right) \cos \left(\frac{\pi(x+2)}{2}\right) d x  \tag{5.28}\\
& +\frac{1}{2} \pi y \cos \left(\frac{\pi(x+2)}{2}\right) \sin \left(\frac{\pi(x+2)}{2}\right) d y
\end{align*}
$$

We mesh the domain using a cut-cell approach. This means that we approximate the curvilinear boundary of the holes by straight line segments and use a Cartesian mesh for the complete domain. We subsequently cut the holes out of the Cartesian mesh. Near the boundary of the holes polygonal cut cells are created in this way. In Figure 5.1 the resulting cut-cell mesh is depicted next to a reconstruction of $\boldsymbol{u}^{(1)}$ in the cell centers. Such a cell-wise reconstruction of $\boldsymbol{u}^{(1)}$ can be determined from its values on the edges of the boundary of the cell. This reconstruction is first order accurate in general. See, for example, [77] for details.

We will use the discrete formulations of Section 5.3 to determine the projection of $\boldsymbol{u}^{(1)}$ on the three subspaces of (5.22) and the three subspaces of (5.23). In contrast to the situation where we want to solve a div-curl problem to find $\boldsymbol{u}^{(1)}$ given the data, here

[^41]

Figure 5.2: An orthonormal basis (with respect to $\mathbb{H}^{(1)}$ ) for $H_{t^{*}}^{(1)}$. For clarity the vectors are shown at half their size.
we already know $\boldsymbol{u}^{(1)}$ and from this derive the data:

$$
\tilde{\boldsymbol{f}}^{(2)}:=\tilde{\mathbb{D}}^{(1)}\left[\begin{array}{c}
\mathbb{H}^{(1)} \boldsymbol{u}^{(1)} \\
\tilde{\boldsymbol{r}}_{\mathrm{b}}^{(1)}
\end{array}\right], \quad \boldsymbol{g}^{(2)}:=\mathbb{D}^{(1)} \boldsymbol{u}^{(1)}, \quad \boldsymbol{r}_{\mathrm{b}}^{(1)}:=\mathbb{T}^{(1)} \boldsymbol{u}^{(1)} \quad \tilde{\boldsymbol{r}}_{\mathrm{b}}^{(1)}:=\mathbf{0}_{\mathrm{b}}
$$

We can also set $\tilde{\boldsymbol{r}}_{\mathrm{b}}^{(1)}$ equal to $\tilde{R}_{\mathrm{b}}^{(1)}\left(\star u^{(1)}\right)$, but this does not alter the projections of $\boldsymbol{u}^{(1)}$ on the different subspaces in the decomposition. So if these projections are all that is of interest, then $\tilde{\boldsymbol{r}}_{\mathrm{b}}^{(1)}=\mathbf{0}_{\mathrm{b}}$ is the simplest choice.

In 2 D the discretization matrices of Section 5.3 simplify because certain terms are absent. We briefly state the linear systems that need to be solved to find the decompositions. In this part we use the MFD-discrete Hodge matrices. We start with the case in which natural boundary conditions apply.

### 5.4.1.2 Linear systems for natural boundary conditions in 2D

In the case of natural boundary conditions we have the following spaces of harmonic forms:

$$
\begin{aligned}
& H_{t^{*}}^{(0)}:=\left\{\boldsymbol{c}^{(0)} \in \mathcal{C}^{(0)}(\Omega) \mid \mathbb{D}^{(0)} \boldsymbol{c}^{(0)}=\mathbf{0}^{(1)}\right\}=\left\{\boldsymbol{c}^{(0)} \in \mathcal{C}^{(0)}(\Omega) \mid \boldsymbol{c}^{(0)} \text { is constant }\right\}, \\
& H_{t^{*}}^{(1)}:=\left\{\boldsymbol{c}^{(1)} \in \mathcal{C}^{(1)}(\Omega) \mid \mathbb{D}^{(1)} \boldsymbol{c}^{(1)}=\mathbf{0}^{(2)}, \mathbb{D}^{(0) T} \mathbb{H}^{(1)} \boldsymbol{c}^{(1)}=\tilde{\boldsymbol{0}}^{(0)}\right\}, \\
& H_{t^{*}}^{(2)}:=\left\{\boldsymbol{c}^{(1)} \in \mathcal{C}^{(1)}(\Omega) \mid \mathbb{D}^{(1) T} \mathbb{H}^{(2)} \boldsymbol{c}^{(2)}=\tilde{\mathbf{0}}^{(1)}\right\}=\varnothing .
\end{aligned}
$$

In general the space $H_{t^{*}}^{(k)}$ is given by the kernel of $\mathbb{D}^{*(k+1)} \mathbb{D}^{(k)}+\mathbb{D}^{(k-1)} \mathbb{D}^{*(k)}$. We calculate a basis for $H_{t^{*}}^{(1)}$ using a singular value decomposition. An example of a basis is shown in Figure 5.2.

The decomposition is given by

$$
\boldsymbol{u}^{(1)}=\underbrace{\mathbb{D}^{(0)} \boldsymbol{a}_{t^{*}}^{(0)}}_{\epsilon B^{(1)}}+\underbrace{\boldsymbol{\psi}_{t^{*}}^{(1)}}_{\epsilon \dot{B}^{*(1)}}+\underbrace{\boldsymbol{u}^{(1)}-\mathbb{D}^{(0)} \boldsymbol{a}_{t^{*}}^{(0)}-\boldsymbol{\psi}_{t^{*}}^{(1)}}_{\epsilon H_{t^{*}}^{(1)}}
$$



Figure 5.3: An orthonormal basis (with respect to $\mathbb{H}^{(1)}$ ) for $H_{t}^{(1)}$. For clarity the vectors are shown at half their size.

The potential $\boldsymbol{a}_{t^{*}}^{(0)}$ is found by solving

$$
\left[\begin{array}{cc}
\mathbb{D}^{(0) T} \mathbb{H}^{(1)} \mathbb{D}^{(0)} & \mathbb{H}^{(0)} \mathbb{M}_{t^{*}}^{(0)} \\
\mathbb{M}_{t^{*}}^{(0) T} \mathbb{H}^{(0)} & 0
\end{array}\right]\left[\begin{array}{l}
\boldsymbol{a}_{t^{*}}^{(0)} \\
\boldsymbol{p}_{t^{*}}^{(0)}
\end{array}\right]=\left[\begin{array}{c}
\tilde{\boldsymbol{f}}^{(2)}-\mathbb{T}^{(0) T} \tilde{\boldsymbol{r}}_{\mathrm{b}}^{(1)} \\
\mathbf{0}
\end{array}\right]
$$

and, $\boldsymbol{\psi}_{t^{*}}^{(1)}$ is found by solving

$$
\left[\begin{array}{cc}
-\mathbb{H}^{(1)} & \mathbb{D}^{(1) T} \mathbb{H}^{(2)} \\
\mathbb{H}^{(2)} \mathbb{D}^{(1) T} & 0
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{\psi}_{t^{*}}^{(1)} \\
\boldsymbol{b}^{(2)}
\end{array}\right]=\left[\begin{array}{c}
\tilde{\mathbf{0}}^{(1)} \\
\mathbb{H}^{(2)} \boldsymbol{g}^{(2)}
\end{array}\right]
$$

Subsequently, the harmonic part is simply given by $\boldsymbol{h}_{t^{*}}^{(1)}:=\boldsymbol{u}^{(1)}-\mathbb{D}^{(0)} \boldsymbol{a}_{t^{\star}}^{(0)}-\boldsymbol{\psi}_{t^{\star}}^{(1)}$. These different components are shown in the left column of Figure 5.4.

### 5.4.1.3 Linear systems for essential boundary conditions in 2D

For essential boundary conditions we have the following harmonic forms:

$$
\begin{aligned}
H_{t}^{(0)} & :=\left\{\boldsymbol{c}^{(0)} \in \mathcal{C}^{(0)}(\Omega) \mid \mathbb{D}^{(0)} \boldsymbol{c}^{(0)}=\mathbf{0}^{(1)}, \mathbb{T}^{(0)} \boldsymbol{c}^{(0)}=\mathbf{0}_{\mathrm{b}}^{(0)}\right\}=\varnothing, \\
H_{t}^{(1)} & :=\left\{\boldsymbol{c}^{(1)} \in \mathcal{C}^{(1)}(\Omega) \mid \mathbb{D}^{(1)} \boldsymbol{c}^{(1)}=\mathbf{0}^{(2)}, \mathbb{T}^{(1)} \boldsymbol{c}^{(1)}=\mathbf{0}_{\mathrm{b}}^{(1)}, \mathbb{P}^{(0)} \mathbb{D}^{(0) T} \mathbb{H}^{(1)} \boldsymbol{c}^{(1)}=\mathbf{0}^{(0)}\right\}, \\
H_{t}^{(2)} & :=\left\{\boldsymbol{c}^{(2)} \in \mathcal{C}^{(2)}(\Omega) \mid \mathbb{P}^{(1)} \mathbb{D}^{(1) T} \mathbb{H}^{(2)} \boldsymbol{c}^{(2)}=\mathbf{0}^{(1)}\right\} \\
& =\left\{\boldsymbol{c}^{(2)} \in \mathcal{C}^{(2)}(\Omega) \mid \mathbb{H}^{(2)} \boldsymbol{c}^{(2)} \text { is constant }\right\} .
\end{aligned}
$$

An example of a basis for $H_{t}^{(1)}$ is shown in Figure 5.3.
The decomposition is now given by

$$
\boldsymbol{u}^{(1)}=\underbrace{\mathbb{D}^{(0)} \boldsymbol{a}_{t}^{(0)}}_{\epsilon B^{(1)}}+\underbrace{\boldsymbol{\psi}_{t}^{(1)}}_{\epsilon B^{*(1)}}+\underbrace{\boldsymbol{u}^{(1)}-\mathbb{D}^{(0)} \boldsymbol{a}_{t}^{(0)}-\boldsymbol{\psi}_{t}^{(1)}}_{\epsilon H_{t}^{(1)}}
$$

The potential $\boldsymbol{a}_{t}^{(0)}$ is found by solving

$$
\left[\begin{array}{cc}
\mathbb{D}^{(0) T} \mathbb{H}^{(1)} \mathbb{D}^{(0)} & \mathbb{T}^{(0) T} \\
\mathbb{T}^{(0)} & 0
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{a}_{t}^{(0)} \\
\tilde{\boldsymbol{\eta}}_{\mathrm{b}, t}^{(1)}
\end{array}\right]=\left[\begin{array}{c}
\tilde{\boldsymbol{f}}^{(2)} \\
\mathbf{0}_{\mathrm{b}}
\end{array}\right]
$$

$\psi_{t}$ is found by solving

$$
\left[\begin{array}{cccc}
-\mathbb{H}^{(1)} & \mathbb{D}^{(1) T} \mathbb{H}^{(2)} & \mathbb{T}^{(1) T} & 0 \\
\mathbb{H}^{(2)} \mathbb{D}^{(1)} & 0 & 0 & \mathbb{H}^{(2)} \mathbb{M}_{t}^{(2)} \\
\mathbb{T}^{(1)} & 0 & 0 & 0 \\
0 & \mathbb{M}_{t}^{(2) T} \mathbb{H}^{(2)} & 0 & 0
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{\psi}_{t}^{(1)} \\
\boldsymbol{b}_{t}^{(2)} \\
\tilde{\boldsymbol{\xi}}_{\mathrm{b}, t}^{(0)} \\
\boldsymbol{q}_{t}^{(2)}
\end{array}\right]=\left[\begin{array}{c}
\tilde{\mathbf{0}} \\
\mathbb{H}^{(2)} \boldsymbol{g}^{(2)} \\
\boldsymbol{r}_{\mathrm{b}}^{(1)} \\
\mathbf{0}
\end{array}\right]
$$

and the harmonic part is given by $\boldsymbol{h}_{t}^{(1)}:=\boldsymbol{u}^{(1)}-\mathbb{D}^{(0)} \boldsymbol{a}_{t}^{(0)}-\boldsymbol{\psi}_{t}^{(1)}$. The three components of the decomposition are depicted in the right column of Figure 5.4.

### 5.4.2 The div-curl problem in 3D

Next, we consider the 3-dimensional div-curl problems (5.3) in the domain $\Omega=[0,1]^{3}$. We consider the problems with the exact solutions $u_{\mathrm{e}}^{(1)}=u_{\mathrm{e}}^{x} d x+u_{\mathrm{e}}^{y} d y+u_{\mathrm{e}}^{z} d z$ and $u_{\mathrm{e}}^{(2)}=$ $u_{\mathrm{e}}^{x} d y \wedge d z+u_{\mathrm{e}}^{y} d z \wedge d x+u_{\mathrm{e}}^{z} d x \wedge d y$. Let us define $x^{\prime}=\pi(2 x-1) / 4, y^{\prime}=2 \pi(2 y-1) / 4$ and $z^{\prime}=3 \pi(2 z-1) / 4$. The components of $u_{\mathrm{e}}^{(1)}$ and $u_{\mathrm{e}}^{(2)}$ are given by

$$
\begin{aligned}
& u_{\mathrm{e}}^{x}(x, y, z)= 3 \sin \left(y^{\prime}\right) \cos ^{2}\left(y^{\prime}\right) \sin ^{3}\left(x^{\prime}\right)+\frac{9}{2} \cos \left(z^{\prime}\right) \sin ^{2}\left(z^{\prime}\right) \cos ^{3}\left(x^{\prime}\right) \\
&+\frac{3}{2} \sin \left(x^{\prime}\right) \cos ^{2}\left(x^{\prime}\right) \cos ^{3}\left(y^{\prime}\right) \cos ^{3}\left(z^{\prime}\right), \\
& u_{\mathrm{e}}^{y}(x, y, z)= \frac{9}{2} \sin \left(z^{\prime}\right) \cos ^{2}\left(z^{\prime}\right) \sin ^{3}\left(y^{\prime}\right)+\frac{3}{2} \cos \left(x^{\prime}\right) \sin ^{2}\left(x^{\prime}\right) \cos ^{3}\left(y^{\prime}\right) \\
&+3 \sin \left(y^{\prime}\right) \cos ^{2}\left(y^{\prime}\right) \cos ^{3}\left(x^{\prime}\right) \cos ^{3}\left(z^{\prime}\right), \\
& u_{\mathrm{e}}^{z}(x, y, z)=\frac{3}{2} \sin \left(x^{\prime}\right) \cos ^{2}\left(x^{\prime}\right) \sin ^{3}\left(y^{\prime}\right)+3 \cos \left(y^{\prime}\right) \sin ^{2}\left(y^{\prime}\right) \cos ^{3}\left(z^{\prime}\right) \\
&+\frac{9}{2} \sin \left(z^{\prime}\right) \cos ^{2}\left(z^{\prime}\right) \cos ^{3}\left(x^{\prime}\right) \cos ^{3}\left(y^{\prime}\right) .
\end{aligned}
$$

The data for the problems is calculated from this as

$$
\begin{array}{llll}
\tilde{\rho}^{(3)}=\star d^{*} u^{(1)}, & \omega^{(2)}=d u^{(1)}, & \tilde{r}_{\mathrm{b}}^{(2)}=\star_{\mathrm{b}} t^{*} u^{(1)}, & r_{\mathrm{b}}^{(1)}=t u^{(1)}, \\
\tilde{\omega}^{(3)}=\star d^{*} u^{(2)}, & \rho^{(3)}=d u^{(2)}, & \tilde{r}_{\mathrm{b}}^{(1)}=\star_{\mathrm{b}} t^{*} u^{(1)}, & r_{\mathrm{b}}^{(2)}=t u^{(2)} .
\end{array}
$$

These provide the discrete data by the de Rham maps as defined in Problem ( $\mathrm{DP}_{t^{*}}$ ) and Problem ( $\mathrm{DP}_{t}$ ). This leads to the following discrete analogues of the div-curl problems


Figure 5.4: The three components in the decomposition of $\boldsymbol{u}^{(1)}$ are shown: in the left column for the decomposition $C^{(1)}(\Omega)=B^{(1)}+\dot{B}^{*(1)}+H_{t^{*}}^{(1)}$ and in the right column for the decomposition $C^{(1)}(\Omega)=$ $\stackrel{\circ}{B}^{(1)}+B^{*(1)}+H_{t}^{(1)}$. The vectors are shown at one-fourth of their actual size.
in (5.3):

$$
\begin{align*}
& \left(\mathrm{DP}_{t^{*}}^{(1)}\right)\left\{\begin{array}{c}
\tilde{\mathbb{D}}^{(2)}\left[\begin{array}{c}
\mathbb{H}^{(1)} \boldsymbol{u}_{t^{*}}^{(1)} \\
\tilde{\boldsymbol{r}}_{\mathrm{b}}^{(2)}
\end{array}\right]=\tilde{\boldsymbol{\rho}}^{(3)}, \\
\mathbb{D}^{(1)} \boldsymbol{u}_{t^{*}}^{(1)}=\boldsymbol{\omega}^{(2)},
\end{array}\right. \\
& \left(\mathrm{DP}_{t}^{(1)}\right)\left\{\begin{array}{r}
\tilde{\mathbb{D}}^{(2)}\left[\begin{array}{c}
\mathbb{H}^{(1)} \boldsymbol{u}_{t}^{(1)} \\
\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(2)}
\end{array}\right]=\tilde{\boldsymbol{\rho}}^{(3)}, \\
\mathbb{D}^{(1)} \boldsymbol{u}_{t}^{(1)}=\boldsymbol{\omega}^{(2)}, \\
\mathbb{T}^{(1)} \boldsymbol{u}_{t}^{(1)}=\boldsymbol{r}_{\mathrm{b}}^{(1)},
\end{array}\right. \\
& \left(\mathrm{DP}_{t^{*}}^{(2)}\right)\left\{\begin{aligned}
\tilde{\mathbb{D}}^{(1)}\left[\begin{array}{c}
\mathbb{H}^{(2)} \boldsymbol{u}_{t^{*}}^{(2)} \\
\tilde{\boldsymbol{r}}_{\mathrm{b}}^{(1)}
\end{array}\right] & =\tilde{\boldsymbol{\omega}}^{(1)}, \\
\mathbb{D}^{(2)} \boldsymbol{u}_{t^{*}}^{(2)} & =\boldsymbol{\rho}^{(3)},
\end{aligned}\right. \\
& \left(\mathrm{DP}_{t}^{(2)}\right)\left\{\begin{aligned}
\tilde{\mathbb{D}}^{(1)}\left[\begin{array}{c}
\mathbb{H}^{(2)} \boldsymbol{u}_{t}^{(2)} \\
\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}
\end{array}\right] & =\tilde{\boldsymbol{\omega}}^{(1)}, \\
\mathbb{D}^{(2)} \boldsymbol{u}_{t}^{(2)} & =\boldsymbol{\rho}^{(3)}, \\
\mathbb{T}^{(2)} \boldsymbol{u}_{t}^{(2)} & =\boldsymbol{r}_{\mathrm{b}}^{(2)} .
\end{aligned}\right. \tag{5.29}
\end{align*}
$$

We consider the convergence of $\boldsymbol{u}_{t^{*}}^{(1)}$ and $\boldsymbol{u}_{t}^{(1)}$, and, $\boldsymbol{u}_{t^{*}}^{(2)}$ and $\boldsymbol{u}_{t}^{(2)}$ to respectively $\boldsymbol{u}_{\mathrm{e}}^{(1)}$ := $R^{(1)}\left(u_{\mathrm{e}}^{(1)}\right)$ and $\boldsymbol{u}_{\mathrm{e}}^{(2)}:=R^{(1)}\left(u_{\mathrm{e}}^{(2)}\right)$ for the two kinds of boundary conditions. We do this for a wide range of meshes and four types of discrete Hodge operators.

In the next section we first test the method using the diagonal discrete Hodge matrices from Discrete Exterior Calculus ${ }^{8}$. The diagonal Hodge matrices can only be formed consistently on Cartesian and simplicial meshes. Subsequently we consider the method using the MFD and DGA discrete Hodge matrices ${ }^{9}$ These are used for all seven mesh types.

### 5.4.2.1 DEC-H

We first consider the diagonal discrete Hodge matrices, namely the ones used in the Discrete Exterior Calculus methods. As we discussed in Chapter 3, these diagonal discrete Hodge matrices depend on a circumcentric dual mesh which, in general, can only be formed for Cartesian meshes and tetrahedral meshes. Examples of these meshes are shown in Figure 5.5.

We analyze the convergence for a sequence of these meshes in two norms. The first norm is provided by the discrete Hodge matrices themselves:

$$
e_{\mathbb{H}}^{(k)}:=\frac{\sqrt{\left(\boldsymbol{u}^{(k)}-\boldsymbol{u}_{\mathrm{e}}^{(k)}\right)^{T} \mathbb{H}{ }^{(k)}\left(\boldsymbol{u}^{(k)}-\boldsymbol{u}_{\mathrm{e}}^{(k)}\right)}}{\sqrt{\boldsymbol{u}_{\mathrm{e}}^{(k) T} \mathbb{H}^{(k)} \boldsymbol{u}_{\mathrm{e}}^{(k)}}} .
$$

The second norm we use is the one defined in (3.18). It is based on a barycentric subdivision of the mesh cells and independent of the discrete Hodge operator. We denote it by $e_{\mathrm{I}}^{(k)}$. Just like $e_{\mathbb{H}}^{(k)}, e_{\mathrm{I}}^{(k)}$ is an approximation of a norm equivalent to the $L^{2}$-norm.

[^42]

Figure 5.5: On the left we see an example of a Cartesian mesh and a simplicial mesh. For clarity only the boundary primal mesh (in blue) and the boundary circumcentric dual mesh (in red) are shown. On the right we see the convergence of $e_{\mathbb{H}}^{(k)}$ (full lines) and $e_{\mathrm{I}}^{(k)}$ (dashed lines) for the four problems.

As can be seen in the convergence plots in Figure 5.5, there is almost no difference between $e_{\mathbb{H}}^{(k)}$ and $e_{\mathrm{I}}^{(k)}$. On the Cartesian meshes the convergence is second order, while on the tetrahedral meshes the convergence is close to first order. For the tetrahedral meshes the errors for the problems that use $\boldsymbol{u}^{(2)}$ are approximately five times larger than the errors for the problems that use $\boldsymbol{u}^{(1)}$. The convergence rate is roughly the same.

The solutions of (5.29) are given by

$$
\begin{array}{ll}
\left(\mathrm{DP}_{t^{*}}^{(1)}\right): & \boldsymbol{u}_{t^{*}}^{(1)}=\mathbb{D}^{(0)} \boldsymbol{a}_{t^{*}}^{(0)}+\left(\mathbb{H}^{(1)}\right)^{-1} \tilde{\mathbb{D}}^{(1)}\left[\begin{array}{c}
\mathbb{H}^{(2)} \boldsymbol{b}_{t^{*}}^{(2)} \\
\mathbf{0}
\end{array}\right], \\
\left(\mathrm{DP}_{t}^{(1)}\right): & \boldsymbol{u}_{t}^{(1)}=\mathbb{D}^{(0)} \boldsymbol{a}_{t}^{(0)}+\left(\mathbb{H}^{(1)}\right)^{-1} \tilde{\mathbb{D}}^{(1)}\left[\begin{array}{c}
\mathbb{H}^{(2)} \boldsymbol{b}_{t}^{(2)} \\
\tilde{\boldsymbol{\xi}}_{b, t}^{(1)}
\end{array}\right], \\
\left(\mathrm{DP}_{t^{*}}^{(2)}\right): & \boldsymbol{u}_{t^{*}}^{(2)}=\mathbb{D}^{(1)} \boldsymbol{a}_{t^{*}}^{(1)}+\left(\mathbb{H}^{(2)}\right)^{-1} \tilde{\mathbb{D}}^{(0)}\left[\begin{array}{c}
\mathbb{H}^{(3)} \boldsymbol{b}_{t^{*}}^{(3)} \\
\mathbf{0}
\end{array}\right], \\
\left(\mathrm{DP}_{t}^{(2)}\right): & \boldsymbol{u}_{t}^{(2)}=\mathbb{D}^{(1)} \boldsymbol{a}_{t}^{(1)}+\left(\mathbb{H}^{(2)}\right)^{-1} \tilde{\mathbb{D}}^{(0)}\left[\begin{array}{c}
\mathbb{H}^{(3)} \boldsymbol{b}_{t}^{(3)} \\
\tilde{\boldsymbol{\xi}}_{b, t}^{(0)}
\end{array}\right],
\end{array}
$$

because there are no harmonic parts as $\Omega$ is smoothly contractible to a point. Theorems 5.4 and 5.5 state that the "vector potentials" are divergence-free, i.e., the $1-/ 2$-cochain potentials in the solutions satisfy, respectively,

$$
\begin{array}{lll}
\left(\mathrm{DP}_{t^{*}}^{(1)}\right): & \mathbb{D}^{(2)} \boldsymbol{b}_{t^{*}}^{(2)}=\mathbf{0}, & \left(\mathrm{DP}_{t}^{(1)}\right): \\
\left(\mathrm{DP}_{t^{*}}^{(2)}\right): & \tilde{\mathbb{D}}^{(2)}\left[\begin{array}{c}
\mathbb{D}^{(2)} \boldsymbol{b}_{t}^{(2)}=\mathbf{0}, \\
\tilde{\boldsymbol{\mu}}_{\mathrm{b}, t^{*}}^{(1)} \boldsymbol{a}_{t^{*}}^{(1)}
\end{array}\right]=\tilde{\mathbf{0}}, & \left(\mathrm{DP}_{t}^{(2)}\right):  \tag{5.30}\\
\tilde{\mathbb{D}}^{(2)}\left[\begin{array}{c}
\mathbb{H}^{(1)} \boldsymbol{a}_{t}^{(1)} \\
\tilde{\boldsymbol{\mu}}_{\mathrm{b}, t}^{(2)}
\end{array}\right]=\tilde{\mathbf{0}},
\end{array}
$$

where we use $\tilde{\boldsymbol{\mu}}_{b, t^{*}}^{(2)}$ and $\tilde{\boldsymbol{\mu}}_{b}^{(2)}$ to denote the boundary variables that are denoted by $\boldsymbol{\mu}_{\mathrm{b}}^{(d-k+1)}$ in Theorem 5.5. We numerically tested these in Figure 5.6 using the DECHodge matrices on the hexahedral and tetrahedral meshes. The extent to which these relations are satisfied depends solely on the accuracy of the quadrature formulas used to implement the de Rham maps which are used for determining the discrete data. ${ }^{10}$ We use a fourth order quadrature rule.

### 5.4.2.2 MFD- $\mathbb{H}$ and DGA- $\mathbb{H}$

Next we consider the MFD-discrete Hodge operators and DGA-discrete Hodge operators. These methods rely on a barycentric dual mesh which can be defined for a very broad range of mesh types. In Figure 5.4.2.2 coarse mesh examples are shown for the seven mesh sequences for which we consider the convergence. The seven mesh types that we consider are hexahedral meshes (HE), tetrahedral meshes (TE), checkerboard meshes (CB), locally refined meshes (LR), the so-called Kershaw meshes (KS), the prismatic tetrahedral meshes (PT) and, finally, the prismatic Voronoi meshes (PV). These meshes are taken from [98]. The convergence behavior is shown for these mesh sequences in Figure 5.7.

[^43]

Figure 5.6: In these plots the maximum absolute value of the vectors in the left-hand side of (5.30) are shown for the method with DEC-Hodge matrices. On the left it is shown for the hexahedral meshes and on the right for the tetrahedral meshes.

We analyze the rate of convergence for the different mesh sequences. The rate of convergence is estimated by

$$
r_{\mathbb{H}}^{(k)}:=-3 \frac{\log \left(e_{\mathbb{H}, i}^{(k)} / e_{\mathbb{H}, i-1}^{(k)}\right)}{\log \left(N_{i}^{(k)} / N_{i-1}^{(k)}\right)},
$$

where $i$ and $i-1$ are two succeeding meshes in the mesh sequence and $N_{i}^{(k)}$ is the dimension of $C^{(k)}(\Omega)$ for mesh $i$, i.e., the number of $k$-dimensional mesh cells. The rate of convergence in the mesh-independent norm, which we denote by $r_{\mathrm{I}}^{(k)}$, is given analogously. We always report the convergence rates based on the two finest meshes in the mesh sequence. The rates of convergences are given in Table 5.1.

The convergence behaviors of $e_{\mathbb{H}}^{(k)}$ and $e_{\mathrm{I}}^{(k)}$ are approximately the same in all cases with as exception the KS mesh sequence for which the convergence in the norm $e_{\mathrm{I}}^{(k)}$ is slightly worse in most cases. For the HE, TE, CB and LR mesh sequences the performances of the DGA method and MFD method are very similar. The convergence on the HE and LR meshes is second order, while the convergence on the TE and CB meshes is roughly first order.

The performance of the two methods on the KS, PT and PV mesh sequences is quite different. On the one hand, for the KS meshes the MFD method outperforms the DGA method for the problems with the 2-form as unknown: $\left(\mathrm{P}_{t}^{(2)}\right)$ and $\left(\mathrm{P}_{t^{*}}^{(2)}\right)$. On the other hand, the DGA method performs significantly better for all problems on the prismatic mesh sequences PT and PV. A convergence analysis of the methods based on a priori error estimates might explain the varying performance of the methods on the different mesh sequences. This is beyond the scope of this thesis.


Figure 5.7: The convergence of the compatible discretization for the four problems in (5.29) using the MFD-discrete Hodge operators and the DGA-discrete Hodge operators.

Table 5.1: The convergence rates $r_{\mathbb{H}}^{(k)}$ (first) and $r_{\mathrm{I}}^{(k)}$ (second) for the four problems in (5.29). Results for the DGA and MFD methods and seven mesh types.

|  | $\left(P_{t}^{(1)}\right)$ |  | $\left(P_{t}^{(2)}\right)$ |  | $\left(P_{t^{*}}^{(1)}\right)$ |  | $\left(P_{t^{*}}^{(2)}\right)$ |  |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Mesh | DGA | MFD | DGA | MFD | DGA | MFD | DGA | MFD |
| HE | $2.1 / 2.1$ | $2.0 / 2.0$ | $2.0 / 2.0$ | $2.0 / 2.0$ | $2.0 / 2.1$ | $1.9 / 1.9$ | $2.0 / 2.0$ | $2.0 / 2.0$ |
| TE | $1.0 / 1.0$ | $0.9 / 1.0$ | $1.1 / 1.1$ | $1.0 / 1.1$ | $1.1 / 1.1$ | $1.0 / 1.1$ | $1.1 / 1.1$ | $1.1 / 1.1$ |
| CB | $1.1 / 1.1$ | $1.0 / 1.0$ | $1.0 / 0.9$ | $0.8 / 0.9$ | $1.3 / 1.3$ | $1.2 / 1.3$ | $1.0 / 1.0$ | $0.9 / 0.9$ |
| LR | $1.9 / 1.9$ | $1.8 / 1.8$ | $2.0 / 2.0$ | $1.9 / 1.9$ | $2.0 / 2.0$ | $1.9 / 1.9$ | $2.0 / 2.0$ | $2.0 / 1.9$ |
| KS | $1.1 / 0.9$ | $0.9 / 1.0$ | $0.7 / 0.4$ | $1.3 / 1.0$ | $1.1 / 0.8$ | $0.9 / 0.8$ | $0.7 / 0.4$ | $1.2 / 0.9$ |
| PT | $1.9 / 1.9$ | $1.8 / 1.6$ | $2.0 / 2.0$ | $1.7 / 1.6$ | $1.9 / 1.9$ | $1.8 / 1.6$ | $2.0 / 1.9$ | $1.8 / 1.6$ |
| PV | $2.0 / 1.9$ | $1.6 / 1.5$ | $2.1 / 2.1$ | $1.6 / 1.5$ | $2.0 / 1.9$ | $1.6 / 1.5$ | $2.0 / 2.0$ | $1.6 / 1.5$ |

### 5.5 Final remarks

In this chapter we showed how the mimetic discretization methods introduced in Chapter 3 can be used to solve div-curl problems with normal or tangential boundary conditions. We showed that by using discrete Helmholtz-Hodge decompositions the solution to the div-curl problems can be found by solving two separate problems for the scalar and vector potential of the solution. Taking into account all terms in the discrete HodgeLaplace problem, full-rank discrete systems are found. This is in contrast to the earlier publication [96], where non-standard linear algebra methods needed to be employed to solve the ill-posed linear systems. Here we showed an alternative approach.

In the future it will be important to extend the method to include no-slip boundary conditions on the vector field. This may be done through again another (discrete) Hilbert complex.

## Dual Discrete Hodge Operators on Simplicial Meshes

For the DEC method the discrete Hodge matrices are diagonal matrices, because the circumcentric dual mesh used is orthogonal to the primal simplicial mesh. The MFD and DGA methods, which can deal with polygonal meshes, use a (non-orthogonal) barycentric dual mesh that results in sparse, but non-diagonal, discrete Hodge matrices. The fact that for the DEC method the discrete Hodge matrices are diagonal implies that their inverses are immediately known. Therefore, it is possible to explicitly interpolate discrete variables from the dual mesh back to the primal mesh. For the MFD and DGA methods discrete Hodge matrices are non-diagonal and require solving a linear system to interpolate from the dual mesh to the primal mesh. This makes the DEC method more attractive on simplicial meshes. In this chapter we restrict our discussion to simplicial meshes.

A disadvantage of the circumcentric dual mesh is the fact that the circumcenter of a simplex, in contrast to its barycenter, may lie outside of the simplex, and the resulting discrete Hodge matrices can lose their positive definiteness. In [49] it is shown that a sufficient requirement for positive definite discrete Hodge matrices is that the primal simplicial mesh is a Delaunay mesh ${ }^{1}$. However, the DEC still shows good convergence for non-Delaunay meshes [99].

For DEC-type schemes, extra requirements can be demanded on the triangulation near (internal) boundaries of the domain. It is often important for simplices that have one side lying in the (internal) boundary of the domain, that their circumcenter lies on the same side of this boundary. If a Delaunay triangulation satisfies this as well, it is called a boundary conforming Delaunay mesh. The dual cells then exactly partition the same domain as the primal cells, which can be a necessity for correct discretization at (internal) boundaries [100]. The generation of boundary conforming Delaunay meshes can be a complicated matter [100].

In contrast to the cicumcentric dual mesh, the barycentric dual mesh can be directly formed for any simplicial primal mesh and the dual vertices always lie inside the primal cells. As a result the lengths, areas and volumes of the dual cells are always positive. However, as remarked before, the discrete Hodge operators for a barycentric dual mesh are not diagonal. The more complicated shape of the barycentric dual cells and the nonorthogonality between the primal and dual cells make it more complicated to interpolate

[^44]

Figure 6.1: A typical circumcentric dual cell (left) and barycentric dual cell (right). All primal tetrahedra that contain a part of dual cell are drawn as well. The dual cells are dual to the same primal vertex in the same mesh and the perspective in both cases is the same. While the faces of the circumcentric dual cell are planar, the faces of the barycentric dual cell consist of simplices which can each have a different orientation.
from the barycentric dual mesh back to the primal mesh. Figure 6.1 shows a circumcentric dual cell and barycentric dual cell side by side.

It is stated in [101] and [99] that up to now no discrete sparse definitions for the inverses of the discrete Hodge matrices $\mathbb{H}^{(1)}$ (mapping from primal edges to dual edges in 2D and dual faces in 3 D ) and $\mathbb{H}^{(2)}$ (mapping from primal faces to dual edges in 3D), i.e., $\tilde{H}^{(2)}$ and $\tilde{\mathbb{H}}^{(1)}$, respectively, have been presented for barycentric dual meshes. This is not correct. Although, they do not seem to have appeared in the computational fluid dynamics literature, an explicit inverse of the matrix $\mathbb{H}^{(1)}$ has been given in the computational electromagnetics literature [102,103]. Furthermore, in 2D the issue was addressed in [104, 105].

In this chapter we will extend these results by showing that such sparse discrete dual Hodge operators $\tilde{\mathbb{H}}^{(k)}$ exist for any $k$, and we will construct them. Moreover, we show them to be symmetric positive-definite. We employ a general definition that covers all different realizations of these matrices up to three dimensions. We expect that the definitions hold in arbitrary dimensions.

In [101] it is also stated that the discrete versions of Hodge-star operators and their inverses may not be each others matrix-inverses and as a result simultaneous use of both in numerically solving a partial differential equation may not be possible. The discrete versions of the inverse Hodge-star operators that we introduce will indeed not be matrixinverses of the discretization of the Hodge-star operators. However, simultaneous use of both discrete Hodge matrices in a linear system does give good results as we will see in Section 6.3.

It should be noted that there are other approaches that determine an inverse mass matrix by using a hybrid formulation, construct a block-diagonal mass matrix and cheaply invert this. This is the approach taken in Hybridizable Discontinuous Galerkin methods [106-109]. If a dual mesh and dual degrees of freedom are used, the inverse of the mass matrix (which can be interpreted as a dual discrete Hodge matrix) can even be explicitly calculated [55,110,111]. See also the related approach in [112]. Here we do not use a hybrid approach but directly construct the dual Hodge matrices geometrically.

### 6.1 Dual to primal mesh interpolation

To derive discrete Hodge matrices $\tilde{\mathbb{H}}^{(l)}: \tilde{C}^{(l)}(\Omega) \rightarrow C^{(d-l)}(\Omega)$ we will repeat the argumentation of Section 3.3.3 but focus on a dual cell $\sharp \sigma_{(0)}$. For each dual cell we will derive a local discrete Hodge matrix $\tilde{\mathbb{H}}_{\mathfrak{\sigma} \sigma_{(0)}^{(0)}}^{(l)}$ and subsequently construct the global discrete Hodge matrix by adding the local contributions.

The above construction of $\mathbb{H}^{(l)}$ works for general polytopal cells. The construction of the matrices $\tilde{H}^{(l)}$ is more difficult because the dual cells have, for example, non-planar faces. The derivation of interpolation formulas turns out to be only possible if the primal mesh is simplicial or Cartesian (or a combination, e.g., simplicial in $(x, y)$-planes and Cartesian in the $z$-direction).

### 6.1.1 Dual cell reconstruction

The interpolation from the dual mesh to the primal mesh is the "dual" construction of the primal-mesh-to-dual-mesh interpolation, which we discussed in Section 3.3.2. In Section 3.3.2 we focussed on a primal cell and derived a consistent approximation of an $l$-form by using the discrete values on the $l$-cells in the boundary of the primal cell (cf. interpolation formula (3.9)).

In this chapter the focus will be on a dual cell instead of on a primal cell. We will derive a consistent approximation of a $(d-l)$-form using the discrete values on the dual ( $d-l$ )-cells in the boundary of the dual $d$-cell under consideration.

In the original interpolation formula (3.9), each term in the sum consists of a product of a discrete scalar value with a $(d-l)$-form $\star_{\sigma_{(d)}} \sigma^{(l)}$ which represents the size and orientation of the part of the dual cell $\approx \sigma_{(l)}$ that lies in $\sigma_{(d)}$. The interpolation formula that we will derive in this chapter will have a similar form. This time the discrete scalar value (the value of a dual cochain on the dual cell) will be multiplied by an $l$-form $\sigma_{(0)} \sigma^{(l)}$, which will represent the size and orientation of the part of the primal cell $\sigma_{(l)}$ that lies in the dual cell to $\sigma_{(0)}$

We will now define these $l$-forms ${ }_{\sigma_{(0)}} \sigma^{(l)}$ in a more general form as ${ }_{\sigma(m)} \sigma_{(l)}$ (just like we did with ${ }_{\boldsymbol{\wedge}_{(m)}} \sigma_{(l)}$ in Section 3.3.2). The more general definition is needed to prove the consistency of the interpolation formula in a recursive way as was the case in Section 3.3.2.

Definition 6.1. The $(l-m)$-vector ${ }_{\sigma_{(m)}} \sigma^{(l)}$ corresponding to the $l$-cell $\sigma_{(l)}$ restricted to the $m$-cell $\sigma_{(m)} \in \partial^{l-m} \sigma_{(l)}$ with $m<l \leq d$ is defined by

$$
:=\sum_{\left(\tau_{(l-1)}, \ldots, \tau_{(m+1)}\right) \in I_{(m)}^{\sigma_{(l)}}} \frac{\sigma_{(l)}}{\sigma_{\sigma_{(l)} \tau_{(l-1)}} \cdots o_{\tau_{(m+1)} \sigma_{(m)}}}\left(x_{(1)}^{\tau_{(l-1)}}-x_{(1)}^{\sigma_{(l)}}\right) \wedge \cdots \wedge\left(x_{(1)}^{\sigma_{(m)}}-x_{(1)}^{\tau_{(m+1)}}\right),
$$

when $m+1<l, \sigma_{(l-1)} \sigma^{(l)}:=o_{\sigma_{(l)} \sigma_{(l-1)}}\left(x_{(1)}^{\sigma_{(l-1)}}-x_{(1)}^{\sigma_{(l)}}\right)$ and $\sigma_{(l)} \sigma_{(l)}=+1$. Furthermore, we
define $\sigma_{(m)} \sigma^{(l)}:=b_{\sigma_{(l)}}\left(\sharp \sigma^{(l)}\right)$. This implies that

$$
\begin{aligned}
& \sigma_{(m)} \sigma^{(l)} \\
& :=\sum_{\left(\tau_{(l-1)}, \ldots, \tau_{(m+1)}\right) \in I_{\sigma_{(m)}}^{\sigma_{(l)}}} \frac{o_{\sigma_{(l)} \tau_{(l-1)}} \cdots o_{\tau_{(m+1)} \sigma_{(m)}}}{(l-m)!}\left(x_{\tau_{(l-1)}^{(1)}}^{(1)}-x_{\sigma_{(l)}}^{(1)}\right) \wedge \cdots \wedge\left(x_{\sigma_{(m)}}^{(1)}-x_{\tau_{(m+1)}}^{(1)}\right) .
\end{aligned}
$$

The local primal cells are the dual analogue to the local dual cells. For example, ${ }^{\star} \sigma_{(d)} \sigma_{(l)}$ is the part of $\approx \sigma_{(l)}$ that lies in $\sigma_{(d)}$, and, analogously, $\sigma_{(0)} \sigma_{(l)}$ is the part of $\sigma_{(l)}$ that lies in $\stackrel{\sim}{n}(0)$.

The local primal cell forms $\sigma_{(m)} \sigma^{(l)}$ satisfy the recursive relation

$$
\begin{equation*}
\sigma_{(m)} \sigma^{(l)}=\sum_{\tau_{(m+1)} \in \mathcal{Z}^{-1} \sigma_{(m)} \cap \partial^{l-m-1} \sigma_{(l)}}\left(\frac{o_{\tau_{(m+1)} \sigma_{(m)}}}{l-m}\right)\left(\tau_{(m+1)} \sigma^{(l)}\right) \wedge\left(x_{\sigma_{(m)}}^{(1)}-x_{\tau_{(m+1)}}^{(1)}\right) . \tag{6.1}
\end{equation*}
$$

This recursive relation will again be an important ingredient for the interpolation formula.
Suppose we have a $(d-l)$-form $\tilde{a}^{(d-l)} \in \tilde{\Lambda}^{(d-l)} V$, which is discretized on the dual $(d-l)$-cells of the mesh as $\tilde{\boldsymbol{a}}^{(d-l)}=\tilde{R}^{(d-l)}\left(\tilde{a}^{(d-l)}\right)$. Let us consider a single $d$-dimensional dual mesh cell $\rightsquigarrow \sigma_{(0)}$ and let $\tilde{a}_{\sharp \sigma_{(l)}}^{(d-l)}$ be the value of $\tilde{\boldsymbol{a}}^{(d-l)}$ on $\sharp \sigma_{(l)}$. The primal cell $l$-forms $\sigma_{(0)} \sigma^{(l)}$ can be used to calculate a consistent approximation of the $l$-form $\tilde{\star} \tilde{a}^{(d-l)}$ in the cell $\star \sigma_{(0)}$. If $\tilde{a}^{(d-l)}$ is constant in $\star \sigma_{(0)}$, then

$$
\begin{equation*}
\tilde{\star} \tilde{a}^{(d-l)}=\frac{1}{\left|\approx \sigma_{(0)}\right|} \sum_{\sigma_{(l)} \in \mathcal{\partial}^{-l} \sigma_{(0)}} \tilde{a}_{\star \sigma_{(l)} \sigma_{(0)}}^{(d-l)} \sigma^{(l)} . \tag{6.2}
\end{equation*}
$$

This formula can be used to derive a dual cell-wise symmetric interpolation formula from the dual mesh to the primal (simplicial) mesh. However, there is one caveat. The formula does not apply to cells $\sigma_{(0)} \in C_{(0)}(\partial \Omega)$. For these cells, the corresponding dual cells are not complete and we get a boundary contribution provided by the boundary condition of the problem under consideration.

We will first derive this interpolation formula while assuming that the primal cell under consideration is not in the boundary mesh. The derivation is again recursively. When we write $\tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}$ for some outer-oriented $(d-l)$-form $\tilde{a}^{(d-l)}=\left\{o^{(d)}\right\} a^{(d-l)} \in \tilde{\Lambda}^{(d-l)} V$ and outer-oriented $k$-simplex $\tilde{s}$, we mean $\tilde{\star}_{\tilde{s}} \tilde{t}_{\tilde{s}} \tilde{a}^{(d-l)}$, where $\tilde{t}_{\tilde{s}}: \tilde{\Lambda}^{(d-l)} V \rightarrow \tilde{\Lambda}^{(d-l)}(\tilde{s})$ is defined according to $\tilde{t}_{\tilde{s}} \tilde{a}^{(d-l)}=\left\{o_{(k)}\right\} t_{\tilde{s}} a^{(d-l)}$, where $t_{\tilde{s}}$ is the usual trace on $\tilde{s}$ and $\left\{o_{(d)}\right\} o_{(k)}$ is the outer orientation of $\tilde{s}$. Furthermore, $\tilde{\star}_{\tilde{s}}$ is the Hodge operator on the space $\tilde{\Lambda}^{(d-l)}(\tilde{s})$.
Theorem 6.1. (Dual mesh interpolation property for the interior.)
Let $\mathcal{G}=\left\{C_{(d)}(\Omega), \ldots, C_{(0)}(\Omega)\right\}$ be a simplicial cell complex. For all $(d-l)$-forms $\tilde{a}^{(d-l)} \epsilon$ $\tilde{\Lambda}^{(d-l)}$ and $\sigma_{(m)} \in C_{(m)}(\Omega) \backslash C_{(m)}(\partial \Omega)$ with $0 \leq m \leq l \leq d \leq 3$, we have

$$
\begin{align*}
& \sum_{\tilde{s} \in \Delta\left(\sharp \sigma_{(m)}\right)} \tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}|\tilde{s}| \\
&=\left\{\begin{array}{r}
\sum_{\sigma_{(l)} \in \partial^{m-l} \sigma_{(m)}}\left\langle\tilde{a}^{(d-l)}, \star \sigma_{(l)}\right\rangle \\
\sigma_{(m)} \sigma^{(l)} \\
(-1)^{l-m} \sum_{\sigma_{(l)} \in \partial^{m-l} \sigma_{(m)}}\left\langle\tilde{a}^{(d-l)}, \sharp \sigma_{(l)}\right\rangle \\
\sigma_{(m)} \sigma^{(l)}
\end{array}\right. \text { if is odd d is even. } \tag{6.3}
\end{align*}
$$

Note that we cannot write $\tilde{\star}_{\boldsymbol{*} \sigma_{(m)}} \tilde{a}^{(d-l)}\left|\sharp \sigma_{(m)}\right|$ for the left-hand side when $m>0$, because $\stackrel{\star}{(m)}$ consists of simplices that do not all lie in the same $(d-m)$-dimensional affine subspace of $\mathbb{R}^{d}$, hence $\tilde{\boldsymbol{x}}_{\boldsymbol{*} \sigma_{(m)}}$ does not make sense. Instead we have a sum over all simplices that make $u p \approx \sigma_{(m)}$. We denote the set of the $(d-m)$-dimensional simplices making up $\approx \sigma_{(m)}$ by $\Delta\left(» \sigma_{(m)}\right)$.

Proof. The proof for the interpolation property on the dual mesh is similar to that of Theorem 3.1. We show that the left- and right-hand side of (6.3) satisfy (up to a sign) the same recursion formula, namely

$$
A_{\sigma_{(m)}}^{(l)}= \pm \sum_{\tau_{(m+1)} \in \partial \sigma_{(m)}}(-1)^{l-m}\left(\frac{o_{\tau_{(m+1)} \sigma_{(m)}}}{l-m}\right)\left(x_{\tau_{(m+1)}}^{(1)}-x_{\sigma_{(m)}}^{(1)}\right) \wedge A_{\tau_{(m+1)}}^{(l)}
$$

together with $A_{\sigma_{(l)}}^{(l)}=\left\langle\tilde{a}^{(d-l)}, \star \sigma_{(l)}\right\rangle$.
From the recursion formula (6.1) it follows that the right-hand side of (6.3) satisfies

$$
\begin{aligned}
& \sum_{\sigma_{(l)} \in \partial^{m-l} \sigma_{(m)}}\left\langle\tilde{a}^{(d-l)}, \sharp \sigma_{(l)}\right\rangle \sigma_{(m)} \sigma^{(l)} \\
& =\sum_{\sigma_{(l)} \in \partial^{m-l} \sigma_{(m)}} \sum_{\substack{\tau_{(m+1)} \\
\epsilon \partial^{-1} \sigma_{(m)} \cap \partial^{l-m-1} \sigma_{(l)}}}(-1)^{l-m} \\
& \cdot\left(\frac{o_{\tau_{(m+1)} \sigma_{(m)}}}{l-m}\right) \cdot\left(x_{\tau_{(m+1)}^{(1)}}^{(1)}-x_{\sigma_{(m)}}^{(1)}\right) \wedge\left(\left\langle\star \tilde{a}^{(d-l)}, \sharp \sigma_{(l)}\right\rangle{ }_{\tau_{(m+1)}} \sigma^{(l)}\right) \\
& =\sum_{\tau_{(m+1)} \in \mathcal{O}^{-1} \sigma_{(m)}} \sum_{\sigma_{(l)} \in \partial^{l-m-1} \tau_{(m+1)}}(-1)^{l-m} \\
& \cdot\left(\frac{o_{\tau_{(m+1)} \sigma_{(m)}}}{l-m}\right)\left(x_{\tau_{(m+1)}^{(1)}}^{(1)}-x_{\sigma_{(m)}^{(1)}}^{(1)}\right) \wedge\left(\left\langle\tilde{a}^{(d-l)}, \sharp \sigma_{(l)}\right\rangle \tau_{\tau_{(m+1)}} \sigma^{(l)}\right) \\
& =\sum_{\tau_{(m+1)} \in \varrho^{-1} \sigma_{(m)}}(-1)^{l-m} \\
& \cdot\left(\frac{o_{\tau_{(m+1)} \sigma_{(m)}}}{l-m}\right)\left(x_{\tau_{(m+1)}^{(1)}}^{(1)}-x_{\sigma_{(m)}}^{(1)}\right) \wedge\left(\sum_{\sigma_{(l)} \in \partial^{l-m-1} \tau_{(m+1)}}\left\langle\tilde{a}^{(d-l)}, \sharp \sigma_{(l)}\right\rangle \tau_{\tau_{(m+1)}} \sigma^{(l)}\right) .
\end{aligned}
$$

The left-hand side again needs more work. We have

$$
\sum_{\tilde{s} \in \Delta\left(\sharp \sigma_{(k)}\right)} \tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}|\tilde{s}|=\sum_{i_{1}<\cdots<i_{l-m}} \sum_{\tilde{s} \in \Delta\left(\sharp \sigma_{(m)}\right)}\left(d x^{i_{1}} \wedge \cdots \wedge d x^{i_{l-m}}, \tilde{\star}_{\tilde{s}}\left(\tilde{a}^{(d-l)}\right)\right)_{\tilde{s}} d x^{i_{1}} \wedge \cdots \wedge d x^{i_{l-m}} .
$$

Similarly, as in the proof of the primal case we manipulate the term for the simplices as
follows

$$
\begin{aligned}
& \sum_{\tilde{s} \in \Delta\left(\notin \sigma_{(m)}\right)}\left(d x^{i_{1}} \cdots d x^{i_{l-m}}, \tilde{\star}_{\tilde{s}}\left(\tilde{a}^{(d-l)}\right)\right)_{\tilde{s}} \\
& =\sum_{\tilde{s} \in \Delta\left(* \sigma_{(m)}\right)}(-1)^{(d-l)(l-m)} \int_{s} d x^{i_{1} \cdots d x^{i_{l-m}} \wedge \tilde{a}^{(d-l)}, ~} \\
& =\left(\frac{(-1)^{(d-l)(l-m)}}{l-m}\right) \sum_{\tilde{s} \in \Delta\left(\nless \sigma_{(m)}\right)}
\end{aligned}
$$

$$
\begin{aligned}
& =\left(\frac{(-1)^{(d-l)(l-m)}}{l-m}\right) \sum_{t=1}^{l-m}(-1)^{t+1} \\
& \sum_{\tilde{s} \in \Delta\left(\notin \sigma_{(m)}\right)} \int_{\partial \tilde{s}}\left(x_{i_{t}}-x_{i_{t}}^{\sigma_{(m)}}\right) d x^{i_{1} \ldots \widehat{d x^{i_{t}}} \ldots d x^{i_{l-m}} \wedge \tilde{a}^{(d-l)}} \\
& =\left(\frac{(-1)^{(d-l)(l-m)}}{l-m}\right) \sum_{t=1}^{l-m}(-1)^{t+1} \sum_{\tau_{(m+1)} \in \mathcal{D}^{-1} \sigma_{(m)}} o_{\star \sigma_{(m)}{ }^{\star} \tau_{(m+1)}} \\
& \sum_{\tilde{s} \in \Delta\left(\stackrel{\star}{ } \tau_{(m+1)}\right)} \int_{\tilde{s}}\left(x_{i_{t}}-x_{i_{t}}^{\sigma_{(m)}}\right) d x^{i_{1}} \ldots \widehat{d x^{i_{t}}} \ldots d x^{i_{l-m}} \wedge \tilde{a}^{(d-l)},
\end{aligned}
$$

where, in the last equality, we used the fact that the integrals over the boundaries $\partial \tilde{s}$ for $\tilde{s} \in \Delta\left(\sharp \sigma_{(m)}\right)$ that lie in the interior of $\sharp \sigma_{(m)}$ cancel. We consider the integral over one of the faces $\star \tau_{(m+1)}$ and rewrite it by applying the midpoint rule (which is exact here because the term to be integrated is linear) on the simplex:

$$
\begin{aligned}
& \sum_{\tilde{s} \in \Delta\left(\stackrel{\star}{ } \tau_{(m+1)}\right)} \int_{\tilde{s}}\left(x_{i_{t}}-x_{i_{t}}^{\sigma_{(m)}}\right) d x^{i_{1} \ldots \widehat{d x^{i_{t}}} \ldots d x^{i_{l-m}} \wedge \tilde{a}^{(d-l)}, ~}
\end{aligned}
$$

$$
\begin{aligned}
& =\sum_{\tilde{s} \in \Delta\left(\star \tau_{(m+1)}\right)} \int_{\tilde{s}}\left(\left(x_{i_{t}}^{\tilde{i_{t}}}-x_{i_{t}}^{\tau_{(m+1)}}\right)+\left(x_{i_{t}}^{\tau_{(m+1)}}-x_{i_{t}}^{\sigma_{(m)}}\right)\right) d x^{i_{1}} \cdots \widehat{d x^{i_{t}}} \ldots d x^{i_{l-m}} \wedge \tilde{a}^{(d-l)} .
\end{aligned}
$$

We introduce now, for notational convenience, the following two quantities

$$
\begin{aligned}
& I_{\dot{\star} \tau_{(m+1)}}=\sum_{i_{1}<\cdots<i_{l-m}}\left(\sum_{t=1}^{l-m}(-1)^{t+1} \sum_{\tilde{s} \in \Delta\left(\dot{\&} \tau_{(m+1)}\right)}\right.
\end{aligned}
$$

$$
\begin{aligned}
& I_{\star \tau_{(m+1)}, \Delta}=\sum_{i_{1}<\cdots<i_{l-m}}\left(\sum_{t=1}^{l-m}(-1)^{t+1} \sum_{\tilde{s} \in \Delta\left(\star \tau_{(m+1)}\right)}\right.
\end{aligned}
$$

Note that we have

$$
\sum_{\tilde{s} \in \Delta\left(\sharp \sigma_{(m)}\right)} \tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}|\tilde{s}|=\left(\frac{(-1)^{(d-l)(l-m)}}{l-m}\right)_{\tau_{(m+1)} \in \partial^{-1} \sigma_{(m)}} o_{\sharp \sigma_{(m)}{ }^{\star} \tau_{(m+1)}}\left(I_{\sharp} \tau_{(m+1)}+I_{\sharp} \tau_{(m+1)}, \Delta\right) .
$$

Similarly as in the proof of Theorem 3.1, we can rewrite $I_{\star \tau_{(m+1)}}$ and $I_{\star \tau_{(m+1)}, \Delta}$ by changing summation order, as

$$
\begin{aligned}
& I_{\star \tau_{(m+1)}}=\left(x_{\tau_{(m+1)}}^{(1)}-x_{\sigma_{(m)}}^{(1)}\right) \\
& \wedge \sum_{\tilde{s} \in \Delta\left(\dot{ } \nmid \tau_{(m+1)}\right)} \sum_{i_{1}<\cdots<i_{l-m-1}}\left(\int_{\tilde{s}} d x^{i_{1}} \cdots d x^{i_{l-m-1}} \wedge \tilde{a}^{(d-l)}\right) d x^{i_{1}} \cdots d x^{i_{l-m-1}}, \\
& I_{\star \tau_{(m+1)}, \Delta}=\sum_{\tilde{s} \in \Delta\left(\underset{ }{\star} \tau_{(m+1)}\right)}\left(x_{\tilde{s}}^{(1)}-x_{\tau_{(m+1)}}^{(1)}\right)
\end{aligned}
$$

We continue with $I_{\sharp \tau_{(m+1)}}$ and find

$$
\begin{aligned}
& I_{\star \tau_{(m+1)}}=\left(x_{\tau_{(m+1)}^{(1)}}^{(1)}-x_{\sigma_{(m)}}^{(1)}\right) \wedge \sum_{i_{1}<\cdots<i_{l-m-1} \tilde{s} \in \Delta\left(* \tau_{(m+1)}\right)}(-1)^{(d-l)(l-m-1)}
\end{aligned}
$$

$$
\begin{aligned}
& =(-1)^{(d-l)(l-m-1)}\left(x_{\tau_{(m+1)}^{(1)}}^{(1)}-x_{\sigma_{(m)}^{(1)}}^{(1)}\right) \wedge\left(\sum_{\tilde{s} \in \Delta\left(\star \tau_{(m+1)}\right)} \tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}|\tilde{s}|\right) \text {. }
\end{aligned}
$$

Similarly,

$$
I_{\star \tau_{(m+1)}, \Delta}=(-1)^{(d-l)(l-m-1)} \sum_{\tilde{s} \in \Delta\left(\star \tau_{(m+1)}\right)}\left(x_{\tilde{s}}^{(1)}-x_{\tau_{(m+1)}^{(1)}}^{(1)}\right) \wedge\left(\tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}|\tilde{s}|\right) .
$$

From this it follows, using $o_{\star \sigma_{(m)} \star \tau_{(m+1)}}=(-1)^{m+1} o_{\tau_{(m+1)} \sigma_{(m)}}$ and $(-1)^{(d-l)(l-m-1)}(-1)^{(d-l)(l-m)}=$ $(-1)^{d-l}$, that we have

$$
\begin{aligned}
\sum_{\tilde{s} \in \Delta\left(\sharp \sigma_{(m)}\right)} \tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}|\tilde{s}|=(-1)^{d+1} & \sum_{\tau_{(m+1)} \in \mathcal{D}^{-1} \sigma_{(m)}}(-1)^{l-m}\left(\frac{o_{\tau_{(m+1)} \sigma_{(m)}}}{l-m}\right) \\
& \cdot\left(x_{\tau_{(m+1)}}^{(1)}-x_{\sigma_{(m)}}^{(1)}\right) \wedge\left(\sum_{\tilde{s} \in \Delta\left(\underset{\star}{ } \tau_{(m+1)}\right)} \tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}|\tilde{s}|\right) \\
+(-1)^{d+1} & \sum_{\tau_{(m+1)} \in \mathcal{D}^{-1} \sigma_{(m)}}(-1)^{l-m}\left(\frac{o_{\tau_{(m+1)} \sigma_{(m)}}}{l-m}\right) \\
& \cdot \sum_{\tilde{s} \in \Delta\left(\star \tau_{(m+1)}\right)}\left(x_{\tilde{s}}^{(1)}-x_{\tau_{(m+1)}}^{(1)}\right) \wedge\left(\tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}|\tilde{s}|\right)
\end{aligned}
$$

Thus we have found that the left-hand side satisfies the same recursion relation (up to a minus sign) as the right-hand side if and only if the second term is zero. This can be shown by geometric arguments. We prove it in Lemma 6.1.

If $d$ is odd then, $\sum_{\tilde{s} \in \Delta\left(\star \sigma_{(m)}\right)} \tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}|\tilde{s}|$ and $\sum_{\sigma_{(l)} \in \partial^{m-l} \sigma_{(m)}}\left\langle\tilde{a}^{(d-l)}, \star \sigma_{(l)}\right\rangle_{\sigma_{(m)}} \sigma^{(l)}$ satisfy the same recursive formula. So, (6.3) holds if $d$ is odd. If $d$ is even, then $\sum_{\tilde{s} \epsilon \Delta\left(\star \sigma_{(m)}\right)} \tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}|\tilde{s}|$ satisfies the recursion formula with an extra minus sign. They are both equal to $\left\langle\tilde{a}^{(d-l)}, \star \sigma_{(l)}\right\rangle$ if $m=l$. However, $\sum_{\tilde{s} \epsilon \Delta\left(\sharp \sigma_{(m)}\right)} \tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}|\tilde{s}|$ switches sign with respect to $\sum_{\sigma_{(l)} \in \partial^{m-l} \sigma_{(m)}}\left\langle\tilde{a}^{(d-l)}, \star \sigma_{(l)}\right\rangle \sigma_{(m)} \sigma^{(l)}$ for every rercursive step, hence we get an extra factor $(-1)^{l-m}$ if $d$ is even. Thus, we have shown that (6.3) holds.

In the proof of Theorem 6.1, we needed that on a simplicial mesh the term

$$
\begin{aligned}
& K_{\sigma_{(m)}}^{l, d}\left(\tilde{a}^{(d-l)}\right) \\
& :=(-1)^{d+1} \sum_{\tau_{(m+1)} \in \mathcal{Q}^{-1} \sigma_{(m)}}(-1)^{l-m}\left(\frac{o_{\tau_{(m+1)} \sigma_{(m)}}}{l-m}\right)_{\tilde{\tilde{s}} \in \Delta\left(\tilde{\star} \tau_{(m+1)}\right)}\left(x_{\tilde{s}}^{(1)}-x_{\tau_{(m+1)}}^{(1)}\right) \wedge\left(\tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}|\tilde{s}|\right),
\end{aligned}
$$

is zero for all primal cells $\sigma_{(m)}$ not in the boundary mesh and any $\tilde{a}^{(d-l)} \in \tilde{\Lambda}^{(d-l)} V$. In the next lemma we show that this indeed holds. Moreover, we determine the value of the term for cells $\sigma_{(m)}$ that are part of the boundary mesh.

Lemma 6.1. Let $\mathcal{G}=\left\{C_{(d)}(\Omega), \ldots, C_{(0)}(\Omega)\right\}$ be a simplicial cell complex and let $\tilde{a}^{(d-l)} \in$
 $0 \leq m<l \leq d \leq 3$.

For $\sigma_{(m)} \in C_{(m)}(\partial \Omega)$ we have $K_{\sigma_{(m)}}^{l, d}\left(\tilde{a}^{(d-l)}\right)=0$ if $l=d$, and, the four remaining cases
satisfying $0 \leq m<l \leq d \leq 3$ are given by

$$
\begin{align*}
& K_{\sigma_{(0)}}^{1,2}\left(\tilde{a}^{(1)}\right)=\frac{1}{2} \sum_{\tau_{(1)} \in \partial_{\mathrm{b}}^{-1} \sigma_{(0)}} o_{\tau_{(1)} \sigma_{(0)}}\left(x_{\sigma_{(0)}^{(1)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\dot{\text { m }}_{\mathrm{b}} \tau_{(1)}}\left(x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\sigma_{(0)}}\right)\right\rangle \text {, }  \tag{6.4}\\
& K_{\sigma_{(1)}}^{2,3}\left(\tilde{a}^{(1)}\right)=\frac{1}{4} \sum_{\tau_{(2)} \in \partial_{\mathrm{b}}^{-1} \sigma_{(1)}} o_{\tau_{(2)} \tau_{(1)}}\left(x_{\sigma_{(1)}}^{(1)}-x_{\tau_{(2)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\dot{\star}_{\mathrm{b}} \tau_{(2)}}\left(x_{(1)}^{\tau_{(2)}}-x_{(1)}^{\sigma_{(1)}}\right)\right\rangle,  \tag{6.5}\\
& K_{\sigma_{(0)}}^{1,3}\left(\tilde{a}^{(2)}\right)=\frac{5}{36} \sum_{\left(\tau_{(2)}, \tau_{(1)}\right) \in I_{(0)}^{\mathrm{b}}} o_{\tau_{(2)} \tau_{(1)}} o_{\tau_{(1)} \sigma_{(0)}}\left(x_{\tau_{(1)}}^{(1)}-x_{\sigma_{(0)}}^{(1)}\right)  \tag{6.6}\\
& \cdot\left\langle\tilde{a}^{(2)}, o_{\dot{\star}_{\mathrm{b}} \tau_{(2)}}\left(x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\sigma_{(0)}}\right) \wedge x_{(1)}^{\tau_{(2)}}\right\rangle, \\
& K_{\sigma_{(0)}}^{2,3}\left(\tilde{a}^{(1)}\right)=\frac{5}{72} \sum_{\left(\tau_{(2)}, \tau_{(1)}\right) \in I_{\sigma_{(0)}^{b}}^{\mathrm{b}}} o_{\tau_{(2)} \tau_{(1)}} o_{\tau_{(1)} \sigma_{(0)}}\left\langle\tilde{a}^{(1)}, o_{\dot{\star}_{\mathrm{b}} \tau_{(2)}}\left(x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\sigma_{(0)}}\right)\right\rangle  \tag{6.7}\\
& \cdot x_{\tau_{(2)}}^{(1)} \wedge\left(x_{\tau_{(1)}}^{(1)}-x_{\sigma_{(0)}}^{(1)}\right),
\end{align*}
$$

where $I_{\sigma_{(0)}}^{\mathrm{b}}$ is the boundary mesh-analogue of $I_{\sigma_{(0)}}$.
Proof. Unfortunately, we have not found a proof yet that covers all the cases at once, hence we consider the 10 cases $0 \leq m<l \leq d \leq 3$ in turn.

## Proof for simple cases

The three cases $m+1=l=d=1, m+1=l=d=2$ and $m+1=l=d=3$ are trivially zero. In these cases $\Delta\left(\stackrel{\star}{*} \tau_{(m+1)}\right)$ contains only one 0 -dimensional simplex $s$ with $x^{\tilde{s}}=x^{\tau_{(m+1)}}$.

Next, we consider the remaining three cases with $l=d$, i.e., $m+2=l=d=2$, $m+2=l=d=3$ and $m+3=l=d=3$. Note that the simplex $\tilde{s} \in \Delta\left(\sharp \tau_{(m+1)}\right)$ in this situation is of the form $\left[x_{(1)}^{\tau_{(m+1)}}, x_{(1)}^{\tau_{(m+2)}}, \ldots, x_{(1)}^{\tau_{(d)}}\right]$. As a result we have

$$
\begin{aligned}
& \sum_{\tilde{s} \in \Delta\left(\dot{\sharp} \tau_{(m+1)}\right)}\left(x_{\tilde{s}}^{(1)}-x_{\tau_{(m+1)}^{(1)}}^{(1)}\right) \wedge\left(\tilde{\star}_{\tilde{s}} \tilde{a}^{(0)}|\tilde{s}|\right)
\end{aligned}
$$

$$
\begin{aligned}
& \wedge\left(x_{\tau_{(m+2)}^{(1)}}^{(1)}-x_{\tau_{(m+1)}}^{(1)}\right) \wedge \cdots \wedge\left(x_{\tau_{(d)}}^{(1)}-x_{\tau_{(d-1)}}^{(1)}\right) \tilde{a}^{(0)}(\stackrel{\wedge}{(d)}),
\end{aligned}
$$

where the barycenter 1-form of the simplex is given by

$$
x_{\tilde{s}}^{(1)}=\left(\frac{1}{d-m-1}\right) \sum_{k=m+1}^{d} x_{\tau_{(k)}}^{(1)} .
$$

We rewrite the first term in the exterior product as

$$
\begin{aligned}
x_{\tilde{s}}^{(1)}-x_{\tau_{(m+1)}}^{(1)}=( & \left.\frac{1}{d-m-1}\right)\left(x_{\tau_{(d)}}^{(1)}-x_{\tau_{(d-1)}}^{(1)}\right)+\left(\frac{2}{d-m-1}\right)\left(x_{\tau_{(d-1)}}^{(1)}-x_{\tau_{(d-2)}}^{(1)}\right) \\
& +\cdots+\left(\frac{d-m-2}{d-m-1}\right)\left(x_{\tau_{(m+2)}}^{(1)}-x_{\tau_{(m+1)}}^{(1)}\right) .
\end{aligned}
$$

Each of these terms already occurs in the other terms of the exterior product and as a result we see that the sum over the simplices is zero and $K_{\sigma_{(m)}}^{l, d}=0$ for all cases $0 \leq m<l=d \leq 3$.

For the above cases we see that each term in the sum over $\tau_{(m+1)} \in \partial^{-1} \sigma_{(m)}$ cancels by itself. For the remaining four cases this does not apply and cancelation of terms only happens once we consider the sum over $\tau_{(m+1)} \in \partial^{-1} \sigma_{(m)}$. The fact that in these four cases the term does not cancel at the boundary is related to this.

## Proof for $K_{\sigma_{(0)}}^{1,2}\left(\tilde{a}^{(1)}\right)$

We consider the four cases in turn and start with the only 2-dimensional case: $K_{\sigma_{(0)}}^{1,2}\left(\tilde{a}^{(1)}\right)$. Using the fact that for $\tilde{s}=\left[x_{(1)}^{\tau_{(1)}}, x_{(1)}^{\tau_{(2)}}\right]$, we have $x_{\tilde{s}}^{(1)}=\left(x_{\tau_{(1)}}^{(1)}+x_{\tau_{(2)}}^{(1)}\right) / 2$ and $\tilde{\star}_{\tilde{s}} \tilde{a}^{(1)}|\tilde{s}|=$ $\left\langle\tilde{a}^{(1)}, o_{\dot{\sharp} \tau_{(1)}{ }^{\star} \tau_{(2)}} o_{\dot{\sharp} \tau_{(2)}}\left(x_{\tau_{(2)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\right\rangle$, we find

$$
\begin{aligned}
& K_{\sigma_{(0)}}^{1,2}\left(\tilde{a}^{(1)}\right) \\
& \quad=\frac{1}{2} \sum_{\tau_{(1)} \in \partial^{-1} \sigma_{(0)}} \sum_{\tau_{(2)} \in \mathcal{\partial}^{-1} \tau_{(1)}} o_{\tau_{(2)} \tau_{(1)}} o_{\tau_{(1)} \sigma_{(0)}}\left(x_{\tau_{(2)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\star \tau_{(2)}}\left(x_{(1)}^{\left.\left.\tau_{(2)}^{(2)}-x_{(1)}^{\tau_{(1)}}\right)\right\rangle .} .\right.\right.
\end{aligned}
$$

We change the order of the sums, such that we can consider one primal $d$-simplex at a time:

$$
\begin{aligned}
& K_{\sigma_{(0)}}^{1,2}\left(\tilde{a}^{(1)}\right) \\
&=\frac{1}{2} \sum_{\tau_{(2)} \in \partial^{-2}} \sigma_{\sigma_{(0)}} \underbrace{\sum_{(1) \in \partial \tau_{(2)} \cap \partial^{-1} \sigma_{(0)}} o_{\tau_{(2)} \tau_{(1)}} o_{\tau_{(1)} \sigma_{(0)}}\left(x_{\tau_{(2)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\sharp \tau_{(2)}}\left(x_{(1)}^{\left.\left.\tau_{(2)}^{(2)}-x_{(1)}^{\tau_{(1)}}\right)\right\rangle} .\right.\right.}_{=: k_{\tau_{(2)}}} .
\end{aligned}
$$

We denote by $\tau_{(1)}^{a}$ and $\tau_{(1)}^{b}$ the two elements of $\partial \tau_{(2)} \cap \partial^{-1} \sigma_{(0)}$. From the definition of induced orientation it can be shown that $o:=o_{\tau_{(2)} \tau_{(1)}^{a}} o_{\tau_{(1)}^{a}}^{a} \sigma_{(0)}=-o_{\tau_{(2)} \tau_{(1)}^{b}} o_{(1)}^{b} \sigma_{(0)}$. The barycenter 1-form of the 2 -simplex $\tau_{(2)}$ is given by

$$
x_{\tau_{(2)}}^{(1)}=\frac{2}{3}\left(x_{\tau_{(1)}^{a}}^{(1)}+x_{\tau_{(1)}^{b}}^{(1)}\right)-\frac{1}{3} x_{\sigma_{(0)}^{(1)}}^{(1)} .
$$

As result, we find, after some algebra,

$$
\begin{aligned}
& k_{\tau_{(2)}} \\
& =\frac{o}{3}\left(\left(\left(x_{\tau_{(1)}^{b}}^{(1)}-x_{\tau_{(1)}^{a}}^{(1)}\right)+\left(x_{\tau_{(1)}^{b}}^{(1)}-x_{\sigma_{(0)}^{(1)}}^{(1)}\right)\right)\left\{\tilde{a}^{(1)}, o_{\star \tau_{(2)}}\left(\left(x_{(1)}^{\tau_{(1)}^{b}}-x_{(1)}^{\tau_{(1)}^{a}}\right)+\left(x_{(1)}^{\tau_{(1)}^{b}}-x_{(1)}^{\sigma_{(0)}}\right)\right)\right)\right. \\
& \left.-\left(\left(x_{\tau_{(1)}^{a}}^{(1)}-x_{\tau_{(1)}^{b}}^{(1)}\right)+\left(x_{\tau_{(1)}^{a}}^{(1)}-x_{\sigma_{(0)}}^{(1)}\right)\right)\left\langle\tilde{a}^{(1)}, o_{\star \tau_{(2)}}\left(\left(x_{(1)}^{\tau_{(1)}^{a}}-x_{(1)}^{\tau_{(1)}^{b}}\right)+\left(x_{(1)}^{\tau_{(1)}^{a}}-x_{(1)}^{\sigma_{(0)}}\right)\right)\right\rangle\right) \\
& =o\left(\left(x_{\sigma_{(0)}^{(1)}}^{(1)}-x_{\tau_{(1)}^{a}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\star \tau_{(2)}}\left(x_{(1)}^{\tau_{(1)}^{a}}-x_{(1)}^{\sigma_{(0)}}\right)\right\rangle\right. \\
& \left.-\left(x_{\sigma_{(0)}}^{(1)}-x_{\tau_{(1)}^{b}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\star \tau}(2)\left(x_{(1)}^{\tau_{(1)}^{b}}-x_{(1)}^{\sigma_{(0)}}\right)\right\rangle\right) .
\end{aligned}
$$

There are no terms mixing $x_{\tau_{(1)}^{(1)}}^{(1)}$ and $x_{(1)}^{\tau_{(1)}^{b}}$ or $x_{\tau_{(1)}^{b}}^{(1)}$ and $x_{(1)}^{\tau_{(1)}^{a}}$. When we go back to the original sum order we obtain

$$
\begin{aligned}
& K_{\sigma_{(0)}}^{1,2}\left(\tilde{a}^{(1)}\right) \\
& \quad=\frac{1}{2} \sum_{\tau_{(1)} \in \partial^{-1} \sigma_{(0)}} \sum_{\tau_{(2)} \in \partial^{-1} \tau_{(1)}} o_{\tau_{(2)} \tau_{(1)}} o_{\tau_{(1)} \sigma_{(0)}}\left(x_{\sigma_{(0)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\star \tau_{(2)}}\left(x_{(1)}^{\tau_{(1)}^{(1)}}-x_{(1)}^{\sigma_{(0)}}\right)\right\rangle .
\end{aligned}
$$

Now, for every $\tau_{(1)}$, there two elements in $\partial^{-1} \tau_{(1)}\left(\right.$ if $\left.\tau_{(1)} \notin C_{(1)}(\partial \Omega)\right)$, let's denote them by $\tau_{(2)}^{a}$ and $\tau_{(2)}^{b}$. Because $o_{\star} \tau_{(2)}^{a} o_{(2)}^{a} \tau_{(1)}=-o_{\sharp \pi \tau_{(2)}^{b}} o_{(2)}^{b} \tau_{(1)}$ we find, for $\tau_{(1)} \notin C_{(1)}(\partial \Omega)$,

$$
\sum_{\tau_{(2)} \in \partial^{-1} \tau_{(1)}} o_{\tau_{(2)} \tau_{(1)}} o_{\tau_{(1)} \sigma_{(0)}}\left(x_{\sigma_{(0)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\sharp \tau_{(2)}}\left(x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\sigma_{(0)}}\right)\right\rangle=0 .
$$

However, for $\tau_{(1)} \in C_{(1)}(\partial \Omega)$, there is only a single cell $\tau_{(2)}$ in $\partial^{-1} \tau_{(1)}$ and the sum does not cancel. Thus, we find, using the fact that $o_{\ddot{\sharp} \tau_{(2)}} o_{\tau_{(2)} \tau_{(1)}}=o_{\ddot{\sharp}_{\mathrm{b}} \tau_{(2)}}$, (6.4).

Proof for $K_{\sigma_{(1)}}^{2,3}\left(\tilde{a}^{(1)}\right)$
We continue with $K_{\sigma_{(1)}}^{2,3}\left(\tilde{a}^{(1)}\right)$, which is given by

$$
\begin{aligned}
& K_{\sigma_{(1)}}^{2,3}\left(\tilde{a}^{(1)}\right) \\
& \quad=\frac{1}{2} \sum_{\tau_{(3)} \in \partial^{-2} \sigma_{(1)}} \underbrace{\sum_{\tau_{(2)} \in \partial \tau_{(3)} \cap \partial^{-1} \sigma_{(1)}} o_{\tau_{(3)} \tau_{(2)}} o_{\tau_{(2)} \tau_{(1)}}\left(x_{\tau_{(2)}}^{(1)}-x_{\tau_{(3)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\star \tau_{(3)}}\left(x_{(1)}^{\tau_{(2)}}-x_{(1)}^{\tau_{(3)}}\right)\right\rangle}_{=k_{\tau_{(3)}}} .
\end{aligned}
$$

The derivation for $K_{\sigma_{(1)}}^{2,3}\left(\tilde{a}^{(1)}\right)$ is completely analogous to the one for $K_{\sigma_{(0)}}^{1,2}\left(\tilde{a}^{(1)}\right)$, only the numbers are different. The barycenter 1-form $x_{\tau_{(3)}}^{(1)}$ is now given in terms of the barycenter 1-forms $x_{\tau_{(2)}^{a}}^{(1)}$ and $x_{\tau_{(2)}^{b}}^{(1)}$ of the two elements of $\partial \tau_{(3)} \cap \partial^{-1} \sigma_{(1)}$ by

$$
x_{\tau_{(3)}}^{(1)}=\frac{3}{4}\left(x_{\tau_{(2)}^{a}}^{(1)}+x_{\tau_{(2)}^{b}}^{(1)}\right)-\frac{1}{2} x_{\sigma_{(1)}}^{(1)} .
$$

This time we find

$$
k_{\tau_{(3)}}=\frac{1}{2} \sum_{\tau_{(2)} \in \partial \tau_{(3)} \cap \partial^{-1} \sigma_{(1)}} o_{\tau_{(3)} \tau_{(2)}} o_{\tau_{(2)} \tau_{(1)}}\left(x_{\sigma_{(1)}}^{(1)}-x_{\tau_{(2)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\sharp \tau_{(3)}}\left(x_{\tau_{(2)}}^{(1)}-x_{\sigma_{(1)}}^{(1)}\right)\right\rangle,
$$

which implies

$$
\begin{aligned}
& K_{\sigma_{(1)}}^{2,3}\left(\tilde{a}^{(1)}\right) \\
& \quad=\frac{1}{4} \sum_{\tau_{(2)} \in \partial^{-1}} \sum_{\sigma_{(1)} \tau_{(3)} \in \partial^{-1} \tau_{(2)}} o_{\tau_{(3)} \tau_{(2)}} o_{\tau_{(2)} \tau_{(1)}}\left(x_{\sigma_{(1)}}^{(1)}-x_{\tau_{(2)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\star \tau_{(3)}}\left(x_{\tau_{(2)}}^{(1)}-x_{\sigma_{(1)}}^{(1)}\right)\right\rangle .
\end{aligned}
$$

This shows that $K_{\sigma_{(1)}}^{2,3}=0$ for $\sigma_{(1)}$ not in the boundary and is given by (6.5) for $\sigma_{(1)} \epsilon$ $C_{(1)}(\partial \Omega)$, where we used $o_{\tau_{(3)} \tau_{(2)}} o_{\dot{\sharp} \tau_{(3)}}=o_{\dot{\sharp}_{\mathrm{b}} \tau_{(2)}}$.

Proof for $K_{\sigma_{(0)}}^{1,3}\left(\tilde{a}^{(2)}\right)$
The derivation of the expression for $K_{\sigma_{(0)}}^{1,3}\left(\tilde{a}^{(2)}\right)$ is more involved. To start, we note that for $\tilde{s}=\left[x_{(1)}^{\tau_{(1)}}, x_{(1)}^{\tau_{(2)}}, x_{(1)}^{\tau_{(3)}}\right]$, we have

$$
\tilde{\star}_{\tilde{s}} \tilde{a}^{(2)}|\tilde{s}|=\frac{1}{2}\left\langle\tilde{a}^{(2)}, o_{\sharp} \tau_{(1)}{ }^{\star} \tau_{(2)} o_{\star<} \tau_{(2)} \tau_{(3)} o_{\star<} \tau_{(3)}\left(x_{(1)}^{\tau_{(2)}}-x_{(1)}^{\tau_{(1)}}\right) \wedge\left(x_{(1)}^{\tau_{(3)}}-x_{(1)}^{\tau_{(2)}^{(2)}}\right)\right\rangle .
$$

From this it follows by (3.4) that

$$
K_{\sigma_{(0)}}^{1,3}\left(\tilde{a}^{(2)}\right)=-\frac{1}{6} \sum_{\tau_{(1)} \in \partial^{-1} \sigma_{(0)}} o_{\tau_{(1)} \sigma_{(0)}} \sum_{\tau_{(3)} \in \partial^{-2} \tau_{(1)}} k_{\tau_{(3)}, \tau_{(1)}},
$$

where the minus sign comes from the fact that $(-1)^{l-m}=-1$, and

$$
\begin{aligned}
& k_{\tau_{(3)}, \tau_{(1)}}:=\sum_{\tau_{(2)} \in \partial \tau_{(3)} \cap \partial^{-1} \tau_{(1)}} o_{\tau_{(3)} \tau_{(2)}} o_{\tau_{(2)} \tau_{(1)}}\left(\left(x_{\tau_{(3)}}^{(1)}-x_{\tau_{(2)}}^{(1)}\right)+2\left(x_{\tau_{(2)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\right) \\
& \cdot\left\langle\tilde{a}^{(2)}, o_{\mathfrak{k} \tau_{(3)}}\left(x_{(1)}^{\tau_{(3)}}-x_{(1)}^{\tau_{(2)}}\right) \wedge\left(x_{(1)}^{\tau_{(2)}}-x_{(1)}^{\tau_{(1)}}\right)\right\rangle .
\end{aligned}
$$

We use again the fact that $\partial \tau_{(3)} \cap \partial^{-1} \tau_{(1)}$ has two elements, $\tau_{(2)}^{a}$ and $\tau_{(2)}^{b}$. We can rewrite everything again in terms of the barycenters of $\tau_{(2)}^{a}, \tau_{(2)}^{b}$ and $\tau_{(1)}$ :

$$
\begin{aligned}
\left(x_{\tau_{(3)}}^{(1)}-x_{\tau_{(2)}^{a}}^{(1)}\right)+2\left(x_{\tau_{(2)}^{a}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right) & =\frac{3}{4}\left(x_{\tau_{(2)}^{b}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)+\frac{7}{4}\left(x_{\tau_{(2)}^{a}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right), \\
\left(x_{\tau_{(3)}}^{(1)}-x_{\tau_{(2)}^{b}}^{(1)}\right)+2\left(x_{\tau_{(2)}^{b}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right) & =\frac{3}{4}\left(x_{\tau_{(2)}^{a}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)+\frac{7}{4}\left(x_{\tau_{(2)}^{b}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right), \\
\left(x_{(1)}^{\tau_{(3)}}-x_{(1)}^{\tau_{(2)}^{a}}\right) & =\frac{3}{4}\left(x_{(2)}^{\tau_{(2)}^{b}}-x_{(1)}^{\tau_{(1)}}\right)-\frac{1}{4}\left(x_{(1)}^{\tau_{(2)}^{a}}-x_{(1)}^{\tau_{(1)}}\right), \\
\left(x_{(1)}^{\tau_{(3)}}-x_{(1)}^{\tau_{(2)}^{b}}\right) & =\frac{3}{4}\left(x_{(1)}^{\tau_{(2)}^{a}}-x_{(1)}^{\tau_{(1)}}\right)-\frac{1}{4}\left(x_{(1)}^{\tau_{(2)}^{b}}-x_{(1)}^{\tau_{(1)}}\right) .
\end{aligned}
$$

We define again $o:=o_{\tau_{(3)} \tau_{(2)}^{a}} o_{\tau_{(2)}}^{a} \tau_{(1)}=-o_{\tau_{(3)} \tau_{(2)}^{b}} o_{\tau_{(2)}^{b} \tau_{(1)}}$. Using this and the above formulas we can write, and rewrite after some algebra ${ }^{2}$,

$$
\begin{aligned}
& k_{\tau_{(3)}, \tau_{(1)}}=\frac{3 o}{16}\left\{\left[7\left(x_{\tau_{(2)}^{a}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)+3\left(x_{\tau_{(2)}^{b}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\right]\right. \\
& \cdot\left\langle\tilde{a}^{(2)}, o_{\star \tau_{(3)}}\left(x_{(1)}^{\tau_{(2)}^{b}}-x_{(1)}^{\tau_{(1)}}\right) \wedge\left(x_{(1)}^{\tau_{(2)}^{a}}-x_{(1)}^{\tau_{(1)}}\right)\right\rangle \\
& -\left[7\left(x_{\tau_{(2)}^{b}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)+3\left(x_{\tau_{(2)}^{(1)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\right] \\
& \left.\cdot\left\{\tilde{a}^{(2)}, o_{\sharp \tau_{(3)}}\left(x_{(1)}^{\tau_{(1)}^{a}}-x_{(1)}^{\tau_{(1)}}\right) \wedge\left(x_{(1)}^{\tau_{(2)}^{b}}-x_{(1)}^{\tau_{(1)}}\right)\right)\right\} \\
& =\frac{30 o}{16}\left[\left(x_{(1)}^{\tau_{(1)}^{a}}-x_{(1)}^{\tau_{(1)}}\right)+\left(x_{(1)}^{\tau_{(1)}^{b}}-x_{(1)}^{\tau_{(1)}}\right)\right]\left\langle\tilde{a}^{(2)}, o_{\star<} \tau_{(3)}\left(x_{(1)}^{\tau_{(2)}^{b}}-x_{(1)}^{\tau_{(1)}}\right) \wedge\left(x_{(1)}^{\tau_{(1)}^{a}}-x_{(1)}^{\tau_{(1)}}\right)\right\rangle .
\end{aligned}
$$

We use the relation

$$
\left(x_{(1)}^{\tau_{(2)}^{a}}-x_{(1)}^{\tau_{(1)}}\right)+\left(x_{(1)}^{\tau_{(2)}^{b}}-x_{(1)}^{\tau_{(1)}}\right)=\frac{4}{3}\left(x_{(1)}^{\tau_{(3)}}-x_{(1)}^{\tau_{(1)}}\right),
$$

to find the somewhat simpler expression

$$
\begin{aligned}
& k_{\tau_{(3)}, \tau_{(1)}}=\frac{5}{3} \sum_{\tau_{(2)} \in \partial \tau_{(3)} \cap \partial^{-1} \tau_{(1)}} o_{\tau_{(3)} \tau_{(2)}} o_{\tau_{(2)} \tau_{(1)}}\left(x_{(1)}^{\tau_{(3)}}-x_{(1)}^{\tau_{(1)}}\right) \\
& \cdot\left\langle\tilde{a}^{(2)}, o_{\star<\tau_{(3)}}\left(x_{(1)}^{\tau_{(3)}}-x_{(1)}^{\tau_{(1)}}\right) \wedge\left(x_{(1)}^{\tau_{(2)}}-x_{(1)}^{\tau_{(1)}}\right)\right\rangle .
\end{aligned}
$$

We write the resulting simpler complete expression for $K_{\sigma_{(0)}}^{1,3}$ as a sum per $d$-simplex to exploit the geometric properties of the simplex:

$$
\begin{aligned}
K_{\sigma_{(0)}}^{1,3}\left(\tilde{a}^{(2)}\right)= & -\frac{5}{18} \sum_{\left(\tau_{(3)}, \tau_{(2)}, \tau_{(1)}\right) \in I_{\sigma_{(0)}}} o_{\tau_{(3)} \tau_{(2)}} o_{\tau_{(2)}} \tau_{(1)} o_{\tau_{(1)} \tau_{(0)}}\left(x_{(1)}^{\tau_{(3)}}-x_{(1)}^{\tau_{(1)}}\right) \\
& \cdot\left\langle\tilde{a}^{(2)}, o_{\star \tau_{(3)}}\left(x_{(1)}^{\tau_{(3)}}-x_{(1)}^{\tau_{(1)}}\right) \wedge\left(x_{(1)}^{\tau_{(2)}}-x_{(1)}^{\tau_{(1)}}\right)\right\rangle \\
= & -\frac{5}{18} \sum_{\tau_{(3)} \in \mathcal{D}^{-3}} k_{\tau_{(3)}},
\end{aligned}
$$

with

$$
\begin{aligned}
k_{\tau_{(3)}}: & \sum_{\left(\tau_{(2)}, \tau_{(1)}\right) \in I_{\sigma}^{(0)}} o_{\tau_{(3)} \tau_{(2)}} o_{\tau_{(2)}} \tau_{(1)} o_{\tau_{(1)} \tau_{(0)}}\left(x_{(1)}^{\tau_{(3)}}-x_{(1)}^{\tau_{(1)}}\right) \\
& \cdot\left\langle\tilde{a}^{(2)}, o_{\star} \tau_{(3)}\left(x_{(1)}^{\tau_{(3)}}-x_{(1)}^{\tau_{(1)}}\right) \wedge\left(x_{(1)}^{\tau_{(2)}}-x_{(1)}^{\tau_{(1)}}\right)\right\rangle .
\end{aligned}
$$

[^45]

Figure 6.2: The mesh elements involved in the expression for $k_{\tau_{(3)}}$ (which itself is part of the sum $\left.K_{\sigma_{(0)}}^{1,3}\left(\tilde{a}^{(2)}\right)\right)$ are shown. In red we see the simplices, lying in $\tau_{(3)}$, that are part of the initial expression for $K_{\sigma_{(0)}}^{1,3}\left(\tilde{a}^{(2)}\right)$ that we started with.

There are 6 terms in the sum $k_{\tau_{(3)}}$. This can be verified in Figure 6.2. We rewrite $k_{\tau_{(3)}}$ in terms of the barycenter vectors and forms of $\sigma_{(0)}, \tau_{(1)}^{a}, \tau_{(1)}^{b}$ and $\tau_{(1)}^{c}$ only. Note that the following relations hold for the barycenter vectors

$$
\begin{gathered}
x_{(1)}^{\tau_{(3)}}-x_{(1)}^{\tau_{(1)}^{a}}=\frac{1}{2}\left(x_{(1)}^{\tau_{(1)}^{b}}-x_{(1)}^{\sigma_{(0)}}\right)+\frac{1}{2}\left(x_{(1)}^{\tau_{(1)}^{c}}-x_{(1)}^{\sigma_{(0)}}\right)-\frac{1}{2}\left(x_{(1)}^{\tau_{(1)}^{a}}-x_{(1)}^{\sigma_{(0)}}\right), \\
x_{(1)}^{\tau_{(3)}}-x_{(1)}^{\tau_{(1)}^{b}}=\frac{1}{2}\left(x_{(1)}^{\tau_{(1)}^{c}}-x_{(1)}^{\sigma_{(0)}}\right)+\frac{1}{2}\left(x_{(1)}^{\tau_{(1)}^{a_{(1)}}}-x_{(1)}^{\sigma_{(0)}}\right)-\frac{1}{2}\left(x_{(1)}^{\tau_{(1)}^{\tau_{(1)}}}-x_{(1)}^{\left.\sigma_{(0)}^{( }\right)}\right), \\
x_{(1)}^{\tau_{(3)}}-x_{(1)}^{\tau_{(1)}^{c}}=\frac{1}{2}\left(x_{(1)}^{\tau_{(1)}^{a}}-x_{(1)}^{\sigma_{(0)}}\right)+\frac{1}{2}\left(x_{(1)}^{\tau_{(1)}^{b}}-x_{(1)}^{\sigma_{(0)}}\right)-\frac{1}{2}\left(x_{(1)}^{\tau_{(1)}^{c}}-x_{(1)}^{\sigma_{(0)}}\right),
\end{gathered}
$$

and

$$
\begin{aligned}
& x_{(1)}^{\tau_{(1)}^{a}}-x_{(1)}^{\tau_{(1)}^{b}}=\frac{2}{3}\left(x_{(1)}^{\tau_{(1)}^{c}}-x_{(1)}^{\sigma_{(0)}}\right)-\frac{1}{3}\left(x_{(1)}^{\tau_{(1)}^{b}}-x_{(1)}^{\sigma_{(0)}}\right), \\
& x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\tau_{(1)}^{c}}=\frac{2}{3}\left(x_{(1)}^{\tau_{(1)}^{b}}-x_{(1)}^{\sigma_{(0)}}\right)-\frac{1}{3}\left(x_{(1)}^{\tau_{(1)}^{c}}-x_{(1)}^{\sigma_{(0)}}\right), \\
& x_{(1)}^{\tau_{(1)}^{b}}-x_{(1)}^{\tau_{(1)}^{a}}=\frac{2}{3}\left(x_{(1)}^{\tau_{(1)}^{c}}-x_{(1)}^{\sigma_{(0)}}\right)-\frac{1}{3}\left(x_{(1)}^{\tau_{(1)}^{a}}-x_{(1)}^{\sigma_{(0)}}\right), \\
& x_{(1)}^{\tau_{(1)}^{b}}-x_{(1)}^{\tau_{(1)}^{c}}=\frac{2}{3}\left(x_{(1)}^{\tau_{(1)}^{a}}-x_{(1)}^{\sigma_{(0)}}\right)-\frac{1}{3}\left(x_{(1)}^{\tau_{(1)}^{c}}-x_{(1)}^{\sigma_{(0)}}\right), \\
& x_{(1)}^{\tau_{(2)}^{c}}-x_{(1)}^{\tau_{(1)}^{a}}=\frac{2}{3}\left(x_{(1)}^{\tau_{(1)}^{b}}-x_{(1)}^{\sigma_{(0)}}\right)-\frac{1}{3}\left(x_{(1)}^{\tau_{(1)}^{a}}-x_{(1)}^{\sigma_{(0)}}\right), \\
& x_{(1)}^{\tau_{(1)}^{c}}-x_{(1)}^{\tau_{(1)}^{b}}=\frac{2}{3}\left(x_{(1)}^{\tau_{(1)}^{a}}-x_{(1)}^{\sigma_{(0)}}\right)-\frac{1}{3}\left(x_{(1)}^{\tau_{(1)}^{b}}-x_{(1)}^{\sigma_{(0)}}\right) .
\end{aligned}
$$

The same expressions hold for the barycenter forms. Moreover, we have
$o:=o_{\tau_{(3)} \tau_{(2)}^{a}} o_{\tau_{(2)}^{a} \tau_{(1)}^{b}} o_{\tau_{(1)}^{b}} \sigma_{(0)}$ and

$$
\begin{aligned}
& o=o_{\tau_{(3)} \tau_{(2)}^{a}} o_{\tau_{(2)}^{a} \tau_{(1)}^{b}} o_{\tau_{(1)}^{b}}^{b} \sigma_{(0)}, \quad-o=o_{\tau_{(3)} \tau_{(2)}^{b}} o_{\tau_{(2)}^{b} \tau_{(1)}^{a}} o_{\tau_{(1)}^{a}}^{a} \sigma_{(0)}, \quad o=o_{\tau_{(3)} \tau_{(2)}^{c}} o_{\tau_{(2)}^{c}} \tau_{(1)}^{a} o_{\tau_{(1)}^{a}}^{a} \sigma_{(0)}, \\
& -o=o_{\tau_{(3)} \tau_{(2)}^{a}} o_{\tau_{(2)}^{a}}^{a} \tau_{(1)}^{c} o_{\tau_{(1)}^{c}}^{c} \sigma_{(0)}, \quad o=o_{\tau_{(3)} \tau_{(2)}^{b}} o_{\tau_{(2)}^{b}} \tau_{(1)}^{c} o_{(1)}^{c} \sigma_{(0)}, \quad-o=o_{\tau_{(3)} \tau_{(2)}^{c}} o_{\tau_{(2)}^{c} \tau_{(1)}^{b}} o_{\tau_{(1)}^{b}} \sigma_{(0)} .
\end{aligned}
$$

We substitute the above in the expression for $k_{\tau_{(3)}}$ and carefully simplify them. After trivial, but tedious algebraic manipulations, which we omit, we find

$$
\begin{aligned}
& k_{\tau_{(3)}}=\frac{1}{2}\left\{\left(x_{(1)}^{\tau_{(1)}^{a}}-x_{(1)}^{\sigma_{(0)}}\right)\left\langle\tilde{a}^{(2)}, o_{\dot{\star} \tau_{(3)}}\left(x_{(1)}^{\tau_{(1)}^{a}}-x_{(1)}^{\sigma_{(0)}}\right) \wedge\left(x_{(1)}^{\tau_{(2)}^{b}}-x_{(1)}^{\tau_{(2)}^{c}}\right)\right)\right. \\
& +\left(x_{(1)}^{\tau_{(1)}^{b}}-x_{(1)}^{\sigma_{(0)}}\right)\left\langle\tilde{a}^{(2)}, o_{\sharp \tau_{(3)}}\left(x_{(1)}^{\tau_{(1)}^{b}}-x_{(1)}^{\sigma_{(0)}}\right) \wedge\left(x_{(1)}^{\tau_{(2)}^{c}}-x_{(1)}^{\tau_{(2)}^{a}}\right)\right) \\
& \left.+\left(x_{(1)}^{\tau_{(1)}^{c}}-x_{(1)}^{\sigma_{(0)}}\right)\left\langle\tilde{a}^{(2)}, o_{\sharp<} \tau_{(3)}\left(x_{(1)}^{\tau_{(1)}^{c}}-x_{(1)}^{\sigma_{(0)}}\right) \wedge\left(x_{(1)}^{\tau_{(2)}^{a}}-x_{(1)}^{\tau_{(2)}^{c}}\right)\right)\right\} \\
& =\frac{1}{2} \sum_{\left(\tau_{(2)}, \tau_{(1)}\right) \in I_{\sigma(0)}^{\tau_{(0)}}} o_{\tau_{(3)} \tau_{(2)}} o_{\tau_{(2)} \tau_{(1)}} o_{\tau_{(1)} \sigma_{(0)}}\left(x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\sigma_{(0)}}\right) \\
& \cdot\left\langle\tilde{a}^{(2)}, o_{\star \tau_{(3)}} x_{(1)}^{\tau_{(2)}} \wedge\left(x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\sigma_{(0)}}\right)\right\rangle .
\end{aligned}
$$

Thus, finally we obtain

$$
\begin{gathered}
K_{\sigma_{(0)}}^{1,3}\left(\tilde{a}^{(2)}\right)=-\frac{5}{36} \sum_{\left(\tau_{(3)}, \tau_{(2)}, \tau_{(1)}\right) \in I_{\sigma_{(0)}}} o_{\tau_{(3)} \tau_{(2)}} o_{\tau_{(2)} \tau_{(1)}} o_{\tau_{(1)} \sigma_{(0)}}\left(x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\sigma_{(0)}}\right) \\
\cdot\left\langle\tilde{a}^{(2)}, o_{\star<\tau_{(3)}} x_{(1)}^{\tau_{(2)}} \wedge\left(x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\sigma_{(0)}}\right)\right\rangle .
\end{gathered}
$$

We see that this sum is independent of $\tau_{(3)}$. Interior faces $\tau_{(2)}$ have two corresponding volumes, $\tau_{(3)}^{a}$ and $\tau_{(3)}^{b}$, with $o_{\sharp \tau_{(3)}^{a}}^{a} o_{(3)}^{a} \tau_{(2)}=-o_{\star \tau_{(3)}^{b}} o_{\tau_{(3)}^{b} \tau_{(2)}}$, and as a result we see that the interior contributions cancel. So, if $\sigma_{(0)} \notin C_{(0)}(\partial \Omega)$, then $K_{\sigma_{(0)}}^{1,3}\left(\tilde{a}^{(2)}\right)=0$, else if $\sigma_{(0)} \in C_{(0)}(\partial \Omega)$ we find (6.6), where we used $o_{\sharp<\tau_{(3)}} o_{\tau_{(3)} \tau_{(2)}}=o_{\sharp_{\mathrm{m}} \tau_{(2)}}$, for $\tau_{(2)} \in C_{(2)}(\partial \Omega)$ and $\tau_{(3)}$ the unique cell in $\partial^{-1} \tau_{(2)}$.

Proof for $K_{\sigma_{(0)}}^{2,3}\left(\tilde{a}^{(1)}\right)$
Finally, we consider $K_{\sigma_{(0)}}^{2,3}\left(\tilde{a}^{(1)}\right)$. It can be verified geometrically that for $\tilde{s}=\left[x_{(1)}^{\tau_{(1)}}, x_{(1)}^{\tau_{(2)}}, x_{(1)}^{\tau_{(3)}}\right]$ we have

$$
\begin{aligned}
& \tilde{\star}_{\tilde{s}} \tilde{a}^{(1)}|\tilde{s}|=\frac{1}{2} o_{\sharp \tau_{(1)}{ }^{\star} \tau_{(2)}} o_{\sharp \tau_{(2)}{ }^{\star} \tau_{(3)}}\left(\left(x_{\tau_{(2)}}^{(1)}-x_{\tau_{(3)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\sharp \tau_{(3)}}\left(x_{\tau_{(1)}}^{(1)}-x_{\left.\tau_{(2)}\right)}^{(1)}\right)\right\rangle\right. \\
& \left.-\left(x_{\tau_{(1)}}^{(1)}-x_{\tau_{(2)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\star \tau_{(3)}}\left(x_{\tau_{(2)}}^{(1)}-x_{\tau_{(3)}}^{(1)}\right)\right\rangle\right) .
\end{aligned}
$$

Furthermore, we have

$$
x_{\tilde{s}}^{(1)}-x_{\tau_{(1)}}^{(1)}=\frac{1}{3}\left(x_{\tau_{(3)}}^{(1)}-x_{\tau_{(2)}}^{(1)}\right)+\frac{2}{3}\left(x_{\tau_{(2)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right) .
$$

From this it follows that

$$
\begin{aligned}
& K_{\sigma_{(0)}}^{2,3}\left(\tilde{a}^{(1)}\right) \\
&=\frac{1}{12} \sum_{\left(\tau_{(3)}, \tau_{(2)}, \tau_{(1)}\right) \in I_{\sigma_{(0)}}} o_{\tau_{(3)} \tau_{(2)}} o_{(2)} \tau_{(1)} o_{\tau_{(1)} \sigma_{(0)}}\{ \left\{\tilde{a}^{(1)}, o_{\sharp \leftarrow \tau_{(3)}}\left(\left(x_{(1)}^{\tau_{(3)}}-x_{(1)}^{\tau_{(2)}}\right) 2\left(x_{(1)}^{\tau_{(2)}}-x_{(1)}^{\tau_{(1)}}\right)\right)\right\rangle \\
&\left.\cdot\left(x_{\tau_{(3)}}^{(1)}-x_{\tau_{(2)}}^{(1)}\right) \wedge\left(x_{\tau_{(2)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\right\} .
\end{aligned}
$$

This expression has a striking similarity with the original expression for $K_{\sigma_{(0)}}^{1,3}\left(\tilde{a}^{(2)}\right)$. The term behind the sum is the same in both expressions with the exception that the 2 -vector that is in the duality pairing for $K_{\sigma_{(0)}}^{1,3}\left(\tilde{a}^{(2)}\right)$ is behind the duality pairing as a 2 -form for $K_{\sigma_{(0)}}^{2,3}\left(\tilde{a}^{(1)}\right)$ and vice versa. Furthermore the scalar in front of the sum is different. However, we can step by step repeat the reasoning of the $K_{\sigma_{(0)}}^{1,3}\left(\tilde{a}^{(2)}\right)$-case and as a result find

$$
\begin{gathered}
K_{\sigma_{(0)}}^{2,3}\left(\tilde{a}^{(1)}\right)=\frac{5}{72} \sum_{\left(\tau_{(3)}, \tau_{(2)}, \tau_{(1)}\right) \in I_{\sigma_{(0)}}} o_{\tau_{(3)} \tau_{(2)}} o_{\tau_{(2)} \tau_{(1)}} o_{\tau_{(1)} \sigma_{(0)}}\left\langle\tilde{a}^{(1)}, o_{\sharp<} \tau_{(3)}\left(x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\sigma_{(0)}}\right)\right\rangle \\
\cdot\left(x_{(1)}^{\tau_{(2)}} \wedge\left(x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\left.\sigma_{(0)}\right)}\right) .\right.
\end{gathered}
$$

By the now familiar arguments it follows that $K_{\sigma_{(0)}}^{2,3}\left(\tilde{a}^{(1)}\right)=0$ when $\sigma_{(0)}$ is not in the boundary mesh and we find (6.7) when $\sigma_{(0)}$ is in the boundary mesh.

We have seen in Theorem 6.1 that the dual interpolation formula (6.2) only applies for the dual $d$-cells $\star \sigma_{(0)}$ for which $\sigma_{(0)} \notin C_{(0)}(\partial \Omega)$. To formulate consistent dual discrete Hodge matrices $\tilde{\mathbb{H}}^{(l)}$ we need to extend this interpolation to the boundary cells. To do this we repeat the proof of Theorem 6.1, but now take into account that the cell may be part of the boundary mesh.

Theorem 6.2. (Dual mesh interpolation property.) Let $\mathcal{G}=\left\{C_{(d)}(\Omega), \ldots, C_{(0)}(\Omega)\right\}$ be a simplicial cell complex. For $(d-l)$-forms $\tilde{a}^{(d-l)} \in \tilde{\Lambda}^{(d-l)}$ and $\sigma_{(m)} \in C_{(m)}(\Omega)$ with $0 \leq m \leq l \leq d \leq 3$,

$$
\begin{aligned}
\sum_{\tilde{s} \in \Delta\left(\sharp \sigma_{(m)}\right)} \tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}|\tilde{s}|= & \sum_{\sigma_{(l)} \in \partial^{m-l} \sigma_{(m)}}\left\langle\tilde{a}^{(d-l)}, \star \sigma_{(l)}\right\rangle \sigma_{(m)} \sigma^{(l)} \\
& +\sum_{\left(\tau_{(d-1)}, \ldots, \tau_{(m+1)}\right) \in I_{\sigma_{(m)}^{\mathrm{b}}}} k_{\sigma_{(m)}^{l, d}, \tau_{(m+1)}, \ldots, \tau_{(d-1)}}^{l, d}\left(\tilde{a}^{(d-l)}\right),
\end{aligned}
$$

when $d$ is odd, and

$$
\begin{aligned}
& \sum_{\tilde{s} \in \Delta\left(\sharp \sigma_{(m)}\right)} \tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}|\tilde{s}|=(-1)^{l-m} \sum_{\sigma_{(l)} \in \partial^{m-l} \sigma_{(m)}}\left\langle\tilde{a}^{(d-l)}, \star \sigma_{(l)}\right\rangle \sigma_{(m)} \sigma^{(l)} \\
&+\sum_{\left(\tau_{(d-1)}, \ldots, \tau_{(m+1)}\right) \in I_{\sigma_{(m)}^{\mathrm{b}}}} k_{\sigma_{(m)}^{l, d}, \tau_{(m+1)}, \ldots, \tau_{(d-1)}}^{l}\left(\tilde{a}^{(d-l)}\right),
\end{aligned}
$$

when $d$ is even. The boundary contribution is only nonzero when $\sigma_{(m)} \in C_{(m)}(\partial \Omega)$ and $(m, l, d)$ is equal to either $(0,1,2),(0,1,3),(1,2,3)$, or, $(0,2,3)$. In these four cases the boundary term is given by

$$
\begin{align*}
& k_{\sigma_{(0)}, \tau_{(1)}}^{1,2}\left(\tilde{a}^{(1)}\right)=o_{\tau_{(1)} \sigma_{(0)}}\left(x_{\tau_{(1)}}^{(1)}-x_{\sigma_{(0)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\boldsymbol{\star}_{\mathrm{b}} \tau_{(1)}}\left(x_{(1)}^{\sigma_{(0)}}-x_{(1)}^{\tau_{(1)}}\right)\right\rangle,  \tag{6.8}\\
& k_{\sigma_{(0)}, \tau_{(1)}, \tau_{(2)}}^{1,3}\left(\tilde{a}^{(2)}\right)=\frac{o}{36}\left(7\left(x_{\tau_{(1)}}^{(1)}-x_{\sigma_{(0)}}^{(1)}\right)+6\left(x_{\tau_{(2)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\right) \\
& \cdot\left\langle\tilde{a}^{(2)}, o_{\dot{\text { в }}_{\mathrm{b}} \tau_{(2)}}\left(x_{(1)}^{\tau_{(2)}}-x_{(1)}^{\tau_{(1)}}\right) \wedge\left(x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\sigma_{(0)}}\right)\right\rangle,  \tag{6.9}\\
& k_{\sigma_{(1)}, \tau_{(2)}}^{2,3}\left(\tilde{a}^{(1)}\right)=\frac{o_{\tau_{(2)} \tau_{(1)}}}{4}\left(x_{\tau_{(2)}}^{(1)}-x_{\sigma_{(1)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\dot{\text { w }}_{\mathrm{b}} \tau_{(2)}}\left(x_{(1)}^{\tau_{(2)}}-x_{(1)}^{\sigma_{(1)}}\right)\right\rangle,  \tag{6.10}\\
& k_{\sigma_{(0)}, \tau_{(1)}, \tau_{(2)}}^{2,3}\left(\tilde{a}^{(1)}\right)=\frac{o}{72}\left(\left(x_{\tau_{(1)}}^{(1)}-x_{\sigma_{(0)}}^{(1)}\right) \wedge\left(x_{\tau_{(2)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\right) \\
& \cdot\left\langle\tilde{a}^{(1)}, o_{\dot{w}_{\mathrm{b}} \tau_{(2)}}\left(7\left(x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\sigma_{(0)}}\right)+15\left(x_{(1)}^{\tau_{(2)}}-x_{(1)}^{\tau_{(1)}}\right)\right)\right\rangle, \tag{6.11}
\end{align*}
$$

where o equals $o_{\tau_{(2)} \tau_{(1)}} o_{\tau_{(1)} \sigma_{(0)}}$.

Proof. When we repeat the proof of Theorem 6.1, but now assume $\sigma_{(m)} \in C_{(m)}(\partial \Omega)$, we obtain

$$
\begin{align*}
& \quad \sum_{\tilde{\tilde{s} \in \Delta\left(\star \sigma_{(m)}\right)}} \tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}|\tilde{s}| \\
& =(-1)^{d+1} \sum_{\tau_{(m+1)} \in \partial^{-1} \sigma_{(m)}}(-1)^{l-m}\left(\frac{\left.o_{\tau_{(m+1)} \sigma_{(m)}}\right)\left(x_{\tau_{(m+1)}}^{(1)}-m\right.}{l-m} x_{\sigma_{(m)}}^{(1)}\right) \\
&  \tag{6.12}\\
& \wedge\left(\sum_{\tilde{s} \in \Delta\left({ }_{\star} \tau_{(m+1)}\right)} \tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}|\tilde{s}|\right) \\
& \quad+(-1)^{d} \sum_{\tilde{s} \in \Delta\left({ }_{\star} \sigma_{(m)}\right)}(-1)^{l-m}\left(\frac{1}{l-m}\right)\left(x_{\tilde{s}}^{(1)}-x_{\left.\sigma_{(m)}\right)}^{(1)}\right) \wedge\left(\tilde{\star}_{\tilde{s}} \tilde{a}^{(d-l)}|\tilde{s}|\right) \\
& \quad+K_{\sigma_{(m)}^{l, d}}^{l, d}\left(\tilde{a}^{(d-l)}\right) .
\end{align*}
$$

The first term is the usual term. The second term is the extra boundary term from Stokes' theorem, because the simplices $\Delta\left({ }_{{ }_{\mathrm{b}}^{\mathrm{b}}} \sigma_{(m)}\right)$ are needed to complement the simplices in $\tilde{s} \in$ $\Delta\left(\tau_{(m+1)}\right)$, for $\tau_{(m+1)}$, to form $\partial \sharp \sigma_{(m)}$. In this term we used the fact that $o_{\sharp<} \sigma_{(m)}{ }^{\star} \sigma_{b} \sigma_{(m)}=$ $(-1)^{m}$ (cf. Lemma 3.11). The third term is the term discussed in Lemma 6.1, which is in general not zero for $\sigma_{(m)} \in C_{(m)}(\partial \Omega)$.

We can use the above formula to recursively deduce (6.8)-(6.11). Note that when $l=m$ we simply have $\left\langle\tilde{a}^{(d-l)}, \star \sigma_{(l)}\right\rangle$. For the remaining cases we start with $d=2$. The
first interesting case is $l=1, m=0$. We then find for (6.12):

$$
\begin{aligned}
& \tilde{\star} \tilde{a}^{(1)}\left|\curvearrowleft \sigma_{(0)}\right|=\sum_{\tau_{(1)} \in \mathcal{\partial}^{-1} \sigma_{(0)}} o_{\tau_{(1)} \sigma_{(0)}}\left(x_{\sigma_{(0)}^{(1)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, \stackrel{\downarrow}{ } \tau_{(1)}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{1}{2} \sum_{\tau_{(1)} \in \partial_{\mathrm{b}}^{-1} \sigma_{(0)}} o_{\tau_{(1)} \sigma_{(0)}}\left(x_{\sigma_{(0)}^{(1)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\boldsymbol{w}_{\mathrm{b}} \tau_{(1)}}\left(x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\sigma_{(0)}}\right)\right\rangle,
\end{aligned}
$$

where we used that $\tilde{s}=o_{\dot{«}_{b} \tau_{(1)}} o_{\ddot{\star}_{\mathrm{b}} \sigma_{(0)}{ }^{\mathrm{m}_{\mathrm{b}} \tau_{(1)}}}\left(x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\sigma_{(0)}}\right)$ and $x_{\tilde{s}}^{(1)}=\left(x_{(1)}^{\tau_{(1)}}+x_{(1)}^{\sigma_{(0)}}\right) / 2$ for $\tilde{s} \in \Delta\left({ }_{\mathrm{\star}}^{\mathrm{b}} \sigma_{(0)}\right)$. Now, because $o_{\dot{\star}_{\mathrm{b}} \tau_{(k)}{ }_{\mathrm{\star} \mathrm{~b}} \tau_{(k+1)}}=(-1)^{k+1} o_{\tau_{(k+1)} \tau_{(k)}}$, we find (6.8).

Next, we consider $l=2$. For $m=1$, ${ }_{\mathrm{b}} \sigma_{(m)}$ is a point when $d=2$, hence $x_{\tilde{s}}^{(1)}=x_{\sigma_{(m)}}^{(1)}$ and the second term in (6.12) is zero. Furthermore, by Lemma 6.1, $K_{\sigma_{(1)}}^{2,2}\left(\tilde{a}^{(0)}\right)=0$. Thus, we get a boundary contribution in this case.

Similarly, when $l=2$ and $m=0$, we have $K_{\sigma_{(0)}}^{2,2}\left(\tilde{a}^{(0)}\right)=0$. The second term in (6.12) is now zero because $x_{\tilde{s}}^{(1)}-x_{\sigma_{(0)}}^{(1)}$ and $\tilde{\star}_{\tilde{s}} \tilde{a}^{(0)}|\tilde{s}|$ are parallel for $\tilde{s} \in \Delta\left({ }_{{ }_{\mathrm{b}}^{\mathrm{b}}} \sigma_{(0)}\right)$, so their exterior product is zero.

We continue with $d=3$. We consider $l=1$ and $m=0$. From (6.12) we get

$$
\begin{aligned}
& \tilde{\star} \tilde{a}^{(2)}\left|\star \sigma_{(0)}\right| \\
& =\sum_{\tau_{(1)} \in \partial^{-1} \sigma_{(0)}} o_{\tau_{(1)} \sigma_{(0)}}\left(x_{\sigma_{(0)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\left\langle\tilde{a}^{(2)}, \sharp \tau_{(1)}\right\rangle \\
& -\frac{1}{6} \sum_{\left(\tau_{(2)}, \tau_{(1)}\right) \in I_{\sigma_{(0)}}^{b}} o_{\tau_{(2)} \tau_{(1)}} o_{\tau_{(1)} \sigma_{(0)}}\left(x_{\tau_{(2)}}^{(1)}+x_{\tau_{(1)}}^{(1)}-2 x_{\sigma_{(0)}}^{(1)}\right) \\
& \cdot\left\langle\tilde{a}^{(2)}, o_{\sharp_{b} \tau_{(2)}}\left(x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\sigma_{(0)}}\right) \wedge\left(x_{(1)}^{\tau_{(2)}}-x_{(1)}^{\tau_{(1)}}\right)\right\rangle \\
& +K_{\sigma_{(0)}}^{1,3}\left(\tilde{a}^{(2)}\right),
\end{aligned}
$$

where we rewrote the second term as follows. We used the fact that $\tilde{s} \in \Delta(\overbrace{\mathrm{~b}} \sigma_{(0)})$ is given by the simplex $\tilde{s}=\left[x_{(1)}^{\sigma_{(0)}}, x_{(1)}^{\tau_{(1)}}, x_{(1)}^{\tau_{(2)}}\right]$ for $\left(\tau_{(2)}, \tau_{(1)}\right) \in I_{\sigma_{(0)}}^{\mathrm{b}}$, and, $x_{\tilde{s}}^{(1)}=\left(x_{\tau_{(2)}}^{(1)}+x_{\tau_{(1)}}^{(1)}+\right.$ $\left.x_{\sigma_{(0)}}^{(1)}\right) / 3$. Furthermore,

$$
\begin{aligned}
& \tilde{\star}_{\tilde{s}} \tilde{a}^{(2)}|\tilde{s}|=\left\langle\tilde{a}^{(2)}, \tilde{s}\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
& =-\frac{1}{2} o_{\tau_{(2)} \tau_{(1)}} o_{\tau_{(1)} \sigma_{(0)}}\left\langle\tilde{a}^{(2)}, o_{\ddot{\star}_{\mathrm{b}} \tau_{(2)}}\left(x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\sigma_{(0)}}\right) \wedge\left(x_{(1)}^{\tau_{(2)}}-x_{(1)}^{\tau_{(1)}}\right)\right\rangle \text {, }
\end{aligned}
$$



Note that we can also write the term $K_{\sigma_{(0)}}^{1,3}\left(\tilde{a}^{(2)}\right)$ (which is given in Lemma 6.1) as

$$
\begin{aligned}
& K_{\sigma_{(0)}}^{1,3}\left(\tilde{a}^{(2)}\right)=\frac{5}{36} \sum_{\left(\tau_{(2)}, \tau_{(1)}\right) \in I_{\sigma_{(0)}}^{\mathrm{b}}} o_{\tau_{(2)} \tau_{(1)}} o_{\tau_{(1)} \sigma_{(0)}}\left(x_{\tau_{(1)}}^{(1)}-x_{\sigma_{(0)}}^{(1)}\right) \\
& \cdot\left\langle\tilde{a}^{(2)}, o_{\dot{\star}_{\mathrm{b}} \tau_{(2)}}\left(x_{(1)}^{\tau_{(1)}}-x_{(1)}^{\sigma_{(0)}}\right) \wedge\left(x_{(1)}^{\tau_{(2)}}-x_{(1)}^{\tau_{(1)}}\right)\right\rangle,
\end{aligned}
$$

because we only added a term independent of $\tau_{(2)}$ which as a result is zero by the same argument we repeatedly used in the proof of Lemma 6.1. Now adding the results together we find (6.9).

Next, we consider $l=2$ and $m=1$. Using again (6.12) we obtain

$$
\begin{aligned}
& \sum_{\tilde{s} \in \Delta\left(\sharp \sigma_{(1)}\right)}{\tilde{\underset{s}{s}}}^{\tilde{a}^{2}} \tilde{a}^{(1)}|\tilde{s}|=\sum_{\tau_{(2)} \in \partial^{-1} \sigma_{(1)}} o_{\tau_{(2)} \sigma_{(1)}}\left(x_{\sigma_{(1)}}^{(1)}-x_{\tau_{(2)}}^{(1)}\right)\left\langle\tilde{a}^{(1)},{ }^{\wedge} \tau_{(2)}\right\rangle \\
& +\frac{1}{2} \sum_{\tau_{(2)} \in \partial_{\mathrm{b}}^{-1} \sigma_{(1)}}\left(x_{\tau_{(2)}}^{(1)}-x_{\sigma_{(1)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\dot{\star}_{\mathrm{b}} \tau_{(2)}} o_{\dot{\star}_{\mathrm{b}} \sigma_{(1)} \star_{\mathrm{b}} \tau_{(2)}}\left(x_{(1)}^{\tau_{(2)}}-x_{(1)}^{\sigma_{(1)}}\right)\right\rangle \\
& +\frac{1}{4} \sum_{\tau_{(2)} \in \partial_{\mathrm{b}}^{-1} \sigma_{(1)}} o_{\tau_{(2)} \sigma_{(1)}}\left(x_{\sigma_{(1)}}^{(1)}-x_{\tau_{(2)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\text {ᄈ }_{\mathrm{b}} \tau_{(2)}}\left(x_{(1)}^{\tau_{(2)}}-x_{(1)}^{\sigma_{(1)}}\right)\right\rangle .
\end{aligned}
$$

Using the fact that $o_{\dot{\star}_{\mathrm{b}} \sigma_{(1)}{ }^{\star_{\mathrm{b}}} \tau_{(2)}}=o_{\tau_{(2)} \sigma_{(1)}}$ and adding the last two terms we find (6.10).
For $l=2$ and $m=0$ we will again use (6.12). Using (6.10) we find that the first term in (6.12) is given by

$$
\begin{aligned}
& \sum_{\tau_{(1)} \in \mathcal{\partial}^{-1} \sigma_{(0)}}( \left.\frac{o_{\tau_{(1)} \sigma_{(0)}}}{2}\right)\left(x_{\tau_{(1)}}^{(1)}-x_{\sigma_{(0)}}^{(1)}\right) \wedge\left(\sum_{\tilde{s} \in \Lambda\left(\stackrel{\star}{*} \tau_{(1)}\right)} \tilde{\star}_{\tilde{s}} \tilde{a}^{(1)}|\tilde{s}|\right) \\
&= \sum_{\tau_{(2)} \in \partial^{-2} \sigma_{(0)}}\left\langle\tilde{a}^{(1)}, \stackrel{\star}{ } \tau_{(2)}\right\rangle \sigma_{(0)} \tau^{(2)} \\
&+\frac{1}{8} \sum_{\left(\tau_{(2)}, \tau_{(1)}\right) \in I_{\sigma_{(0)}^{\mathrm{b}}}} o_{\tau_{(2)} \sigma_{(1)}} o_{\tau_{(1)} \sigma_{(0)}}\left(x_{\tau_{(1)}}^{(1)}-x_{\sigma_{(0)}}^{(1)}\right) \wedge\left(x_{\tau_{(2)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right) \\
& \quad \cdot\left\langle\tilde{a}^{(1)}, o_{\dot{«}_{\mathrm{b}} \tau_{(2)}}\left(x_{(1)}^{\tau_{(2)}}-x_{(1)}^{\tau_{(1)}}\right)\right\rangle .
\end{aligned}
$$

For the second term in (6.12), we note that for $\tilde{s}=\left[x_{(1)}^{\sigma_{(0)}}, x_{(1)}^{\tau_{(1)}}, x_{(1)}^{\tau_{(2)}}\right] \in \Delta\left(\star_{\mathrm{b}} \sigma_{(0)}\right)$, we have

$$
\begin{aligned}
& \tilde{\star}_{\tilde{s}} \tilde{a}^{(1)}|\tilde{s}|=\frac{1}{2} o_{\dot{\star}_{\mathrm{b}} \sigma_{(0)} \boldsymbol{\star}_{\mathrm{b}} \tau_{(1)}} o_{\boldsymbol{\star}_{\mathrm{b}} \tau_{(1)} \boldsymbol{\star}_{\mathrm{b}} \tau_{(2)}}\left(\left(x_{\sigma_{(0)}^{(1)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\dot{\star}_{\mathrm{b}} \tau_{(2)}}\left(x_{\tau_{(1)}}^{(1)}-x_{\tau_{(2)}}^{(1)}\right)\right\rangle\right. \\
& \left.-\left(x_{\tau_{(1)}}^{(1)}-x_{\tau_{(2)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\dot{\text { w }}_{\mathrm{b}} \tau_{(2)}}\left(x_{\sigma_{(0)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\right\rangle\right) \\
& =\frac{1}{2} o_{\tau_{(2)} \tau_{(1)}} o_{\tau_{(1)} \sigma_{(0)}}\left(\left(x_{\tau_{(1)}}^{(1)}-x_{\tau_{(2)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\dot{\boldsymbol{k}} \tau_{(3)}}\left(x_{\sigma_{(0)}^{(1)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\right\rangle\right. \\
& \left.-\left(x_{\sigma_{(0)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)\left\langle\tilde{a}^{(1)}, o_{\text {\# }_{\mathrm{b}} \tau_{(2)}}\left(x_{\tau_{(1)}}^{(1)}-x_{\tau_{(2)}}^{(1)}\right)\right\rangle\right),
\end{aligned}
$$

and

$$
x_{\tilde{s}}^{(1)}-x_{\sigma_{(0)}}^{(1)}=\frac{1}{3}\left(\left(x_{\tau_{(2)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right)+2\left(x_{\tau_{(1)}}^{(1)}-x_{\sigma_{(0)}}^{(1)}\right)\right) .
$$

Therefore, the second term in (6.12) is given by

$$
\begin{aligned}
& -\frac{1}{2} \sum_{\tilde{s} \in \Delta\left(\mathfrak{«}_{\mathrm{b}} \sigma_{(0)}\right)}\left(x_{\tilde{s}}^{(1)}-x_{\sigma_{(0)}}^{(1)}\right) \wedge\left(\tilde{\star}_{\tilde{s}} \tilde{a}^{(1)}|\tilde{s}|\right) \\
& =\frac{1}{12} \sum_{\left(\tau_{(2)}, \tau_{(1)}\right) \in I_{\sigma_{(0)}}^{\mathrm{b}}} o_{\tau_{(2)} \tau_{(1)}} o_{\tau_{(1)} \sigma_{(0)}}\left(x_{\tau_{(1)}^{(1)}}^{(1)}-x_{\sigma_{(0)}}^{(1)}\right) \wedge\left(x_{\tau_{(2)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right) \\
& \cdot\left\langle\tilde{a}^{(1)}, o_{\ddot{\star}_{\mathrm{b}} \tau_{(2)}}\left(x_{(1)}^{\tau_{(2)}^{(2)}}+x_{(1)}^{\tau_{(1)}}-2 x_{(1)}^{\sigma_{(0)}^{(0)}}\right)\right\rangle .
\end{aligned}
$$

Finally, the third term is given by Lemma 6.1

$$
\begin{gathered}
K_{\sigma_{(0)}}^{2,3}\left(\tilde{a}^{(1)}\right)=\frac{5}{72} \sum_{\left(\tau_{(2)}, \tau_{(1)}\right) \in I_{(0)}^{\mathrm{b}}} o_{\tau_{(0)}} \tau_{(1)} o_{\tau_{(1)} \sigma_{(0)}}\left(x_{\tau_{(1)}}^{(1)}-x_{\sigma_{(0)}}^{(1)}\right) \wedge\left(x_{\tau_{(2)}}^{(1)}-x_{\tau_{(1)}}^{(1)}\right) \\
\cdot\left\langle\tilde{a}^{(1)}, o_{\dot{\star}_{\mathrm{b}} \tau_{(2)}}\left(x_{(1)}^{\left.\left.\sigma_{(0)}^{(0)}-x_{(1)}^{\tau_{(1)}}\right)\right\rangle,}\right.\right.
\end{gathered}
$$

where we added again a term independent of $\tau_{(2)}$ which is zero. Adding the three terms gives (6.11).

By Theorem 6.2 we know how to reconstruct the differential forms from the dual mesh cochains in each of the dual cells. In the next section we will use this interpolation formula to construct the dual discrete Hodge matrices.

### 6.1.2 Dual discrete Hodge matrices

The global discrete dual Hodge matrices $\tilde{\mathbb{H}}^{(d-l)}: \tilde{C}^{(d-l)}(\Omega) \rightarrow C^{(l)}(\Omega)$ are constructed by adding the contributions of local dual cell matrices. For simplicity we first consider the construction of the local matrices in the dual cells away from $\partial \Omega$. Let us consider a dual cell $\approx \sigma_{(0)}$, for $\sigma_{(0)} \in C_{(0)}(\Omega) \backslash C_{(0)}(\partial \Omega)$, and the restriction of the primal mesh to $\star \sigma_{(0)}$. Let us denote by $\tilde{C}^{(d-l)}\left(\AA \sigma_{(0)}\right)$ the restriction of $\tilde{C}^{(d-l)}(\Omega)$ to the $(d-l)$-cochain space corresponding to the $(d-l)$-cells $\approx \sigma_{(l)}$ with $\sigma_{(l)} \in \partial^{-l} \sigma_{(0)}$. Furthermore, we denote by $C^{(l)}\left(\sharp \sigma_{(0)}\right)$ the cochain space corresponding to the primal mesh in $\approx \sigma_{(0)}$.
 Suppose we have a constant form $\tilde{a}^{(d-l)} \in \tilde{\Lambda}^{(d-l)} V$. By the dual interpolation formula (6.3) it follows that

$$
\left\langle\tilde{a}^{(d-l)}, \sigma_{(0)} \tau_{(l)}\right\rangle=\sum_{\sigma_{(l)} \in \partial^{d-l} \sigma_{(d)}} \frac{\left\langle\sigma_{(0)} \sigma^{(l)}, \sigma_{(0)} \tau_{(l)}\right\rangle}{\left|\approx \sigma_{(0)}\right|}\left\langle\tilde{a}^{(d-l)}, \star \sigma_{(l)}\right\rangle .
$$

This shows that we can define, analogous to (3.12), the local discrete dual Hodge matrices as

$$
\left[\tilde{\mathbb{H}}_{* \sigma_{(0)}}^{(d-l)}\right]_{\tau_{(l)}, * \sigma_{(l)}}=\frac{\left\langle\sigma_{(0)} \sigma^{(l)}, \sigma_{(0)} \tau_{(l)}\right\rangle}{\left|\approx \sigma_{(0)}\right|}
$$

From the definition it is clear that this matrix is symmetric. To see that it is consistent, let $\tilde{\boldsymbol{a}}_{\sharp \sigma_{(0)}}^{(d-l)}:=\tilde{R}_{\sharp \sigma_{(0)}}^{(d-l)}\left(\tilde{a}^{(d-l)}\right)$, with $\tilde{R}_{\star \sigma_{(0)}}^{(d-l)}: \tilde{\Lambda}^{(d-l)}\left(\sharp \sigma_{(0)}\right) \rightarrow \tilde{C}^{(d-l)}\left(\sharp \sigma_{(0)}\right)$ the local dual de Rham map and, similarly, $\boldsymbol{a}_{\star \sigma_{(0)}}^{(l)}:=R_{\star \sigma_{(0)}}^{(l)}\left(\tilde{\star} \tilde{a}^{(d-l)}\right)$, with $R_{\star \sigma_{(0)}}^{(l)}: \Lambda^{(l)}\left(\star \sigma_{(0)}\right) \rightarrow$ $C^{(l)}\left(\star \sigma_{(0)}\right)$ the local primal De Rham map. From the dual interpolation property (6.3) it follows that

$$
\begin{equation*}
\boldsymbol{a}_{\sharp \sigma_{(0)}}^{(l)}=\tilde{\mathbb{H}}_{\sharp=\sigma_{(0)}}^{(d-l)} \tilde{\boldsymbol{a}}_{\sharp \sigma_{(0)}}^{(d-l)} . \tag{6.13}
\end{equation*}
$$

Thus, for a form $\tilde{a}^{(d-l)}$ that is constant in $\sharp \sigma_{(0)}$ the interpolation from the primal to the dual mesh by the local discrete dual Hodge matrix $\tilde{\mathbb{H}}_{\boldsymbol{\sharp} \sigma \sigma_{(0)}^{(d)}}^{(d)}$ is exact.

So the matrix $\tilde{\mathbb{H}}_{* \sigma_{(0)}}^{(d-l)}$ as defined above is symmetric and consistent. It is in general not positive definite. To make it positive definite, we again add a stability matrix just as we did for the primal discrete Hodge matrices $\mathbb{H}_{\sigma_{(d)}}^{(l)}$. First define two matrices $\tilde{\mathbb{S}}_{\star \sigma_{(0)}}^{(d-l)}$ : $\tilde{C}^{(d-l)}\left(\sharp \sigma_{(0)}\right) \rightarrow \Lambda^{(l)} V$ and $\mathbb{S}_{\star \sigma_{(0)}}^{(l)}: C^{(l)}\left(\approx \sigma_{(0)}\right) \rightarrow \tilde{\Lambda}^{(d-l)} V$ by

$$
\begin{aligned}
& \mathbb{S}_{\sharp \sigma_{(0)}}^{(l)} \boldsymbol{b}_{\sharp \sigma_{(0)}}^{(l)}:=\sum_{\sigma_{(l)} \in \mathcal{D}^{-l} \sigma_{(0)}} b_{\sigma_{(0)}}^{(l)} \sigma_{(l)} \approx \sigma^{(l)}, \\
& \tilde{\mathbb{S}}_{\sharp \sigma_{(0)}}^{(d-l)} \tilde{\boldsymbol{a}}_{\sharp \sigma_{(0)}}^{(d-l)}:=\left\{\begin{array}{lll}
\sum_{\sigma_{(l)} \in \mathcal{D}^{-l} \sigma_{(0)}} & \chi \tilde{a}_{\sharp \sigma_{(l)}}^{(d-l)} & \sigma_{(0)} \sigma^{(l)} \\
\sum_{\sigma_{(l)} \in \mathcal{D}^{-l} \sigma_{(0)}} \chi(-1)^{l} \tilde{a}_{\sharp \sigma_{(l)}}^{(d-l)} & \sigma_{(0)} \sigma^{(l)} & \text { if } d \text { is even, },
\end{array}\right.
\end{aligned}
$$

where $\tilde{a}_{\mathfrak{\hbar} \sigma_{(l)}}^{(d-l)}$ is the component of $\tilde{\boldsymbol{a}}_{\mathfrak{\sharp} \sigma_{(0)}}^{(d-l)}$ corresponding to $\sharp \sigma_{(l)}$, and, similarly, $b_{\sigma_{(0)}}^{(l)} \sigma_{(l)}$ is the component of $\boldsymbol{b}_{\sharp \dot{*} \sigma_{(0)}}^{(l)}$ corresponding to ${ }_{\sigma_{(0)}} \sigma_{(l)}$. Furthermore, $\chi$ is any symmetric positive definite tensor. Note that the matrices $\tilde{\mathbb{S}}_{\sharp \sigma_{(0)}}^{(d-l)}$ and $\mathbb{S}_{\left.\sharp \sigma_{(0)}\right)}^{(l)}$ both have as many columns as there are $(d-l)$-cells $\sharp \sigma_{(l)}$ in $\partial^{-l} \sigma_{(0)}$ and $\operatorname{dim}\left(\tilde{\Lambda}^{(d-l)} V\right)=\operatorname{dim}\left(\Lambda^{(l)} V\right)=$ $d!/(l!(d-l)!)$.

From the dual interpolation property (6.3) it follows that, analogous to Lemma 3.7,

$$
\tilde{\mathbb{S}}_{\star \sigma_{(0)}}^{(d-l)} \mathbb{S}_{\star \sigma_{(0)}}^{(l) T}=\left|\not \approx \sigma_{(0)}\right| \tilde{\star}_{\chi} \tilde{a}^{(d-l)}, \quad \forall \tilde{a}^{(d-l)} \in \tilde{\Lambda}^{(d-l)}(V)
$$

We define a symmetric, consistent and positive definite local dual discrete Hodge matrix according to

$$
\begin{aligned}
\tilde{\mathbb{H}}_{\sharp \sigma_{(0)}}^{(d-l)}:= & \frac{1}{\left|\sharp \sigma_{(0)}\right|} \tilde{\mathbb{S}}_{\sharp \sigma_{(0)}}^{(d-l) T} \tilde{\mathbb{S}}_{\sharp \sigma_{(0)}}^{(d-l)} \\
& +\gamma \operatorname{Tr}\left(\frac{1}{\left|\neq \sigma_{(0)}\right|} \tilde{\mathbb{S}}_{\sharp \sigma_{(0)}}^{(d-l) T} \tilde{\mathbb{S}}_{\sharp \sigma_{(0)}}^{(d-l)}\right)\left(\mathbb{I}-\mathbb{S}_{\sharp \sigma_{(0)}}^{(l) T}\left(\mathbb{S}_{\sharp \sigma_{(0)}}^{(l)} \mathbb{S}_{\sharp \sigma_{(0)}}^{(l) T}\right)^{-1} \mathbb{S}_{\sharp \sigma_{(0)}}^{(l)}\right),
\end{aligned}
$$

where we again take $\gamma=1 / 3$. The stabilization part makes sure that the matrix is symmetric positive definite. However, there are, just as in the primal case, alternatives that also achieve this. Further research should be done to find the best choice for a given problem.

The global dual discrete Hodge matrix $\tilde{\mathbb{H}}_{\chi}^{(d-l)}: \tilde{C}^{(d-l)}(\Omega) \rightarrow C^{(l)}(\Omega)$ is formed by adding the local contributions, also for the cells $\sigma_{(0)} \in C_{(0)}(\partial \Omega)$. Although the resulting
matrix is symmetric positive definite, it is not consistent in the cells $\sharp \sigma_{(0)}$ for $\sigma_{(0)} \in$ $C_{(1)}(\partial \Omega)$, as a result of Theorem 6.2. To interpolate from the dual mesh we need boundary contributions. We define

$$
\boldsymbol{r}_{\mathrm{b}, \star \sigma_{(0)}}^{(l)}\left(\tilde{a}^{(d-l)}\right):=\frac{1}{\left|\sharp \sigma_{(0)}\right|} \sum_{\left(\tau_{(d-1)}, \ldots, \tau_{(1)}\right) \in I_{\sigma_{(m)}^{\mathrm{b}}}} \tilde{\mathbb{S}}_{\neq \sigma_{(0)}}^{(d-l) T} \chi k_{\sigma_{(0)}, \tau_{(1)}, \ldots, \tau_{(d-1)}}^{l d}\left(\tilde{a}^{(d-l)}\right),
$$

where $k_{\sigma_{(0)}^{l, d}, \tau_{(1)}, \ldots, \tau_{(d-1)}}\left(\tilde{a}^{(d-l)}\right)$ is given in Theorem 6.2. For $\sigma_{(0)} \in C_{(1)}(\partial \Omega),(6.13)$ does not hold, but instead we have

$$
\boldsymbol{a}_{\star \sigma_{(0)}}^{(l)}=\tilde{\mathbb{H}}_{\mathfrak{\sharp} \sigma_{(0)}}^{(d-l)} \tilde{\boldsymbol{a}}_{\mathfrak{k} \sigma_{(0)}}^{(d-l)}+\boldsymbol{r}_{\mathrm{b}, \star \sigma_{(0)}}^{(l)}\left(\tilde{a}^{(d-l)}\right) .
$$

We collect these boundary contributions in a vector $\boldsymbol{r}_{\chi, \mathrm{b}}^{(l)}\left(\tilde{a}^{(d-l)}\right)$. In our implementation we build the vector $\boldsymbol{r}_{\chi, \mathrm{b}}^{(l)}\left(\tilde{a}^{(d-l)}\right)$ for $\tilde{a}^{(d-l)} \in \tilde{\Lambda}^{(d-l)}(\Omega)$ by using the value $\tilde{a}^{(d-l)}\left(x_{(1)}^{\sigma_{(0)}}\right)$ in each dual cell $\star \sigma_{(0)}$ at the boundary.

Note that to determine $\boldsymbol{r}_{\chi, \mathbf{b}}^{(l)}\left(\tilde{a}^{(d-l)}\right)$ only the trace $\tilde{t}^{( }{ }^{(d-l)}$ of $\tilde{a}^{(d-l)}$ on $\partial \Omega$ needs to be known. To stress this we will write $\boldsymbol{r}_{\chi, \mathrm{b}}^{(l)}\left(\tilde{t}_{\tilde{a}} \tilde{a}^{(d-l)}\right)$. In the case that $\chi$ is just the identity we simply write $\tilde{\mathbb{H}}^{(d-l)}$ and $\boldsymbol{r}_{\chi, \mathrm{b}}^{(l)}\left(\tilde{a}^{(d-l)}\right)$.

The consistency of the primal and dual Hodge matrices implies that for $\boldsymbol{a}^{(k)}:=$ $R^{(k)}\left(a^{(k)}\right)$ with $a^{(k)} \in \Lambda^{(k)}(\Omega)$ and $\tilde{\boldsymbol{a}}^{(d-k)}:=\tilde{R}^{(d-k)}\left(\tilde{a}^{(d-k)}\right)$ with $\tilde{a}^{(d-k)} \in \tilde{\Lambda}^{(d-k)}(\Omega)$, we have

$$
\begin{align*}
\boldsymbol{a}^{(k)} & =\tilde{\mathbb{H}}^{(d-k)} \mathbb{H}^{(k)} \boldsymbol{a}^{(k)}+\boldsymbol{r}_{\mathrm{b}}^{(k)}\left(\tilde{t} \star a^{(k)}\right)+\mathcal{O}(h),  \tag{6.14a}\\
\tilde{\boldsymbol{a}}^{(d-k)} & =\mathbb{H}^{(k)}\left(\tilde{\mathbb{H}}^{(d-k)} \tilde{\boldsymbol{a}}^{(d-k)}+\boldsymbol{r}_{\mathrm{b}}^{(k)}\left(\tilde{t}^{(d-k)}\right)\right)+\mathcal{O}(h), \tag{6.14b}
\end{align*}
$$

where $\mathcal{O}(h)$ is a term that vanishes as the cell diameter $h$ goes to zero. This shows that $\mathbb{H}^{(k)}$ and $\tilde{\mathbb{H}}^{(d-k)}$ are each others approximate inverses for $k$-cochains $\boldsymbol{a}^{(k)}$ that are a discretization of a $k$-form satisfying $\tilde{t} \star a^{(k)}=0$, and, $(d-k)$-cochains $\tilde{\boldsymbol{a}}^{(d-k)}$ that are the discretization of a $(d-k)$-form satisfying $\tilde{t} \tilde{a}^{(d-k)}=0$.

### 6.2 Numerical analysis of the discrete Hodge operators

We test the numerical convergence of equations (6.14) in three-dimensional space on the sequence of tetrahedral meshes from Table 6.4. For $a^{(k)}$ and $\tilde{a}^{(3-k)}$, we take the forms $u^{(1)}, \tilde{u}^{(1)}, u^{(2)}$ and $\tilde{u}^{(2)}$, and $p^{(0)}, \tilde{p}^{(0)}, p^{(3)}$ and $\tilde{p}^{(3)}$ that have component functions in the canonical basis:

$$
\begin{align*}
u_{x}(x, y, z) & =\sin (2 \pi x) \cos (2 \pi y) \cos (2 \pi z) / 2, \\
u_{y}(x, y, z) & =\cos (2 \pi x) \sin (2 \pi y) \cos (2 \pi z) / 2,  \tag{6.15}\\
u_{z}(x, y, z) & =-\cos (2 \pi x) \cos (2 \pi y) \sin (2 \pi z), \\
p(x, y, z) & =\sin (2 \pi x) \sin (2 \pi y) \sin (2 \pi z)
\end{align*}
$$



Figure 6.3: On the left: $e_{\infty}^{(k)}$ (solid line) and $\tilde{e}_{\infty}^{(k)}$ (dashed line). On the right: $e_{2}^{(k)}$ (solid line) and $\tilde{e}_{2}^{(k)}$ (dashed line).

We consider the quantities

$$
\begin{aligned}
\boldsymbol{e}^{(k)} & =\boldsymbol{a}^{(k)}-\tilde{\mathbb{H}}^{(d-k)} \mathbb{H}^{(k)} \boldsymbol{a}^{(k)}-\boldsymbol{r}_{\mathrm{b}}^{(k)}\left(\tilde{t} \star a^{(k)}\right), \\
\tilde{\boldsymbol{e}}^{(d-k)} & =\tilde{\boldsymbol{a}}^{(d-k)}-\mathbb{H}^{(k)}\left(\tilde{\mathbb{H}}^{(d-k)} \tilde{\boldsymbol{a}}^{(d-k)}+\boldsymbol{r}_{\mathrm{b}}^{(k)}\left(\tilde{t} \tilde{a}^{(d-k)}\right)\right),
\end{aligned}
$$

Let $\mathbb{L}^{(k)}$ be the diagonal matrix on $\mathbb{R}^{N_{(k)}}$ with on the diagonal the size of the mesh elements, i.e., $\mathbb{L}_{\sigma_{(k)}, \sigma_{(k)}}^{(k)}=\left|\sigma_{(k)}\right|$. Similarly let $\tilde{\mathbb{L}}^{(d-k)}$ be the diagonal matrix on $\mathbb{R}^{N_{(k)}}$ with the size of the dual mesh elements, i.e., $\tilde{\mathbb{L}}_{\mathfrak{k} \sigma_{(k)}(d-k)}^{\left(d \sigma_{(k)}\right.}=\left|\star \sigma_{(k)}\right|$. We then consider the convergence of

$$
\begin{align*}
e_{2}^{k}:=\frac{\left\|\mathbb{L}^{(k)} \boldsymbol{e}^{(k)}\right\|_{2}}{\left\|\mathbb{L}^{(k)} \boldsymbol{a}^{(k)}\right\|_{2}}, \quad \tilde{e}_{2}^{k}:=\frac{\left\|\tilde{\mathbb{L}}^{(d-k)} \tilde{\boldsymbol{e}}^{(d-k)}\right\|_{2}}{\left\|\tilde{\mathbb{L}}^{(d-k)} \tilde{\boldsymbol{e}}^{(d-k)}\right\|_{2}} \\
e_{\infty}^{k}:=\frac{\left\|\mathbb{L}^{(k)} \boldsymbol{e}^{(k)}\right\|_{\infty}}{\left\|\mathbb{L}^{(k)} \boldsymbol{a}^{(k)}\right\|_{\infty}}, \quad \tilde{e}_{2}^{k}:=\frac{\left\|\tilde{\mathbb{L}}^{(d-k)} \tilde{\boldsymbol{e}}^{(d-k)}\right\|_{\infty}}{\left\|\tilde{\mathbb{L}}^{(d-k)} \tilde{\boldsymbol{e}}^{(d-k)}\right\|_{\infty}}, \tag{6.16}
\end{align*}
$$

where $\left\|\boldsymbol{a}^{(k)}\right\|_{2}$ and $\left\|\boldsymbol{a}^{(k)}\right\|_{\infty}$ are the usual norms of $\boldsymbol{a}^{(k)}$ on $\mathbb{R}^{N_{(k)}}$. Note that, e.g., $e_{2}^{1}$ and $\tilde{e}_{2}^{1}$ both involve the matrices $\mathbb{H}^{(1)}$ and $\tilde{\mathbb{H}}^{(2)}$.

The convergence behavior is plotted in Figure 6.3. We see that the convergence for especially $e_{\infty}^{k}$ and $\tilde{e}_{\infty}^{k}$ for $k=1,2$ is rather irregular, while this is not the case for $k=0,3$. As $k=0,3$ deal with scalar quantities, one of the discrete Hodge matrices involved is diagonal and no boundary contributions are involved. One of these aspects makes the convergence of $e_{\infty}^{k}$ and $\tilde{e}_{\infty}^{k}$ for $k=0,3$ less capricious than for $k=1,2$. It is a topic of further research to pin down the precise source of this difference.

The convergence of the quantities $e_{2}^{k}$ and $\tilde{e}_{2}^{k}$ is far more regular. This allows us to determine the rates of converge. These are given in Table 6.1. We see that the convergence rate for the quantities related to 0 - and 3 -cochains is again significantly higher than for the quantities related to 1 - and 2 -cochains.

To compare the behavior of the primal and dual discrete Hodge matrices, we consider

$$
\tilde{\boldsymbol{f}}^{(d-k)}=\tilde{\boldsymbol{a}}^{(d-k)}-\mathbb{H}^{(k)} \boldsymbol{a}^{(k)}, \quad \boldsymbol{f}^{(k)}=\boldsymbol{a}^{(k)}-\left(\tilde{\mathbb{H}}^{(d-k)} \tilde{\boldsymbol{a}}^{(d-k)}+\boldsymbol{r}_{\mathrm{b}}^{(k)}\left(\tilde{t}_{\boldsymbol{a}}(d-k)\right)\right)
$$



Figure 6.4: On the left: $f_{\infty}^{(k)}$ (solid line) and $\tilde{f}_{\infty}^{(k)}$ (dashed line). On the right: $f_{2}^{(k)}$ (solid line) and $\tilde{f}_{2}^{(k)}$ (dashed line).

We investigate the convergence of

$$
\begin{aligned}
& \tilde{f}_{2}^{k}:=\frac{\left\|\mathbb{L}^{(k)} \boldsymbol{f}^{(k)}\right\|_{2}}{\left\|\mathbb{L}^{(k)} \boldsymbol{a}^{(k)}\right\|_{2}}, \quad f_{2}^{k}:=\frac{\left\|\tilde{\mathbb{L}}^{(d-k)} \tilde{\boldsymbol{f}}^{(d-k)}\right\|_{2}}{\left\|\tilde{\mathbb{L}}^{(d-k)} \tilde{\boldsymbol{a}}^{(d-k)}\right\|_{2}}, \\
& \tilde{f}_{\infty}^{k}:=\frac{\left\|\mathbb{L}^{(k)} \boldsymbol{f}^{(k)}\right\|_{\infty}}{\left\|\mathbb{L}^{(k)} \boldsymbol{a}^{(k)}\right\|_{\infty}}, \quad f_{\infty}^{k}:=\frac{\left\|\tilde{\mathbb{L}}^{(d-k)} \tilde{\boldsymbol{f}}^{(d-k)}\right\|_{\infty}}{\left\|\tilde{\mathbb{L}}^{(d-k)} \tilde{\boldsymbol{a}}^{(d-k)}\right\|_{\infty}}
\end{aligned}
$$

Note that, e.g. $\tilde{f}_{2}^{2}$ involves $\tilde{\mathbb{H}}^{(1)}$ and $f_{2}^{2}$ involves $\mathbb{H}^{(2)}$. The convergence behavior is plotted in Figure 6.4. The convergence for the primal discrete Hodge matrices is significantly better than for the dual ones when 0 - or 3 -cochains are considered. On the other hand, for 1 - and 2 -cochains the dual operators perform better than the primal ones. In the maximum norm the convergence for $\mathbb{H}^{(1)}$ and $\mathbb{H}^{(2)}$ is almost unnoticeable, while $\tilde{H}^{(1)}$ and $\tilde{\mathbb{H}}^{(2)}$ appear to do a lot better. The same difference in behavior, although more nuanced, can be observed in the rates of convergence for the 2-norm given in Table 6.2.

Finally, we compare the sparsity and condition of the Hodge matrices in Table 6.3. We see that the primal discrete Hodge matrices are significantly sparser than the dual discrete Hodge matrices. This is not surprising because there are about 4.5 times more primal cells than dual cells, e.g. $N_{(3)} / N_{(0)} \approx 4.55$ for the tetrahedral mesh with refinement level RL equal to 2 in the mesh sequence in Table 6.4. The resulting stencil for the dual discrete Hodge matrices is significantly wider. On the other hand, the conditioning of the dual discrete Hodge matrices is better than that of the primal discrete Hodge matrices.

| $k$ | 0 | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: |
| $e_{2}^{k}$ | 1.60 | 0.95 | 1.14 | 1.62 |
| $\tilde{e}_{2}^{k}$ | 1.65 | 1.11 | 1.17 | 1.62 |


| $k$ | 0 | 1 | 2 | 3 |
| :---: | :---: | :---: | :---: | :---: |
| $f_{2}^{k}$ | 2.13 | 0.98 | 1.10 | 2.12 |
| $\tilde{f}_{2}^{k}$ | 1.59 | 1.10 | 1.40 | 1.58 |

Table 6.1: The rate of convergence of $e_{2}^{k}$ and $\tilde{e}_{2}^{k}$. Table 6.2: The rate of convergence of $f_{2}^{k}$ and $\tilde{f}_{2}^{k}$.

Further research should be done to provide explanations for the observations made. Moreover, it should be analyzed to what extent the results depend on our specific choice for the stabilization part in the discrete Hodge matrices.

|  | $\mathbb{H}^{(0)}$ |  |  | $\mathbb{H}^{(1)}$ | $\mathbb{H}^{(2)}$ |  | $\mathbb{H}^{(3)}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| RL | $\mathrm{NNZ} / N_{(0)}$ | CN | $\mathrm{NNZ} / N_{(1)}$ | CN | $\mathrm{NNZ} / N_{(2)}$ | CN | $\mathrm{NNZ} / N_{(3)}$ | CN |
| 0 | 10.1 | 95.3 | 12.8 | 28.6 | 6.2 | 21.9 | 1 | 8.1 |
| 1 | 12.4 | 93.6 | 14.6 | 40.8 | 6.6 | 23.3 | 1 | 11.5 |
| 2 | 13.1 | 41.2 | 15.0 | 77.7 | 6.7 | 23.5 | 1 | 14.0 |
|  |  |  |  |  |  |  |  |  |
| RL | $\mathrm{NNZ} / N_{(0)}$ | CN | $\mathrm{NNZ} / N_{(1)}$ | CN | $\mathrm{NNZ} / N_{(2)}$ | CN | $\mathrm{NNZ} / N_{(3)}$ | CN |
| 0 | 1 | 73.9 | 20.2 | 13.1 | 64.2 | 5.3 | 46.6 | 4.5 |
| 1 | 1 | 76.3 | 24.2 | 12.6 | 81.3 | 5.0 | 61.2 | 5.4 |
| 2 | 1 | 30.7 | 25.2 | 11.0 | 85.6 | 4.9 | 64.7 | 5.8 |

Table 6.3: For both the primal and dual discrete Hodge operators we present the sparsity in terms of the number of nonzero elements divided by the dimension of the matrix (NNZ/ $N_{(k)}$ ) and the condition number (CN), for the three coarsest meshes in the tetrahedral mesh sequence detailed in Table 6.4.

### 6.3 Application to boundary value problems

We apply the newly derived dual discrete Hodge matrices to solve boundary value problems coming from electromagnetism and fluid dynamics. We use discretizations based on the dual discrete Hodge matrices and compare their results with those of the discretizations based on the primal discrete Hodge matrices. Moreover, we formulate discretizations that use a combination of primal and dual discrete Hodge matrices. We test the convergence of these discretization on the sequence of tetrahedral meshes given in Table 6.4. An example of one mesh in this sequence is shown in Figure 6.5.


Figure 6.5: Primal mesh for refinement level 1.

| RL | $N_{(0)}$ | $N_{(1)}$ | $N_{(2)}$ | $N_{(3)}$ |
| :---: | ---: | ---: | ---: | ---: |
| 0 | 80 | 364 | 500 | 215 |
| 1 | 488 | 2792 | 4308 | 2003 |
| 2 | 857 | 5206 | 8248 | 3898 |
| 3 | 1601 | 10037 | 16148 | 7711 |
| 4 | 2997 | 19421 | 31691 | 15266 |
| 5 | 5692 | 37998 | 62787 | 30480 |
| 6 | 10994 | 74929 | 124988 | 61052 |

Table 6.4: Tetrahedral mesh sequence with 7 refinement levels (RL).

### 6.3.1 Magnetostatic boundary value problems

Let us consider Maxwell's equations in a domain $\Omega \subset \mathbb{R}^{3}$ :

$$
\begin{align*}
\partial_{t} b^{(2)}+d e^{(1)} & =0,  \tag{6.17a}\\
-\partial_{t} \tilde{d}^{(2)}+\tilde{d} \tilde{h}^{(1)} & =\tilde{j}^{(2)}, \tag{6.17b}
\end{align*}
$$

$$
\begin{aligned}
d b^{(2)} & =0 \\
\tilde{d} \tilde{d}^{(2)} & =\tilde{q}^{(3)}
\end{aligned}
$$

together with the constitutive laws

$$
\begin{equation*}
b^{(2)}=\tilde{\star}_{\mu} \tilde{h}^{(1)}, \quad \tilde{d}^{(2)}=\star_{\epsilon} e^{(1)} \tag{6.18}
\end{equation*}
$$

where $\tilde{\star}_{\mu}:=\mu \tilde{\star}$ and $\star_{\epsilon}:=\epsilon \star$ and $\mu$ and $\epsilon$ are either constant (in vacuum) or placedependent tensors determined by the material that fills $\Omega$. We assume that the current density $\tilde{j}^{(2)}$ and charge density $\tilde{q}^{(3)}$ are given. Let us consider a situation in which $\tilde{j}^{(2)}$ is stationary and all fields are time-independent. The equations concerned with the magnetic fields are then given by

$$
\begin{equation*}
d b^{(2)}=0, \quad \tilde{d} \tilde{h}^{(1)}=\tilde{j}^{(2)}, \quad \tilde{h}^{(1)}=\star_{\mu^{-1}} b^{(2)} . \tag{6.19}
\end{equation*}
$$

This system can be solved by introducing a potential $u^{(1)} \in \Lambda^{(1)}(\Omega)$ such that $b^{(2)}=d u^{(1)}$. We assume that $\Omega$ is a simply connected domain and without voids. In such a situation $u^{(1)}$ is unique up to a gradient $d q^{(0)}$ for some $q^{(0)} \in \Lambda^{(0)}(\Omega)$. It is then possible to choose $q^{(0)}$ such that the potential is "divergence-free": $\tilde{d} \star u^{(1)}=0$. This is known as the Coulomb gauge.

We do not enforce the divergence constraint on $u^{(1)}$ in its function space, but instead through a Lagrange multiplier $p^{(0)} \in \Lambda^{(0)}(\Omega)$. This results in the following strong formulation:

$$
\begin{align*}
\tilde{d} \star_{\mu^{-1}} d u^{(1)}+\star d p^{(0)} & =\tilde{j}^{(2)} & & \text { in } \Omega,  \tag{6.20a}\\
\tilde{d} \star u^{(1)} & =0 & & \text { in } \Omega . \tag{6.20b}
\end{align*}
$$

Alternatively we can set $u^{(1)}=\tilde{\star} \tilde{u}^{(2)}$, with $\tilde{u}^{(2)} \in \tilde{\Lambda}^{(2)}(\Omega)$, as potential. This leads to a different formulation:

$$
\begin{align*}
-\tilde{\star}_{\mu} \tilde{h}^{(1)}+d \tilde{\star}_{\tilde{u}^{(2)}} & =0 & & \text { in } \Omega,  \tag{6.21a}\\
\tilde{\star} \tilde{d} \tilde{h}^{(1)}+d p^{(0)} & =\tilde{\star} \tilde{j}^{(2)} & & \text { in } \Omega,  \tag{6.21b}\\
\tilde{d} \tilde{u}^{(2)} & =0 & & \text { in } \Omega . \tag{6.21c}
\end{align*}
$$

For simplicity we restrict our discussion to natural boundary conditions. The natural boundary conditions for both problems are given by $\tilde{t}^{(2)}=\tilde{u}_{\mathrm{b}}^{(2)}$ and $\tilde{t}^{(1)}=\tilde{h}_{\mathrm{b}}^{(1)}$, where $\tilde{u}_{\mathrm{b}}^{(2)} \in \tilde{\Lambda}^{(2)}(\partial \Omega)$ and $\tilde{h}_{\mathrm{b}}^{(1)} \in \tilde{\Lambda}^{(1)}(\partial \Omega)$ are given. The solution to these problems is unique up to a constant that can be added to $p^{(0)}$. To fix uniqueness we can, for example, set it to zero in some point in $\Omega$.

In fact it is possible to show that if we fix the uniqueness of $p^{(0)}$ in this way we must have $p^{(0)}=0$. To see this we multiply (6.20a) by $\tilde{d}$. The first term in (6.20a) immediately vanishes, as does the right-hand side, because $\tilde{j}^{(2)}$ should satisfy $\tilde{d} \tilde{j}^{(2)}=0$ for (6.19) to make sense. So, we end up with $\tilde{d} \star d p^{(0)}=0$. Now taking the $L^{2}$-inner product of this with $\star p^{(0)}$ on $\Omega$ we get

$$
\begin{aligned}
0 & =\left(\tilde{d} \star d p^{(0)}, \star p^{(0)}\right)_{\Omega} \\
& =\int_{\Omega}\left(\tilde{d} \star d p^{(0)}\right) \wedge p^{(0)} \\
& =(-1)^{d-1}\left(d p^{(0)}, d p^{(0)}\right)_{\Omega}+\int_{\partial \Omega} \tilde{t}\left(\star d p^{(0)} \wedge p^{(0)}\right) .
\end{aligned}
$$

The original formulation (6.19) implies that we must have $\tilde{t} \star d p^{(0)}=\tilde{t}\left(\tilde{j}^{(2)}-\tilde{d} \tilde{h}^{(1)}\right)=0$, hence we find $d p^{(0)}=0$. Now, because we require $p^{(0)}$ to equal zero in some point in $\Omega$, and $\Omega$ has a single component, we find that actually $p^{(0)}=0$. We shall see in what follows that this carries over to the discrete setting.

### 6.3.1.1 Discrete formulations

The boundary value problems (6.20) and (6.21) are discretized as

$$
\left[\begin{array}{cc}
\mathbb{D}^{(1) T} \mathbb{H}_{\mu^{-1}}^{(2)} \mathbb{D}^{(1)} & \mathbb{H}^{(1)} \mathbb{D}^{(0)}  \tag{6.22}\\
\mathbb{D}^{(0) T} \mathbb{H}^{(1)} & 0
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{u}_{\mathrm{VP}}^{(1)} \\
\boldsymbol{p}_{\mathrm{VP}}^{(0)}
\end{array}\right]=\left[\begin{array}{c}
\tilde{\boldsymbol{f}}_{\mathrm{VP}}^{(1)} \\
\tilde{\boldsymbol{f}}_{\mathrm{VP}}^{(3)}
\end{array}\right]
$$

and

$$
\left[\begin{array}{ccc}
-\tilde{\mathbb{H}}_{\mu}^{(1)} & \mathbb{D}^{(1)} \tilde{\mathbb{H}}^{(2)} & 0  \tag{6.23}\\
\tilde{\mathbb{H}}^{(2)} \mathbb{D}^{(1) T} & 0 & \mathbb{D}^{(0)} \\
0 & \mathbb{D}^{(0) T} & 0
\end{array}\right]\left[\begin{array}{c}
\tilde{\boldsymbol{h}}_{\mathrm{CD}}^{(1)} \\
\tilde{\boldsymbol{u}}_{\mathrm{CD}}^{(2)} \\
\boldsymbol{p}_{\mathrm{CD}}^{(0)}
\end{array}\right]=\left[\begin{array}{c}
\boldsymbol{f}_{\mathrm{CD}}^{(2)} \\
\boldsymbol{f}_{\mathrm{CD}}^{(1)} \\
\tilde{\boldsymbol{f}}_{\mathrm{CD}}^{(3)}
\end{array}\right]
$$

We call (6.22) the Vertex-based Primal (VP) scheme, because the scheme uses the primal discrete Hodge matrices and the divergence condition for the potential is discretized via one equation for every primal vertex. Similarly, we call (6.23) the Cell-based Dual (CD)
scheme, because it is built using the dual discrete Hodge matrices and the divergence condition is discretized via one equation for every dual cell. The right-hand sides of the systems are given by

$$
\begin{array}{ll}
\tilde{\boldsymbol{f}}_{\mathrm{VP}}^{(1)}:=\tilde{\boldsymbol{j}}^{(2)}-\mathbb{T}^{(1) T} \tilde{\boldsymbol{h}}_{\mathrm{b}}^{(1)}, & \tilde{\boldsymbol{f}}_{\mathrm{VP}}^{(3)}:=\mathbb{T}^{(0) T} \tilde{\boldsymbol{u}}_{\mathrm{b}}^{(2)}, \\
\boldsymbol{f}_{\mathrm{CD}}^{(2)}:=\boldsymbol{r}_{\mu, \mathrm{b}}^{(2)}\left(\tilde{h}_{\mathrm{b}}^{(1)}\right)-\mathbb{D}^{(1)} \boldsymbol{r}_{\mathrm{b}}^{(1)}\left(\tilde{u}_{\mathrm{b}}^{(2)}\right), & \boldsymbol{f}_{\mathrm{CD}}^{(1)}:=\tilde{\mathbb{H}}^{(2)} \tilde{\boldsymbol{j}}^{(2)}-\tilde{\mathbb{H}}^{(2)} \mathbb{T}^{(1) T} \tilde{\boldsymbol{h}}_{\mathrm{b}}^{(1)}, \\
\tilde{\boldsymbol{f}}_{\mathrm{CD}}^{(3)}:=\mathbb{T}^{(0) T} \tilde{\boldsymbol{u}}_{\mathrm{b}}^{(2)}, &
\end{array}
$$

where $\tilde{\boldsymbol{j}}^{(2)}:=\tilde{R}^{(2)}\left(\tilde{j}^{(2)}\right), \tilde{\boldsymbol{h}}_{\mathrm{b}}^{(1)}:=\tilde{R}_{\mathrm{b}}^{(1)}\left(\tilde{h}_{\mathrm{b}}^{(1)}\right)$ and $\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(2)}:=\tilde{R}_{\mathrm{b}}^{(2)}\left(\tilde{u}_{\mathrm{b}}^{(2)}\right)$. Note that in the second equation of (6.23) we used $\tilde{d}_{\mathrm{b}} \tilde{h}_{\mathrm{b}}^{(1)}=\tilde{t}^{(2)}$ and as a result the expected boundary contributions $\boldsymbol{r}_{\mathrm{b}}^{(1)}\left(\tilde{d}_{\mathrm{b}} \tilde{h}_{\mathrm{b}}^{(1)}\right)$ and $\boldsymbol{r}_{\mathrm{b}}^{(1)}\left(\tilde{t}^{(2)}\right)$ cancel each other.

The discrete setting presents more possibilities. Up to now the primal mesh has an inner orientation and the dual mesh an outer orientation. We can switch these, because this choice of orientation is rather arbitrary. ${ }^{3}$ Consequentially, the location of the discrete variables will be switched and we find a second set of discrete formulations. From now on the tilde ~on top of a discrete variable, or operator, will indicate that it is located on the dual mesh (which can be inner-oriented).

This switching of the orientations will imply a different set of natural boundary conditions. In fact, the essential boundary conditions for (6.20) and (6.21), which are given by $t u^{(1)}=u_{\mathrm{b}}^{(1)}$ and $t p^{(0)}=p_{\mathrm{b}}^{(0)}$, will now appear as natural boundary conditions. ${ }^{4}$ The second set of discrete formulations is given by

$$
\left[\begin{array}{cc}
\mathbb{D}^{(1)} \tilde{\mathbb{H}}_{\mu^{-1}}^{(2)} \mathbb{D}^{(1) T} & \tilde{\mathbb{H}}^{(1)} \mathbb{D}^{(2) T}  \tag{6.24}\\
\mathbb{D}^{(2)} \tilde{\mathbb{H}}^{(1)} & 0
\end{array}\right]\left[\begin{array}{c}
\tilde{\boldsymbol{u}}_{\mathrm{VD}}^{(1)} \\
\tilde{\boldsymbol{p}}_{\mathrm{VD}}^{(0)}
\end{array}\right]=\left[\begin{array}{c}
\boldsymbol{f}_{\mathrm{VD}}^{(2)} \\
\boldsymbol{f}_{\mathrm{VD}}^{(3)}
\end{array}\right],
$$

and

$$
\left[\begin{array}{ccc}
-\mathbb{H}_{\mu}^{(1)} & \mathbb{D}^{(1) T} \mathbb{H}^{(2)} & 0  \tag{6.25}\\
\mathbb{H}^{(2)} \mathbb{D}^{(1)} & 0 & \mathbb{D}^{(2) T} \\
0 & \mathbb{D}^{(2)} & 0
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{h}_{\mathrm{CP}}^{(1)} \\
\boldsymbol{u}_{\mathrm{CP}}^{(2)} \\
\tilde{\boldsymbol{p}}_{\mathrm{CP}}^{(0)}
\end{array}\right]=\left[\begin{array}{c}
\tilde{\boldsymbol{f}}_{\mathrm{CP}}^{(2)} \\
\tilde{\boldsymbol{f}}_{\mathrm{CP}}^{(1)} \\
\mathbf{0}^{(3)}
\end{array}\right]
$$

with right-hand sides given by

$$
\begin{aligned}
& \boldsymbol{f}_{\mathrm{VD}}^{(3)}:=-\mathbb{D}^{(2)} \boldsymbol{r}_{\mathrm{b}}^{(2)}\left(u_{\mathrm{b}}^{(1)}\right), \\
& \boldsymbol{f}_{\mathrm{VD}}^{(2)}:=\boldsymbol{j}^{(2)}+\mathbb{D}^{(1)}\left(\tilde{\mathbb{H}}_{\mu^{-1}}^{(2)} \mathbb{T}^{(1) T} \tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}-\boldsymbol{r}_{\mu^{-1}, \mathrm{~b}}^{(1)}\left(d_{\mathrm{b}} u_{\mathrm{b}}^{(1)}\right)\right)+\left(\tilde{\mathbb{H}}^{(1)} \tilde{\mathbb{T}}^{(2) T} \tilde{\boldsymbol{p}}_{\mathrm{b}}^{(0)}-\boldsymbol{r}_{\mathrm{b}}^{(2)}\left(d_{\mathrm{b}} p_{\mathrm{b}}^{(0)}\right)\right), \\
& \tilde{\boldsymbol{f}}_{\mathrm{CP}}^{(2)}:=\mathbb{T}^{(1) T} \tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}, \\
& \tilde{\boldsymbol{f}}_{\mathrm{CP}}^{(1)}:=\mathbb{H}^{(2)} \boldsymbol{j}^{(2)}+\mathbb{T}^{(2) T} \tilde{\boldsymbol{p}}_{\mathrm{b}}^{(0)},
\end{aligned}
$$

[^46]where $\boldsymbol{j}^{(2)}:=R^{(2)}\left(\tilde{j}^{(2)}\right)$ (where the map $R^{(2)}$ integrates the outer-oriented form $\tilde{j}^{(2)}$ on the outer-oriented primal mesh), $\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}:=\tilde{R}_{\mathrm{b}}^{(1)}\left(u_{\mathrm{b}}^{(1)}\right)$ and $\tilde{\boldsymbol{p}}_{\mathrm{b}}^{(0)}:=\tilde{R}_{\mathrm{b}}^{(0)}\left(p_{\mathrm{b}}^{(0)}\right)$. We call these schemes the Vertex-based Dual (VD) scheme and the Cell-based Primal (CP) scheme, respectively.

Note that discrete formulation (6.24) also contains boundary contributions involving $d_{\mathrm{b}} u_{\mathrm{b}}^{(1)}$ and $d_{\mathrm{b}} p_{\mathrm{b}}^{(0)}$. These are not extra independent boundary conditions, because they can be calculated from $u_{\mathrm{b}}^{(1)}$ and $p_{\mathrm{b}}^{(0)}$ within $\partial \Omega$. In the discrete setting this can be done (approximately) as well, but we did not do this in our implementation.

Formulations (6.22) and (6.25) are the familiar direct and mixed formulations. They are also studied in [54]. Formulation (6.22) is also investigated for the magnetostatic boundary value problem (with essential boundary conditions) in [113]. Formulations (6.23) and (6.24) are built using the dual discrete Hodge matrices and are new.

The mixed formulations (6.23) and (6.25) have significantly larger linear systems compared to the direct formulations (6.22) and (6.24). However, the mixed formulations do have the advantage that the divergence condition for the potential is a purely topological relation involving only an incidence matrix. We will propose a third set of discrete formulations that tries to maintain this aspect, while using a linear system of size equal to the direct formulations.

This third pair of formulations combines the primal and dual discrete Hodge matrices. The first formulation, which we call the Dual-Primal (DP) scheme, is closely related to (6.22). It is given by

$$
\left[\begin{array}{cc}
\tilde{H}^{(2)} \mathbb{D}^{(1) T} \mathbb{H}_{\mu^{-1}}^{(2)} \mathbb{D}^{(1)} \tilde{H}^{(2)} & \mathbb{D}^{(0)}  \tag{6.26}\\
\mathbb{D}^{(0) T} & 0
\end{array}\right]\left[\begin{array}{c}
\tilde{\boldsymbol{u}}_{\mathrm{DP}}^{(2)} \\
\boldsymbol{p}_{\mathrm{DP}}^{(0)}
\end{array}\right]=\left[\begin{array}{c}
\boldsymbol{f}_{\mathrm{DP}}^{(1)} \\
\tilde{\boldsymbol{f}}_{\mathrm{DP}}^{(3)}
\end{array}\right] .
$$

The second formulation, called the Primal-Dual (PD) scheme, is related to (6.24) and given by

$$
\left[\begin{array}{cc}
\mathbb{H}^{(2)} \mathbb{D}^{(1)} \tilde{\mathbb{H}}^{(2)} \mathbb{D}^{(1) T} \mathbb{H}^{(2)} & \mathbb{D}^{(2) T}  \tag{6.27}\\
\mathbb{D}^{(2)} & 0
\end{array}\right]\left[\begin{array}{c}
\boldsymbol{u}_{\mathrm{PD}}^{(2)} \\
\tilde{\boldsymbol{p}}_{\mathrm{PD}}^{(0)}
\end{array}\right]=\left[\begin{array}{c}
\tilde{\boldsymbol{f}}_{\mathrm{PD}}^{(1)} \\
\mathbf{0}^{(3)}
\end{array}\right] .
$$

The right-hand sides are given by

$$
\begin{aligned}
& \boldsymbol{f}_{\mathrm{DP}}^{(1)}:=\tilde{\mathbb{H}}^{(2)} \tilde{\boldsymbol{j}}^{(2)}-\tilde{\mathbb{H}}^{(2)} \mathbb{D}^{(1) T} \mathbb{H}_{\mu^{-1}}^{(2)} \mathbb{D}^{(1)} \tilde{\boldsymbol{r}}_{\mathrm{b}}^{(2)}\left(\tilde{u}_{\mathrm{b}}^{(2)}\right)+\mathbb{T}^{(1) T} \tilde{\boldsymbol{h}}_{\mathrm{b}}^{(1)}, \\
& \tilde{\boldsymbol{f}}_{\mathrm{DP}}^{(3)}:=\mathbb{T}^{(0) T} \tilde{\boldsymbol{u}}_{\mathrm{b}}^{(2)}, \\
& \tilde{\boldsymbol{f}}_{\mathrm{PD}}^{(1)}:=\mathbb{H}^{(2)} \boldsymbol{j}^{(2)}+\mathbb{T}^{(2) T} \tilde{\boldsymbol{p}}_{\mathrm{b}}^{(0)}+\mathbb{H}^{(2)} \mathbb{D}^{(1)}\left(\tilde{\mathbb{H}}_{\mu^{-1}}^{(2)} \mathbb{T}^{(1) T} \tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}-\tilde{\boldsymbol{r}}_{\mu^{-1}, \mathrm{~b}}^{(2)}\left(\tilde{d}_{\mathrm{b}} \tilde{\mathrm{u}}_{\mathrm{b}}^{(1)}\right)\right) .
\end{aligned}
$$

### 6.3.2 Boundary value problem with constant permeability

We consider the same test case as in [113]. We take $\Omega=[0,1]^{3}$ and consider the potential $\left(u^{(1)}\right.$ or $\left.\tilde{u}^{(2)}\right)$ and Lagrange multiplier $\left(p^{(0)}\right)$, given by

$$
\begin{aligned}
& u_{x}(x, y, z)=2 \pi x^{3} \sin (2 \pi y) \cos (2 \pi z) \\
& u_{y}(x, y, z)=-3 x^{2} \cos (2 \pi y) \cos (2 \pi z), \\
& u_{z}(x, y, z)=-6 x^{2} \sin (2 \pi y) \cos (2 \pi z)
\end{aligned}
$$

|  | Eq. | Boundary conditions | NDF |
| :--- | :---: | :---: | :---: |
| VP | $(6.22)$ | $\tilde{t} \tilde{h}^{(1)}=\tilde{h}_{\mathrm{b}}^{(1)}, \tilde{t} \tilde{u}^{(2)}=\tilde{u}_{\mathrm{b}}^{(2)}$ | $N_{(1)}+N_{(0)}$ |
| CD | $(6.23)$ | $\tilde{t}^{(1)}=\tilde{h}_{\mathrm{b}}^{(1)}, \tilde{t} \tilde{u}^{(2)}=\tilde{u}_{\mathrm{b}}^{(2)}$ | $N_{(2)}+N_{(1)}+N_{(0)}$ |
| VD | $(6.24)$ | $\tilde{t} \tilde{u}^{(1)}=\tilde{u}_{\mathrm{b}}^{(1)}, \tilde{t} \tilde{p}^{(0)}=\tilde{p}_{\mathrm{b}}^{(0)}$ | $N_{(2)}+N_{(3)}$ |
| CP | $(6.25)$ | $\tilde{t} \tilde{u}^{(1)}=\tilde{u}_{\mathrm{b}}^{(1)}, \tilde{t} \tilde{p}^{(0)}=\tilde{p}_{\mathrm{b}}^{(0)}$ | $N_{(1)}+N_{(2)}+N_{(3)}$ |
| DP | $(6.26)$ | $\tilde{t} \tilde{h}^{(1)}=\tilde{h}_{\mathrm{b}}^{(1)}, \tilde{t} \tilde{u}^{(2)}=\tilde{u}_{\mathrm{b}}^{(2)}$ | $N_{(1)}+N_{(0)}$ |
| PD | $(6.27)$ | $\tilde{t}^{(1)}=\tilde{u}_{\mathrm{b}}^{(1)}, \tilde{t} \tilde{p}^{(0)}=\tilde{p}_{\mathrm{b}}^{(0)}$ | $N_{(2)}+N_{(3)}$ |

Table 6.5: For each of the six schemes the boundary conditions used and the number of degrees of freedom (NDF) are shown.

Furthermore, we take the permeability tensor $\mu$ equal to the identity. We use the boundary conditions and $\tilde{j}^{(2)}$ in accordance with this as data.

The convergence of the discrete variables is measured in the norms provided by the discrete Hodge matrices. For discrete variables $\boldsymbol{a}^{(k)} \in C^{(k)}(\Omega)$ and $\tilde{\boldsymbol{a}}^{(k)} \in \tilde{C}^{(k)}(\Omega)$ we use the relative norms

$$
\begin{align*}
& E\left(\boldsymbol{a}^{(k)}\right):=\frac{\sqrt{\left(\boldsymbol{a}^{(k)}-\boldsymbol{a}_{\mathrm{e}}^{(k)}\right)^{T} \mathbb{H}_{\chi}^{(k)}\left(\boldsymbol{a}^{(k)}-\boldsymbol{a}_{\mathrm{e}}^{(k)}\right)}}{\sqrt{\boldsymbol{a}_{\mathrm{e}}^{(k) T} \mathbb{H}_{\chi}^{(k)} \boldsymbol{a}_{\mathrm{e}}^{(k)}}}, \quad N\left(\boldsymbol{a}^{(k)}\right):=N_{(k)},  \tag{6.28}\\
& \tilde{E}\left(\tilde{\boldsymbol{a}}^{(k)}\right):=\frac{\sqrt{\left(\tilde{\boldsymbol{a}}^{(k)}-\tilde{\boldsymbol{a}}_{\mathrm{e}}^{(k)}\right)^{T} \tilde{\mathbb{H}}_{\chi}^{(k)}\left(\tilde{\boldsymbol{a}}^{(k)}-\tilde{\boldsymbol{a}}_{\mathrm{e}}^{(k)}\right)}}{\sqrt{\tilde{\boldsymbol{a}}_{\mathrm{e}}^{(k) T} \tilde{\mathbb{H}}_{\chi}^{(k)} \tilde{\boldsymbol{a}}_{\mathrm{e}}^{(k)}}}, \quad N\left(\tilde{\boldsymbol{a}}^{(k)}\right):=N_{(3-k)},
\end{align*}
$$

where $\boldsymbol{a}_{\mathrm{e}}^{(k)}:=R^{(k)}\left(a^{(k)}\right)$ or $\boldsymbol{a}_{\mathrm{e}}^{(k)}:=R^{(k)}\left(\tilde{a}^{(k)}\right)$ (when the primal mesh has outer orientation) and $\tilde{\boldsymbol{a}}_{\mathrm{e}}^{(k)}:=\tilde{R}^{(k)}\left(\tilde{a}^{(k)}\right)$ or $\tilde{\boldsymbol{a}}_{\mathrm{e}}^{(k)}:=\tilde{R}^{(k)}\left(a^{(k)}\right)$ (when the dual mesh has inner orientation) are the discrete representations of the exact solutions. Furthermore, $\chi$ is the relevant tensor, i.e., either $\mu$ or $\mu^{-1}$, depending on the variable $a^{(k)}$. If $a^{(k)}=b^{(2)}$, then $\chi=\mu^{-1}$, if $a^{(k)}=h^{(1)}$, then $\chi=\mu$ and if $a^{(k)}=u^{(1)}$, then $\chi$ is just the identity. The analogous situation holds for $\tilde{E}\left(\tilde{\boldsymbol{a}}^{(k)}\right)$.

Note that $E\left(\boldsymbol{a}^{(k)}\right)$ is an approximation of the relative error in the $L^{2}$-norm on $\Omega$ when $\boldsymbol{a}^{(k)}$ represents the vector potential and an approximation of the relative error of magnetic energy when $\boldsymbol{a}^{(k)}$ represents $\boldsymbol{b}^{(2)}$ or $\boldsymbol{h}^{(1)}$. However, this does not hold for $\tilde{E}\left(\tilde{\boldsymbol{a}}^{(k)}\right)$ because at the boundary $\tilde{\mathbb{H}}^{(k)}$ is not consistent without boundary contributions. Nevertheless, the dual discrete Hodge matrices are symmetric positive definite matrices also and therefore do define an inner product and corresponding norm.

We consider the convergence of the six schemes for the four quantities $u^{(1)}, \tilde{u}^{(2)}, b^{(2)}$ and $\tilde{h}^{(1)}$. Some of these are approximated directly by the unknowns of the discretization, others we calculate explicitly from these unknowns. In Table 6.6 it is listed for each of the schemes how the discrete variables are determined that approximate the discrete representation of $u^{(1)}, \tilde{u}^{(2)}, b^{(2)}$ and $\tilde{h}^{(1)}$. The exact solution $\left(p^{(0)}=0\right)$ of the Lagrange

|  | $u^{(1)}$ | $b^{(2)}$ |
| :---: | :---: | :---: |
| VP | $\boldsymbol{u}^{(1)}$ | $\mathbb{D}^{(1)} \boldsymbol{u}^{(1)}$ |
| CD | $\tilde{\mathbb{H}}^{(2)} \tilde{\boldsymbol{u}}^{(2)}+\boldsymbol{r}_{\mathrm{b}}^{(1)}\left(\tilde{u}_{\mathrm{b}}^{(2)}\right)$ | $\tilde{\mathbb{H}}_{\mu}^{(1)} \tilde{\boldsymbol{h}}^{(1)}+\boldsymbol{r}_{\mu, \mathbf{b}}^{(2)}\left(\tilde{h}_{\mathrm{b}}^{(1)}\right)$ |
| VD | $\tilde{\boldsymbol{u}}^{(1)}$ | $\mathbb{D}^{(1) T} \tilde{\boldsymbol{u}}^{(1)}-\mathbb{T}^{(1) T} \tilde{\boldsymbol{u}}^{(1)}$ |
| CP | $\mathbb{H}^{(2)} \boldsymbol{u}^{(2)}$ | $\mathbb{H}_{\mu}^{(1)} \boldsymbol{h}^{(1)}$ |
| DP | $\tilde{\mathbb{H}}^{(2)} \tilde{\boldsymbol{u}}^{(2)}+\boldsymbol{r}_{\mathrm{b}}^{(1)}\left(\tilde{u}_{\mathrm{b}}^{(2)}\right)$ | $\mathbb{D}^{(1)}\left(\tilde{\mathbb{H}}^{(2)} \tilde{\boldsymbol{u}}^{(2)}+\boldsymbol{r}_{\mathrm{b}}^{(1)}\left(\tilde{u}_{\mathrm{b}}^{(2)}\right)\right)$ |
| PD | $\mathbb{H}^{(2)} \boldsymbol{u}^{(2)}$ | $\mathbb{D}^{(1) T} \mathbb{H}^{(2)} \boldsymbol{u}^{(2)}-\mathbb{T}^{(1) T} \tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}$ |
|  | $\tilde{u}^{(2)}$ | $\tilde{h}^{(1)}$ |
| VP | $\mathbb{H}^{(1)} \boldsymbol{u}^{(1)}$ | $\mathbb{H}_{\mu^{-1}}^{(2)} \mathbb{D}^{(1)} \boldsymbol{u}^{(1)}$ |
| CD | $\tilde{\boldsymbol{u}}^{(2)}$ | $\tilde{\boldsymbol{h}}^{(1)}$ |
| VD | $\tilde{\mathbb{H}}^{(1)} \tilde{\boldsymbol{u}}^{(1)}+\boldsymbol{r}_{\mathrm{b}}^{(2)}\left(u_{\mathrm{b}}^{(1)}\right)$ | $\tilde{\mathbb{H}}_{\mu^{-1}}^{(2)}\left(\mathbb{D}{ }^{(1) T} \tilde{\boldsymbol{u}}^{(1)}-\mathbb{T}^{(1) T} \tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}\right)+\boldsymbol{r}_{\mu^{-1}, \mathrm{~b}}^{(1)}\left(d_{\mathrm{b}} u_{\mathrm{b}}^{(1)}\right)$ |
| CP | $\boldsymbol{u}^{(2)}$ | $\boldsymbol{h}^{(1)}$ |
| DP | $\tilde{\boldsymbol{u}}^{(2)}$ | $\mathbb{H}_{\mu^{-1}}^{(2)} \mathbb{D}^{(1)}\left(\tilde{\mathbb{H}}^{(2)} \tilde{\boldsymbol{u}}^{(2)}+\boldsymbol{r}_{\mathrm{b}}^{(1)}\left(\tilde{u}_{\mathrm{b}}^{(2)}\right)\right)$ |
| PD | $\boldsymbol{u}^{(2)}$ | $\tilde{\mathbb{H}}_{\mu^{-1}}^{(2)}\left(\mathbb{D}^{(1) T} \mathbb{H}^{(2)} \boldsymbol{u}^{(2)}-\mathbb{T}^{(1) T} \tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}\right)+\boldsymbol{r}_{\mu^{-1}, \mathrm{~b}}^{(1)}\left(d_{\mathrm{b}} u_{\mathrm{b}}^{(1)}\right)$ |

Table 6.6: For each of the schemes it is shown how the discrete variable is calculated that approximates the discrete representations of $u^{(1)}, \tilde{u}^{(2)}, b^{(2)}$ and $\tilde{h}^{(1)}$.
multiplier is found by all schemes on all meshes up to the accuracy of the quadrature rule used in the discretization of $\tilde{j}^{(2)}$ and the boundary conditions.

In Figure 6.6, the convergence behavior is shown. The close relation between the VP and DP schemes results in the fact that they find the same approximations for $b^{(2)}$ and $\tilde{h}^{(1)}$. Similarly, the close relation between the VD and PD schemes implies that they find the same approximations for $b^{(2)}$ and $\tilde{h}^{(1)}$.

The convergence behavior for the different schemes seem to be quite comparable. For the potential $u^{(1)}$ the CD and DP schemes outperform the other schemes. Similarly, the VD and PD schemes outperform the other schemes for $\tilde{h}^{(1)}$. For the potential $\tilde{u}^{(2)}$ and $b^{(2)}$ the results are closer together. However, for $\tilde{u}^{(2)}$ the VD scheme converges at a better rate than the other and for $b^{(2)}$ the CD scheme converges better than the others. These observations are confirmed when we look at the rates of convergence for the various schemes and variables. They can be found in Table 6.7.

We see that the schemes using only the dual discrete Hodge operators have a higher convergence rate than the schemes using only the primal discrete Hodge operators. Furthermore, the behavior of the DP and PD schemes that mix the two types of operators is somewhere in between. The DP scheme shows the same relatively high convergence for $u^{(1)}$ as the CD scheme, but for the rest has convergence rates close to the VP scheme.


Figure 6.6: The convergence behavior for the six schemes for the potentials $u^{(1)}$ and $\tilde{u}^{(2)}$, and, the magnetic fields $b^{(2)}$ and $\tilde{h}^{(1)}$. The relevant error $E$ is plotted against the relevant number of degrees of freedom $N$ as defined in (6.28). For $b^{(2)}$ and $\tilde{h}^{(1)}$ the lines for the VD and PD schemes are exactly on top of each other and also the lines for the VP and DP schemes are on top of each other. The approximations found for the discrete representations for $b^{(2)}$ and $\tilde{h}^{(1)}$ are the same for these pairs, respectively. Errors based on primal discrete Hodge matrices are drawn with solid line and errors based on dual discrete Hodge matrices are drawn with dashed line.

Similarly, the PD scheme shares the same good convergence for $\tilde{h}^{(1)}$ with the VD scheme, but for the other variables the rates are significantly lower.

### 6.3.3 Boundary value problem with variable permeability

As a second test we take the same exact solutions for $u^{(1)}$ and $p^{(0)}$, but the permeability is variable. This problem is again taken from [113]. The inverse of the permeability is
given by

$$
\mu^{-1}:=\left[\begin{array}{ccc}
1+y^{2}+z^{2} & -x y & -x z \\
-x y & 1+x^{2}+z^{2} & -y z \\
-x z & -y z & 1+x^{2}+z^{2}
\end{array}\right]
$$

We calculate the boundary data and $\tilde{j}^{(2)}$ in accordance with this.
The convergence behavior for this problem is shown in Figure 6.7 and the rates of convergence are given in Table 6.8. We see that for $u^{(1)}, \tilde{u}^{(2)}$ and $b^{(2)}$ the convergence is roughly similar to the convergence for the problem with constant permeability. It is for $\tilde{h}^{(1)}$ that the convergence is significantly worse. Especially, the rates of convergence of the schemes that performed best for $\tilde{h}^{(1)}$ when $\mu$ was constant, the VD and PD schemes, suffer. However, the size of the error for $\tilde{h}^{(1)}$ is still smallest for these schemes as can be seen in Figure 6.7.

|  | VP | CD | VD | CP | DP | PD |  | VP | CD | VD | CP | DP | PD |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $u^{(1)}$ | 1.2 | 1.6 | 1.2 | 1.2 | 1.5 | 1.0 | $u^{(1)}$ | 1.2 | 1.8 | 1.2 | 1.3 | 1.5 | 1.0 |
| $\tilde{u}^{(2)}$ | 1.1 | 1.1 | 1.5 | 1.1 | 1.1 | 1.0 | $\tilde{u}^{(2)}$ | 1.1 | 1.2 | 1.5 | 1.2 | 1.1 | 1.0 |
| $b^{(2)}$ | 0.9 | 1.7 | 1.1 | 1.2 | 0.9 | 1.1 | $b^{(2)}$ | 1.0 | 1.5 | 0.9 | 1.2 | 1.0 | 0.9 |
| $\tilde{h}^{(1)}$ | 1.0 | 1.1 | 1.8 | 1.3 | 1.0 | 1.8 | $\tilde{h}^{(1)}$ | 0.9 | 1.1 | 0.7 | 1.1 | 0.9 | 0.7 |

Table 6.7: Convergence rates for problem with constant $\mu$.

Table 6.8: Convergence rates for the problem with variable $\mu$.

### 6.3.4 Stokes flow

As a final test consider Stokes flow, given by

$$
\begin{align*}
\tilde{d} \star d u^{(1)}+\star d p^{(0)} & =\tilde{f}^{(2)} & & \text { in } \Omega,  \tag{6.29a}\\
\tilde{d} \star u^{(1)} & =0 & & \text { in } \Omega, \tag{6.29b}
\end{align*}
$$

where $u^{(1)} \in \Lambda^{(1)}(\Omega)$ is the inner-oriented velocity form describing flow along curves, $p^{(0)} \in \Lambda^{(0)}(\Omega)$ the pressure $\left(p^{(0)}\right.$ will now be nonzero) and $\tilde{f}^{(2)} \in \tilde{\Lambda}^{(2)}(\Omega)$ a given external load. Alternatively, Stokes flow can be described by

$$
\begin{align*}
-\tilde{\star} \tilde{\omega}^{(1)}+d \tilde{\star}^{\tilde{u}^{(2)}} & =0 & & \text { in } \Omega,  \tag{6.30a}\\
\tilde{\star} \tilde{d} \tilde{h}^{(1)}+d p^{(0)} & =\tilde{\star} \tilde{j}^{(2)} & & \text { in } \Omega,  \tag{6.30b}\\
\tilde{d} \tilde{u}^{(2)} & =0 & & \text { in } \Omega,
\end{align*}
$$

where now $\tilde{u}^{(2)} \in \tilde{\Lambda}^{(2)}(\Omega)$ is the outer-oriented velocity form describing flux through surfaces and $\tilde{\omega}^{(1)} \in \tilde{\Lambda}^{(1)}(\Omega)$ is the outer-oriented vorticity form describing the circulation of the flow around curves. These formulations are in the language of vector calculus known as curl-curl formulations, because they correspond to formulations where the


Figure 6.7: Convergence behavior for the boundary value problem with variable permeability.
diffusive form is written as $\Delta u_{(1)}=\nabla\left(\nabla \cdot u_{(1)}\right)-\nabla \times \nabla \times u_{(1)}=-\nabla \times \nabla \times u_{(1)}$, where it is used that the velocity field is divergence free. Earlier discussion of Stokes' equations in this form can be found in [31,114-116].

Of course (6.29) and (6.30) are simply the problems (6.20) and (6.21) again, but the variables have a different physical meaning this time. We take $\Omega=[0,1]^{3}$ and assume the solution to be given by (6.15). We calculate the external load and boundary conditions in accordance with this. This problem was also considered in [31].

We consider the same six schemes with the natural boundary conditions. Note that this implies that for the VP, CD and DP schemes the tangential vorticity ( $\tilde{t}_{\tilde{\omega}}{ }^{(1)}=\tilde{\omega}_{\mathrm{b}}^{(1)}$ ) and the normal velocity $\left(\tilde{t} \tilde{u}^{(2)}=\tilde{u}_{\mathrm{b}}^{(2)}\right)$ are given on $\partial \Omega$, and, similarly for the VD, CP and PD schemes the tangential velocity $\left(\tilde{t} \tilde{u}^{(1)}=\tilde{u}_{\mathrm{b}}^{(1)}\right)$ and the pressure $\left(\tilde{t}^{(0)}=\tilde{p}_{\mathrm{b}}^{(0)}\right)$ are given on $\partial \Omega$. The discrete variables approximating $u^{(1)}, \tilde{u}^{(2)}, \tilde{\omega}^{(2)}$ and $\omega^{(1)}$ are calculated just like the variables in the magnetostatic boundary value problem, given in Table 6.6.

|  | $p^{(0)}$ | $\tilde{p}^{(3)}$ |
| :---: | :---: | :---: |
| VP | $\boldsymbol{p}^{(0)}$ | $\mathbb{H}^{(0)} \boldsymbol{p}^{(0)}$ |
| CD | $\boldsymbol{p}^{(0)}$ | $\mathbb{H}^{(0)} \boldsymbol{p}^{(0)}$ |
| VD | $\tilde{\boldsymbol{p}}^{(0)}$ | $\tilde{\mathbb{H}}^{(0)} \tilde{\boldsymbol{p}}^{(0)}$ |
| CP | $\tilde{\boldsymbol{p}}^{(0)}$ | $\tilde{H}^{(0)} \tilde{\boldsymbol{p}}^{(0)}$ |
| DP | $\boldsymbol{p}^{(0)}$ | $\mathbb{H}^{(0)} \boldsymbol{p}^{(0)}$ |
| PD | $\tilde{\boldsymbol{p}}^{(0)}$ | $\tilde{\mathbb{H}}^{(0)} \tilde{\boldsymbol{p}}^{(0)}$ |

Table 6.9: For each scheme it is shown how the discrete variables are calculated that approximate the discrete representation of $p^{(0)}$ and $\tilde{p}^{(3)}=\star p^{(0)}$.

|  | VP | CD | VD | CP | DP | PD |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $u^{(1)}$ | 1.0 | 1.8 | 1.2 | 1.4 | 1.5 | 1.2 |
| $\tilde{u}^{(2)}$ | 1.1 | 1.2 | 1.6 | 1.2 | 1.1 | 1.1 |
| $\omega^{(2)}$ | 1.0 | 1.6 | 1.2 | 1.1 | 1.0 | 1.2 |
| $\tilde{\omega}^{(1)}$ | 1.0 | 1.2 | 1.5 | 1.0 | 1.0 | 1.5 |
| $p^{(0)}$ | 2.2 | 2.1 | 1.8 | 2.2 | 2.1 | 2.2 |
| $\tilde{p}^{(0)}$ | 2.0 | 1.9 | 1.5 | 1.5 | 1.9 | 1.5 |

Table 6.10: The convergence rates, determined by using the results on the two finest meshes (RL=6 and $\mathrm{RL}=7$ ), with the exception of the convergence rates for the pressure where we used the meshes with $\mathrm{RL}=3$ and $\mathrm{RL}=7$, because the convergence for the pressure is jerky (see Figure 6.8).

Similarly the discrete variables approximating $p^{(0)}$ and $\tilde{p}^{(3)}:=\star p^{(0)}$ are calculated as given in Table 6.9.

The convergence graphs are shown in Figure 6.8 and the convergence rates are given in Table 6.10. The relative performance of the different schemes is roughly the same as for the previous two tests. We again see that the VP and DP schemes find the same vorticity, just as the VD and PD schemes. Similarly, it can be observed that the CD and DP schemes find the same pressure and also the CP and PD schemes find the same pressure. This is not entirely surprising as the DP scheme combines features from the VP and CD scheme and the PD scheme similarly combines features from the VD and CP scheme.

The convergence of the pressure is for most schemes close to second order. Furthermore it can be remarked that the VP scheme performs best with respect to the pressure and the VD scheme worst. However, as can be seen in Figure 6.8, the results are quite close together especially for $p^{(0)}$.

(a) Convergence to $u^{(1)}: e\left(\boldsymbol{u}_{\mathrm{VP}}^{(1)}\right), e\left(\boldsymbol{u}_{\mathrm{CD}}^{(1)}\right)$, $\tilde{e}\left(\tilde{\boldsymbol{u}}_{\mathrm{VD}}^{(1)}\right), \tilde{e}\left(\tilde{\boldsymbol{u}}_{\mathrm{CP}}^{(1)}\right), e\left(\boldsymbol{u}_{\mathrm{DP}}^{(1)}\right), \tilde{e}\left(\tilde{\boldsymbol{u}}_{\mathrm{PD}}^{(1)}\right)$.

(c) Convergence to $\omega^{(2)}: e\left(\boldsymbol{\omega}_{\mathrm{VP}}^{(2)}\right), e\left(\boldsymbol{\omega}_{\mathrm{CD}}^{(2)}\right)$, $\tilde{e}\left(\tilde{\boldsymbol{\omega}}_{\mathrm{VD}}^{(2)}\right), \tilde{e}\left(\tilde{\boldsymbol{\omega}}_{\mathrm{CP}}^{(2)}\right), e\left(\boldsymbol{\omega}_{\mathrm{DP}}^{(2)}\right), \tilde{e}\left(\tilde{\boldsymbol{\omega}}_{\mathrm{PD}}^{(2)}\right)$.

(e) Convergence to $p^{(0)}: e\left(\boldsymbol{p}_{\mathrm{VP}}^{(0)}\right), e\left(\boldsymbol{p}_{\mathrm{CD}}^{(0)}\right)$, $\tilde{e}\left(\tilde{\boldsymbol{p}}_{\mathrm{VD}}^{(0)}\right), \tilde{e}\left(\tilde{\boldsymbol{p}}_{\mathrm{CP}}^{(0)}\right), e\left(\boldsymbol{p}_{\mathrm{DP}}^{(0)}\right), \tilde{e}\left(\tilde{\boldsymbol{p}}_{\mathrm{PD}}^{(0)}\right)$.

(b) Convergence to $\tilde{u}^{(2)}: \tilde{e}\left(\tilde{\boldsymbol{u}}_{\mathrm{VP}}^{(2)}\right), \tilde{e}\left(\tilde{\boldsymbol{u}}_{\mathrm{CD}}^{(2)}\right)$, $e\left(\boldsymbol{u}_{\mathrm{VD}}^{(2)}\right), e\left(\boldsymbol{u}_{\mathrm{CP}}^{(2)}\right), \tilde{e}\left(\tilde{\boldsymbol{u}}_{\mathrm{DP}}^{(2)}\right), e\left(\boldsymbol{u}_{\mathrm{PD}}^{(2)}\right)$.

(d) Convergence to $\tilde{\omega}^{(1)}: \tilde{e}\left(\tilde{\omega}_{\mathrm{VP}}^{(1)}\right), \tilde{e}\left(\tilde{\omega}_{\mathrm{CD}}^{(1)}\right)$, $e\left(\boldsymbol{\omega}_{\mathrm{VD}}^{(1)}\right), e\left(\boldsymbol{\omega}_{\mathrm{CP}}^{(1)}\right), \tilde{e}\left(\tilde{\boldsymbol{\omega}}_{\mathrm{DP}}^{(1)}\right), e\left(\boldsymbol{\omega}_{\mathrm{PD}}^{(1)}\right)$.

(f) Convergence to $\tilde{p}^{(3)}: \tilde{e}\left(\tilde{\boldsymbol{p}}_{\mathrm{VP}}^{(3)}\right), \tilde{e}\left(\tilde{\boldsymbol{p}}_{\mathrm{CD}}^{(3)}\right)$, $e\left(\boldsymbol{p}_{\mathrm{VD}}^{(3)}\right), e\left(\boldsymbol{p}_{\mathrm{CP}}^{(3)}\right), \tilde{e}\left(\tilde{\boldsymbol{p}}_{\mathrm{DP}}^{(3)}\right), e\left(\boldsymbol{p}_{\mathrm{PD}}^{(3)}\right)$.

Figure 6.8: Convergence behavior for the Stokes flow boundary value problem.

### 6.4 Final remarks

We have derived consistent dual discrete Hodge matrices $\tilde{\mathbb{H}}^{(k)}$ that interpolate $k$-cochains from a barycentric dual mesh to $(d-k)$-cochains on the corresponding simplicial primal mesh. The dual discrete Hodge matrices are symmetric positive definite, just as the primal discrete Hodge matrices $\mathbb{H}^{(k)}$. To make the dual discrete Hodge matrices consistent in dual cells touching the boundary $\partial \Omega$, extra boundary contributions are needed which we also derived.

In contrast to the diagonal discrete Hodge matrices used in the DEC method, the discrete Hodge matrices discussed in this chapter are always symmetric positive definite, irrespective of the primal simplicial mesh. Moreover, in contrast to the cirumcentric dual cells, the barycentric dual cells always exactly partition the same domain as their corresponding primal mesh cells. Hence, extra difficulties with (internal) boundaries (as discussed in [100]) are avoided.

Next, we used the dual and primal discrete Hodge matrices to discretize Stokes' equations in various ways, resulting in six numerical schemes. Two of these only use primal discrete Hodge matrices, two only use dual discrete Hodge matrices and two combine the primal and dual discrete Hodge matrices in their formulation. The schemes using only primal discrete Hodge matrices have been studied before [31,113], the other four schemes are new.

We applied the schemes to a magnetostatic boundary value problem and a Stokes flow problem. For the magnetostatic case we tested the schemes with constant and with varying permeability tensor. In most cases the schemes using the dual discrete Hodge matrices performed somewhat better than the schemes using only primal discrete Hodge matrices. However, the schemes using the dual discrete Hodge matrices are more expensive as they have a wider stencil.

In future research it should be investigated to what extent the convergence behavior of the dual discrete Hodge matrices is influenced by the stability term used in their definition. Just like for the primal discrete Hodge matrices, for the dual ones a variety of forms for the stability term are possible. It should be determined which choice is optimal.

The schemes using the dual discrete Hodge matrices should be compared to the same schemes using DEC Hodge matrices. The DEC primal discrete Hodge matrices are diagonal and therefore the dual discrete Hodge matrices are given by their inverses which are explicitly known. The schemes discussed in this chapter should be compared to their DEC equivalents to analyze their behavior and see how each performs on simplicial meshes of varying regularity. It would be interesting to see whether for non-Delaunay meshes it is beneficial to use the less sparse, but symmetric positive definite, discrete Hodge matrices discussed in this chapter.

Finally, the discretizations for Stokes' equations can be extended to energy-conserving discretizations for the incompressible Navier-Stokes equations suitable for simulations of turbulent flows. The reconstruction operators used to derive the consistency part can be used to discretize the nonlinear convective term, which is either in the divergence form or in the rotational form. The newly derived dual cell reconstruction operators $\mathbb{S}_{\star \sigma_{(0)}}^{(d-l)}$ allow for new energy-conserving discretizations of the convective term which can be combined with the discretizations for Stokes' equations introduced in this chapter.

Conclusions and Future Research

### 7.1 Conclusions

At the start of this thesis we set out to find an easily implementable energy-conserving extension of the MAC method to cut-cell meshes that allows for local mesh refinement.

In Chapter 3 we reformulated the MAC method in the framework of mimetic discretizations. This gave us insight in the possible generalizations of the method to nonCartesian meshes through the use of a barycentric dual mesh and corresponding discrete Hodge matrices.

In Chapter 4 we used this framework to discretize the incompressible Navier-Stokes equations. We first discretized the Stokes equations and saw that there are two ways to do this based on the location of the variables and the corresponding type of orientation chosen for the primal mesh. The inner-oriented scheme employs velocity variables located on the mesh edges and pressure variables in the vertices, and, the outer-oriented scheme uses velocity variables at the mesh faces, vorticity variables at the mesh edges and pressure variables in the cell centers.

The main difficulty in the step from the Stokes equations to the Navier-Stokes equations is the discretization of the nonlinear convective term. For the outer-oriented scheme we used an earlier proposed discretization of the convection in the divergence form that leads to a momentum-, energy- and vorticity-conserving scheme. However, the stencil of this scheme is rather big and we proposed two alternative discretizations for the convection based on the rotational form, one for the inner-oriented scheme and one for the outer-oriented scheme. These discretizations are both energy- and vorticity-conserving, but do not conserve momentum, with Cartesian meshes as the exception for which all methods considered are momentum-conserving too.

We tested the three numerical methods using three different discrete Hodge matrices on a wide range of meshes. We found that for the Cartesian mesh sequence the methods show second order convergence for the velocity. The convergence of the pressure and vorticity is around first order for the outer-oriented schemes while it is close to second order for the inner-oriented scheme. On the other mesh sequences the convergence rate of the velocity is roughly between first and second order depending on the irregularity of the meshes. The inner-oriented scheme generally shows the best convergence rates, but often also has the largest error. Based on the convergence results and stencil width, the inner-oriented scheme seems to be the most efficient of the three.

In the last part of Chapter 4 we applied the outer-oriented scheme with convection term in divergence form on two flow problems where the geometry was discretized using
a locally refined cut-cell mesh. We calculated the benchmark flow around a cylinder in a channel and found good predictions of the lift and drag forces on the cylinder. Next, we calculated the impulsively started flow around the NACA0012 airfoil and found a very good qualitative agreement of the streamlines with those determined by PIV measurements.

These results allow us to draw the main conclusion of this thesis, namely, that using the framework of mimetic methods the MAC method can be extended to locally refined cut-cell meshes, while retaining (most of) its good properties: the method is mass-, energy- and vorticity-conserving. Moreover, the mimetic methods are easily implementable. The only point of Section 1.4 that is not fully met is the one pertaining to its simplification to the original MAC method on Cartesian meshes, as for all three methods the discrete convection term does not coincide with the original MAC convection term on Cartesian meshes in 3D.

In the second part of the thesis we considered two separate but related problems. In Chapter 5 we consider div-curl problems with either normal or tangential boundary conditions and discretized these in the mimetic framework introduced in Chapter 3. We showed that using appropriate discrete Helmholtz-Hodge decompositions (one for each boundary condition) these problems can be solved.

Furthermore, we showed that, when an appropriate discrete formulation is used, the potentials in the discrete Helmholtz-Hodge decompositions can be determined by solving non-singular linear systems. This shows that the non-standard linear algebra methods needed for ill-posed linear systems for the potentials derived in the recent paper [96] can be avoided.

Finally, in Chapter 6 we considered mimetic methods on simplicial meshes. We proved that not only for the circumcentric dual mesh, but also for the barycentric dual mesh there exist consistent symmetric positive-definite dual discrete Hodge operators (for all mesh element dimensions) that interpolate from the dual to the primal mesh. Good conditioning of these barycentric dual discrete Hodge operators does not require the stringent mesh-regularity requirements that the circumcentric ones require. However, this is paid for by the sparsity of the barycentric dual discrete Hodge matrices, which is significantly worse than that of the diagonal circumcentric Hodge matrices.

### 7.2 Future research directions

Many directions for follow up research can be imagined. Here we mention two topics that we did some preliminary research on: an energy- and helicity-conserving discretization and moving immersed boundaries.

### 7.2.1 Energy- and helicity-conserving discretization

The helicity of a flow is defined as

$$
\begin{equation*}
\mathcal{H}:=\int_{\Omega} \underline{u} \cdot \underline{\omega} d V \tag{7.1}
\end{equation*}
$$

with $\underline{u}$ the velocity field and $\underline{\omega}$ the vorticity field. It can be shown that this is a secondary conserved quadratic invariant in the absence of viscosity and body forces.

The helicity of a flow and its conservation are considered to be important for turbulent flows. The helicity is related to the flow topology as it can be related to the linkage of vortex lines [117]. Furthermore, there is a joint simultaneous cascade of energy and helicity to small scales [118].

Just as kinetic energy, helicity is in general not conserved discretely. In recent years finite element methods have been developed that do conserve helicity [119, 120]. The discrete conservation of helicity for finite difference and spectral methods was studied in [121]. Here it was found that the MAC scheme, although it conserves mass, momentum, vorticity and energy (in the inviscid case), does not conserve helicity (in the inviscid case). Moreover, the definition of discrete helicity is ambiguous as it can be defined (using simple central averages) either in the cell centers or the vertices of the mesh. Both of these discrete helicities are not conserved.

The definition of the discrete helicity for the MAC scheme is ambiguous, because the velocity variables are located at the mesh faces and the vorticity variables are located at the mesh edges. It is not clear how to define a discrete inner product between the two. The two approximations defined in [121] are obviously not symmetric.

In Chapter 4 we saw that the MAC scheme can be generalized in two ways. One generalization corresponds to the MAC scheme on the primal mesh and resulted in the outer-oriented discretization (4.11). The other generalization can be viewed as a scheme on the dual mesh and resulted in the inner-oriented discretization (4.12). Combining the discrete variables of these schemes it is possible to define two discrete helicities:

$$
\begin{equation*}
H_{\mathrm{io}}:=\boldsymbol{u}_{\mathrm{i}}^{(1) T} \mathbb{H}^{(1)} \boldsymbol{\omega}_{\mathrm{o}}^{(1)}, \quad \text { and } \quad H_{\mathrm{oi}}:=\boldsymbol{u}_{\mathrm{o}}^{(2) T} \mathbb{H}^{(2)} \boldsymbol{\omega}_{\mathrm{i}}^{(2)} \tag{7.2}
\end{equation*}
$$

where $\boldsymbol{u}_{\mathrm{i}}^{(1)}$ and $\boldsymbol{\omega}_{\mathrm{i}}^{(2)}$ are the variables of the inner-oriented scheme and $\boldsymbol{u}_{\mathrm{o}}^{(2)}$ and $\boldsymbol{\omega}_{\mathrm{o}}^{(1)}$ are the variables of the outer-oriented scheme. Similarly, we can define the two discrete energies

$$
\begin{equation*}
K_{\mathrm{ii}}:=\frac{\rho}{2} \boldsymbol{u}_{\mathrm{i}}^{(1) T} \mathbb{H}^{(1)} \boldsymbol{u}_{\mathrm{i}}^{(1)}, \quad \text { and } \quad K_{\mathrm{oo}}:=\frac{\rho}{2} \boldsymbol{u}_{\mathrm{o}}^{(2) T} \mathbb{H}^{(2)} \boldsymbol{u}_{\mathrm{o}}^{(2)} \tag{7.3}
\end{equation*}
$$

An energy- and helicity-conserving generalization of the MAC scheme can be formulated by combining the inner- and outer-oriented discretizations through the rotationalform convection term. We define the combination of the inner- and outer-oriented scheme by

$$
\begin{align*}
& \rho \mathbb{H}^{(2)} \frac{\partial \boldsymbol{u}_{\mathrm{o}}^{(2)}}{\partial t}+\mathbb{N}_{\mathrm{o}}\left(\boldsymbol{\omega}_{\mathrm{i}}^{(2)}, \boldsymbol{u}_{\mathrm{o}}^{(2)}\right)+ \mu \mathbb{H}^{(2)} \mathbb{D}^{(1)} \boldsymbol{\omega}_{\mathrm{o}}^{(1)}+\tilde{\mathbb{D}}^{(0)}\left[\begin{array}{c}
\tilde{\boldsymbol{q}}^{(0)} \\
\tilde{\boldsymbol{q}}_{\mathrm{b}}^{(0)}
\end{array}\right]  \tag{7.4a}\\
& \mathbb{H}^{(1)} \boldsymbol{\omega}_{\mathrm{o}}^{(1)}-\tilde{\mathbb{D}}^{(1)}\left[\begin{array}{c}
\mathbb{H}^{(2)} \boldsymbol{f}^{(2)}, \\
\boldsymbol{u}_{\mathrm{o}}^{(2)} \\
\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(1)}
\end{array}\right]=\tilde{\mathbf{0}}^{(2)},  \tag{7.4b}\\
& \mathbb{D}^{(2)} \boldsymbol{u}_{\mathrm{o}}^{(2)}=\mathbf{0}^{(3)}, \tag{7.4c}
\end{align*}
$$

and

$$
\begin{align*}
\rho \mathbb{H}^{(1)} \frac{\partial \boldsymbol{u}_{\mathrm{i}}^{(1)}}{\partial t}+\mathbb{N}_{\mathrm{i}}\left(\boldsymbol{\omega}_{\mathrm{o}}^{(1)}, \boldsymbol{u}_{\mathrm{i}}^{(1)}\right)+\mu \tilde{\mathbb{D}}^{(1)}\left[\begin{array}{c}
\mathbb{H}^{(2)} \boldsymbol{\omega}_{\mathrm{i}}^{(2)} \\
\tilde{\boldsymbol{\omega}}_{\mathrm{b}}^{(1)}
\end{array}\right]+\mathbb{H}^{(1)} \mathbb{D}^{(0)} \boldsymbol{q}^{(0)} & =\mathbb{H}^{(1)} \boldsymbol{f}^{(1)},  \tag{7.5a}\\
\boldsymbol{\omega}_{\mathrm{i}}^{(2)}-\mathbb{D}^{(1)} \boldsymbol{u}_{\mathrm{i}}^{(1)} & =0  \tag{7.5b}\\
\tilde{\mathbb{D}}^{(2)}\left[\begin{array}{c}
\mathbb{H}^{(1)} \boldsymbol{u}_{\mathrm{i}}^{(1)} \\
\tilde{\boldsymbol{u}}_{\mathrm{b}}^{(2)}
\end{array}\right] & =\tilde{\mathbf{0}}^{(3)} \tag{7.5c}
\end{align*}
$$

Note that the only difference with respect to (4.11) and (4.12) is in the convection terms through which (7.5) and (7.4) are now linked. It can be shown that this formulation conserves both discrete helicities (7.2) and both discrete energies in (7.3). (See Appendix B.)

We have seen in Chapter 4 that the inner- and outer-orientations are also momentumconserving on Cartesian meshes. This may also be the case for the above discretization, but should be verified. Furthermore, it should be investigated how this formulation can by discretized in time. An efficient choice would be to use leapfrog time integration with the variables of the inner-oriented scheme located at the intermediate time levels of the outer-oriented scheme. This then leads to a linear system as the vorticity in the convection term is then already given by the previous time step. The resulting leapfrog scheme can easily be checked to be energy-conserving also in time. However, the helicity conservation in time should be analyzed too.

### 7.2.2 Moving immersed boundaries

An obvious extension of the cut-cell method described in Chapter 4 is the extension to moving immersed bodies. A first step in this direction was made in [122]. ${ }^{1}$ Here the 2D discretization with the convection in divergence form was extended to moving objects with a prescribed velocity.

After a time step from $t_{n}$ to $t_{n+1}$ the object acquires a new position and as a result the mesh has to be adjusted. The new mesh is calculated just as described in Section 4.4.1. Subsequently, the Hodge matrices and convection discretization have to be adjusted to the new mesh. In [122] the time integration is performed by the implicit Euler method. The new velocity, vorticity, and pressure at $t_{n+1}$ are calculated (in 2D) according to

$$
\begin{array}{r}
\frac{\rho}{\Delta t} \mathbb{H} \mathbb{H}_{n+1}^{(1)} \boldsymbol{u}_{n+1}^{(1)}+\mathbb{N}_{n+1, \mathrm{~d}}\left(\boldsymbol{u}_{n+1}^{(1)}\right)+\mu \mathbb{H}_{n+1}^{(1)} \mathbb{D}_{n+1}^{(0)} \boldsymbol{\omega}_{n+1}^{(0)}+\tilde{\mathbb{D}}_{n+1}^{(0)}\left[\begin{array}{c}
\tilde{\boldsymbol{p}}_{n+1}^{(0)} \\
\tilde{\boldsymbol{p}}_{n+1, \mathrm{~b}}^{(0)}
\end{array}\right]=\frac{\rho}{\Delta t} \mathbb{H}_{n+1}^{(1)} \hat{\boldsymbol{u}}_{n}^{(1)}, \\
\mathbb{H}_{n+1}^{(0)} \boldsymbol{\omega}_{n+1}^{(0)}+\tilde{\mathbb{D}}^{(1)}\left[\begin{array}{c}
\mathbb{H}_{n+1}^{(1)} \boldsymbol{u}_{n+1}^{(1)} \\
\tilde{\boldsymbol{u}}_{n+1, \mathrm{~b}}^{(1)}
\end{array}\right]=\tilde{\mathbf{0}}^{(2)}, \\
\mathbb{D}_{n+1}^{(1)} \boldsymbol{u}_{n+1}^{(1)}=\mathbf{0}^{(3)},
\end{array}
$$

where the subscripts indicated the time level, $\Delta t$ is the time step, and $\hat{\boldsymbol{u}}_{n}^{(1)}$, is the discrete velocity at $t_{n}$, but on the mesh at $t_{n+1}$. Thus, to complete the discretization an

[^47]interpolation from the velocity $\boldsymbol{u}_{n}^{(1)}$ given at the mesh at $t_{n}$ to the velocity $\hat{\boldsymbol{u}}_{n}^{(1)}$ defined at the mesh at $t_{n+1}$ must be chosen.

At $t_{n+1}$ there are four types of mesh edges:
(1) edges that remained unchanged from $t_{n}$ to $t_{n+1}$,
(2) edges that lay in the object at $t_{n}$ and appear at $t_{n+1}$,
(3) edges that exist at both $t_{n}$ and $t_{n+1}$ but changed length,
(4) edges that were open at $t_{n}$ and disappear at $t_{n+1}$ because they now lie in the object.

For the type (1) edges nothing has to be done. For the type (2) edges the speed of the object at $t_{n}$ is prescribed. The velocity values for the type (3) edges are rescaled to account for the change in the length of the edge, while assuming a constant velocity along the edge. The velocity variables for the type (4) edges are simply not needed in $\hat{\boldsymbol{u}}_{n}^{(1)}$.

In [122] the method is applied to the flow around a cylinder cf. Section 4.4.2. In this case no-slip boundary conditions are prescribed at each side of the domain and the cylinder is moved. The cylinder is prescribed a movement through the closed channel, starting at $x=-1.5$ at $t=0$, moving at constant speed up to $x=1.5$ at $t=3$, and, (instantly) reversing and ending up at its starting point again at $t=6$. The Reynolds number was set at $\operatorname{Re}=100$, a mesh of $200 \times 50$ with no refinement was used and time step $\Delta t=0.01$. The calculated vorticity field is shown in Figure 7.1.

In the future the method should be equipped with a time integration method that is less dissipative. Furthermore, the method should be extended to the situation where the movement of the object interacts with the fluid by Newton's second law. Finally, an extension to three-dimensional problems remains to be done. For this only the calculation of the cut-cell mesh becomes more cumbersome. The 3D discretizations described in Chapter 4 can readily be applied to three-dimensional cut-cell meshes. To be able to apply the method to the calculation of the flow through a wind farm the method should be combined with a mesh refinement algorithm that adaptively refines the mesh near the position of the turbine blades. With these extensions the last steps can be made in the direction set in Chapter 1: to an energy-conserving cut-cell method for the simulation of wind-turbine wakes in wind farms.


Figure 7.1: Snapshots of the vorticity field for a cylinder moving through a canal with no-slip boundaries at $\operatorname{Re}=100$. Calculated using the method described in [122].

## Appendix

## Well-posedness of Problem ( $\mathbf{P}_{t}^{b}$ )

In this appendix we show the well-posedness of Problem $\left(\mathrm{P}_{t}^{b}\right)$ stated in Section 5.1.3.2. We first show that Problem $\left(\mathrm{P}_{t}^{b}\right)$ is equivalent to the strong formulation

$$
\begin{aligned}
\left(d d^{*}+d^{*} d\right) b^{(k+1)} & =g^{(k+1)}, \\
t d^{*} b^{(k+1)} & =r_{\mathrm{b}}^{(k)}, \\
t b^{(k+1)} & =0,
\end{aligned}
$$

together with the requirement $b^{(k+1)} \in \mathcal{H}_{t}^{(k+1) \perp}$. By multiplying with the functions from the appropriate function spaces we easily see that a solution to this strong formulation solves $\left(\mathrm{P}_{t}^{b}\right)$.

We now show that a solution of $\left(\mathrm{P}_{t}^{b}\right)$ also solves the strong formulation. We use a reasoning similar to the one used in [123, p.41]. Restricting the first equation in the problem statement of $\left(\mathrm{P}_{t}^{b}\right)$ to $\chi \in H \Lambda^{(k)}(d, \Omega)$ shows that $b \in H \Lambda^{(k)}\left(d^{*}, \Omega\right)$ and $d^{*} b=\psi$. Furthermore, the last three equations state that $t b=0, t d^{*} \psi=r_{\mathrm{b}}$ and $b \in$ $\mathcal{H}_{t}^{(k+1) \perp}$. Substituting this in the second equation and restricting it to the function space $\dot{H} \Lambda^{(k+1)}(d, \Omega) \cap H \Lambda^{(k+1)}\left(d^{*}, \Omega\right)$ gives

$$
(d b, d v)+\left(d^{*} b, d^{*} v\right)+(q, v)=(g, v)+\left(r_{\mathrm{b}}, t^{*} v\right)_{\mathrm{b}}, \quad \forall v \in \stackrel{\circ}{H} \Lambda^{(k+1)}(d, \Omega) \cap H \Lambda^{(k+1)}\left(d^{*}, \Omega\right) .
$$

Taking $v=q$ leaves $(q, q)=0$ so $q=0$ and we have

$$
(d b, d v)+\left(d^{*} b, d^{*} v\right)=(g, v)+\left(r_{\mathrm{b}}, t^{*} v\right)_{\mathrm{b}}, \quad \forall v \in \stackrel{\circ}{H} \Lambda^{(k+1)}(d, \Omega) \cap H \Lambda^{(k+1)}\left(d^{*}, \Omega\right) .
$$

We use this equation to show that $d b \in H \Lambda^{(k+2)}\left(d^{*}, \Omega\right)$ and $d^{*} b \in H \Lambda^{(k)}(d, \Omega)$. Note that by Lemma 2.3 we can write $v \in \stackrel{\circ}{H} \Lambda^{(k+1)}(d, \Omega) \cap H \Lambda^{(k+1)}\left(d^{*}, \Omega\right)$ as $v=v_{1}+v_{2}$ with $v_{1} \in \dot{\mathcal{Z}}^{(k+1)}, v_{2} \in \mathcal{B}^{*(k+1)}$ which satisfy

$$
\begin{aligned}
(d b, d v) & =\left(d b, d v_{2}\right) & \left(d b, d v_{1}\right) & =0, \\
\left(d^{*} b, d^{*} v\right) & =\left(d^{*} b, d^{*} v_{1}\right) & \left(d^{*} b, d^{*} v_{2}\right) & =0 .
\end{aligned}
$$

Using the fact that there exists a $c>0$ such that $(g, v)+\left(r_{\mathrm{b}}, t^{*} v\right)_{\mathrm{b}} \leq c\|v\|$ for all $v \in$ $\stackrel{\circ}{H} \Lambda^{(k+1)}(d, \Omega) \cap H \Lambda^{(k+1)}\left(d^{*}, \Omega\right)$ we find

$$
(d b, d v)=\left(d b, d v_{2}\right)=\left(d b, d v_{2}\right)+\left(d^{*} b, d^{*} v_{2}\right)=\left(g, v_{2}\right)+\left(r_{\mathrm{b}}, t^{*} v_{2}\right)_{\mathrm{b}} \leq c\left\|v_{2}\right\| \leq c\|v\| .
$$

This shows by Definition 2.32 that $d b \in H \Lambda^{(k+2)}\left(d^{*}, \Omega\right)$. By a similar argument it follows that $d^{*} b \in H \Lambda^{(k)}(d, \Omega)$. Integration by parts now gives $\left(\left(d^{*} d+d d^{*}\right) b, v\right)=(g, v)$ for all $v \in H \Lambda^{(k+1)}(d, \Omega)$ and because $g \in H \Lambda^{(k+1)}(d, \Omega)$ we see that the $b$ also solves the strong formulation.

In similar ways it is possible to show that the strong formulation is equivalent to the weak mixed problem: Find $\left(\psi^{(k+2)}, b^{(k+1)}, q^{(k+1)}\right) \in H \Lambda^{(k+2)}\left(d^{*}, \Omega\right) \times H \Lambda^{(k+1)}\left(d^{*}, \Omega\right) \times$ $\mathcal{H}_{t}^{(k+1)}$ such that

$$
\begin{aligned}
(\psi, \chi)-\left(b, d^{*} \chi\right) & =0 & & \forall \chi \in H \Lambda^{(k+2)}\left(d^{*}, \Omega\right), \\
\left(d^{*} b, d^{*} v\right)+\left(d^{*} \psi, v\right)+(q, v) & =(g, v)-\left(r_{\mathrm{b}}, t^{*} v\right)_{\mathrm{b}} & & \forall v \in H \Lambda^{(k+1)}\left(d^{*}, \Omega\right), \\
(b, h) & =0 & & \forall \mathcal{H}_{t}^{(k+1)} .
\end{aligned}
$$

This problem is again a version of Problem (GP) (with reversed numbering, because the spaces and operators form a chain complex) and is therefore well-posed. This implies the well-posedness of the strong formulation and $\left(\mathrm{P}_{t}^{b}\right)$.

We show that in the absence of viscosity, and with a conservative force term, (7.4) and (7.5) conserve the discrete helicities and energies given in (7.2) and (7.3), respectively. For simplicity we assume a periodic domain. Thus, we can rewrite (7.4) and (7.5) as

$$
\begin{align*}
\rho \mathbb{H}^{(2)} \frac{\partial \boldsymbol{u}_{\mathrm{o}}^{(2)}}{\partial t}+\mathbb{N}_{\mathrm{o}}\left(\boldsymbol{\omega}_{\mathrm{i}}^{(2)}, \boldsymbol{u}_{\mathrm{o}}^{(2)}\right)-\mathbb{D}^{(2) T} \tilde{\boldsymbol{q}}^{(0)} & =\tilde{\boldsymbol{f}}^{(1)},  \tag{B.1a}\\
\mathbb{H}^{(1)} \boldsymbol{\omega}_{\mathrm{o}}^{(1)}-\mathbb{D}^{(1) T} \mathbb{H}^{(2)} \boldsymbol{u}_{\mathrm{o}}^{(2)} & =\tilde{\mathbf{0}}^{(2)},  \tag{B.1b}\\
\mathbb{D}^{(2)} \boldsymbol{u}_{\mathrm{o}}^{(2)} & =\mathbf{0}^{(3)}, \tag{B.1c}
\end{align*}
$$

and

$$
\begin{align*}
\rho \mathbb{H}^{(1)} \frac{\partial \boldsymbol{u}_{\mathrm{i}}^{(1)}}{\partial t}+\mathbb{N}_{\mathrm{i}}\left(\boldsymbol{\omega}_{\mathrm{o}}^{(1)}, \boldsymbol{u}_{\mathrm{i}}^{(1)}\right)+\mathbb{H}^{(1)} \mathbb{D}^{(0)} \boldsymbol{q}^{(0)} & =\mathbb{H}^{(1)} \boldsymbol{f}^{(1)},  \tag{B.2a}\\
\boldsymbol{\omega}_{\mathrm{i}}^{(2)}-\mathbb{D}^{(1)} \boldsymbol{u}_{\mathrm{i}}^{(1)} & =0,  \tag{B.2b}\\
\mathbb{D}^{(2) T} \mathbb{H}^{(1)} \boldsymbol{u}_{\mathrm{i}}^{(1)} & =\tilde{\mathbf{0}}^{(3)} . \tag{B.2c}
\end{align*}
$$

The forces are conservative, hence we have $\mathbb{D}^{(1) T} \tilde{\boldsymbol{f}}^{(1)}=\mathbb{D}^{(1)} \boldsymbol{f}^{(1)}=\mathbf{0}$.
By the product rule we have

$$
\frac{\partial H_{\mathrm{io}}}{\partial t}=\boldsymbol{\omega}_{\mathrm{o}}^{(1) T} \mathbb{H}^{(1)} \frac{\partial \boldsymbol{u}_{\mathrm{i}}^{(1)}}{\partial t}+\boldsymbol{u}_{\mathrm{i}}^{(1) T} \mathbb{H}^{(1)} \frac{\partial \boldsymbol{\omega}_{\mathrm{o}}^{(1)}}{\partial t}
$$

The first term cancels as a result of (B.2a), (B.1b), and, the fact that $\boldsymbol{\omega}_{\mathrm{o}}^{(1) T} \mathbb{N}_{\mathrm{o}}\left(\boldsymbol{\omega}_{\mathrm{i}}^{(2)}, \boldsymbol{u}_{\mathrm{o}}^{(2)}\right)=$ 0 , since we have

$$
\begin{aligned}
\boldsymbol{\omega}_{\mathrm{o}}^{(1) T} \mathbb{H}^{(1)} \frac{\partial \boldsymbol{u}_{\mathrm{i}}^{(1)}}{\partial t} & =-\rho^{-1} \boldsymbol{\omega}_{\mathrm{o}}^{(1) T} \mathbb{N}_{\mathrm{o}}\left(\boldsymbol{\omega}_{\mathrm{i}}^{(2)}, \boldsymbol{u}_{\mathrm{o}}^{(2)}\right)-\rho^{-1} \boldsymbol{\omega}_{\mathrm{o}}^{(1) T} \mathbb{H}^{(1)} \mathbb{D}^{(0)} \boldsymbol{q}^{(0)}+\rho^{-1} \boldsymbol{\omega}_{\mathrm{o}}^{(1) T} \mathbb{H}^{(1)} \boldsymbol{f}^{(1)} \\
& =\rho^{-1} \boldsymbol{u}_{\mathrm{o}}^{(2) T} \mathbb{H}^{(2)} \mathbb{D}^{(1)} \mathbb{D}^{(0)} \boldsymbol{q}^{(0)}+\rho^{-1} \boldsymbol{u}_{\mathrm{o}}^{(2) T} \mathbb{H}^{(2)} \mathbb{D}^{(1)} \boldsymbol{f}^{(1)} \\
& =0
\end{aligned}
$$

For the second term, we use (B.1a), (B.1b), and, (B.2b) to find

$$
\begin{aligned}
\boldsymbol{u}_{\mathrm{i}}^{(1) T} & \mathbb{H}^{(1)} \\
& \frac{\partial \boldsymbol{\omega}_{\mathrm{o}}^{(1)}}{\partial t} \\
& =\boldsymbol{u}_{\mathrm{i}}^{(1) T} \mathbb{D}^{(1) T} \mathbb{H}^{(2)} \frac{\partial \boldsymbol{u}_{\mathrm{o}}^{(2)}}{\partial t} \\
& =-\rho^{-1} \boldsymbol{\omega}_{\mathrm{i}}^{(2) T} \mathbb{N}_{\mathrm{i}}\left(\boldsymbol{\omega}_{\mathrm{o}}^{(1)}, \boldsymbol{u}_{\mathrm{i}}^{(1)}\right)+-\rho^{-1} \boldsymbol{u}_{\mathrm{i}}^{(1) T} \mathbb{D}^{(1) T} \mathbb{D}^{(2) T} \tilde{\boldsymbol{q}}^{(0)}+-\rho^{-1} \boldsymbol{u}_{\mathrm{i}}^{(1) T} \mathbb{D}^{(1) T} \tilde{\boldsymbol{f}}^{(1)} \\
& =-\rho^{-1} \boldsymbol{\omega}_{\mathrm{i}}^{(2) T} \mathbb{N}_{\mathrm{i}}\left(\boldsymbol{\omega}_{\mathrm{o}}^{(1)}, \boldsymbol{u}_{\mathrm{i}}^{(1)}\right) \\
& =0,
\end{aligned}
$$

where the last step again follows by definition of $\mathbb{N}_{\mathrm{i}}\left(\boldsymbol{\omega}_{\mathrm{O}}^{(1)}, \boldsymbol{u}_{\mathrm{i}}^{(1)}\right)$. Similarly, we have for $H_{\mathrm{oi}}$,

$$
\frac{\partial H_{\mathrm{oi}}}{\partial t}=\boldsymbol{\omega}_{\mathrm{i}}^{(2) T} \mathbb{H}^{(2)} \frac{\partial \boldsymbol{u}_{\mathrm{o}}^{(2)}}{\partial t}+\boldsymbol{u}_{\mathrm{o}}^{(2) T} \mathbb{H}^{(2)} \frac{\partial \boldsymbol{\omega}_{\mathrm{i}}^{(2)}}{\partial t}
$$

For the first term we find this time by (B.1a)

$$
\begin{aligned}
\boldsymbol{\omega}_{\mathrm{i}}^{(2) T} \mathbb{H}^{(2)} \frac{\partial \boldsymbol{u}_{\mathrm{o}}^{(2)}}{\partial t} & =-\rho^{-1} \boldsymbol{\omega}_{\mathrm{i}}^{(2) T} \mathbb{N}_{\mathrm{o}}\left(\boldsymbol{\omega}_{\mathrm{i}}^{(2)}, \boldsymbol{u}_{\mathrm{o}}^{(2)}\right)+\rho^{-1} \boldsymbol{\omega}_{\mathrm{i}}^{(2) T} \mathbb{D}^{(2) T} \tilde{\boldsymbol{q}}^{(0)}+\rho^{-1} \boldsymbol{\omega}_{\mathrm{i}}^{(2) T} \tilde{\boldsymbol{f}}^{(1)} \\
& =\rho^{-1} \boldsymbol{u}_{\mathrm{i}}^{(1) T} \mathbb{D}^{(1) T} \mathbb{D}^{(2) T} \tilde{\boldsymbol{q}}^{(0)}+\rho^{-1} \boldsymbol{u}_{\mathrm{i}}^{(1) T} \mathbb{D}^{(1) T} \tilde{\boldsymbol{f}}^{(1)} \\
& =0 .
\end{aligned}
$$

Finally, for the second term we have

$$
\begin{aligned}
\boldsymbol{u}_{\mathrm{o}}^{(2) T} \mathbb{H}^{(2)} \frac{\partial \boldsymbol{\omega}_{\mathrm{i}}^{(2)}}{\partial t}= & \boldsymbol{u}_{\mathrm{o}}^{(2) T} \mathbb{H}^{(2)} \mathbb{D}^{(1)} \frac{\partial u_{\mathrm{i}}^{(1)}}{\partial t} \\
= & -\rho^{-1} \boldsymbol{u}_{\mathrm{o}}^{(2) T} \mathbb{H}^{(2)} \mathbb{D}^{(1)}\left(\mathbb{H}^{(1)}\right)^{-1} \mathbb{N}_{\mathrm{i}}\left(\boldsymbol{\omega}_{\mathrm{o}}^{(1)}, \boldsymbol{u}_{\mathrm{i}}^{(1)}\right)-\rho^{-1} \boldsymbol{u}_{\mathrm{o}}^{(2) T} \mathbb{H}^{(2)} \mathbb{D}^{(1)} \mathbb{D}^{(0)} \boldsymbol{q}^{(0)} \\
& \quad+\rho^{-1} \boldsymbol{u}_{\mathrm{o}}^{(2) T} \mathbb{H}^{(2)} \mathbb{D}^{(1)} \boldsymbol{f}^{(1)} \\
= & -\rho^{-1} \boldsymbol{\omega}_{\mathrm{o}}^{(2) T} \mathbb{N}_{\mathrm{i}}\left(\boldsymbol{\omega}_{\mathrm{o}}^{(1)}, \boldsymbol{u}_{\mathrm{i}}^{(1)}\right) \\
= & 0
\end{aligned}
$$

This shows that both $H_{\text {io }}$ and $H_{\text {oi }}$ are conserved in time.
The conservation of $K_{\mathrm{ii}}$ and $K_{\mathrm{oo}}$, defined in (7.3), can be proved, as in Theorem 4.2.
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## Curriculum Vitae

René Beltman was born on 23 July 1991 in Zevenaar, the Netherlands. After finishing VWO in 2009 at the Liemers College in Zevenaar, he studied Physics and Mathematics at the Radboud University in Nijmegen. Upon finishing the double bachelor degree Physics and Mathematics in 2013, he moved on to start the Industrial and Applied Mathematics master program, with the specialization Computational Science and Engineering, at the Eindhoven University of Technology. In 2015 he graduated cum laude with a thesis on a numerical approach for solving the Monge-Ampère equation, an equation important for free-form reflector and lens design. The graduation research was carried out at Philips Lighting in Eindhoven.

In September 2015, he started a PhD project at the Eindhoven University of Technology, part of the NWO-TTW research program EUROS (Excellence in Uncertainty Reduction for Offshore Wind Systems) and under the supervision of prof.dr.ir Barry Koren and dr.ir. Martijn Anthonissen. The results obtained during this project are presented in this dissertation.

Since December 2019 he is employed at Signify (formerly known as Philips Lighting) in Eindhoven, where he is a member of the Optical Design group.

## List of Publications

Publications of research in this thesis:

- R. Beltman, M. Anthonissen, and B. Koren, "Compatible discretizations of divcurl problems and corresponding discrete Helmholtz-Hodge decompositions," in preparation.
- R. Beltman, "New mimetic discretizations based on discrete Hodge matrices interpolating from dual mesh to primal mesh," CASA Report 20-X, 2020.
- R. Beltman, M. Anthonissen, and B. Koren, "A locally refined cut-cell method with exact conservation for the incompressible Navier-Stokes equations," in 6th ECCOMAS European Conference on Computational Mechanics: Solids, Structures and Coupled Problems, ECCM 2018 and 7th ECCOMAS European Conference on Computational Fluid Dynamics, ECFD 2018, pp. 4087-4098, International Centre for Numerical Methods in Engineering, CIMNE, 2020.
- R. Beltman, M. Anthonissen, and B. Koren, "Conservative polytopal mimetic discretization of the incompressible Navier-Stokes equations," Journal of Computational and Applied Mathematics, vol. 340, pp. 443-473, 2018.
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- R. Beltman, M. J. H. Anthonissen, and B. Koren, "Mimetic staggered discretization of incompressible Navier-Stokes for barycentric dual mesh," in International Conference on Finite Volumes for Complex Applications, pp. 467-475, Springer, 2017.

Other publications:

- R. Beltman, J. ten Thije Boonkkamp, W. IJzerman, "A least-squares method for the inverse reflector problem in arbitrary orthogonal coordinates," in Journal of Computational Physics, vol. 367, pp. 347-373, 2018.
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## Summary

## Mimetic Discretizations of the Incompressible Navier-Stokes Equations for Polyhedral Meshes

The main method of simulating turbulent flows is by solving the Navier-Stokes equations. Due to the highly turbulent nature and the high Reynolds number of many flows, the full range of length scales involved cannot often be resolved by direct numerical simulation (DNS). A turbulence model may be used then to model the effect of the unresolved small scales on the flow. State-of-the-art turbulence modeling is provided by Large Eddy Simulation (LES).

Flow simulations by DNS and LES require a careful approach when discretizing the Navier-Stokes equations. For instance, upwind discretizations of the convective term often result in stable numerical methods but also introduce numerical dissipation. This unphysical numerical energy dissipation can have a detrimental effect on the simulation as it can interfere with the modeled turbulent dissipation or even overwhelm it. A solution to this problem is provided by energy-conserving discretizations. An energy-conserving discretization is nonlinearly stable and does not introduce numerical dissipation.

Even for an LES simulation the computational requirements are often beyond the available resources and, as a result, the computational mesh used is often still too coarse to resolve all flow features. Simulations involving a time-dependent geometry require even more computational resources because the mesh has to be adapted to account for the geometry change. An example of such a highly turbulent flow with time-dependent geometry is the flow through a wind farm. To avoid the extra computational cost of adapting the mesh every time step to the moving turbine blades, the interaction between the turbine blades and flow is often modeled through an extra force term in the momentum equation. This allows for the use of a simple Cartesian mesh.

In this thesis the energy-conserving Marker-and-Cell (MAC) method is extended to cut-cell meshes. A cut-cell mesh is a Cartesian mesh that is locally adjusted to conform to the geometry of the flow domain (e.g. involving the turbine blades and turbine towers in a wind farm). This approach allows for a direct modeling of the geometry by the mesh (not through an extra force term), while maintaining the computational efficiency of the Cartesian mesh.

In Chapters 2, 3 and 4 of this thesis it is shown how the recent developments of mimetic discretization methods can be used to extend the MAC discretization to the polyhedral cells that occur in a cut-cell mesh. In Chapters 2 and 3 the necessary concepts from exterior calculus and mimetic discretization methods are collected. In Chapter 4
these concepts are used to define two new extensions of the MAC method to meshes with polyhedral cells, one with velocity variables located on the faces of the mesh and the other with velocity variables located on the edges of the mesh. A new discretization of the convection term in rotational form is introduced, which leads to a significantly smaller discretization stencil. Both extensions of the MAC method are demonstrated to be energy-conserving but only the method using the velocity variables located on the mesh faces, together with a discretization of the convection term in divergence form, is also momentum-conserving.

Chapters 5 and 6 of this thesis deal with topics closely related to mimetic discretization methods from the preceding chapters, but not directly with the cut-cell methods of Chapter 4. In Chapter 5 we show how the velocity field can be found when the vorticity field is known. Using the mimetic discretization techniques introduced in the foregoing chapters, it is shown how a discrete Helmholtz-Hodge decomposition can be calculated and how this can be used to determine the velocity field from the vorticity field. The Helmholtz-Hodge decomposition plays an important role in projection methods for the time integration of the Navier-Stokes equations.

Chapter 6 is concerned with mimetic methods on simplicial meshes. Mimetic discretizations using a circumcentric dual mesh allow for an explicit interpolation from the dual mesh to the primal mesh. This avoids the introduction of additional degrees of freedom that are needed when such an explicit interpolation from dual to primal mesh is not available. The use of the circumcentric dual mesh imposes stringent conditions on the simplicial primal mesh. If these conditions are not met the interpolation matrix loses its positive definiteness and this jeopardizes the method's stability. In Chapter 6 we show that explicit interpolation matrices can also be derived when the barycentric dual mesh is used. This relieves the stringent condition on the regularity of the simplicial primal mesh but also results in a discretization with a wider stencil.


[^0]:    ${ }^{1}$ Compressibility effects can be neglected everywhere except near the tips of the turbine blades. In this thesis the flow is assumed to be incompressible. However, with the increasing turbine size and hence increasing tip speed the compressibility effects get more significant.
    ${ }^{2}$ The Reynolds number, determining the ratio between the largest and smallest turbulent scales, for the flow through a wind farm is if the order $10^{8}$ [3].

[^1]:    ${ }^{3}$ This is most easily seen when written in components: $u_{j} \partial_{i}\left(u_{j}\left(\rho u_{i}\right)\right)=\frac{1}{2} \rho \partial_{j}\left(u_{i} u_{i} u_{j}\right)$ (sum over repeated indices). Here we used the incompressibility, i.e., $\partial_{i} u_{i}=0$.

[^2]:    ${ }^{1}$ Outer-oriented $k$-vectors are also sometimes called twisted $k$-vectors [33], pseudo $k$-vectors, or axial $k$-vectors. Similarly, inner $k$-vectors are also sometimes called normal $k$-vectors or polar $k$-vectors.

[^3]:    ${ }^{2}$ We call a $k$-vector simple when it can be written as an exterior product of $k$ vectors.

[^4]:    ${ }^{3}$ This is independent of the parametrization. See, for example, [36, Theorem 1.46].

[^5]:    ${ }^{4}$ A map $\phi: M_{(m)} \rightarrow N_{(n)}$ between two manifolds is smooth if for any coordinate system $\phi_{M_{(m)}}$ on $M_{(m)}$ and $\phi_{N_{(n)}}$ on $N_{(n)}$, the map $\phi_{N_{(n)}} \circ \phi \circ \phi_{M_{(m)}}^{-1}$ is a smooth map from an open subset of $\mathbb{R}^{m}$ to an open subset of $\mathbb{R}^{n}$ whenever it is defined.

[^6]:    ${ }^{5}$ Here we assume that $M_{(k)}$ is parametrized by a single map $\phi$, however, this is often not the case. If multiple maps $\phi_{i}: U_{i} \rightarrow M_{(k)}$ are involved $\left(\phi_{i}\left(U_{i}\right)\right.$ are a finite open cover of the support of $\left.a^{(k)}\right), a^{(k)}$ is written as a sum $\sum_{i} f_{i} a^{(k)}$, where the maps $f_{i}$ are a partition of unity such that $f_{i} a^{(k)}$ is compactly supported on $\phi_{i}\left(U_{i}\right)$. The integral is the defined as $\int_{M_{(k)}} a^{(k)}=\sum_{i} \int_{U_{i}} \phi_{i}^{*}\left(f_{i} a^{(k)}\right)$.

[^7]:    ${ }^{6}$ Outer orientation is also known as transverse orientation [39].

[^8]:    ${ }^{7}$ Formally, we should write $\left\langle M_{(1)}, t_{M_{(1)}}^{(1)} u^{(1)}\right\rangle$, where $t_{M_{(1)}}^{(1)}: \Lambda^{(1)}\left(\Omega_{(3)}\right) \rightarrow \Lambda^{(1)}\left(M_{(1)}\right)$ is the trace operator on $M_{(1)}$. However, for simplicity we often omit this trace when it is clear how the expression should be interpreted.

[^9]:    ${ }^{8}$ For the general case, write $a^{(m-1)}$ as a sum using a partition of unity such that each term of the sum is supported in one coordinate chart only. See [36].

[^10]:    ${ }^{9}$ It can be shown algebraically as follows. For a monomial $k$-form we have $d d\left(a d x^{i_{1}} \wedge \cdots \wedge d x^{i_{k}}\right)=$ $\sum_{j=1}^{d} \sum_{k=1}^{d}\left(\partial_{k} \partial_{j} a\right) d x^{k} \wedge d x^{j} \wedge d x^{i_{1}} \wedge \cdots \wedge d x^{i_{k}}=\sum_{j<k}\left(\partial_{k} \partial_{j} a-\partial_{j} \partial_{k} a\right) d x^{k} \wedge d x^{j} \wedge d x^{i_{1}} \wedge \cdots \wedge d x^{i_{k}}=0$. The general result for any $k$-form follows by linearity.

[^11]:    ${ }^{10}$ We write from now on simply $\Omega$ for our $d$-dimensional domain $\Omega_{(d)}$.

[^12]:    ${ }^{11}$ The Hodge star operators extend naturally to operators $\star^{(k)}: L^{2} \Lambda^{(k)}(\Omega) \rightarrow L^{2} \tilde{\Lambda}^{(d-k)}(\Omega)$ and $\tilde{\star}^{(k)}: L^{2} \tilde{\Lambda}^{(k)}(\Omega) \rightarrow L^{2} \Lambda^{(d-k)}(\Omega)$.

[^13]:    ${ }^{12}$ The notation is also partly taken from [45].

[^14]:    ${ }^{13}$ The (unbounded) operator $A^{(k)}: V^{(k)} \subset W^{(k)} \rightarrow W^{(k+1)}$ is called closed if its graph $\left\{\left(v, A^{(k)} v\right) \mid v \in\right.$ $\left.V^{(k)}\right\}$ is closed in $W^{(k)} \times W^{(k+1)}$.
    ${ }^{14}$ I.e., $V^{(k)}$ is dense in $W^{(k)}$.

[^15]:    ${ }^{15}$ A manifold $\Omega$ is called smoothly contractible to a point $x \in \Omega$ if there is a smooth function $H$ : $\Omega \times[0,1] \rightarrow \Omega$ such that $H(y, 1)=y$ and $H(y, 0)=x$ for any $y \in \Omega$.

[^16]:    ${ }^{1}$ For this formula it does not matter what center is taken, as long as the center of $\tau_{(l)}$ lies somewhere in the plane of $\tau_{(l)}$. The barycenter can be used, but any other point can be used as well. It does not matter if $x_{(1)}^{\tau_{(l)}}$ does not lie in $\tau_{(l)}$, the formula still holds. However, it is of course crucial that $x_{(1)}^{\tau_{(0)}}$ is the location of $\tau_{(0)}$. (The point $x_{(1)}^{\tau_{(0)}}$ can lie anywhere in the plane of $\tau_{(0)}$, which is 0 -dimensional and therefore leaves only one option.)

[^17]:    ${ }^{2}$ An abelian group is called free if it has a basis, i.e., a generating set of linearly independent elements [41].

[^18]:    ${ }^{3}$ For simplicity we assume in this section that the de Rham maps have the spaces of smooth differential forms as their domain.

[^19]:    ${ }^{4}$ The barycenter is also known as the centroid or geometric center.
    ${ }^{5}$ The circumcenter of a cell $\sigma_{(l)}$ is given by the center of the $l$-dimensional sphere that precisely intersects all the corners/vertices $\sigma_{(l)}$. Note that such a sphere, and therefore the circumcenter, in general, only exists for simplices or regular polytopes.

[^20]:    ${ }^{6} \mathrm{~A}$ mesh is called well-centered if for every mesh cell $\sigma_{(l)}$ the circumcenter $x_{(1)}^{\sigma_{(l)}}$ lies in $\sigma_{(l)}$. The formula for $\sharp \sigma_{(k)}$ is, in the circumcentric case, only correct when the primal mesh is well-centered. If the primal mesh is not well-centered but circumcentra do exist, then some of the terms in the union for $\hbar \sigma_{(k)}$ should be interpreted with "negative sign". Further on in (3.5) we define the dual cell vector that takes this into account and is therefore not restricted to well-centered primal meshes. This can be considered as the proper definition of the circumcentric dual cell.

[^21]:    ${ }^{7}$ If for every $d$-cell the corresponding circumscribed ( $d-1$ )-sphere only contains the circumcenter of the $d$-cell itself then the mesh is known as Delaunay.

[^22]:    ${ }^{8}$ It does not necessarily have to be normal to $\partial \Omega$, it should be transverse in the outward direction.

[^23]:    ${ }^{9}$ We take here $\beta$ such that it is equal to $\beta$ as used in [54]. That is the reason it appears squared in this definition.

[^24]:    ${ }^{10}$ If the mesh is well-centered the ratio $d\left(\star^{\star} \sigma_{(d)} \sigma_{(l)}, \sharp^{\star} \sigma_{(d)} \sigma_{(l)}\right) /\left(\star \sigma_{(l)}, \star^{\star} \sigma_{(d)} \sigma_{(l)}\right)$ is always positive.

[^25]:    ${ }^{11}$ The ratio of the inscribed sphere $r_{\Delta}$ of each subsimplex $\Delta$ and the diameter $h_{\Delta}$ should be bounded from below, i.e., $r_{\Delta} / h_{\Delta} \geq \rho>0$ should hold for the subsimplices in every $d$-dimensional mesh cell with $\rho$ fixed [56]. If a sequence of meshes is considered, then $\rho$ should be the same for the complete sequence.

[^26]:    ${ }^{1}$ Note that the discretization of the boundary conditions for these two formulations will be different as a result. For example, while the normal velocity is an essential boundary condition for the discretization of (2.21), the tangential velocity is an essential boundary condition for the discretization of (2.22), and vice versa.
    ${ }^{2}$ We call this the CDO-SUSHI Hodge matrix because it is related to the stabilization used in the SUSHI scheme discussed in [55]. Note that in this paper the dual Hodge matrix $\tilde{\mathbb{H}}_{\mathrm{b}}^{(1)}$ with stability term given by (3.17) with $\beta=1 / \sqrt{d}$ is used in a hybrid approach.

[^27]:    ${ }^{3}$ See the requirements in Proposition 3.13.
    ${ }^{4}$ The only element of the kernel should be a constant pressure field.

[^28]:    ${ }^{5}$ Note that the vorticity $\boldsymbol{\omega}^{(2)}$ can be computed from $\boldsymbol{u}^{(1)}$ explicitly (without solving a linear system). This discrete variable and (4.12b) are therefore not needed in the linear system.

[^29]:    $\sigma_{\star} \tilde{\varkappa}^{(2)}$ is represented by the identity matrix in Cartesian components.
    ${ }^{7}$ Here $\mathbb{T}_{\sigma_{(3)}}^{(2)}$ is the matrix such that $\boldsymbol{u}_{\sigma_{(3)}}^{(2)}=\mathbb{T}_{\sigma_{(3)}}^{(2)} \boldsymbol{u}^{(2)}$, i.e., it restricts the discrete velocity vector to the faces of $\sigma_{(3)}$.

[^30]:    ${ }^{8}$ This is not the same approximation as used in Section 4.2.2 to calculate the pressure, because the formula for $k_{i, j}^{(0)}$ used here holds for a Cartesian mesh only.

[^31]:    ${ }^{9}$ For example, $e_{x}^{(1)}:=d x$.

[^32]:    ${ }^{10}$ For simplicity we assume $\left\{\sigma_{(3)}\right\}$ has a right handed orientation for all cells $\sigma_{(3)} \in C_{(3)}(\Omega)$.

[^33]:    ${ }^{11}$ Due to computer memory limitations we were not able to perform the test for the finest mesh in the PH mesh sequence that we considered in Section 4.1.2.

[^34]:    ${ }^{12}$ There is of course a very small time interval in which the airfoil accelerates. For details see [90].

[^35]:    ${ }^{13}$ The PIV pictures are adaptations of the originals which can be found in [90]. For better comparison we mirrored the PIV images and inserted the shape of the airfoil.

[^36]:    ${ }^{1}$ The equations can be generalized to the anisotropic case by including a permittivity or permeability tensor.

[^37]:    ${ }^{2}$ The inner products are defined in Definition 2.27 and (2.8). Note that the required relation between $f^{(k-1)}$ and $r^{(k-1)}$ is simply a consistency requirement that follows from $\left(f^{(k-1)}, z^{(k-1)}\right)=\left(d^{*} u^{(k)}, z\right)=$ $-\left(t^{*} u^{(k)}, t z^{(k-1)}\right)_{\mathrm{b}}=-\left(r_{\mathrm{b}}^{(k-1)}, t z^{(k-1)}\right)_{\mathrm{b}}$ by (2.9). That is $f^{(k-1)}$ and $r_{\mathrm{b}}^{(k-1)}$ are such that they are consistent with (2.9). The same holds for $g^{(k+1)}$ and $r_{\mathrm{b}}^{(k)}$.

[^38]:    ${ }^{3}$ The boundary term is contained in $F(v)$.

[^39]:    ${ }^{4}$ In the case of a compatible discretization method without dual mesh $\tilde{\boldsymbol{b}}^{(d-k-1)}=\mathbb{H}^{(k+1)} \boldsymbol{b}^{(k+1)}$, with $\boldsymbol{b}^{(k+1)} \in C^{(k+1)}(\Omega)$.
    ${ }^{5}$ We assume in the rest of this chapter that the data is such that the De Rham maps make sense. In our numerical tests they will be smooth.

[^40]:    ${ }^{6}$ Here the first equation should be considered with the zero rows (the rows corresponding to the boundary cells) removed.

[^41]:    ${ }^{7}$ See Section 2.3.4.

[^42]:    ${ }^{8}$ Section 3.3.1.
    ${ }^{9}$ Section 3.3.2.

[^43]:    ${ }^{10}$ This is true because the proof of these relations hinges on, for example when considering $\left(\mathrm{DP}_{t}^{(1)}\right)$, the equalities $\mathbb{D}^{(2)} \boldsymbol{\omega}^{(2)}=\mathbf{0}$ and $\mathbb{T}^{(2)} \boldsymbol{\omega}^{(2)}=\mathbb{D}_{\mathrm{b}}^{(1)} \boldsymbol{r}_{\mathrm{b}}^{(1)}$. These equalities are in turn satisfied up to the accuracy of the de Rham maps used in determining $\boldsymbol{\omega}^{(2)}$ and $\boldsymbol{r}_{\mathrm{b}}^{(1)}$.

[^44]:    ${ }^{1}$ A simplex is said to be Delaunay if its smallest circumscribed sphere does not contain any other vertices of the mesh which are not part of the simplex. A mesh that consists of Delaunay simplices is called a Delaunay mesh.

[^45]:    ${ }^{2}$ We use square brackets and accolades as alternative parentheses to increase readability.

[^46]:    ${ }^{3}$ One way to do this is by choosing an orientation for the ambient space $\mathbb{R}^{d}$, say $\left\{o_{(d)}\right\}$, and then transforming the orientation of the outer-oriented and inner-oriented cells according to $\left\{o_{(d)}\right\} o_{(k)} \mapsto o_{(k)}$ and $o_{(k)} \mapsto\left\{o_{(d)}\right\} o_{(k)}$, respectively.
    ${ }^{4}$ Vice versa, if we would have considered essential boundary conditions instead, then these would be the natural boundary conditions for (6.20) and (6.21).

[^47]:    ${ }^{1}$ The research reported in [122] was carried out in collaboration with the author of this thesis.

